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QUASIEQUATIONAL THEORIES OF FLAT ALGEBRAS

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Abstract. We prove that finite flat digraph algebras and, more generally, finite compatible flat algebras satisfying a certain condition are finitely q-based (possess a finite basis for their quasiequations). We also exhibit an example of a twelve-element compatible flat algebra that is not finitely q-based.

Keywords: quasiequation, flat algebra

MSC 2000: 08C15, 08B05

1. Introduction

For a finite directed graph (V, E) one can define an algebra with the underlying set $V \cup E \cup \{0\}$, one constant 0 and two binary operations $\wedge$, $\cdot$ in this way: $a \wedge a = a$ and $a \wedge b = 0$ whenever $a \neq b$; $ab = c$ whenever $a, c \in V$ and $b = (a, c) \in E$; $ab = 0$ in all other cases. Algebras obtained from finite directed graphs in this way are called finite *flat digraph algebras*. One particular six-element flat digraph algebra (inherently non-finitely based for equations) played a significant role in the proof of undecidability of the existence of a finite basis for the equational theory of a finite algebra ([2], [3] and [4]). It was plausible to expect that it could serve a similar purpose in an attempt to prove that also the existence of a finite basis for the quasiequations of a finite algebra is undecidable. However, in this paper we are going to show that all finite flat digraph algebras are finitely q-based (possess a finite basis for their uasiequations), which makes them unsuitable. We will investigate a more

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general class of finite compatible flat algebras, in which (under a modest assumption on the signature) every algebra can be embedded both into a finitely q-based and into a non-finitely q-based algebra.

For the terminology and basic concepts of universal algebra the reader is referred to the monograph [5]. For the literature on quasiequational theories see, e.g., [1] and [6].

2. Compatible 0-semilattice algebras

Let σ be a finite signature containing (among other symbols) a binary symbol $\wedge$ (the meet) and a nullary symbol 0.

By a 0-*semilattice algebra* we mean a σ-algebra satisfying the equations

(1) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$,

(2) $x \wedge y = y \wedge x$,

(3) $x \wedge x = x$,

(4) $f(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) = 0$ for every n-ary operation f of σ and every $i \in \{1, \ldots, n\}$.

A 0-semilattice algebra is said to be *compatible* if it satisfies the equations

(5) $f(z_1, \ldots, z_{i-1}, x \wedge y, z_{i+1}, \ldots, z_n) = f(z_1, \ldots, z_{i-1}, x, z_{i+1}, \ldots, z_n) \wedge f(z_1, \ldots, z_{i-1}, y, z_{i+1}, \ldots, z_n)$ for every n-ary operation f of σ and every $i \in \{1, \ldots, n\}$.

So, the class of compatible 0-semilattice σ-algebras is a variety.

For a variable x, *basic x-terms* of depth n are defined as follows. The term x is the only basic x-term of depth 0. For $n > 0$, basic x-terms of depth n are the terms $f(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n)$ such that f is an n-ary operation of σ, $1 \leqslant i \leqslant n$, t is a basic x-term of depth $n-1$ and $x_1, \ldots$ are variables different from x. A basic x-term t will be usually denoted by $t(x)$, in which case $t(u)$ stands for the term resulting from t by substituting u for x (where u is any term).

For a σ-algebra B and a basic x-term t of depth n, any interpretation of the variables different from x by elements of B gives rise to a unary polynomial of B. The unary polynomials obtained in this way will be called the *basic polynomials* of B of depth n.

