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Bethe Ansatz and Current Distribution for the TASEP with Particle-Dependent Hopping Rates

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Abstract. Using the Bethe ansatz we obtain in a determinant form the exact solution of the master equation for the conditional probabilities of the totally asymmetric exclusion process with particle-dependent hopping rates on \mathbf{Z} . From this we derive a determinant expression for the time-integrated current for a step-function initial state.

KEYWORDS: current distribution, random matrices, asymmetric simple exclusion process, Bethe ansatz, disorder

AMS SUBJECT CLASSIFICATION: 82C23, 82C22, 60K35, 15A52

1. Introduction

In the usual one-dimensional totally asymmetric simple exclusion process (TASEP) particles hop randomly in continuous time on the integer lattice \mathbf{Z} . A hopping event occurs independently for each particle after an exponentially distributed random time with parameter 1 to its right neighboring site, provided this site is empty; otherwise the attempted move is rejected [1, 2]. This is the exclusion interaction between biased random walkers, which physically models a short-range hard-core repulsion of identical driven diffusive particles. Here we consider the case of distinct particles where each particle i has its own intrinsic hopping rate v_i .

This model has been studied earlier by other authors. To our knowledge it was first introduced (with a random choice of rates v_i , drawn from some given distribution) by Benjamini et al. [3] who proved the existence of a product invariant distribution (as seen from a tagged particle) above some critical

density ρ^* of particles. They also proved for this case the hydrodynamic limit of an associated zero-range process (ZRP) which is obtained from the TASEP by identifying particles with the sites of a new 1-D lattice and the interparticle distance (number of empty sites between particles $i, i + 1$) as occupation number at site i of that lattice. Subsequently it was shown for periodic boundary conditions that below the critical density ρ^* a macroscopic traffic jam, trailing the slowest particle, develops [4, 5]. The macroscopic dynamical signature of the transition under Eulerian scaling is a discontinuity in the coarse-grained density at the front of a rarefaction wave that develops out of an initial step function profile [6]. The stationary distribution of an open system where particles are injected and extracted was obtained numerically [7]. Somewhat surprisingly, the theory of boundary-induced phase transitions developed by Kolomeisky et al. [8] accounts for the stationary phase diagram of the system as a function of the effective boundary densities.

The jamming transition occurring in this model is a classical analog of Bose–Einstein condensation and one of the motivations to study this system in more detail. An analogous transition occurs also if the rates are identical for each particle, but dependent on the lattice distance to the next particle, see [9] for a recent review (in terms of the ZRP) and [10–15] for the current developments. A second motivation comes from the recent interest in particle systems with several conservation laws [16]. By choosing the v_i in some way out of p distinct values the model indeed becomes a particle system with p conservation laws where, unlike in a similar class of models introduced by Karimipour [17–19], no overtaking of particles is permitted. Exact results for such multi-component systems are scarce. Our contribution aims at providing some tools for future exact and rigorous analysis. We first solve the master equation for an N -particle system defined on \mathbf{Z} with the Bethe ansatz generalizing the approach taken by one of us earlier [20]. This yields a determinant formula for the conditional probabilities which we then use to derive an exact expression for the distribution of the time-integrated current for the step initial configuration envisaged in [6]. Our result generalizes that obtained recently for the TASEP with constant rates [21, 22].

2. Solution of the master equation

We consider initial states such that there is no particle beyond a lattice point $x_{\max} < \infty$ at time $t = 0$. Since the process is totally asymmetric and defined on \mathbf{Z} the N rightmost particles are not influenced by particles to their left. Without loss of generality we may therefore consider N -particle initial states where the particles are labelled from 1 through N from left to right. Therefore at any finite time $t \geq 0$ a configuration of the system can be represented by an increasing sequence of integer numbers (x_1, x_2, \dots, x_N) , x_i denoting the position of the i th particle with hopping rate v_i . We assume $0 < v_i \leq v_{\max}$ where

$v_{\max} < \infty$ sets the microscopic time scale. We remark that allowing for $v_{\max} = \infty$ is equivalent to considering particles covering more than 1 (neighboring) site [23–28]. Such a system can be mapped to an exclusion process with standard particles (of lattice size 1) by an appropriate coordinate shift. The case $v_i = 0$ is trivial in so far as this amounts to cutting the lattice \mathbf{Z} into finite segments in which after some time all particles pile up at the right boundary of the segment and the dynamics freeze.

