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PERCOLATION-BASED APPROACHES FOR RAY-OPTICAL PROPAGATION IN INHOMOGENEOUS RANDOM DISTRIBUTIONS OF DISCRETE SCATTERERES

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ABSTRACT

We address the problem of optical ray propagation in an inhomogeneous half-plane lattice, where each cell can be occupied according to a known one-dimensional obstacles density distribution. A monochromatic plane wave impinges on the random grid with a known angle and undergoes specular reflections on the occupied cells. We present two different approaches for evaluating the propagation depth inside the lattice. The former is based on the theory of the Martingale random processes, while in the latter ray propagation is modelled in terms of a Markov chain. A numerical validation assesses the proposed solutions, while validation through experimental data shows that the percolation model, in spite of its simplicity, can be applied to model real propagation problems.

1. INTRODUCTION

In principle, the electromagnetic wave propagation is fully described by the classic electromagnetic theory. However, in several fields of applied science, such as mobile communication, remote sensing and radar engineering, we often have to deal with very complex propagation media, whose characteristics fluctuate both in time and space. Accordingly, providing an analytical solution by applying the Maxwell equations turns out to be an impracticable or, at least, very time-expensive way to solve the problem and the use of approximate methods is mandatory.

In the framework of mobile communication, many site-specific numerical methods have been proposed (for a general overview, see [1] and references cited therein). Such solvers, normally based on a ray-tracing procedure [2], exhibit several limitations, mainly related to the required amount of processing time and information on the propagation environment. Moreover, such models are ad-hoc tailored on a specific area.

A complementary approach consists in developing random models of the area of interest, characterized by few meaningful parameters. Waves in such environments vary randomly in amplitude and phase and they are accordingly described in terms of statistical averages and probability densities. As far as electromagnetic propagation in urban areas and indoor environments is concerned, several models have been proposed (see as examples [3]-[5]). Such models do not provide very accurate propagation predictions but they turn out to be extremely simple and allow figuring out closed-form analytical formulas that mathematically explain how media affect propagation.

In such a framework, we consider the problem of the optical ray propagation in a half-plane percolation lattice [6]. The idea of applying such model for characterizing a medium of disordered scatterers was originally proposed in [7], where the urban environment has been described in terms of a uniform random lattice. Now, we focus on the inhomogeneous case, where each cell of the lattice can be occupied according to a known obstacles density distribution, $q_j = 1 - p_j$, j being the row index. Assuming grid cell dimension to be large compared to the wave-length, we model the incident plane wave impinging on the half-plane grid with an angle θ in terms of a collection of parallel rays that undergo specular reflections on occupied cells. In such a situation, our aim is estimating the probability $\Pr\{0 \to k\}$ that a ray penetrates up to a prescribed level *k* inside the lattice before being reflected back in the above empty half-plane, see Fig. 1.

We present and compare two different mathematical approaches, leading to closed-form analytical formulas. The former, presented in Section 2 and referred to as Martingale approach (MTG), is an extension to the non-uniform case of that proposed in [7]. Such an approach exhibits some limitations, especially when abrupt discontinuities in the density distributions of the obstacles are taken into account. In Section 3, we propose an alternative solution, i.e., an innovative approach based on the Markov chains theory [8] and accordingly referred to as Markov approach (MKV).

Selected numerical results, reported in Section 4, assess the effectiveness of the proposed methods, demonstrating that the MKV approach outperforms the MTG approach. Section 5 deals with experiments carried out in a real controlled environment, showing that in spite of its simplicity the percolation model can be applied to real propagation problems. Final comments and conclusions are drawn in Section 6.

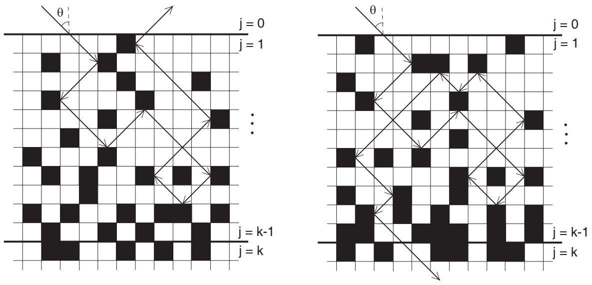

Figure 1. Examples of propagating rays in a random half-plane lattice.

