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Torsion-free crystallographic groups with indecomposable holonomy group. II

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Abstract. Let K be a principal ideal domain, G a finite group, and M a KG-module which is a free K-module of finite rank on which G acts faithfully. A *generalized crystallographic group* is a non-split extension $\mathfrak{C}$ of M by G such that conjugation in $\mathfrak{C}$ induces the G-module structure on M. (When $K = \mathbb{Z}$, these are just the classical crystallographic groups.) The *dimension* of $\mathfrak{C}$ is the K-rank of M, the *holonomy group* of $\mathfrak{C}$ is G, and $\mathfrak{C}$ is *indecomposable* if M is an indecomposable KG-module.

We study indecomposable torsion-free generalized crystallographic groups with holonomy group G when K is $\mathbb{Z}$, or its localization $\mathbb{Z}_{(p)}$ at the prime p, or the ring $\mathbb{Z}_p$ of p-adic integers. We prove that the dimensions of such groups with G non-cyclic of order p^2 are unbounded. For $K = \mathbb{Z}$, we show that there are infinitely many non-isomorphic such groups with G the alternating group of degree 4 and we study the dimensions of such groups with G cyclic of certain orders.

1 Introduction

Zassenhaus developed algebraic methods in [11] for studying the classical crystallographic groups and he pointed out the close connection between them and the theory of integral representations of finite groups. Historical overviews and an account of the present state of the theory of crystallographic groups and its connections to other branches of mathematics are given in [9], [10].

In general, the classification of the crystallographic groups is a problem of wild type, in the sense that it is related to the classical unsolvable problem of describing the canonical forms of pairs of linear operators acting on finite-dimensional vector spaces (see [5], [7]). One may however focus on special classes of crystallographic groups, for example, on groups whose translation group affords an irreducible (or indecomposable) integral representation of the holonomy group. In this direction, Hiss and Szczepański [6] proved that there are no torsion-free crystallographic groups with irreducible holonomy group. On the other hand, Kopcha and Rudko [7] showed

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that the problem of describing torsion-free crystallographic groups with indecomposable cyclic holonomy group of order p^n with $n \geqslant 5$ is still of wild type.

The *generalized crystallographic groups* introduced in [3] are defined as follows. Let K be a principal ideal domain, G a finite group, and M a KG-module which is a free K-module of finite rank on which G acts faithfully. A generalized crystallographic group is a group $\mathfrak{C}$ which has a normal subgroup isomorphic to M with quotient G, such that conjugation in $\mathfrak{C}$ induces the G-module structure on M and such that the extension does not split. The K-rank of M is called the *dimension* of $\mathfrak{C}$, and the *holonomy group* of $\mathfrak{C}$ is G. (When $K = \mathbb{Z}$, this agrees with one of the usual descriptions of crystallographic groups; for emphasis, we sometimes refer to them as *classical* crystallographic groups.)

In [3], we studied indecomposable generalized crystallographic groups when K is $\mathbb{Z}$, or its localization $\mathbb{Z}_{(p)}$ at the prime p, or the ring $\mathbb{Z}_p$ of p-adic integers, and either G is a cyclic p-group or $p = 2$ and G is non-cyclic of order 4. Retaining this restriction on the choice of K but allowing p to be arbitrary, we consider here indecomposable torsion-free generalized crystallographic groups with holonomy group non-cyclic of order p^2 and we prove in Theorem 2 that the dimensions of such groups are unbounded.

For the classical case (when $K = \mathbb{Z}$), we show in Theorem 3 that there are infinitely many non-isomorphic indecomposable torsion-free crystallographic groups with holonomy group the alternating group of degree 4. In Theorem 1, we consider G cyclic of order satisfying the following condition: p^2 divides G for all prime divisors p of $|G|$ and p^3 divides $|G|$ for at least one p. We prove that then every product of $|G|$ with a positive integer coprime to it is the dimension of an indecomposable torsion-free crystallographic group with holonomy group G.

2 The main results

Let K be a principal ideal domain, F be a field containing K and let G be a finite group. Let M be a K-free KG-module, with a finite K-basis affording a faithful representation Γ of G by matrices over K. Let FM be the F-space spanned by this K-basis of M, so that M becomes a full lattice in FM. Let $\hat{M} = FM^+/M^+$ be the quotient group of the additive group FM^+ of the linear space FM by the additive group M^+ of the module M. Then FM is an FG-module and $\hat{M}$ is a KG-module with operations defined by

$$g(\alpha m) = \alpha g(m), \quad g(x + M) = g(x) + M,$$

for $g \in G$, $\alpha \in F$, $m \in M$, $x \in FM$.

Let $T : G \to \hat{M}$ be a 1-cocycle of G with values in $\hat{M}$. Elements of $\hat{M}$ being cosets in FM^+ modulo M^+, we regard each value $T(g)$ of T as a subset of FM, and define the group

$$\mathfrak{Crys}(G; M; T) = \{(g, x) \mid g \in G, x \in T(g)\}$$

with the operation

$$(g, x)(g', x') = (gg', gx' + x),$$

for $g, g' \in G$, $x \in T(g)$, $x' \in T(g')$.

The K-rank of M will be called the K-dimension of $\mathfrak{Crys}(G; M; T)$. When T is not cohomologous to 0, the group $\mathfrak{Crys}(G; M; T)$ is called *indecomposable* if M is an indecomposable KG-module. If $K = \mathbb{Z}$ and $F = \mathbb{R}$, then the abstract group $\mathfrak{Crys}(G; M; T)$ is a classical crystallographic group.

Let $C^1(G, \hat{M})$ and $B^1(G, \hat{M})$ be the groups of 1-cocycles and 1-coboundaries of G with values in $\hat{M}$, so that $H^1(G, \hat{M}) = C^1(G, \hat{M})/B^1(G, \hat{M})$. The group $\mathfrak{Crys}(G; M; T)$ is an extension of M^+ by G; it is torsion-free if and only if for each subgroup H of G of prime order the restriction $T|_H$ is not a coboundary.