In case of equal hopping rates the exchange of particles is irrelevant as they are indistinguishable. However, in this generalized version one has to define the corresponding exchange rules separately since the particles have their own identity. In our model particles cannot exchange so one can interpret it as a one-lane traffic model with cars having their own preferred speed.

The master equation for the probability of finding particles at sites x_i is the following (the t -dependence is dropped for simplicity of notation):

$$\begin{aligned} \frac{d}{dt} \mathbb{P}(x_1, x_2, \dots, x_N) & \quad (2.1) \\ & = v_1 \mathbb{P}(x_1 - 1, x_2, \dots, x_N) + v_2(1 - \delta_{x_2-1, x_1}) \mathbb{P}(x_1, x_2 - 1, \dots, x_N) + \dots \\ & \quad + v_N(1 - \delta_{x_N-1, x_{N-1}}) \mathbb{P}(x_1, x_2 - 1, \dots, x_N - 1) \\ & \quad - (v_1(1 - \delta_{x_2-1, x_1}) + v_2(1 - \delta_{x_3-1, x_2}) + \dots + v_N) \mathbb{P}(x_1, x_2, \dots, x_N), \end{aligned}$$

where $\mathbb{P}(x_1, x_2, \dots, x_N)$ is the probability of the configuration (x_1, x_2, \dots, x_N) . In the spirit of [20] we extend the range of definition of $\mathbb{P}(x_1, x_2, \dots, x_N)$ from the physical domain $x_i < x_{i+1}$ to \mathbf{Z}^N . By requiring the boundary condition

$$\begin{aligned} v_{k+1} \mathbb{P}(x_1, \dots, x_{k-1}, x_k, x_k, x_{k+2}, \dots, x_N) & \quad (2.2) \\ & = v_k \mathbb{P}(x_1, \dots, x_{k-1}, x_k, x_k + 1, x_{k+2}, \dots, x_N) \end{aligned}$$

to be valid for all $t > 0$ the master equation (2.1) reduces to the simple form

$$\begin{aligned} \frac{d}{dt} \mathbb{P}(x_1, x_2, \dots, x_N) & \quad (2.3) \\ & = \sum_{i=1}^N v_i \mathbb{P}(x_1, \dots, x_{i-1}, x_i - 1, x_{i+1}, \dots, x_N) - \sum_{i=1}^N v_i \mathbb{P}(x_1, x_2, \dots, x_N) \\ & \equiv -\hat{H} \mathbb{P}(x_1, x_2, \dots, x_N). \end{aligned}$$

The second equality defines the linear operator \hat{H} generating the time evolution of the probability distribution. For given initial configuration (y_1, y_2, \dots, y_N) this quantity is the conditional probability $\mathbb{P}(x_1, \dots, x_N \mid y_1, \dots, y_N; t)$.

It is obvious that (2.3) can be solved by Fourier transformation with a “momentum” variable k_i associated with each particle coordinate x_i . However, a straightforward Fourier ansatz does not satisfy the boundary condition (2.2).

This problem can be resolved by noting that there is some freedom by choosing a linear combination of Fourier transforms over all permutations of the dummy momentum variables k_i with complex coefficients (wave amplitudes) that depend on the k_i . It is the essence of the Bethe ansatz to take this approach and to implement it by factorizing the wave amplitudes for the N -particle problem into amplitudes for the two-particle problem. It is done such that for each permutation of the momentum variables the N -particle amplitude picks up the corresponding two-particle amplitude as extra factor.

