On Sugeno Integral as an Aggregation Function

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Revised version

Abstract

The Sugeno integral, for a given fuzzy measure, is studied under the viewpoint of aggregation. In particular, we give some equivalent expressions of it. We also give an axiomatic characterization of the class of all the Sugeno integrals. Some particular subclasses, such as the weighted maximum and minimum functions are investigated as well.

Keywords: fuzzy measures; Sugeno integral; aggregation functions; multicriteria decision making; pseudo-Boolean functions; max-min algebra; ordinal scales.

1 Introduction

Aggregation refers to the process of combining numerical values $x_1, \ldots, x_m$ into a single one $M^{(m)}(x_1, \ldots, x_m)$, so that the final result of aggregation takes into account all the individual values. In decision making, values to be aggregated are typically preference or satisfaction degrees and thus belong to the unit interval $[0, 1]$. For more details, see [9].

This paper aims at investigating the Sugeno integral (see [18, 19]) which can be regarded as an aggregation function (see Section 2). In particular, we show that any Sugeno integral is a weighted max-min function, that is, setting $X = \{1, \ldots, m\}$, a function of the form

$$M^{(m)}(x_1, \ldots, x_m) = \bigvee_{T\subseteq X} [a_T \wedge (\bigwedge_{i\in T} x_i)], \quad a_T \in [0, 1],$$

where a is a set function satisfying $a_\emptyset = 0$ and $\bigvee_{T\subseteq X} a_T = 1$. Such functions are investigated in Section 3. We then show that those functions can also be written as

$$M^{(m)}(x_1, \ldots, x_m) = \bigwedge_{T\subseteq X} [b_T \vee (\bigvee_{i\in T} x_i)], \quad b_T \in [0, 1],$$

(weighted min-max functions) where b is a set function satisfying $b_\emptyset = 1$ and $\bigwedge_{T\subseteq X} b_T = 0$. The correspondance formulae $b = b(a)$ and $a = a(b)$ are given as well. For instance, we have

$$(0.1 \wedge x_1) \vee (0.3 \wedge x_2) \vee (x_2 \wedge x_3) = (0.1 \vee x_2) \wedge (0.3 \vee x_3) \wedge (x_1 \vee x_2).$$

We also propose an axiomatic characterization of this class of functions based on some aggregation properties: the increasingness and the stability for minimum and maximum with the same unit.

Most of these results are applied to the Sugeno integral in Section 4. In particular, we can derive equivalent expressions and characterize the family of all the Sugeno integrals.

In Section 5, we consider particular weighted max-min functions: Boolean max-min functions, weighted maximum and minimum functions, ordered weighted maximum and minimum functions, partial maximum and minimum functions, order statistics and associative medians. Of course, all these functions are Sugeno integrals.

2 The Sugeno integral as an aggregation function

We first want to define the concept of aggregation function. Without loss of generality, we will assume that the information to be aggregated consists of numbers belonging to the interval $[0, 1]$ as required in most applications. In fact, all the definitions and results presented in this paper can be defined on any closed interval $[a, b]$ of the real line.

Let m denote any strictly positive integer.

Definition 2.1 *An* aggregation function *defined on* $[0,1]^m$ *is a function* $M^{(m)} : [0,1]^m \to \mathbb{R}$.

We consider a discrete set of m elements $X = \{1, \ldots, m\}$, which could be players of a co-operative game, criteria, attributes or voters in a decision making problem. $\mathcal{P}(X)$ indicates the power set of X, i.e. the set of all subsets in X.

In order to avoid heavy notations, we introduce the following terminology. It will be used all along this paper.

- We set $\mathbb{B} := \{0, 1\}$ and $\mathbb{I} := [0, 1]$.
- For all $T \subseteq X$, the characteristic vector of T in $\mathbb{B}^m$ is defined by

  $$e_T := (x_1, \ldots, x_m) \in \mathbb{B}^m \text{ with } x_i = 1 \Leftrightarrow i \in T.$$

  Of course, the e_T's ($T \subseteq X$) are the 2^m vertices of the hypercube $\mathbb{I}^m$. Then we set $\theta_T := M^{(m)}(e_T)$. The expressions $e_{\{i\}}$ and $\theta_{\{i\}}$ will be denoted e_i and θ_i respectively.
- Given a vector $(x_1, \ldots, x_m) \in \mathbb{I}^m$, let $(\cdot)$ be the permutation on X which arranges the elements of this vector by increasing values: that is, $x_{(1)} \leq \ldots \leq x_{(m)}$.
- The notation $K \subsetneq T$ means $K \subset T$ and $K \neq T$.

In order to define the Sugeno integral, we use the concept of *fuzzy measure*.

Definition 2.2 *A* (discrete) fuzzy measure *on* X *is a set function* $\mu : \mathcal{P}(X) \to \mathbb{I}$ *satisfying the following conditions:*
 (i) $\mu(\emptyset) = 0, \mu(X) = 1,$
 (ii) $R \subseteq S \subseteq X \Rightarrow \mu(R) \leq \mu(S).$

$\mu(R)$ can be viewed as the weight of importance of the set of elements R. In the sequel we will write μ_R instead of $\mu(R)$.

Definition 2.3 *A* pseudo-Boolean function *is a function* $f : \mathbb{B}^m \to \mathbb{R}$.

Hammer and Rudeanu [14] showed that any pseudo-Boolean function can be put under a multilinear polynomial in m variables:

$$f(x) = \sum_{T \subseteq X} a_T \prod_{i \in T} x_i$$

with $a_T \in \mathbb{R}$ and $x = (x_1, \ldots, x_m) \in \mathbb{B}^m$. It is easy to see that a fuzzy measure is a particular case of pseudo-Boolean function: simply remark that for any $R \subseteq X$, R is equivalent to the point $e_R \in \mathbb{B}^m$. We then have,

$$\mu_R = f(e_R) = \sum_{T \subseteq R} a_T \quad \forall R \subseteq X.$$

Now, let us introduce the concept of $\mathbb{I}$-valued pseudo-Boolean function as follows:

Definition 2.4 *An* $\mathbb{I}$-valued pseudo-Boolean function *is a function* $f : \mathbb{B}^m \to \mathbb{I}$. *It is said to be* increasing *if* f *is increasing in each argument.*

It is easy to see that any increasing $\mathbb{I}$-valued pseudo-Boolean function f fulfilling $f(0, \ldots, 0) = 0$ and $f(1, \ldots, 1) = 1$ can be put under the following forms:

$$f(x) = \bigvee_{T \subseteq X} [a_T \wedge (\bigwedge_{i \in T} x_i)] = \bigwedge_{T \subseteq X} [b_T \vee (\bigvee_{i \in T} x_i)]$$

with, for instance, $a_T = f(e_T) \in \mathbb{I}$ and $b_T = f(e_{X \setminus T}) \in \mathbb{I}$ for all $T \subseteq X$. Indeed, we then have, for all $R \subseteq X$,

$$f(e_R) = \bigvee_{T \subseteq R} a_T = \bigvee_{T \subseteq R} f(e_T)$$

and

$$f(e_R) = \bigwedge_{T \subseteq X \setminus R} b_T = \bigwedge_{T \subseteq X \setminus R} f(e_{X \setminus R}).$$

Any fuzzy measure can be regarded as an increasing $\mathbb{I}$-valued pseudo-Boolean function for which $f(0, \ldots, 0) = 0$ and $f(1, \ldots, 1) = 1$. Conversely, any increasing $\mathbb{I}$-valued pseudo-Boolean function f satisfying these boundary conditions define a fuzzy measure:

$$\mu_R = f(e_R) = \bigvee_{T \subseteq R} a_T = \bigwedge_{T \subseteq X \setminus R} b_T \quad \forall R \subseteq X.$$

We introduce now the concept of discrete Sugeno integral, viewed as an aggregation function. For this reason, we will adopt a connective-like notation instead of the usual integral form, and the integrand will be a set of m values $x_1, \ldots, x_m$ of $\mathbb{I}$. For theorical developments, see [13, 18, 19].

Definition 2.5 *Let* $(x_1, \ldots, x_m) \in \mathbb{I}^m$, *and* μ *a fuzzy measure on* X. *The* (discrete) Sugeno integral of $(x_1, \ldots, x_m)$ with respect to μ *is defined by*

$$\mathcal{S}_\mu^{(m)}(x_1, \ldots, x_m) := \bigvee_{i=1}^{m} [x_{(i)} \wedge \mu_{\{(i), \ldots, (m)\}}].$$

For instance, if $x_3 \leq x_1 \leq x_2$, we have

$$\mathcal{S}_\mu^{(3)}(x_1, x_2, x_3) = (x_3 \wedge \mu_{\{3,1,2\}}) \vee (x_1 \wedge \mu_{\{1,2\}}) \vee (x_2 \wedge \mu_{\{2\}}).$$

Of course, given a fuzzy measure μ, the Sugeno integral $\mathcal{S}_\mu^{(m)}$ can be regarded as an aggregation function defined on $\mathbb{I}^m$. We will show that it can be written in the form of a *weighted max-min function* to be introduced next.

3 Weighted max-min and min-max functions

This section is devoted to weighted max-min and min-max functions. Although the coefficients involved in these functions are not really weights, but rather thresholds or aspiration degrees, we will talk in terms of weights. It is shown that any weighted max-min function is a weighted min-max function and conversely.

The formal analogy between the weighted max-min function and the multilinear polynomial is obvious: minimum corresponds to product, maximum does to sum. Moreover, it is emphasized that weighted max-min functions can be calculated as medians, i.e., the qualitative counterparts of multilinear polynomials.

Finally, we give an axiomatic characterization of the family of weighted max-min functions.

3.1 Weighted max-min functions

Definition 3.1 *For any set function $a : \mathcal{P}(X) \to \mathbb{I}$ such that $a_\emptyset = 0$ and $\bigvee_{T\subseteq X} a_T = 1$, the* weighted max-min *aggregation function $WMAXMIN_a^{(m)}$ associated to a is defined by*

$$WMAXMIN_a^{(m)}(x_1,\ldots,x_m) = \bigvee_{T\subseteq X} [a_T \wedge (\bigwedge_{i\in T} x_i)] \quad \forall (x_1,\ldots,x_m)\in \mathbb{I}^m.$$

Observe first that for any $\mathrm{WMAXMIN}_a^{(m)}$, we have

$$\theta_R = \bigvee_{T\subseteq R} a_T \quad \forall R\subseteq X.$$

Moreover, the set function a which define $\mathrm{WMAXMIN}_a^{(m)}$ is not uniquely determined: indeed, we have, for instance, $x_1 \vee (x_1\wedge x_2) = x_1$. The next proposition precises conditions under which two weighted max-min functions are identical.