Lemma 2.1. *Let A be a compatible 0-semilattice algebra. Then $p(a \wedge b) = p(a) \wedge p(b)$ for all basic polynomials p of A and all elements $a, b \in A$.*

P r o o f. It is easy. (Observe that the statement is not true for all unary polynomials p.) □

Lemma 2.2. *Let A be a compatible 0-semilattice algebra and F be a proper filter of A (i.e., a nonempty subset closed under meet, not containing 0 and such that $b \in F$ whenever $a \in F$ and $a \leqslant b$). Then for every basic polynomial p of A, $p^{-1}(F)$ is either empty or a proper filter of A.*

Proof. It follows easily from Lemma 2.1. □

By a *flat* algebra we mean a 0-semilattice algebra A such that $a \wedge b = 0$ for all pairs of distinct elements $a, b \in A$. Observe that a flat algebra is monotonic, i.e., satisfies $x \leqslant y \rightarrow f(z_1, \ldots, z_{i-1}, x, z_{i+1}, \ldots, z_n) \leqslant f(z_1, \ldots, z_{i-1}, y, z_{i+1}, \ldots, z_n)$ for every n-ary operation f of σ and every $i \in \{1, \ldots, n\}$.

One can easily see that a flat algebra is compatible if and only if

(5′) $f(c_1, \ldots, c_{i-1}, a, c_{i+1}, \ldots, c_n) = f(c_1, \ldots, c_{i-1}, b, c_{i+1}, \ldots, c_n) \neq 0$ implies $a = b$ for every n-ary operation f of σ and every $i \in \{1, \ldots, n\}$.

For every partial algebra G of a signature τ not containing $\wedge$ and 0 we can define a flat $\tau \cup \{\wedge, 0\}$-algebra A, called the *flat algebra over* G, by $A = G \cup \{0\}$, $f(a_1, \ldots, a_n) = a$ in A whenever $f(a_1, \ldots, a_n) = a$ in G, and $f(a_1, \ldots, a_n) = 0$ otherwise. This flat algebra is not necessarily compatible. For example, if G is a finite groupoid, then the flat algebra over G is compatible if and only if G is a quasigroup. Finite flat digraph algebras are all compatible.

Observation 2.3. *For every finite compatible flat algebra A there exists a first-order sentence Φ such that the finite models of Φ are precisely the finite algebras belonging to the quasivariety generated by A.*

Proof. Put $K = |A|$. It is easy to see that the following are equivalent for a finite compatible 0-semilattice algebra B:

(e1) B belongs to the quasivariety generated by A;

(e2) every two elements b_0, b_1 of B such that $b_0 < b_1$ can be separated by a congruence of B, the factor by which is isomorphic to a subalgebra of A;

(e3) for every $b_0, b_1 \in B$ with $b_0 < b_1$ there exist elements $c_1, \ldots, c_r \in B$ for some $r < K$ such that the principal filters $F_1, \ldots, F_r$ generated by $c_1, \ldots, c_r$ are pairwise disjoint, $b_1 \in F_1$, b_0 belongs to the complement O of $F_1 \cup \ldots \cup F_r$ in B, the equivalence R with blocks $O, F_1, \ldots, F_r$ is a congruence of B and the factor B/R is isomorphic to a subalgebra of A.

Clearly, the condition (e3) can be rewritten as a first-order sentence. □

3. The quasivariety Q'_A

In the following let A be a finite compatible, flat algebra. Put $K = |A|$.

Denote by Q'_A the quasivariety determined by the equations (1)–(5) and the following quasiequations:

(6) $x_0 \leqslant x_1$ & $t(x) \geqslant x_1$ & $u(x) \geqslant x_1$& $t(y) \geqslant x_1$ & $u(y) \wedge x_1 \leqslant x_0 \to x_0 = x_1$ for every pair of basic x-terms t, u of depth $\leqslant K$;

(7) $x_0 \leqslant x_1$ & $H_{t_1,\dots,t_K} \to x_0 = x_1$ for every K-tuple of basic x-terms $t_1, \dots, t_K$ of depth $\leqslant K$, where $H_{t_1,\dots,t_K}$ is the conjunction of the following equations:

$$t_i(x_i) \geqslant x_1 \quad (i = 1, \dots, K),$$
$$t_i(x_j) \wedge x_1 \leqslant x_0 \quad (i, j = 1, \dots, K \text{ and } i \neq j).$$

