

INTEGRAL GROUP RING OF THE FIRST MATHIEU SIMPLE GROUP

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Abstract

We investigate the classical Zassenhaus conjecture for the normalized unit group of the integral group ring of the simple Mathieu group M_{11} . As a consequence, for this group we confirm the conjecture by Kimmerle about prime graphs.

1 Introduction and main results

Let $V(\mathbb{Z}G)$ be the normalized unit group of the integral group ring $\mathbb{Z}G$ of a finite group G . The following famous conjecture was formulated by H. Zassenhaus in [15]:

Conjecture 1 (ZC) Every torsion unit $u \in V(\mathbb{Z}G)$ is conjugate within the rational group algebra $\mathbb{Q}G$ to an element of G .

This conjecture is already confirmed for several classes of groups but, in general, the problem remains open, and a counterexample is not known.

Various methods have been developed to deal with this conjecture. One of the original ones was suggested by I. S. Luthar and I. B. S. Passi [12, 13], and it was improved further by M. Hertweck [9]. Using this method, the conjecture was proved for several new classes of groups, in particular for S_5 and for some finite simple groups (see [4, 9, 10, 12, 13]).

The Zassenhaus conjecture appeared to be very hard, and several weakened variations of it were formulated (see, for example, [3]). One of the most interesting modifications was suggested by W. Kimmerle [11]. Let us briefly introduce it now.

Let G be a finite group. Denote by $\#(G)$ the set of all primes dividing the order of G . Then the *Gruenberg–Kegel graph* (or the *prime graph*) of G is a graph $\pi(G)$ with vertices labelled by primes from $\#(G)$, such that vertices p and q are adjacent if and only if there is an element of order pq in the group G . Then the conjecture by Kimmerle can be formulated in the following way:

Conjecture 2 (KC) If G is a finite group, then $\pi(G) = \pi(V(\mathbb{Z}G))$.

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For Frobenius groups and solvable groups this conjecture was confirmed in [11]. In the present paper we continue the investigation of **(KC)**, and confirm it for the first simple Mathieu group M_{11} , using the Luthar–Passi method. Moreover, this allows us to give a partial solution of **(ZC)** for M_{11} .

Our main results are the following:

Theorem 1 *Let $V(\mathbb{Z}G)$ be the normalized unit group of the integral group ring $\mathbb{Z}G$, where G is the simple Mathieu group M_{11} . Let u be a torsion unit of $V(\mathbb{Z}G)$ of order $|u|$. We have:*

- (i) *if $|u| \neq 12$, then $|u|$ coincides with the order of some element $g \in G$;*
- (ii) *if $|u| \in \{2, 3, 5, 11\}$, then u is rationally conjugate to some $g \in G$;*
- (iii) *if $|u| = 4$, then the tuple of partial augmentations of u belongs to the set*

$$\{(\nu_{2a}, \nu_{3a}, \nu_{4a}, \nu_{6a}, \nu_{5a}, \nu_{8a}, \nu_{8b}, \nu_{11a}, \nu_{11b}) \in \mathbb{Z}^9 \mid \nu_{kx} = 0, \\ kx \notin \{2a, 4a\}, (\nu_{2a}, \nu_{4a}) \in \{(0, 1), (2, -1)\}\};$$

- (iv) *if $|u| = 6$, then the tuple of partial augmentations of u belongs to the set*

$$\{(\nu_{2a}, \nu_{3a}, \nu_{4a}, \nu_{6a}, \nu_{5a}, \nu_{8a}, \nu_{8b}, \nu_{11a}, \nu_{11b}) \in \mathbb{Z}^9 \mid \nu_{kx} = 0, \\ kx \notin \{2a, 3a, 6a\}, (\nu_{2a}, \nu_{3a}, \nu_{6a}) \in \{(-2, 3, 0), (0, 0, 1), \\ (0, 3, -2), (2, -3, 2), (2, 0, -1)\}\};$$

- (v) *if $|u| = 8$, then the tuple of partial augmentations of u belongs to the set*

$$\{(\nu_{2a}, \nu_{3a}, \nu_{4a}, \nu_{6a}, \nu_{5a}, \nu_{8a}, \nu_{8b}, \nu_{11a}, \nu_{11b}) \in \mathbb{Z}^9 \mid \nu_{kx} = 0, \\ kx \notin \{4a, 8a, 8b\}, (\nu_{4a}, \nu_{8a}, \nu_{8b}) \in \{(0, 0, 1), (0, 1, 0), \\ (2, -1, 0), (2, 0, -1)\}\};$$

