Semilattices with a group of automorphisms

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Abstract

We investigate semilattices expanded by a group **F** of automorphisms acting as new unary basic operations. We describe up to isomorphism all simple algebras of this kind in case that **F** is commutative. Finally, we present an example of a simple algebra that does not fit in the previous description, if **F** is not commutative.

Introduction

Algebras with a semilattice operation, which commutes with all other operations, have been studied in various forms. In many respects these algebras behave similarly to modules. For example, it is proved in [6] that if a locally finite variety of type-set $\{\mathbf{5}\}$. satisfies a term-condition similar to the term-condition for abelian algebras, then it has a semilattice term that commutes with all other term operations.

Within the class of modes—that is, idempotent algebras such that every operation commutes with all (term) operations—those having a semilattice term operation play an important role (see [7], [8]); these algebras are called semilattice modes. The structure of locally finite varieties of semilattice modes is described in [5].

An interesting class of algebras with a commuting semilattice operation arises if we add automorphisms, as basic operations, to a semilattice. This is a special case of the construction studied in [2]. In general, one can expand any variety $\mathcal{V}$ by a fixed monoid of endomorphisms **F** in a natural way. The expanded variety is the variety of $\mathcal{V}$-algebras **A** equipped with new unary basic operations, acting as endomorphisms on **A**. In this construction we keep **F** fixed, and do the same when **F** is a group. We remark that there is a different approach, when the group **F** is not kept fixed; what one gets then is the theory of varieties of group representations, where the objects are groups acting on some semilattices (see [1]).

In a number of different cases the simple and subdirectly irreducible algebras of the expanded variety have been determined. In [3] J. Ježek described all simple algebras in the variety of semilattices expanded by two commuting automorphisms. In this case the monoid **F** is the free commutative group with two generators. In [4] he also described all subdirectly irreducible semilattices with a single distinguished automorphism.

In this paper we generalize the main result of [3] to arbitrary commutative group **F**, that is, we describe all simple algebras in the variety of semilattices expanded by an abelian group of automorphisms.

The same results were discovered independently by T. Kepka and R. El Bashir. In fact, their results are slightly more general: they study simple semimodules over commutative semirings, where addition is a semilattice operation.

1. F-semilattices

In [2] one can find the definition of the expansion of a variety by a fixed monoid of endomorphisms, and also some basic properties of this construction. In this paper we need only the following special case.

Definition 1.1. An algebra $\mathbf{S} = \langle S; \wedge, F\rangle$ with a binary operation $\wedge$ and a set F of unary operations is an $\mathbf{F}$-*semilattice*, if $\mathbf{F} = \langle F; \cdot, ^{-1}, \mathrm{id}\rangle$ is a group and $\mathbf{S}$ satisfies the following identities:

(a) the operation $\wedge$ is a semilattice operation,
(b) $\mathrm{id}(x) = x$,
(c) $f(g(x)) = (f \cdot g)(x)$ for all $f, g \in F$, and
(d) $f(x \wedge y) = f(x) \wedge f(y)$ for all $f \in F$.

In other words, an $\mathbf{F}$-semilattice is a semilattice expanded with a set F of new operations which forms an automorphism group of the semilattice. Usually the group $\mathbf{F}$ will be fixed. Note that every semilattice can be considered as an $\mathbf{F}$-semilattice in a trivial way: every unary operation of F acts as the identity function. Now we give a much more typical example of an $\mathbf{F}$-semilattice.

Definition 1.2. Let $\mathbf{P}(F) = \langle P(F); \wedge, F\rangle$ be the $\mathbf{F}$-semilattice which is defined on the set $P(F)$ of all subsets of F by setting

(a) $A \wedge B = A \cap B$ for all $A, B \subseteq F$, and
(b) $f(A) = A \cdot f^{-1}$ for all $f \in F$ and $A \subseteq F$.

Thus the meet operation is intersection, and every unary operation $f \in F$ acts by taking complex product with f^{-1} on the right hand side. We show that $\mathbf{P}(F)$ contains all subdirectly irreducible $\mathbf{F}$-semilattices.

Proposition 1.3. *Every subdirectly irreducible* $\mathbf{F}$*-semilattice can be embedded in* $\mathbf{P}(F)$*.*

Proof. Let $\mathbf{S}$ be a subdirectly irreducible $\mathbf{F}$-semilattice. For every element $s \in S$ we define a homomorphism φ_s from $\mathbf{S}$ to $\mathbf{P}(F)$ as follows:

$$\varphi_s\colon S \to P(F); \quad \varphi_s(x) = \{\, f \in F \mid f(x) \geq s \,\}. \tag{1}$$

This function is indeed a homomorphism, since

$$\begin{aligned}\varphi_s(x \wedge y) &= \{\, f \in F \mid f(x \wedge y) \geq s \,\} \\ &= \{\, f \in F \mid f(x) \wedge f(y) \geq s \,\} \\ &= \{\, f \in F \mid f(x) \geq s \text{ and } f(y) \geq s \,\} \\ &= \{\, f \in F \mid f(x) \geq s \,\} \cap \{\, f \in F \mid f(y) \geq s \,\} \\ &= \varphi_s(x) \wedge \varphi_s(y),\end{aligned}$$

and for any unary operation $g \in F$ we have

$$\begin{aligned}
\varphi_s(g(x)) &= \{\, f \in F \mid f(g(x)) \geq s \,\} \\
&= \{\, f \in F \mid (f \cdot g)(x) \geq s \,\} \\
&= \left\{\, h \cdot g^{-1} \in F \;\middle|\; h(x) \geq s \,\right\} \\
&= \{\, h \in F \mid h(x) \geq s \,\} \cdot g^{-1} \\
&= \varphi_s(x) \cdot g^{-1} \\
&= g(\varphi_s(x)).
\end{aligned}$$

Now we show that at least one of these homomorphisms is an embedding from $\mathbf{S}$ to $\mathbf{P}(F)$. Let $\langle x, y \rangle \in \bigcap_{s \in S} \ker \varphi_s$ be an arbitrary pair of elements. Since $\langle x, y \rangle \in \ker \varphi_x$, therefore $\varphi_x(x) = \varphi_x(y)$. We have $\mathrm{id} \in \varphi_x(x)$ by (1), so $\mathrm{id} \in \varphi_x(y)$, thus again by (1) we conclude that $y \geq x$. A similar argument shows that $x \geq y$, thus $x = y$. This proves that the congruence $\bigcap_{s \in S} \ker \varphi_s$ is the equality relation on $\mathbf{S}$. But $\mathbf{S}$ is subdirectly irreducible, therefore for at least one element $s \in S$ the kernel of φ_s is the equality relation. Hence φ_s is an embedding. $\square$

We have seen that every subdirectly irreducible $\mathbf{F}$-semilattice is isomorphic to some subalgebra of $\mathbf{P}(F)$. So it is natural to ask which subalgebras of $\mathbf{P}(F)$ are in fact subdirectly irreducible. The following corollary states that all finite subalgebras of $\mathbf{P}(F)$ are subdirectly irreducible. However, it is not hard to construct an example showing that the infinite subalgebras of $\mathbf{P}(F)$ are not necessarily subdirectly irreducible.

Proposition 1.4. *The finite subdirectly irreducible* $\mathbf{F}$*-semilattices are exactly the nontrivial finite subalgebras of* $\mathbf{P}(F)$.

Proof. We already know from Proposition 1.3 that the finite subdirectly irreducible $\mathbf{F}$-semilattices are subalgebras of $\mathbf{P}(F)$. Conversely, we must show that each nontrivial finite subalgebra of $\mathbf{P}(F)$ is indeed subdirectly irreducible. Let $\mathbf{U}$ be a finite subalgebra of $\mathbf{P}(F)$, and suppose that $\mathbf{U}$ has more than one element. First we will define a pair of elements in U, and subsequently we will show that every nontrivial congruence of $\mathbf{U}$ contains this pair. Clearly, this is enough to ensure that $\mathbf{U}$ is subdirectly irreducible.

Consider the pair $\langle M, \emptyset \rangle$ where

$$M = \bigcap \{\, A \in U \mid \mathrm{id} \in A \,\}. \tag{2}$$

The set on the right hand side of (2) is not empty. In order to verify this, we choose an element A of U different from the empty set. This can be done, since U has more than one element. Let a be an arbitrary element of A. From Definition 1.2 we see that $a(A) = A \cdot a^{-1}$, and since $\mathrm{id} \in A \cdot a^{-1}$, we conclude that $\mathrm{id} \in a(A)$. Therefore the set on the right hand side of (2) contains the element $a(A)$ of U, thus it is nonempty. Furthermore, this set is finite, since U is finite. Finally, if we use the meet operation of $\mathbf{P}(F)$, we get that the set M is in U. With a similar argument it is easy to verify that the empty set is also in U. To this end we need to take the intersection of all elements of U.

Now we show that M is a subgroup of $\mathbf{F}$. It is obvious that $\mathrm{id} \in M$. Let m be an arbitrary element of M. Then $\mathrm{id} \in M \cdot m^{-1} = m(M)$, so by (2) we get $M \cdot m^{-1} \supseteq M$. If we multiply this inclusion by m on the right, we conclude that $M \supseteq M \cdot m$ for every element m of M. Therefore M is closed under the multiplication of $\mathbf{F}$. To prove that M is closed under taking inverses also, consider the sets $M \cdot m^k$ where k is a nonnegative integer. Since $M \in U$ and $M \cdot m^k = m^{-k}(M)$, we see that these sets are elements of U. But U is finite, so there exist two distinct integers k and l, such that $M \cdot m^k = M \cdot m^l$. We can assume without loss of generality that $k > l$. Since $k - l - 1 \geq 0$ and M is a monoid, we get $m^{k-1} = m^{k-l-1} \cdot m^l \in M \cdot m^l = M \cdot m^k$, that is, $m^{-1} \in M$. Now we are ready to complete our proof.