Proposition 3.1 *Let a and a' be set functions defining $WMAXMIN_a^{(m)}$ and $WMAXMIN_{a'}^{(m)}$ respectively. Then the following four assertions are equivalent:*

$$
\begin{array}{ll}
(i) & WMAXMIN_{a'}^{(m)} = WMAXMIN_a^{(m)} \\
(ii) & \forall T\subseteq X : \bigvee_{K\subseteq T} a'_K = \bigvee_{K\subseteq T} a_K \\
(iii) & \forall T\subseteq X, T\neq\emptyset : \begin{cases} a'_T = a_T & \text{if } a_T > \bigvee_{K\subsetneq T} a_K \\ 0\le a'_T \le \bigvee_{K\subseteq T} a_K & \text{otherwise} \end{cases} \\
(iv) & \forall T\subseteq X, T\neq\emptyset : \inf\{z \mid (\bigvee_{K\subsetneq T} a_K)\vee z \ge a_T\} \le a'_T \le \bigvee_{K\subseteq T} a_K.
\end{array}
$$

Proof. $(i)\Rightarrow(ii)$. We simply have, for all $T\subseteq X$,

$$\bigvee_{K\subseteq T} a'_K = \theta_T = \bigvee_{K\subseteq T} a_K.$$

$(ii)\Rightarrow(iii)$. Let $T\subseteq X, T\neq\emptyset$. On the one hand, we have

$$0\le a'_T \le \bigvee_{K\subseteq T} a'_K = \bigvee_{K\subseteq T} a_K.$$

On the other hand, assuming that $a_T > \bigvee_{K\subsetneq T} a_K$, we obtain

$$a_T = \bigvee_{K\subseteq T} a_K = \bigvee_{K\subseteq T} a'_K$$

implying $a_T = a'_T$: indeed, otherwise there would exist $K^* \subsetneq T$ such that

$$a_T = a'_{K^*} \leq \bigvee_{L\subseteq K^*} a'_L = \bigvee_{L\subseteq K^*} a_L \leq \bigvee_{K\subsetneq T} a_K < a_T$$

which is a contradiction.

$(iii) \Rightarrow (i)$. Assume $a_T \leq \bigvee_{K\subsetneq T} a_K$ and let $K^* \subsetneq T$ such that a_{K^*} is maximum. Then we have $a_{K^*} \geq a_T$ and

$$[a_{K^*} \wedge (\bigwedge_{i\in K^*} x_i)] \geq [a_T \wedge (\bigwedge_{i\in T} x_i)]$$

and so a_T can be replaced by any number lying between 0 and $a_{K^*} = \bigvee_{K\subseteq T} a_K$ without changing $\mathrm{WMAXMIN}_a^{(m)}$.

$(iii) \Leftrightarrow (iv)$. Trivial. ■

Let a be any set function defining $\mathrm{WMAXMIN}_a^{(m)}$. By the third assertion of the previous proposition, each a_T is either uniquely determined or can lie in a closed interval. If a is such that

$$\forall T \subseteq X, T \neq \emptyset : \; a_T = 0 \Leftrightarrow a_T \leq \bigvee_{K\subsetneq T} a_K$$

then the a_T's are the smallest and we say that $\mathrm{WMAXMIN}_a^{(m)}$ is put in its *canonical* form. On the other hand, if a is such that

$$\forall T \subseteq X : \; a_T = \bigvee_{K\subseteq T} a_K$$

then the a_T's are the largest and we say that $\mathrm{WMAXMIN}_a^{(m)}$ is put in its *complete* form. In this case, a is a fuzzy measure since it is increasing (by inclusion).

It should be noted that we can determine the complete form of any function $\mathrm{WMAXMIN}_a^{(m)}$ by taking $a_T = \theta_T$ for all $T \subseteq X$. We then get its canonical form by considering successively the T's in the decreasing cardinality order and setting $a_T = 0$ whenever $T \neq \emptyset$ and

$$a_T \leq \bigvee_{k\in T} a_{T\setminus\{k\}}.$$

3.2 Weighted min-max functions

By exchanging the position of the max and min operations in Definition 3.1, we can define the weighted min-max functions as follows.

Definition 3.2 *For any set function $b : \mathcal{P}(X) \to \mathbb{I}$ such that $b_\emptyset = 1$ and $\bigwedge_{T\subseteq X} b_T = 0$, the* weighted min-max *aggregation function $WMINMAX_b^{(m)}$ associated to b is defined by*

$$WMINMAX_b^{(m)}(x_1, \ldots, x_m) = \bigwedge_{T\subseteq X} [b_T \vee (\bigvee_{i\in T} x_i)] \;\; \forall (x_1, \ldots, x_m) \in \mathbb{I}^m.$$

Observe first that for any $\text{WMINMAX}_b^{(m)}$, we have

$$\theta_R = \bigwedge_{T\subseteq X\setminus R} b_T \quad \forall R \subseteq X.$$

Moreover, the set function b which define $\text{WMINMAX}_b^{(m)}$ is not uniquely determined: indeed, we have, for instance, $x_1 \wedge (x_1 \vee x_2) = x_1$. We then have a result similar to Proposition 3.1.

Proposition 3.2 *Let b and b' be set functions defining $WMINMAX_b^{(m)}$ and $WMINMAX_{b'}^{(m)}$ respectively. Then the following four assertions are equivalent:*

$$\begin{array}{ll}
(i) & WMINMAX_{b'}^{(m)} = WMINMAX_b^{(m)} \\
(ii) & \forall T \subseteq X: \bigwedge_{K\subseteq T} b'_K = \bigwedge_{K\subseteq T} b_K \\
(iii) & \forall T \subseteq X, T \neq \emptyset: \begin{cases} b'_T = b_T & \text{if } b_T < \bigwedge_{K\subsetneq T} b_K \\ \bigwedge_{K\subseteq T} b_K \leq b'_T \leq 1 & \text{otherwise} \end{cases} \\
(iv) & \forall T \subseteq X, T \neq \emptyset: \bigwedge_{K\subseteq T} b_K \leq b'_T \leq \sup\{z | (\bigwedge_{K\subsetneq T} b_K) \wedge z \leq b_T\}.
\end{array}$$

Let b be any set function defining $\text{WMINMAX}_b^{(m)}$. By the third assertion of the previous proposition, each b_T is either uniquely determined or can lie in a closed interval. If b is such that

$$\forall T \subseteq X, T \neq \emptyset: \; b_T = 1 \Leftrightarrow b_T \geq \bigwedge_{K\subsetneq T} b_K$$

then the b_T's are the largest and we say that $\text{WMINMAX}_b^{(m)}$ is put in its *canonical* form. On the other hand, if b is such that

$$\forall T \subseteq X: \; b_T = \bigwedge_{K\subseteq T} b_K$$

then the b_T's are the smallest and we say that $\text{WMINMAX}_b^{(m)}$ is put in its *complete* form. In this case, b is decreasing (by inclusion).

It should be noted that we can determine the complete form of any function $\text{WMINMAX}_b^{(m)}$ by taking $b_T = \theta_{X\setminus T}$ for all $T \subseteq X$. We then get its canonical form by considering successively the T's in the decreasing cardinality order and setting $b_T = 1$ whenever $T \neq \emptyset$ and

$$b_T \geq \bigwedge_{k\in T} b_{T\setminus\{k\}}.$$

3.3 Correspondance formulae and equivalent forms

As announced at the beginning of this section, any weighted max-min function can be put under the form of a weighted min-max function and conversely. The next proposition gives the correspondance formulae.

Proposition 3.3 *Let a and b be set functions defining $WMAXMIN_a^{(m)}$ and $WMINMAX_b^{(m)}$ respectively. Then we have*

$$WMAXMIN_a^{(m)} = WMINMAX_b^{(m)} \Leftrightarrow \bigvee_{K\subseteq T} a_K = \bigwedge_{K\subseteq X\setminus T} b_K \quad \forall T \subseteq X.$$

Proof. (Necessity) We simply have, for all $T \subseteq X$,

$$\bigvee_{K\subseteq T} a_K = \theta_T = \bigwedge_{K\subseteq X\setminus T} b_K.$$

(Sufficiency). Let b be any set function defining $\text{WMINMAX}_b^{(m)}$. Using classical distributivity, we can find a set function a' defining $\text{WMAXMIN}_{a'}^{(m)}$ such that

$$\text{WMAXMIN}_{a'}^{(m)} = \text{WMINMAX}_b^{(m)}.$$

We then observe that, for all $T \subseteq X$,

$$\bigvee_{K\subseteq T} a'_K = \theta_T = \bigwedge_{K\subseteq X\setminus T} b_K = \bigvee_{K\subseteq T} a_K.$$

By Proposition 3.1, we simply have $\text{WMAXMIN}_{a'}^{(m)}=\text{WMAXMIN}_a^{(m)}$. ■

When the $\text{WMAXMIN}_a^{(m)}$ and $\text{WMINMAX}_b^{(m)}$ functions are put in their complete forms, the correspondance formulae become simpler.

Corollary 3.1 *For any increasing set function a defining $\mathit{WMAXMIN}_a^{(m)}$ and any decreasing set function b defining $\mathit{WMINMAX}_b^{(m)}$, we have*

$$\mathit{WMINMAX}_b^{(m)} = \mathit{WMAXMIN}_a^{(m)} \Leftrightarrow b_T = a_{X\setminus T} \quad \forall T \subseteq X.$$

The following example illustrates the use of the correspondance formulae.

Example 3.1 *Let $X = \{1, 2, 3\}$. We have*

$$(0.1 \wedge x_1) \vee (0.3 \wedge x_2) \vee (x_2 \wedge x_3) = (0.1 \vee x_2) \wedge (0.3 \vee x_3) \wedge (x_1 \vee x_2).$$

Indeed, starting from the left-hand side (a canonical form), we can compute its complete form then its dual complete form and finally its dual canonical form as follows:

$$\begin{aligned}
& (0.1 \wedge x_1) \vee (0.3 \wedge x_2) \vee (x_2 \wedge x_3) \\
= \; & 0 \vee (0.1 \wedge x_1) \vee (0.3 \wedge x_2) \vee (0 \wedge x_3) \vee (0.3 \wedge x_1 \wedge x_2) \vee (0.1 \wedge x_1 \wedge x_3) \vee (1 \wedge x_2 \wedge x_3) \\
& \vee (1 \wedge x_1 \wedge x_2 \wedge x_3) \\
= \; & 1 \wedge (1 \vee x_1) \wedge (0.1 \vee x_2) \wedge (0.3 \vee x_3) \wedge (0 \vee x_1 \vee x_2) \wedge (0.3 \vee x_1 \vee x_3) \wedge (0.1 \vee x_2 \vee x_3) \\
& \wedge (0 \vee x_1 \vee x_2 \vee x_3) \\
= \; & (0.1 \vee x_2) \wedge (0.3 \vee x_3) \wedge (x_1 \vee x_2).
\end{aligned}$$

Now, we show that any $\text{WMAXMIN}_a^{(m)}$ function can be written under equivalent forms involving at most m variable coefficients. These coefficients only depend on the order of the x_i's. In order to present this, we need a technical lemma which was established by Dubois and Prade [6].

Lemma 3.1 *Let $(x_1, \ldots, x_m)$, $(x'_1, \ldots, x'_m) \in \mathbb{I}^m$ with $x_1 \leq \ldots \leq x_m$ and $x'_1 \geq \ldots \geq x'_m$.*

(i) If $x'_1 = 1$ then

$$\bigvee_{i=1}^{m} (x_i \wedge x'_i) = \mathit{median}(x_1, \ldots, x_m, x'_2, \ldots, x'_m).$$

(ii) If $x'_m = 0$ then

$$\bigwedge_{i=1}^{m} (x_i \vee x'_i) = \mathit{median}(x_1, \ldots, x_m, x'_1, \ldots, x'_{m-1}).$$

Now, we can state the result as follows.