Using results from [1], [2], [8] we prove the following two theorems.

Theorem 1. *Let G be a cyclic group of order $|G| = p_1^{n_1} \dots p_s^{n_s}$, where $p_1, \dots, p_s$ are distinct primes and suppose that $n_1 \geqslant 3$ and that $n_2 \geqslant 2, \dots, n_s \geqslant 2$ if $s \geqslant 2$. Let m be a natural number coprime to $|G|$ and put $d = m|G|$. Then there exists a torsion-free indecomposable classical crystallographic group of dimension d with holonomy group isomorphic to G.*

Theorem 2. *Let K be $\mathbb{Z}, \mathbb{Z}_{(p)}$ or $\mathbb{Z}_p$, and let $G \cong C_p \times C_p$. Then the K-dimensions of the indecomposable torsion-free groups $\mathfrak{Crys}(G; M; T)$ are unbounded.*

In [3] we described completely the indecomposable torsion-free crystallographic groups with holonomy group $C_2 \times C_2$. We proved that there exist at least $2p - 3$ torsion-free crystallographic groups having cyclic indecomposable holonomy group of order p^2. Note that the holonomy group of an indecomposable torsion-free crystallographic group can never have prime order. Therefore we have the following result.

Theorem 3. *There exist infinitely many non-isomorphic indecomposable torsion-free classical crystallographic groups with holonomy group isomorphic to the alternating group A_4 of degree* 4.

3 Preliminary results and the proof of Theorem 1

Let $K = \mathbb{Z}, \mathbb{Z}_{(p)}$ or $\mathbb{Z}_p$ as above, $H_{p^n} = \langle a \mid a^{p^n} = 1 \rangle$ be a cyclic group of order p^n ($n \geqslant 2$), ξ_s be a primitive p^sth root of unity, with $\xi_s^p = \xi_{s-1}$ for $s \geqslant 1$, and $\xi_0 = 1$. Define ordered bases B_i for the free K-modules $\mathfrak{R}_i = K[\xi_i]$ by setting

$$B_1 = \{1, \xi_1, \dots, \xi_1^{p-2}\},$$

$$B_2 = \{1, \xi_1, \dots, \xi_1^{p-2}, \xi_2, \xi_2\xi_1, \dots, \xi_2^{p-1}\xi_1^{p-2}\},$$

and in general (for $i > 1$)

$$B_i = B_{i-1} \cup \xi_i B_{i-1} \cup \xi_i^2 B_{i-1} \cup \cdots \cup \xi_i^{p-1} B_{i-1},$$

ordered as indicated. Obviously $|B_i| = \phi(p^i)$ (where ϕ is the Euler function). Each $\mathfrak{R}_i$ with $i \leqslant n$ is a KH_{p^n}-module with action defined by

$$a(\alpha) = \xi_i \cdot \alpha \quad (\alpha \in \mathfrak{R}_i). \tag{1}$$

We note that $\mathfrak{R}_i$ is only a K-submodule of $\mathfrak{R}_{i+1}$, not a KH_{p^n}-submodule. Let $\tilde{\xi}_i$ be the matrix representing multiplication by ξ_i in the ring $\mathfrak{R}_i$ with respect to the K-basis B_i for each $i \geqslant 0$ (where $\mathfrak{R}_0 = K$). Note that

$$\tilde{\xi}_i^p = E_p \otimes \tilde{\xi}_{i-1} \quad (i > 1)$$

where E_p is the identity matrix of degree p and $\otimes$ is the Kronecker product of matrices.

Let δ_i be the matrix representation of H_{p^n} with respect to the K-basis B_i of the KH_{p^n}-module $\mathfrak{R}_i$. From (1) it follows that

$$\delta_i(a) = \tilde{\xi}_i \quad (i \geqslant 0)$$

and $\delta_0, \dots, \delta_n$ are irreducible K-representations of H_{p^n}.

Let $0 \leqslant i \leqslant j \leqslant n$. For each $\alpha \in \mathfrak{R}_i$ we denote by $\langle \alpha \rangle_j^i$ the matrix with $\phi(p^i)$ rows and $\phi(p^j)$ columns in which all columns are zero except the last which is the coordinate vector of $\alpha \in \mathfrak{R}_i$ in the basis B_i. Thus

$$\begin{aligned} \tilde{\xi}_i \cdot \langle \alpha \rangle_j^i &= \langle \xi_i \alpha \rangle_j^i; \\ \langle \alpha \rangle_j^i &= (\langle 0 \rangle_{j-1}^i, \dots, \langle 0 \rangle_{j-1}^i, \langle \alpha \rangle_{j-1}^i); \\ \langle \alpha \rangle_j^i \cdot \tilde{\xi}_j^k &= (\langle \alpha_1(k) \rangle_{j-1}^i, \dots, \langle \alpha_{p-1}(k) \rangle_{j-1}^i, \langle \alpha_p(k) \rangle_{j-1}^i), \end{aligned} \tag{2}$$

for $0 \leqslant k < p$, where $\alpha_{p-k}(k) = \alpha$ and $\alpha_s(k) = 0$ for $s \neq p - k$. The matrix $\langle \alpha \rangle_j^i$ defines an extension of the KH_{p^n}-module $\mathfrak{R}_i$ by the KH_{p^n}-module $\mathfrak{R}_j$ realizing the following K-representation of H_{p^n}:

$$a \mapsto \begin{pmatrix} \tilde{\xi}_i & \langle \alpha \rangle_j^i \\ 0 & \tilde{\xi}_j \end{pmatrix}. \tag{3}$$

