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Some Applications of Bifurcation Formulae to the Period Map of Delay Differential Equations

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Abstract

Our purpose is to present some applications of the bifurcation formulae derived in [13] for periodic delay differential equations. We prove that a sequence of Neimark-Sacker bifurcations occurs as the parameter increases. For some special classes of equations, easily checkable conditions are given to determine the direction of the bifurcation of the time-one map.

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1 Introduction

Consider the nonautonomous scalar delay differential equation

$$\varepsilon\dot{x}(t) = a(t)x(t) + f(t, x(t-1)), \tag{1}$$

where $a : \mathbb{R} \to \mathbb{R}$ is a continuous and $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a C^4-smooth function satisfying

$$a(t+1) = a(t),$$

$$f(t+1, \xi) = f(t, \xi)$$

and

$$f(t, 0) = 0$$

for all $t, \xi \in \mathbb{R}$, $\varepsilon \neq 0$ is a real parameter. Such equations arise naturally from the growth dynamics of a single species population with two stages (immature and mature), where 1 is the normalized maturation time, from the synchronized activities of a network of identical neurons that can be described by a scalar delay differential equation with delayed feedback, and from other delayed feedback models, where the environment fluctuates periodically. See the monographs [7, 8, 18] and [19] for related references.

The Banach space C of continuous real functions on the interval $[-1, 0]$ equipped with the norm

$$\|\phi\| = \sup_{-1\le t\le 0} |\phi(t)|$$

serves as state space. Every $\phi \in C$ determines a unique continuous function $x^\phi : [-1, \infty) \to \mathbb{R}$, which is differentiable on $(0, \infty)$, satisfies (1) for all $t > 0$ and $x^\phi(t) = \phi(t)$ for all $t \in [-1, 0]$. We call such a function x^ϕ the solution of (1) with the initial value ϕ. The time-one map $T : C \to C$ is defined by the relations

$$T(\phi) = x_1^\phi, \quad x_t(s) = x(t + s), \quad s \in [-1, 0].$$

For simplicity, we deal often with $v := \frac{1}{\varepsilon}$. The time-one map T depends also on v, when we want to emphasize this fact, we write T_v. Varying the parameter, Floquet multipliers cross the complex unit circle, that is a Neimark-Sacker bifurcation (it is also known as Hopf-bifurcation for maps). A bifurcation of the fixed point 0 is considered with respect to the map T. The main point is that in the critical case, using spectral projection and Riesz-Dunford calculus, it is possible to reduce the system to the center manifold, introducing new variables on the decomposition of C into center and stable-unstable subspaces. When the delay equals the period, then the restricted system can be computed explicitly and we can perform the usual bifurcation analysis. The details can be found in [13], we briefly summarize those results in Section 2. Section 3 is devoted to a Mackey-Glass equation with periodic coefficients. A modification of the Nicholson blowflies model is considered in Section 4. Section 5 concerns the Krisztin-Walther equation and a model of a periodically forced single, self-excitatory neuron with graded delayed response.

2 Preliminaries

Denote by $C_{\mathbb{C}}$ the Banach space of continuous complex valued functions on the interval $[-1, 0]$ with the sup-norm. The behavior of solutions close to the equilibrium 0 is determined by the spectrum $\sigma(U)$ of the monodromy operator U. That is the derivative of the time-one map T at 0. The monodromy operator is a linear continuous map and with the relation $U(\psi) = U(\mathrm{Re}\ \psi) + iU(\mathrm{Im}\ \psi)$ considered as an operator $C_{\mathbb{C}} \to C_{\mathbb{C}}$ and given by $U(\psi) = y_1^\psi$, where $y^\psi : [-1, 0] \to C_{\mathbb{C}}$ is the solution of the linear variational equation

$$\varepsilon\dot{y}(t) = a(t)y(t) + f_\xi(t, 0)y(t - 1) \tag{2}$$

with $y_0^\psi = \psi \in C_{\mathbb{C}}$. The operator U is compact, therefore all the non-zero points of the spectrum are isolated points and eigenvalues of finite multiplicity with finite dimensional range of the associated eigenprojection $P_\mu : C_{\mathbb{C}} \to C_{\mathbb{C}}$, where $\mu \in \sigma(U)$, $\mu \neq 0$. The spectral theory of delay differential equations is extensively studied in [1] and [16]. A non-zero point μ of the spectrum of the monodromy operator U is called a Floquet multiplier of equation (2) and any λ for which $\mu = e^\lambda$ is called a Floquet exponent of equation (2). Throughout the paper we use the notation $G(t) = \int_{-1}^t g(s)\,ds$ and $G = G(0)$ for any function g. Let $b(t) = f_\xi(t,0)$. By the Floquet theory ([5]), one can conclude that $\mu = e^\lambda$ is a Floquet multiplier if and only if there is a nonzero solution $y(t) = p(t)e^{\lambda t}$ of equation (2), where $p(t)$ is a periodic function. The eigenfunction corresponding to μ is

$$\chi_\mu(t) : [-1,0] \ni t \mapsto e^{\int_{-1}^t [va(s)+vf_\xi(s,0)e^{-\lambda}]\,ds} = e^{vA(t)+vB(t)e^{-\lambda}} \in \mathbb{C}.$$

The Floquet multipliers coincide with the roots of the characteristic function

$$h(\lambda) = vA + vBe^{-\lambda} - \lambda. \tag{3}$$

Characteristic functions of the same type occur linearizing around a periodic orbit of an autonomous equation. With this object these characteristic functions were widely discussed. A detailed analysis can be found, *e.g.*, in [1, chapter XI.] . We recall some basic facts. If $v \neq 0$, then the characteristic function has a root on the unit circle if and only if $|A|/|B| < 1$ and

$$v_{\pm n} = -\frac{\pm \arccos(-\frac{A}{B}) + 2n\pi}{\pm B \sin\big(\arccos(-\frac{A}{B})\big)} = -\frac{\pm \arccos(-\frac{A}{B}) + 2n\pi}{\pm B\sqrt{1-\frac{A^2}{B^2}}}\,, \quad n \in \mathbb{N}. \tag{4}$$

