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FUNCTIONAL CENTRAL LIMIT THEOREMS ON LIE GROUPS. A SURVEY

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ABSTRACT

The general solution of the functional central limit problems for triangular arrays of random variables with values in a Lie group is described. The role of processes of finite variation is clarified. The special case of processes with independent increments having Markov generator is treated. Connections with Hille–Yosida theory for two–parameter evolution families of operators and with the martingale problem are explained.

1. INTRODUCTION

The functional central limit problem on a Lie group G can be formulated as follows. There is given a rowwise independent array $\{\xi_{n\ell} : (n, \ell) \in \mathbb{N}^2\}$ of G–valued random variables which for a sequence $\{k_n : n \in \mathbb{N}\}$ of increasing right–continuous scaling functions $k_n : \mathbb{R}_+ \to \mathbb{Z}_+$ with $k_n(0) = 0$ satisfies the infinitesimality condition

$$\lim_{n\to\infty} \max_{1\leqslant\ell\leqslant k_n(t)} \mathbb{P}(\xi_{n\ell} \notin U) = 0$$

for all Borel neighbourhoods U of the neutral element e of G and for all $t \in \mathbb{R}_+$. One forms the products

$$\xi_n(t) := \prod_{\ell=1}^{k_n(t)} \xi_{n\ell} := \xi_{n1}\xi_{n2}\cdots\xi_{n,k_n(t)}$$

and considers the sequence $\{\xi_n : n \in \mathbb{N}\}$ of stochastic processes with paths in the Skorokhod space $\mathbb{D}(\mathbb{R}_+, G)$ of càdlàg functions. One searches for conditions on the array and the scaling functions so that convergence $\xi_n \to \xi$ in distribution in the Skorokhod space holds, where $\xi = \{\xi(t) : t \in \mathbb{R}_+\}$ is a process, necessarily having independent (but not necessarily stationary) left–increments, i.e., $\xi(0) = e$ and for any sequence $0 \leqslant t_1 \leqslant t_2 \leqslant \ldots \leqslant t_n$ the increments $\xi(t_1)$, $(\xi(t_1))^{-1}\xi(t_2)$, ..., $(\xi(t_{n-1}))^{-1}\xi(t_n)$ are independent. We are interested in stochastically continuous limit processes (equivalently, in limit processes without fixed time of discontinuity), for which we can always choose a càdlàg version.

The problems are, more precisely,

- to parametrize the set $\mathrm{PII_c}(G)$ of the distributions $\mathbb{P}_\xi$ on $\mathbb{D}(\mathbb{R}_+, G)$ of stochastically continuous processes ξ with independent increments, i.e., to give a bijection between $\mathrm{PII_c}(G)$ and an appropriate parameter set $\mathcal{P}(\mathbb{R}_+, G)$;
- to relate suitable quantities K_n to the rows $\{\xi_{n\ell} : 1 \leqslant \ell \leqslant k_n(t)\}$, $n \in \mathbb{N}$, $t \in \mathbb{R}_+$, such that
$$\xi_n \to \xi \quad \Longleftrightarrow \quad K_n \to K,$$
where $K \in \mathcal{P}(\mathbb{R}_+, G)$ is the parameter corresponding to the limit process ξ, and the convergence $K_n \to K$ is meant in an appropriate sense.

One way to solve the first problem is to parametrize the *evolution family* $\{T(s,t) : 0 \leqslant s \leqslant t\}$ of the *convolution operators*

$$T(s,t)f(x) := \mathbb{E}f\big(x\xi(s)^{-1}\xi(t)\big), \qquad x \in G, \quad 0 \leqslant s \leqslant t,$$

initiated in (Siebert, 1982), where the class of Lipschitz continuous processes in $\mathrm{PII_c}(G)$ has been parametrized by the help of Lipschitz continuous functions $A : \mathbb{R}_+ \to \mathbb{A}(G)$ with values in the set of the generating functionals of stochastically continuous G–valued processes with stationary independent increments. We describe a solution to both problems for the whole set $\mathrm{PII_c}(G)$ (the proofs are given in (Pap, 1997)). The main idea is that for any process $\{\xi(t) : t \in \mathbb{R}_+\}$ from the set $\mathrm{PII_c}(G)$ there exists a continuous function $m : \mathbb{R}_+ \to G$ such that the shifted process $m(t)^{-1}\xi(t)$ is of continuous finite variation. The class of processes of continuous finite variation in $\mathrm{PII_c}(G)$ have been parametrized and sufficient conditions for convergence $\xi_n \to \xi$ toward a process of continuous finite variation in $\mathrm{PII_c}(G)$ have been given in (Heyer and Pap, 1997) if G is a Lie group (see Section 4), and in (Heyer and Pap, 1998) if G is a Lie projective group (in these cases the set of the functions $A : \mathbb{R}_+ \to \mathbb{A}(G)$ of continuous finite variation can be chosen as the parameter set). We remark that in (Feinsilver, 1978) another approach has been used, based on the characterization of Markov processes by associated martingales (see Section 6).

2. SOLUTION FOR $(\mathbb{R}^d, +)$

In order to formulate the answers in case of the group $(\mathbb{R}^d, +)$ we need some notations.

Let $h : \mathbb{R}^d \to \mathbb{R}^d$ be a *truncation function*, that is, a continuous function with compact support such that $h(x) = x$ in a neighborhood of 0.

Let $\mathbb{L}(\mathbb{R}_+, \mathbb{R}^d)$ be the set of (nonnegative) measures η on $\mathbb{R}_+ \times \mathbb{R}^d$ such that $\eta(\mathbb{R}_+ \times \{0\}) = 0$, $\int_{\mathbb{R}^d} (|y|^2 \wedge 1)\, \eta([0,t] \times dy) < \infty$ for all $t > 0$ and the mapping $t \mapsto \int_{\mathbb{R}^d} (|y|^2 \wedge 1)\, \eta([0,t] \times dy)$ is continuous.