Theorem 3.1 *(i) For any increasing set function a defining $WMAXMIN_a^{(m)}$, we have, for all $(x_1, \ldots, x_m) \in \mathbb{I}^m$,*

$$\begin{aligned} WMAXMIN_a^{(m)}(x_1, \ldots, x_m) &= \bigvee_{i=1}^{m} [x_{(i)} \wedge a_{\{(i),\ldots,(m)\}}] \\ &= median(x_1, \ldots, x_m, a_{\{(2),\ldots,(m)\}}, a_{\{(3),\ldots,(m)\}}, \ldots, a_{\{(m)\}}). \end{aligned}$$

(ii) For any decreasing set function b defining $WMINMAX_b^{(m)}$, we have, for all $(x_1, \ldots, x_m) \in \mathbb{I}^m$,

$$\begin{aligned} WMINMAX_b^{(m)}(x_1, \ldots, x_m) &= \bigwedge_{i=1}^{m} [x_{(i)} \vee b_{\{(1),\ldots,(i)\}}] \\ &= median(x_1, \ldots, x_m, b_{\{(1)\}}, b_{\{(1),(2)\}}, \ldots, b_{\{(1),\ldots,(m-1)\}}). \end{aligned}$$

Proof. (i) Let $(x_1, \ldots, x_m) \in \mathbb{I}^m$. Since a is increasing, we have

$$\begin{aligned} \bigvee_{i=1}^{m} [x_{(i)} \wedge a_{\{(i),\ldots,(m)\}}] &= \bigvee_{i=1}^{m} \bigvee_{\substack{T \subseteq \{(i),\ldots,(m)\} \\ T \ni (i)}} [a_T \wedge x_{(i)}] \\ &= \bigvee_{i=1}^{m} \bigvee_{\substack{T \subseteq \{(i),\ldots,(m)\} \\ T \ni (i)}} [a_T \wedge (\bigwedge_{j \in T} x_j)] \\ &= \bigvee_{T \subseteq X} [a_T \wedge (\bigwedge_{j \in T} x_j)] \end{aligned}$$

which prove the first equality. The second one follows from Lemma 3.1.

(ii) Let a be an increasing set function defined by $a_T = b_{X \setminus T}$ for all $T \subseteq X$. By Corollary 3.1, we have $\text{WMINMAX}_b^{(m)} = \text{WMAXMIN}_a^{(m)}$, and hence, for all $(x_1, \ldots, x_m) \in \mathbb{I}^m$,

$$\begin{aligned} & \text{WMINMAX}_b^{(m)}(x_1, \ldots, x_m) = \text{WMAXMIN}_a^{(m)}(x_1, \ldots, x_m) \\ = \ & \text{median}(x_1, \ldots, x_m, a_{\{(2),\ldots,(m)\}}, a_{\{(3),\ldots,(m)\}}, \ldots, a_{\{(m)\}}) \quad (\text{by } (i)) \\ = \ & \text{median}(x_1, \ldots, x_m, b_{\{(1)\}}, b_{\{(1),(2)\}}, \ldots, b_{\{(1),\ldots,(m-1)\}}) \\ = \ & \bigwedge_{i=1}^{m} [x_{(i)} \vee b_{\{(1),\ldots,(i)\}}] \quad (\text{Lemma 3.1}). \end{aligned}$$

■

3.4 Axiomatic characterization of the family of weighted max-min functions

According to Proposition 3.3, the set of weighted max-min functions and the set of weighted min-max functions represent the same family of functions. This family can be characterized with the help of some selected properties. These are presented in the next definition.

Definition 3.3 *The aggregation function $M^{(m)}$ defined on $\mathbb{I}^m$ is*

- increasing (In) *if* $M^{(m)}$ *is increasing in each argument, i.e. if, for all* $(x_1,\dots,x_m)$, $(x'_1,\dots,x'_m)\in \mathbb{I}^m$, *we have*

$$x_i \le x'_i \ \forall i\in X \Rightarrow M^{(m)}(x_1,\dots,x_m)\le M^{(m)}(x'_1,\dots,x'_m).$$

- idempotent (I) *if, for all* $x\in\mathbb{I}$,

$$M^{(m)}(x,\dots,x)=x.$$

- stable for minimum with the same unit (SMINU) *if, for all* $(x_1,\dots,x_m)\in\mathbb{I}^m$ *and all* $r\in\mathbb{I}$, *we have*

$$M^{(m)}(x_1\wedge r,\dots,x_m\wedge r)=M^{(m)}(x_1,\dots,x_m)\wedge r.$$

- stable for maximum with the same unit (SMAXU) *if, for all* $(x_1,\dots,x_m)\in\mathbb{I}^m$ *and all* $r\in\mathbb{I}$, *we have*

$$M^{(m)}(x_1\vee r,\dots,x_m\vee r)=M^{(m)}(x_1,\dots,x_m)\vee r.$$

The first two properties seem natural enough. The (In) property imposes that the functions present a nonnegative response to any increase of the arguments, and (I) clearly expresses the *unanimity* principle.

The other two ones are stability properties written in a functional equation form. They were introduced by Fodor and Roubens [10] and are visibly related to an algebra which uses min and max operations instead of classical sum and product operations.

They are respectively to be compared with stability for admissible similarities (SSI)

$$M^{(m)}(rx_1,\dots,rx_m)=rM^{(m)}(x_1,\dots,x_m)$$

and stability for admissible translations (STR)

$$M^{(m)}(x_1+r,\dots,x_m+r)=M^{(m)}(x_1,\dots,x_m)+r$$

which were investigated by Aczél and Roberts [1], Aczél *et al.* [2], Fodor and Roubens [9] and Marichal *et al.* [15], in the framework of the measurement theory for ratio scales, difference scales and interval scales.

For instance, the (SMAXU) property can be written as

$$M^{(m)}(f_r(x_1),\dots,f_r(x_m))=f_r(M^{(m)}(x_1,\dots,x_m))$$

where $f_r(x)=x\vee r$ (maxitive translation) is such that $f_r=f_s\Leftrightarrow r=s$ (see [4, §2.2]).

We also have the following result.

Proposition 3.4 *For any function* $M^{(m)}$ *defined on* $\mathbb{I}^m$, *we have (SMINU, SMAXU)⇒(I).*

Proof. For all $x\in\mathbb{I}$, we have, by (SMINU, SMAXU),

$$M^{(m)}(x,\dots,x)=M^{(m)}(x,\dots,x)\wedge x\le M^{(m)}(x,\dots,x)\vee x=M^{(m)}(x,\dots,x),$$

and thus $M^{(m)}(x,\dots,x)=x$. ∎

We have a comparable result in the case where sum and product operations are considered (see [15]): (SSI, STR)⇒(I).

Now, we show that the family of $\mathrm{WMAXMIN}_a^{(m)}$ functions can be characterized with the help of only three properties: (In), (SMINU) and (SMAXU).

Theorem 3.2 *Let $M^{(m)}$ be any aggregation function defined on $\mathbb{I}^m$. Then the following three assertions are equivalent:*

$$\begin{array}{rl} (i) & M^{(m)} \text{ fulfils (In, SMINU, SMAXU)} \\ (ii) & \text{There exists a set function } a \text{ such that } M^{(m)} = WMAXMIN_a^{(m)} \\ (iii) & \text{There exists a set function } b \text{ such that } M^{(m)} = WMINMAX_b^{(m)} \end{array}$$

Proof. $(ii) \Rightarrow (i)$. Easy.

$(i) \Rightarrow (ii)$. Let $(x_1, \ldots, x_m) \in \mathbb{I}^m$. On the one hand, for all $T \subseteq X$, we have

$$M^{(m)}(x_1, \ldots, x_m) \overset{(In)}{\geq} M^{(m)}[(\bigwedge_{i \in T} x_i) e_T] \overset{(SMINU)}{=} \theta_T \wedge (\bigwedge_{i \in T} x_i)$$

and thus

$$M^{(m)}(x_1, \ldots, x_m) \geq \bigvee_{T \subseteq X} [\theta_T \wedge (\bigwedge_{i \in T} x_i)].$$

On the other hand, let $T^* \subseteq X$ such that $\theta_{T^*} \wedge (\bigwedge_{i \in T^*} x_i)$ is maximum and set

$$Y := \{j \in X | x_j \leq \theta_{T^*} \wedge (\bigwedge_{i \in T^*} x_i)\}.$$

We should have $Y \neq \emptyset$. Indeed, if $x_j > \theta_{T^*} \wedge (\bigwedge_{i \in T^*} x_i)$ for all $j \in X$, we have, since $\theta_X = 1$,

$$\theta_X \wedge (\bigwedge_{i \in X} x_i) > \theta_{T^*} \wedge (\bigwedge_{i \in T^*} x_i)$$

which contradicts the definition of T^*. Then, we have,

$$\begin{array}{rcl} M^{(m)}(x_1, \ldots, x_m) & \overset{(In)}{\leq} & M^{(m)}[(\theta_{T^*} \wedge (\bigwedge_{i \in T^*} x_i)) e_Y + e_{X \setminus Y}] \\ & \overset{(SMAXU)}{=} & [\theta_{T^*} \wedge (\bigwedge_{i \in T^*} x_i)] \vee \theta_{X \setminus Y} \\ & = & \theta_{T^*} \wedge (\bigwedge_{i \in T^*} x_i) = \bigvee_{T \subseteq X} [\theta_T \wedge (\bigwedge_{i \in T} x_i)]. \end{array}$$

Indeed, we have $\theta_{X \setminus Y} \leq \theta_{T^*} \wedge (\bigwedge_{i \in T^*} x_i)$, for otherwise we would have, by definition of Y,

$$\theta_{X \setminus Y} \wedge (\bigwedge_{i \in X \setminus Y} x_i) > [\theta_{T^*} \wedge (\bigwedge_{i \in T^*} x_i)] \wedge [\theta_{T^*} \wedge (\bigwedge_{i \in T^*} x_i)] = \theta_{T^*} \wedge (\bigwedge_{i \in T^*} x_i)$$

which contradicts the definition of T^*.

$(ii) \Leftrightarrow (iii)$. See Proposition 3.3. ∎

When $m = 2$, we can propose an other characterization. It involves properties which are not directly related to an algebra endowed with min and max operations. These properties are given in the next definition.

Definition 3.4 *The aggregation function $M^{(m)}$ defined on $\mathbb{I}^m$ is*

- continuous (Co) *if $M^{(m)}$ is a continuous function on $\mathbb{I}^m$.*

- associative (A) *if* $m = 2$ *and*

$$M^{(2)}(M^{(2)}(x_1, x_2), x_3) = M^{(2)}(x_1, M^{(2)}(x_2, x_3)) \tag{1}$$

for all $x_1, x_2, x_3 \in \mathbb{I}$

These properties are classical enough. If we are searching for functions which do not present any chaotic reaction to a small change of the arguments, we restrict to *smooth* functions i.e. functions fulfilling (Co). Associativity (A) is a well-known algebraic property which allows to omit "parentheses" in an aggregation of three elements.

The following characterization, restricted to the case of $m = 2$, shows that, under (In), the (A) property combined with (Co) and (I) produces exactly the same effect as that of (SMINU, SMAXU).