The number of Floquet multipliers situated outside the unit circle remains the same as we vary v in the interval (v_n, v_{n+1}) or (v_{-n-2}, v_{-n-1}), that is, $2n+2$ and $2n+1$, respectively. In other words, a bifurcation can occur only at the critical values $v_{\pm n}$. We remark that the sign of $v_{\pm n}$ is not necessarily the same as the sign of its index. If $B < 0$ and $-|A| < B < |A|$, then all the eigenvalues of U have moduli less than one, and we have asymptotic stability of the equilibrium 0. The smoothness of T is guaranteed by the smoothness of $a(t)$ and $f(t,\xi)$. The transversality condition $|\frac{\partial|\mu(v)|}{\partial v}|_{v_k} \neq 0$ is always fulfilled for such equations, the nonresonance conditions $\mu^4 \neq 1$ and $\mu^3 \neq 1$ are also satisfied if $A \neq 0$ and $B \neq 2A$. If $|\mu| = 1$, then $\mu = e^\lambda = e^{iv\sqrt{B^2-A^2}} = -\frac{A}{B} - i\sqrt{1-\frac{A^2}{B^2}}$ and $\bar\mu = e^{\bar\lambda} = e^{-iv\sqrt{B^2-A^2}} = -\frac{A}{B} + i\sqrt{1-\frac{A^2}{B^2}}$. It is clear that the conjugate of a Floquet multiplier is also a Floquet multiplier, moreover, the algebraic multiplicity of μ as an eigenvalue equals the multiplicity of λ as a root of the characteristic function. In our case this number is always 1 for a critical Floquet multiplier. Applying the Riesz-Schauder Theorem to a simple eigenvalue of the compact operator U, one has that there are two closed subspaces E_μ and Q_μ of $C_{\mathbb{C}}$ such that E_μ is one-dimensional; $E_\mu \oplus Q_\mu = C_{\mathbb{C}}$; $U(E_\mu) \subset E_\mu$ and $U(Q_\mu) \subset Q_\mu$; $\sigma(U|E_\mu) = \{\mu\}$, $\sigma(U|Q_\mu) = \sigma(U)\backslash\{\mu\}$; and the spectral projection P_μ onto E_μ along Q_μ can be represented by a Riesz-Dunford integral

$$P_\mu = \frac{1}{2\pi i}\int_{\Gamma_\mu} (zI - U)^{-1}\, dz = \operatorname*{Res}_{z=\mu}\, (zI - U)^{-1},$$

where Γ_μ is a small circle around μ such that μ is the only singularity of the resolvent $(zI-U)^{-1}$ inside Γ_μ. By the variation-of-constants formula for ordinary differential equations we find the following representation of the time-one map

$$T(\phi)(t) = e^{vA(t)}\left(\phi(0) + \int_{-1}^{t} e^{-vA(s)} v f(s,\phi(s))\, ds\right), \quad t \in [-1,0], \tag{5}$$

which implies for the monodromy operator

$$U(\phi)(t) = e^{vA(t)}\left(\phi(0) + \int_{-1}^{t} e^{-vA(s)} v b(s)\phi(s)\, ds\right), \quad t \in [-1,0], \tag{6}$$

and for the higher order derivatives $V := D^2T(0)$, $W := D^3T(0)$:

$$V(\phi_1,\phi_2)(t) = e^{vA(t)}\int_{-1}^{t} e^{-vA(s)} v f_{\xi\xi}(s,0)\phi_1(s)\phi_2(s)\, ds, \tag{7}$$

$$W(\phi_1,\phi_2,\phi_3)(t) = e^{vA(t)}\int_{-1}^{t} e^{-vA(s)} v f_{\xi\xi\xi}(s,0)\phi_1(s)\phi_2(s)\phi_3(s)\, ds. \tag{8}$$

The resolvent of the monodromy operator can be expressed as

$$\begin{aligned}(zI - U)^{-1}(\psi)(t) = e^{[A(t)+\frac{B(t)}{z}]}\Bigg(&\Bigg(\frac{1}{z}\psi(0) + e^{[A+\frac{B}{z}]} \\ &\times \int_{-1}^{0} \frac{1}{z^2} e^{-[A(s)+\frac{B(s)}{z}]} b(s)\psi(s)\, ds\Bigg)\left(z - e^{[A+\frac{B}{z}]}\right)^{-1} \\ &+ \frac{1}{z}e^{-[A(t)+\frac{B(t)}{z}]}\psi(t) + \int_{-1}^{t} \frac{1}{z^2} e^{-[A(s)+\frac{B(s)}{z}]} b(s)\psi(s)\, ds\Bigg).\end{aligned} \tag{9}$$

This formula is very important in the bifurcation analysis *per se* and allows us to find the following representation of the spectral projection:

$$P_\mu(\psi) = \chi_\mu \mathcal{R}_\mu(\psi),$$

where

$$\mathcal{R}_\mu(\psi) = \left(\frac{1}{\mu + vB}\right)\left(\psi(0) + \int_{-1}^{0} \frac{b(s)\psi(s)}{\chi_\mu(s)}\, ds\right). \tag{10}$$