Let ϑ be a congruence of $\mathbf{U}$ different from the equality relation. Hence we can choose a pair $\langle A, B \rangle \in \vartheta$ with $A \neq B$. Without loss of generality we can assume that $A \not\subseteq B$, thus we can choose an element $a \in A \setminus B$. For this element a we have $\mathrm{id} \in a(A)$, but $\mathrm{id} \notin a(B)$. Let

$$\langle C, D \rangle = \langle a(A) \cap M, a(B) \cap M \rangle.$$

Clearly, this pair belongs to ϑ. Furthermore, we have $\mathrm{id} \in C$ and $C \subseteq M$, thus by (2) we conclude that C equals M. On the other hand, $\mathrm{id} \notin D$ and $D \subseteq M$. We show that D must be equal to the empty set. In order to verify this suppose that d is an arbitrary element of D. Then $\mathrm{id} \in d(D)$, and since M is a subgroup of $\mathbf{F}$, $d(D)$ is also a subset of M. In view of (2) this means that $d(D) = M$, thus D equals M. Hence $\mathrm{id} \in D$, a contradiction. So we have shown that $\langle C, D \rangle = \langle M, \emptyset \rangle$. $\square$

We remark that we have proved more than what we stated in Proposition 1.4. Namely, in the last paragraph of the proof we have also shown that M is an atom of $\mathbf{U}$. Since the unary operations of $\mathbf{U}$ are automorphisms of the semilattice reduct of $\mathbf{U}$, we conclude that the atoms of $\mathbf{U}$ are exactly the right cosets of M. So the above proof yields also a proof for the following lemma.

Lemma 1.5. *If a subalgebra* $\mathbf{U}$ *of* $\mathbf{P}(F)$ *contains the empty set and the set*

$$M = \bigcap \{\, A \in U \mid \mathrm{id} \in A \,\},$$

where M *is a subgroup of* $\mathbf{F}$*, then* $\mathbf{U}$ *is subdirectly irreducible, and the atoms in* $\mathbf{U}$ *are exactly the right cosets of* M*.* $\square$

In view of (2) one can define the set M for each subalgebra $\mathbf{U}$ of $\mathbf{P}(F)$, but in general M will be neither a subgroup of $\mathbf{F}$ nor an element of U. However, if $\mathbf{U}$ is the image of a subdirectly irreducible $\mathbf{F}$-semilattice under the embedding described in the proof of Proposition 1.3, then M does enjoy similar properties. Later on we will need these technical properties which are summarized in the following lemma.

Lemma 1.6. *If* $\mathbf{S}$ *is a subdirectly irreducible* $\mathbf{F}$*-semilattice, then* $\mathbf{S}$ *is isomorphic to a subalgebra* $\mathbf{U}$ *of* $\mathbf{P}(F)$*. The algebra* $\mathbf{U}$ *can be selected so that it has a unique element* $M \subseteq F$ *with the following properties:*

(a) $\mathrm{id} \in M$ *and* $M \cdot M = M$,
(b) $A = M \cdot A$ *for all* $A \in U$, *and*
(c) $M = \bigcap \{ A \in U \mid \mathrm{id} \in A \}$.

This means that the element M of $\mathbf{U}$—considered as a subset of F—is a submonoid of $\mathbf{F}$, and every element in $\mathbf{U}$ is closed with respect to taking complex product with M. Furthermore, the element M is the least element in $\mathbf{U}$ which contains the element $\mathrm{id} \in \mathbf{F}$.

Proof. We will use the embedding φ_s which was defined in the proof of Proposition 1.3. So suppose that $\mathbf{S}$ is a subdirectly irreducible $\mathbf{F}$-semilattice, s is a fixed element of S, and φ_s is an embedding of $\mathbf{S}$ into $\mathbf{P}(F)$. Let $\mathbf{U} = \varphi_s(\mathbf{S})$ and $M = \varphi_s(s)$. Now we show that for any element $A \in U$ the equality

$$A = \{ f \in F \mid A \supseteq M \cdot f \} \tag{3}$$

holds. To verify this, let $a \in S$ be an element such that $\varphi_s(a) = A$. Since φ_s is an isomorphism from $\mathbf{S}$ to $\mathbf{U}$, we have

$$\begin{aligned} A &= \varphi_s(a) \\ &= \{ f \in F \mid f(a) \geq s \} \\ &= \{ f \in F \mid \varphi_s(f(a)) \supseteq \varphi_s(s) \} \\ &= \{ f \in F \mid f(\varphi_s(a)) \supseteq \varphi_s(s) \} \\ &= \{ f \in F \mid f(A) \supseteq M \} \\ &= \left\{ f \in F \;\middle|\; A \cdot f^{-1} \supseteq M \right\} \\ &= \{ f \in F \mid A \supseteq M \cdot f \}. \end{aligned}$$

Since $M \supseteq M \cdot \mathrm{id}$, it follows from (3) that $\mathrm{id} \in M$. Again by (3) it is easy to see that $A \supseteq M \cdot A$ for every element $A \in U$. Finally, since $\mathrm{id} \in M$, we conclude that $A = M \cdot A$. In particular, for the element $M \in U$ this means that $M = M \cdot M$. In order to prove (c), let A be an element of U containing the element id. Then we have $A = M \cdot A \supseteq M \cdot \mathrm{id} = M$. This proves the inclusion $\subseteq$. The reverse inclusion is obvious, as M is one of the sets that are intersected on the right hand side. $\square$

Corollary 1.7. *If* $\mathbf{F}$ *is a locally finite group, then, up to isomorphism, the subdirectly irreducible* $\mathbf{F}$*-semilattices are exactly those nontrivial subalgebras* $\mathbf{U}$ *of* $\mathbf{P}(F)$ *for which the set* $M = \bigcap \{ A \in U \mid \mathrm{id} \in A \}$ *is an element of* $\mathbf{U}$. *Furthermore, if* $\mathbf{U}$ *satisfies this condition, then it also has the following properties:*

(a) $\emptyset \in U$,
(b) M *is a subgroup of* $\mathbf{F}$, *and*
(c) *the atoms of* $\mathbf{U}$ *are exactly the right cosets of* M.

Proof. In Lemma 1.6 we have proved that each subdirectly irreducible $\mathbf{F}$-semilattice is isomorphic to a subalgebra $\mathbf{U}$ of $\mathbf{P}(F)$ such that $M \in \mathbf{U}$.

For the converse statement let $\mathbf{U}$ be a nontrivial subalgebra of $\mathbf{P}(F)$ such that $M \in \mathbf{U}$. We must show that $\mathbf{U}$ is subdirectly irreducible. From now on we will use similar ideas as in the proof of Proposition 1.4. In the same way as in that proof, we see that M is a submonoid of $\mathbf{F}$. But $\mathbf{F}$ is locally finite, therefore M must be a subgroup of $\mathbf{F}$.

Now we show that U contains the empty set. We will repeatedly use the fact that M and the elements generated by M in $\mathbf{P}(F)$ are in $\mathbf{U}$. Suppose first that $M = F$. Then for any element A of U different from the empty set and for any element $a \in A$, the set $a(A)$ is in U and $\mathrm{id} \in a(A)$. By the definition of M this means that $A = F$. But U contains more than one element, so in this case we conclude that $U = \{F, \emptyset\}$. Now let us consider the case where M is a proper subgroup of $\mathbf{F}$. Then for any element $f \in F \setminus M$ we have $f^{-1}(M) \wedge M = \emptyset$, that is, the empty set is again in U.

So far we have verified the properties (a) and (b). Now we can apply Lemma 1.5 to obtain that $\mathbf{U}$ is subdirectly irreducible and has property (c) as well. □

2. Simple F-semilattices

In the previous section we have proved that every subdirectly irreducible $\mathbf{F}$-semilattice is isomorphic to some subalgebra of $\mathbf{P}(F)$. Furthermore, we have seen that the nontrivial finite subalgebras of $\mathbf{P}(F)$ are subdirectly irreducible, and if $\mathbf{F}$ is locally finite, then we have described a family of subalgebras of $\mathbf{P}(F)$ which represents all subdirectly irreducible $\mathbf{F}$-semilattices. In both of these special cases it turned out that these subdirectly irreducible subalgebras of $\mathbf{P}(F)$ contain the empty set and some subgroup M of $\mathbf{F}$. Now we will show that such an algebra is simple if and only if it consists of the empty set and the right cosets of M.

Definition 2.1. If $\mathbf{F}$ is a fixed group and M is a subgroup of $\mathbf{F}$, then let $\mathbf{S}_M$ denote the subalgebra of $\mathbf{P}(F)$, the elements of which are the empty set and the right cosets of M.

Thus the empty set is the least element in $\mathbf{S}_M$, and all the right cosets of M are atoms. The set F of unary operations of $\mathbf{S}_M$ acts as a transitive permutation group on the set of atoms. It is easy to see that each $\mathbf{S}_M$ is a simple subalgebra of $\mathbf{P}(F)$ which has a least element and some atoms. The following lemma shows that the converse statement is also true.

Lemma 2.2. *The subalgebras $\mathbf{S}_M$ of $\mathbf{P}(F)$ are, up to isomorphism, exactly those simple $\mathbf{F}$-semilattices that have a least element and some atoms.*

Proof. It is easy to verify that each subalgebra $\mathbf{S}_M$ is simple, and clearly contains a least element and some atoms. For the converse, let $\mathbf{S}$ be a simple $\mathbf{F}$-semilattice with a least element 0 and an atom a. By Lemma 1.6, $\mathbf{S}$ is isomorphic to some subalgebra $\mathbf{U}$ of $\mathbf{P}(F)$. From the definition of this embedding we see that the image of 0 is the empty set. Let us denote the image of a by A. We can assume that $\mathrm{id} \in A$, because A is nonempty and for any element $f \in A$ the element $f(A)$ of $\mathbf{U}$ is an atom containing the identity. On the other hand, we also know from 1.6 that in $\mathbf{U}$ there exists a unique element M with properties (a)–(c). In particular, M is a submonoid of $\mathbf{F}$. Since $\mathrm{id} \in A$, we have $M \subseteq A$ by Lemma 1.6 (c). But A is an atom, therefore A must be equal to M, so the submonoid M is an atom in $\mathbf{U}$. Now we show that this submonoid M is actually a subgroup.

For every element $m \in M$, the set $m^{-1}(M) = M \cdot m$ is an element of $\mathbf{U}$ and a subset of M. But it cannot be a proper subset of M, because M is an atom, so $M \cdot m = M$. Therefore M is a subgroup of $\mathbf{F}$. The right cosets of M are the atoms of $\mathbf{U}$, and together with the empty set they form a subalgebra of $\mathbf{U}$ which is exactly the algebra $\mathbf{S}_M$. Our last task is now to show that $\mathbf{S}_M$ coincides with $\mathbf{U}$.

