A CLASSIFICATION OF BISYMMETRIC POLYNOMIAL FUNCTIONS OVER INTEGRAL DOMAINS OF CHARACTERISTIC ZERO

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Abstract. We describe the class of n-variable polynomial functions that satisfy Aczél's bisymmetry property over an arbitrary integral domain of characteristic zero with identity.

1. Introduction

Let $\mathcal{R}$ be an integral domain of characteristic zero (hence $\mathcal{R}$ is infinite) with identity and let $n \geqslant 1$ be an integer. In this paper we provide a complete description of all the n-variable polynomial functions over $\mathcal{R}$ that satisfy the (Aczél) bisymmetry property. Recall that a function $f{:}\mathcal{R}^n \to \mathcal{R}$ is *bisymmetric* if the n^2-variable mapping

$$\big(x_{11},\ldots,x_{1n};\ldots;x_{n1},\ldots,x_{nn}\big) \mapsto f\big(f(x_{11},\ldots,x_{1n}),\ldots,f(x_{n1},\ldots,x_{nn})\big)$$

does not change if we replace every x_{ij} by x_{ji}.

The bisymmetry property for n-variable real functions goes back to Aczél [1, 2]. It has been investigated since then in the theory of functional equations by several authors, especially in characterizations of mean functions and some of their extensions (see, e.g., [3, 5–7]). This property is also studied in algebra where it is called *mediality*. For instance, an algebra (A,f) where f is a bisymmetric binary operation is called a *medial groupoid* (see, e.g., [8, 9, 11]).

We now state our main result, which provides a description of the possible bisymmetric polynomial functions from $\mathcal{R}^n$ to $\mathcal{R}$. Let $\mathrm{Frac}(\mathcal{R})$ denote the fraction field of $\mathcal{R}$ and let $\mathbb{N}$ be the set of nonnegative integers. For any n-tuple $\mathbf{x} = (x_1,\ldots,x_n)$, we set $|\mathbf{x}| = \sum_{i=1}^n x_i$.

Main Theorem. *A polynomial function $P{:}\mathcal{R}^n \to \mathcal{R}$ is bisymmetric if and only if it is*

(*i*) *univariate, or*

(*ii*) *of degree $\leqslant 1$, that is, of the form*

$$P(\mathbf{x}) = a_0 + \sum_{i=1}^n a_i\, x_i\,,$$

where $a_i \in \mathcal{R}$ for $i = 0,\ldots,n$, or

Date: March 23, 2012.
2010 *Mathematics Subject Classification.* Primary 39B72; Secondary 13B25, 26B35.
Key words and phrases. Aczél's bisymmetry, mediality, polynomial function, integral domain.

(*iii*) *of the form*

$$P(\mathbf{x}) = a\prod_{i=1}^{n}(x_i + b)^{\alpha_i} - b\,,$$

where $a \in \mathcal{R}$, $b \in \mathrm{Frac}(\mathcal{R})$, *and* $\boldsymbol{\alpha} \in \mathbb{N}^n$ *satisfy* $ab^k \in \mathcal{R}$ *for* $k = 1,\ldots,|\boldsymbol{\alpha}|-1$ *and* $ab^{|\boldsymbol{\alpha}|} - b \in \mathcal{R}$.

The following example, borrowed from [10], gives a polynomial function of class (*iii*) for which $b \notin \mathcal{R}$.

Example 1. The third-degree polynomial function $P\colon\mathbb{Z}^3 \to \mathbb{Z}$ defined on the ring $\mathbb{Z}$ of integers by

$$P(x_1,x_2,x_3) = 9\,x_1x_2x_3 + 3\,(x_1x_2 + x_2x_3 + x_3x_1) + x_1 + x_2 + x_3$$

is bisymmetric since it is the restriction to $\mathbb{Z}$ of the bisymmetric polynomial function $Q\colon\mathbb{Q}^3 \to \mathbb{Q}$ defined on the field $\mathbb{Q}$ of rationals by

$$Q(x_1,x_2,x_3) = 9\prod_{i=1}^{3}\Big(x_i + \frac{1}{3}\Big) - \frac{1}{3}\,.$$

Since polynomial functions usually constitute the most basic functions, the problem of describing the class of bisymmetric polynomial functions is quite natural. On this subject it is noteworthy that a description of the class of bisymmetric lattice polynomial functions over bounded chains and more generally over distributive lattices has been recently obtained [4,5] (there bisymmetry is called self-commutation), where a lattice polynomial function is a function representable by combinations of variables and constants using the fundamental lattice operations $\wedge$ and $\vee$.