From this point on in the paper μ always means a critical Floquet multiplier corresponding to some critical value $v = v_n$, *i.e.*, $|\mu| = 1$. Sometimes we omit the index for simplicity. The main theorem of [13], proposed just after the next formula, illuminates the significance of the coefficient $\delta(v_n)$ defined by

$$\begin{aligned}\delta(v_n) = \frac{1}{2}\mathrm{Re}\,\Bigg(\frac{1}{\mu}\mathcal{R}_\mu\Big(&W(\chi_\mu, \chi_\mu, \bar{\chi}_\mu) + 2V\big(\chi_\mu, (1-U)^{-1}V(\chi_\mu, \bar{\chi}_\mu)\big)\\ &+ V\big(\bar{\chi}_\mu, (\mu^2 - U)^{-1}V(\chi_\mu, \chi_\mu)\big)\Big)\Bigg).\end{aligned} \tag{11}$$

Theorem A. *Suppose that the one-parameter family of time-one maps $T_v : C \to C$ corresponding to the equation* (1) *has at the critical value $v = v_j$ the fixed point $\phi = 0$ with exactly two simple Floquet multipliers $e^{i\theta}, e^{-i\theta}$ on the unit circle. Then there is a neighborhood of* 0 *in which a unique closed invariant curve bifurcates from* 0 *as v passes through v_j, providing that the nonresonance conditions*

$$\mu_j^4 \neq 1, \quad \mu_j^3 \neq 1$$

hold. The direction of the appearance of the invariant curve is determined by the sign of the coefficient $\delta(v_j)$: the cases $\delta(v_j) < 0$ and $\delta(v_j) > 0$ are called supercritical and subcritical Neimark-Sacker bifurcations. In the supercritical case a stable (only in a restricted sense, inside the invariant center manifold) invariant curve appears for $v > v_j$, while in the subcritical case an unstable invariant curve disappears when v increasingly crosses v_j.

The evolutionary system associated with the translation along the solutions of the equation (1) is given by the relation

$$E(t,s)\varphi = x_t^{\varphi,s},$$

where $t > s$, $E(t,s) : C \to C$ and $x^{\varphi,s}$ is the solution of (1) satisfying $x_s^{\varphi,s} = \varphi$. Let $T^\tau = E(\tau+1, \tau)$, then $T^0 = T$. The periodicity of (1) yields that $T^\tau = T^{\tau+1}$ and the system can be considered in the space $C \times S^1$ as an autonomous system with the solution maps

$$S(t) :\ C \times S^1 \ni (\varphi, s) \mapsto (x_t^{\varphi,s}, t + s \mod 1) \in C \times S^1.$$

The characteristic equation and the Floquet multipliers of the monodromy operators $U^\tau = DT^\tau(0)$ are independent of τ, hence an invariant curve occurs at the same parameter values for all T^τ. Denote these curves by Γ^τ. While $T(t,s)(\Gamma^s)$ is an invariant curve with respect to F^t, the uniqueness property shows that $T(t,s)(\Gamma^s) = \Gamma^t$ and the set

$$\mathbb{T} = \bigcup_{\tau\in[0,1)} (\Gamma^\tau, \tau)$$

forms an invariant torus in the space $C \times S^1$ under the dynamics generated by the associated maps $S(t)$.

3 Mackey-Glass equation

The famous Mackey-Glass equation

$$\dot{x}(t) = -mx(t) + \frac{qx(t-\tau)}{1 + x(t-\tau)^n} \tag{12}$$

was proposed as a model for the production of white blood cells and later to describe certain periodic deseases. It was studied for different values of the parameters m, q, τ, n, and became well known for its chaotic behavior (see [2] for history, [11] for more general recent results).

We consider a more realistic, periodic version of a Mackey-Glass equation

$$\varepsilon\dot{x}(t) = -m(t)x(t) + \frac{q(t)x(t-1)}{1 + x(t-1)^2}, \tag{13}$$

where $\varepsilon > 0$ is a real parameter, $m, q : \mathbb{R} \to \mathbb{R}$ are satisfying $m(t+1) = m(t) > 0$ and $q(t+1) = q(t) > 0$ for all $t \in \mathbb{R}$. Fix $v = v_n$. Using the notations of the previous sections we have

$$a(t) = -m(t), \quad f(t,\xi) = \frac{q(t)\xi}{1+\xi^2},$$

$$f_\xi(t,0) = q(t), \quad f_{\xi\xi}(t,0) = 0, \quad f_{\xi\xi\xi}(t,0) = -6q(t),$$

$$A = -M, \quad B = Q,$$

$$\chi_\mu(s) = e^{v(-M(s)+\bar{\mu}Q(s))},$$

$$\lambda = v(-M + Qe^{-\lambda}),$$

$$\mu = e^\lambda = e^{iv\sqrt{B^2-A^2}} = \frac{M}{Q} - i\sqrt{1 - \left(\frac{M}{Q}\right)^2},$$

where μ, λ, χ_μ corresponds to v_n. Bifurcation phenomena appear only if $\frac{M}{Q} < 1$.

Theorem 1 *Assume that* $0.9 < \frac{M}{Q} < 1$. *Then the dynamical system generated by the time-one map related to equation* (13) *undergoes a supercritical Neimark-Sacker bifurcation as the parameter* v *passes through* v_n *increasingly.*

Proof. Using (8), we conclude

$$\begin{aligned} W(\chi_\mu,\chi_\mu,\bar{\chi}_\mu)(t) &= e^{-vM(t)}\int_{-1}^{t} e^{vM(s)}(-6)vq(s)\chi_\mu(s)\chi_\mu(s)\bar{\chi}_\mu(s)\,ds \\ &= -\,e^{-vM(t)}\int_{-1}^{t} e^{v(-2M(s)+(2\bar{\mu}+\mu)Q(s))}(-6)vq(s)\,ds. \end{aligned} \tag{14}$$