Lemma 3.1. *Q'_A is a finitely q-based quasivariety containing A.*

Proof. The set of quasiequations (6)–(7) is essentially finite, as it contains only finitely many quasiequations that differ by not only renaming their variables. Consequently, Q'_A is finitely q-based. It remains to prove that (6) and (7) are satisfied in A. Suppose that (6) fails in A by some interpretation $v \mapsto v'$ of variables. Then $x'_0 < x'_1$, so that $x'_0 = 0$; now $t(x') \geqslant x'_1$ implies $t(x') = x'_1$. Similarly we get $u(x') = x'_1$ and $t(y') = x'_1$. But A satisfies (5′), so $t(x') = t(y') \neq 0$ implies $x' = y'$; hence $x'_1 = u(x') \wedge x'_1 = u(y') \wedge x'_1 = 0$, a contradiction. Using the fact that A cannot contain K nonzero, pairwise distinct elements, one can similarly prove that A satisfies the quasiequations (7). □

Lemma 3.2. *Let $B \in Q'(A)$ and $b_0, b_1 \in B$ be two elements such that $b_1 \nleqslant b_0$; let F be a maximal filter of B such that $b_1 \in F$ and $b_0 \notin F$. For any two basic polynomials p, q of B of depth $\leqslant K$, the sets $p^{-1}(F)$ and $q^{-1}(F)$ are either disjoint or equal.*

Proof. The two basic polynomials p and q correspond to two basic terms t and u of depth $\leqslant K$. Suppose that there exist elements x', y' such that $p(x') \in F$, $p(y') \in F$, $q(x') \in F$ and $q(y') \notin F$. It follows from the maximality of F that there exists an element $e \in F$ with $q(y') \wedge e \leqslant b_0$. Put $x'_1 = p(x') \wedge p(y') \wedge q(x') \wedge e$, so that $x'_1 \in F$. Put $x'_0 = b_0 \wedge x'_1$, so that $x'_0 < x'_1$. But the quasiequation (e6) interpreted by $x \mapsto x'$, $y \mapsto y'$, $x_0 \mapsto x'_0$, $x_1 \mapsto x'_1$ gives $x'_0 = x'_1$, a contradiction. □

Lemma 3.3. *Let $B \in Q'(A)$ and $b_0, b_1 \in B$ be two elements such that $b_1 \nleqslant b_0$; let F be a maximal filter of B such that $b_1 \in F$ and $b_0 \notin F$. There are at most $K-1$ nonempty subsets of B that can be expressed as $q^{-1}(F)$ for a basic polynomial q of B, and they can be arranged into a sequence $F_1, \ldots, F_r$ (for some $r < K$) in such a way that $F_1 = F$ and for every $i \in \{2, \ldots, r\}$ there are an index $j \in \{1, \ldots, i-1\}$ and a basic polynomial p_i of B of depth 1 with $F_i = p_i^{-1}(F_j)$. The collection $F_1, \ldots, F_r$, together with the complement of their union, is a partition and the corresponding equivalence is a congruence of B.*

Proof. Let us define a (finite or infinite) sequence $F_1, p_1, F_2, p_2, \ldots$ of filters F_i and basic polynomials p_i of depth $\leqslant 1$ by induction in this way: $F_1 = F$ and p_1 is the identity on B; if F_i, p_i have been defined and if there exist an element $a \notin F_1 \cup \ldots \cup F_i$ and a basic polynomial p of depth 1 such that $p(a) \in F_j$ for some $j \leqslant i$, take one such pair a, p and put $p_{i+1} = p$ and $F_{i+1} = p_{i+1}^{-1}(F_j)$; if there is no such pair a, p, the sequence already constructed will have no continuation. Clearly (by induction on i), $F_i = q_i^{-1}(F)$ for a basic polynomial q_i of B of depth $< i$. The sets F_i are pairwise disjoint filters according to Lemmas 2.2 and 3.2.