If $\alpha \equiv 0 \pmod{p\mathfrak{R}_i}$ this K-representation is completely reducible and the corresponding extension of modules is split, i.e.

$$p \operatorname{Ext}_{KH_{p^n}}(\mathfrak{R}_j, \mathfrak{R}_i) = 0 \quad (i > j). \tag{4}$$

Let m be a natural number and let A be an $m \times m$ matrix over K. Consider the K-representations of the cyclic group $H_{p^n} = \langle a \mid a^{p^n} = 1 \rangle$, with $n > 2$, defined by

$$\Delta_1 = E_m \otimes \delta_0 + E_m \otimes \delta_1\colon \quad a \mapsto \begin{pmatrix} E_m & 0 \\ 0 & E_m \otimes \tilde{\xi}_1 \end{pmatrix};$$

$$\Delta_2 = E_m \otimes \delta_2 + \cdots + E_m \otimes \delta_n\colon \quad a \mapsto \begin{pmatrix} E_m \otimes \tilde{\xi}_2 & & 0 \\ & \ddots & \\ 0 & & E_m \otimes \tilde{\xi}_n \end{pmatrix};$$

$$\Gamma_{p,A}^{(m)} = \begin{pmatrix} \Delta_1 & U \\ 0 & \Delta_2 \end{pmatrix}\colon \quad a \mapsto \begin{pmatrix} \Delta_1(a) & U(a) \\ 0 & \Delta_2(a) \end{pmatrix},$$

where

$$U(a) = \begin{pmatrix} A \otimes \langle 1 \rangle_2^0 & E_m \otimes \langle 1 \rangle_3^0 & \cdots & E_m \otimes \langle 1 \rangle_n^0 \\ E_m \otimes \langle 1 \rangle_2^1 & E_m \otimes \langle 1 \rangle_3^1 & \cdots & E_m \otimes \langle 1 \rangle_n^1 \end{pmatrix}$$

is the *intertwining* matrix.

For $n = 2$ we define the following K-representation of $H_{p^2} = \langle a \,|\, a^{p^2} = 1 \rangle$:

$$\Gamma_p^{(1)}\colon \quad a \mapsto \begin{pmatrix} 1 & 0 & \langle 1 \rangle_2^0 \\ & \tilde{\xi}_1 & \langle 1 \rangle_2^1 \\ 0 & & \tilde{\xi}_2 \end{pmatrix}. \tag{5}$$

Lemma 1. *Let J_m be the lower triangular Jordan block of degree m with entries* 1 *on the main diagonal. Then $\Gamma_{p,J_m}^{(m)}$ (resp. $\Gamma_p^{(1)}$) is an indecomposable K-representation of degree $m|H_{p^n}|$ of H_{p^n} for $n \geqslant 2$ (resp. of degree $|H_{p^2}|$ of the group H_{p^2}).*

Proof. Representations depending on matrix parameters in this way were studied in [1], [2]. Using methods and results from these papers, it is not difficult to show that for $n > 2$ the K-representations $\Gamma_{p,A}^{(m)}$ and $\Gamma_{p,B}^{(m)}$ are equivalent if and only if

$$C^{-1}AC - B \equiv 0 \pmod{p}, \tag{6}$$

for some invertible matrix C. Moreover, the K-representation $\Gamma_{p,A}^{(m)}$ is decomposable if and only if there is a decomposable matrix B which satisfies (6). In particular, $\Gamma_{p,J_m}^{(m)}$ is an indecomposable K-representation of H_{p^n}. The case of the representation $\Gamma_p^{(1)}$ follows from [1].

Put

$$\Gamma_p^{(m)} = \begin{cases} \Gamma_{p,J_m}^{(m)} & \text{for } n > 2,\ m > 1; \\ \Gamma_{p,1}^{(1)} & \text{for } n > 2,\ m = 1; \\ \Gamma_p^{(1)} & \text{for } n = 2. \end{cases} \tag{7}$$

Lemma 2. *Let L_p be a KH_{p^n}-module affording the K-representation $\Gamma_p^{(m)}$ of H_{p^n} (for $n \geqslant 2$) and $\{v_1, v_2, \dots, v_t\}$ be a K-basis corresponding to this representation in L_p. Then Kv_1 is a KH_{p^n}-submodule in L_p, and over K it has a direct complement L_p' invariant under a^p with K-basis $\{w_2, \dots, w_t\}$ where $w_i = v_i + \lambda_i v_1$ with $\lambda_i \in K$ for $i = 2, \dots, t$.*

Proof. Let $n > 2$. Clearly $a \cdot v_1 = v_1$, i.e. Kv_1 is a KH_{p^n}-submodule in L_p. Using (2) it is easy to check that in the matrix $\Gamma_p^{(m)}(a^p)$ the intertwining matrix

$$U(a^p) = \sum_{t=0}^{p-1} \Delta_1^{p-t-1}(a) \cdot U(a) \cdot \Delta_2^t(a)$$

has the form

$$U(a^p) = \begin{pmatrix} J_m \otimes U_{11} & \cdots & E_m \otimes U_{1\ n-1} \\ E_m \otimes U_{21} & \cdots & E_m \otimes U_{2\ n-1} \end{pmatrix},$$

where

$$U_{1\ i} = (\langle 1 \rangle_i^0, \dots, \langle 1 \rangle_i^0), \quad U_{2\ i} = (\langle 1 \rangle_i^1, \langle \xi_1 \rangle_i^1, \dots, \langle \xi_1^{p-1} \rangle_i^1) \quad (i = 1, \dots, n-1).$$

We change the basis elements v_{m+i} to $w_{m+i} = v_{m+i} + v_1$ for $i = 1, \dots, p-1$. Since the sum $-(v_{m+1} + \cdots + v_{m+p-1}) + v_1$ is replaced by $-(w_{m+1} + \cdots + w_{m+p-1}) + pv_1$, the effect on the first row of the matrix $U(a^p)$ is to make its elements either 0 or non-zero multiples of p. From (4) with $i = 0$ we can change the basis elements by setting $w_{m+i} = v_{m+i} + \lambda_i v_1$ for $p \leqslant i$ with each $\lambda_i \in K$ and so we get a K-module L_p' invariant under a^p such that $L_p = Kv_1 \oplus L_p'$.