Denote the above function by $W_0(t)$ for short, that can be written as

$$W_0(t) = -e^{-vM(t)}\int_{-1}^{t} g(s)e^{iv\sqrt{1-\left(\frac{M}{Q}\right)^2}Q(s)}\,ds, \tag{15}$$

where $g(s)$ is a positive real valued function, we have collected the positive terms into $g(s)$. Let $x \cong y$ if and only if $xy > 0$. $f_{\xi\xi}(t,0) = 0$ implies that $V \equiv 0$. Now by Theorem A, (10) and (11) we get

$$\begin{aligned} \delta(v) =& \frac{1}{2}\mathrm{Re}\left(\frac{1}{\mu}R_\mu\Big(W_0(t)\Big)\right) \\ \cong& \mathrm{Re}\left(\frac{1}{\mu}\Big(\frac{1}{\mu+vQ}\Big)\Big(W_0(0)+\int_{-1}^{0}\frac{vq(s)W_0(s)}{\chi_\mu(s)}\,ds\Big)\right) \\ \cong& -\mathrm{Re}\left(\bar{\mu}\Big(\bar{\mu}+vQ\Big)\Big(e^{-vM}\int_{-1}^{0} g(s)e^{iv\sqrt{1-\left(\frac{M}{Q}\right)^2}Q(s)}\,ds\right. \\ &\left. + \int_{-1}^{0} vq(s)\int_{-1}^{s} g(u)e^{iv\sqrt{1-\left(\frac{M}{Q}\right)^2}Q(u)}\,du\; e^{-iv\sqrt{1-\left(\frac{M}{Q}\right)^2}Q(s)}\,ds\Big)\right). \end{aligned}$$

Now we show that

$$\mathrm{Re}\;(\bar{\mu}^2+\bar{\mu}vQ)e^{-vM}g(s)e^{iv\sqrt{1-\left(\frac{M}{Q}\right)^2}Q(s)} \cong (\bar{\mu}^2+\bar{\mu}vQ)e^{iv\sqrt{1-\left(\frac{M}{Q}\right)^2}Q(s)} > 0 \tag{16}$$

and

$$\begin{aligned} &\mathrm{Re}\;(\bar{\mu}^2+\bar{\mu}vQ)vq(s)\int_{-1}^{s} g(u)e^{iv\sqrt{1-\left(\frac{M}{Q}\right)^2}Q(u)}\,du\; e^{-iv\sqrt{1-\left(\frac{M}{Q}\right)^2}Q(s)} \\ &\quad\cong \mathrm{Re}\;(\bar{\mu}^2+\bar{\mu}vQ)\int_{-1}^{s} g(u)e^{iv\sqrt{1-\left(\frac{M}{Q}\right)^2}\big(Q(u)-Q(s)\big)}\,du > 0 \end{aligned} \tag{17}$$

for all $s \in [-1, 0]$. For the latter inequality (17),

$$\text{Re } (\bar{\mu}^2 + \bar{\mu} v Q) e^{iv\sqrt{1-\left(\frac{M}{Q}\right)^2}\left(Q(u)-Q(s)\right)} du > 0$$

for any $0 \geq s \geq u \geq -1$ is sufficient. Since the range of $Q(s)$ and $Q(u) - Q(s)$ is in $[-Q, Q]$,

$$\text{Re } (\bar{\mu}^2 + \bar{\mu} v Q) e^{i\theta v\sqrt{1-\left(\frac{M}{Q}\right)^2} Q} du > 0$$

for all $\theta \in [-1, 1]$ implies (16) and (17). Introduce $w = \frac{M}{Q}$. Then $\bar{\mu} = w + i\sqrt{1 - w^2}$ and by (4), $vQ = \frac{\arccos w}{\sqrt{1-w^2}}$. With this notation

$$\begin{aligned} &\text{Re } \left((w + i\sqrt{1 - w^2})^2 + (w + i\sqrt{1 - w^2}) \frac{\arccos w}{\sqrt{1 - w^2}} \right) e^{i\theta \arccos w} = \\ &\text{Re } \left(2w^2 + \frac{w \arccos w}{\sqrt{1 - w^2}} - 1 + i\left(2w\sqrt{1 - w^2} + \arccos w\right) \right) e^{i\theta \arccos w} > 0 \end{aligned} \tag{18}$$

is to be proved for all $\theta \in [-1, 1]$. Since $2w\sqrt{1 - w^2} + \arccos w \geq 0$, $2w^2 + \frac{w \arccos w}{\sqrt{1-w^2}} - 1 \geq 0$ and $\frac{\pi}{2} \geq \arccos w \geq 0$ if $w \in [\frac{9}{10}, 1]$, if (18) is fulfilled for $\theta = 1$, then it is fulfilled for all $\theta \in [-1, 1]$. In the case of $\theta = 1$, we have $e^{i\theta \arccos w} = w + i\sqrt{1 - w^2}$. Now computing directly the real part in (18) gives

$$3w^3 - 3w + \arccos w \frac{2w^2 - 1}{\sqrt{1 - w^2}} > 3w^3 + 2w^2 - 3w - 1 > 0$$

for all $0.9 < w < 1$.

Here we used the well known inequality $\frac{t}{\sin t} > 1$ $(t > 0)$, the facts that $2w^2 - 1 > 0$ for $w > 0.9$, the polynomial $3w^3 + 2w^2 - 3w - 1 > 0$ at $w = 0.9$, and has the derivative $6w^2 + 4w - 3$ which is positive for $w > 0.9$. □

4 Nicholson's blowflies

In this section we consider the equation

$$\dot{N}(t) = qr(t)\left(- dN(t) + pN(t-1)e^{-aN(t-1)} \right), \tag{19}$$

where $q > 0$ is a real parameter, $d > 0$, $p > 0$, $a > 0$ are given constants, $r : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying $r(t+1) = r(t) > 0$ for all $t \in \mathbb{R}$. When $qr(t) \equiv 1$, (19) becomes the celebrated Nicholson's blowflies equation. Here $N(t)$ is the size of the population, δ is the death rate, p is the per capita daily egg production rate and $\frac{1}{a}$ is the size at which the population reproduces at the maximum rate. Nicholson's model was widely discussed, see the monographs [3] or [15]. There exist also very recent papers: a new oscillation theorem is presented in [4], several earlier results are performed for the case of periodic coefficients in [14], the Hopf-bifurcation is studied with respect to the delay as a parameter in [10].