From the Main Theorem we can derive the following test to determine whether a non-univariate polynomial function $P\colon\mathcal{R}^n \to \mathcal{R}$ of degree $p \geqslant 2$ is bisymmetric. For $k \in \{p-1,p\}$, let P_k be the homogenous polynomial function obtained from P by considering the terms of degree k only. Then P is bisymmetric if and only if P_p is a monomial and $P_p(\mathbf{x}) = P(\mathbf{x}-b\mathbf{1})+b$, where $\mathbf{1} = (1,\ldots,1)$ and $b = P_{p-1}(\mathbf{1})/(p\,P_p(\mathbf{1}))$.

Note that the Main Theorem does not hold for an infinite integral domain $\mathcal{R}$ of characteristic $r > 0$. As a counterexample, the bivariate polynomial function $P(x_1,x_2) = x_1^r + x_2^r$ is bisymmetric.

In the next section we provide the proof of the Main Theorem, assuming first that $\mathcal{R}$ is a field and then an integral domain.

2. Technicalities and proof of the Main Theorem

We observe that the definition of $\mathcal{R}$ enables us to identify the ring $\mathcal{R}[x_1,\ldots,x_n]$ of polynomials of n indeterminates over $\mathcal{R}$ with the ring of polynomial functions of n variables from $\mathcal{R}^n$ to $\mathcal{R}$.

It is a straightforward exercise to show that the n-variable polynomial functions given in the Main Theorem are bisymmetric. We now show that no other n-variable polynomial function is bisymmetric. We first consider the special case when $\mathcal{R}$ is a field. We then prove the Main Theorem in the general case (i.e., when $\mathcal{R}$ is an integral domain of characteristic zero with identity).

Unless stated otherwise, we henceforth assume that $\mathcal{R}$ is a field of characteristic zero. Let $p \in \mathbb{N}$ and let $P\colon\mathcal{R}^n \to \mathcal{R}$ be a polynomial function of degree p. Thus P

can be written in the form

$$P(\mathbf{x}) = \sum_{|\boldsymbol{\alpha}|\leqslant p} c_{\boldsymbol{\alpha}}\, \mathbf{x}^{\boldsymbol{\alpha}}, \quad \text{with } \mathbf{x}^{\boldsymbol{\alpha}} = \prod_{i=1}^{n} x_i^{\alpha_i}\,,$$

where the sum is taken over all $\boldsymbol{\alpha} \in \mathbb{N}^n$ such that $|\boldsymbol{\alpha}| \leqslant p$.

The following lemma, which makes use of formal derivatives of polynomial functions, will be useful as we continue.

Lemma 2. *For every polynomial function $B\colon \mathcal{R}^n \to \mathcal{R}$ of degree p and every $\mathbf{x}_0, \mathbf{y}_0 \in \mathcal{R}^n$, we have*

$$B(\mathbf{x}_0 + \mathbf{y}_0) = \sum_{|\boldsymbol{\alpha}|\leqslant p} \frac{\mathbf{y}_0^{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!}\, (\partial_{\mathbf{x}}^{\boldsymbol{\alpha}} B)(\mathbf{x}_0)\,, \tag{1}$$

where $\partial_{\mathbf{x}}^{\boldsymbol{\alpha}} = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$ and $\boldsymbol{\alpha}! = \alpha_1! \cdots \alpha_n!$.

Proof. It is enough to prove the result for monomial functions since both sides of (1) are additive on the function B. We then observe that for a monomial function $B(\mathbf{x}) = c\,\mathbf{x}^{\boldsymbol{\beta}}$ the identity (1) reduces to the multi-binomial theorem. □

As we will see, it is useful to decompose P into its homogeneous components, that is, $P = \sum_{k=0}^{p} P_k$, where

$$P_k(\mathbf{x}) = \sum_{|\boldsymbol{\alpha}|=k} c_{\boldsymbol{\alpha}}\, \mathbf{x}^{\boldsymbol{\alpha}}$$

is the unique homogeneous component of degree k of P. In this paper the homogeneous component of degree k of a polynomial function R will often be denoted by $[R]_k$.

Since $P_p \neq 0$, the polynomial function $Q = P - P_p$, that is

$$Q(\mathbf{x}) = \sum_{|\boldsymbol{\alpha}|<p} c_{\boldsymbol{\alpha}}\, \mathbf{x}^{\boldsymbol{\alpha}},$$

is of degree $q < p$ and its homogeneous component $[Q]_q$ of degree q is P_q.