For $n = 2$ the statement of the lemma is clear.

For the rest of this section we suppose that $K = \mathbb{Z}$. Let G be cyclic of order $q_1 \dots q_s$ where $q_i = p_i^{n_i}$ for each i, with $p_1, \dots, p_s$ distinct primes, and with $n_i \geqslant 2$ for each i and $n_1 \geqslant 3$. Write $G = H_{q_1} \times \cdots \times H_{q_s}$ with H_{q_i} cyclic of order q_i for each i.

Let $\Gamma^{(m)}$ be the tensor product of the $\mathbb{Z}$-representation $\Gamma_{p_1}^{(m)}$ of H_{q_1} and the $\mathbb{Z}$-representations Γ_{q_j} of the groups H_{q_j} for $m \in \mathbb{N}$ and $j = 2, \dots, s$. Then $\Gamma^{(m)}$ is a $\mathbb{Z}$-representation of the group G in which

$$\Gamma^{(m)}(a_1^{t_1}, \dots, a_s^{t_s}) = \Gamma_{p_1}^{(m)}(a_1^{t_1}) \otimes \Gamma_{p_2}^{(1)}(a_2^{t_2}) \otimes \cdots \otimes \Gamma_{p_s}^{(1)}(a_s^{t_s}).$$

Lemma 3. *If $(m, |G|) = 1$ then $\Gamma^{(m)}$ is an indecomposable $\mathbb{Z}$-representation of G.*

Proof. Let $\Gamma^{(m)}|_{H_{q_i}}$ be the restriction of the representation $\Gamma^{(m)}$ to H_{q_i}. By Lemma 1 the degree of each indecomposable summand of $\Gamma^{(m)}|_{H_{q_i}}$ is $m|H_{q_1}|$ for $i = 1$ and $|H_{q_i}|$ for $i > 1$.

If Γ is a non-zero summand in $\Gamma^{(m)}$, then its degree of is divisible by $m|H_{q_1}|$ and by $|H_{q_2}|, \dots, |H_{q_s}|$ if $s \geqslant 2$ (see Lemma 1 for the case $K = \mathbb{Z}_p$). Thus since $(m, |G|) = 1$ we have $\Gamma = \Gamma^{(m)}$, as required.

Now we construct a cocycle for G. Let M be a $\mathbb{Z}G$-module of the $\mathbb{Z}$-representation $\Gamma^{(m)}$ affording the group G and

$$M = L_{p_1} \otimes_K \cdots \otimes_K L_{p_s}, \tag{8}$$

where L_{p_i} is a $\mathbb{Z}H_{p_i^n}$-submodule for $\Gamma_{p_i}^{(m)}$ for each i. If $g = a_1^{t_1} \dots a_s^{t_s} \in G$ and $l = l_1 \otimes \cdots \otimes l_s \in M$, then

$$g \cdot l = a_1^{t_1} \cdot l_1 \otimes \cdots \otimes a_s^{t_s} \cdot l_s,$$

where $l_i \in L_{p_i}$, $t_i \in \mathbb{Z}$ for each i.

We can suppose that $M \subset \mathbb{R}^d$. Each $\mathbb{Z}$-basis for M is also an $\mathbb{R}$-basis in $\mathbb{R}^d$ and an $\mathbb{R}^+/\mathbb{Z}^+$-basis in $\hat{M} = \mathbb{R}^{d+}/M^+$, where $d = m|G| = \deg(\Gamma^{(m)})$.

Let $v = v_1^{(1)} \otimes \cdots \otimes v_s^{(1)}$ be the tensor product of the first $\mathbb{Z}$-basis elements of the modules $L_1, \dots, L_{p_s}$. Obviously $a \cdot v = v$. Define $f : G \to \hat{M}$ by

$$f(g) = \left(\frac{t_1}{q_1} + \cdots + \frac{t_s}{q_s}\right) \cdot v + M, \tag{9}$$

where $g = a_1^{t_1} \dots a_s^{t_s} \in G$ with $t_1, \dots, t_s \in \mathbb{Z}$. Since $g_1 \cdot v = v$ and

$$f(g_1 \cdot g_2) = f(g_1) + f(g_2) \quad \text{for } g_1, g_2 \in G,$$

we obtain

$$f(g_1 \cdot g_2) = f(g_2) + f(g_1) = g_1 \cdot f(g_2) + f(g_1)$$

and therefore f is a 1-cocycle of G in $\hat{M}$. The lemma is proved.

Lemma 4. *The restriction of f to each subgroup of G of prime order subgroup is not a coboundary.*

Proof. Let $1 \leqslant i \leqslant s$ and let $b = a^r$ where $r = p_i^{n_i - 1}$. From Lemma 2 and (8) the $\mathbb{Z}$-module M can be decomposed as $M = \mathbb{Z}v \oplus M'$, where $\mathbb{Z}v$ is a $\mathbb{Z}G$-module and M' is a $\mathbb{Z}$-module which is invariant under $a_i^{p_i}$ and hence under b. Thus $\hat{M} = Fv \oplus \hat{M}'$ and $b(\hat{M}') = \hat{M}'$. If $z \in \hat{M}$, then $z = \alpha v + z_1$ for some $\alpha \in F$, $z_1 \in \hat{M}'$. From (9) and since $b(z_1) \in \hat{M}'$, it follows that

$$f(b) = p_i^{-1} v + M \neq (b - 1)z + M$$

for any $z \in \hat{M}$. Therefore the restriction of f to $\langle b \rangle$ is not a coboundary, which proves the lemma.