Theorem 3.3 *Let* $M^{(2)}$ *be any aggregation function defined on* $\mathbb{I}^2$*. Then the following three assertions are equivalent:*

(*i*) $M^{(2)}$ *fulfils (In, I, Co, A)*
(*ii*) *There exists a set function a such that* $M^{(2)} = WMAXMIN_a^{(2)}$
(*iii*) *There exists a set function b such that* $M^{(2)} = WMINMAX_b^{(2)}$

Proof. $(ii) \Rightarrow (i)$. Only associativity is not immediate. For all $x_1, x_2 \in \mathbb{I}$, we have

$$\begin{aligned} M^{(2)}(x_1, x_2) &= (\overline{\theta} \wedge x_1) \vee (\theta \wedge x_2) \vee (1 \wedge x_1 \wedge x_2) \\ &= \begin{cases} x_1 \vee (\theta \wedge x_2) & \text{if } x_1 \leq x_2 \\ (\overline{\theta} \wedge x_1) \vee x_2 & \text{if } x_1 \geq x_2 \end{cases} \end{aligned}$$

where $\theta = M^{(2)}(0, 1)$ and $\overline{\theta} = M^{(2)}(1, 0)$. Let $x_1, x_2, x_3 \in \mathbb{I}$. We will show that (1) holds.

- If $x_1 \leq x_2 \leq x_3$ then (1) holds trivially.
- If $x_2 \leq x_1 \leq x_3$ then

$$M^{(2)}(M^{(2)}(x_1, x_2), x_3) = (\overline{\theta} \wedge x_1) \vee x_2 \vee (\theta \wedge x_3).$$

If $x_1 \leq M^{(2)}(x_2, x_3)$, that is $x_1 = x_2$ or $\theta \geq x_1$, then

$$M^{(2)}(x_1, M^{(2)}(x_2, x_3)) = x_1 \vee (\theta \wedge x_2) \vee (\theta \wedge x_3)$$

and (1) holds. Otherwise, if $x_1 \geq M^{(2)}(x_2, x_3)$ then (1) holds trivially.

One proceeds in the same manner when $x_2 \leq x_3 \leq x_1$ or $x_3 \leq x_2 \leq x_1$ or $x_1 \leq x_3 \leq x_2$ or $x_3 \leq x_1 \leq x_2$.

$(i) \Rightarrow (ii)$. Let $x_1, x_2 \in \mathbb{I}$ and assume that $x_1 \leq x_2$. The other case can be treated similarly. Let us show that

$$M^{(2)}(x_1, x_2) = x_1 \vee (\theta \wedge x_2)$$

where $\theta = M^{(2)}(0, 1)$. On the one hand, we have

$$M^{(2)}(0, \theta) = M^{(2)}(\theta, 1) = \theta \tag{2}$$

Indeed, we have, for instance,

$$M^{(2)}(0,\theta) = M^{(2)}(0, M^{(2)}(0,1)) \stackrel{(A)}{=} M^{(2)}(M^{(2)}(0,0),1) \stackrel{(I)}{=} M^{(2)}(0,1) = \theta.$$

By (2) and (In), we also have

$$M^{(2)}(x_1,x_2) = \theta \;\text{ if } x_1 \le \theta \le x_2. \tag{3}$$

On the other hand, we have

$$M^{(2)}(x,1) = x \;\; \forall x \in [\theta,1] \tag{4}$$
$$M^{(2)}(0,x) = x \;\; \forall x \in [0,\theta] \tag{5}$$

Indeed, if z increases from θ to 1, $M^{(2)}(z,1)$ increases continuously from $M^{(2)}(\theta,1)=\theta$ to $M^{(2)}(1,1)=1$. By (Co), this implies that: $\forall x \in [\theta,1], \exists z \in [\theta,1]$ such that $x = M^{(2)}(z,1)$. We then have

$$M^{(2)}(x,1) = M^{(2)}(M^{(2)}(z,1),1) \stackrel{(A)}{=} M^{(2)}(z, M^{(2)}(1,1)) \stackrel{(I)}{=} M^{(2)}(z,1) = x$$

which proves (4). Likewise, if z increases from 0 to θ, $M^{(2)}(0,z)$ increases continuously from $M^{(2)}(0,0)=0$ to $M^{(2)}(0,\theta)=\theta$. By (Co), this implies that: $\forall x \in [0,\theta], \exists z \in [0,\theta]$ such that $x = M^{(2)}(0,z)$. We then have

$$M^{(2)}(0,x) = M^{(2)}(0, M^{(2)}(0,z)) \stackrel{(A)}{=} M^{(2)}(M^{(2)}(0,0),z) \stackrel{(I)}{=} M^{(2)}(0,z) = x$$

which proves (5). To conclude, we note that

$$\begin{aligned} M^{(2)}(x_1,x_2) &= x_1 \;\text{ if } \theta \le x_1 \le x_2 \\ M^{(2)}(x_1,x_2) &= x_2 \;\text{ if } x_1 \le x_2 \le \theta \end{aligned}$$

Indeed,

$$x_1 \stackrel{(I)}{=} M^{(2)}(x_1,x_1) \stackrel{(In)}{\le} M^{(2)}(x_1,x_2) \stackrel{(In)}{\le} M^{(2)}(x_1,1) \stackrel{(4)}{=} x_1$$

and

$$x_2 \stackrel{(5)}{=} M^{(2)}(0,x_2) \stackrel{(In)}{\le} M^{(2)}(x_1,x_2) \stackrel{(In)}{\le} M^{(2)}(x_2,x_2) \stackrel{(I)}{=} x_2.$$

$(ii) \Leftrightarrow (iii)$. See Proposition 3.3. ∎

4 Back to the Sugeno integral

According to some results from the previous section, we can see that the class of the Sugeno integrals coincides with the family of weighted max-min functions. By using Theorem 3.1, we are then allow to derive equivalent forms of the Sugeno integral. The next theorem deals with this issue.

Theorem 4.1 *Let $(x_1,\ldots,x_m) \in \mathbb{I}^m$ and μ a fuzzy measure on X. Then we have*

$$\begin{aligned} \mathcal{S}_\mu^{(m)}(x_1,\ldots,x_m) &= \bigvee_{i=1}^{m} [x_{(i)} \wedge \mu_{\{(i),\ldots,(m)\}}] = \bigwedge_{i=1}^{m} [x_{(i)} \vee \mu_{\{(i+1),\ldots,(m)\}}] \\ &= \bigvee_{T \subseteq X} [\mu_T \wedge (\bigwedge_{i\in T} x_i)] = \bigwedge_{T \subseteq X} [\mu_{X\setminus T} \vee (\bigvee_{i \in T} x_i)] \\ &= \mathit{median}(x_1,\ldots,x_m, \mu_{\{(2),\ldots,(m)\}}, \mu_{\{(3),\ldots,(m)\}}, \ldots, \mu_{\{(m)\}}). \end{aligned}$$

Proof. Since μ is an increasing set function, we can conclude by Corollary 3.1, Theorem 3.1 and Definition 2.5. ■

According to Theorem 4.1, we can observe that, as an aggregation function, the Sugeno integral with respect to a measure μ is an extension on the entire hypercube $\mathbb{I}^m$ of any increasing $\mathbb{I}$-valued pseudo-Boolean function which define μ (see Section 2). The same conclusion has been obtain for the Choquet integral by Chateauneuf and Jaffray [3].

In addition to the previous result, Theorem 3.2 leads us to an axiomatic characterization of the class of Sugeno integrals. We state it as follows:

Theorem 4.2 *The aggregation function $M^{(m)}$ defined on $\mathbb{I}^m$ fulfils (In, SMINU, SMAXU) if and only if there exists a fuzzy measure μ on X such that $M^{(m)} = \mathcal{S}_\mu^{(m)}$.*

Proof. By Theorem 3.2, $M^{(m)}$ fulfils (In, SMINU, SMAXU) if and only if there exists a set function a such that $M^{(m)} =\mathrm{WMAXMIN}_a^{(m)}$. According to Proposition 3.1, a can be chosen increasing and thus be assimilated to a fuzzy measure. Theorem 4.1 then allows to conclude. ■

An important topic in multicriteria decision making is the concept of *veto*. Suppose that $M^{(m)}$ is an aggregation function being used for a problem. A criterion k is a veto for this problem if for any m-uple $(x_1,\ldots,x_m) \in \mathbb{I}^m$ of scores,

$$M^{(m)}(x_1,\ldots,x_m) \leq x_k.$$

This means that if the score on criterion k is high, it has no effect on the evaluation, but if it is low, the global score will be low too, whatever the values of the other scores. Similarly, criterion k is said to be a *favor* if for any m-uple $(x_1,\ldots,x_m) \in \mathbb{I}^m$ of scores,

$$M^{(m)}(x_1,\ldots,x_m) \geq x_k.$$

The next proposition shows that the Sugeno integral can model these veto and favor effects by using a suitable fuzzy measure.

Proposition 4.1 *For the Sugeno integral $\mathcal{S}_\mu^{(m)}$, a veto effect on $k \in X$ is obtained if and only if $\mu_T = 0$ whenever $k \notin T$. Similarly, a favor effect on $k \in X$ is obtained if and only if $\mu_T = 1$ whenever $k \in T$.*

Proof. (Necessity). Trivial since $\mu_T = \mathcal{S}_\mu^{(m)}(e_T)$ for all $T \subseteq X$.

(Sufficiency). Let $(x_1,\ldots,x_m) \in \mathbb{I}^m$. If $\mu_T = 0$ whenever $k \notin T$, we have, by Theorem 4.1,

$$\mathcal{S}_\mu^{(m)}(x_1,\ldots,x_m) = \bigvee_{T\subseteq X\setminus\{k\}} [\mu_{T\cup\{k\}} \wedge (\bigwedge_{i\in T} x_i) \wedge x_k] \leq x_k.$$

If $\mu_T = 1$ whenever $k \in T$, we have, by Theorem 4.1,

$$\mathcal{S}_\mu^{(m)}(x_1,\ldots,x_m) = \bigwedge_{T\subseteq X\setminus\{k\}} [\mu_{X\setminus(T\cup\{k\})} \vee (\bigvee_{i\in T} x_i) \vee x_k] \geq x_k.$$

■

It is possible to generalize the concept of veto to several criteria: a veto for criteria $K \subseteq X$, which means $\mathcal{S}_\mu^{(m)}(x_1,\ldots,x_m) \leq \bigwedge_{k\in K} x_k$, is obtained by any fuzzy measure μ such that $\mu_T = 0$ whenever $K \nsubseteq T$. Similarly, a favor for criteria $K \subseteq X$, which means $\mathcal{S}_\mu^{(m)}(x_1,\ldots,x_m) \geq \bigvee_{k\in K} x_k$, is obtained by any fuzzy measure μ such that $\mu_T = 1$ whenever $K \cap T \neq \emptyset$.

5 Subfamilies of weighted max-min functions

This section aims at introducing some subclasses of weighted max-min functions. The results from Section 3 are then applied to the functions of these subclasses in order to derive equivalent expressions. All aggregation functions introduced in this section are particular weighted max-min functions and thus particular Sugeno integrals. In order to check this, it suffices to use Theorem 3.2.

We also give an axiomatic characterization of each of those subsets of functions. To do this, we introduce hereafter some properties, in addition to Definitions 3.3 and 3.4.

In the sequel, Φ denotes the set of all strictly increasing functions $\phi : \mathbb{I} \to \mathbb{I}$.