We now assume that P is a bisymmetric polynomial function. This means that the polynomial identity

$$P\big(P(\mathbf{r}_1), \ldots, P(\mathbf{r}_n)\big) - P\big(P(\mathbf{c}_1), \ldots, P(\mathbf{c}_n)\big) = 0 \tag{2}$$

holds for every $n \times n$ matrix

$$X = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} \in \mathcal{R}_n^n\,, \tag{3}$$

where $\mathbf{r}_i = (x_{i1}, \ldots, x_{in})$ and $\mathbf{c}_j = (x_{1j}, \ldots, x_{nj})$ denote its ith row and jth column, respectively. Since all the polynomial functions of degree $\leqslant 1$ are bisymmetric, we may (and henceforth do) assume that $p \geqslant 2$.

From the decomposition $P = P_p + Q$ it follows that

$$P\big(P(\mathbf{r}_1), \ldots, P(\mathbf{r}_n)\big) = P_p\big(P(\mathbf{r}_1), \ldots, P(\mathbf{r}_n)\big) + Q\big(P(\mathbf{r}_1), \ldots, P(\mathbf{r}_n)\big),$$

where $Q(P(\mathbf{r}_1), \ldots, P(\mathbf{r}_n))$ is of degree pq.

To obtain necessary conditions for P to be bisymmetric, we will equate the homogeneous components of the same degree $> pq$ of both sides of (2). By the previous observation this amounts to considering the equation

$$\Big[P_p\big(P(\mathbf{r}_1), \ldots, P(\mathbf{r}_n)\big) - P_p\big(P(\mathbf{c}_1), \ldots, P(\mathbf{c}_n)\big)\Big]_d = 0\,, \quad \text{for } pq < d \leqslant p^2. \tag{4}$$

By applying (1) to the polynomial function P_p and the n-tuples

$$\mathbf{x}_0 = (P_p(\mathbf{r}_1), \ldots, P_p(\mathbf{r}_n)) \quad \text{and} \quad \mathbf{y}_0 = (Q(\mathbf{r}_1), \ldots, Q(\mathbf{r}_n)),$$

we obtain

$$P_p(P(\mathbf{r}_1), \ldots, P(\mathbf{r}_n)) = \sum_{|\boldsymbol{\alpha}| \leqslant p} \frac{\mathbf{y}_0^{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!} \, \partial_{\mathbf{x}}^{\boldsymbol{\alpha}} P_p(\mathbf{x}_0) \tag{5}$$

and similarly for $P_p(P(\mathbf{c}_1), \ldots, P(\mathbf{c}_n))$. We then observe that in the sum of (5) the term corresponding to a fixed $\boldsymbol{\alpha}$ is either zero or of degree

$$q\,|\boldsymbol{\alpha}| + (p - |\boldsymbol{\alpha}|)\,p = p^2 - (p - q)\,|\boldsymbol{\alpha}|$$

and its homogeneous component of highest degree is obtained by ignoring the components of degrees $< q$ in Q, that is, by replacing $\mathbf{y}_0$ by $(P_q(\mathbf{r}_1), \ldots, P_q(\mathbf{r}_n))$.

Using (4) with $d = p^2$, which leads us to consider the terms in (5) for which $|\boldsymbol{\alpha}| = 0$, we obtain

$$P_p(P_p(\mathbf{r}_1), \ldots, P_p(\mathbf{r}_n)) - P_p(P_p(\mathbf{c}_1), \ldots, P_p(\mathbf{c}_n)) = 0. \tag{6}$$

Thus, we have proved the following claim.

Claim 3. *The polynomial function P_p is bisymmetric.*

We now show that P_p is a monomial function.

Proposition 4. *Let $H\colon \mathcal{R}^n \to \mathcal{R}$ be a bisymmetric polynomial function of degree $p \geqslant 2$. If H is homogeneous, then it is a monomial function.*

Proof. Consider a bisymmetric homogeneous polynomial $H\colon \mathcal{R}^n \to \mathcal{R}$ of degree $p \geqslant 2$. There is nothing to prove if H depends on one variable only. Otherwise, assume for the sake of a contradiction that H is the sum of at least two monomials of degree p, that is,

$$H(\mathbf{x}) = a\,\mathbf{x}^{\boldsymbol{\alpha}} + b\,\mathbf{x}^{\boldsymbol{\beta}} + \sum_{|\boldsymbol{\gamma}|=p} c_{\boldsymbol{\gamma}}\,\mathbf{x}^{\boldsymbol{\gamma}},$$

where $a\,b \neq 0$ and $|\boldsymbol{\alpha}| = |\boldsymbol{\beta}| = p$. Using the lexicographic order $\preccurlyeq$ over $\mathbb{N}^n$, we can assume that $\boldsymbol{\alpha} \succ \boldsymbol{\beta} \succ \boldsymbol{\gamma}$. Applying the bisymmetry property of H to the $n \times n$ matrix whose (i,j)-entry is $x_i y_j$, we obtain