$N_0 = 0$ is always an equilibrium of (19). The positive equilibrium $N_1 = \frac{\ln(\frac{p}{d})}{a}$ exists if and only if $a > 0$ and $p > d$. Setting $N(t) = N_1 + \frac{1}{a}x(t)$, $v = dq$, $k = \ln(\frac{p}{d}) > 0$, $x(t)$ satisfies

$$\dot{x}(t) = vr(t)\big(-x(t) - k(1 - e^{-x(t-1)}) + x(t-1)e^{-x(t-1)}\big). \tag{20}$$

Due to the biological interpretation, only the positive solutions are taken into account and we are interested in the bifurcation of the positive steady state, which is transformed into the null solution of (20). Using the notations of the previous sections we have

$$a(t) = -r(t), \quad f(t,\xi) = r(t)(-k(1-e^{-\xi}) + \xi e^{-\xi}),$$

$$f_\xi(t,0) = b(t) = r(t)(1-k), \quad f_{\xi\xi}(t,0) = r(t)(k-2),$$
$$f_{\xi\xi\xi}(t,0) = r(t)(3-k), \quad A = -R, \quad B = R(1-k).$$

Without loss of generality we may suppose that $R = \int_{-1}^{0} r(s)\,ds = 1$. If $k \le 2$, then all the Floquet multipliers are lying inside the unit circle and the steady solution 0 of (20) is asymptotically stable. We have resonance if $k = 3$, since then $B = 2A$. Assume $2 < k \neq 3$.

Lemma 1 *Suppose that $|\mu| = 1$. Then the eigenfunction corresponding to the center subspace satisfies $|\chi_\mu(t)| = 1$ and $\chi_\mu(t)\bar{\chi}_\mu(t) = 1$ for all $t \in [-1,0]$.*

Proof. The characteristic equation of (20) is

$$\lambda = -v + (1-k)ve^{-\lambda} \tag{21}$$

and the eigenfunctions have the form

$$\chi_\mu(s) = e^{vR(s)(-1+(1-k)e^{-\lambda})} = e^{\lambda R(s)}, \tag{22}$$

and $\lambda R(s) = i\theta R(s)$ is a purely imaginary number. □

Theorem 2 *The dynamical system generated by the time-one map related to equation* (20) *undergoes a supercritical Neimark-Sacker bifurcation as the parameter v passes through v_n increasingly for any $n \ge 1$.*

Proof. The simple identity $\int_{-1}^{t} e^{F(s)} f(s)\, ds = e^{F(t)} - 1$ is used several times in the sequel.

Using (7), (8) and (22), we conclude that

$$\begin{aligned} W(\chi_\mu,\chi_\mu,\bar{\chi}_\mu)(t) = & e^{-vR(t)} \int_{-1}^{t} e^{vR(s)} v(3-k) r(s) \chi_\mu(s)\chi_\mu(s)\bar{\chi}_\mu(s)\, ds \\ =& v(3-k)e^{-vR(t)} \int_{-1}^{t} e^{(v+\lambda)R(s)} r(s)\, ds \\ =& v(3-k)e^{-vR(t)} \left(e^{(v+\lambda)R(t)} - 1 \right) \frac{1}{v+\lambda} \\ =& \frac{(k-3)v}{v+\lambda} \left(e^{-vR(t)} - \chi_\mu(t) \right), \end{aligned} \tag{23}$$

$$\begin{aligned} V(\chi_\mu,\bar{\chi}_\mu)(t) = & e^{-vR(t)} \int_{-1}^{t} e^{vR(s)} v(k-2) r(s) \chi_\mu(s)\bar{\chi}_\mu(s)\, ds \\ =& (k-2)e^{-vR(t)} \int_{-1}^{t} e^{vR(s)} v r(s)\, ds \\ =& (2-k)(e^{-vR(t)} - 1), \end{aligned} \tag{24}$$

and

$$\begin{aligned} V(\chi_\mu,\chi_\mu)(t) = & e^{-vR(t)} \int_{-1}^{t} e^{vR(s)} v(k-2) r(s) \chi_\mu(s)\chi_\mu(s)\, ds \\ =& (k-2)ve^{-vR(t)} \int_{-1}^{t} e^{(2\lambda+v)R(s)} r(s)\, ds \\ =& \frac{(k-2)v}{2\lambda+v} e^{-vR(t)} (e^{(2\lambda+v)R(t)} - 1) \\ =& \frac{(2-k)v}{2\lambda+v} (e^{-vR(t)} - \chi_\mu(t)^2). \end{aligned} \tag{25}$$