By $\mathbb{M}_d$ we denote the space of $d \times d$ real matrices endowed with the supremum norm. By $\mathbb{M}_d^+$ we denote the subspace of symmetric positive semidefinite matrices in $\mathbb{M}_d$. A function $B : \mathbb{R}_+ \to \mathbb{M}_d$ is said to be *increasing* if $B(t) - B(s) \in \mathbb{M}_d^+$ for all $0 \leqslant s \leqslant t$. For a matrix $B \in \mathbb{M}_d$ and for a (column) vector $a \in \mathbb{R}^d$ we denote by $B^\top$ and $a^\top$ the transpose of B and a, respectively.

Let $\mathcal{P}(\mathbb{R}_+, \mathbb{R}^d)$ be the set of triples (a, B, η) where $a : \mathbb{R}_+ \to \mathbb{R}^d$ is continuous with $a(0) = 0$, $B : \mathbb{R}_+ \to \mathbb{M}_d^+$ is increasing and continuous with $B(0) = 0$ and $\eta \in \mathbb{L}(\mathbb{R}_+, \mathbb{R}^d)$.

The set $\mathrm{PII_c}(\mathbb{R}^d)$ of probability measures on $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$ generated by stochastically continuous processes $\xi = \{\xi(t) : t \in \mathbb{R}_+\}$ with independent increments can be parametrized by the parameter set $\mathcal{P}(\mathbb{R}_+, \mathbb{R}^d)$ giving the Fourier transform of the increments $\{\xi(t) - \xi(s) : 0 \leqslant s \leqslant t\}$ as follows (see, for instance, (Jacod and Shiryaev, 1987, II.5.2 Theorem)).

2.1 Theorem. *The relation*

$$\mathrm{PII_c}(\mathbb{R}^d) \ni \mathbb{P}_\xi \overset{\mathrm{F}}{\sim} (a, B, \eta) \in \mathcal{P}(\mathbb{R}_+, \mathbb{R}^d)$$

given by

$$\mathbb{E}\left[e^{\sqrt{-1}\langle u, \xi(t) - \xi(s)\rangle}\right] = \exp\Bigg\{ i\langle u, a(t) - a(s)\rangle - \frac{1}{2}\langle u, (B(t) - B(s))u\rangle + \int_{\mathbb{R}^d} \left(e^{i\langle u, y\rangle} - 1 - i\langle u, h(y)\rangle\right) \eta(]s,t] \times dy) \Bigg\}$$

for all $u \in \mathbb{R}^d$ *and* $0 \leqslant s \leqslant t$, *defines a bijection.*

Let $\mathcal{C}_0(\mathbb{R}^d)$ denote the space of real valued bounded continuous functions vanishing in some neighborhood of 0.

The answer to the second question, that is, a functional central limit theorem can be formulated as follows (see, for instance, (Jacod and Shiryaev, 1987, VII.3.4 Theorem)).

2.2 Theorem. *Let* $\{\xi_{n\ell} : (n, \ell) \in \mathbb{N}^2\}$ *be a rowwise independent array of* $\mathbb{R}^d$*–valued random variables. For all* $n \in \mathbb{N}$, *let* $k_n : \mathbb{R}_+ \to \mathbb{Z}_+$ *be an increasing right–continuous function with* $k_n(0) = 0$ *and* $k_n(\mathbb{R}_+) = \mathbb{Z}_+$. *Suppose that for all* $t \in \mathbb{R}_+$ *the system* $\{\xi_{n\ell} : n \in \mathbb{N}, 1 \leqslant \ell \leqslant k_n(t)\}$ *is*

infinitesimal. Let $\xi = \{\xi(t) : t \in \mathbb{R}_+\}$ *be an* $\mathbb{R}^d$*–valued process such that* $\mathbb{P}_\xi \overset{\mathrm{F}}{\sim} (a, B, \eta)$. *Let* D *be a dense set in* $\mathbb{R}_+$.
Then the following statements are equivalent:

(i) $\displaystyle\sum_{\ell=1}^{k_n(\cdot)} \xi_{n\ell} \to \xi.$

(ii) (a) $\displaystyle\sum_{\ell=1}^{k_n(t)} \mathbb{E}\,[h(\xi_{n\ell})] \to a(t)$ *uniformly in* $t \in [0, T]$ *for all* $T > 0$,

(b) $\displaystyle\sum_{\ell=1}^{k_n(t)} \operatorname{Cov}\,[h(\xi_{n\ell})] \to B(t) + \int_{\mathbb{R}^d} h(y)h(y)^\top \eta([0,t] \times dy)$ *for all* $t \in D$,

(c) $\displaystyle\sum_{\ell=1}^{k_n(t)} \mathbb{E}\,[f(\xi_{n\ell})] \to \int_{\mathbb{R}^d} f(y)\,\eta([0,t] \times dy)$ *for all* $t \in D$, $f \in \mathfrak{C}_0(\mathbb{R}^d)$.

The aim of the paper is to describe the generalization of the above theorems for Lie groups.