- (vi) *if $|u| = 12$, then the tuple of partial augmentations of u cannot belong to the set*

$$\mathbb{Z}^9 \setminus \{(\nu_{2a}, \nu_{3a}, \nu_{4a}, \nu_{6a}, \nu_{5a}, \nu_{8a}, \nu_{8b}, \nu_{11a}, \nu_{11b}) \in \mathbb{Z}^9 \mid \nu_{kx} = 0, \\ kx \notin \{2a, 4a, 6a\}, (\nu_{2a}, \nu_{4a}, \nu_{6a}) \in \{(-1, 1, 1), (1, 1, -1)\}\}.$$

Corollary 1 *Let $V(\mathbb{Z}G)$ be the normalized unit group of the integral group ring $\mathbb{Z}G$, where G is the simple Mathieu group M_{11} . Then $\pi(G) = \pi(V(\mathbb{Z}G))$, where $\pi(G)$ and $\pi(V(\mathbb{Z}G))$ are prime graphs of G and $V(\mathbb{Z}G)$, respectively. Thus, for M_{11} the conjecture by Kimmerle is true.*

2 Notation and preliminaries

Let $u = \sum \alpha_g g$ be a normalized torsion unit of order k and let $\nu_i = \varepsilon_{C_i}(u)$ be a partial augmentation of u . By S. D. Berman's Theorem [2] we have that $\nu_1 = 0$ and

$$\nu_2 + \nu_3 + \cdots + \nu_m = 1. \tag{2.1}$$

For any character χ of G of degree n , we have that $\chi(u) = \sum_{i=2}^m \nu_i \chi(h_i)$, where h_i is a representative of the conjugacy class C_i .

We need the following results:

Proposition 1 (see [12]) *Suppose that u is an element of $\mathbb{Z}G$ of order k . Let z be a primitive k -th root of unity. Then for every integer l and any character χ of G , the number*

$$\mu_l(u, \chi) = \frac{1}{k} \sum_{d|k} \text{Tr}_{\mathbb{Q}(z^d)/\mathbb{Q}} \{ \chi(u^d) z^{-dl} \} \quad (2.2)$$

is a non-negative integer.

Proposition 2 (see [6]) *Let u be a torsion unit in $V(\mathbb{Z}G)$. Then the order of u divides the exponent of G .*

Proposition 3 (see [12] and Theorem 2.7 in [14]) *Let u be a torsion unit of $V(\mathbb{Z}G)$. Let C be a conjugacy class of G . If p is a prime dividing the order of a representative of C but not the order of u then the partial augmentation $\varepsilon_C(u) = 0$.*

M. Hertweck (see [10], Proposition 3.1; [9], Lemma 5.6) obtained the next result:

Proposition 4 *Let G be a finite group and let u be a torsion unit in $V(\mathbb{Z}G)$.*

- (i) *If u has order p^n , then $\varepsilon_x(u) = 0$ for every x of G whose p -part is of order strictly greater than p^n .*
- (ii) *If x is an element of G whose p -part, for some prime p , has order strictly greater than the order of the p -part of u , then $\varepsilon_x(u) = 0$.*

Note that the first part of Proposition 4 gives a partial answer to the conjecture by A. Bovdi (see [1]). Also M. Hertweck ([9], Lemma 5.5) gives a complete answer to the same conjecture in the case when $G = \text{PSL}(2, \mathbb{F})$, where $\mathbb{F} = \text{GF}(p^k)$.

In the rest of the paper, for the partial augmentation ν_i we shall also use the notation ν_{kx} , where k is the order of the representative of the i -th conjugacy class, and x is a distinguishing letter for this particular class with elements of order k .

Proposition 5 (see [12] and Theorem 2.5 in [14]) *Let u be a torsion unit of $V(\mathbb{Z}G)$ of order k . Then u is conjugate in $\mathbb{Q}G$ to an element $g \in G$ if and only if for each d dividing k there is precisely one conjugacy class C_{i_d} with partial augmentation $\varepsilon_{C_{i_d}}(u^d) \neq 0$.*

Proposition 6 (see [6]) *Let p be a prime, and let u be a torsion unit of $V(\mathbb{Z}G)$ of order p^n . Then for $m \neq n$ the sum of all partial augmentations of u with respect to conjugacy classes of elements of order p^m is divisible by p .*

The Brauer character table modulo p of the group M_{11} will be denoted by $\mathfrak{BCI}(p)$.