Denote the above functions by $W_0(t)$, $V_0(t)$ and $V_1(t)$ for short, respectively. Now we compute the resolvents we need to get $\delta(v)$:

$$\begin{aligned} (I-U)^{-1}(V_0)(t) = e^{-kvR(t)} \Bigg(\Bigg(& V_0(0) \\ + e^{-kv} & \int_{-1}^{0} e^{kvR(s)} v(1-k) r(s) V_0(s)\, ds \Bigg) \\ \times \left(1 - e^{-kv}\right)^{-1} & + e^{kvR(t)} V_0(t) + \int_{-1}^{t} e^{kvR(s)} v(1-k) r(s) V_0(s)\, ds \Bigg) \end{aligned}$$

$$
\begin{aligned}
&= (2-k)e^{-kvR(t)}\Bigg(\bigg(e^{-v}-1 \\
&+ e^{-kv}\int_{-1}^{0} e^{kvR(s)}v(1-k)r(s)(e^{-vR(s)}-1)\,ds\bigg) \\
&\times \big(1-e^{-kv}\big)^{-1} + e^{kvR(t)}(e^{-vR(t)}-1)+ \\
&\int_{-1}^{t} e^{kvR(s)}v(1-k)r(s)(e^{-vR(s)}-1)\,ds\Bigg),
\end{aligned}
$$

using our simple identity this becomes

$$
\begin{aligned}
&(2-k)e^{-kvR(t)}\Bigg(\bigg(e^{-v}-1+ \\
&e^{-kv}\big(-(e^{(k-1)v}-1)+\frac{k-1}{k}(e^{kv}-1)\big)\bigg) \\
&\times \big(1-e^{-kv}\big)^{-1} + e^{kvR(t)}(e^{-vR(t)}-1)-(e^{(k-1)vR(t)}-1) \\
&+\frac{k-1}{k}(e^{kvR(t)}-1)\Bigg) \\
&= (2-k)e^{-kvR(t)}\Bigg(-\frac{1}{k}(1-e^{kv})(1-e^{kv})^{-1}-\frac{1}{k}(e^{kvR(t)}-1)\Bigg) \\
&= \frac{k-2}{k}.
\end{aligned}
\tag{26}
$$

Let $\omega = v(-1+(1-k)\bar{\mu}^2)$, with this notation we have

$$
\begin{aligned}
&(\mu^2 I - U)^{-1}(V_1)(t) = e^{\omega R(t)}\Bigg(\bigg(\bar{\mu}^2 V_1(0) + e^{\omega} \\
&\times \int_{-1}^{0} \bar{\mu}^4 e^{-\omega R(s)}v(1-k)r(s)V_1(s)\,ds\bigg)\big(\mu^2-e^{\omega}\big)^{-1} \\
&+\bar{\mu}^2 e^{-\omega R(t)}V_1(t) + \int_{-1}^{t} \bar{\mu}^4 e^{-\omega R(s)}v(1-k)r(s)V_1(s)\,ds\Bigg) \\
&= \frac{(2-k)v}{2\lambda+v}e^{\omega R(t)}\Bigg(\bigg(\bar{\mu}^2(e^{-v}-\mu^2)+e^{\omega} \\
&\times \int_{-1}^{0} \bar{\mu}^4 e^{-\omega R(s)}v(1-k)r(s)\big(e^{-vR(s)}-\chi_{\mu}(s)^2\big)\,ds\bigg)\big(\mu^2-e^{\omega}\big)^{-1}
\end{aligned}
$$

$$+ \bar{\mu}^2 e^{-\omega R(t)} \big(e^{-vR(t)} - \chi_\mu(t)^2 \big)$$
$$+ \int_{-1}^{t} \bar{\mu}^4 e^{-\omega R(s)} v(1-k) r(s) \big(e^{-vR(s)} - \chi_\mu(s)^2 \big)\, ds \Bigg),$$

and using our simple identity again to simplify integral terms, the preceding expression equals

$$\begin{aligned}
&\frac{(2-k)v}{2\lambda+v} e^{\omega R(t)} \Bigg(\bigg(\Big(\bar{\mu}^2 e^{-v} - 1 + e^{\omega} \bar{\mu}^4 v(1-k) \\
&\times \Big(\frac{e^{-\omega-v}-1}{-\omega-v} - \frac{e^{2\lambda-\omega}-1}{2\lambda-\omega} \Big) \bigg) \big(\mu^2 - e^{\omega} \big)^{-1} + \bar{\mu}^2 e^{-\omega R(t)} \big(e^{-vR(t)} - \chi_\mu(t)^2 \big) \\
&+ \bar{\mu}^4 v(1-k) \Big(\frac{e^{(-\omega-v)R(t)}-1}{-\omega-v} - \frac{e^{(2\lambda-\omega)R(t)}-1}{2\lambda-\omega} \Big) \Bigg) \\
&= \frac{(2-k)v}{2\lambda+v} \left(e^{\omega R(t)} \bar{\mu}^2 \left(\frac{\omega+v}{\omega-2\lambda} - 1 \right) + \bar{\mu}^2 \left(e^{2\lambda R(t)} - e^{\omega R(t)} \right) \right. \\
&\left. \times \left(\frac{\omega+v}{\omega-2\lambda} - 1 \right) \right) \\
&= \frac{(2-k)v}{\omega-2\lambda} \bar{\mu}^2 e^{2\lambda R(t)}.
\end{aligned} \tag{27}$$

Furthermore,

$$\begin{aligned}
V\big(\chi_\mu, (1-U)^{-1} V(\chi_\mu, \bar{\chi}_\mu)\big) &= e^{-vR(t)} \int_{-1}^{t} e^{vR(s)} v(k-2) r(s) e^{\lambda R(s)}\, ds\, \frac{k-2}{k} \\
&= -\frac{(k-2)^2 v}{k(v+\lambda)} \big(e^{-vR(t)} - e^{\lambda R(t)} \big),
\end{aligned} \tag{28}$$

and

$$\begin{aligned}
&V\big(\bar{\chi}_\mu, (\mu^2 - U)^{-1} V(\chi_\mu, \chi_\mu)\big) \\
&\quad = e^{-vR(t)} \int_{-1}^{t} e^{vR(s)} v(k-2) r(s) e^{-\lambda R(s)} \frac{(2-k)v}{\omega-2\lambda} \bar{\mu}^2 e^{2\lambda R(s)}\, ds \\
&\quad = \frac{(k-2)^2 v^2 \bar{\mu}^2}{(\omega-2\lambda)(v+\lambda)} \big(e^{-vR(t)} - e^{\lambda R(t)} \big).
\end{aligned} \tag{29}$$