Definition 5.1 *The aggregation function $M^{(m)}$ defined on $\mathbb{I}^m$ is*

- weakly idempotent (WI) *if $M^{(m)}(0,\ldots,0) = 0$ and $M^{(m)}(1,\ldots,1) = 1$.*
- symmetric (Sy) *if $M^{(m)}$ is a symmetric function on $\mathbb{I}^m$, i.e. if, for all permutation σ of X and all $(x_1,\ldots,x_m) \in \mathbb{I}^m$, we have*

$$M^{(m)}(x_1,\ldots,x_m) = M^{(m)}(x_{\sigma(1)}\ldots,x_{\sigma(m)}).$$

- unanimously increasing (UIn) *if $M^{(m)}$ fulfils (In) and if, for all $(x_1,\ldots,x_m),(x'_1,\ldots,x'_m) \in \mathbb{I}^m$, we have*

$$x_i < x'_i \ \forall i \in X \Rightarrow M^{(m)}(x_1,\ldots,x_m) < M^{(m)}(x'_1,\ldots,x'_m).$$

- comparison meaningful (CM) *if, for all $\phi \in \Phi$ and all $(x_1,\ldots,x_m),(x'_1,\ldots,x'_m) \in \mathbb{I}^m$, we have that*

$$M^{(m)}(x_1,\ldots,x_m) \le M^{(m)}(x'_1,\ldots,x'_m)$$

implies

$$M^{(m)}(\phi(x_1),\ldots,\phi(x_m)) \le M^{(m)}(\phi(x'_1),\ldots,\phi(x'_m)).$$

- ordinally stable (OS) *if, for all $\phi \in \Phi$ and all $(x_1,\ldots,x_m) \in \mathbb{I}^m$, we have*

$$M^{(m)}(\phi(x_1),\ldots,\phi(x_m)) = \phi(M^{(m)}(x_1,\ldots,x_m)).$$

- minitive (MIN) *if for all $(x_1,\ldots,x_m)$, $(x'_1,\ldots,x'_m) \in \mathbb{I}^m$, we have*

$$M^{(m)}(x_1 \wedge x'_1,\ldots,x_m \wedge x'_m) = M^{(m)}(x_1,\ldots,x_m) \wedge M^{(m)}(x'_1,\ldots,x'_m).$$

- maxitive (MAX) *if for all $(x_1,\ldots,x_m)$, $(x'_1,\ldots,x'_m) \in \mathbb{I}^m$, we have*

$$M^{(m)}(x_1 \vee x'_1,\ldots,x_m \vee x'_m) = M^{(m)}(x_1,\ldots,x_m) \vee M^{(m)}(x'_1,\ldots,x'_m).$$

Let us comment on the properties from Definition 5.1. The (Sy) property leads us to *neutral* functions i.e. independent of the labels. The (UIn) property is a requirement stronger than (In), imposing a positive response whenever all the arguments increase. We introduce it in this paper for our needs. For instance, observe that the maximum function

$$M^{(m)}(x_1,\ldots,x_m) = \bigvee_{i=1}^{m} x_i$$

fulfils (UIn) whereas the bounded sum

$$M^{(m)}(x_1, \dots, x_m) = (\sum_{i=1}^{m} x_i) \wedge 1$$

does not.

The (CM) property was introduced by Ovchinnikov [17]. He studied the meaningfulness (stability) of means comparison in the framework of ordinal measurement. The (OS) property is closely linked to (CM) as the next proposition shows.

Proposition 5.1 *We have*

$$\begin{array}{rl} (i) & (OS) \Rightarrow M^{(m)}(x_1, \dots, x_m) \in \{x_1, \dots, x_m\} \quad \forall x_1, \dots, x_m \in \mathbb{I}, \\ (ii) & (CM, I) \Leftrightarrow (OS). \end{array}$$

Proof. (i) Consider $(x_1, \dots, x_m) \in \mathbb{I}^m$ reordered as $x_{(1)} \leq \dots \leq x_{(m)}$ and set $x_0 := M^{(m)}(x_1, \dots, x_m)$. Suppose the result false. We then have three exclusive cases:

- If $x_{(i)} < x_0 < x_{(i+1)}$ for one $i \in \{1, \dots, m-1\}$ then there are elements $u, v \in \mathbb{I}$ and a function $\phi \in \Phi$ such that $x_{(i)} < u < x_0 < v < x_{(i+1)}$, $\phi(x) = x$ on $\mathbb{I} \setminus [x_{(i)}, x_{(i+1)}]$ and $\phi(u) = v$. This implies $\phi(x_0) > x_0$ which is impossible since

$$\begin{aligned} \phi(x_0) &= \phi(M^{(m)}(x_1, \dots, x_m)) = M^{(m)}(\phi(x_1), \dots, \phi(x_m)) \\ &= M^{(m)}(x_1, \dots, x_m) = x_0. \end{aligned}$$

- If $0 \leq x_0 < x_{(1)}$ then there are $v \in \mathbb{I}$ and a function $\phi \in \Phi$ such that $x_0 < v < x_{(1)}$, $\phi(x) = x$ on $[x_{(1)}, 1]$ and $\phi(x_0) = v$. This implies $\phi(x_0) > x_0$, a contradiction.

- The case $x_{(m)} < x_0 \leq 1$ can be treated as the previous one.

(ii) (Necessity) Let $x_1, \dots, x_m \in \mathbb{I}$ and set $x_0 = M^{(m)}(x_1, \dots, x_m)$. By (I), we have

$$M^{(m)}(x_1, \dots, x_m) = M^{(m)}(x_0, \dots, x_0)$$

and thus, for all $\phi \in \Phi$,

$$\begin{aligned} M^{(m)}(\phi(x_1), \dots, \phi(x_m)) &\overset{(CM)}{=} M^{(m)}(\phi(x_0), \dots, \phi(x_0)) \\ &\overset{(I)}{=} \phi(x_0) = \phi(M^{(m)}(x_1, \dots, x_m)). \end{aligned}$$

and $M^{(m)}$ fulfils (OS).

(Sufficiency) For all $\phi \in \Phi$ and all $(x_1, \dots, x_m), (x'_1, \dots, x'_m) \in \mathbb{I}^m$, we have, by (OS),

$$\begin{aligned} & M^{(m)}(x_1, \dots, x_m) \leq M^{(m)}(x'_1, \dots, x'_m) \\ \Rightarrow \; & \phi(M^{(m)}(x_1, \dots, x_m)) \leq \phi(M^{(m)}(x'_1, \dots, x'_m)) \\ \Rightarrow \; & M^{(m)}(\phi(x_1), \dots, \phi(x_m)) \leq M^{(m)}(\phi(x'_1), \dots, \phi(x'_m)), \end{aligned}$$

and $M^{(m)}$ fulfils (CM). Moreover, by (i), it fulfils (I). ■

The (MIN) and (MAX) properties are related to an algebra using min and max operations. Of course, they are to be compared with classical *additivity*, that is

$$M^{(m)}(x_1 + x'_1, \dots, x_m + x'_m) = M^{(m)}(x_1, \dots, x_m) + M^{(m)}(x'_1, \dots, x'_m).$$

The following lemma gives a description of the aggregation functions fulfilling (MIN) or (MAX) (see also [7]).

Lemma 5.1 *(i) The aggregation function $M^{(m)}$ defined on $\mathbb{I}^m$ fulfils (MIN) if and only if there exists increasing functions $g_i : \mathbb{I} \to \mathbb{R}$, $i \in X$, such that*

$$M^{(m)}(x_1, \ldots, x_m) = \bigwedge_{i=1}^{m} g_i(x_i) \quad \forall (x_1, \ldots, x_m) \in \mathbb{I}^m.$$

(ii) The aggregation function $M^{(m)}$ defined on $\mathbb{I}^m$ fulfils (MAX) if and only if there exists increasing functions $h_i : \mathbb{I} \to \mathbb{R}$, $i \in X$, such that

$$M^{(m)}(x_1, \ldots, x_m) = \bigvee_{i=1}^{m} h_i(x_i) \quad \forall (x_1, \ldots, x_m) \in \mathbb{I}^m.$$

Proof. (i) (Necessity) Let $(x_1, \ldots, x_m) \in \mathbb{I}^m$. By (MIN), we have

$$M^{(m)}(x_1, \ldots, x_m) = \bigwedge_{i=1}^{m} M^{(m)}(x_i e_i + e_{X\setminus\{i\}}) = \bigwedge_{i=1}^{m} g_i(x_i)$$

where $g_i(x) = M^{(m)}(x e_i + e_{X\setminus\{i\}})$, $i \in X$. Moreover, for all $i \in X$, g_i is increasing: indeed, if $x, x' \in \mathbb{I}$, $x \leq x'$, we have that

$$g_i(x) = g_i(x \wedge x') = g_i(x) \wedge g_i(x')$$

implies $g_i(x) \leq g_i(x')$.

(Sufficiency) We have

$$g_i(x \wedge x') = g_i(x) \wedge g_i(x')$$

for all $x, x' \in \mathbb{I}$ and all $i \in X$: indeed, if $x \leq x'$, we have $g_i(x) \leq g_i(x')$ and

$$g_i(x \wedge x') = g_i(x) = g_i(x) \wedge g_i(x').$$

We then can conclude.

(ii). Similar to (i). ■

A valued binary relation R on a set A of alternatives is transitive (resp. negatively transitive) if, for all $a, b, c \in A$,

$$R(a,c) \wedge R(c,b) \leq R(a,b) \quad (\text{resp. } R(a,b) \leq R(a,c) \vee R(c,b)).$$

The next proposition shows that it is useful to assume the (MIN) and (MAX) properties when we consider aggregation of transitive (or negatively transitive) valued binary relations (see also [9, §7.3.1]).

Proposition 5.2 *Let $M^{(m)}$ be an aggregation function defined on $\mathbb{I}^m$ and fulfilling (In). Let A be a set of alternatives and $R_1, \ldots, R_m$ be transitive (resp. negatively transitive) valued binary relations on A. Then the aggregated valued relation R defined as*

$$R(a,b) = M^{(m)}(R_1(a,b), \ldots, R_m(a,b)) \quad \forall a, b \in A$$

is a transitive (resp. negatively transitive) valued binary relation if and only if $M^{(m)}$ fulfils (MIN) (resp. (MAX)).

Proof. Consider the case of transitivity. The other one can be treated similarly.
(Necessity). Set $x_i^{ab} = R_i(a,b)$ for all $a,b \in A$ and all $i \in X$. By hypothesis, whenever $x_i^{ac} \wedge x_i^{cb} \leq x_i^{ab}$ for all $a,b,c \in A$ and all $i \in X$, we have

$$M^{(m)}(x_1^{ac},\dots,x_m^{ac}) \wedge M^{(m)}(x_1^{cb},\dots,x_m^{cb}) \leq M^{(m)}(x_1^{ab},\dots,x_m^{ab})$$

for all $a,b,c \in A$. In the particular case where $x_i^{ac} \wedge x_i^{cb} = x_i^{ab}$ for all $a,b,c \in A$ and all $i \in X$, since $M^{(m)}$ fulfils (In), we obtain that:

$$\begin{aligned} M^{(m)}(x_1^{ab},\dots,x_m^{ab}) &= M^{(m)}(x_1^{ac} \wedge x_1^{cb},\dots,x_m^{ac} \wedge x_m^{cb}) \\ &\leq M^{(m)}(x_1^{ac},\dots,x_m^{ac}) \wedge M^{(m)}(x_1^{cb},\dots,x_m^{cb}) \end{aligned}$$

for all $a,b,c \in A$. Finally, we have that:

$$M^{(m)}(x_1^{ac} \wedge x_1^{cb},\dots,x_m^{ac} \wedge x_m^{cb}) = M^{(m)}(x_1^{ac},\dots,x_m^{ac}) \wedge M^{(m)}(x_1^{cb},\dots,x_m^{cb})$$

for all $a,b,c \in A$. Therefore, $M^{(m)}$ fulfils (MIN).