Proof of Theorem 1. By Lemma 4 the group $\mathfrak{Crys}(G; M; T)$ is torsion-free. Moreover, according to Lemma 3, $\Gamma^{(m)}(G)$ is an indecomposable subgroup in $\mathrm{GL}(d, K)$, where $d = m|G|$ and $(m, |G|) = 1$. So the proof is complete.

4 Proof of Theorem 2

Let $K = \mathbb{Z}, \mathbb{Z}_{(p)}$ or $\mathbb{Z}_p$ as above and let $\varepsilon = \xi$ be a primitive pth root of unity (where $p > 2$). Then $B_1 = \{1, \varepsilon, \dots, \varepsilon^{p-2}\}$ is an $\mathfrak{F}$-basis in the field $\mathfrak{F}(\varepsilon)$ and a K-basis in the ring $K[\varepsilon]$, where $\mathfrak{F}$ is the field of fractions of the ring K.

We write $\langle\alpha\rangle$ for the column co-ordinate vector the element $\alpha \in \mathfrak{F}(\varepsilon)$ in the basis B_1 and $\tilde{\alpha}$ for the matrix representing the operation of multiplication by α in the $\mathfrak{F}$-basis B_1 of the field $\mathfrak{F}(\varepsilon)$. Clearly $\tilde{\varepsilon} \cdot \langle\alpha\rangle = \langle\varepsilon\alpha\rangle$.

The group $G = \langle a, b\rangle \cong C_p \times C_p$ (where $p > 2$) has the following $p + 2$ irreducible K-representations, which are pairwise inequivalent over the field $\mathfrak{F}$:

$$\begin{aligned}
&\gamma_0 : a \mapsto 1, \quad b \mapsto 1;\\
&\gamma_1 : a \to \tilde{1}, \quad b \to \tilde{\varepsilon};\\
&\gamma_2 : a \mapsto \tilde{\varepsilon}, \quad b \mapsto \tilde{1};\\
&\gamma_3 : a \mapsto \tilde{\varepsilon}, \quad b \mapsto \tilde{\varepsilon};\\
&\rho_i : a \mapsto \tilde{\varepsilon}, \quad b \mapsto \tilde{\varepsilon}^i,
\end{aligned} \tag{10}$$

for $i = 2, \dots, p-1$, where $\tilde{1} = E_{p-1}$ is the $(p-1) \times (p-1)$ identity matrix.

Put $\tau = \rho_{p-1} \oplus \cdots \oplus \rho_2 \oplus \gamma_3 \oplus \gamma_2 \oplus \gamma_1$. Define the K-representation Γ_0 of the group $G = \langle a, b\rangle$ by

$$a \mapsto \begin{pmatrix} \tau(a) & U(a) \\ 0 & \gamma_0(a) \end{pmatrix}, \quad b \mapsto \begin{pmatrix} \tau(b) & U(b) \\ 0 & \gamma_0(b) \end{pmatrix};$$

where the intertwining matrix U satisfies:

$$U(a) = \begin{pmatrix} \langle 1\rangle \\ \vdots \\ \langle 1\rangle \\ 0 \end{pmatrix}, \quad U(b) = \begin{pmatrix} \langle\alpha_1\rangle \\ \vdots \\ \langle\alpha_p\rangle \\ \langle 1\rangle \end{pmatrix};$$

and $\alpha_i = (\varepsilon^{p-i} - 1)/(\varepsilon - 1)$ for $i = 1, 2, \dots, p$.

Lemma 5. Γ_0 *is a faithful indecomposable K-representation of $G = \langle a, b\rangle$.*

Proof. Using $\tilde{\varepsilon} \cdot \langle\alpha\rangle = \langle\varepsilon\alpha\rangle$ and $1 + \varepsilon + \cdots + \varepsilon^{p-1} = 0$ it is easy to see that Γ_0 is a K-representation. Since $\mathbb{Z} \subset \mathbb{Z}_{(p)} \subset \mathbb{Z}_p$, it is now enough to complete the proof of the lemma for $K = \mathbb{Z}_p$. For this it is sufficient to prove that the centralizer

$$E(\Gamma_0) = \{X \in M(p^2, K) \mid X\Gamma_0(g) = \Gamma_0(g)X \text{ for all } g \in G\}$$

of Γ_0 is a local ring. Let δ, δ' be representations from (10) and let V be a K-matrix such that $\delta(g)V = V\delta'(g)$ for all $g \in G$. Then $V = 0$ if $\delta \neq \delta'$ and $V = \tilde{x}$ with $x \in K[\varepsilon]$ if $\delta = \delta' \neq \gamma_0$. It follows that each $X \in E(\Gamma_0)$ has the form

$$\begin{pmatrix} \tilde{x}_1 & 0 & \cdots & \cdots & 0 & \langle y_1 \rangle \\ & \tilde{x}_2 & 0 & \cdots & 0 & \langle y_2 \rangle \\ & & \ddots & \ddots & \vdots & \vdots \\ & & & \tilde{x}_p & 0 & \langle y_p \rangle \\ & & & & \tilde{x}_{p+1} & \langle y_{p+1} \rangle \\ 0 & & & & & x_0 \end{pmatrix},$$

where $x_i = x_0 + (\varepsilon - 1)y_i$, $x_0 \in K$ and $y_i \in K[\varepsilon]$ for $i = 1, 2, \ldots, p+1$. From the form of the matrix X and the condition $K = \mathbb{Z}_p$ we see that X is an invertible matrix if and only if x_0 is a unit in K. Since K is a local ring, it follows that $E(\Gamma_0)$ is also local, as required.