In the present case this strategy also turns out to be successful, but requires some modification due to the occurrence of the different hopping rates v_i . In order to obtain the eigenvalues and eigenfunctions of the ‘‘Hamiltonian’’ \hat{H} with the boundary condition (2.2) we write

$$\frac{d}{dt} \mathbf{P}_{\mathbf{k}}(x_1, \dots, x_N) = -E_{\mathbf{k}} \mathbf{P}_{\mathbf{k}}(x_1, \dots, x_N). \quad (2.4)$$

where \mathbf{k} represents the conjugate momenta k_1, \dots, k_N . Modifying the Bethe ansatz to account for the v_i leads to eigenfunctions of the form [18]

$$\mathbf{P}_{k_1, \dots, k_N}(x_1, \dots, x_N) = \prod_{i=1}^N v_i^{x_i} \sum_{\pi} S_{\pi} \exp\left(i \sum_{j=1}^N x_j k_{\pi(j)}\right), \quad (2.5)$$

where the sum is taken over all permutations π of $(1, 2, 3, \dots, N)$. The above eigenfunctions correspond to the eigenvalues

$$E_{k_1, \dots, k_N} = \sum_{j=1}^N (v_j - \exp(-ik_j)). \quad (2.6)$$

With the inverse Fourier transformation

$$\mathbf{P}(x_1, \dots, x_N) = \prod_{i=1}^N \int \frac{dk_i}{2\pi} \exp(-E_{\mathbf{k}} t) \mathbf{P}_{\mathbf{k}}(x_1, \dots, x_N) \quad (2.7)$$

it is trivial to show that this ansatz satisfies (2.3) with arbitrary wave amplitudes $S_{\pi}(\mathbf{k})$.

On the other hand from the boundary condition (2.2) it follows that

$$S_{T_k \pi} = -\frac{1 - v_k \exp(ik_{\pi(k+1)})}{1 - v_k \exp(ik_{\pi(k)})} S_{\pi}, \quad (2.8)$$

where the operator T_k changes the k th and $k + 1$ st element of π and $S_{(1,2,\dots,N)}$ is an arbitrary function defined by the initial condition to be satisfied at $t = 0$, see next section.

In order to obtain the N -particle amplitude S_π for a given permutation π one has to decompose this permutation into a sequence of nearest neighbour exchanges and take the product of factors appearing in (2.8). Although this decomposition is not unique the resulting S_π is well-defined if certain relations, known as Yang–Baxter equations, are satisfied. Instead of checking this explicitly one can easily show that the r.h.s. of (2.5) with an S_π satisfying (2.8) can be written in the following determinant form

$$\begin{aligned}
 & P_{k_1, \dots, k_N}(x_1, \dots, x_N) & (2.9) \\
 & = S_{(1, \dots, N)} \prod_{i=1}^N v_i^{x_i} \det \left[\prod_{j=1}^{n-1} (1 - v_j e^{ik_m})^{-1} \prod_{j=1}^{m-1} (1 - v_j e^{ik_m}) e^{ik_m x_n} \right]_{m, n}.
 \end{aligned}$$

This verifies uniqueness by construction and therefore also implies the Yang–Baxter equations.

3. Conditional probability

In order to obtain the conditional probability for the initial configuration (y_1, \dots, y_N) we require

$$P(\mathbf{x}; t = 0) = \delta_{\mathbf{x}, \mathbf{y}}. \tag{3.1}$$

In the following we show that the choice

$$S_{(1, \dots, N)} = \exp \left(-i \sum_{j=1}^N y_j k_j \right) \prod_{i=1}^N v_i^{-y_i} \tag{3.2}$$

with an appropriate (see later) definition of the contours of integration in the complex k_i planes (2.7) indeed reproduces the required initial condition.