$$H(\mathbf{x})^p\,H(\mathbf{y}^p) = H(\mathbf{y})^p\,H(\mathbf{x}^p),$$

where $\mathbf{x}^p = (x_1^p, \ldots, x_n^p)$. Regarding this equality as a polynomial identity in $\mathbf{y}$ and then equating the coefficients of its monomial terms with exponent $p\,\boldsymbol{\alpha}$, we obtain

$$H(\mathbf{x})^p = a^{p-1}\,H(\mathbf{x}^p). \tag{7}$$

Since $\mathcal{R}$ is of characteristic zero, there is a nonzero monomial term with exponent $(p-1)\,\boldsymbol{\alpha} + \boldsymbol{\beta}$ in the left-hand side of (7) while there is no such term in the right-hand side since $p\,\boldsymbol{\alpha} \succ (p-1)\,\boldsymbol{\alpha} + \boldsymbol{\beta} \succ p\,\boldsymbol{\beta}$ (since $p \geqslant 2$). Hence a contradiction. □

The next claim follows immediately from Proposition 4.

Claim 5. *P_p is a monomial function.*

By Claim 5 we can (and henceforth do) assume that there exist $c \in \mathcal{R} \smallsetminus \{0\}$ and $\boldsymbol{\gamma} \in \mathbb{N}^n$, with $|\boldsymbol{\gamma}| = p$, such that

$$P_p(\mathbf{x}) = c\,\mathbf{x}^{\boldsymbol{\gamma}}. \tag{8}$$

A polynomial function $F\colon\mathcal{R}^n\to\mathcal{R}$ is said to *depend on* its ith variable x_i (or x_i is *essential* in F) if $\partial_{x_i}F\neq 0$. The following claim shows that P_p determines the essential variables of P.

Claim 6. *If P_p does not depend on the variable x_j, then P does not depend on x_j.*

Proof. Suppose that $\partial_{x_j}P_p=0$ and fix $i\in\{1,\dots,n\}$, $i\neq j$, such that $\partial_{x_i}P_p\not\equiv 0$. By taking the derivative of both sides of (2) with respect to x_{ij}, the (i,j)-entry of the matrix X in (3), we obtain

$$(\partial_{x_i}P)(P(\mathbf{r}_1),\dots,P(\mathbf{r}_n))(\partial_{x_j}P)(\mathbf{r}_i)=(\partial_{x_j}P)(P(\mathbf{c}_1),\dots,P(\mathbf{c}_n))(\partial_{x_i}P)(\mathbf{c}_j). \tag{9}$$

Suppose for the sake of a contradiction that $\partial_{x_j}P\neq 0$. Thus, neither side of (9) is the zero polynomial. Let R_j be the homogeneous component of $\partial_{x_j}P$ of highest degree and denote its degree by r. Since P_p does not depend on x_j, we must have $r<p-1$. Then the homogeneous component of highest degree of the left-hand side in (9) is given by

$$(\partial_{x_i}P_p)(P_p(\mathbf{r}_1),\dots,P_p(\mathbf{r}_n))\,R_j(\mathbf{r}_i)$$

and is of degree $p(p-1)+r$. But the right-hand side in (9) is of degree at most $rp+p-1=(r+1)(p-1)+r<p(p-1)+r$, since $r<p-1$ and $p\geqslant 2$. Hence a contradiction. Therefore $\partial_{x_j}P=0$. □

We now give an explicit expression for $P_q=[P-P_p]_q$ in terms of P_p. We first present an equation that is satisfied by P_q.

Claim 7. *P_q satisfies the equation*

$$\sum_{i=1}^{n}P_q(\mathbf{r}_i)(\partial_{x_i}P_p)(P_p(\mathbf{r}_1),\dots,P_p(\mathbf{r}_n))=\sum_{i=1}^{n}P_q(\mathbf{c}_i)(\partial_{x_i}P_p)(P_p(\mathbf{c}_1),\dots,P_p(\mathbf{c}_n)) \tag{10}$$

for every matrix X as defined in (3).

Proof. By (6) and (8) we see that the left-hand side of (4) for $d=p^2$ is zero. Therefore, the highest degree terms in the sum of (5) are of degree $p^2-(p-q)>pq$ (because $(p-1)(p-q)>0$) and correspond to those $\boldsymbol{\alpha}\in\mathbb{N}^n$ for which $|\boldsymbol{\alpha}|=1$. Collecting these terms and then considering only the homogeneous component of highest degree (that is, replacing Q by P_q), we see that the identity (4) for $d=p^2-(p-q)$ is precisely (10). □

Claim 8. *We have*

$$P_q(\mathbf{x})=\frac{P_q(\mathbf{1})}{cp}\,P_p(\mathbf{x})\sum_{j=1}^{n}\frac{\gamma_j}{x_j^{p-q}}\,. \tag{11}$$

Moreover, $P_q=0$ if there exists $j\in\{1,\dots,n\}$ such that $0<\gamma_j<p-q$.