Consider the equivalence relation ϑ on U which has only one nontrivial equivalence class, namely the set S_M. We check that ϑ is a congruence relation of $\mathbf{U}$. It is clear that every unary operation of $\mathbf{U}$ preserves this relation, since S_M is a subuniverse of $\mathbf{U}$. On the other hand, we know that the elements of S_M are the least element and the atoms in $\mathbf{U}$, therefore the meet operation also preserves ϑ. Since $\mathbf{U}$ is simple, we conclude that ϑ must be the full relation on $\mathbf{U}$, so U must be equal to S_M. □

We have characterized the subdirectly irreducible $\mathbf{F}$-semilattices in two special cases in 1.4 and 1.7. In view of the previous lemma we can now easily characterize the simple $\mathbf{F}$-semilattices in these cases. It is enough to observe that in these cases the simple $\mathbf{F}$-semilattices contain a least element and some atoms. But this is trivial in the first case, and in the second case it follows from Corollary 1.7.

Corollary 2.3. *The finite simple* $\mathbf{F}$*-semilattices are, up to isomorphism, exactly the subalgebras* $\mathbf{S}_M$ *of* $\mathbf{P}(F)$ *where* M *runs over the subgroups of finite index of* $\mathbf{F}$. □

Corollary 2.4. *If* $\mathbf{F}$ *is a locally finite group, then the simple* $\mathbf{F}$*-semilattices are, up to isomorphism, exactly the subalgebras* $\mathbf{S}_M$ *of* $\mathbf{P}(F)$. □

The rest of this paper is devoted to the description of all simple $\mathbf{F}$-semilattices in the case when $\mathbf{F}$ is a fixed commutative group. We will see that there are two types of simple $\mathbf{F}$-semilattices in this case. One of the types are the algebras isomorphic to $\mathbf{S}_M$, as in Corollaries 2.3 and 2.4. The other type of simple $\mathbf{F}$-semilattices will turn out to be representable by an $\mathbf{F}$-semilattice of real numbers where the unary operations act as translations. First we consider the simple $\mathbf{F}$-semilattices which have a least element.

Proposition 2.5. *If* $\mathbf{F}$ *is a commutative group, then the simple* $\mathbf{F}$*-semilattices containing a least element are, up to isomorphism, exactly the subalgebras* $\mathbf{S}_M$ *of* $\mathbf{P}(F)$.

Proof. Let $\mathbf{S}$ be a simple $\mathbf{F}$-semilattice which contains a least element. We have to prove that $\mathbf{S}$ is isomorphic to some subalgebra $\mathbf{S}_M$ of $\mathbf{P}(F)$. By Lemma 1.6, $\mathbf{S}$ is isomorphic to some subalgebra $\mathbf{U}$ of $\mathbf{P}(F)$. Moreover, we know that $\mathbf{U}$ can be selected in such a way that it contains the empty set and a unique element M with properties (a)–(c). In particular, the element M is a submonoid of $\mathbf{F}$. Our aim is to prove that M is not only a submonoid of $\mathbf{F}$ but also a subgroup of $\mathbf{F}$. Once this is done, we can use Lemma 1.5

to show that M is actually an atom of $\mathbf{U}$, and we can complete the proof using the same argument as in the last paragraph of Lemma 2.2.

In order to prove that M is a subgroup of $\mathbf{F}$, let us introduce the notation M^{-1} for the set of inverses of the elements in M. We define a homomorphism φ from $\mathbf{U}$ to $\mathbf{P}(F)$ as follows:

$$\varphi: U \to P(F); \quad \varphi(A) = M^{-1} \cdot A. \tag{4}$$

This mapping is compatible with all unary operations, because

$$\begin{aligned} \varphi(f(A)) &= M^{-1} \cdot f(A) \\ &= M^{-1} \cdot A \cdot f^{-1} \\ &= \varphi(A) \cdot f^{-1} \\ &= f(\varphi(A)). \end{aligned}$$

Now we have to show that

$$M^{-1} \cdot (A \cap B) = (M^{-1} \cdot A) \cap (M^{-1} \cdot B).$$

The inclusion $\subseteq$ is trivial. To prove the reverse inclusion, let us choose an element from the right hand side. So there exist elements $a \in A$, $b \in B$, $m, n \in M$ such that $m^{-1} \cdot a = n^{-1} \cdot b$. Since $\mathbf{F}$ is commutative this is equivalent to the equality $n \cdot a = m \cdot b$. But by Lemma 1.6 (b) we know that $A = M \cdot A$ and $B = M \cdot B$, hence both A and B contain the element $n \cdot a = m \cdot b$. Therefore our original element $m^{-1} \cdot a$ can be expressed in the way of $(m^{-1} \cdot n^{-1}) \cdot (n \cdot a) \in M^{-1} \cdot (A \cap B)$. So we have shown that φ is a homomorphism from $\mathbf{U}$ to $\mathbf{P}(F)$.

Clearly, $\varphi(\emptyset) = \emptyset$ and $\varphi(M) = M^{-1} \cdot M$. Since $M^{-1} \cdot M \neq \emptyset$, the kernel of the homomorphism φ cannot be the full relation. But $\mathbf{U}$ is simple, hence φ must be an embedding. Now let m be an arbitrary element of M. Since M is a submonoid, we have $M^{-1} \cdot M \cdot m = M^{-1} \cdot M$, that is, $\varphi(M \cdot m) = \varphi(M)$. However, φ is an embedding, hence $M \cdot m = M$. This means that M is a subgroup of $\mathbf{F}$, so the proof is complete. □

From now on we will discuss simple $\mathbf{F}$-semilattices that have no least element. We will see that they can be embedded in a special algebra which we define now.

Definition 2.6. Let $\mathbf{F}$ be a fixed commutative group. Then for every nonconstant homomorphism β from $\mathbf{F}$ to the additive group $\langle \mathbb{R}; + \rangle$ of the real numbers let us define an $\mathbf{F}$-semilattice $\mathbf{R}_\beta = \langle \mathbb{R}; \min, F \rangle$ on the set of real numbers as follows:

(a) $\min(a, b)$ is taken with respect to the natural order of $\mathbb{R}$, and
(b) $f(a) = a - \beta(f)$ for all $f \in F$ and $a, b \in \mathbb{R}$.

As we will see later, not every subalgebra of this algebra is simple; however the algebras $\mathbf{R}_\beta$ contain, up to isomorphism, all the simple $\mathbf{F}$-semilattices without least element.

Lemma 2.7. *If $\mathbf{F}$ is a fixed commutative group then every simple $\mathbf{F}$-semilattice without least element can be embedded in $\mathbf{R}_\beta$ for an appropriate nonconstant homomorphism β.*

Proof. The first step of the proof is to represent the given simple **F**-semilattice, according to Lemma 1.6, as a subalgebra of $\mathbf{P}(F)$. So we have a simple subalgebra $\mathbf{U}$ of $\mathbf{P}(F)$ without least element. Furthermore, $\mathbf{U}$ has an element M, which is actually a submonoid of $\mathbf{F}$, and in addition M has the properties described in 1.6. We will see that in this situation M must have $M \cup M^{-1} = F$. This will lead us to the proof that the semilattice order of $\mathbf{U}$ is linear, and that any two distinct element of U can be separated by a shifted image $f(M) = M \cdot f^{-1}$ of M. At this point we will choose a unit shift $e \in F$. The number $0 \in \mathbf{R}_\beta$ will correspond to M, and the integers $k \in \mathbf{R}_\beta$ to $M \cdot e^k$. After this, we will extend this correspondence to the rational numbers and then to the real numbers. In the meanwhile the homomorphism β will also be discoverd.

Now let us see the details.

Claim 1. $\emptyset, F \notin U$.

Since $\mathbf{U}$ has no least element, $\mathbf{U}$ cannot contain the empty set. In order to prove that $F \notin U$, suppose the contrary. To this end let ϑ be the equivalence relation on U which has only two blocks $\{F\}$ and $U \setminus \{F\}$. Clearly, ϑ is compatible with the unary operations as well as with intersection, so it is a congruence relation. Since $\mathbf{U}$ has no least element, U must be infinte. So the block $U \setminus \{F\}$ contains more than one element, and hence the congruence relation ϑ is not trivial. But this contradicts the assumption that $\mathbf{U}$ is simple.

Claim 2. *The submonoid M of* $\mathbf{F}$ *is not a subgroup.*

In the second last paragraph of Corollary 1.7 we have shown that if M were a subgroup of $\mathbf{F}$, then $\mathbf{U}$ would contain the empty set. But the empty set would be a least element in $\mathbf{U}$, and we know that $\mathbf{U}$ has none, therefore M cannot be a subgroup of $\mathbf{F}$.

Claim 3. $M^{-1} \cdot M = F$.

We will use the homomorphism φ defined in (4) (the proof that φ is indeed a homomorphism works here, as well). Here the kernel of φ is not the equality relation. This is because of the fact that the submonoid M is not a subgroup. To see this, choose an element m from $M \setminus M^{-1}$. We will examine the images of M and $M \cdot m^{-1}$ under φ. Since $m^{-1} \notin M$ and $m^{-1} \in M \cdot m^{-1}$, we have $M \neq M \cdot m^{-1}$. On the other hand, the images under φ are $\varphi(M) = M^{-1} \cdot M$ and $\varphi(M \cdot m^{-1}) = M^{-1} \cdot M \cdot m^{-1}$, respectively. But the set $M^{-1} \cdot M$ is a subgroup of $\mathbf{F}$, because $\mathbf{F}$ is commutative and M is a submonoid. Therefore $\varphi(M) = \varphi(M \cdot m^{-1})$. This shows that φ cannot be an embedding, hence it must be a constant homomorphism, since its domain is the simple algebra $\mathbf{U}$.

If f is an arbitrary element of F, the sets M and $f(M) = M \cdot f^{-1}$ are elements of U, so their images under φ are equal. This means that $M^{-1} \cdot M = M^{-1} \cdot M \cdot f^{-1}$ for every element $f \in F$. Since $M^{-1} \cdot M$ is a subgroup of $\mathbf{F}$, $f^{-1} \in M^{-1} \cdot M$ for every element $f \in F$, that is, $M^{-1} \cdot M = F$.

Claim 4. $F = M \cup M^{-1}$.