Suppose that the sequence has at least K members $F_1, \ldots, F_K$. For any $i = 1, \ldots, K$ take an element $x'_i \in F_i$, so that $q_i(x'_i) \in F$. For every $i \neq j$ we have $x'_j \notin F_i$, i.e., $q_i(x'_j) \notin F$, so that there exists an element $e_{i,j} \in F$ with $q_i(x'_j) \wedge e_{i,j} \leqslant b_0$. There is an element $x'_1 \in F$ such that $x'_1 \leqslant q_i(x'_i)$ for all i and $x'_1 \leqslant e_{i,j}$ for all $i \neq j$. Put $x'_0 = b_0 \wedge x'_1$, so that $x'_0 < x'_1$. But the quasiequation (e7), interpreted in the obvious way, says that $x'_0 = x'_1$, a contradiction.

So, the sequence $F_1, p_1, \ldots$ ends with F_r, p_r for some $r \leqslant K-1$. Clearly, every subset of the form $q^{-1}(F)$ for a basic polynomial q can be found among $F_1, \ldots, F_r$. Put $O = B - (F_1 \cup \ldots \cup F_r)$, so that $0 \in O$ and $F_1, \ldots, F_r, O$ is a partition of B. It remains to prove that the corresponding equivalence is a congruence of B.

Suppose that there exist an n-ary operation f in σ and an n-tuple $a_1, \ldots, a_n$ of elements of B such that $a_j \in O$ for some j but $f(a_1, \ldots, a_n) \in F_i$ for some i. Then $p(a_j) \in F_i$ where $p(x) = f(a_1, \ldots, a_{j-1}, x, a_{j+1}, \ldots, a_n)$ is a basic polynomial of depth 1 and $a_j \notin F_1 \cup \ldots \cup F_r$, so that $(q_i p)^{-1}(F)$ is nonempty and different from all $F_1, \ldots, F_r$, a contradiction. We have proved that if at least one of the elements $a_1, \ldots, a_n$ belongs to O, then $f(a_1, \ldots, a_n) \in O$.

Now it remains to show that if f is n-ary, $f(a_1, \ldots, a_n) \in F_j$ and $a_i, a'_i \in F_k$ for some $j, k \in \{1, \ldots, r\}$ and $i \in \{1, \ldots, n\}$, then $f(a_1, \ldots, a_{i-1}, a'_i, a_{i+1}, \ldots, a_n) \in F_j$. Put $q(x) = q_j(f(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n))$, so that q is a basic polynomial of B of depth at most K. We have $q(a_i) \in F$ and $q_k(a_i) \in F$, so that $q^{-1}(F) = q_k^{-1}(F)$. Since a'_i belongs to this set, we get $q(a'_i) \in F$, i.e., $f(a_1, \ldots, a_{i-1}, a'_i, a_{i+1}, \ldots, a_n) \in F_j$. $\square$

Theorem 3.4. *Let A be a finite compatible, flat algebra with K elements. Then Q'_A is a finitely q-based and locally finite quasivariety containing A; every algebra in Q'_A is isomorphic to a subdirect product of algebras of cardinality at most K. Consequently, A is not inherently nonfinitely q-based.*

P r o o f. Let $B \in Q'_A$. For every pair b_0, b_1 of distinct elements of B (we can assume that $b_1 \nleq b_0$) there exists a maximal filter of B containing b_1 but not b_0, so that by Lemma 3.3 these two elements can be separated by a congruence with at most K blocks. It follows that every algebra from B is isomorphic to a subdirect product of algebras of cardinality at most K. Thus Q'_A is contained in a finitely generated variety and hence it is locally finite. According to Lemma 3.1, Q'_A is finitely q-based and contains A. $\square$

4. Finitely q-based compatible flat algebras

Let A be a finite compatible flat algebra. By a *segment* of A we will mean a nonempty subset of A, the elements of which can be arranged into a finite sequence $0, c_1, \ldots, c_r$ in such a way that $c_1 \neq 0$ and for every $i = 2, \ldots, r$ there exists a basic polynomial p of A of depth 1 with $p(c_i) = c_j$ for some $j \in \{1, \ldots, i-1\}$.