Let $M_0 = K^{p^2}$ be the K-module of the K-representation Γ_0 of G consisting of p^2-dimensional columns over K. It is convenient to condense each element of M_0, regarding it as a column vector of length $p+2$ with $p+1$ entries from $K^{p-1} \cong K[\varepsilon]$ and final entry from K. We will do the same with elements of FM_0 (the space of column vectors of length p^2 over F).

Lemma 6. *Let $\alpha = (\varepsilon - 1)^{-1}$ and let X, Y be the following elements from FM_0:*

$$X = \begin{pmatrix} \langle 0 \rangle \\ \vdots \\ \langle 0 \rangle \\ \langle \alpha \rangle \\ 0 \end{pmatrix}; \quad Y = \begin{pmatrix} \langle \alpha \rangle \\ \vdots \\ \langle \alpha \rangle \\ \langle 0 \rangle \\ 0 \end{pmatrix}. \tag{11}$$

There exists a 1-cocycle $f : G = \langle a, b \rangle \cong C_p \times C_p \to \hat{M}_0 = FM_0^+/M_0^+$ such that

$$f(a) = X + M_0 \quad \textit{and} \quad f(b) = Y + M_0.$$

Moreover, this cocycle f is special, i.e. on each non-trivial subgroup of G it is not cohomologous to the zero cocycle.

Proof. Note that $\alpha = (\varepsilon - 1)^{-1} \in \mathfrak{F}(\varepsilon)$ does not belong to $K[\varepsilon]$, but $p\alpha \in K[\varepsilon]$. It is easy to see that the initial $p+1$ diagonal quadratic blocks of the matrix

$$(\Gamma_0^{p-1} + \Gamma_0^{p-2} + \cdots + \Gamma_0 + E_{p^2})(g) \quad (g \in G)$$

are either zero or have the form $p\tilde{1}$, and that the final 1-dimensional block is equal to p. It follows that

$$
\begin{aligned}
(\Gamma_0^{p-1}(a) + \Gamma_0^{p-2}(a) + \cdots + \Gamma_0(a) + E_{p^2})X &\in M_0, \\
(\Gamma_0^{p-1}(b) + \Gamma_0^{p-2}(b) + \cdots + \Gamma_0(b) + E_{p^2})Y &\in M_0, \\
(\Gamma_0(a) - E_{p^2})Y - (\Gamma_0(b) - E_{p^2})X &\in M_0.
\end{aligned} \tag{12}
$$

The third condition follows since $(\tilde{\varepsilon} - \tilde{1})\langle\alpha\rangle = \langle 1\rangle \in K^{p-1}$.

Define a function $f : G = \langle a, b\rangle \cong C_p \times C_p \to \hat{M}_0$ by

$$
\begin{aligned}
&f(1) = M_0; \\
&f(a^i) = (a^{i-1} + \cdots + a + 1)X + M_0, \\
&f(b^j) = (b^{j-1} + \cdots + b + 1)Y + M_0, \quad f(a^i b^j) = a^i f(b^j) + f(a^i),
\end{aligned} \tag{13}
$$

for $i, j = 1, \dots, p-1$.

According to (11)–(13) we get that f is a cocycle from G to $\hat{M}_0$. To prove the rest of the statement it is sufficient to consider generating elements $\{a, a^i b \mid i = 0, \dots, p-1\}$ for all non-trivial cyclic subgroups of G.

For each $x \in FM_0$ and for $1 \leqslant s \leqslant p+1$, write $x_{(s)}$ for sth condensed co-ordinate of the vector x. We will do the same with elements of $\hat{M}_0$. Then by (13) we obtain that

$$
f_{(s)}(a^s b) = \langle \varepsilon^s \alpha\rangle + K^{p-1} \quad (s = 1, \dots, p), \quad f_{(p+1)}(a) = \langle\alpha\rangle + K^{p-1}. \tag{14}
$$

It is easy to see that

$$
\Gamma_0(a^s) = \begin{pmatrix}
\tilde{\varepsilon}^s & 0 & \cdots & \cdots & 0 & \langle\beta_s\rangle \\
 & \tilde{\varepsilon}^s & 0 & \cdots & 0 & \langle\beta_s\rangle \\
 & & \ddots & \ddots & \vdots & \vdots \\
 & & & \tilde{\varepsilon}^s & 0 & \langle\beta_s\rangle \\
0 & & & & \tilde{1} & 0 \\
 & & & & & 1
\end{pmatrix},
$$

where $\beta_s = (\varepsilon^s - 1)/(\varepsilon - 1)$ for $s = 1, 2, \dots, p$. Since $\varepsilon^s \alpha_s + \beta_s = 0$ for $s = 1, 2, \dots, p$ (see the notation before Lemma 5), $p-1$ rows of the matrix $\Gamma_0(a^s b) - E_{p^2}$ corresponding to the sth diagonal block will be 0. Moreover the final p rows of this matrix are also 0. Thus for any vector $z \in FM_0$ the sth condensed coordinate of the vector $(\Gamma_0(a^s b) - E_{p^2})z$ for $s = 1, 2, \dots, p$ will be equal to 0. The $(p+1)$st coordinate in $(\Gamma_0(a) - E_{p^2})z$ will also be 0.

Hence, from (14) and the condition $\alpha = (\varepsilon - 1)^{-1} \notin K[\varepsilon]$ it follows that

$$
(\Gamma_0(a^s b) - E_{p^2})z + f(a^s b) \neq M_0 \quad \text{and} \quad (\Gamma_0(a) - E_{p^2})z + f(a) \neq M_0
$$

for any $z \in FM_0$ and for $s = 1, 2, \dots, p$. The lemma is proved.