2. MARTINGALE APPROACH

We model the propagation of a single ray as a realization of the stochastic process

$$r_n = \sum_{m=1}^{n} x_m, \quad n \geq 1, \qquad (1)$$

where r_n is the level where the n-th reflection takes place, completely defined by the succession of ray-jumps x_m, $m = 1, \ldots, n$.
Now, let us express $\Pr\{0 \to k\}$ in terms of the probability mass function of the first jump $\Pr\{r_1 = i\}$ and the probability $\Pr\{i \to k | i\}$ that the ray reaches level k before escaping from the grid given that it starts moving from level i. We get

$$\Pr\{0 \to k\} = \sum_{i=0}^{\infty} \Pr\{r_1 = i\} \Pr\{i \to k | i\}. \qquad (2)$$

As far as $\Pr\{r_1 = i\}$ is concerned, we easily obtain

$$\Pr\{r_1 = i\} = \begin{cases} q_1, & i = 0 \\ p_1 q_{e_i}^+ \prod_{j=1}^{i-1} p_{e_j}^+, & i \geq 0 \end{cases} \qquad (3)$$

In Eq. 3 $p_{e_j}^+ = p_j^{\tan\theta} p_{j+1} = 1 - q_{e_j}^+$ is the effective probability that a ray, travelling with positive direction, freely crosses level j and reaches next level j+1.
As far as the second term in Eq. 2 is concerned, in the trivial cases $r_1 = 0$ and $r_1 \geq k$ we get $\Pr\{i \to k | i\} = 0$ and $\Pr\{i \to k | i\} = 1$, respectively. When the first reflection takes place at a generic level between 0 and k a deeper analysis is needed. In [7], the authors applied the theory of the Martingale random processes [9] and the so-called Wald approximation, showing that under the assumptions of independent, identically distributed and zero-mean jumps,

$$\Pr\{i \to k | i\} = i/k, \quad 0 < i < k. \qquad (4)$$

A little thought shows that this means evaluating the unknown by using a distance criterion. As a matter of fact, $\Pr\{i \to k | i\}$ is assumed to be directly proportional to the distance between the level i where the first reflection takes

place and the absorbing level *k*. On the other hand, $\Pr\{i \to 0|i\}$, i.e., the probability that the ray escapes from the grid, is assumed to be directly proportional to *i*, the distance between the first reflection row and the above empty half-plane. Thanks to the mutual exclusivity of the two events, Eq. 4 follows.
Now, we observe that in the problem at hand if the ray is travelling with negative direction inside level 1, then it surely escapes from the grid, since there are not occupied horizontal faces between level 1 and level 0 (see left-hand side of Fig. 1). Accordingly, Eq. 4 is modified as follows

$$\Pr\{i \to k|i\} = \frac{i-1}{k-1}, \quad 1 < i < k. \qquad (5)$$

Clearly, Eq. 5 does not hold for the case $r_1 = 1$, that must be handled separately. We introduce the process

$$\tilde{r}_n = \sum_{m=1}^{n} \tilde{x}_m, \quad n \geq 1, \qquad (6)$$

where $\tilde{r}_n = r_{S+n}$, $S \in [0, \tan\theta + 1]$ being the number of ray jumps occurring in the first level, and we write

$$\Pr\{1 \to k|1\} = \sum_{i=2}^{\infty} \Pr\{\tilde{r}_1 = i|r_1 = 1\}\Pr\{i \to k|i\}. \qquad (7)$$

To evaluate the first term of Eq. 7, we observe that the ray reaches level 2 with at least one reflection in the first level if the vertical face between level 1 and 2 is free (event with probability p_2) and at least one of the $\tan\theta$ vertical cells in level 1 is occupied (event with probability $1 - p_1^{\tan\theta}$). According to such considerations, we can write

$$\Pr\{\tilde{r}_1 = i|r_1 = 1\} = \frac{p_1 p_2 \left(1 - p_1^{\tan\theta}\right)}{p_1 q_{e_1}^+} q_{e_i}^+ \prod_{j=2}^{i-1} p_{e_j}^+. \qquad (8)$$