3 Proof of the Theorem

It is well known [7, 8] that $|G| = 2^4 \cdot 3^2 \cdot 5 \cdot 11$ and $\exp(G) = 1320$. The character table of G , as well as the Brauer character tables $\mathfrak{BC}\mathfrak{T}(p)$, where $p \in \{2, 3, 5, 11\}$, can be found using the computational algebra system GAP [7].

Since the group G possesses elements of orders 2, 3, 4, 5, 6, 8 and 11, first of all we shall investigate units with these orders. After this, by Proposition 2, the order of each torsion unit divides the exponent of G , so it will be enough to consider units of orders 10, 12, 15, 22, 24, 33 and 55, because if u will be a unit of another possible order, then there is $t \in \mathbb{N}$ such that u^t has an order from this list. We shall prove that units of all these orders except 12 do not appear in $V(\mathbb{Z}G)$. For units of order 12 we are not able to prove this, but we reduce this question to only two cases.

Let $u \in V(\mathbb{Z}G)$ have order k . By S. D. Berman's Theorem [2] and Proposition 3 we have $\nu_{1a} = 0$ and

$$\begin{array}{ll}
 \nu_{3a} = \nu_{5a} = \nu_{6a} = \nu_{11a} = \nu_{11b} = 0 & \text{when } k = 2, 4, 8; \\
 \nu_{2a} = \nu_{4a} = \nu_{5a} = \nu_{6a} = \nu_{8a} = \nu_{8b} = \nu_{11a} = \nu_{11b} = 0 & \text{when } k = 3; \\
 \nu_{2a} = \nu_{3a} = \nu_{4a} = \nu_{6a} = \nu_{8a} = \nu_{8b} = \nu_{11a} = \nu_{11b} = 0 & \text{when } k = 5; \\
 \nu_{5a} = \nu_{11a} = \nu_{11b} = 0 & \text{when } k = 6; \\
 \nu_{2a} = \nu_{3a} = \nu_{4a} = \nu_{5a} = \nu_{6a} = \nu_{8a} = \nu_{8b} = 0 & \text{when } k = 11; \\
 \nu_{3a} = \nu_{6a} = \nu_{11a} = \nu_{11b} = 0 & \text{when } k = 10; \\
 \nu_{5a} = \nu_{11a} = \nu_{11b} = 0 & \text{when } k = 12; \\
 \nu_{2a} = \nu_{4a} = \nu_{6a} = \nu_{8a} = \nu_{8b} = \nu_{11a} = \nu_{11b} = 0 & \text{when } k = 15; \\
 \nu_{2a} = \nu_{3a} = \nu_{4a} = \nu_{5a} = \nu_{6a} = 0 & \text{when } k = 22; \\
 \nu_{5a} = \nu_{11a} = \nu_{11b} = 0 & \text{when } k = 24; \\
 \nu_{2a} = \nu_{4a} = \nu_{5a} = \nu_{6a} = \nu_{8a} = \nu_{8b} = 0 & \text{when } k = 33; \\
 \nu_{2a} = \nu_{3a} = \nu_{4a} = \nu_{6a} = \nu_{8a} = \nu_{8b} = 0 & \text{when } k = 55.
 \end{array} \tag{3.1}$$

It follows immediately by Proposition 5 that the units of orders 3 and 5 are rationally conjugate to some element of G .

Now we consider each case separately:

- Let u be an involution. Then using (3.1) and Proposition 4 we obtain that $\nu_{4a} = \nu_{8a} = \nu_{8b} = 0$, so $\nu_{2a} = 1$.
- Let u be a unit of order 4. Then by (3.1) and Proposition 4 we have $\nu_{2a} + \nu_{4a} = 1$. By (2.2), $\mu_0(u, \chi_3) = \frac{1}{4}(-4\nu_{2a} + 8) \geq 0$ and $\mu_2(u, \chi_3) = \frac{1}{4}(4\nu_{2a} + 8) \geq 0$, so $\nu_{2a} \in \{-2, -1, 0, 1, 2\}$. Now using the inequalities

$$\begin{aligned}
 \mu_0(u, \chi_5) &= \frac{1}{4}(6\nu_{2a} - 2\nu_{4a} + 14) \geq 0; \\
 \mu_2(u, \chi_5) &= \frac{1}{4}(-6\nu_{2a} + 2\nu_{4a} + 14) \geq 0,
 \end{aligned}$$

we get that there are only two integral solutions $(\nu_{2a}, \nu_{4a}) \in \{(0, 1), (2, -1)\}$ satisfying (2.1) and Proposition 6, such that all $\mu_i(u, \chi_j)$ are non-negative integers.