Let us suppose the contrary. So, we can take an element $r \in F$ such that neither r nor r^{-1} is in M. This will lead to a contradiction. First of all we will define a sequence a_i $(i = 1, 2, \ldots)$ in M. Since $M^{-1} \cdot M = F$, we know that for an arbitrary element $f \in F$

there exist elements $a, b \in M$ such that $f = a^{-1} \cdot b$. In other words, for every element $f \in F$ there exists an element $a \in M$ such that $f \cdot a \in M$. We can apply this argument several times to define the elements a_i $(i = 1, 2, \ldots)$ in such a way that

$$\begin{array}{ccr} a_1 \in M & \text{with} & r \cdot a_1 \in M, \\ a_2 \in M & \text{with} & r^2 \cdot a_1 \cdot a_2 \in M, \\ \vdots & & \vdots \\ a_i \in M & \text{with} & r^i \cdot a_1 \cdot \ldots \cdot a_i \in M, \\ \vdots & & \vdots \end{array}$$

Furthermore, we require that the choice $a_i = \mathrm{id}$ has to be made whenever $r^i \cdot a_1 \cdot \ldots \cdot a_{i-1} \in M$.

Now we define a homomorphism $\psi \colon \mathbf{U} \to \mathbf{P}(F)$ by setting

$$\psi(A) = \left\{ f \in F \;\middle|\; f \cdot (r^i \cdot a_1 \cdot \ldots \cdot a_i) \in A \text{ for almost all natural numbers } i \right\}.$$

It is easy to see that this mapping is compatible with the unary operations as well as with intersection. Now we show that this mapping is not injective; namely, we have

$$\psi(M) = \psi(M \cap M \cdot r).$$

The sets M and $M \cap M \cdot r$ are distinct elements of U, because the first one contains id, while the other one does not, since $r^{-1} \notin M$.

In order to see that the images are equal, take an element f from $\psi(M)$. This means that $f \cdot (r^i \cdot a_1 \cdot \ldots \cdot a_i) \in M$ for almost all i. Therefore there exists a natural number k such that this condition holds for every $i \geq k$. If $f \cdot (r^i \cdot a_1 \cdot \ldots \cdot a_i) \in M$, then let us multiply each side by $r \cdot a_{i+1}$, and we get $f \cdot (r^{i+1} \cdot a_1 \cdot \ldots \cdot a_i \cdot a_{i+1}) \in M \cdot a_{i+1} \cdot r$. But we know that $a_{i+1} \in M$ and M is a submonoid, so we get $M \cdot a_{i+1} \subseteq M$. Therefore $f \cdot (r^{i+1} \cdot a_1 \cdot \ldots \cdot a_{i+1}) \in M \cdot r$. To sum it up, we know that $f \cdot (r^i \cdot a_1 \cdot \ldots \cdot a_i) \in M \cdot r$ if $i > k$. But the element $f \cdot (r^i \cdot a_1 \cdot \ldots \cdot a_i)$ is in M, hence it is in $M \cap M \cdot r$, too. This proves the inclusion $\psi(M) \subseteq \psi(M \cap M \cdot r)$. The reverse inclusion is trivial, since $M \cap M \cdot r$ is a subset of M.

So far we have proven that ψ is a homomorphism from $\mathbf{U}$ to $\mathbf{P}(F)$, and it is not an embedding. Since the algebra $\mathbf{U}$ is simple, we conclude that ψ must be a constant mapping. The question is which element of $\mathbf{P}(F)$ is assigned by ψ to the elements of U. Since ψ is a homomorphism, this element must form a one element subalgebra of $\mathbf{P}(F)$. But because of the unary operations, there are only two such subalgebras of $\mathbf{P}(F)$, namely $\{\emptyset\}$ and $\{F\}$. From the definition of ψ we see that $\psi(M)$ contains id, hence we conclude that $\psi(M) = F$.

In particular, the element r is in $\psi(M)$. This means that there exists a natural number k such that $r \cdot (r^i \cdot a_1 \cdot \ldots \cdot a_i) \in M$ for every $i \geq k$. But this shows that we have choosen id when we defined the element a_{i+1}. Therefore we conclude that $a_{k+1} = a_{k+2} = \ldots = \mathrm{id}$. Since $\psi(M) = F$, the element $(a_1 \cdot a_2 \cdot \ldots \cdot a_k)^{-1}$ is also in $\psi(M)$. This means that there exists a natural number l such that $(a_1 \cdot a_2 \cdot \ldots \cdot a_k)^{-1} \cdot (r^i \cdot a_1 \cdot \ldots \cdot a_i) \in M$ for every $i \geq l$. We can assume without loss of generality that $l > k$. This shows that $M \ni (a_1 \cdot a_2 \cdot \ldots \cdot a_k)^{-1} \cdot (r^i \cdot a_1 \cdot \ldots \cdot a_i) = r^i \cdot a_{k+1} \cdot \ldots \cdot a_i = r^i$ for every $i \geq l$.

Up to this point we have proved the following statement. If neither r nor r^{-1} is in M, then there exists a natural number l such that $r^l, r^{l+1}, \ldots \in M$. If we switch the role of r and r^{-1}, we get in the same way another natural number j such that $r^{-j}, r^{-j-1}, \ldots \in M$. Now choose a natural number i greater than both k and l. Then r^{i+1} and r^{-i} are elements of M, and since M is a submonoid of $\mathbf{F}$, we get $r = r^{i+1} \cdot r^{-i} \in M$. But this contradicts our assumption that M contains neither r nor r^{-1}.

Claim 5. *Set inclusion yields a linear order on* $\mathbf{U}$. *Furthermore if A and B are two elements of U such that $A \nsubseteq B$, then for any element $a \in A \setminus B$ we have $B \subseteq M \cdot a \subseteq A$.*

Clearly, the second statement implies the first. In order to prove the second statement, consider elements $A, B \in U$ and $a \in A \setminus B$. From Lemma 1.6 we know that $M \cdot A = A$, so $M \cdot a \subseteq A$. Now suppose that the other inclusion does not hold, that is, there exists an element $b \in B \setminus M \cdot a$. Thus $b \cdot a^{-1} \in B \cdot a^{-1} \setminus M$. By Claim 4 we get that the element $b \cdot a^{-1}$ must be in M^{-1}, so $a \cdot b^{-1} \in M$. Thus $a = (a \cdot b^{-1}) \cdot b \in M \cdot B = B$, and this is a contradiction. So we conclude that $B \subseteq M \cdot a$.

At this stage of the proof we can indicate how the homomorphism $\beta \colon \mathbf{F} \to \langle \mathbb{R}; + \rangle$ will be defined. We have the subset M of F which divides F into two parts. Those elements of F which lie in M will be mapped by β to nonpositive real numbers; and those which lie in M^{-1}, to the positive ones. The kernel of β will be the subgroup $M \cap M^{-1}$ of $\mathbf{F}$. Now we take an element e which will be mapped by β to the number 1. So let us choose and fix an element e from $M^{-1} \setminus M$. This can be done, since M is not a subgroup of F. Since β is to be a homomorphism, for any integer k the elements of $M \cdot e^k$ must correspond to real numbers not greater than k. This suggests the conjecture that every element of F will be an element of $M \cdot e^k$ for some integer k.

Claim 6. $\displaystyle\bigcup_{k \in \mathbb{Z}} M \cdot e^k = F$.

We will define again a homomorphism η from $\mathbf{U}$ to $\mathbf{P}(F)$. For any element $A \in U$ let

$$\eta(A) = \bigcup_{k \in \mathbb{Z}} A \cdot e^k.$$

It is easy to see that this mapping is compatible with the unary operations, since $\mathbf{F}$ is commutative. To prove that η is compatible with the intersection as well, we have to show that

$$\bigcup_{k \in \mathbb{Z}} (A \cap B) \cdot e^k = \left(\bigcup_{k \in \mathbb{Z}} A \cdot e^k \right) \cap \left(\bigcup_{k \in \mathbb{Z}} B \cdot e^k \right).$$

The inclusion $\subseteq$ is trivial. To prove the reverse inclusion, take an arbitrary element from the right hand side. So there exist elements $a \in A, b \in B$ and integers $k, l \in \mathbb{Z}$ such that $a \cdot e^k = b \cdot e^l$. We can assume that $k \le l$. Since $e \in M^{-1}$, we get that $e^{k-l} \in M$. From Lemma 1.6 we know that $A = M \cdot A$, hence the element $a \cdot e^{k-l}$ belongs to A. But from the equality $a \cdot e^{k-l} \cdot e^l = a \cdot e^k = b \cdot e^l$ we get $a \cdot e^{k-l} = b \in A \cap B$, therefore the element $a \cdot e^{k-l} \cdot e^l = b \cdot e^l$ is in $(A \cap B) \cdot e^l$.

The homomorphism η cannot be injective, since $\eta(M) = \eta(M \cdot e)$ but $M \neq M \cdot e$ (as $e \in M \cdot e \setminus M$). But $\mathbf{U}$ is simple, therefore η is a constant mapping. The same argument as before for ψ yields that η maps each element of U to F. In particular, $\eta(M) = F$, which is what we wanted to prove.

Claim 7. *For any integer k we have $\mathrm{id} \in M \cdot e^k$ iff $k \geq 0$.*

This claim is an immediate consequence of the facts that $e \in M^{-1} \setminus M$ and that M is a submonoid of $\mathbf{F}$.

Now we can define the homomorphism $\beta\colon \mathbf{F} \to \langle \mathbb{R}; + \rangle$. For any element $a \in F$ let

$$\beta(a) = \inf \left\{ \frac{k}{l} \in \mathbb{Q} \;\middle|\; k \in \mathbb{Z},\, l \in \mathbb{N} \text{ and } a^l \in M \cdot e^k \right\}. \tag{5}$$

Claim 8. *The mapping β is a nonconstant homomorphism from $\mathbf{F}$ to $\langle \mathbb{R}; + \rangle$. Furthermore,*
(a) *$\beta(e^i) = i$ for any integer i, and*
(b) *$\beta(m) \leq 0$ for every element $m \in M$.*

To see that β is well defined, we have to check that for every element $a \in F$ the set on the right hand side of (5) is nonempty, and has a lower bound. So let a be an arbitrary element of F. By Claim 6 we get an integer k such that $a \in M \cdot e^k$, therefore the set on the right hand side of (5) contains k. Again by Claim 6 we get another integer i such that $a^{-1} \in M \cdot e^i$. Since M is closed under multiplication, we get $a^{-l} \in M \cdot e^{il}$ for any natural number l. If for some integer k we have $a^l \in M \cdot e^k$, then $\mathrm{id} = a^l \cdot a^{-l} \in M \cdot e^k \cdot M \cdot e^{il} = M \cdot e^{k+il}$, hence by Claim 7 the exponent $k + il$ is nonnegative. This implies that $k/l \geq -i$, therefore the integer $-i$ is a lower bound for the rational numbers belonging to the set in (5).