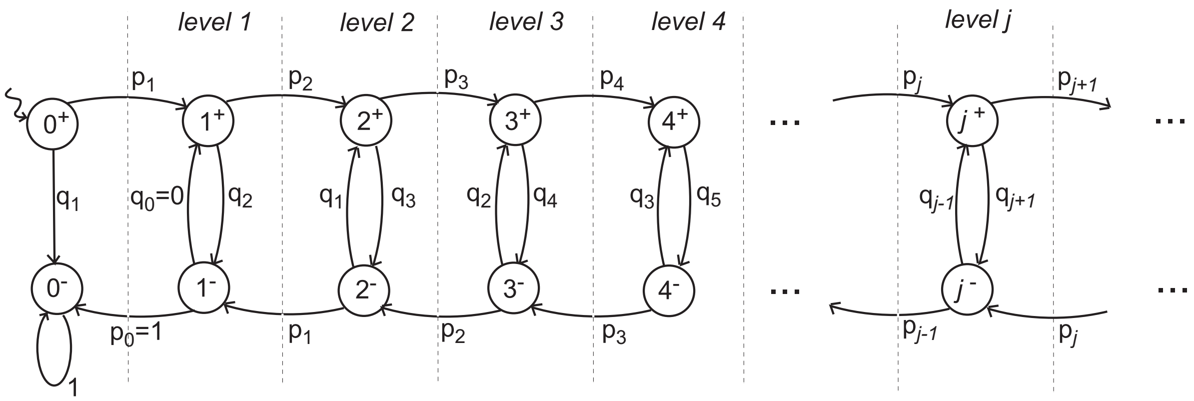

Figure 2. Markov Chain modeling ray propagation inside a non-uniform random lattice.

By substituting Eqs. 3, 5, and 7-8 into Eq. 2, after similar algebra as in [10] we obtain as final result

$$\Pr\{0 \to k\} = \begin{cases} p_1, & k = 1 \\ p_1 p_2 \left[\sum_{i=2}^{k-1} q_{e_i}^+ \left(\frac{i-1}{k-1} \right) \prod_{j=2}^{i-1} p_{e_j}^+ + \prod_{j=2}^{k-1} p_{e_j}^+ \right], & k \geq 2 \end{cases} \qquad (9)$$

In the particular case of a uniform random grid, Eq. 9 takes the form

$$\Pr\{0 \to k\} = \begin{cases} p, & k = 1 \\ p^2 \dfrac{1 - p_e^{k-1}}{q_e(k-1)}, & k \ge 2 \end{cases} \qquad (10)$$

that is a slightly different result than that in [7].

Some key-issues on the range of validity of Eqs. 9 and 10 must be pointed out. As stated before, in order to apply the Martingale random processes theory, ray jumps following the first one must be independent, identically distributed and zero-mean. By analyzing the probability mass function of the jumps $x_n, n \ge 2,$ it can be shown that this assumption holds true when either (*a.1*) $\theta \cong 45^o$ or (*a.2*) $n \to \infty$ and (*b*) if the occupancy profile does not present abrupt changes and significant variation in the levels of the lattice [10].

However, we can intuitively get the same conclusions by observing that ray jumps following the first one are considered as a unique mathematical entity, see Eq. 2, and by remembering that this quantity is evaluated by using a distance criterion. It is easy to guess that these assumptions do not allow to detect abrupt discontinuities and changes in the obstacles density profile and they satisfactorily perform when we consider symmetric configurations, where the ray hits vertical and horizontal faces with the same probability ($\theta \cong 45^o$) or the grid is so dense that a large number of reflections is required before the ray reaches level *k* or it escapes from the grid.

3. THE MARKOV APPROACH

We proceed by transforming the two-dimensional ray propagation problem in a simple one-dimensional random walk problem that does not depend on the incidence angle θ.

Firstly, we observe that whenever a ray hits an occupied vertical face, its vertical direction of propagation does not change. Thus, since our aim is estimating the propagation depth, we just consider reflections on horizontal faces. Now, we assume that the ray never crosses cells that it has already gone through. Accordingly, propagation in the vertical direction occurs with steps that are independent of each other. In particular, a ray travelling into a generic level *j* either remains in the same level, changing its direction of propagation, or it reaches a new level, keeping the same direction of propagation. If the ray is travelling with positive (negative) direction, such events occur with probability q_{j+1} (q_{j-1}) and p_{j+1} (p_{j-1}), respectively. This situation is formally described by the Markov Chain [8] depicted in Fig. 2, where states j^+ and j^- denote a ray travelling in level *j* with positive and negative direction, respectively.