- Let u be a unit of order 6. Then by (2.1), (2.2) and Proposition 4 we obtain

$$\nu_{2a} + \nu_{3a} + \nu_{6a} = 1.$$

Now, using $\mathfrak{BCI}(11)$ from the system of inequalities $\mu_0(u, \chi_6) = \frac{1}{6}(-4\nu_{3a} + 12) \geq 0$ and $\mu_3(u, \chi_6) = \frac{1}{6}(4\nu_{3a} + 12) \geq 0$, we have that $\nu_{3a} \in \{-3, 0, 3\}$. Furthermore, from the system of inequalities

$$\begin{aligned}\mu_3(u, \chi_2) &= \frac{1}{6}(-2\nu_{2a} + 4\nu_{6a} + 8) \geq 0; \\ \mu_0(u, \chi_2) &= \frac{1}{6}(2\nu_{2a} - 4\nu_{6a} + 10) \geq 0; \\ \mu_1(u, \chi_2) &= \frac{1}{6}(\nu_{2a} - 2\nu_{6a} + 8) \geq 0,\end{aligned}$$

we get that $\nu_{2a} - 2\nu_{6a} \in \{-2, 4\}$, so $\nu_{6a} \in \{-2, -1, 0, 1, 2\}$. Using the inequalities

$$\begin{aligned}\mu_0(u, \chi_3) &= \frac{1}{6}(-4\nu_{2a} + 2\nu_{3a} + 2\nu_{6a} + 10) \geq 0; \\ \mu_2(u, \chi_3) &= \frac{1}{6}(2\nu_{2a} - \nu_{3a} - \nu_{6a} + 7) \geq 0,\end{aligned}$$

we obtain only the following integral solutions $(\nu_{2a}, \nu_{3a}, \nu_{6a})$:

$$\{(-2, 3, 0), (0, 0, 1), (0, 3, -2), (2, -3, 2), (2, 0, -1)\}, \quad (3.2)$$

such that all $\mu_i(u, \chi_j)$ are non-negative integers.

Using the GAP package LAGUNA [5], we tested all possible $\mu_i(u, \chi_j)$ for all tuples $(\nu_{2a}, \nu_{3a}, \nu_{6a})$ from (3.2), and were not able to produce a contradiction. Thus, in this case, as well as in the case of elements of order 4, the Luthar–Passi method is not enough to prove the rational conjugacy.

- Let u be a unit of order 8. By (3.1) and Proposition 4 we have

$$\nu_{2a} + \nu_{4a} + \nu_{8a} + \nu_{8b} = 1.$$

Since $|u^2| = 4$, by (3.2) it yields that $\chi_j(u^2) = \bar{\nu}_{2a}\chi_j(2a) + \bar{\nu}_{4a}\chi_j(4a)$. Now using $\mathfrak{BCI}(3)$, by (2.2) in the case when $(\bar{\nu}_{2a}, \bar{\nu}_{4a}) = (0, 1)$ we obtain

$$\begin{aligned}\mu_0(u, \chi_5) &= \frac{1}{8}(-8\nu_{2a} + 8) \geq 0; & \mu_4(u, \chi_5) &= \frac{1}{8}(8\nu_{2a} + 8) \geq 0; \\ \mu_0(u, \chi_4) &= \frac{1}{8}(8\nu_{2a} + 8\nu_{4a} + 16) \geq 0; & \mu_4(u, \chi_4) &= \frac{1}{8}(-8\nu_{2a} - 8\nu_{4a} + 16) \geq 0; \\ \mu_1(u, \chi_2) &= \frac{1}{8}(4\nu_{8a} - 4\nu_{8b} + 4) \geq 0; & \mu_4(u, \chi_7) &= \frac{1}{8}(-8\nu_{8a} - 8\nu_{8b} + 24) \geq 0; \\ \mu_5(u, \chi_2) &= \frac{1}{8}(-4\nu_{8a} + 4\nu_{8b} + 4) \geq 0; & \mu_0(u, \chi_7) &= \frac{1}{8}(8\nu_{8a} + 8\nu_{8b} + 24) \geq 0.\end{aligned}$$