Proof. Considering Eq. (10) for a matrix X such that $\mathbf{r}_i=\mathbf{x}$ for $i=1,\dots,n$, we obtain

$$cp\,P_q(\mathbf{x})\,P_p(\mathbf{x})^{p-1}=P_q(\mathbf{1})\sum_{i=1}^{n}x_i^q(\partial_{x_i}P_p)(c\,x_1^p,\dots,c\,x_n^p).$$

Since $\partial_{x_i}P_p(\mathbf{x})=\gamma_i\,P_p(\mathbf{x})/x_i$, the previous equation becomes

$$cp\,P_q(\mathbf{x})\,P_p(\mathbf{x})^{p-1}=P_q(\mathbf{1})\,P_p(\mathbf{x})^p\sum_{i=1}^{n}\frac{\gamma_i}{x_i^{p-q}} \tag{12}$$

from which Eq. (11) follows. Now suppose that $P_q \neq 0$ and let $j \in \{1,\ldots,n\}$. Comparing the lowest degrees in x_j of both sides of (12), we obtain

$$(p-1)\,\gamma_j \leqslant \begin{cases} p\,\gamma_j - p + q\,, & \text{if } \gamma_j \neq 0,\\ p\,\gamma_j\,, & \text{if } \gamma_j = 0.\end{cases}$$

Therefore, we must have $\gamma_j = 0$ or $\gamma_j \geqslant p-q$, which ensures that the right-hand side of (11) is a polynomial. □

If $\varphi\colon\mathcal{R}\to\mathcal{R}$ is a bijection, we can associate with every function $f\colon\mathcal{R}^n\to\mathcal{R}$ its *conjugate* $\varphi(f)\colon\mathcal{R}^n\to\mathcal{R}$ defined by

$$\varphi(f)(x_1,\ldots,x_n) = \varphi^{-1}\big(f(\varphi(x_1),\ldots,\varphi(x_n))\big).$$

It is clear that f is bisymmetric if and only if so is $\varphi(f)$. We then have the following fact.

Fact 9. *The class of n-variable bisymmetric functions is stable under the action of conjugation.*

Since the Main Theorem involves polynomial functions over a ring, we will only consider conjugations given by translations $\varphi_b(x) = x+b$.

We now show that it is always possible to conjugate P with an appropriate translation φ_b to eliminate the terms of degree $p-1$ of the resulting polynomial function $\varphi_b(P)$.

Claim 10. *There exists $b \in R$ such that $\varphi_b(P)$ has no term of degree $p-1$.*

Proof. If $q < p-1$, we take $b = 0$. If $q = p-1$, then using (1) with $\mathbf{y}_0 = b\mathbf{1}$, we get

$$\big[\varphi_b(P)\big]_{p-1} = P_{p-1} + b\sum_{i=1}^{n}\partial_{x_i}P_p\,.$$

On the other hand, by (11) we have

$$P_{p-1} = \frac{P_{p-1}(\mathbf{1})}{c\,p}\sum_{i=1}^{n}\partial_{x_i}P_p\,.$$

It is then enough to choose $b = -P_{p-1}(\mathbf{1})/(c\,p)$ and the result follows. □

We can now prove the Main Theorem for polynomial functions of degree $\leqslant 2$.

Proposition 11. *The Main Theorem is true when $\mathcal{R}$ is a field of characteristic zero and P is a polynomial function of degree $\leqslant 2$.*

Proof. Let P be a bisymmetric polynomial function of degree $p \leqslant 2$. If $p \leqslant 1$, then P is in class (ii) of the Main Theorem. If $p = 2$, then by Claim 10 we see that P is (up to conjugation) of the form $P(\mathbf{x}) = c_2\,x_i\,x_j + c_0$. If $i = j$, then by Claim 6 we see that P is a univariate polynomial function, which corresponds to the class (i). If $i \neq j$, then by Claim 8 we have $c_0 = 0$ and hence P is a monomial (up to conjugation). □

By Proposition 11 we can henceforth assume that $p \geqslant 3$. We also assume that P is a bivariate polynomial function. The general case will be proved by induction on the number of essential variables of P.