We state our main result (see [11] for a detailed proof) as follows,

$$\Pr\{0 \to k\} = \frac{p_1 p_2}{1 + p_1 p_2 \sum\limits_{i=3}^{k} \dfrac{q_i}{p_i p_{i-1}}}. \qquad (11)$$

In the case of a uniform grid, Eq. 11 reduces to

$$\Pr\{0 \to k\} = \frac{p^2}{(k-2)q + 1}. \qquad (12)$$

We note that Eqs. 11 and 12 are much simpler than the corresponding Eqs. 9 and 10, being independent on the incident angle θ.

As far as the range of validity of Eqs. 11 and 12 is concerned, some observations are appropriate. In order to describe ray-propagation by means of the Markov Chain depicted in Fig. 2, we have assumed that the ray does not go through already encountered cells. Cleary, such an assumption does not always hold true. In particular, we expect that Eq. 11 is more accurate as the obstacles are more sparse and the incident angle θ is closer to 45^o. As a matter of fact, when

θ is far from 45^o, the ray is more likely to cross the same cells whenever a reflection occurs, while if the lattice is dense, repeated reflections over the same few cells occur with a high probability.

4. NUMERICAL VALIDATION

In order to validate and compare the proposed solutions, an exhaustive set of numerical experiments has been carried out. As a reference, the propagation depth has been estimated in the first $K = 32$ levels by computer-based ray-tracing experiments [7].
We define prediction error δ_k and mean error $\langle \delta \rangle$ as

$$\delta_k = \frac{\left| \Pr_R \{0 \to k\} - \Pr_P \{0 \to k\} \right|}{\max_k \left[\Pr_R \{0 \to k\} \right]} \times 100 , \qquad (13)$$

$$\langle \delta \rangle = \frac{1}{K} \sum_{k=1}^{K} \delta_k , \qquad (14)$$

where the sub-scripts *R* and *P* indicate the values estimated with the reference approach and by means of either Eq. 9 or Eq. 11, respectively.
The first test case is aimed at analyzing the role of the incidence angle and of the scatterers density in determining the solution accuracy. Towards this end, we consider two homogenous occupancy profiles, the first modelling very sparse media ($q = 0.05$) and the other very dense ones ($q = 0.35$). With reference to Figs. 3 and 4, we observe that the accuracy of the MTG approach increases when we consider occupancy probability values approaching the percolation threshold [6] (for instance, when $\theta = 45^o$ and $\theta = 75^o$, we get $\langle \delta \rangle_{q=0.05} = 3.87\%$ vs. $\langle \delta \rangle_{q=0.35} = 0.56\%$ and $\langle \delta \rangle_{q=0.05} = 19.57\%$ vs. $\langle \delta \rangle_{q=0.35} = 1.45\%$, respectively). As a matter of fact, in high dense media a ray tends to be reflected many times before reaching level *k* or escaping from the grid. On the other hand, the MKV approach describes very well the propagation when *q*=0.05 (for $\theta = 45^o$ and $\theta = 75^o$ we get $\langle \delta \rangle = 0.23\%$ and $\langle \delta \rangle = 0.8\%$, respectively), while performances get worse for *q*=0.35 ($\langle \delta \rangle = 2.2\%$ for $\theta = 45^o$ and $\langle \delta \rangle = 3.99\%$ for $\theta = 75^o$).

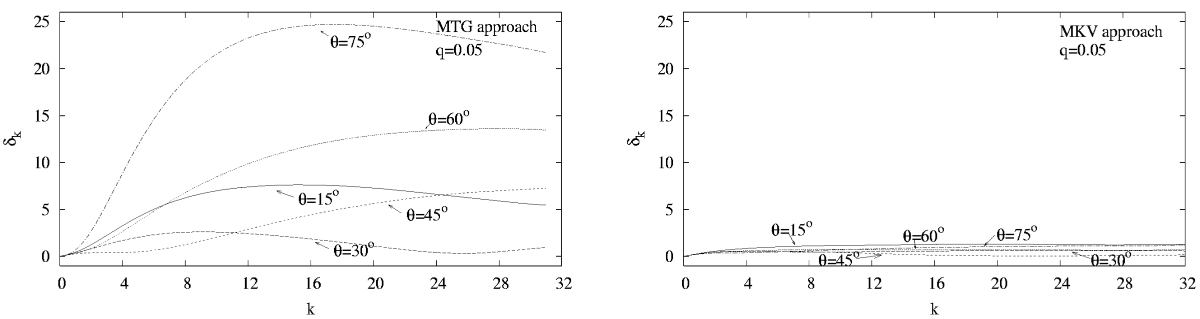

Figure 3. Uniform random lattice with q=0.05: prediction error δ_k versus k for different θ values.