3. SOLUTION FOR LIE GROUPS

Let G be a σ–compact Lie group of dimension d with identity e. By $\mathfrak{L}(G)$ we denote the Lie algebra of G. Let $\exp_G : \mathfrak{L}(G) \to G$ be the exponential mapping. By $\mathfrak{C}^b(G)$ we denote the space of real valued bounded continuous functions on G furnished with the supremum norm $\|\cdot\|$. Let $\mathfrak{C}^0(G)$ and $\mathfrak{C}_e(G)$ be the subspaces of functions in $\mathfrak{C}^b(G)$ vanishing at infinity and vanishing in some neighbourhood of the identity, respectively. By $\mathfrak{D}(G)$ we denote the space of infinitely differentiable real–valued fuctions with compact support on G.
If $f \in \mathfrak{C}^b(G)$ is continuously differentiable in some neighbourhood of a $y \in G$ then for every $X \in \mathfrak{L}(G)$ there exist the left and right derivatives of f in y with respect to X defined by

$$Xf(y) := \lim_{t\to 0} \frac{f(\exp_G(tX)y) - f(y)}{t},$$
$$\widetilde{X}f(y) := \lim_{t\to 0} \frac{f(y\exp_G(tX)) - f(y)}{t},$$

respectively.
Let $\{X_1, \ldots, X_d\}$ be a basis of $\mathfrak{L}(G)$. Let $x_1, \ldots, x_d \in \mathfrak{D}(G)$ be a system of skew–symmetric canonical local coordinates of the first kind adapted to the basis $\{X_1, \ldots, X_d\}$ and valid in a compact neighbourhood U_0 of e,

i.e.,

$$y = \exp_G\left(\sum_{i=1}^{d} x_i(y) X_i\right) \qquad \text{for all } \ y \in U_0,$$

and $x_i(y^{-1}) = -x_i(y)$ for $i = 1, \ldots, d$. Let $\varphi : G \to [0,1]$ be a *Hunt function* for G, i.e., $1 - \varphi \in \mathbf{\mathfrak{D}}(G)$ and

$$\varphi(y) = \sum_{i=1}^{d} x_i(y)^2 \qquad \text{for all } \ y \in U_0.$$

Let $\mathbb{L}(\mathbb{R}_+, G)$ be the set of (nonnegative) measures η on $\mathbb{R}_+ \times G$ such that $\eta(\mathbb{R}_+ \times \{e\}) = 0$, $\int_G \varphi(y)\, \eta([0,t] \times dy) < \infty$ for all $t > 0$ and the mapping $t \mapsto \int_G \varphi(y)\, \eta([0,t] \times dy)$ is continuous.

Let $\mathcal{P}(\mathbb{R}_+, G)$ be the set of triples (m, B, η) where $m : \mathbb{R}_+ \to G$ is continuous with $m(0) = e$, $B : \mathbb{R}_+ \to \mathbb{M}_d^+$ is increasing and continuous with $B(0) = 0$ and $\eta \in \mathbb{L}(\mathbb{R}_+, G)$.

The set $\mathrm{PII_c}(G)$ of probability measures on $\mathbb{D}(\mathbb{R}_+, G)$ generated by stochastically continuous processes $\xi = \{\xi(t) : t \in \mathbb{R}_+\}$ with independent increments can be parametrized by the parameter set $\mathbb{P}(\mathbb{R}_+, G)$ giving the expectations

$$\left\{\mathbb{E}\left[f(z\widetilde{\xi}(s)^{-1}\widetilde{\xi}(t))\right] : 0 \leqslant s \leqslant t,\ z \in G,\ f \in \mathbf{\mathfrak{D}}(G)\right\}$$

of functionals of increments of the shifted process $\widetilde{\xi}(t) := \xi(t) m(t)^{-1}$ as follows (see (Pap, 1997)).

3.1 Theorem. *The relation*

$$\mathrm{PII_c}(G) \ni \mathbb{P}_\xi \sim (m, B, \eta) \in \mathcal{P}(\mathbb{R}_+, G)$$

given by

$$\begin{aligned} g_{s,t}(e) - g_{t,t}(e) = {} & \frac{1}{2} \sum_{i,j=1}^{d} \int_{]s,t]} X_i X_j g_{\tau,t}(e)\, B_{i,j}(d\tau) \\ & + \iint\limits_{]s,t] \times G} \left(g_{\tau,t}(y) - g_{\tau,t}(e) - \sum_{i=1}^{d} X_i g_{\tau,t}(e) x_i(y) \right) \eta(d\tau \times dy) \end{aligned}$$

for all $0 \leqslant s \leqslant t$ *and* $f \in \mathbf{\mathfrak{D}}(G)$, *where*

$$g_{s,t}(y) := \mathbb{E}\left[f(m(s) y m(s)^{-1} \widetilde{\xi}(s)^{-1} \widetilde{\xi}(t))\right],$$

$$\widetilde{\xi}(t) := \xi(t)m(t)^{-1},$$

defines a bijection.

A random variable ξ with values in G is said to have *local mean* $\mathrm{Mean}(\xi) \in U_0$ (with respect to the local coordinates $x_1, \dots, x_d$) if

$$x_i(\mathrm{Mean}(\xi)) = \mathbb{E}\, x_i(\xi) \qquad \text{for all } \ i = 1, \dots, d.$$

The answer to the second question, that is, a functional central limit theorem can be formulated as follows (see (Feinsilver, 1978, 3d, 3f) with martingale method and (Pap, 1997) with evolution equations).

3.2 Theorem. *Let* $\{\xi_{n\ell} : (n, \ell) \in \mathbb{N}^2\}$ *be a rowwise independent array of* G*–valued random variables. For all* $n \in \mathbb{N}$, *let* $k_n : \mathbb{R}_+ \to \mathbb{Z}_+$ *be an increasing right–continuous function with* $k_n(0) = 0$ *and* $k_n(\mathbb{R}_+) = \mathbb{Z}_+$. *Suppose that for all* $t \in \mathbb{R}_+$ *the system* $\{\xi_{n\ell} : n \in \mathbb{N}, 1 \leqslant \ell \leqslant k_n(t)\}$ *is infinitesimal. Let* $\xi = \{\xi(t) : t \in \mathbb{R}_+\}$ *be a* G*–valued process such that* $\mathbb{P}_\xi \sim (m, B, \eta)$. *Let* D *be a dense set in* $\mathbb{R}_+$.
Then the following statements are equivalent:

(i) $\displaystyle\prod_{\ell=1}^{k_n(\cdot)} \xi_{n\ell} \to \xi.$

(ii) (a) $\displaystyle\prod_{\ell=1}^{k_n(t)} \mathrm{Mean}(\xi_{n\ell}) \to m(t)$ *uniformly in* $t \in [0, T]$ *for all* $T > 0$,

(b) $\displaystyle\sum_{\ell=1}^{k_n(t)} \mathrm{Cov}(x_i(\xi_{n\ell}), x_j(\xi_{n\ell})) \to B_{i,j}(t) + \int_G x_i(y)x_j(y)\, \eta([0, t] \times dy)$ *for all* $t \in D$,

(c) $\displaystyle\sum_{\ell=1}^{k_n(t)} \mathbb{E}\,[f(\xi_{n\ell})] \to \int_G f(y)\, \eta([0, t] \times dy)$ *for all* $t \in D$, $f \in \mathfrak{C}_e(G)$.