It follows that $-1 \leq \nu_{2a} \leq 1$, $-3 \leq \nu_{4a} \leq 3$, $-2 \leq \nu_{8a}, \nu_{8b} \leq 2$. Considering the additional inequality

$$\mu_0(u, \chi_2) = \frac{1}{8}(4\nu_{2a} - 4\nu_{4a} - 4\nu_{8a} - 4\nu_{8b} + 4) \geq 0,$$

and using Proposition 6, it is easy to check that this system has the following integral solutions $(\nu_{2a}, \nu_{4a}, \nu_{8a}, \nu_{8b})$:

$$\left\{ \begin{array}{lll} (0, 2, 0, -1), & (0, 2, -1, 0), & (0, -2, 1, 2), \\ (0, -2, 2, 1), & (0, 0, 1, 0), & (0, 0, 0, 1) \end{array} \right\}. \quad (3.3)$$

In the case when $(\bar{\nu}_{2a}, \bar{\nu}_{4a}) = (2, -1)$ using $\mathfrak{BCI}(3)$, by (2.2) we obtained that $\mu_0(u, \chi_5) = -\mu_4(u, \chi_5) = -\nu_{2a} = 0$ and

$$\begin{aligned}\mu_0(u, \chi_4) &= \frac{1}{8}(8\nu_{4a} + 16) \geq 0; & \mu_4(u, \chi_4) &= \frac{1}{8}(-8\nu_{4a} + 16) \geq 0; \\ \mu_1(u, \chi_2) &= \frac{1}{8}(4\nu_{8a} - 4\nu_{8b} + 4) \geq 0; & \mu_5(u, \chi_2) &= \frac{1}{8}(-4\nu_{8a} + 4\nu_{8b} + 4) \geq 0.\end{aligned}$$

It is easy to check that $-2 \leq \nu_{4a}, \nu_{8a}, \nu_{8b} \leq 2$, and this system has the following integral solutions:

$$\left\{ \begin{array}{lll} (0, 2, -1, 0), & (0, 2, 0, -1), & (0, 0, 0, 1), \\ (0, -2, 2, 1), & (0, 0, 1, 0), & (0, -2, 1, 2) \end{array} \right\}. \quad (3.4)$$

Now using $\mathfrak{BCI}(11)$ in the case when $(\bar{\nu}_{2a}, \bar{\nu}_{4a}) = (0, 1)$, by (2.2) we get

$$\begin{aligned}\mu_0(u, \chi_3) &= \frac{1}{8}(-8\nu_{2a} + 8) \geq 0; & \mu_4(u, \chi_3) &= \frac{1}{8}(8\nu_{2a} + 8) \geq 0; \\ \mu_1(u, \chi_3) &= \frac{1}{8}(4t + 12) \geq 0; & \mu_5(u, \chi_3) &= \frac{1}{8}(-4t + 12) \geq 0; \\ \mu_0(u, \chi_2) &= \frac{1}{8}(4v - 4w + 12) \geq 0; & \mu_4(u, \chi_2) &= \frac{1}{8}(-4v + 4w + 12) \geq 0; \\ \mu_0(u, \chi_5) &= \frac{1}{8}(4z - 4w + 12) \geq 0; & \mu_4(u, \chi_5) &= \frac{1}{8}(-4z + 4w + 12) \geq 0,\end{aligned}$$

where $t = \nu_{8a} - \nu_{8b}$, $z = 3\nu_{2a} - \nu_{4a}$, $v = \nu_{2a} + \nu_{4a}$ and $w = \nu_{8a} + \nu_{8b}$. From this it follows that $-1 \leq \nu_{2a} \leq 1$, $-8 \leq \nu_{2a} \leq 10$, $-4 \leq \nu_{8a}, \nu_{8b} \leq 4$, and, using Proposition 6, it is easy to check that this system has the following integral solutions $(\nu_{2a}, \nu_{4a}, \nu_{8a}, \nu_{8b})$:

$$\left\{ \begin{array}{llll} (0, 2, 1, -2), & (0, 2, -2, 1), & (0, 2, 0, -1), & (0, 2, -1, 0), \\ (0, 0, 1, 0), & (0, 0, 2, -1), & (0, 0, 0, 1), & (0, 0, -1, 2) \end{array} \right\}. \quad (3.5)$$