In order to write the conditional probability in a compact form we define the functions

$$F_{k, l}(x, t) = \frac{1}{2\pi i} \oint e^{t/z} z^{x-1} \prod_{i=1}^{l-1} (1 - v_i z)^{-1} \prod_{i=1}^{k-1} (1 - v_i z) dz. \tag{3.3}$$

Here, as throughout this paper, the empty product is defined as unity. The integral has to be taken along a circle of radius ϵ around the origin of the complex plane. Notice that if all the rates are less than a given number v_{\max} (as we assume here) then the integral can be taken just as well along the circle with radius $1/v_{\max} - 0$. Now we exploit the determinant expression (2.9) to obtain

Theorem 3.1 (Conditional probability). *Let $P(\mathbf{x} \mid \mathbf{y}; t)$ be the conditional probability for the TASEP with N particles on the infinite one-dimensional*

lattice with right hopping rates v_1, v_2, \dots, v_N (from left to right), starting at sites y_1, y_2, \dots, y_N at time $t = 0$. The conditional probability of finding these particles on sites x_1, x_2, \dots, x_N at time t can be written as the determinant

$$P(\mathbf{x} | \mathbf{y}; t) = \prod_{i=1}^N (e^{-tv_i} v_i^{x_i - y_i}) \quad (3.4)$$

$$\times \begin{vmatrix} F_{1,1}(x_1 - y_1, t) & F_{1,2}(x_2 - y_1, t) & \cdots & F_{1,N}(x_N - y_1, t) \\ F_{2,1}(x_1 - y_2, t) & F_{2,2}(x_2 - y_2, t) & \cdots & F_{2,N}(x_N - y_2, t) \\ \vdots & \vdots & & \vdots \\ F_{N,1}(x_1 - y_N, t) & F_{N,2}(x_2 - y_N, t) & \cdots & F_{N,N}(x_N - y_N, t) \end{vmatrix}$$

Remark 3.1. The expression (3.4) is a generalization of the result found in [20] for the case of equal hopping rates $v_i = 1$. The functions $F_{k,l}(x, t)$ then reduce to $\exp(t)F_{l-k}(x, t)$ of that work.

Proof. One has to show that $P(\mathbf{x} | \mathbf{y}; t)$ satisfies (2.2), (2.3) and (3.1). Using the specific form of $\mathbf{P}_{\mathbf{k}}$ given by (2.9) and (3.2) the inverse Fourier transformation (2.7) leads to

$$\prod_{i=1}^N v_i^{x_i - y_i} \int \frac{dk_i}{2\pi} \exp\{-E_k t\} \quad (3.5)$$

$$\times \det \left[\prod_{j=1}^{n-1} (1 - v_j \exp\{ik_m\})^{-1} \prod_{j=1}^{m-1} (1 - v_j \exp\{ik_m\}) \exp\{ik_m x_n\} \right]_{m,n}.$$

This expression satisfies (2.2) and (2.3) by construction, independently of the contour of integration. Notice that the different rows of the determinant contain different k variables which enables us to perform the integration over k_i separately on each element of row i . After changing the variable k to $z = \exp(ik)$ (3.5) indeed reduces to the r.h.s. of (3.4). By adopting the argument of [20] one can easily show that the specific choice of the contour of integration in (3.3) indeed yields (3.1). \square

In the following $F(x)$ has to be understood as $F(x, t)$.

Corollary 3.1. Consider the probability

$$\tilde{P}(\mathbf{z} | \mathbf{y}; t) = \text{Prob}(x_1 \geq z_1, x_2 \geq z_2, \dots, x_N \geq z_N | \mathbf{y}; t)$$

that all N particles, starting on \mathbf{y} , have reached “at least” the configuration \mathbf{z} where $z_1 < z_2 < \dots < z_N$. This quantity can be written as a determinant

$$\tilde{P}(\mathbf{z} | \mathbf{y}; t) = \prod_{i=1}^N (e^{-tv_i} v_i^{z_i - y_i}) \tag{3.6}$$

$$\times \begin{vmatrix} F_{1,2}(z_1 - y_1) & F_{1,3}(z_2 - y_1) & \cdots & F_{1,N+1}(z_N - y_1) \\ F_{2,2}(z_1 - y_2) & F_{2,3}(z_2 - y_2) & \cdots & F_{2,N+1}(z_N - y_2) \\ \vdots & \vdots & & \vdots \\ F_{N,2}(z_1 - y_N) & F_{N,3}(z_2 - y_N) & \cdots & F_{N,N+1}(z_N - y_N) \end{vmatrix}.$$