It is interesting to note that the local coordinate functions $x_1, \dots, x_d$ play the role of truncation functions as well.

4. LIMIT PROCESSES OF FINITE VARIATION

The Banach space $\mathfrak{C}_2(G)$ of differentiable functions with norm

$$|f|_2 := \|f\| + \sum_{i=1}^{d} \|X_i f\| + \sum_{i,j=1}^{d} \|X_i X_j f\|$$

is defined as in (Heyer, 1977, 4.1.6) but with the notable difference to contain only functions vanishing at infinity.

Let $\mathbb{S} := \{(s,t) \in \mathbb{R}^2 : 0 \leqslant s \leqslant t\}$.

4.1 DEFINITION. *A function* $f : \mathbb{S} \to \mathbb{R}$ *is said to be of* (continuous) finite variation *if for all* $t \in \mathbb{R}_+$,

$$V_f(t) := \sup \left\{ \sum_{i=1}^{m} \|f(\tau_{i-1}, \tau_i)\| : 0 \leqslant \tau_0 < \tau_1 < \ldots < \tau_m \leqslant t,\, m \in \mathbb{N} \right\} < \infty$$

(and V_f *is continuous). A function* $g : \mathbb{R}_+ \to \mathbb{R}$ *is said to be of* (continuous) finite variation *if the function* $(s,t) \mapsto g(t) - g(s)$ *from* $\mathbb{S}$ *into* $\mathbb{R}$ *enjoys the corresponding property.*

A process $\xi = \{\xi(t) : t \in \mathbb{R}_+\}$ *is said to be of* weak (continuous) finite variation *if for all* $f \in \mathfrak{C}_2(G)$, *the function* $(s,t) \mapsto \mathbb{E}\left[f(\xi(s)^{-1}\xi(t))\right] - f(e)$ *from* $\mathbb{S}$ *into* $\mathbb{R}$ *is of (continuous) finite variation.* $\mathrm{PII}_{\mathrm{c,fv}}(G)$ *denotes the subset of* $\mathrm{PII}_{\mathrm{c}}(G)$ *containing the distributions of processes of weak continuous finite variation.*

For more information on functions and processes of finite variation see, for example, (Heyer and Pap, 1997; Born, 1990).

We remark that a process of weak (continuous) finite variation does not have necessarily trajectories of finite variation, but we have the following simple characterization for processes with independent increments (see Pap, 1997).

4.2 THEOREM. *Let* $\xi = \{\xi(t) : t \in \mathbb{R}_+\}$ *be a* G*–valued process such that* $\mathbb{P}_\xi \sim (m, B, \eta)$. *Then* ξ *is of weak continuous finite variation if and only if the function* m *is of finite variation.*

Note that an $\mathbb{R}^d$–valued process $\xi = \{\xi(t) : t \in \mathbb{R}_+\}$ with $\mathbb{P}_\xi \sim (m, B, \eta)$ is of weak continuous finite variation if and only it is a semimartingale.

Let $\mathcal{P}_{\mathrm{fv}}(\mathbb{R}_+, G)$ be the set of triples (a, B, η) where $a : \mathbb{R}_+ \to \mathbb{R}^d$ is of finite variation and continuous with $a(0) = 0$, $B : \mathbb{R}_+ \to \mathbb{M}_d^+$ is increasing and continuous with $B(0) = 0$ and $\eta \in \mathbb{L}(\mathbb{R}_+, G)$.

The set $\mathrm{PII}_{\mathrm{c,fv}}(G)$ of probability measures on $\mathbb{D}(\mathbb{R}_+, G)$ generated by stochastically continuous processes $\xi = \{\xi(t) : t \in \mathbb{R}_+\}$ of weak finite variation with independent increments can be parametrized by the parameter set $\mathcal{P}_{\mathrm{fv}}(\mathbb{R}_+, G)$ giving the expectations

$$\{\mathbb{E}\left[f(y\xi(s)^{-1}\xi(t))\right] : 0 \leqslant s \leqslant t,\ y \in G,\ f \in \mathfrak{D}(G)$$

of functionals of increments of the process ξ as follows (see (Heyer, Pap, 1997, Theorems 6 and 7)).

4.3 Theorem. *The relation*

$$\mathrm{PII}_{\mathrm{c,fv}}(G) \ni \mathbb{P}_\xi \overset{\mathrm{fv}}{\sim} (a, B, \eta) \in \mathcal{P}_{\mathrm{fv}}(\mathbb{R}_+, G)$$

given by

$$\begin{aligned} & h_{s,t}(e) - h_{t,t}(e) \\ &= \sum_{i=1}^d \int_{]s,t]} X_i h_{\tau,t}(e)\, a_i(d\tau) + \frac{1}{2} \sum_{i,j=1}^d \int_{]s,t]} X_i X_j h_{\tau,t}(e)\, B_{i,j}(d\tau) \\ &\quad + \iint_{]s,t]\times G} \Big(h_{\tau,t}(y) - h_{\tau,t}(e) - \sum_{i=1}^d X_i h_{\tau,t}(e) x_i(y) \Big) \eta(d\tau \times dy) \end{aligned}$$

for all $0 \leqslant s \leqslant t$ *and* $f \in \mathfrak{D}(G)$, *where*

$$h_{s,t}(y) := \mathbb{E}\left[f(y\xi(s)^{-1}\xi(t)) \right],$$

defines a bijection.