(Sufficiency). Suppose that $R_i(a,c) \wedge R_i(c,b) \leq R_i(a,b)$ for all $a,b,c \in A$ and all $i \in X$. We have, using (MIN) and (In) successively,

$$\begin{aligned} & M^{(m)}(R_1(a,c),\dots,R_m(a,c)) \wedge M^{(m)}(R_1(c,b),\dots,R_m(c,b)) \\ = \ & M^{(m)}(R_1(a,c) \wedge R_1(c,b),\dots,R_m(a,c) \wedge R_m(c,b)) \\ \leq \ & M^{(m)}(R_1(a,b),\dots,R_m(a,b)) \end{aligned}$$

for all $a,b,c \in A$. Therefore, R is transitive. ∎

Now turn to the announced subfamilies of weighted max-min functions. We start with Boolean max-min functions.

5.1 Boolean max-min functions

Definition 5.2 *For any set function $a : \mathcal{P}(X) \to \mathbb{B}$ such that $a_\emptyset = 0$ and $\bigvee_{T \subseteq X} a_T = 1$, the* Boolean max-min *function $BMAXMIN_a^{(m)}$ associated to a is defined by*

$$BMAXMIN_a^{(m)} = WMAXMIN_a^{(m)}.$$

For any set function $b : \mathcal{P}(X) \to \mathbb{B}$ such that $b_\emptyset = 1$ and $\bigwedge_{T \subseteq X} b_T = 0$, the Boolean min-max *function $BMINMAX_b^{(m)}$ associated to b is defined by*

$$BMINMAX_b^{(m)} = WMINMAX_b^{(m)}.$$

Thus defined, a Boolean max-min function (resp. Boolean min-max function) is nothing less than a weighted max-min function (resp. weighted min-max function) whose canonical and complete forms are defined by set functions taking their values in $\mathbb{B}$. Moreover, we can write, for any $(x_1,\dots,x_m) \in \mathbb{I}^m$,

$$\begin{aligned} \mathrm{BMAXMIN}_a^{(m)}(x_1,\dots,x_m) &= \bigvee_{T \subseteq X, a_T = 1} \bigwedge_{i \in T} x_i \in \{x_1,\dots,x_m\}, \\ \mathrm{BMINMAX}_b^{(m)}(x_1,\dots,x_m) &= \bigwedge_{T \subseteq X, b_T = 0} \bigvee_{i \in T} x_i \in \{x_1,\dots,x_m\}. \end{aligned}$$

In terms of fuzzy measures, if the set function a is increasing, it represents a *0-1 fuzzy measure*. Murofushi and Sugeno [16] have proved that, if μ is a 0-1 fuzzy measure then we have $\mathcal{S}_\mu^{(m)}$ =BMAXMIN$_\mu^{(m)}$. It is a particular case of Theorem 4.1. Therefore, any Sugeno integral $\mathcal{S}_\mu^{(m)}$ is a Boolean max-min function if and only if μ is a 0-1 fuzzy measure.

The following result shows that the Boolean max-min functions are exactly those weighted max-min functions (or Sugeno integrals) which fulfils (UIn). They are also appropriate to aggregate ordinal values.

Theorem 5.1 *Let $M^{(m)}$ be any aggregation function defined on $\mathbb{I}^m$. Then the following four assertions are equivalent:*

(i) *$M^{(m)}$ fulfils (UIn, SMINU, SMAXU)*
(ii) *$M^{(m)}$ fulfils (In, I, Co, CM)*
(iii) *There exists a set function a such that $M^{(m)}$=BMAXMIN$_a^{(m)}$*
(iv) *There exists a set function b such that $M^{(m)}$=BMINMAX$_b^{(m)}$*

Proof. $(iii) \Rightarrow (i)$. Easy.

$(iii) \Rightarrow (ii)$. According to Proposition 5.1, it suffices to observe that any Boolean max-min function fulfils (OS). This is true since, for all $\phi \in \Phi$ and all $x, x' \in \mathbb{I}$, we have $\phi(x \vee x') = \phi(x) \vee \phi(x')$ and $\phi(x \wedge x') = \phi(x) \wedge \phi(x')$.

$(i) \Rightarrow (iii)$. By Theorem 3.2, there exists a set function a such that $M^{(m)}$ =WMAXMIN$_a^{(m)}$ and we can assume a increasing. Suppose that there exists $T \subseteq X$ such that $a_T \in (0,1)$. We can write $X = \{t_1, \ldots, t_m\}$ and $T = \{t_k, \ldots, t_m\}$, with $k \in \{2, \ldots, m\}$. Let $(x_1, \ldots, x_m) \in \mathbb{I}^m$ such that

$$x_{t_1} \leq \ldots \leq x_{t_{k-1}} < a_T < x_{t_k} \leq \ldots \leq x_{t_m}.$$

By Theorem 3.1, we always have

$$\text{WMAXMIN}_a^{(m)}(x_1, \ldots, x_m) = \text{median}(x_1, \ldots, x_m, a_{\{t_2,\ldots,t_m\}}, \ldots, a_T, \ldots, a_{\{t_m\}}) = a_T.$$

This means that WMAXMIN$_a^{(m)}$ does not fulfils (UIn).

$(ii) \Rightarrow (iii)$. By Proposition 5.1, $M^{(m)}$ fulfils (OS).

Let $r \in \mathbb{I}$ and define a sequence $\phi_i \in \Phi$, $i \in \mathbb{N}_0$ as

$$\phi_i(x) = \begin{cases} x & \text{on } [0, r] \\ r + (x - r)/i & \text{on } [r, 1] \end{cases}$$

We then have, for all $x_1, \ldots, x_m \in \mathbb{I}$,

$$\begin{aligned} M^{(m)}(x_1 \wedge r, \ldots, x_m \wedge r) &= M^{(m)}(\lim_{i\to+\infty} \phi_i(x_1), \ldots, \lim_{i\to+\infty} \phi_i(x_m)) \\ &\overset{(Co)}{=} \lim_{i\to+\infty} M^{(m)}(\phi_i(x_1), \ldots, \phi_i(x_m)) \\ &\overset{(OS)}{=} \lim_{i\to+\infty} \phi_i(M^{(m)}(x_1, \ldots, x_m)) = M^{(m)}(x_1, \ldots, x_m) \wedge r \end{aligned}$$

and $M^{(m)}$ fulfils (SMINU). We can show similarly that it fulfils (SMAXU). By Theorem 3.2, there exists a set function a such that $M^{(m)}$ =WMAXMIN$_a^{(m)}$ and we can assume a increasing. Finally, by Proposition 5.1, we have $a_T = \theta_T \in \mathbb{B}$ for all $T \subseteq X$.

$(iii) \Leftrightarrow (iv)$. See Proposition 3.3. ∎

5.2 Weighted minimum and maximum functions

The weighted minimum and maximum functions were introduced and investigated by Dubois and Prade [6]. They are to be compared with weighted arithmetic mean functions.

Definition 5.3 *For any weight vector* $\omega = (\omega_1, \ldots, \omega_m) \in \mathbb{I}^m$ *such that* $\bigvee_{i=1}^m \omega_i = 1$, *the* weighted maximum *function* $WMAX_\omega^{(m)}$ *associated to* ω *is defined by*

$$WMAX_\omega^{(m)}(x_1, \ldots, x_m) = \bigvee_{i=1}^m (\omega_i \wedge x_i) \quad \forall (x_1, \ldots, x_m) \in \mathbb{I}^m.$$

For any weight vector $\omega = (\omega_1, \ldots, \omega_m) \in \mathbb{I}^m$ *such that* $\bigwedge_{i=1}^m \omega_i = 0$, *the* weighted minimum *function* $WMIN_\omega^{(m)}$ *associated to* ω *is defined by*

$$WMIN_\omega^{(m)}(x_1, \ldots, x_m) = \bigwedge_{i=1}^m (\omega_i \vee x_i) \quad \forall (x_1, \ldots, x_m) \in \mathbb{I}^m.$$

Any $\mathrm{WMAX}_\omega^{(m)}$ function is a $\mathrm{WMAXMIN}_a^{(m)}$ function whose canonical form is defined by:

$$\begin{cases} a_i = \omega_i & \forall i \in X \\ a_T = 0 & \forall T \subseteq X \text{ such that } |T| \neq 1 \end{cases}$$

and complete form by:

$$a_T = \bigvee_{i \in T} \omega_i \quad \forall T \subseteq X.$$

When a is increasing then it represents a *possibility measure* π which is characterized by the following property:

$$\pi(R \cup S) = \pi(R) \vee \pi(S) \quad \forall R, S \subseteq X.$$

Likewise, any $\mathrm{WMIN}_\omega^{(m)}$ function is a $\mathrm{WMINMAX}_b^{(m)}$ function whose canonical form is defined by:

$$\begin{cases} b_i = \omega_i & \forall i \in X \\ b_T = 1 & \forall T \subseteq X \text{ such that } |T| \neq 1 \end{cases}$$

and complete form by:

$$b_T = \bigwedge_{i \in T} \omega_i \quad \forall T \subseteq X.$$

When b is decreasing then the set function a', defined by $a'_T = b_{X \setminus T}$ for all $T \subseteq X$, represents a *necessity measure* $\mathcal{N}$ which is characterized by the following property:

$$\mathcal{N}(R \cap S) = \mathcal{N}(R) \wedge \mathcal{N}(S) \quad \forall R, S \subseteq X.$$

Therefore, any Sugeno integral $\mathcal{S}_\mu^{(m)}$ is a weighted maximum function (resp. weighted minimum function) if and only if μ is a possibility measure (resp. necessity measure).

The aggregation functions $\mathrm{WMAX}_\omega^{(m)}$ and $\mathrm{WMIN}_\omega^{(m)}$ can be characterized in the following way (see also [10]).

Theorem 5.2 (*i*) *The aggregation function* $M^{(m)}$ *defined on* $\mathbb{I}^m$ *fulfils (WI, MAX, SMINU) if and only if there exists a weight vector* ω *such that* $M^{(m)} = WMAX_\omega^{(m)}$.
(*ii*) *The aggregation function* $M^{(m)}$ *defined on* $\mathbb{I}^m$ *fulfils (WI, MIN, SMAXU) if and only if there exists a weight vector* ω *such that* $M^{(m)} = WMIN_\omega^{(m)}$.

Proof. (i) (Sufficiency). Trivial.

(Necessity). For all $i \in X$, we have $\theta_i \in \mathbb{I}$. Indeed, if $\theta_i \geq 1$ then, by (SMINU), $\theta_i = \theta_i \wedge 1 = 1$, and if $\theta_i \leq 0$ then, by (MAX), $\theta_i = \theta_i \vee M^{(m)}(0, \ldots, 0) = \theta_i \vee 0 = 0$. On the other hand, for all $(x_1, \ldots, x_m) \in \mathbb{I}^m$, we have, setting $\omega_i = \theta_i \in \mathbb{I}$,

$$M^{(m)}(x_1, \ldots, x_m) \overset{(MAX)}{=} \bigvee_{i=1}^{m} M^{(m)}(x_i e_i) \overset{(SMINU)}{=} \bigvee_{i=1}^{m} (\omega_i \wedge x_i).$$

Moreover, $\bigvee_{i=1}^{m} \omega_i = M^{(m)}(1, \ldots, 1) = 1$ as required.