Now we show that $\beta(e^i) = i$ for any integer i. It is clear that $e^i \in M \cdot e^i$, hence from (5) we get $\beta(e^i) \leq i/1 = i$. Now suppose that $(e^i)^l \in M \cdot e^k$ for some integer k and natural number l. Thus $\mathrm{id} \in M \cdot e^{k-il}$, and by Claim 7 we get $k/l \geq i$. This shows that $\beta(e^i) = i$.

From the definition of β it is clear that $\beta(m) \leq 0$ for every element $m \in M$, since we can choose 0 for k and 1 for l.

Now we prove that $\beta(a^{-1}) \leq -\beta(a)$ for every element $a \in F$. To this end let ε be an arbitrary small positive real number, and let us choose the numbers k and l such that $\beta(a) - \varepsilon \leq k/l < \beta(a)$. From $k/l < \beta(a)$ we know that $a^l \notin M \cdot e^k$. Using the facts that M is a submonoid of the commutatice group F and that $M \cup M^{-1} = F$, we get

$$\begin{aligned} a^l \notin M \cdot e^k &\Rightarrow a^l \cdot e^{-k} \notin M \\ &\Rightarrow a^l \cdot e^{-k} \in M^{-1} \\ &\Rightarrow a^{-l} \cdot e^k \in M \\ &\Rightarrow a^{-l} \in M \cdot e^{-k} \\ &\Rightarrow (a^{-1})^l \in M \cdot e^{-k}. \end{aligned}$$

But according to (5) this means that $\beta(a^{-1}) \leq -k/l$. Since we have chosen the numbers k and l such that $-k/l \leq -\beta(a) + \varepsilon$, we get $\beta(a^{-1}) \leq -\beta(a) + \varepsilon$. But this holds for every positive real number ε, therefore it must hold for zero, that is, $\beta(a^{-1}) \leq -\beta(a)$.

To prove that β is compatible with multiplication, let $a, b \in F$ and let ε be an arbitrary small positive real number. From the definition of β we get two pairs k_1, l_1 and k_2, l_2 of integers such that $l_1, l_2 > 0$ and

$$\beta(a) \leq \frac{k_1}{l_1} \leq \beta(a) + \varepsilon \quad \text{where} \quad a^{l_1} \in M \cdot e^{k_1}, \text{ and}$$
$$\beta(b) \leq \frac{k_2}{l_2} \leq \beta(b) + \varepsilon \quad \text{where} \quad b^{l_2} \in M \cdot e^{k_2}.$$

Since M is a submonoid, raising $a^{l_1} \in M \cdot e^{k_1}$ to the l_2th power yields $a^{l_1 l_2} \in M \cdot e^{k_1 l_2}$. Similarly $b^{l_1 l_2} \in M \cdot e^{l_1 k_2}$, and by multiplication we conculde that $(a \cdot b)^{l_1 l_2} \in M \cdot e^{k_1 l_2 + l_1 k_2}$. By the definition of β this means that

$$\beta(a \cdot b) \leq \frac{k_1 l_2 + l_1 k_2}{l_1 l_2} = \frac{k_1}{l_1} + \frac{k_2}{l_2} \leq \beta(a) + \beta(b) + 2\varepsilon.$$

But ε was again an arbitrary positive real number, hence it follows that $\beta(a \cdot b) \leq \beta(a) + \beta(b)$. Finally, the inequalities below prove that $\beta(a \cdot b) = \beta(a) + \beta(b)$ and $\beta(a^{-1}) = -\beta(a)$:

$$\begin{aligned}
\beta(a \cdot b) &\leq \beta(a) + \beta(b) \\
&= \beta(a) + \beta(a^{-1} \cdot a \cdot b) \\
&\leq \beta(a) + \beta(a^{-1}) + \beta(a \cdot b) \\
&\leq \beta(a) - \beta(a) + \beta(a \cdot b) \\
&= \beta(a \cdot b).
\end{aligned}$$

Now we can define an embedding $\xi \colon \mathbf{U} \to \mathbf{R}_\beta$ which will complete the proof of the lemma. For any element $A \in U$ let

$$\xi(A) = \sup \{ \beta(a) \mid a \in A \} . \tag{6}$$

Claim 9. *The mapping ξ is an embedding of $\mathbf{U}$ in $\mathbf{R}_\beta$. Furthermore, $\xi(M \cdot f) = \beta(f)$ for every element $f \in F$.*

To see that ξ is well defined, we have to check that for every element $A \in U$ the set on the right hand side of (6) has an upper bound. By Claim 1 we have $A \neq \emptyset$, and we can choose an element $f \in F \setminus A$ for every element $A \in U$. Since $f \in M \cdot f \setminus A$, by Claim 5 we get $A \subseteq M \cdot f$. So we conclude that if $\xi(M \cdot f)$ exists, then $\xi(A)$ also exists, by the definition of ξ, and $\xi(A) \leq \xi(M \cdot f)$.

Now we prove that $\xi(M \cdot f) = \beta(f)$. From Claim 8 we know that β is a homomorphism, and $\beta(m) \leq 0$ for every element $m \in M$. Hence for every element $m \cdot f$ of $M \cdot f$ we have

$$\beta(m \cdot f) = \beta(m) + \beta(f) \leq 0 + \beta(f) = \beta(f).$$

Therefore $\xi(M \cdot f) \leq \beta(f)$. But $f \in M \cdot f$, so $\xi(M \cdot f) \geq \beta(f)$, hence $\xi(M \cdot f) = \beta(f)$.

In order to prove that ξ is complatible with the semilattice operation, let A, B be arbitrary elements in U. By Claim 5 we can assume that $A \subseteq B$. But from this we get $\xi(A) \leq \xi(B)$, and hence

$$\xi(A \cap B) = \xi(A) = \min(\xi(A), \xi(B)).$$

Now we are going to show that ξ is compatible with the unary operations as well. For arbitrary elements $A \in U$ and $f \in F$ we have

$$\begin{aligned}
\xi(f(A)) &= \xi(A \cdot f^{-1}) \\
&= \sup \{ \beta(b) \mid b \in A \cdot f^{-1} \} \\
&= \sup \{ \beta(a \cdot f^{-1}) \mid a \in A \} \\
&= \sup \{ \beta(a) + \beta(f^{-1}) \mid a \in A \} \\
&= \sup \{ \beta(a) \mid a \in A \} + \beta(f^{-1}) \\
&= \xi(A) + \beta(f^{-1}) \\
&= \xi(A) - \beta(f) \\
&= f(\xi(A)).
\end{aligned}$$

So we conclude that ξ is a homomorphism from $\mathbf{U}$ to $\mathbf{R}_\beta$. Since $\xi(M \cdot e^k) = \beta(e^k) = k$ for every integer k, ξ cannot be a constant mapping. But $\mathbf{U}$ is simple, hence ξ is an embedding. $\square$

The next example shows that for a homomorphism $\beta: \mathbf{F} \to \mathbb{R}$, a subalgebra $\mathbf{S}$ of $\mathbf{R}_\beta$ is not necessarily simple. This will help us to describe the simple subalgebras of $\mathbf{R}_\beta$.

Example 2.8. Let $\mathbf{F}$ be the additive group $\langle \mathbb{Z}; + \rangle$ of the integers and $\beta: \mathbf{F} \to \langle \mathbb{R}; + \rangle$ be the identical embedding. Then the subalgebra $\mathbf{S}$ of $\mathbf{R}_\beta$ with the underlying set

$$S = \left\{ \frac{a}{2} \in \mathbb{R} \;\middle|\; a \in \mathbb{Z} \right\}$$

is not simple.

Proof. Clearly, the subset S of $\mathbb{R}$ is closed under the operation of subtracting any integer $\beta(i) = i$ ($i \in \mathbb{Z}$), that is, it is closed under the unary operations of $\mathbf{R}_\beta$. In addition, S is also closed under the binary operation of taking the minimum. Therefore $\mathbf{S}$ is a subalgebra of $\mathbf{R}_\beta$.

Now we construct a nontrivial congruence relation ϑ on S which will yield that $\mathbf{S}$ is not simple. Let ϑ be the equivalence relation on S whose blocks are the two-element sets $\{i, i + 1/2\}$, where $i \in \mathbb{Z}$. Clearly, this relation is compatible with the operations of $\mathbf{S}$. $\square$

Now we will show that if the image of β contains arbitrary small positive real numbers, then every subalgebra of $\mathbf{R}_\beta$ is simple. Let us define this property of β exactly.

Definition 2.9. A homomorphism $\beta\colon \mathbf{F} \to \langle \mathbb{R}; + \rangle$ is called *dense* if for each real number $\varepsilon > 0$ there exists an element $f \in F$ such that $0 < \beta(f) \leq \varepsilon$.

Lemma 2.10. *If* $\mathbf{F}$ *is a commutative group and* $\beta\colon \mathbf{F} \to \langle \mathbb{R}; + \rangle$ *is a dense homomorphism, then every subalgebra of* $\mathbf{R}_\beta$ *is simple.*

Proof. Let $\mathbf{S}$ be a subalgebra of $\mathbf{R}_\beta$. We will prove that every pair of two distinct real numbers x, y in $\mathbf{S}$ generates the full congruence of $\mathbf{S}$. Clearly, this ensures that $\mathbf{S}$ is simple. So let ϑ denote the congruence relation on $\mathbf{S}$ generated by the pair $\langle x, y \rangle$. We may assume that $x < y$. Since β is dense, there exists an element $e \in F$ such that $0 < \beta(e) \leq y - x$. Let $\varepsilon = \beta(e) > 0$, and $z = x + \varepsilon$. The number z is in S, because

$$z = x + \varepsilon = x + \beta(e) = x - \beta(e^{-1}) = e^{-1}(x) \in S.$$

Since the pairs $\langle x, y \rangle$ and $\langle z, z \rangle$ are in ϑ, we have

$$\vartheta \ni \langle \min(x, z), \min(y, z) \rangle = \langle x, z \rangle = \langle x, x + \varepsilon \rangle.$$

If we apply the unary operation $e^{-k} \in F$ to the pair $\langle x, x + \varepsilon \rangle$ for some integer k, we get

$$\begin{aligned} \vartheta &\ni \langle e^{-k}(x), e^{-k}(x + \varepsilon) \rangle \\ &= \langle x - \beta(e^{-k}), x + \varepsilon - \beta(e^{-k}) \rangle \\ &= \langle x + \beta(e^{k}), x + \varepsilon + \beta(e^{k}) \rangle \\ &= \langle x + k\varepsilon, x + \varepsilon + k\varepsilon \rangle \\ &= \langle x + k\varepsilon, x + (k+1)\varepsilon \rangle. \end{aligned}$$

Since ϑ is transitive, we conclude that $\langle x + k\varepsilon, x + l\varepsilon \rangle \in \vartheta$ for any two integers k, l.