Substituting the above computed functions into (11) gives

$$\delta(v) = \frac{1}{2}\mathrm{Re}\left(\frac{1}{\mu}\mathcal{R}_\mu\Big(\big(e^{-vR(t)} - e^{\lambda R(t)}\big)\big(\frac{(k-3)v}{v+\lambda} + \frac{(k-2)^2 v^2 \bar{\mu}^2}{(\omega-2\lambda)(v+\lambda)} - 2\frac{(k-2)^2 v}{k(v+\lambda)}\big)\Big)\right). \tag{30}$$

Take into account the relations

- $\mu = \frac{1}{1-k} - i\sqrt{1-\frac{1}{(1-k)^2}}$,
- $\lambda = iv(1-k)\sqrt{1-\frac{1}{(1-k)^2}}$,
- $\mu^2 = \frac{2}{(1-k)^2} - 1 - i2\frac{\sqrt{1-\frac{1}{(1-k)^2}}}{1-k}$,
- $v + \lambda = v(1-k)\bar{\mu}$.

Let $z = \frac{v\bar{\mu}^2}{(\omega-2\lambda)} = z_1 + iz_2 = \frac{w_1 - iw_2}{w_1^2 + w_2^2}$, where $\frac{1}{z} = w_1 + iw_2$. Let us compute $\frac{1}{z}$:

$$\begin{aligned}\frac{1}{z} &= (\omega-2\lambda)\mu^2 = -\mu^2 + (1-k) - \mu^2\Big(i2(1-k)\sqrt{1-\frac{1}{(1-k)^2}}\ \Big) \\ &= -k - 2 + \frac{2}{(1-k)^2} + i2\sqrt{1-\frac{1}{(1-k)^2}}\Big((1-k) - \frac{1}{(1-k)}\Big).\end{aligned} \tag{31}$$

By (10), we find

$$\begin{aligned}\mathcal{R}(e^{-vR(t)}) &= \left(\frac{1}{\mu + v(1-k)}\right)\left(e^{-v} + \int_{-1}^{0} e^{-vR(s)}\bar{\chi}_\mu(s) v(1-k) r(s)\,ds\right) \\ &= \left(\frac{1}{\mu + v(1-k)}\right)\left(\ e^{-v} + \frac{v(1-k)(e^{-v}\bar{\mu} - 1)}{-v-\lambda}\right) \\ &= \frac{\mu}{\mu + v(1-k)} = \frac{1}{1+v+\lambda},\end{aligned}$$

thus

$$\mathcal{R}_\mu\big(e^{-vR(t)}\big) - 1 = \frac{1}{1+v+\lambda} - 1 = -\frac{v+\lambda}{1+v+\lambda} = -\frac{v+\lambda}{1+v+\lambda},$$

and (30) is simplified to

$$\delta(v) \cong -v\mathrm{Re}\left(\Big(\bar{\mu}(1+v-\lambda)\Big)\Big(z + \frac{k-3}{(k-2)^2} - \frac{2}{k}\Big)\right). \tag{32}$$

Recall that $v > 0$. The first term can be expressed as

$$\begin{aligned}
&\bar{\mu}(1+v-\lambda)\\
&= \Big(\frac{1}{1-k} + i\sqrt{1-\frac{1}{(1-k)^2}}\Big)\Big(1+v-iv(1-k)\sqrt{1-\frac{1}{(1-k)^2}}\Big)\\
&= \frac{1}{1-k} + v(1-k) + i\sqrt{1-\frac{1}{(1-k)^2}},
\end{aligned} \tag{33}$$

with this, and taking apart the real and imaginary parts of z, (32) takes the form

$$\begin{aligned}
\delta(v) &\cong -\bigg(\Big(\frac{1}{1-k} + v(1-k)\Big)\Big(z_1 + \frac{k-3}{(k-2)^2} - \frac{2}{k}\Big) - z_2\sqrt{1-\frac{1}{(1-k)^2}}\bigg)\\
&\cong -\bigg(\Big(\frac{1}{1-k} + v(1-k)\Big)\Big(w_1 + \big(\frac{k-3}{(k-2)^2} - \frac{2}{k}\big)(w_1^2+w_2^2)\Big)\\
&\quad + \sqrt{1-\frac{1}{(1-k)^2}}w_2\bigg).
\end{aligned} \tag{34}$$

We use the following relations for $2 < k \in \mathbb{R}$ and $1 \le n \in \mathbb{N}$:

- $\frac{1}{(1-k)^2} < 1$ yields $w_1 = -k - 2 + \frac{2}{(1-k)^2} < -k$,
- $\sqrt{1-\frac{1}{(1-k)^2}}w_2 = 2\sqrt{1-\frac{1}{(1-k)^2}}\sqrt{1-\frac{1}{(1-k)^2}}\Big((1-k) - \frac{1}{(1-k)}\Big)$
  $= 2(1-k)\big(1-\frac{1}{(1-k)^2}\big)^2 > 2(1-k)$,
- $v = v_n = -\frac{\arccos(\frac{1}{1-k})+2n\pi}{(1-k)\sqrt{1-\frac{1}{(1-k)^2}}}$,
- $\frac{k-3}{(k-2)^2} - \frac{2}{k} = \frac{-k^2+5k-8}{(k-2)^2k} < 0$,