Proof. One has to write $\tilde{P}(\mathbf{z} | \mathbf{y}; t)$ in the form

$$\tilde{P}(\mathbf{z} | \mathbf{y}; t) = \sum_{x_N=z_N}^{\infty} \sum_{x_{N-1}=z_{N-1}}^{x_N-1} \cdots \sum_{x_3=z_3}^{x_4-1} \sum_{x_2=z_2}^{x_3-1} \sum_{x_1=z_1}^{x_2-1} P(\mathbf{x}, \mathbf{y}; t) \tag{3.7}$$

then use (3.4). Applying the identities

$$\sum_{x=x_1}^{x_2} v_l^x F_{k,l}(x) = v_l^{x_1} F_{k,l+1}(x_1) - v_l^{x_2+1} F_{k,l+1}(x_2 + 1) \tag{3.8}$$

$$\sum_{x=x_1}^{x_2} v_{k-1}^x F_{k,l}(x) = v_{k-1}^{x_1} F_{k-1,l}(x_1) - v_{k-1}^{x_2+1} F_{k-1,l}(x_2 + 1) \tag{3.9}$$

and standard determinant-manipulations then leads to (3.6). The identities (3.8,3.9) follow by straightforward computation from the definitions (3.3). \square

4. Current distribution

We consider now the step function initial state where the lattice is fully occupied up to the lattice site $x = 0$ and empty for $x > 0$. The time evolution of the mean density (under Eulerian scaling) has been obtained in [6] for a large class of random distributions of rates v_i . An interesting problem are the fluctuations of the current which were first obtained in [29] for the homogeneous system with $v_i = 1$. As a step towards characterizing the long-time properties of these fluctuations (which we do not consider) we derive here for finite times an exact expression for the probability that the time-integrated current across a given bond (i.e. the number of particles that have crossed that bond up to time t) equals some number N . This quantity is related to the distribution of the distance travelled by the first particle with the initial condition where N particles sit on consecutive sites. More precisely,

Theorem 4.1 (Current distribution). *Choosing as initial sites the set $\mathbf{y}_N = (1 - N, \dots, 0)$ the probability that all particles have crossed the bond $x - 1, x$ up to time t can be written as*

$$Q_N(x, t) = \prod_{i=1}^N (e^{-tv_i} v_i^{x+N-1}) \tag{4.1}$$

$$\times \begin{vmatrix} F_{1,2}(x+N-1) & F_{1,3}(x+N) & \cdots & F_{1,N+1}(x+2N-2) \\ F_{2,2}(x+N-2) & F_{2,3}(x+N-1) & \cdots & F_{2,N+1}(x+2N-3) \\ \vdots & \vdots & & \vdots \\ F_{N,2}(x) & F_{N,3}(x+1) & \cdots & F_{N,N+1}(x+N-1) \end{vmatrix}.$$

Remark 4.1. In the case of the initial condition \mathbf{y}_∞ when the lattice is fully occupied on the left and empty on the right of the origin $Q_N(x, t)$ is the probability that *at least* N particles have crossed the bond $x - 1, x$ up to time t . On the other hand this is equal to the probability that the N th particle (from the right) reached *at least* the site x up to time t .