Corollary 1. *The group* $\mathfrak{Crys}(G; M_0; f)$ *is torsion-free.*

Let us define a K-representation of the group $G = \langle a, b \rangle$ as follows. Set

$$\Delta_n = \begin{pmatrix} E_n \otimes \gamma_3 & 0 & u_{11} & u_{12} \\ & E_n \otimes \gamma_2 & u_{21} & u_{22} \\ & & E_n \otimes \gamma_1 & 0 \\ 0 & & & E_n \otimes \gamma_0 \end{pmatrix},$$

where

$$u_{11}(a) = u_{21}(a) = -u_{21}(b) = E_n \otimes \tilde{1}, \quad u_{11}(b) = u_{22}(b) = 0,$$
$$u_{12}(a) = u_{12}(b) = J_n \otimes \langle 1 \rangle, \qquad u_{22}(a) = E_n \otimes \langle 1 \rangle,$$

and J_n is the upper triangular Jordan block of degree n.

Lemma 7. *The K-representation* Δ_n *of the group* $G = \langle a, b \rangle$ *is indecomposable.*

Proof. See [1], [5].

Using Γ_0 we define the following K-representation of G:

$$\Gamma_n = \begin{pmatrix} \Gamma_0 & V_n \\ 0 & \Delta_n \end{pmatrix},$$

where V_n is the matrix whose entries are intertwining functions of the composition factors in Γ_0 with the composition factors in Δ_n. All of these intertwining functions are 0 except the function v which intertwines γ_3 in Γ_0 with the first representation γ_0 in $E_n \otimes \gamma_0$ and $v(a) = v(b) = \langle 1 \rangle$. Thus

$$V_n(a) = V_n(b) = \begin{pmatrix} 0\dots0 & 0 & 0 & \cdots & 0 \\ \dots\dots & \dots & \dots & \dots & \dots \\ 0\dots0 & 0 & 0 & \cdots & 0 \\ & & & & \\ 0\dots0 & \langle 1 \rangle & \langle 0 \rangle & \cdots & \langle 0 \rangle \\ 0\dots0 & \langle 0 \rangle & \langle 0 \rangle & \cdots & \langle 0 \rangle \\ 0\dots0 & \langle 0 \rangle & \langle 0 \rangle & \cdots & \langle 0 \rangle \\ 0\dots0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Lemma 8. Γ_n *is an indecomposable K-representation of* $G = \langle a, b \rangle$.

Proof. Clearly Γ_n is a K-representation equivalent to the K-representation

$$\Gamma_n' = \begin{pmatrix} \rho_{p-1} + \cdots + \rho_2 & V_n' \\ 0 & \Delta_{n+1}' \end{pmatrix}, \tag{15}$$

of the group G, where Δ_{n+1}' differs from Δ_{n+1} only by the intertwining matrix $U' = (u_{ij}')$ (the notation for Δ_n, u_{ij} was introduced after Corollary 1, and that for Γ_0, before Lemma 5):

$$\begin{aligned} &u_{12}'(a) = u_{12}'(b) = J_{n+1}' \otimes \langle 1 \rangle, \\ &u_{22}'(a) = E_{n+1} \otimes \langle 1 \rangle, \\ &u_{11}'(b) = u_{22}'(b) = 0, \\ &u_{11}'(a) = u_{21}'(a) = -u_{21}'(b) = \begin{pmatrix} \tilde{0} & 0 \\ 0 & E_n \otimes \tilde{1} \end{pmatrix}. \end{aligned} \tag{16}$$

Moreover, in the representation Δ_{n+1}' there is a non-zero intertwining between the first γ_1 and the first γ_0: we have $u(a) = 0$, $u(b) = \langle 1 \rangle$. Note that we obtained Γ_n' from Γ_n by a permutation of the indecomposable components, and intertwining functions of Γ_n' were obtained from the corresponding ones of Γ_n. If Γ_n' is decomposable, then either the representations $\rho_{p-1}, \dots, \rho_2$ or their sum cannot be components in Γ_n'. Each of these representations has non-zero intertwining with γ_0, which cannot be changed without changing the zero intertwining for $\rho_{p-1}, \dots, \rho_2$. Thus if Γ_n' is decomposable then so is the representation Δ_{n+1}'.

However the Δ_{n+1}' of G is indecomposable. Indeed, the additive group of the intertwining functions for any pairs of different irreducible K-representations (10) of the group G is isomorphic to the additive group of the field $K_p = K/pK$. Any equivalence transformation (over K) acting on Δ_{n+1}' will change the intertwining functions of the different pairs of the irreducible components of Δ_{n+1}'. If we change the intertwining functions by elements of the field K_p then the effect on the functions is to change the elements of the field K_p. As a consequence the K-representation Δ_{n+1}' can be parametrized by the following matrix over K_p:

$$C = \begin{pmatrix} E_n' & J_{n+1}' \\ E_n' & E_{n+1} \end{pmatrix}, \quad \text{where } E_n' = \begin{pmatrix} 0 & 0 \\ 0 & E_n \end{pmatrix}.$$

We recall that the notation for Δ_n was introduced before Lemma 7 and in (15)–(16).

The representation Δ_{n+1}' is decomposable over K if and only if there exist matrices $S_i \in \mathrm{GL}(n+1, K_p)$ for $i = 1, \dots, 4$ such that

$$\begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}^{-1} C \begin{pmatrix} S_3 & 0 \\ 0 & S_4 \end{pmatrix} = \begin{pmatrix} E_n' & X \\ E_n' & E_{n+1} \end{pmatrix}, \tag{17}$$

where

$$X = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \tag{18}$$

is decomposable over K_p, and X_1, X_2 are square matrices.