to deduce

$$\begin{aligned}
&\bigg(\Big(\frac{1}{1-k} + v(1-k)\Big)\Big(w_1 + \big(\frac{k-3}{(k-2)^2} - \frac{2}{k}\big)(w_1^2+w_2^2)\Big)\\
&+ \sqrt{1-\frac{1}{(1-k)^2}}w_2\bigg) > \bigg(\Big(\frac{1}{1-k} + v(1-k)\Big)w_1 + 2(1-k)\bigg)\\
&> \frac{k}{k-1} + 2n\pi k - 2(k-1) > 0
\end{aligned} \tag{35}$$

for $n \geq 1$, hence $\delta(v_n) < 0$, which completes the proof. □

One can obtain a similar result for the case $n = 0$. The computations are mechanical from this point, but very circuitous, and for that reason omitted. We used very rough estimates, but so far it was enough for $n \geq 1$. If $n = 0$, then by the simple estimate $\arccos(\frac{1}{1-k}) > \frac{\pi}{2}$, we obtain immediately the same inequality for $k < 7$. Substituting the expressions given for v, w_1, w_2 into (34), we get a rational function of k including an arccos-term, and we can detect its sign. However, this function is rather complicated to handle analytically in general, it is quite easy to compute the sign of $\delta(v_0)$ for any given k. Finally, the critical value q_n of the original equation (19) equals $\frac{v_n}{d}$. The bifurcation from the equilibrium 0 of (20) means a bifurcation from the positive equilibrium of the original equation (19).

5 Krisztin-Walther equation and a periodically forced neuron

The delay differential equation

$$\dot{x}(t) = -mx(t) + \alpha \tanh(\beta x(t-1)) \tag{36}$$

with $\alpha > 0$, $\beta > 0$ and $m > 0$ models the behavior of a single, self-excitatory neuron with graded delayed response (see [7, 19] for related references). The more general Krisztin-Walther equation (the denomination is originated from [12])

$$\dot{x}(t) = -mx(t) + f(x(t-1))$$

was studied in [6, 7] and [17], with the assumptions of monotone positive feedback, and the structure of the global attractor was described. The nonmonotonicity of the nonlinearity can induce complicated behavior (see [9]), the Mackey-Glass equation is also an example of that.

Consider the parametrized, periodic version of (36)

$$\varepsilon \dot{x}(t) = -m(t)x(t) + \alpha(t) \tanh(\beta x(t-1)), \tag{37}$$

where $m(t) = m(t+1) > 0$ and $\alpha(t) = \alpha(t+1) > 0$ for all $t \in \mathbb{R}$. Notice that Theorem 1 can be applied to equation (37). Everything can be done in the same way as for the Mackey-Glass equation, the only difference is that now we have $f_{\xi\xi\xi}(t, 0) = -2\beta^3\alpha(t)$ instead of $-6q(t)$. But during the proof the exact value was not used, only its sign, hence we immediately obtain the following theorem.

Theorem 3 *Assume that* $0.9 < \frac{M}{\beta \int_{-1}^{0} \alpha(s)\, ds} < 1$. *Then the dynamical system generated by the time-one map related to equation* (37) *undergoes a supercritical Neimark-Sacker bifurcation as the parameter* v *passes through* v_n *increasingly.*

Glancing at the proof of Theorem 1, one can see that the computations can be performed similarly in the case when the time-periodicity is involved inside the nonlinearity, and when $f_{\xi\xi\xi}(t,0) > 0$ for all $t \in \mathbb{R}$, but it changes the sign of the coefficient δ to the opposite. Now we state the theorem in the general form.

Theorem 4 *Consider*

$$\varepsilon \dot{x}(t) = -a(t)x(t) + f(t, x(t-1)), \tag{38}$$

where $a : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are C^4-smooth functions satisfying $a(t+1) = a(t)$, $f(t+1,\xi) = f(t,\xi)$ and $f(t,0) = 0$ for all $t, \xi \in \mathbb{R}$, $\varepsilon \neq 0$ is a real parameter. Assume that $f_\xi(t,0) > 0$ for all $t \in \mathbb{R}$ and $0.9 < \frac{A}{\int_{-1}^{0} f_\xi(s,0)\,ds} < 1$. Then in the case of $f_{\xi\xi\xi}(t,0) < 0$ for all $t \in \mathbb{R}$, the dynamical system generated by the time-one map related to equation (38) undergoes a supercritical, in the case of $f_{\xi\xi\xi}(t,0) > 0$ for all $t \in \mathbb{R}$ a subcritical Neimark-Sacker bifurcation as the parameter $v = \frac{1}{\varepsilon}$ passes through v_n increasingly.

Notice that the assumption $0.9 < \frac{A}{\int_{-1}^{0} f_\xi(s,0)\,ds} < 1$ excludes the resonant case.

The presence of a periodic force is very common in neural network theory. A periodic force tranforms the autonomous system into a time-periodic one. Consider the model of a periodically forced single neuron:

$$\varepsilon \dot{x}(t) = -mx(t) + f(x(t-1)) + p(t), \tag{39}$$

where $p(t+1) = p(t)$ for all $t \in \mathbb{R}$. Via the change of variables $y(t) = x(t) - \int_{-\infty}^{t} e^{-a(t-s)} p(s)\,ds$, the periodicity of the integral term yields that (39) is transformed into the periodic equation

$$\varepsilon \dot{y}(t) = -my(t) + f\big(y(t-1) + \int_{-\infty}^{t} e^{-a(t-s)} p(s)\,ds\big). \tag{40}$$

Our final remark is that if we know any particular solution, for example a periodic solution of (39) with period 1, that can be transformed into a periodic solution of (40). In the case of a control problem we have the choice to choose $p(t)$ to obtain a prescribed solution. Then varying the parameter, we can study the Neimark-Sacker bifurcation of an invariant curve for the system (40) in the same way as before, linearizing along the periodic solution. In this case the picture sketched at the end of Section 2 is more impressive, namely, we can think about it as a bifurcation of an invariant torus from the periodic solution.

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