Let S be a segment of A. The algebra obtained from S, considered as a partial subalgebra of A, by setting all the undefined operations to 0 will be called the 0-*completion* of S.

Let S be a segment of A and S' be the subalgebra of A generated by S. The segment S is said to be *regular* if the equivalence on S' with the only non-singleton block $\{0\} \cup (S' - S)$ is a congruence of S'. In that case, the factor of S' by this congruence is isomorphic to the 0-completion of S.

Theorem 4.1. *Let A be a finite compatible flat algebra such that the* 0*-completion of every regular segment of A belongs to the quasivariety generated by A. Then A is finitely q-based.*

P r o o f. Denote by Q''_A the subquasivariety of Q'_A determined by the quasiequations (1)–(7) and all quasiequations in at most K variables that are satisfied in A. (Here $K = |A|$.) Since Q'_A is locally finite by Theorem 3.4, Q''_A is locally finite. Since only finitely many equations are needed to reduce the terms in at most K variables to a finite set T_0 of such terms, and then quasiequations in at most K variables correspond to subsets of T_0^2 with distinguished elements, Q''_A is finitely q-based. Of course, $A \in Q''_A$. We are going to prove that Q''_A is the quasivariety generated by A. It is sufficient to show that every finite algebra from Q''_A belongs to the quasivariety generated by A.

Let B be a finite algebra from Q''_A; let $b_0, b_1 \in B$ be such that $b_1 \nleqslant b_0$. By 3.3 there is a congruence with at most K blocks $O, F_1, \dots, F_r$, yielding a quotient algebra C, such that $F_1, \dots, F_r$ are filters (now they are principal filters), $F_1 = F$, $b_1 \in F_1$, $b_0 \in O$, and for every $i \in \{2, \dots, r\}$ there exist an index $j < i$ and a basic polynomial p_i of length 1 with $F_i = p_i^{-1}(F_j)$. But all the coefficients occurring in p_i belong to $F_1 \cup \dots \cup F_r$, so there exists a basic x-term $u_i(x, x_1, \dots, x_r)$ of depth 1 such that $u_i(F_i, F_1, \dots, F_r) \subseteq F_j$. Now we can combine these terms u_i together to obtain, for each i, a basic x-term $t_i(x, x_1, \dots, x_r)$ such that $t_i(F_i, F_1, \dots, F_r) \subseteq F$, i.e., $t_i^C(F_i, F_1, \dots, F_r) = F_1$. (We take $t_1 = x$.) For any term u denote by $t_i(u)$ the term obtained from t_i by replacing the only occurrence of x with u. Now consider the quasiequation

$$x_0 \leqslant x_1 \ \& \ D \to x_0 = x_1$$

where D is the conjunction of all these equations:

(i) $t_i(x_i) \geqslant x_1$, for any $i = 1, \dots, r$;

(ii) $t_i(x_j) \wedge x_1 \leqslant x_0$, for any $i, j \in \{1, \dots, r\}$ with $i \neq j$;

(iii) $t_i(f(x_{i_1}, \dots, x_{i_n})) \geqslant x_1$, for any n-ary operation f of σ and any $i, i_1, \dots, i_n$ with $f^C(F_{i_1}, \dots, F_{i_n}) = F_i$;

(iv) $t_i(u) \wedge x_1 \leqslant x_0$, for any $i = 1, \dots, r$ and any term u in variables $x_1, \dots, x_r$ containing a subterm $f(x_{i_1}, \dots, x_{i_n})$ with $f^C(F_{i_1}, \dots, F_{i_n}) = O$ (it is possible to consider only finitely many such terms u).