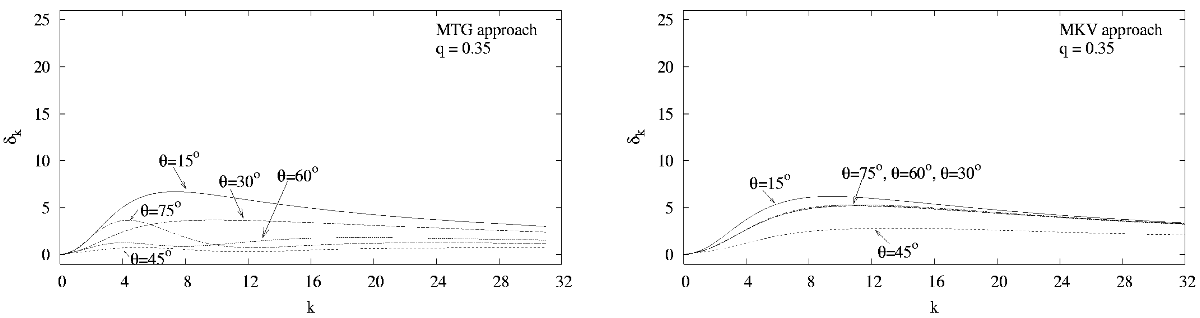

Figure 4. Uniform random lattice with q=0.35: prediction error δ_k *versus k for different* θ *values.*

Other relevant observations are the following. Both approaches get the best results when $\theta = 45^o$. For *q*=0.05 the MTG approach evidently outperforms the MKV approach, while when *q*=0.35 the performances are comparable. Finally, the MKV approach is more stable with respect to both the incidence angle θ and the lattice depth *k*, especially for low *q* values.

A second test case is devoted at showing the role of the slope intensity and discontinuities in the obstacles density profile. Towards this end, we fix the incidence angle to $\theta = 45^o$ and we consider two representative profiles. The first one is a double-exponential profile

$$q_j = \begin{cases} 0.4\exp\left[(j-16)\cdot 13.86\cdot 10^{-2}\right], & j \le 16 \\ 0.4\exp\left[(16-j)\cdot 13.86\cdot 10^{-2}\right], & j > 16 \end{cases} \quad (15)$$

with parameters chosen in order to get with a significant variation along the lattice depth, i.e, $q_1 = q_{31} = 0.05$ and $q_{16} = 0.4$. The second one is a step-like profile that presents an abrupt discontinuity,

$$q_j = \begin{cases} 0.05, & j \le 8 \\ 0.35, & j > 8 \end{cases} \quad (16)$$

The plots of the obtained results are shown in Figs. 5 and 6. It is evident that the MKV approach performances do not depend on significant variations of the density profile along the lattice levels and they are not even affected by discontinuities. As a matter of fact, the mean errors obtained in correspondence with the step profile ($\langle \delta \rangle = 1.64\%$) and the exponential profile ($\langle \delta \rangle = 1.09\%$) are lower than that of the uniform profile with $q = 0.35$ ($\langle \delta \rangle = 2.2\%$). On the contrary, the MTG approach completely fails ($\langle \delta \rangle = 11.28\%$ and $\langle \delta \rangle = 17.86\%$ for the exponential and the step, respectively), since in such a case the ray jumps following the first one are considered as a single mathematical entity, while in the MKV approach each single vertical jump is taken into account.

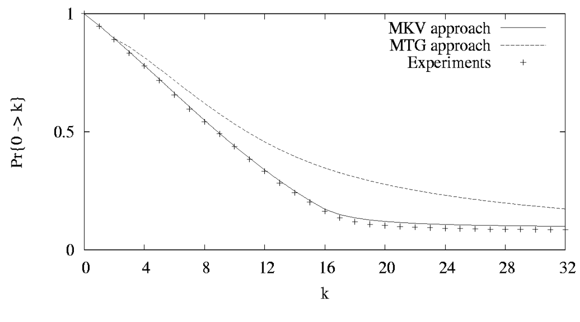

Figure 5. Double-exponential profile: $\Pr\{0 \to k\}$ *versus k for* $\theta = 45^o$.