In (Heyer, Pap, 1997, Theorem 4) the following sufficient conditions have been proved for convergence toward a process of weak continuous finite variation.

4.4 Theorem. *Let* $\{\xi_{n\ell} : (n, \ell) \in \mathbb{N}^2\}$ *be a rowwise independent array of* G*–valued random variables. For all* $n \in \mathbb{N}$, *let* $k_n : \mathbb{R}_+ \to \mathbb{Z}_+$ *be an increasing right–continuous function with* $k_n(0) = 0$ *and* $k_n(\mathbb{R}_+) = \mathbb{Z}_+$. *Let* D *be a dense set in* $\mathbb{R}_+$.

Suppose that

(i) *there is a measure* $\eta \in \mathbb{L}(\mathbb{R}_+, G)$ *such that for all* $t \in D$ *and* $f \in \mathfrak{C}_e(G)$,

$$\sum_{\ell=1}^{k_n(t)} \mathbb{E}\left[f(\xi_{n\ell}) \right] \to \int_G f(y)\, \eta([0,t] \times dy),$$

(ii) *there is a continuous function* $a : \mathbb{R}_+ \to \mathbb{R}^d$ *such that for all* $t \in D$ *and* $i = 1, \dots, d$,

$$\sum_{\ell=1}^{k_n(t)} \mathbb{E}\left[x_i(\xi_{n,\ell}) \right] \to a_i(t),$$

(iii) *there is a continuous function* $B : \mathbb{R}_+ \to \mathbb{M}^d$ *such that for all* $t \in D$ *and* $i, j = 1, \dots, d$,

$$\sum_{\ell=1}^{k_n(t)} \mathrm{Cov}(x_i(\xi_{n\ell}), x_j(\xi_{n\ell})) \to B_{i,j}(t) + \int_G x_i(y) x_j(y)\, \eta([0,t] \times dy),$$

(iv) *for all* $T > 0$ *and* $i = 1, \dots, d$,

$$\lim_{\delta \to 0} \limsup_{n \to \infty} \sup_{\substack{|t-s| \leqslant \delta \\ 0 \leqslant s \leqslant t \leqslant T}} \sum_{\ell = k_n(s)+1}^{k_n(t)} \left| \mathbb{E}\left[x_i(\xi_{n,\ell}) \right] \right| = 0.$$

Then $(a, B, \eta) \in \mathcal{P}_{\mathrm{fv}}(\mathbb{R}_+, G)$ *and*

$$\prod_{\ell=1}^{k_n(\cdot)} \xi_{n\ell} \xrightarrow{\mathcal{L}} \xi,$$

where $\xi = \{\xi(t) : t \in \mathbb{R}_+\}$ *is a* G*–valued process such that* $\mathbb{P}_\xi \overset{\mathrm{fv}}{\sim} (a, B, \eta)$.

For $y \in G$ let $\mathrm{Ad}_y : \mathfrak{L}(G) \to \mathfrak{L}(G)$ be the linear mapping defined by

$$\exp(t \mathrm{Ad}_y(X)) = y \exp(tX) y^{-1}, \qquad X \in \mathfrak{L}(G),\ t \in \mathbb{R}.$$

With respect to the basis $\{X_1, \dots, X_d\}$ the linear mapping Ad_y can be expressed as a matrix $(\mathrm{Ad}_y^{ij})_{i,j=1,\dots,d}$ satisfying

$$\mathrm{Ad}_y(X_j) = \sum_{i=1}^{d} \mathrm{Ad}_y^{ij} \cdot X_i.$$

The following theorem explains the role of the shift in Theorem 3.1 (see (Pap, 1997)).

4.5 THEOREM. *If* $\xi = \{\xi(t) : t \in \mathbb{R}_+\}$ *is a* G*–valued process such that* $\mathbb{P}_\xi \sim (m, B, \eta) \in \mathcal{P}(\mathbb{R}_+, G)$ *then the shifted process* $\widetilde{\xi}(t) := \xi(t) m(t)^{-1}$, $t \geqslant 0$, *is of weak continuous finite variation such that* $\mathbb{P}_{\widetilde{\xi}} \overset{\mathrm{fv}}{\sim} (\widetilde{a}, \widetilde{B}, \widetilde{\eta}) \in \mathcal{P}_{\mathrm{fv}}(\mathbb{R}_+, G)$, *where*

$$\begin{aligned}
\widetilde{a}(i)(d\tau) &:= \int_G \left(x_i(m(\tau) y m(\tau)^{-1}) - \sum_{j=1}^{d} x_j(y) \mathrm{Ad}_{m(\tau)}^{ij} \right) \eta(dy \times d\tau), \\
\widetilde{B}(d\tau) &:= \mathrm{Ad}_{m(\tau)}\, B(d\tau)\, \mathrm{Ad}_{m(\tau)}^{\top}, \\
\widetilde{\eta}(dy \times d\tau) &:= \eta(m(\tau)^{-1}\, dy\, m(\tau) \times d\tau).
\end{aligned}$$

4.6 COROLLARY. *If* $\xi = \{\xi(t) : t \in \mathbb{R}_+\}$ *is an* $\mathbb{R}^d$*–valued process such that* $\mathbb{P}_\xi \sim (m, B, \eta) \in \mathcal{P}(\mathbb{R}_+, \mathbb{R}^d)$ *then* $\mathbb{P}_\xi \overset{\mathrm{F}}{\sim} (m, B, \eta)$ *and the shifted process* $\widetilde{\xi}(t) := \xi(t) - m(t)$, $t \geqslant 0$, *is of weak continuous finite variation such that* $\mathbb{P}_{\widetilde{\xi}} \overset{\mathrm{fv}}{\sim} (0, B, \eta) \in \mathcal{P}_{\mathrm{fv}}(\mathbb{R}_+, \mathbb{R}^d)$.