Clearly, this quasiequation fails in B; since it is a quasiequation in at most K variables $x_0, \dots, x_r$, it must fail in A by some elements $a_0, a_1, \dots, a_r$. But then the subset $\{a_0, a_1, \dots, a_r\}$ is a regular segment of A, and the 0-completion of this subset is isomorphic to C. Since C belongs to the quasivariety generated by A, the elements b_0, b_1 were separated by a congruence, the factor by which belongs to the quasivariety. □

Corollary 4.2. *Every finite flat digraph algebra is finitely q-based.*

Proof. In this case, all segments are subalgebras. □

Corollary 4.3. *The flat algebra over any finite quasigroup (considered as a groupoid) is finitely q-based.*

Proof. In this case, all regular segments are subalgebras. □

Corollary 4.4. *If σ is the signature containing only one unary symbol in addition to $\wedge$ and 0, then every finite compatible flat σ-algebra is finitely q-based.*

Proof. In this case, the 0-completion of every segment is isomorphic to a subalgebra. □

Theorem 5.1. *Let σ be a finite signature containing, in addition to $\wedge$ and 0, at least two unary symbols f and g (and, possibly, some other operation symbols). Then every finite compatible flat σ-algebra can be embedded into two finite compatible flat σ-algebras, one finitely q-based and the other one not finitely q-based.*

Proof. Let G be a finite compatible flat algebra.

Denote by $S_1, \ldots, S_r$ all the segments of G. (It would be sufficient to take just those with the 0-completions not belonging to the quasivariety generated by G.) For every $i = 1, \ldots, r$ let us take an isomorphic copy T_i of the partial algebra $S_i - \{0\}$, in such a way that the sets $G, T_1, \ldots, T_r$ are pairwise disjoint. Denote by G' the flat algebra with the underlying set $G \cup T_1 \cup \ldots \cup T_r$, with the operations evaluated to 0 in all cases except when needed to define them in such a way that G is a subalgebra and T_i are partial subalgebras. It follows from Theorem 4.1 that G' is finitely q-based.

Next we are going to construct a non-finitely q-based extension of G. Let us take one fixed positive integer k such that $k \geqslant 2$ and there is no sequence $u_0, u_1, \ldots, u_k$ of pairwise distinct elements of $G - \{0\}$ such that $g(u_{i-1}) = u_i$ for $i = 1, \ldots, k$. Denote by A the flat algebra, with G as a subalgebra, containing $k + 10$ additional elements $u_0, u_1, \ldots, u_k, a, b, c, v_2, a_2, b_2, v_3, a_3, c_3$ with all operations not inside G evaluated to 0 except for

$$g(u_{i-1}) = u_i \quad \text{for } i = 1, \ldots, k,$$
$$f(u_0) = a, \quad f(a) = b, \quad g(a) = c,$$
$$f(v_2) = a_2, \quad f(a_2) = b_2, \quad f(v_3) = a_3, \quad g(a_3) = b_3.$$

(Fig. 1, in which the elements not belonging to G are pictured for $k = 2$, may help to understand this definition.)

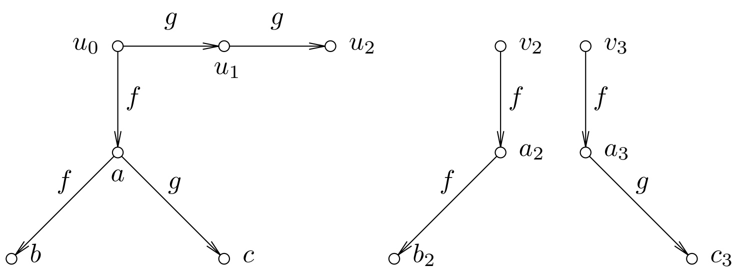

Fig. 1

Denote by Q the quasivariety generated by A. A σ-algebra B belongs to Q if and only if every two distinct elements of B can be separated by a homomorphism of B into A.