Now we prove that every pair $\langle r, s \rangle \in S \times S$ belongs to ϑ. Since $\varepsilon > 0$, we can choose two integers k, l such that $x + k\varepsilon < r, s < x + l\varepsilon$. Hence we have

$$\begin{aligned} &\vartheta \ni \langle \min(x + k\varepsilon, r), \min(x + l\varepsilon, r) \rangle = \langle x + k\varepsilon, r \rangle, \text{ and} \\ &\vartheta \ni \langle \min(x + k\varepsilon, s), \min(x + l\varepsilon, s) \rangle = \langle x + k\varepsilon, s \rangle. \end{aligned}$$

Finally, the symmetry and the transitivity of ϑ yields $\langle r, s \rangle \in \vartheta$. $\square$

Let us examine the case when β is not dense. It turns out that in this case $\mathbf{R}_\beta$ contains, up to isomorphism, only one simple subalgebra, which has the following structure.

Definition 2.11. Let $\mathbf{F}$ be a fixed commutative group. Then for every surjective homomorphism α from $\mathbf{F}$ onto the additive group $\langle \mathbb{Z}; + \rangle$ of the integers let $\mathbf{Z}_\alpha = \langle \mathbb{Z}; \min, F \rangle$ be the $\mathbf{F}$-semilattice defined on the set of integers as follows:

(a) $\min(a, b)$ is taken with respect to the natural order of $\mathbb{Z}$, and
(b) $f(a) = a - \alpha(f)$ for all $f \in F$ and $a, b \in \mathbb{Z}$.

Lemma 2.12. *If* $\mathbf{F}$ *is a commutative group and* $\alpha\colon \mathbf{F} \to \langle \mathbb{Z}; + \rangle$ *is a surjective homomorphism, then the* $\mathbf{F}$*-semilattice* $\mathbf{Z}_\alpha$ *is simple.*

Proof. The proof is the same as that of Lemma 2.10, except that the role of ε should be now played by 1. Since α is surjective, we can find an element $e \in F$ such that $\alpha(e) = \varepsilon = 1$. □

Lemma 2.13. *If* $\beta\colon \mathbf{F} \to \langle \mathbb{R}; + \rangle$ *is a nonconstant and nondense homomorphism, then there exists a positive real number* ε *such that the mapping* $\alpha\colon f \mapsto \beta(f)/\varepsilon$ *is a surjective homomorphism of* $\mathbf{F}$ *onto* $\langle \mathbb{Z}; + \rangle$*. Furthermore, every simple subalgebra of* $\mathbf{R}_\beta$ *is isomorphic to* $\mathbf{Z}_\alpha$*.*

Proof. Since β is not constant, the real number

$$\varepsilon = \inf \{\, \beta(f) \mid f \in F \text{ and } \beta(f) > 0 \,\} \tag{7}$$

is well defined. The homomorphism β is not dense, thus $\varepsilon > 0$.

Now we show that there exists an element $e \in F$ such that $\beta(e) = \varepsilon$. Since $\varepsilon > 0$, by (7) we can choose an element $e \in F$ such that $\varepsilon \le \beta(e) < 2\varepsilon$. If $\beta(e) \ne \varepsilon$, then we can again find an element $f \in F$ such that $\varepsilon \le \beta(f) < \beta(e) < 2\varepsilon$. Hence $0 < \beta(e) - \beta(f) < \varepsilon$. But β is a homomorphism, therefore $\beta(e \cdot f^{-1}) = \beta(f) - \beta(e)$. So we have found an element $e \cdot f^{-1} \in F$ such that $0 < \beta(e \cdot f^{-1}) < \varepsilon$, which contradicts the definition of ε. Therefore we conclude that $\beta(e) = \varepsilon$.

Now we prove that $\beta(f)/\varepsilon \in \mathbb{Z}$ for every element $f \in F$. This will ensure that α assigns integers to the elements of F. Let $f \in F$ be an arbitrary element. Since $\varepsilon > 0$, we can choose an integer k such that $k\varepsilon \le \beta(f) < (k+1)\varepsilon$. But $\beta(e^k) = k\beta(e) = k\varepsilon$, therefore $0 \le \beta(f) - k\varepsilon = \beta(a \cdot e^{-k}) < \varepsilon$. By the definition of ε we get $0 = \beta(f) - k\varepsilon$, that is, $\beta(f)/\varepsilon \in \mathbb{Z}$. Since $\beta(e^k) = k\varepsilon$ for every integer k, we conclude that the mapping α of F into $\mathbb{Z}$ is surjective. Finally, since β is a homomorphism, α is also a homomorphism.

To complete the proof, it remains to check that if $\mathbf{S}$ is a simple subalgebra of $\mathbf{R}_\beta$, then it is isomorphic to $\mathbf{Z}_\alpha$. To this end let us choose an arbitrary element $s \in S$. Now we define a mapping φ of $\mathbf{S}$ into $\mathbf{Z}_\alpha$, and subsequently we show that φ is a surjective homomorphism. For any number $a \in S$ let

$$\varphi(a) = \lfloor (a - s)/\varepsilon \rfloor.$$

Clearly, this mapping is order preserving, and for any integer k we have

$$\begin{aligned}
\varphi(e^k(a)) &= \varphi(a - \beta(e^k)) \\
&= \varphi(a - k\varepsilon) \\
&= \lfloor (a - k\varepsilon - s)/\varepsilon \rfloor \\
&= \lfloor (a - s)/\varepsilon - k \rfloor \\
&= \lfloor (a - s)/\varepsilon \rfloor - k.
\end{aligned}$$

Thus φ is surjective. Since $\alpha(f) \in \mathbb{Z}$ for any unary operation $f \in F$,

$$\begin{aligned} f(\varphi(a)) &= f(\lfloor (a-s)/\varepsilon \rfloor) \\ &= \lfloor (a-s)/\varepsilon \rfloor - \alpha(f) \\ &= \lfloor (a-s)/\varepsilon - \alpha(f) \rfloor \\ &= \lfloor (a-s)/\varepsilon - \beta(f)/\varepsilon \rfloor \\ &= \lfloor (a-\beta(f)-s)/\varepsilon \rfloor \\ &= \lfloor (f(a)-s)/\varepsilon \rfloor \\ &= \varphi(f(a)). \end{aligned}$$

Hence φ is a surjective homomorphism of $\mathbf{S}$ onto $\mathbf{Z}_\alpha$. But $\mathbf{S}$ is simple, therefore φ is an isomorphism. $\square$

Now we can summarize the results in Proposition 2.5 and Lemmas 2.7 through 2.13 to give a characterization of all simple $\mathbf{F}$-semilattices for a commutative group $\mathbf{F}$.

Theorem 2.14. *If $\mathbf{F}$ is a commutative group, then every simple $\mathbf{F}$-semilattice is isomorphic to one of the following algebras:*

(a) $\mathbf{S}_M$*, where M is a subgroup of $\mathbf{F}$,*
(b) $\mathbf{Z}_\alpha$*, where $\alpha\colon \mathbf{F} \to \langle \mathbb{Z}; + \rangle$ is a surjective group homomorphism, and*
(c) *the subalgebras of $\mathbf{R}_\beta$, where $\beta\colon \mathbf{F} \to \langle \mathbb{R}; + \rangle$ is a dense group homomorphism.*

Furthermore, these simple $\mathbf{F}$-semilattices are pairwise nonisomorphic, except for the case when β_1, β_2 are dense homomorphisms, $\mathbf{S}_1, \mathbf{S}_2$ are subalgebras of $\mathbf{R}_{\beta_1}, \mathbf{R}_{\beta_2}$ respectively, and there exist real numbers $t > 0$ and d such that $\beta_2 = t\beta_2$ and $S_2 = tS_1 + d$.

Proof. Let $\mathbf{F}$ be a fixed commutative group. First of all we know that the $\mathbf{F}$-semilattices listed in (a), (b) and (c) are simple (use Lemmas 2.2, 2.12 and 2.10, respectively). Our task is now to prove that each simple $\mathbf{F}$-semilattice $\mathbf{S}$ is isomorphic to one of these.

If $\mathbf{S}$ has a least element, then according to Proposition 2.5, $\mathbf{S}$ is isomorphic to some algebra listed in (a). Now suppose that $\mathbf{S}$ has no least element. By Lemma 2.7 we can assume that $\mathbf{S}$ is a subalgebra of $\mathbf{R}_\beta$ for an appropriate nonconstant homomorphism $\beta\colon \mathbf{F} \to \langle \mathbb{R}; + \rangle$. If β is dense, then $\mathbf{S}$ is one of the algebras listed in (c). If β is not dense, then by Lemma 2.13, $\mathbf{S}$ is isomorphic to some algebra listed in (b). So we have proved the first part of the theorem.

In the rest of the proof we will show that the given algebras in (a), (b) and (c) are pairwise nonisomorphic, except for the special cases indicated above. Since the algebras in (a) have a least element, while the algebras in (b) and (c) have not, the algebras in (a) cannot be isomorphic to any algebra in (b) or (c). Now we show that distinct algebras in (a) are nonisomorphic; that is, if M_1 and M_2 are distinct subgroups of $\mathbf{F}$, then $\mathbf{S}_{M_1} \not\cong \mathbf{S}_{M_2}$. We can assume that $M_2 \not\subseteq M_1$, so we can choose an element $f \in M_1 \setminus M_2$. We recall that if two algebras are isomorphic then the same identities hold in them. The identity $x \wedge f(x) = x$ holds in $\mathbf{S}_{M_1}$, since $\emptyset \cap f(\emptyset) = \emptyset$, and for each element $M_1 \cdot a \in S_{M_1}$ we have

$$\begin{aligned} M_1 \cdot a \cap f(M_1 \cdot a) &= M_1 \cdot a \cap M_1 \cdot a \cdot f^{-1} \\ &= M_1 \cdot a \cap M_1 \cdot f^{-1} \cdot a \\ &= M_1 \cdot a \cap M_1 \cdot a \\ &= M_1 \cdot a. \end{aligned}$$

On the other hand, the identity $x \wedge f(x) = x$ does not hold in $\mathbf{S}_{M_2}$, since $M_2 \neq M_2 \cdot f^{-1}$ and $M_2 \cap f(M_2) = M_2 \cap M_2 \cdot f^{-1} = \emptyset$. Hence we conclude that $\mathbf{S}_{M_1}$ and $\mathbf{S}_{M_2}$ are nonisomorphic.