In the case when $(\bar{\nu}_{2a}, \bar{\nu}_{4a}) = (2, -1)$, first using $\mathfrak{BCI}(11)$, by (2.2) we obtain $\mu_0(u, \chi_3) = -\mu_4(u, \chi_3) = -\nu_{2a} = 0$ and

$$\begin{aligned}\mu_1(u, \chi_3) &= \frac{1}{8}(4\nu_{8a} - 4\nu_{8b} + 12) \geq 0; \\ \mu_5(u, \chi_3) &= \frac{1}{8}(-4\nu_{8a} + 4\nu_{8b} + 12) \geq 0; \\ \mu_0(u, \chi_2) &= \frac{1}{8}(4\nu_{4a} - 4\nu_{8a} - 4\nu_{8b} + 12) \geq 0; \\ \mu_4(u, \chi_2) &= \frac{1}{8}(-4\nu_{4a} + 4\nu_{8a} + 4\nu_{8b} + 12) \geq 0; \\ \mu_0(u, \chi_5) &= \frac{1}{8}(-4\nu_{4a} - 4\nu_{8a} - 4\nu_{8b} + 28) \geq 0; \\ \mu_4(u, \chi_5) &= \frac{1}{8}(4\nu_{4a} + 4\nu_{8a} + 4\nu_{8b} + 28) \geq 0.\end{aligned}$$

It is easy to check that $-7 \leq \nu_{4a} \leq 9$, $-4 \leq \nu_{8a}, \nu_{8b} \leq 4$, and the system has the following integral solutions $(\nu_{2a}, \nu_{4a}, \nu_{8a}, \nu_{8b})$:

$$\left\{ \begin{array}{llll} (0, 2, -1, 0), & (0, 0, -1, 2), & (0, 2, 0, -1), & (0, 0, 0, 1), \\ (0, 0, 2, -1), & (0, 2, -2, 1), & (0, 0, 1, 0), & (0, 2, 1, -2) \end{array} \right\}. \quad (3.6)$$

It follows from (3.3)–(3.6) that the only four solutions which appear in both cases when $p = 3$ and $p = 11$ are the following ones: $\nu_{2a} = 0$ and

$$(\nu_{4a}, \nu_{8a}, \nu_{8b}) \in \{(0, 0, 1), (0, 1, 0), (2, -1, 0), (2, 0, -1)\}.$$

Again, we were not able to produce a contradiction computing all possible $\mu_i(u, \chi_j)$ for all above listed tuples $(\nu_{4a}, \nu_{8a}, \nu_{8b})$ for the ordinary character table of G as well as for $\mathfrak{BCI}(p)$, where $p \in \{3, 5, 11\}$.

- Let u be a unit of order 11. Then using $\mathfrak{BCI}(3)$, by (2.2) we have

$$\begin{aligned}\mu_1(u, \chi_2) &= \frac{1}{11}(6\nu_{11a} - 5\nu_{11b} + 5) \geq 0; \\ \mu_2(u, \chi_2) &= \frac{1}{11}(-5\nu_{11a} + 6\nu_{11b} + 5) \geq 0,\end{aligned}$$

which has only the following trivial solutions $(\nu_{11a}, \nu_{11b}) = \{(1, 0), (0, 1)\}$.

For all the above mentioned cases except elements of orders 4, 6 and 8 we see that there is precisely one conjugacy class with non-zero partial augmentation. Thus, by Proposition 5, part (ii) of the Theorem is proved.

It remains to prove parts (i) and (vi), considering units of $V(\mathbb{Z}G)$ of orders 10, 12, 15, 22, 24, 33 and 55. Now we treat each of these cases separately:

- Let u be a unit of order 10. Then by (2.1), (3.1) and Proposition 4 we get $\nu_{2a} + \nu_{5a} = 1$. Using $\mathfrak{BCI}(3)$, by (2.2) we have the system of inequalities

$$\begin{aligned}\mu_5(u, \chi_4) &= \frac{1}{10}(-8\nu_{2a} + 8) \geq 0; \\ \mu_0(u, \chi_4) &= \frac{1}{10}(8\nu_{2a} + 12) \geq 0; \\ \mu_2(u, \chi_2) &= \frac{1}{10}(-\nu_{2a} + 6) \geq 0,\end{aligned}$$

which has no integral solutions such that $\mu_5(u, \chi_4), \mu_0(u, \chi_4), \mu_2(u, \chi_2) \in \mathbb{Z}$.