The next theorem describes the connection between the two parametrization in case of processes of weak finite variation.

4.7 THEOREM. *If* $\xi = \{\xi(t) : t \in \mathbb{R}_+\}$ *is a* G*–valued process such that* $\mathbb{P}_\xi \overset{\text{fv}}{\sim} (\widetilde{a}, \widetilde{B}, \widetilde{\eta})$ *then* $\mathbb{P}_\xi \sim (m, B, \eta) \in \mathcal{P}(\mathbb{R}_+, G)$ *where* $B = \widetilde{B}$, $\eta = \widetilde{\eta}$ *and the function* $m : \mathbb{R}_+ \to G$ *is determined by*

$$f(m(t)) - f(e) = \sum_{i=1}^{d} \int_0^t \widetilde{X}_i f(m(\tau))\, a_i(d\tau), \qquad t \geqslant 0,\ f \in \mathbf{\mathfrak{D}}(G).$$

4.8 COROLLARY. *If* $\xi = \{\xi(t) : t \in \mathbb{R}_+\}$ *is an* $\mathbb{R}^d$*–valued process such that* $\mathbb{P}_\xi \overset{\text{fv}}{\sim} (\widetilde{a}, \widetilde{B}, \widetilde{\eta}) \in \mathcal{P}_{\text{fv}}(\mathbb{R}_+, \mathbb{R}^d)$ *then* $\mathbb{P}_\xi \sim (\widetilde{a}, \widetilde{B}, \widetilde{\eta}) \in \mathcal{P}(\mathbb{R}_+, \mathbb{R}^d)$.

A G–valued process $\xi = \{\xi(t) : t \in \mathbb{R}_+\}$ with independent increments is a (not necessarily time–homogeneous) Markov process, which has, under some differentiability condition, (time–dependent) *infinitesimal generator*

$$N(t)f(y) := \lim_{h \to 0} \frac{\mathbb{E}\left[f(y\xi^{-1}(t)\xi(t+h))\right] - f(y)}{h}, \qquad t \geqslant 0.$$

Applying Theorem 4.3 we obtain the following subclass of processes of weak continuous finite variation with independent increments possessing Markov generators, which was also described in (Kunita, 1997, Theorem 3.1).

4.9 THEOREM. *Let* $a' : \mathbb{R}_+ \to \mathbb{R}^d$ *and* $B' : \mathbb{R}_+ \to \mathbb{M}_+^d$ *be continuous functions. For all* $t \geqslant 0$, *let* η'_t *be a (nonnegative) measure on* G *such that* $\eta'_t(\{e\}) = 0$, $\int_G \varphi(y)\, \eta'_t(dy) < \infty$ *and the mapping* $t \mapsto \int_G \varphi(y)\, \eta'_t(dy)$ *is continuous. For all* $t \geqslant 0$, *define*

$$\begin{aligned} a(t) &:= \int_0^t a'(\tau)\, d\tau, \\ B(t) &:= \int_0^t B'(\tau)\, d\tau, \\ \eta([0,t] \times dy) &:= \int_0^t \eta'_\tau(dy)\, d\tau. \end{aligned}$$

Then $(a, B, \eta) \in \mathcal{P}_{\text{fv}}(\mathbb{R}_+, G)$ *and a* G*–valued process* $\xi = \{\xi(t) : t \in \mathbb{R}_+\}$

with $\mathbb{P}_\xi \overset{\text{fv}}{\sim} (\widetilde{a}, \widetilde{B}, \widetilde{\eta})$ *has the (time–dependent) Markov generator*

$$N(t)f(y) := \sum_{i=1}^{d} a_i'(t)\widetilde{X}_i f(y) + \frac{1}{2}\sum_{i,j=1}^{d} B_{i,j}'(t)\widetilde{X}_i\widetilde{X}_j f(y) + \int_G \left(f(yz) - f(y) - \sum_{i=1}^{d} x_i(z)\widetilde{X}_i f(y) \right) \eta_t'(dz)$$

for all $f \in \mathfrak{D}(G)$.

The special case of processes with stationary independent increments is described in the following proposition.

4.10 Proposition. *Let* $\xi = \{\xi(t) : t \in \mathbb{R}_+\}$ *be a* G*–valued process with stationary independent increments, i.e.,* $\mathbb{P}_{\xi(s)^{-1}\xi(t)} = \mathbb{P}_{\xi(t-s)}$ *for all* $0 \leqslant s \leqslant t$. *Then* ξ *is of weak continuous finite variation and* $\mathbb{P}_\xi \overset{\text{fv}}{\sim} (a, B, \eta)$ *where* (a, B, η) *is linear, i.e.,* $a(t) = ta^{(0)}$ *with some* $a^{(0)} \in \mathbb{R}^d$, $B(t) = tB^{(0)}$ *with some* $B^{(0)} \in \mathbb{M}_+^d$ *and* $\eta([0,t] \times dy) = t\eta^{(0)}(dy)$ *with some measure* $\eta^{(0)}$ *on* G *such that* $\eta^{(0)}(\{e\}) = 0$ *and* $\int_G \varphi(y)\,\eta^{(0)}(dy) < \infty$. *Moreover, the process* ξ *is a time–homogeneous Markov process with (time–independent) infinitesimal generator*

$$Nf(y) := \sum_{i=1}^{d} a_i^{(0)}\widetilde{X}_i f(y) + \frac{1}{2}\sum_{i,j=1}^{d} B_{i,j}^{(0)}\widetilde{X}_i\widetilde{X}_j f(y) + \int_G \left(f(yz) - f(y) - \sum_{i=1}^{d} x_i(z)\widetilde{X}_i f(y) \right) \eta^{(0)}(dz)$$

for all $f \in \mathfrak{D}(G)$.