Let $\mathbf{Z}_\alpha$ be an arbitrary algebra from (b). Then for any two integers $a, b \in Z_\alpha$ there are only finitely many elements in Z_α that are between a and b with respect to the natural order induced by the meet operation. However, if $\mathbf{S}$ is a subalgebra of $\mathbf{R}_\beta$, that is, if $\mathbf{S}$ is an algebra from (c), then for any two different real numbers $a, b \in R_\beta$ there are infinitely many elements in R_β that are between a and b, because β is dense. This implies that the algebras in (b) are not isomorphic to any algebra in (c).

Now we will show that if α_1 and α_2 are distinct surjective homomorphisms of $\mathbf{F}$ onto $\langle \mathbb{Z}; + \rangle$, then $\mathbf{Z}_{\alpha_1} \ncong \mathbf{Z}_{\alpha_2}$. Since $\alpha_1 \neq \alpha_2$, we can choose an element $f \in F$ such that $\alpha_1(f) \neq \alpha_2(f)$. If either $\alpha_1(f)$ or $\alpha_2(f)$ is zero, then we can assume that $\alpha_1(f) = 0 \neq \alpha_2(f)$, and we will examine the idenitity $f(x) = x$. If $\alpha_1(f)$ and $\alpha_2(f)$ have opposite signs, then we can assume that $\alpha_1(f) < 0 < \alpha_2(f)$, and we examine the identity $x \wedge f(x) = x$. It is easy to check that in these cases the identities hold in $\mathbf{Z}_{\alpha_1}$ but they fail in $\mathbf{Z}_{\alpha_2}$, thus $\mathbf{Z}_{\alpha_1} \ncong \mathbf{Z}_{\alpha_2}$. For example the identity $x \wedge f(x) = x$ holds in $\mathbf{Z}_{\alpha_1}$, because $\alpha_1(f) < 0$ and $f(x) = x - \alpha_1(f) > x$.

Now let us examine the case when $\alpha_1(f)$ and $\alpha_2(f)$ have the same sign. We can assume that $|\alpha_1(f)| > |\alpha_2(f)| > 0$. Since α_1 is surjective, we can choose an element $e \in F$ such that $\alpha_1(e) = 1$. We know that $\alpha_1(f)$ is an integer, so let $g = f \cdot e^{-\alpha_1(f)} \in F$. Hence we have

$$\begin{aligned} \alpha_1(g) &= \alpha_1(f \cdot e^{-\alpha_1(f)}) \\ &= \alpha_1(f) + \alpha_1(e^{-\alpha_1(f)}) \\ &= \alpha_1(f) + (-\alpha_1(f))\alpha_1(e) \\ &= \alpha_1(f) - \alpha_1(f)\alpha_1(e) \\ &= \alpha_1(f) - \alpha_1(f) \\ &= 0, \end{aligned}$$

and similarly

$$\alpha_2(g) = \alpha_2(f) - \alpha_1(f)\alpha_2(e).$$

It is easy to see that $\alpha_2(g) \neq 0$, since $|\alpha_1(f)| > |\alpha_2(f)| > 0$ and $\alpha_2(e)$ is an integer. So we have found again an element $g \in F$ such that $\alpha_1(g) = 0 \neq \alpha_2(g)$. Hence the identity $g(x) = x$ holds in $\mathbf{Z}_{\alpha_1}$ but fails in $\mathbf{Z}_{\alpha_2}$. Thus we conclude that the algebras listed in (b) are pairwise nonisomorphic.

In order to complete the proof, we have to show that the only possibility for two algebras from (c) to be isomorphic is the case indicated in the statement of the theorem.

Let β_1, β_2 be dense homomorphisms from $\mathbf{F}$ to $\langle \mathbb{R}; + \rangle$, and $\mathbf{S}_1$, $\mathbf{S}_2$ be subalgebras of $\mathbf{R}_{\beta_1}$, $\mathbf{R}_{\beta_1}$, respectively. If $t > 0$ and d are real numbers such that $\beta_2 = t\beta_1$ and $S_2 = tS_1 + d$, then let us define an isomorphism $\tau \colon \mathbf{R}_{\beta_1} \to \mathbf{R}_{\beta_2}$ by

$$\tau(x) = tx + d.$$

This mapping is indeed an isomorphism, since it is bijective, it preserves the natural order, and for any unary operation $f \in F$ and real number x we have

$$\begin{aligned} f(\tau(x)) &= \tau(x) - \beta_2(f) \\ &= tx + d - t\beta_1(f) \\ &= t(x - \beta_1(f)) + d \\ &= t \cdot f(x) + d \\ &= \tau(f(x)). \end{aligned}$$

Now it is easy to see that the restriction of τ to $\mathbf{S}_1$ is an isomorphism between $\mathbf{S}_1$ and $\mathbf{S}_2$.

Conversely, let $\iota \colon \mathbf{S}_1 \to \mathbf{S}_2$ be an isomorphism. Since β_1 is dense, we can choose an element $f \in F$ such that $\beta_1(f) > 0$. If $\beta_2(f) \leq 0$, then in $\mathbf{S}_2$ we have $f(x) = x - \beta_2(f) \geq x$, thus the identity $x \wedge f(x) = x$ holds in $\mathbf{S}_2$. However, this identity does not hold in $\mathbf{S}_1$, since in $\mathbf{S}_1$ we have $f(x) = x - \beta_1(x) < x$. This contradicts our assumption $\mathbf{S}_1 \cong \mathbf{S}_2$, hence we conclude that $\beta_2(f) > 0$. Put

$$t = \frac{\beta_2(f)}{\beta_1(f)},$$

thus $t > 0$. If $\beta_2 \neq t\beta_1$ then we have an element $g \in F$ such that $\beta_2(g) \neq t\beta_2(g)$. Suppose $\beta_2(g) < t\beta_1(g)$. Since $\beta_2(f) > 0$, we can choose integers $p, q \in \mathbb{Z}$ such that $q > 0$ and

$$\beta_2(g) < \frac{p}{q}\beta_2(f) < t\beta_1(g).$$

Multiplying by $q > 0$ and using $\beta_2(f) = t\beta_1(f)$, we get

$$q\beta_2(g) < p\beta_2(f) = tp\beta_1(f) < tq\beta_1(g).$$

Since β_1, β_2 are homomorphisms and $t > 0$, we get

$$\beta_2(g^q) < \beta_2(f^p) \quad \text{and} \quad \beta_1(f^p) < \beta_1(g^q).$$

This implies that the identity $f^p(x) \wedge g^q(x) = f^p(x)$ holds in $\mathbf{S}_2$, but fails in $\mathbf{S}_1$. It is not hard to see that this method works also for the other case when $\beta_2(g) > t\beta_1(g)$. Thus we get a contradiction, which shows that $\beta_2 = t\beta_1$.

Now let us choose an arbitrary real number $s_1 \in S_1$, and put $s_2 = \iota(s_1) \in S_2$ and $d = s_2 - ts_1$. Furthermore, let

$$Q_i = \{\, f(s_i) \mid f \in F \,\} = \{\, s_i - \beta_i(f) \mid f \in F \,\} \quad \text{for } i = 1, 2.$$

Clearly $Q_i \subseteq S_i$ for $i = 1, 2$, and since β_i is a dense homomorphism, Q_i is a dense subset of $\mathbb{R}$, and hence of S_i, too. Now we will show that the isomorphisms $\tau: \mathbf{R}_{\beta_1} \to \mathbf{R}_{\beta_2}$ and ι coincide on the set $Q_1 \subseteq \mathbb{R}$. For each element $f(s_1) \in Q_1$ we have $\iota(f(s_1)) = f(\iota(s_1)) = f(s_2)$ and $\tau(f(s_1)) = f(\tau(s_1)) = f(ts_1 + d) = f(s_2)$. Thus τ yields a bijection between Q_1 and Q_2.

Since both τ and ι preserve the natural order of the real numbers, and they coincide on Q_1, and the sets Q_1, Q_2 are dense in S_1, S_2, respectively, the isomorphisms τ and ι coincide on the whole set S_1. Thus $\tau(S_1) = S_2$, that is, $tS_1 + d = S_2$. □

Up to this point we have seen two types of simple $\mathbf{F}$-semilattices. The first type consists of the algebras $\mathbf{S}_M$, while the other type contains the algebras $\mathbf{Z}_\alpha$ and $\mathbf{R}_\beta$. The simple $\mathbf{F}$-semilattices of the first type have a least element and some atoms, while the algebras of the second type are linear. Now we will present an example of a simple $\mathbf{F}$-semilattice which has a least element but no atoms and its semilattice order is not linear. From Corollary 2.4 we know that if we want to succeed, then the group $\mathbf{F}$ cannot be locally finite. By Lemma 2.7 we also know that $\mathbf{F}$ cannot be commutative. Therefore it is natural to let $\mathbf{F}$ be an appropriate infinite subgroup of the symmetric group $\operatorname{Sym} \mathbb{Z}$ on the set of integers. Since $\mathbf{F}$ cannot be locally finite, $\mathbf{F}$ must contain a permutation which moves infinitely many integers. Following Lemma 1.6 we will construct this simple $\mathbf{F}$-semilattice as a subalgebra $\mathbf{U}$ of $\mathbf{P}(F)$. Furthermore, we have to provide a submonoid M of the group $\mathbf{F}$ with the properties described in 1.6. From the proof of Corollary 1.7 we also see that M cannot be a subgroup of $\mathbf{F}$, because then $\mathbf{U}$ would contain $\emptyset$ and M, thus the element M would be an atom in $\mathbf{U}$. Now let us see the construction of this simple subalgebra $\mathbf{U}$ of $\mathbf{P}(F)$.

We will use the cycle notation in the usual way. For example, the element $(012) \in \operatorname{Sym} \mathbb{Z}$ maps $0 \mapsto 1 \mapsto 2 \mapsto 0$, and leaves every other integer fixed. We write permutations to the right hand side of the element on which it acts. Thus $1(012) = 2$, hence $(12)(23) = (132)$. We will need a special element $(+1) \in \operatorname{Sym} \mathbb{Z}$ which maps every integer k to $k + 1$, thus $k(+1) = k + 1$. We will denote the ith power of $(+1)$ by

$$(+i) = (+1)^i.$$

Example 2.15. Let $\mathbf{F}$ be the subgroup of $\operatorname{Sym} \mathbb{Z}$ generated by the elements $\{(012), (+3)\}$, and let M be the submonoid of $\mathbf{F}$ generated by the elements $\{(+3)(012), (+3)\}$. Furthermore, let $\mathbf{U}$ be the subalgebra of the $\mathbf{F}$-semilattice $\mathbf{P}(F)$ generated by M. Then $\mathbf{U}$ is simple but not linear, and has a least element but no atoms.