For every positive integer n let A_n be the σ-algebra with elements $0, u_0, \ldots, u_k$, $\alpha_{i,j}$, β_i, γ_j ($0 \leqslant i \leqslant n$, $0 \leqslant j \leqslant n-1$, $i-1 \leqslant j \leqslant i$) and with operations defined in this way: A_n is a semilattice with the only comparabilities $0 < u_i$ ($i = 0, \ldots, k$), $0 < \beta_n < \beta_{n-1} < \ldots < \beta_0$, $0 < \gamma_{n-1} < \gamma_{n-2} < \ldots < \gamma_0$, $0 < \alpha_{n,n-1} < \alpha_{n-1,n-1} < \alpha_{n-1,n-2} < \ldots < \alpha_{1,0} < \alpha_{0,0}$; the other operations evaluate to 0 except that $g(u_{i-1}) = u_i$ ($i = 1, \ldots, k$), $f(u_0) = \alpha_{0,0}$, $f(\alpha_{i,j}) = \beta_i$, $g(\alpha_{i,j}) = \gamma_j$. (Fig. 2, in which the situation is illustrated for $k = 2$ and $n = 3$, may help to understand this definition. In the picture lines with arrows indicate unary operations, while the other lines represent coverings but the covers between 0 and the elements u_i are not indicated.)

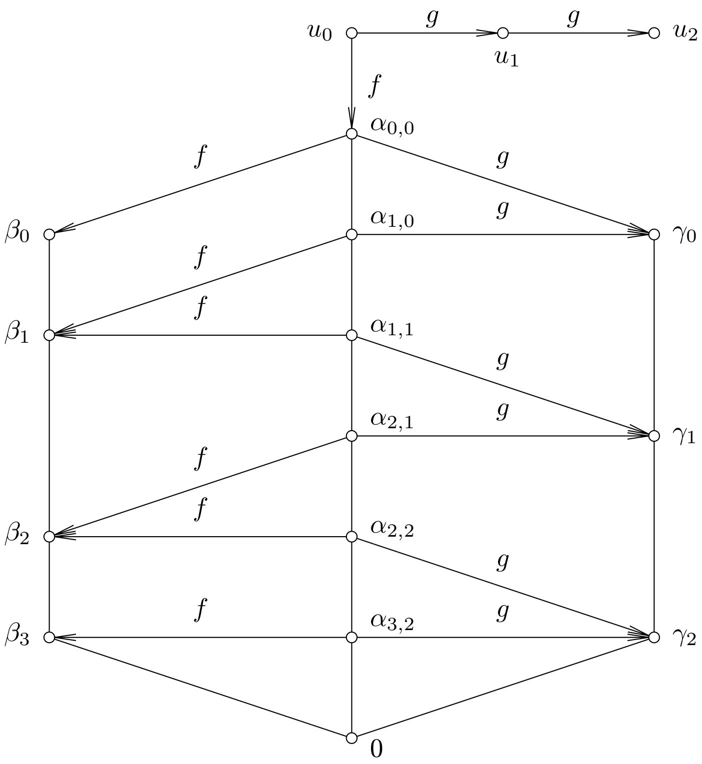

Fig. 2

Denote by r_n the equivalence on A_n with the only non-singleton block $\{0, \beta_n\}$. Clearly, r_n is a congruence of A_n. Denote the factor A_n/r_n by B_n. For $a \in A_n - \{0, \beta_n\}$, the element a/r_n will be identified with a.