5. HILLE–YOSIDA THEORY FOR TWO–PARAMETER EVOLUTION FAMILIES

The *convolution operator* of a G–valued random variable ξ is the linear bounded operator T_ξ on the Banach space $\mathfrak{C}^0(G)$ defined by

$$T_\xi f(y) := \mathbb{E}\left[f(y\xi)\right], \qquad y \in G.$$

If $\xi = \{\xi(t) : t \in \mathbb{R}_+\}$ is a G–valued process of independent increments then the convolution operators

$$T(s,t) := T_{\xi(s)^{-1}\xi(t)}, \qquad 0 \leqslant s \leqslant t,$$

form an *evolution family* of bounded linear operators on the Banach space $\mathfrak{C}^0(G)$ in the sense of the following definition.

5.1 Definition. *A family* $\{T(s,t) : 0 \leqslant s \leqslant t\}$ *of bounded linear operators on a Banach space is called an* evolution family *if* $T(s,r)T(r,t) = T(s,t)$ *for all* $0 \leqslant s \leqslant r \leqslant t$, $T(t,t) = I$ *for all* $t \geqslant 0$, *and the mapping* $(s,t) \mapsto T(s,t)$ *is strongly continuous.*

Moreover, in this way we obtain a bijection between $\mathrm{PII}_{\mathrm{c}}(G)$ and the set of positive (left–) invariant bounded linear operators with norm 1 on the Banach space $\mathfrak{C}^0(G)$. Hence the parametrization problem can be viewed as a question of characterization of special evolution families. In contrast to the theory of continuous one–parameter semigroups of linear operators there are only partial results concerning evolution families. (See (Herod and McKelvey, 1980); application for characterization of processes of strong finite variation with independent increments and with values in a locally compact goup can be found in (Born, 1990).)

Theorem 3.1 gives the solution for the special evolution families on the Banach space $\mathfrak{C}^0(G)$ related to processes with independent increments. The equation in Theorem 3.1 can be viewed as a *shifted weak backward evolution equation* in the following way. For a triple $(a, B, \eta) \in \mathcal{P}_{\mathrm{fv}}(\mathbb{R}_+, G)$ and for a family $\{g_\tau : \tau \in \mathbb{R}_+\} \subset \mathfrak{D}(G)$ of functions let

$$\begin{aligned}\int_{]s,t]} A_{a,B,\eta}(d\tau)(g_\tau) := &\sum_{i=1}^{d} \int_{]s,t]} X_i g_\tau(e)\, a_i(d\tau) \\ &+ \frac{1}{2} \sum_{i,j=1}^{d} \int_{]s,t]} X_i X_j g_\tau(e)\, B_{i,j}(d\tau) \\ &+ \iint\limits_{]s,t]\times G} \Big(g_\tau(y) - g_\tau(e) - \sum_{i=1}^{d} X_i g_\tau(e) x_i(y) \Big) \eta(d\tau \times dy),\end{aligned}$$

provided that the integrals exist. (In fact, this integral can also be interpreted as a Riemann–Stieltjes integral with respect to a function $A_{a,B,\eta} : \mathbb{R}_+ \to \mathbb{A}(G)$ as in (Born, 1990).) Now Theorem 4.3 says that $\mathrm{PII}_{\mathrm{c,fv}}(G) \ni \mathbb{P}_\xi \overset{\mathrm{fv}}{\sim} (a, B, \eta) \in \mathcal{P}_{\mathrm{fv}}(\mathbb{R}_+, G)$ if and only if

$$(T_{\xi(s)^{-1}\xi(t)} - I) f(e) = \int_{]s,t]} A_{a,B,\eta}(d\tau)(T_{\xi(\tau)^{-1}\xi(t)} f).$$

Theorem 3.1 has a more complicated form, namely,

$$\mathrm{PII}_{\mathrm{c}}(G) \ni \mathbb{P}_\xi \sim (m, B, \eta) \in \mathcal{P}(\mathbb{R}_+, G)$$

is equivalent to

$$(T_{\widetilde{\xi}(s)^{-1}\widetilde{\xi}(t)} - I)f(e) = \int_{]s,t]} A_{0,B,\eta}(d\tau)(g_{\tau,t}),$$

where

$$\widetilde{\xi}(t) := \xi(t)m(t)^{-1},$$
$$g_{\tau,t}(y) := T_{\widetilde{\xi}(\tau)^{-1}\widetilde{\xi}(t)} f(m(\tau)ym(\tau)^{-1}),$$

or, equivalently,

$$(T_{\widetilde{\xi}(s)^{-1}\widetilde{\xi}(t)} - I)f(e) = \int_{]s,t]} A_{0,B,\eta}(d\tau)(L_{m(\tau)}R_{m(\tau)^{-1}}T_{\widetilde{\xi}(\tau)^{-1}\widetilde{\xi}(t)}f),$$

where for $f : G \to \mathbb{R}$ and $z \in G$ the functions $L_z f$ and $R_z f$ are defined by $L_z f(y) := f(zy)$, $R_z f(y) := f(yz)$, $y \in G$.

We note that the appropriate *forward evolution equations* also hold. The natural way is using the *right–invariant convolution operator* $\widetilde{T}_\xi$ of a random variable, defined on $\mathfrak{C}^0(G)$ by

$$\widetilde{T}_\xi f(y) := \mathbb{E}\left[f(\xi y)\right], \qquad y \in G.$$

For example, if $\mathrm{PII}_{\mathrm{c,fv}}(G) \ni \mathbb{P}_\xi \overset{\mathrm{fv}}{\sim} (a, B, \eta) \in \mathcal{P}_{\mathrm{fv}}(\mathbb{R}_+, G)$ then

$$(\widetilde{T}_{\xi(s)^{-1}\xi(t)} - I)f(e) = \int_{]s,t]} A_{a,B,\eta}(d\tau)(\widetilde{T}_{\xi(s)^{-1}\xi(\tau)}f).$$

Clearly, we have also

$$(\widetilde{T}_{\xi(s)^{-1}\xi(t)} - I)f(z) = \int_{]s,t]} A_{a,B,\eta}(d\tau)(R_z\widetilde{T}_{\xi(s)^{-1}\xi(\tau)}f)$$

for all $z \in G$. Using the arguments in (Siebert, 1982, Theorem 4.3 and Lemma 2.10) one can show that the above equation holds not only pointwise but also in the stronger sense defining the right hand side (as a function of $z \in G$) as a Bochner integral for functions with values in the Banach space $\mathfrak{C}^0(G)$.