Proof. First of all, it is clear that $(012)(+3) = (+3)(345)$ and $(012)(345) = (345)(012)$. Therefore, every element of $\mathbf{F}$ can be written in a canonical form

$$(+3a) \ldots (-3\ -2\ -1)^{e(-1)}(012)^{e(0)} \ldots (3i\ 3i+1\ 3i+2)^{e(i)} \ldots \tag{8}$$

where $a \in \mathbb{Z}$, $e: \mathbb{Z} \to \mathbb{Z}_3$ is a mapping to the residue classes modulo 3, and $e(i) = 0$ for almost all integers i. It is not hard to see that the elements of M in canonical form are exactly

$$(+3a)(012)^{e(0)}(345)^{e(1)} \ldots (3a-3\ 3a-2\ 3a-1)^{e(a-1)}$$

where $a \geq 0$; $\ldots, e(-2), e(-1), e(a), e(a+1), \ldots = 0$; and $e(0), e(1), \ldots, e(a-1) \in \{0,1\}$. We will use the abbreviation (a, e) for the canonical form (8).

Now we introduce operations on mappings from $\mathbb{Z}$ to $\mathbb{Z}_3$. Namely, for every mapping $e\colon \mathbb{Z} \to \mathbb{Z}_3$ and integers k, l we define the mappings $e^{(-k)}, e|_{[k,l)}$ and $e|^{[k,l)}$ of $\mathbb{Z}$ into $\mathbb{Z}_3$ as

$$e^{(-k)}(i) = e(i-k);$$

$$e|_{[k,l)}(i) = \begin{cases} e(i), & \text{if } k \leq i < l, \\ 0, & \text{otherwise; and} \end{cases}$$

$$e|^{[k,l)}(i) = \begin{cases} 0, & \text{if } k \leq i < l, \\ e(i), & \text{otherwise.} \end{cases}$$

Furthermore, for any two mappings e, f we denote their coordinate-wise sum by $e+f$, and we say that $e \leq f$ if this holds in all coordinates. Using these notations we could represent the elements of the monoid M as

$$M = \left\{ (a, e) \in F \;\middle|\; 0 \leq a,\ e|_{[0,a)} \leq 1 \text{ and } e|^{[0,a)} = 0 \right\},$$

and we have the following formula for multiplication in $\mathbf{F}$:

$$(a, e) \cdot (b, f) = (a+b, e^{(-b)} + f).$$

Since $\mathbf{U}$ is generated by M, we see from Definition 1.2 that U contains the right cosets of M. Now we give the elements of the right cosets of M explicitly.

Claim 1. *For every two elements $(a, e), (b, f) \in M$ we have $(b, f) \in M \cdot (a, e)$ iff $a \leq b$, $(f-e)|^{[a,b)} = 0$ and $(f-e)|_{[a,b)} \leq 1$.*

Suppose that $(b, f) \in M \cdot (a, e)$, hence we have an element $(c, g) \in M$ such that $(b, f) = (c, g) \cdot (a, e)$. Thus $b = c + a$, and $f = g^{(-a)} + e$. Since $(c, g) \in M$, we have $c \geq 0$, therefore $b \leq a$. Furthermore, the following equalities yield the missing properties:

$$\begin{aligned} (f-e)|^{[a,b)} &= g^{(-a)}|^{[a,b)} \\ &= (g|^{[a-a,b-a)})^{(-a)} \\ &= (g|^{[0,c)})^{(-a)} \\ &= 0^{(-a)} \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} (f-e)|_{[a,b)} &= g^{(-a)}|_{[a,b)} \\ &= (g|_{[0,c)})^{(-a)} \\ &\leq 1. \end{aligned}$$

Conversely, it is not hard to see that if we put $c = b - a$ and $g = (f-e)^{(+a)}$, then we get $(c, g) \in M$ and $(b, f) = (c, g) \cdot (a, f)$.

Claim 2. *For every two elements $(a,e),(b,f)\in F$ such that $a\le b$ the condition $M\cdot(a,e)\cap M\cdot(b,f)\neq\emptyset$ holds iff $(f-e)|^{[a,\infty)}=0$ and $(f-e)|_{[a,b)}\le 1$.*

Suppose $(c,g)\in M\cdot(a,e)\cap M\cdot(b,f)$. Since $(1,0)\in M$, for every integer $d\ge c$ the element $(d,g)=(d-c,0)\cdot(c,g)$ belongs also to $M\cdot(a,e)\cap M\cdot(b,f)$. By Claim 1 we get $(g-e)|^{[a,d)}=(g-f)|^{[b,d)}=0$ for every integer $d\ge c$, which implies that $(g-e)|^{[a,\infty)}=(g-f)|^{[b,\infty)}=0$. Since $a\le b$, we conclude that $(f-e)|^{[a,\infty)}=0$. Now let us compute the mapping $(f-e)|_{[a,b)}$. By Claim 1 we get $a,b\le c$, $(g-e)|_{[a,c)}\le 1$ and $(g-f)|^{[b,c)}=0$. Since we have assumed $a\le b$, the condition $(g-f)|^{[b,c)}=0$ implies that $(g-f)|_{[a,b)}=0$. Therefore, $(f-e)|_{[a,b)}=(-(g-f)+(g-e))|_{[a,b)}\le -0+1=1$.

Conversely, since e and f assign 0 to almost all integers, we can choose an integer $c\ge a,b$ such that $e|^{[a,c)}=f|^{[a,c)}$. It is not hard to see that if we put $g=f|^{[b,\infty)}+(3-e-f)|_{[b,\infty)}$, then we get $(c,g)\in M\cdot(a,e)\cap M\cdot(b,f)$, thus this set is nonempty.

In the rest of the proof we will show that if ϑ is a congruence of $\mathbf{U}$ containing a pair $\langle A,B\rangle$ of two distinct elements of U, then ϑ contains the pair $\langle\emptyset,M\rangle$ as well. Clearly, this ensures that $\mathbf{U}$ is simple. We can assume that A is a proper subset of B.

Claim 3. *If A is a proper subset of B and the congruence ϑ of $\mathbf{U}$ contains the pair $\langle A,B\rangle$, then $\langle\emptyset,C\rangle\in\vartheta$ for an appropriate element $C\neq\emptyset$ of U.*

Since $\mathbf{U}$ is generated by M, every element of U equals to the intersection of some right cosets of M. Since $A\subset B$, we can choose two integers $n<m$ and right cosets $M\cdot(a_i,e_i)$ $(i=1,\dots,m)$, such that

$$\begin{aligned}&A=M\cdot(a_1,e_1)\cap\ldots\cap M\cdot(a_n,e_n)\cap\ldots\cap M\cdot(a_m,e_m),\ \text{and}\\&B=M\cdot(a_1,e_1)\cap\ldots\cap M\cdot(a_n,e_n).\end{aligned}$$

Furthermore, since $A\subset B$, we can choose an element $(b,f)\in B\subseteq A$. Thus $(b,f)\in M\cdot(a_1,e_1),\dots,M\cdot(a_n,e_n)$, and (b,f) does not belong to all of the right cosets $M\cdot(a_i,e_i)$. We can assume that $(b,f)\notin M\cdot(a_{n+1},e_{n+1})$. If $a_{n+1}\le b$, then by Claim 1 either $(f-e)|^{[a_{n+1},b)}=0$ or $(f-e)|_{[a_{n+1},b)}\le 1$ fails. Therefore from Claim 2 we conclude that $M\cdot(b,f)\cap M\cdot(a_{n+1},e_{n+1})=\emptyset$, hence $M\cdot(b,f)\cap A=\emptyset$. However, $(b,f)\in M\cdot(b,f)\cap B$, hence $M\cdot(b,f)\cap B\neq\emptyset$. This shows that the pair $\langle\emptyset,M\cdot(b,f)\rangle$ belongs to ϑ.

Now let us consider the other case, when $a_{n+1}>b$. Put $g=1|_{[b,b+1)}$. By Claim 2 it is easy to check that $M\cdot(a_{n+1},e_{n+1})\cap M\cdot(a_{n+1},e_{n+1}+g)=\emptyset$. Thus $A\cap A\cdot(0,g)\subseteq M\cdot(a_{n+1},e_{n+1})\cap M\cdot(a_{n+1},e_{n+1})\cdot(0,g)=M\cdot(a_{n+1},e_{n+1})\cap M\cdot(a_{n+1},e_{n+1}+g)=\emptyset$. However, by Claim 1 the element $(b+1,f+g)$ belongs to the sets $M\cdot(a_i,e_i)$ and $M\cdot(a_i,e_i+g)$ for $i=1,\dots,n$. Thus

$$M\cdot(a_1,e_1+g)\cap\ldots\cap M\cdot(a_n,e_n+g)=(M\cdot(a_1,e_1)\cap\ldots\cap M\cdot(a_n,e_n))\cdot(0,g)=B\cdot(0,g).$$

Therefore the element $(b+1,f+g)$ belongs to $B\cap B\cdot(0,g)$, hence $B\cap B\cdot(0,g)\neq\emptyset$. Clearly, this completes the proof of this claim, since the pair $\langle\emptyset,B\cap B\cdot(0,g)\rangle$ belongs to ϑ.

Claim 4. *If $C\in U$ is a nonempty set and the congruence ϑ of $\mathbf{U}$ contains the pair $\langle\emptyset,C\rangle$, then the pair $\langle\emptyset,M\rangle$ is also in ϑ.*

Let (a,e) be an arbitrary element in C. Since $\mathbf{U}$ is generated by M, every element of U is closed under multiplication on the left hand side by the elements of M. Thus $M\cdot(a,e) \subseteq C$, hence $M \cap C\cdot(a,e)^{-1} = M$. So we conclude that the pair $\langle M \cap \emptyset\cdot(a,e)^{-1}, M \cap C\cdot(a,e)^{-1}\rangle = \langle \emptyset, M\rangle$ is in ϑ.

This proves that $\mathbf{U}$ is simple. Clearly, the inclusion order of $\mathbf{U}$ is not linear. To see that $\mathbf{U}$ has no atoms, let $A \in U$ be an arbitrary nonempty set. Now one can check that $\emptyset \subset A\cdot(1,0) \subset A$. □

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