Proposition 12. *The Main theorem is true when $\mathcal{R}$ is a field of characteristic zero and P is a bivariate polynomial function.*

Proof. Let P be a bisymmetric bivariate polynomial function of degree $p \geqslant 3$. We know that P_p is of the form $P_p(x,y) = c\,x^{\gamma_1} y^{\gamma_2}$. If $\gamma_1\gamma_2 = 0$, then by Claim 6 we see that P is a univariate polynomial function, which corresponds to the class (i).

Conjugating P, if necessary, we may assume that $P_{p-1} = 0$ (by Claim 10) and it is then enough to prove that $P = P_p$ (i.e., $P_q = 0$). If $\gamma_1 = 1$ or $\gamma_2 = 1$, then the result follows immediately from Claim 8 since $p - q \geqslant 2$. We may therefore assume that $\gamma_1 \geqslant 2$ and $\gamma_2 \geqslant 2$. We now prove that $P = P_p$ in three steps.

Step 1. $P(x,y)$ is of degree $\leqslant \gamma_1$ in x and of degree $\leqslant \gamma_2$ in y.

Proof. We prove by induction on $r \in \{0,\ldots,p-1\}$ that $P_{p-r}(x,y)$ is of degree $\leqslant \gamma_1$ in x and of degree $\leqslant \gamma_2$ in y. The result is true by our assumptions for $r = 0$ and $r = 1$ and is obvious for $r = p$. Considering Eq. (4) for $d = p^2 - r > pq$, with $\mathbf{r_1} = \mathbf{r_2} = (x,y)$, we obtain

$$[P(x,y)^p]_{p^2-r} = [P(x,x)^{\gamma_1}\,P(y,y)^{\gamma_2}]_{p^2-r}\,. \tag{13}$$

Clearly, the right-hand side of (13) is a polynomial function of degree $\leqslant p\gamma_1$ in x and $\leqslant p\gamma_2$ in y. Using the multinomial theorem, the left-hand side of (13) becomes

$$[P(x,y)^p]_{p^2-r} = \left[\left(\sum_{k=0}^{p} P_{p-k}(x,y)\right)^p\right]_{p^2-r} = \sum_{\boldsymbol{\alpha}\in A_{p,r}} \binom{p}{\boldsymbol{\alpha}} \prod_{k=0}^{p} P_{p-k}(x,y)^{\alpha_k}\,,$$

where

$$A_{p,r} = \Big\{\boldsymbol{\alpha} = (\alpha_0,\ldots,\alpha_p) \in \mathbb{N}^{p+1} : \sum_{k=0}^{p} k\,\alpha_k = r,\ |\boldsymbol{\alpha}| = p\Big\}.$$

Observing that for every $\boldsymbol{\alpha} \in A_{p,r}$ we have $\alpha_k = 0$ for $k > r$ and $\alpha_r \neq 0$ only if $\alpha_r = 1$ and $\alpha_0 = p-1$, we can rewrite (13) as

$$p\,P_p(x,y)^{p-1}\,P_{p-r}(x,y) = [P(x,x)^{\gamma_1}P(y,y)^{\gamma_2}]_{p^2-r} - \sum_{\substack{\boldsymbol{\alpha}\in A_{p,r}\\ \alpha_r=\cdots=\alpha_p=0}} \binom{p}{\boldsymbol{\alpha}} \prod_{k=0}^{r-1} P_{p-k}(x,y)^{\alpha_k}\,.$$

By induction hypothesis, the right-hand side is of degree $\leqslant p\gamma_1$ in x and of degree $\leqslant p\gamma_2$ in y. The result then follows by analyzing the highest degree terms in x and y of the left-hand side. $\square$

Step 2. $P(x,y)$ factorizes into a product $P(x,y) = U(x)\,V(y)$.

Proof. By Step 1, we can write

$$P(x,y) = \sum_{k=0}^{\gamma_1} x^k\,V_k(y)\,,$$

where V_k is of degree $\leqslant \gamma_2$ and $V_{\gamma_1}(y) = \sum_{j=0}^{\gamma_2} c_{\gamma_2-j}\,y^j$, with $c_0 = c \neq 0$ and $c_1 = 0$ (since $P_{p-1} = 0$). Equating the terms of degree γ_1^2 in z in the identity

$$P(P(z,t),P(x,y)) = P(P(z,x),P(t,y))\,,$$

we obtain

$$V_{\gamma_1}(t)^{\gamma_1}\,V_{\gamma_1}(P(x,y)) = V_{\gamma_1}(x)^{\gamma_1}\,V_{\gamma_1}(P(t,y)).$$

Equating now the terms of degree $\gamma_1\gamma_2$ in t in the latter identity, we obtain

$$c^{\gamma_1}\,V_{\gamma_1}(P(x,y)) = c\,V_{\gamma_1}(x)^{\gamma_1}\,V_{\gamma_1}(y)^{\gamma_2}\,. \tag{14}$$

We now show by induction on $r \in \{0,\ldots,\gamma_1\}$ that every polynomial function V_{γ_1-r} is a multiple of V_{γ_1} (the case $r = 0$ is trivial), which is enough to prove the result.