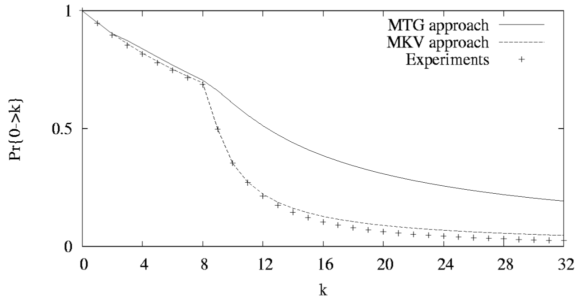

Figure 6. Step-like profile: $\Pr\{0 \to k\}$ *versus k for* $\theta = 45^o$.

5. EXPERIMENTAL VALIDATION

In order to assess the applicability of the percolation model to real propagation problems, our results have been compared to experimental data collected in an anechoic chamber. We have placed polystyrene cylinders filled with a metallic film in a grid of K=10 rows and *I*=10 columns, realizing S=20 different grid configurations described by the following linear obstacles' density profile

$$q_j = 0.1 + 3.125 \cdot 10^{-3} j \qquad (17)$$

and we have illuminated the scenario with a horn antenna. Detailed description of the experimental setup is provided in [10]. We explicitly note that Eq. 17 is representative of a slowly variable profile neither too dense nor too sparse, and thus, both MTG and MKV are expected to satisfactorily perform.

For comparison purposes, we fit our probability distributions to actual measurements data as follows

$$\langle PL(k) \rangle = -10 \log_{10} \left[\frac{1}{P_T} \left\langle P_R^{fs}(k) \right\rangle \Pr\{0 \to k\} \right] [dB], \qquad (18)$$

where P_T is the transmitted power and $\left\langle P_R^{fs}(k) \right\rangle$ is the free space received power at row *k* averaged with respect to *I*. On the other hand, we can estimate the expected path loss from the collection of measurements performed when the cylinders are placed in the grid

$$\langle PL(k) \rangle = -10 \log_{10} \left[\frac{1}{P_T} \frac{1}{IS} \sum_{i=1}^{I} \sum_{s=1}^{S} P_R^m(k,i;s) \right] [dB], \qquad (19)$$

$P_R^m(k,i;s)$ being the power measured at site (*k,i*) in the *s*-th grid realization.

The results obtained by applying Eqs. 18 and 19 are plotted in Fig. 7. Whatever the approach, we observe a good fitting between theoretical and reference data, despite inevitable measurements inaccuracies and the experimental setup that turns out to be very different from the ideal geometry assumed in deriving Eqs. 9 and 11 [10].

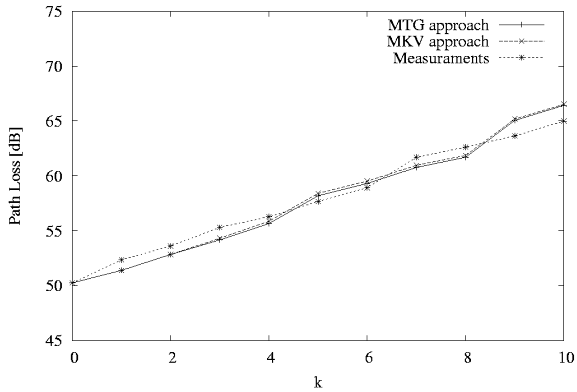

Figure 7. Estimated and reference path loss values.

6. CONCLUSIONS

We have studied ray propagation in inhomogeneous half-plane random lattices by presenting two different analytical approaches, the first one based on the theory of the Martingale random processes and the second one modeling ray propagation in terms of a Markov chain. Both mathematical considerations and numerical Monte-Carlo-like experiments have shown that the MKV approach outperforms the MTG approach, the latter one working better only in correspondence with highly dense and slowly variable profiles. Finally, a validation through experimental data collected in a real controlled environment has assessed the applicability of the percolation model and of the assumed propagation mechanisms to real propagation problems.

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