Now suppose that Δ'_{n+1} is decomposable and satisfies (17)–(18). It follows that

$$S_1 = \begin{pmatrix} t_1^{-1} & 0 \\ * & S \end{pmatrix}, \quad S_2 = S_4 = \begin{pmatrix} t_2 & 0 \\ * & S \end{pmatrix},$$

where $t_1 t_2 \neq 0$, $S \in \mathrm{GL}(n, K_p)$ and

$$X = S_1^{-1} J'_{n+1} S_2 = T^{-1} Y T, \tag{19}$$

where

$$T = \begin{pmatrix} 1 & 0 \\ 0 & S \end{pmatrix}, \quad Y = \begin{pmatrix} t_1 \cdot t_2 & y_{12} \\ y_{21} & Y_n \end{pmatrix}. \tag{20}$$

Here Y has the following description: $t_1 \cdot t_2 \neq 0$, $y_{12} = (t_1, 0, \ldots, 0)$, y_{21} is a column vector of length n, Y_n is a matrix obtained from the matrix J'_n by changing the first column by a column vector over K_p. Since $y_{12}S \neq 0$ we cannot have $X_1 = t_1 t_2$, $X_2 = S^{-1} Y_n S$, where X_1, X_2 are defined in (18). Thus

$$S^{-1} Y_n S = \begin{pmatrix} * & 0 \\ 0 & X_2 \end{pmatrix}. \tag{21}$$

Let $\overline{K_p}$ be the algebraic closure of the K_p. The equivalence transformation with T over $\overline{K_p}$ given in (19) can be used to decompose further the matrix X so that X_2 splits into Jordan blocks over $\overline{K_p}$ (see (17)–(21)).

Of course, we can arrange that X_2 is $J_s(\alpha)$, the Jordan block with entries α on the main diagonal. We regard Y as a linear operator on the space $\overline{K_p}^{n+1}$ of column vectors. Thus from (18) it follows that X is the matrix of the operator Y in that basis of the space $\overline{K_p}^{n+1}$, consisting of the columns of the matrix T. The Jordan block $X_2 = J_s(\alpha)$ corresponds to the eigenvector $e \in \overline{K_p}^{n+1}$ of the operator Y; thus $Ye = \alpha e$. Since X_2 does not include the first column of X, the vector e is a column of T, different from the first column, i.e. the first component of e is equal to 0. Using the description of Y in (19), it is easy to show that the equation $Ye = \alpha e$ (with $\alpha \in \overline{K_p}$) is impossible for a vector $e = (0, \gamma_1, \ldots, \gamma_n)^T \neq 0$. This contradicts the decomposability of the K-representation Δ'_{n+1} of G and the lemma is proved.

Proof of Theorem 2. We can suppose that $M \subset F^{d_n}$. Clearly each K-basis in M is also an F-basis in F^{d_n} and an (F^+/K^+)-basis in $\hat{M} = F^{d_n +}/M^+$, where $d_n = \deg(\Gamma_n) = (3p-2)n + p^2$. Thus $\mathfrak{Crys}(G; M_n; T_n)$ has dimension equal to the degree d_n of the representation Γ_n, and since d_n is not bounded as a function of n the theorem is proved.

5 Proof of Theorem 3

We take $K = \mathbb{Z}$ and consider classical crystallographic groups. Let

$$A_4 = \langle a, b \mid a^2 = b^3 = (ab)^3 = 1 \rangle$$

be the alternating group of degree 4. We begin with the following $\mathbb{Z}$-representations of A_4:

$$\begin{aligned}
&\Delta_1\colon \quad a \to 1, && b \to 1;\\
&\Delta_2\colon \quad a \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, && b \to \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix};\\
&\Delta_3\colon \quad a \to \begin{pmatrix} 0 & -1 & 1 \\ 0 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix}, && b \to \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix};\\
&\Delta_4\colon \quad a \to \begin{pmatrix} 1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, && b \to \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.
\end{aligned}$$

Now consider the representations Δ, Γ_n defined by

$$\Delta = \begin{pmatrix} \Delta_3 & 0 & X_1 & X_3 \\ & \Delta_3 & X_2 & 0 \\ & & \Delta_2 & 0 \\ 0 & & & \Delta_4 \end{pmatrix}, \quad \Gamma_n = \begin{pmatrix} E_n \otimes \Delta_1 & U \\ 0 & E_n \otimes \Delta \end{pmatrix},$$

where

$$X_1(a) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \end{pmatrix}, \quad X_2(a) = \begin{pmatrix} 0 & 1 \\ -1 & 1 \\ -1 & 0 \end{pmatrix}, \quad X_3(a) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix},$$

$$X_i(b) = 0 \quad (i = 1, 2, 3),$$

$$U(a) = E_n \otimes \alpha + J_n(0) \otimes \beta, \quad U(b) = 0,$$

$$\alpha = (0, 0, 0, 0, 2, 0, 1, -1, 0, 0, 0), \quad \beta = (0, -2, 0, 0, 0, 0, 0, 1, -1, -1, 0),$$

and $J_n(v)$ is $n \times n$ the Jordan block with entries v on the main diagonal. It was proved in [8] that the representations $\Delta_1, \Delta_2, \Delta_3$ and Δ_4 are irreducible and Δ and Γ_n are indecomposable $\mathbb{Z}$-representations.

Let M_n be a $\mathbb{Z}$-module affording the representation Γ_n of A_4 consisting of column vectors of length d_n over $\mathbb{Z}$, where $\deg(\Gamma_n) = d_n = 12n$. It is easy to check that $f_n : A_4 \to \hat{M}_n$ defined by

$$f_n(a) = \big(\underbrace{0, \ldots, 0}_{n+3}, \tfrac{1}{2}, \tfrac{1}{2}, 0, \ldots, 0\big)^T + M_n, \quad f_n(b) = \big(\tfrac{1}{3}, 0, \ldots, 0\big)^T + M_n$$

is a 1-cocycle which is special. Therefore we obtain

Corollary 2. *The classical crystallographic group* $\mathfrak{Crys}(A_4; M_n; f_n)$ *is torsion-free.*

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