To do so, we equate the terms of degree $\gamma_1\gamma_2 - r$ in x in (14) (by using the explicit form of V_{γ_1} in the left-hand side). Note that terms with such a degree in x can appear in the expansion of $V_{\gamma_1}(P(x,y))$ only when $P(x,y)$ is raised to the highest power γ_2 (taking into account the condition $c_1 = 0$ when $r = \gamma_1$). Thus, we obtain

$$c^{\gamma_1+1}\left[\left(\sum_{k=0}^{\gamma_1} x^{\gamma_1-k}\, V_{\gamma_1-k}(y)\right)^{\gamma_2}\right]_{\gamma_1\gamma_2-r} = c\,[V_{\gamma_1}(x)^{\gamma_1}]_{\gamma_1\gamma_2-r}V_{\gamma_1}(y)^{\gamma_2}\,,$$

(here the notation $[\cdot]_{\gamma_1\gamma_2-r}$ concerns only the degree in x). By computing the left-hand side (using the multinomial theorem as in the proof of Step 1) and using the induction on r, we finally obtain an identity of the form

$$a\,V_{\gamma_1}(y)^{\gamma_2-1}\,V_{\gamma_1-r}(y) = a'\,V_{\gamma_1}(y)^{\gamma_2}, \qquad a,a' \in \mathcal{R},\ a \neq 0,$$

from which the result immediately follows. □

Step 3. *U and V are monomial functions.*

Proof. Using (14) with $P(x,y) = U(x)\,V(y)$ and $V_{\gamma_1} = V$, we obtain

$$c^{\gamma_1}\sum_{j=0}^{\gamma_2} c_{\gamma_2-j}\,(U(x)\,V(y))^j = c\,V(x)^{\gamma_1}\,V(y)^{\gamma_2}. \tag{15}$$

Equating the terms of degree γ_2^2 in y in (15), we obtain

$$c^{\gamma_1+\gamma_2+1}\,U(x)^{\gamma_2} = c^{\gamma_2+1}\,V(x)^{\gamma_1}\,. \tag{16}$$

Therefore, (15) becomes

$$\sum_{j=0}^{\gamma_2-1} c_{\gamma_2-j}\,(U(x)\,V(y))^j = 0,$$

which obviously implies $c_k = 0$ for $k = 1,\ldots,\gamma_2$, which in turn implies $V(x) = c\,x^{\gamma_2}$. Finally, from (16) we obtain $U(x) = x^{\gamma_1}$. □

Steps 2 and 3 together show that $P = P_p$, which establishes the proposition. □

Recall that the action of the symmetric group $\mathfrak{S}_n$ on functions from $\mathcal{R}^n$ to $\mathcal{R}$ is defined by

$$\sigma(f)(x_1,\ldots,x_n) = f(x_{\sigma(1)},\ldots,x_{\sigma(n)}), \qquad \sigma \in \mathfrak{S}_n.$$

It is clear that f is bisymmetric if and only if so is $\sigma(f)$. We then have the following fact.

Fact 13. *The class of n-variable bisymmetric functions is stable under the action of the symmetric group $\mathfrak{S}_n$.*

Consider also the following action of identification of variables. For $f\colon\mathcal{R}^n \to \mathcal{R}$ and $i < j \in [n]$ we define the function $I_{i,j}f\colon\mathcal{R}^{n-1} \to \mathcal{R}$ as

$$(I_{i,j}f)(x_1,\ldots,x_{n-1}) = f(x_1,\ldots,x_{j-1},x_i,x_j,\ldots,x_{n-1}).$$

This action amounts to considering the restriction of f to the "subspace of equation $x_i = x_j$" and then relabeling the variables. By Fact 13 it is enough to consider the identification of the first and second variables, that is,

$$(I_{1,2}f)(x_1,\ldots,x_{n-1}) = f(x_1,x_1,x_2\ldots,x_{n-1}).$$

Proposition 14. *The class of n-variable bisymmetric functions is stable under identification of variables.*

Proof. To see that $I_{1,2}f$ is bisymmetric, it is enough to apply the bisymmetry of f to the $n \times n$ matrix
$$\begin{pmatrix} x_{1,1} & x_{1,1} & \cdots & x_{1,n-1} \\ x_{1,1} & x_{1,1} & \cdots & x_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n-1,1} & x_{n-1,1} & \cdots & x_{n-1,n-1} \end{pmatrix}.$$
To see that $I_{i,j}f$ is bisymmetric, we can similarly consider the matrix whose rows i and j are identical and the same for the columns (or use Fact 13). □

We now prove the Main Theorem by using both a simple induction on the number of essential variables of P and the action of identification of variables.