- Let u be a unit of order 12. By (2.1), (3.1) and Proposition 4, we obtain that

$$\nu_{2a} + \nu_{3a} + \nu_{4a} + \nu_{6a} = 1.$$

Since $|u^2| = 6$ and $|u^3| = 4$, by (3.2) it yields that

$$\chi_j(u^2) = \bar{\nu}_{2a}\chi_j(2a) + \bar{\nu}_{3a}\chi_j(3a) + \bar{\nu}_{6a}\chi_j(6a)$$

and $\chi_j(u^3) = \tilde{\nu}_{2a}\chi_j(2a) + \tilde{\nu}_{4a}\chi_j(4a)$.

Consider the following four cases from parts (ii) and (iii) of the Theorem:

1. Let $(\bar{\nu}_{2a}, \bar{\nu}_{3a}, \bar{\nu}_{6a}) \in \{(0, 0, 1), (2, -3, 2), (-2, 3, 0)\}$ and suppose that

$$(\tilde{\nu}_{2a}, \tilde{\nu}_{4a}) \in \{(0, 1), (2, -1)\}.$$

Then by (2.2) we have $\mu_0(u, \chi_2) = \frac{1}{2} \notin \mathbb{Z}$, a contradiction.

2. Let $(\bar{\nu}_{2a}, \bar{\nu}_{3a}, \bar{\nu}_{6a}) = (2, 0, -1)$ and $(\tilde{\nu}_{2a}, \tilde{\nu}_{4a}) \in \{(0, 1), (2, -1)\}$. Then by (2.2) we have $\mu_1(u, \chi_3) = \frac{1}{2} \notin \mathbb{Z}$, a contradiction.

3. Let $(\bar{\nu}_{2a}, \bar{\nu}_{3a}, \bar{\nu}_{6a}) = (0, 3, -2)$ and $(\tilde{\nu}_{2a}, \tilde{\nu}_{4a}) = (0, 1)$. According to (2.2), $\mu_6(u, \chi_6) = -\mu_0(u, \chi_6) = \frac{2}{3}\nu_{3a} = 0$, so $\nu_{3a} = 0$, and we have the system

$$\begin{aligned}\mu_2(u, \chi_5) &= \frac{1}{12}(6\nu_{2a} - 2\nu_{4a} + 8) \geq 0; \\ \mu_4(u, \chi_5) &= \frac{1}{12}(-6\nu_{2a} + 2\nu_{4a} + 4) \geq 0; \\ \mu_2(u, \chi_3) &= \frac{1}{12}(-4\nu_{2a} + 2\nu_{6a} + 6) \geq 0; \\ \mu_4(u, \chi_3) &= \frac{1}{12}(4\nu_{2a} - 2\nu_{6a} + 6) \geq 0; \\ \mu_4(u, \chi_2) &= \frac{1}{12}(-4\nu_{2a} - 4\nu_{4a} + 2\nu_{6a} + 10) \geq 0; \\ \mu_2(u, \chi_2) &= \frac{1}{12}(4\nu_{2a} + 4\nu_{4a} - 2\nu_{6a} + 2) \geq 0,\end{aligned}$$

that has only two solutions $\{(-1, 0, 1, 1), (1, 0, 1, -1)\}$ with $\mu_i(u, \chi_j) \in \mathbb{Z}$.

4. Let $(\bar{\nu}_{2a}, \bar{\nu}_{3a}, \bar{\nu}_{6a}) = (0, 3, -2)$ and $(\tilde{\nu}_{2a}, \tilde{\nu}_{4a}) = (2, -1)$. Using (2.2), we get $\mu_6(u, \chi_6) = -\mu_0(u, \chi_6) = \frac{2}{3}\nu_{3a} = 0$, so $\nu_{3a} = 0$. Put $t = 2\nu_{2a} - \nu_{6a}$. Then by (2.2)

$$\mu_0(u, \chi_3) = \frac{1}{12}(-4t + 4) \geq 0; \quad \mu_4(u, \chi_3) = \frac{1}{12}(2t - 2) \geq 0,$$

so $2\nu_{2a} - \nu_{6a} = 1$. Now by (2.2) we have

$$\begin{aligned} \mu_2(u, \chi_5) &= \frac{1}{12}(2(3\nu_{2a} - \nu_{4a}) - 8) \geq 0, \\ \mu_0(u, \chi_9) &= \frac{1}{12}(-4(3\nu_{2a} - \nu_{4a}) + 28) \geq 0, \end{aligned}$$

and $3\nu_{2a} - \nu_{4a} = 4$. Using (2.1) we obtain that $\nu_{2a} = \nu_{4a} = -\nu_{6a} = 1$. Finally, $\mu_4(u, \chi_9) = \frac{1}{12}(6\nu_{2a} - 2\nu_{4a} + 28) = \frac{8}{3} \notin \mathbb{Z}$, a contradiction. Thus, part (vi) of the Theorem is proved.