(ii) Similar to (i). ■

5.3 Ordered weighted maximum and minimum functions

If, in Definition 5.3, weights ω_i are associated with a particular rank rather than a particular element, then we define ordered weighted maximum and minimum functions. Dubois *et al.* [8] used them for modelling soft partial matching. They are defined as follows.

Definition 5.4 *For any weight vector* $\omega = (\omega_1, \ldots, \omega_m) \in \mathbb{I}^m$ *such that* $1 = \omega_1 \geq \ldots \geq \omega_m$, *the* ordered weighted maximum *function* $OWMAX_{\omega}^{(m)}$ *associated to* ω *is defined by*

$$OWMAX_{\omega}^{(m)}(x_1, \ldots, x_m) = \bigvee_{i=1}^{m} (\omega_i \wedge x_{(i)}) \quad \forall (x_1, \ldots, x_m) \in \mathbb{I}^m.$$

For any weight vector $\omega' = (\omega'_1, \ldots, \omega'_m) \in \mathbb{I}^m$ *such that* $\omega'_1 \geq \ldots \geq \omega'_m = 0$, *the* ordered weighted minimum *function* $OWMIN_{\omega'}^{(m)}$ *associated to* ω' *is defined by*

$$OWMIN_{\omega'}^{(m)}(x_1, \ldots, x_m) = \bigwedge_{i=1}^{m} (\omega'_i \vee x_{(i)}) \quad \forall (x_1, \ldots, x_m) \in \mathbb{I}^m.$$

In Definition 5.4, the inequalities $\omega_1 \geq \ldots \geq \omega_m$ and $\omega'_1 \geq \ldots \geq \omega'_m$ are not restrictive. Indeed, if there exists $i \in \{1, \ldots, m-1\}$ such that $\omega_i \leq \omega_{i+1}$ and $\omega'_i \leq \omega'_{i+1}$ then we have

$$(\omega_i \wedge x_{(i)}) \vee (\omega_{i+1} \wedge x_{(i+1)}) = \omega_{i+1} \wedge x_{(i+1)},$$
$$(\omega'_i \vee x_{(i)}) \wedge (\omega'_{i+1} \vee x_{(i+1)}) = \omega'_i \vee x_{(i)}.$$

This means that ω_i can be replaced by ω_{i+1} in $\mathrm{OWMAX}_{\omega}^{(m)}$ and ω'_{i+1} by ω'_i in $\mathrm{OWMIN}_{\omega'}^{(m)}$.

Any $\mathrm{OWMAX}_{\omega}^{(m)}$ function is a $\mathrm{WMAXMIN}_{a}^{(m)}$ function whose canonical form is defined by:

$$\forall T \subseteq X, T \neq \emptyset : a_T = \begin{cases} 0 & \text{if } \omega_{m-|T|+1} = \omega_{m-|T|+2} \\ \omega_{m-|T|+1} & \text{otherwise} \end{cases}$$

and complete form by:

$$\forall T \subseteq X, T \neq \emptyset : a_T = \omega_{m-|T|+1}.$$

Likewise, any $\mathrm{OWMIN}_{\omega'}^{(m)}$ function is a $\mathrm{WMINMAX}_{b}^{(m)}$ function whose canonical form is defined by:

$$\forall T \subseteq X, T \neq \emptyset : b_T = \begin{cases} 1 & \text{if } \omega'_{|T|} = \omega'_{|T|-1} \\ \omega'_{|T|} & \text{otherwise} \end{cases}$$

and complete form by:

$$\forall T \subseteq X, T \neq \emptyset : b_T = \omega'_{|T|}.$$

The next proposition shows that any ordered weighted maximum function can be put in the form of an ordered weighted minimum function and conversely.

Proposition 5.3 *Let ω and ω' be weight vectors defining $OWMAX_{\omega}^{(m)}$ and $OWMIN_{\omega'}^{(m)}$ respectively. Then we have*

$$OWMIN_{\omega'}^{(m)} = OWMAX_{\omega}^{(m)} \Leftrightarrow \omega'_i = \omega_{i+1} \quad \forall i \in \{1, \dots, m-1\}.$$

Proof. If the fuzzy measure μ define the complete form of $\mathrm{OWMAX}_{\omega}^{(m)}$ then we have $\mu_{\{(i+1),\dots,(m)\}} = \omega_{i+1}$ for all $i \in \{1, \dots, m-1\}$. Theorem 4.1 then allows to conclude. ∎

It is interesting to note that, according to Lemma 3.1, we have, for all $(x_1, \dots, x_m) \in \mathbb{I}^m$,

$$\begin{aligned} \mathrm{OWMAX}_{\omega}^{(m)}(x_1, \dots, x_m) &= \mathrm{median}(x_1, \dots, x_m, \omega_2, \dots, \omega_m), \\ \mathrm{OWMIN}_{\omega'}^{(m)}(x_1, \dots, x_m) &= \mathrm{median}(x_1, \dots, x_m, \omega'_1, \dots, \omega'_{m-1}). \end{aligned}$$

We now show that the $\mathrm{OWMAX}_{\omega}^{(m)}$ and $\mathrm{OWMIN}_{\omega'}^{(m)}$ functions are exactly those weighted max-min functions (or Sugeno integrals) which fulfil (Sy). To do this, we need a lemma which is due to Grabisch [11].

Lemma 5.2 *Let $\mathcal{S}_{\mu}^{(m)}$ be a Sugeno integral. Then $\mathcal{S}_{\mu}^{(m)}$ fulfils (Sy) if and only if*

$$\mu_R = \mu_S \textit{ for all } R, S \subseteq X \textit{ such that } |R| = |S|.$$

We then have the following characterization (see also [10]).

Theorem 5.3 *Let $M^{(m)}$ be any aggregation function defined on $\mathbb{I}^m$. Then the following three assertions are equivalent:*

(*i*) *$M^{(m)}$ fulfils (Sy, In, SMINU, SMAXU)*
(*ii*) *There exists a weight vector ω such that $M^{(m)} = OWMAX_{\omega}^{(m)}$*
(*iii*) *There exists a weight vector ω' such that $M^{(m)} = OWMIN_{\omega'}^{(m)}$*

Proof. $(ii) \Rightarrow (i)$. Easy.

$(i) \Rightarrow (ii)$. By Theorem 3.2, there exists a set function a such that $M^{(m)} = \mathrm{WMAXMIN}_{a}^{(m)}$. If a is increasing, we can write $M^{(m)} = \mathcal{S}_{a}^{(m)}$. Let $(x_1, \dots, x_m) \in \mathbb{I}^m$. By Lemma 5.2, for all $i \in X$, $a_{\{(i),\dots,(m)\}}$ depends only on $|\{(i), \dots, (m)\}| = m - i + 1$ and hence on i. Setting $\omega_i = a_{\{(i),\dots,(m)\}}$ for all $i \in X$, we have, by Theorem 4.1,

$$M^{(m)}(x_1, \dots, x_m) = \bigvee_{i=1}^{m} (x_{(i)} \wedge \omega_i).$$

$(ii) \Leftrightarrow (iii)$. See Proposition 5.3. ∎

Theorem 5.3 shows that the class of $\mathrm{OWMAX}_{\omega}^{(m)}$ functions and the class of $\mathrm{OWMIN}_{\omega'}^{(m)}$ functions represent the same family of functions and coincide with the class of commutative Sugeno integrals. Moreover, according to Lemma 5.2, any Sugeno integral $\mathcal{S}_{\mu}^{(m)}$ is an ordered weighted maximum (or minimum) function if and only if μ is a fuzzy measure depending only on the cardinal of subsets.

5.4 Partial maximum and minimum functions

Definition 5.5 *For any nonempty subset $N \subseteq X$, the* partial minimum *function $MIN_N^{(m)}$ and the* partial maximum *function $MAX_N^{(m)}$ associated to N, are respectively defined by*

$$\begin{aligned} MIN_N^{(m)}(x_1,\ldots,x_m) &= \bigwedge_{i\in N} x_i \ \ \forall (x_1,\ldots,x_m)\in \mathbb{I}^m, \\ MAX_N^{(m)}(x_1,\ldots,x_m) &= \bigvee_{i\in N} x_i \ \ \forall (x_1,\ldots,x_m)\in \mathbb{I}^m. \end{aligned}$$

Any $\mathrm{MIN}_N^{(m)}$ function is a $\mathrm{WMAXMIN}_a^{(m)}$ function whose canonical form is defined by:

$$\forall T\subseteq X, T\neq\emptyset : a_T = \begin{cases} 1 & \text{if } T = N \\ 0 & \text{otherwise} \end{cases}$$

and complete form by:

$$\forall T\subseteq X, T\neq\emptyset : a_T = \begin{cases} 1 & \text{if } T \supseteq N \\ 0 & \text{otherwise.} \end{cases}$$

Any $\mathrm{MAX}_N^{(m)}$ function is a $\mathrm{WMINMAX}_b^{(m)}$ function whose canonical form is defined by:

$$\forall T\subseteq X, T\neq\emptyset : b_T = \begin{cases} 0 & \text{if } T = N \\ 1 & \text{otherwise} \end{cases}$$

and complete form by:

$$\forall T\subseteq X, T\neq\emptyset : b_T = \begin{cases} 0 & \text{if } T \supseteq N \\ 1 & \text{otherwise.} \end{cases}$$

Moreover, for all $N\subseteq X$, $N\neq\emptyset$, we have

$$\mathrm{MIN}_N^{(m)} = \mathrm{WMIN}_{e_{X\setminus N}}^{(m)} \text{ and } \mathrm{MAX}_N^{(m)} = \mathrm{WMAX}_{e_N}^{(m)}.$$

It follows that any Sugeno integral $\mathcal{S}_\mu^{(m)}$ is a partial maximum function (resp. partial minimum function) if and only if μ is a 0-1 possibility measure (resp. 0-1 necessity measure).

In multiperson game theory, a fuzzy measure defines a game. A unanimity game u_N for subset $N\subseteq X$ is such that $u_N(T)=1$ if and only if $T\supseteq N$, and is zero otherwise. This fuzzy measure thus defines a partial minimum function.

The following theorem gives a characterization of the class of partial maximum functions and of the class of partial minimum functions.

Theorem 5.4 (i) *The aggregation function $M^{(m)}$ defined on $\mathbb{I}^m$ fulfils (WI, UIn, MAX, SMINU) if and only if there exists a nonempty subset $N\subseteq X$ such that $M^{(m)} = MAX_N^{(m)}$.*
(ii) *The aggregation function $M^{(m)}$ defined on $\mathbb{I}^m$ fulfils (WI, UIn, MIN, SMAXU) if and only if there exists a nonempty subset $N\subseteq X$ such that $M^{(m)} = MIN_N^{(m)}$.*

Proof. (i) (Sufficiency). Trivial.