Proof. Notice that $Q_N(x, t) = \tilde{P}(x, x+1, \dots, x+N-1 \mid \mathbf{y}_N; t)$. By using (3.6) one arrives at (4.1). □

Remark 4.2. The determinant appearing in (4.1) has several equivalent forms. One can derive these by using the identities

$$F_{k,l}(x) = F_{k,l+1}(x) - v_l F_{k,l+1}(x+1) \tag{4.2}$$

$$F_{k,l}(x) = F_{k-1,l}(x) - v_{k-1} F_{k-1,l}(x+1), \tag{4.3}$$

which are special cases of (3.8,3.9) corresponding to $x_1 = x_2$. An example is the following

$$Q_N(x, t) = \prod_{i=1}^N (e^{-tv_i} v_i^{x+N-1}) \tag{4.4}$$

$$\times \begin{vmatrix} F_{1,2}(x+N-1) & F_{1,3}(x+N) & \cdots & F_{1,N+1}(x+2N-2) \\ F_{1,2}(x+N-2) & F_{1,3}(x+N-1) & \cdots & F_{1,N+1}(x+2N-3) \\ \vdots & \vdots & & \vdots \\ F_{1,2}(x) & F_{1,3}(x+1) & \cdots & F_{1,N+1}(x+N-1) \end{vmatrix}.$$

Another equivalent expression is

$$\begin{aligned}
 \mathbf{Q}_N(x, t) &= \prod_{i=1}^N (e^{-tv_i} v_i^{x+N-1}) \\
 &\times \begin{vmatrix} F_{1,N+1}(x+N-1) & F_{1,N+1}(x+N) & \cdots & F_{1,N+1}(x+2N-2) \\ F_{1,N+1}(x+N-2) & F_{1,N+1}(x+N-1) & \cdots & F_{1,N+1}(x+2N-3) \\ \vdots & \vdots & & \vdots \\ F_{1,N+1}(x) & F_{1,N+1}(x+1) & \cdots & F_{1,N+1}(x+N-1) \end{vmatrix}.
 \end{aligned}
 \tag{4.5}$$

5. Final remarks

The study of current fluctuations provides insight into universal properties of far-from-equilibrium systems and therefore has attracted a considerable amount of attention in recent years [14, 21, 22, 29–37]. There is, however, to our knowledge no work on current fluctuations in particle systems with more than one conservation law. The result (4.5) is a first step in this direction. The form of (4.5) indicates that the distribution $\mathbf{Q}_N(x, t)$ is independent of the order of the hopping rates of particles (e.g. $v_1 = a, v_2 = b, v_3 = c$ would lead to the same $\mathbf{Q}_N(x, t)$ as $v_1 = c, v_2 = a, v_3 = b$) since these hopping rates enter symmetrically in $F_{1,N+1}(x, t)$. This at first sight perhaps counterintuitive property becomes obvious in the growth presentation of the exclusion process [6] and also in the last passage percolation picture [29, 33] where the hopping rates v_i represent line defects with a different distribution of random energies. For the statistical weight of paths crossing these defects it is clearly irrelevant in which order they occur. In the interpretation of the TASEP with particle-dependent rates as an N -component lattice gas this means that the nonergodicity of the process implied in the absence of passing is irrelevant for the fluctuations of the total current.

A next step would be to study the long time behaviour of the current fluctuations. This was calculated recently for the homogeneous ASEP [21, 22, 29, 33, 35, 37] and the polynuclear growth model (PNG) [32, 34, 36]. It was found that the current fluctuations in the long time limit are characterized by universal scaling forms, which are given explicitly in terms of Tracy–Widom distributions known from random matrix theory. It would be of interest to see whether this universality is preserved in the case of particle-wise disorder.

Recently Priezzhev [38] has extended the Bethe ansatz solution for the usual TASEP on \mathbf{Z} to periodic boundary conditions. It would be interesting to generalize this analysis to the present case. It would also be interesting to investigate

whether determinant solutions can be obtained for integrable models with passing [17–19, 39].

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Note added

After acceptance of the manuscript two recent papers [40, 41] on the long time asymptotics of current distributions of closely related systems were brought to our attention.

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