- Let u be a unit of order 15. Then by (2.1) and (3.1) we have $\nu_{3a} + \nu_{5a} = 1$. Now using the character table of G , by (2.2) we get the system of inequalities

$$\mu_0(u, \chi_2) = \frac{1}{15}(8\nu_{3a} + 12) \geq 0; \quad \mu_5(u, \chi_2) = \frac{1}{15}(-4\nu_{3a} + 9) \geq 0,$$

that has no integral solutions such that $\mu_0(u, \chi_2), \mu_5(u, \chi_2) \in \mathbb{Z}$.

- Let u be a unit of order 22. Then by (2.1), (3.1) and Proposition 4 we obtain that

$$\nu_{2a} + \nu_{11a} + \nu_{11b} = 1.$$

In (2.2) we need to consider two cases: $\chi(u^2) = \chi(11a)$ and $\chi(u^2) = \chi(11b)$, but in both cases by (2.2) we have

$$\mu_0(u, \chi_2) = -\mu_{11}(u, \chi_2) = \frac{1}{22}(20\nu_{2a} - 10\nu_{11a} - 10\nu_{11b} + 2) = 0.$$

It yields $10\nu_{2a} - 5\nu_{11a} - 5\nu_{11b} = -1$, that has no integral solutions.

- Let u be a unit of order 24. Then by (2.1) and (3.1) we have

$$\nu_{2a} + \nu_{3a} + \nu_{4a} + \nu_{6a} + \nu_{8a} + \nu_{8b} = 1.$$

Since $|u^2| = 12$, $|u^4| = 6$, $|u^3| = 8$, $|u^6| = 4$, and G has two conjugacy classes of elements of order 8, we need to consider 40 various cases defined by parts (iii)–(vi) of the Theorem.

Let $(\bar{\nu}_{2a}, \bar{\nu}_{3a}, \bar{\nu}_{6a}) \in \{(0, 3, -2), (2, 0, -1)\}$, where

$$\chi_j(u^4) = \bar{\nu}_{2a}\chi_j(2a) + \bar{\nu}_{3a}\chi_j(3a) + \bar{\nu}_{6a}\chi_j(6a).$$

According to (2.2) we have that $\mu_1(u, \chi_2) = \frac{1}{2} \notin \mathbb{Z}$, a contradiction. In the remaining cases similarly we obtain that $\mu_1(u, \chi_2) = \frac{1}{4} \notin \mathbb{Z}$, which is also a contradiction.

- Let u be a unit of order 33. Then by (2.1) and (3.1) we have $\nu_{3a} + \nu_{11a} + \nu_{11b} = 1$. Again, in (2.2) we need to consider two cases: $\chi(u^3) = \chi(11a)$ and $\chi(u^3) = \chi(11b)$, but both cases lead us to the same system of inequalities

$$\mu_1(u, \chi_5) = \frac{1}{33}(2\nu_{3a} + 9) \geq 0; \quad \mu_{11}(u, \chi_5) = \frac{1}{33}(-20\nu_{3a} + 9) \geq 0,$$

which has no integral solutions with $\mu_1(u, \chi_5), \mu_{11}(u, \chi_5) \in \mathbb{Z}$.

- Let u be a unit of order 55. Then by (2.1) and (3.1) we have $\nu_{5a} + \nu_{11a} + \nu_{11b} = 1$. Considering two cases when $\chi(u^5) = \chi(11a)$ and $\chi(u^5) = \chi(11b)$, we get the same systems of inequalities

$$\mu_5(u, \chi_8) = \frac{1}{55}(4\nu_{5a} + 40) \geq 0; \quad \mu_5(u, \chi_5) = \frac{1}{55}(-4\nu_{5a} + 15) \geq 0,$$

which also has no integral solutions such that $\mu_5(u, \chi_5), \mu_5(u, \chi_8) \in \mathbb{Z}$.

Thus, the theorem is proved, and now the corollary follows immediately.

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