(Necessity). By Theorem 5.2, there exists a weight vector ω such that

$$M^{(m)} = \mathrm{WMAX}_\omega^{(m)} = \mathrm{WMAXMIN}_a^{(m)}$$

with $a_T = \bigvee_{i\in T}\omega_i$ for all $T\subseteq X$. By Theorem 5.1, $\omega_i = a_i \in \mathbb{B}$ for all $i\in X$.

Setting $N = \{i\in X|\omega_i = 1\}$, we obtain $M^{(m)} = \mathrm{WMAX}_{e_N}^{(m)} = \mathrm{MAX}_N^{(m)}$.

(ii) Similar to (i). ∎

5.5 Order statistics

Definition 5.6 *For any $k \in X$, the* order statistic *function $OS_k^{(m)}$ associated to the k-th argument is defined by*

$$OS_k^{(m)}(x_1, \ldots, x_m) = x_{(k)} \quad \forall (x_1, \ldots, x_m) \in \mathbb{I}^m.$$

Any $\mathrm{OS}_k^{(m)}$ function is a $\mathrm{WMAXMIN}_a^{(m)}$ function whose canonical form is defined by:

$$\forall T \subseteq X, T \neq \emptyset : a_T = \begin{cases} 1 & \text{if } |T| = m-k+1 \\ 0 & \text{otherwise} \end{cases}$$

and complete form by:

$$\forall T \subseteq X, T \neq \emptyset : a_T = \begin{cases} 1 & \text{if } |T| \geq m-k+1 \\ 0 & \text{otherwise.} \end{cases}$$

Of course, it is also a $\mathrm{WMINMAX}_b^{(m)}$ function whose canonical form is defined by:

$$\forall T \subseteq X, T \neq \emptyset : b_T = \begin{cases} 0 & \text{if } |T| = k \\ 1 & \text{otherwise} \end{cases}$$

and complete form by:

$$\forall T \subseteq X, T \neq \emptyset : b_T = \begin{cases} 0 & \text{if } |T| \geq k \\ 1 & \text{otherwise.} \end{cases}$$

Therefore, we have, for all $k \in X$,

$$\begin{aligned} x_{(k)} &= \bigvee_{T \subseteq X, |T| = m-k+1} \big(\bigwedge_{i \in T} x_i\big) = \bigvee_{1 \leq i_1 < \ldots < i_{m-k+1} \leq m} (x_{i_1} \wedge \ldots \wedge x_{i_{m-k+1}}) \\ &= \bigwedge_{T \subseteq X, |T| = k} \big(\bigvee_{i \in T} x_i\big) = \bigwedge_{1 \leq i_1 < \ldots < i_k \leq m} (x_{i_1} \vee \ldots \vee x_{i_k}) \end{aligned}$$

and by Theorem 4.1,

$$x_{(k)} = \mathrm{median}(x_1, \ldots, x_m, \underbrace{1, \ldots, 1}_{k-1}, \underbrace{0, \ldots, 0}_{m-k}).$$

In particular, if $x_1, \ldots, x_{2k-1} \in \mathbb{I}$, we have

$$\begin{aligned} \mathrm{median}(x_1, \ldots, x_{2k-1}) = x_{(k)} &= \bigvee_{1 \leq i_1 < \ldots < i_k \leq 2k-1} (x_{i_1} \wedge \ldots \wedge x_{i_k}) \\ &= \bigwedge_{1 \leq i_1 < \ldots < i_k \leq 2k-1} (x_{i_1} \vee \ldots \vee x_{i_k}). \end{aligned}$$

For instance, we have

$$\begin{aligned} \mathrm{median}(x_1, x_2, x_3) &= (x_1 \wedge x_2) \vee (x_1 \wedge x_3) \vee (x_2 \wedge x_3) \\ &= (x_1 \vee x_2) \wedge (x_1 \vee x_3) \wedge (x_2 \vee x_3). \end{aligned}$$

The next characterization shows that the order statistics are exactly those $\mathrm{OWMAX}_\omega^{(m)}$ which fulfil (UIn) or those $\mathrm{BMAXMIN}_a^{(m)}$ which fulfils (Sy).

Theorem 5.5 *Let $M^{(m)}$ be any aggregation function defined on $\mathbb{I}^m$. Then the following three assertions are equivalent:*

$$\begin{array}{rl} (i) & M^{(m)} \text{ fulfils (Sy, UIn, SMINU, SMAXU)} \\ (ii) & M^{(m)} \text{ fulfils (Sy, I, Co, CM)} \\ (iii) & \text{There exists } k \in X \text{ such that } M^{(m)} = OS_k^{(m)}. \end{array}$$

Proof. $(iii) \Rightarrow (i)$ and (ii). Easy.

$(i) \Rightarrow (iii)$. By Theorem 5.3, there exists a weight vector ω such that

$$M^{(m)} = \mathrm{OWMAX}_{\omega}^{(m)} = \mathrm{WMAXMIN}_{a}^{(m)}$$

with $a_T = \omega_{m-|T|+1}$ for all $T \subseteq X$. By Theorem 5.1, $a_T \in \mathbb{B}$ for all $T \subseteq X$. This implies that there exists $k \in X$ such that $\omega = e_{\{1,\dots,k\}}$. We then have, for all $(x_1, \dots, x_m) \in \mathbb{I}^m$,

$$M^{(m)}(x_1, \dots, x_m) = \bigvee_{i=1}^{k} x_{(i)} = x_{(k)}.$$

$(ii) \Rightarrow (iii)$. Let $z_1, \dots, z_m \in \mathbb{I}$ such that $z_1 < \dots < z_m$. By Proposition 5.1, there exists $k \in X$ such that $M^{(m)}(z_1, \dots, z_m) = z_k$. Let $x_1, \dots, x_m \in \mathbb{I}$ reordered as $x_{(1)} \le \dots \le x_{(m)}$ and let us consider $\psi(x)$, a non decreasing function on $\mathbb{I}$ such that $\psi(z_i) = x_{(i)}$ for all $i \in X$. It is always possible to build a sequence $\phi_i \in \Phi$, $i \in \mathbb{N}_0$, such that $\lim_{i\to+\infty} \phi_i(x) = \psi(x)$ for all $x \in \mathbb{I}$. Moreover, by Proposition 5.1, $M^{(m)}$ fulfils (OS). We then have, using (Sy), (Co) and (OS),

$$\begin{aligned} M^{(m)}(x_1, \dots, x_m) &= M^{(m)}(x_{(1)}, \dots, x_{(m)}) = M^{(m)}(\psi(z_1), \dots, \psi(z_m)) \\ &= M^{(m)}(\lim_{i\to+\infty} \phi_i(z_1), \dots, \lim_{i\to+\infty} \phi_i(z_m)) \\ &= \lim_{i\to+\infty} \phi_i(M^{(m)}(z_1, \dots, z_m)) = \psi(z_k) = x_{(k)}. \end{aligned}$$

■

According to Theorem 5.5, we can readily see that any Sugeno integral $\mathcal{S}_{\mu}^{(m)}$ is an order statistic if and only if μ is a 0-1 fuzzy measure depending only on the cardinal of subsets.

5.6 Associative medians

Definition 5.7 *For any $\theta \in \mathbb{I}$, the* associative median *function $AMED_{\theta}^{(m)}$ associated to θ is defined by*

$$AMED_{\theta}^{(m)}(x_1, \dots, x_m) = median(\bigwedge_{i=1}^{m} x_i, \bigvee_{i=1}^{m} x_i, \theta) \quad \forall (x_1, \dots, x_m) \in \mathbb{I}^m.$$

Observe that, for all $\theta \in \mathbb{I}$ and all $(x_1, \dots, x_m) \in \mathbb{I}^m$, we have

$$\mathrm{median}(\bigwedge_{i=1}^{m} x_i, \bigvee_{i=1}^{m} x_i, \theta) = \mathrm{median}(x_1, \dots, x_m, \underbrace{\theta, \dots, \theta}_{m-1}).$$

Any $\mathrm{AMED}_\theta^{(m)}$ function is a $\mathrm{WMAXMIN}_a^{(m)}$ function whose canonical form is defined by:

$$\begin{cases} a_i = \theta & \text{for all } i \in X \\ a_T = 0 & \text{for all } T \subseteq X \text{ such that } |T| > 1 \text{ and } T \neq X \\ a_X = \begin{cases} 1 & \text{if } \theta < 1 \\ 0 & \text{if } \theta = 1 \end{cases} & \end{cases}$$

and complete form by:

$$\begin{cases} a_T = \theta & \text{for all } T \subseteq X \text{ such that } T \neq \emptyset, T \neq X \\ a_X = 1. & \end{cases}$$

Moreover, from Theorem 4.1, we immediately have, for all $(x_1, \ldots, x_m) \in \mathbb{I}^m$,

$$\mathrm{AMED}_\theta^{(m)}(x_1, \ldots, x_m) = x_{(1)} \vee [\bigvee_{i=2}^{m} (x_{(i)} \wedge \theta)] = x_{(1)} \vee (x_{(m)} \wedge \theta) = x_{(m)} \wedge (x_{(1)} \vee \theta).$$

It follows that any Sugeno integral $\mathcal{S}_\mu^{(m)}$ is an associative median if and only if the fuzzy measure μ is constant on $\mathcal{P}(X) \setminus \{\emptyset, X\}$.

Coming back to associative functions defined on $\mathbb{I}^2$, we have the following characterization.

Theorem 5.6 *Let $M^{(2)}$ be any aggregation function defined on $\mathbb{I}^2$. Then the following four assertions are equivalent:*

(i) *$M^{(2)}$ fulfils (In, I, Co, Sy, A)*
(ii) *There exists $\theta \in \mathbb{I}$ such that $M^{(2)}$=$AMED_\theta^{(2)}$*
(iii) *There exists $\omega \in \mathbb{I}^2$ such that $M^{(2)}$=$OWMAX_\omega^{(2)}$*
(iv) *There exists $\omega' \in \mathbb{I}^2$ such that $M^{(2)}$=$OWMIN_{\omega'}^{(2)}$*

Proof. $(i) \Leftrightarrow (iii) \Leftrightarrow (iv)$. See Theorems 3.3 and 5.3.

$(ii) \Leftrightarrow (iii)$. We simply have, if $(x_1, x_2) \in \mathbb{I}^2$,

$$\mathrm{AMED}_\theta^{(2)}(x_1, x_2) = x_{(1)} \vee (\theta \wedge x_{(2)}).$$

■

In Theorem 5.6, the equivalence $(i) \Leftrightarrow (ii)$ was already established by Dubois and Prade [5]. We also have the following corollary.

Corollary 5.1 *Let $(M^{(m)})_{m \in \mathbb{N} \setminus \{0,1\}}$ be a sequence of aggregation functions defined on $\mathbb{I}^m$. If the functions of this sequence are linked by the classical associative property and if there exists $\theta \in \mathbb{I}$ such that $M^{(2)}$ =$AMED_\theta^{(2)}$, then $M^{(m)}$ =$AMED_\theta^{(m)}$ for all $m \in \mathbb{N} \setminus \{0, 1\}$.*

6 Conclusion

We have investigated the Sugeno integral under the viewpoint of aggregation. In particular, it has been shown that this integral can be written under the form of a weighted max-min function, which has been introduced and studied in Section 3. An axiomatic characterization of the class of those functions has also been given with the help of some stability properties related to an algebra endowed with min and max operations.

The results of this paper contribute to the theory of fuzzy MCDM and offer a best understanding of the nature of the Sugeno integrals as fuzzy connectives.

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