Suppose that there exists a homomorphism $H\colon B_n \to A$ such that $H(u_k) \neq H(0/r_n)$, i.e., $H(u_k) \neq 0$. Since $g^k(u_0) = u_k$ in B_n and there is no other element e

in A with $g^k(e) \neq 0$ and $g^{k+1}(e) = 0$ other than u_0, we get $H(u_0) = u_0$ and then $H(u_i) = H(g^i(u_0)) = g^i(H(u_0)) = g^i(u_0) = u_i$ for all i. Now $H(\alpha_{0,0}) = H(f(u_0)) = f(H(u_0)) = f(u_0) = a$. Consequently, $H(\beta_0) = b$ and $H(\gamma_0) = c$. Since $g(\alpha_{1,0}) = \gamma_0$ and a is the only element of A with $g(a) = c$, it follows that $H(\alpha_{1,0}) = a$. If $H(\alpha_{i,i-1}) = a$ for some $i < n$, then using f in a similar way we can show that $H(\alpha_{i,i}) = a$, and then using g to show that $H(\alpha_{i+1,i}) = a$. By induction we get $H(\alpha_{n,n-1}) = a$. But then $H(0/r_n) = H(\beta_n/r_n) = H(f(\alpha_{n,n-1})) = f(a) = b$, a contradiction.

Since the element u_k cannot be separated from $0/r_n$ by a homomorphism of B_n into A, we conclude that B_n does not belong to Q.

Let $\alpha_{m,m'}$ be an element of B_n such that $0 < m < n$. Clearly, the set $C = B_n - \{\alpha_{m,m'}\}$ is a subalgebra of B_n. We are going to prove that C belongs to Q. For this purpose, it is sufficient to show that whenever e, e' are two elements of C such that e is covered by e', then e, e' can be separated by a homomorphism of C into A.

For every $i \leqslant n-1$ define a mapping ψ_i of B_n into A by $\psi_i(u_0) = v_2$, $\psi_i(e) = a_2$ for $e \geqslant \alpha_{i,i}$, $\psi_i(e) = b_2$ for $e \geqslant \beta_i$ and $\psi_i(e) = 0$ for all other elements e. Also, for every $i \leqslant n-1$ define a mapping χ_i of B_n into A by $\chi_i(u_0) = v_3$, $\chi_i(e) = a_3$ for $e \geqslant \alpha_{i+1,i}$, $\chi_i(e) = c_3$ for $e \geqslant \gamma_i$ and $\chi_i(e) = 0$ for all other elements e. It is easy to check that both ψ_i and χ_i are homomorphisms. Consequently, their restrictions to C are homomorphisms of C into A. The only pairs of covers not separated by any of these homomorphisms are the pairs $(0, u_1), \dots, (0, u_k)$. So, it remains to separate these pairs of elements.

If $m = m'$, then these pairs are separated by the homomorphism φ defined in this way: $\varphi(u_0) = u_0, \dots, \varphi(u_k) = u_k$, $\varphi(e) = a$ for $e \geqslant \alpha_{m,m-1}$, $\varphi(e) = b$ for $e \geqslant \beta_m$, $\varphi(e) = c$ for $e \geqslant \gamma_{m-1}$ and $\varphi(e) = 0$ for all other elements e. If $m' = m-1$, then they are separated by the homomorphism φ' defined in this way: $\varphi'(u_0) = u_0, \dots, \varphi'(u_k) = u_k$, $\varphi'(e) = a$ for $e \geqslant \alpha_{m',m'}$, $\varphi'(e) = b$ for $e \geqslant \beta_{m'}$, $\varphi'(e) = c$ for $e \geqslant \gamma_{m'}$ and $\varphi'(e) = 0$ for all other elements e.

We have proved that C belongs to Q. Since every subalgebra of B_n generated by at most $n-k$ elements is contained in at least one such C, it follows that every subalgebra generated by at most $n-k$ elements belongs to Q. Consequently, there is no base for the quasiequations of Q that would contain only quasiequations in at most $n-k$ variables. Since k was fixed while n was arbitrary, there is no finite base at all. □

Remark 5.2. In the above construction of the algebra A it was not essential that the elements b_2 and c_3 are distinct.

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