Proof of the Main Theorem when $\mathcal{R}$ is a field. We proceed by induction on the number of essential variables of P. By Proposition 12 the result holds when P depends on 1 or 2 variables only. Let us assume that the result also holds when P depends on $n-1$ variables ($n-1 \geqslant 2$) and let us prove that it still holds when P depends on n variables. By Proposition 11 we may assume that P is of degree $p \geqslant 3$. We know that $P_p(\mathbf{x}) = c\mathbf{x}^{\gamma}$, where $c \neq 0$ and $\gamma_i > 0$ for $i = 1, \ldots, n$ (cf. Claim 6). Up to a conjugation we may assume that $P_{p-1} = 0$ (cf. Claim 10). Therefore, we only need to prove that $P = P_p$. Suppose on the contrary that $P - P_p$ has a polynomial function $P_q \neq 0$ as the homogeneous component of highest degree. Then the polynomial function $I_{1,2}\,P$ has $n-1$ essential variables, is bisymmetric (by Proposition 14), has $I_{1,2}\,P_p$ as the homogeneous component of highest degree (of degree $p \geqslant 3$), and has no component of degree $p-1$. By induction hypothesis, $I_{1,2}\,P$ is in class (iii) of the Main Theorem with $b = 0$ (since it has no term of degree $p-1$) and hence it should be a monomial function. However, the polynomial function $[I_{1,2}\,P]_q = I_{1,2}\,P_q$ is not zero by (11), hence a contradiction. □

Proof of the Main Theorem when $\mathcal{R}$ is an integral domain. Using the identification of polynomials and polynomial functions, we can extend every bisymmetric polynomial function over an integral domain $\mathcal{R}$ with identity to a polynomial function on $\mathrm{Frac}(\mathcal{R})$. The latter function is still bisymmetric since the bisymmetry property for polynomial functions is defined by a set of polynomial equations on the coefficients of the polynomial functions. Therefore, every bisymmetric polynomial function over $\mathcal{R}$ is the restriction to $\mathcal{R}$ of a bisymmetric polynomial function over $\mathrm{Frac}(\mathcal{R})$. We then conclude by using the Main Theorem for such functions. □

Acknowledgments

The authors wish to thank J. Dascăl and E. Lehtonen for fruitful discussions. This research is supported by the internal research project F1R-MTH-PUL-12RDO2 of the University of Luxembourg.

References

[1] J. Aczél. The notion of mean values. *Norske Videnskabers Selskabs Forhandlinger*, 19:83–86, 1946.

[2] J. Aczél. On mean values. *Bull. of the Amer. Math. Soc.*, 54:392–400, 1948.

[3] J. Aczél and J. Dhombres. *Functional Equations in Several Variables.* Encyclopedia of Mathematics and Its Applications, vol. 31, Cambridge University Press, Cambridge, UK, 1989.

[4] M. Behrisch, M. Couceiro, K. A. Kearnes, E. Lehtonen, and Á. Szendrei. Commuting polynomial operations of distributive lattices. *Order*, to appear.

[5] M. Couceiro and E. Lehtonen. Self-commuting lattice polynomial functions on chains. *Aeq. Math.*, 81(3):263–278, 2011.

[6] J. Fodor and J.-L. Marichal. On nonstrict means. *Aeq. Math.*, 54(3):308-327, 1997.

[7] M. Grabisch, J.-L. Marichal, R. Mesiar, and E. Pap. *Aggregation Functions*. Encyclopedia of Mathematics and its Applications, vol. 127, Cambridge University Press, Cambridge, UK, 2009.

[8] J. Ježek and T. Kepka. Equational theories of medial groupoids. *Algebra Universalis*, 17: 174–190, 1983.

[9] J. Ježek and T. Kepka. Medial groupoids. *Rozpravy Československé Akad. Věd, Řada Mat. Přírod. Věd*, 93: 93 pp., 1983.

[10] J.-L. Marichal and P. Mathonet. A description of n-ary semigroups polynomial-derived from integral domains. *Semigroup Forum*, in press.

[11] J.-P. Soublin. Étude algébrique de la notion de moyenne. *J. Math. Pure Appl.*, 50:53–264, 1971. 53264.

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