Note that all of these evolution equations are, in fact, Volterra type integral equations.

In the special case of processes possessing Markov generators (see Theorem 4.9) we obtain Kolmogorov's forward and backward differential equations

$$\frac{\partial^+}{\partial t} T_{\xi(s)^{-1}\xi(t)} f = T_{\xi(s)^{-1}\xi(t)} N(t) f,$$
$$\frac{\partial^-}{\partial s} T_{\xi(s)^{-1}\xi(t)} f = -N(s) T_{\xi(s)^{-1}\xi(t)} f.$$

The above evolution equations are the integrated form of these equations.

6. MARTINGALE PROBLEM

6.1 Definition. *A* G*–valued process* $\xi = \{\xi(t) : t \in \mathbb{R}_+\}$ *and a triple* $(a, B, \eta) \in \mathcal{P}_{\mathrm{fv}}(\mathbb{R}_+, G)$ correspond to each other via associated martingales *if for all* $f \in \mathbf{D}(G)$ *the process*

$$f(\xi(t)) - \int_{[0,t]} A_{a,B,\eta}(d\tau)(L_{\xi(\tau)} f)$$

is a martingale (with the natural filtration).

In (Feinsilver, 1978, 3b) the following theorem is proved about martingale characterization of processes with independent increments.

6.2 Theorem. *For all triple* $(a, B, \eta) \in \mathcal{P}_{\mathrm{fv}}(\mathbb{R}_+, G)$ *there exists a* G*–valued process* $\xi = \{\xi(t) : t \in \mathbb{R}_+\}$ *with càdlàg paths which corresponds via associated martingale to the triple* (a, B, η)*. Moreover* ξ *is a stochastically continuous process with independent increments, and its distribution in* $\mathbb{D}(\mathbb{R}_+, G)$ *is uniquely determined by the triple* (a, B, η)*.*

The next statement describes the connection between characterization of processes with independent increments via associated martingales and by the parameter set $\mathcal{P}_{\mathrm{fv}}(\mathbb{R}_+, G)$ (see (Pap, 1997)).

6.3 Proposition. *A* G*–valued process* $\xi = \{\xi(t) : t \in \mathbb{R}_+\}$ *and a triple* $(a, B, \eta) \in \mathcal{P}_{\mathrm{fv}}(\mathbb{R}_+, G)$ *correspond to each other via associated martingales if and only if*

$$(\widetilde{T}_{\xi(s)^{-1}\xi(t)} - I) f(z) = \int_{]s,t]} A_{a,B,\eta}(d\tau)(R_z \widetilde{T}_{\xi(s)^{-1}\xi(\tau)} f)$$

holds for all $0 \leqslant s \leqslant t$*, for all* $f \in \mathbf{D}(G)$ *and for* $\mathbb{P}_{\xi(s)}$*–almost all* $z \in G$*. (In other words, the forward evolution equation holds* $\mathbb{P}_{\xi(s)}$*–almost everywhere.)*

6.4 Definition. *A* G*–valued process* $\xi = \{\xi(t) : t \in \mathbb{R}_+\}$ *and a triple* $(m, B, \eta) \in \mathcal{P}(\mathbb{R}_+, G)$ correspond to each other via associated shifted martingales *if for all* $f \in \mathbf{D}(G)$ *the process*

$$f(\xi(t)m(t)^{-1}) - \int_{[0,t]} A_{0,B,\eta}(d\tau)(L_{\xi(\tau)} R_{m(\tau)^{-1}} f)$$

is a martingale (with the natural filtration).

In (Feinsilver, 1978, 3e) the following theorem is proved about shifted martingale characterization of processes with independent increments.

6.5 THEOREM. *For all triple* $(m, B, \eta) \in \mathcal{P}(\mathbb{R}_+, G)$ *there exists a* G*–valued process* $\xi = \{\xi(t) : t \in \mathbb{R}_+\}$ *with càdlàg paths which corresponds via associated shifted martingale to the triple* (m, B, η). *Moreover* ξ *is a stochastically continuous process with independent increments, and its distribution in* $\mathbb{D}(\mathbb{R}_+, G)$ *is uniquely determined by the triple* (m, B, η).

The connection between characterization of processes with independent increments via associated shifted martingales and by the parameter set $\mathcal{P}(\mathbb{R}_+, G)$ can be handled exactly as in the non–shifted case (see (Pap, 1997)).

6.6 PROPOSITION. *A* G*–valued process* $\xi = \{\xi(t) : t \in \mathbb{R}_+\}$ *and a triple* $(m, B, \eta) \in \mathcal{P}(\mathbb{R}_+, G)$ *correspond to each other via associated shifted martingales if and only if*

$$(\widetilde{T}_{\xi(s)^{-1}\xi(t)} - I)f(z) = \int_{]s,t]} A_{a,B,\eta}(d\tau)(L_{m(\tau)} R_{m(\tau)^{-1}z} T_{\widetilde{\xi}(s)^{-1}\widetilde{\xi}(\tau)} f)$$

holds for all $0 \leqslant s \leqslant t$, *for all* $f \in \mathfrak{D}(G)$ *and for* $\mathbb{P}_{\xi(s)}$*–almost all* $z \in G$. *(In other words, the shifted forward evolution equation holds* $\mathbb{P}_{\xi(s)}$*–almost everywhere.)*

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