A NOTE ON AN ADDITIVE PROPERTY OF PRIMES

LUCA GOLDONI

ABSTRACT. An elementary construction is given of an infinite sequence of natural numbers, having at least two different decompositions as sum of primes and no prime number appears in more than one of them.

1. INTRODUCTION

Prime numbers are the "building blocks" for the multiplicative structure of the set of all natural numbers. They arise in a very natural way when one deals with factorization of integers and one of the most important theorem about them is the so called "Fundamental theorem of Arithmetic"(FTA) witch states that each natural number $n > 1$ can be written in an essentially unique way as a product of distinct primes factors. But what is the behavior of primes in the " additive context"? It is easy to check that a theorem like FTA can not be true. In fact all the integers of the form $p + 5$, being p a prime, can be written as $p + 2 + 3$ as well. In this way one provides an infinite family of integers admitting at least two different decompositions. But in all the numbers is present a "fixed" summand: the number 5. What if fixed summands are not allowed? Does there exist an infinite family of integers admitting multiple "essentially different" decomposition? Here by "essentially different" we mean that the primes used in the decompositions of one of these integers are not used in the decompositions of the others ones. A positive answer to the question can be given with the use of standard tools of Analytic Number Theory. Using Chebishev bounds of $\pi(x)$ it is possible to prove the existence of such a family of integers. The aim of this note is just to give an elementary construction of such a sequence, requiring only very basic notions of Calculus. To this purpose the proof is based on a combinatorial result by Erdos and Benkoski [1], that in spite its depth, admits a nice proof using nothing more than High School algebra.

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Dipartimento di matematica. Università di Trento. Dottorato di ricerca XXIII ciclo.

2. The main result

Theorem 1. *There exist an infinite set* $A = \{a_1, a_2, \cdots a_n \cdots\} \subseteq \mathbb{N}$ *such that for every* $a_n \in A$ *there exist two different sets* A_n,B_n *of primes such that*$A_n \cap B_n = \emptyset$ *and*

$$a_n = \sum_{p \in A_n} p$$

$$a_n = \sum_{p \in B_n} p$$

Moreover, if $D_n = A_n \cup B_n$ *then* $D_n \cap D_m = \emptyset \ \forall n \neq m$

The last claim tell us that all the primes used for a_n are smaller than the primes used for a_{n+1} so the representations are really different. For the sake of completion it will be given even an proof based on standard techniques of Analytic Number Theory witch ultimately can be regarded as a corollary of a work of Chebyshev [2]. He proved namely the following theorem:

Theorem 2 (Chebyshev bounds)**.** *If* $\pi(x)$ *denotes the number of primes less or equal to a given real number* x*, then there are two positives constants, a and A such that*

$$\frac{ax}{\log x} \leqslant \pi(x) \leqslant \frac{Ax}{\log x}$$

for all x *large enough.*

On the basis of this theorem an analytical proof runs as follow:

Proof. Let x large enough,then if we consider the interval $[x/100, x/2]$ the number of primes in such interval is

$$\pi(x/2) - \pi(x/100) \gg \frac{x}{\log x}$$

as follows from Chebyshev's bounds. If P denotes the set of all primes and

$$A = \{(p, q) : p, q \in \mathsf{P}, \quad p < q, \quad p,\ q \in [x/100, x/2]\}$$

we have

$$|A| \gg \frac{x^2}{\log^2 x}$$

But for each $(p, q) \in A$ it is $p + q \leq x$ so ,for x large enough, not all sums can be distinct, therefore there must exist two different pairs of primes having the same sum. Varying x, we get infinitely many such examples. ∎

As one can see, it is possible achieve very quickly the main result along this way, but it require the knowledge of the Chebyshev's bounds and even though they are not too difficult to prove, they requires a quite long preparatory work.

3. The pillar of the elementary proof

In 1974 P.Erdös and S. Benkoski found the following result:

Lemma 1 (Erdös-Benkoski)**.** *Let A a finite set of positive integers such that for all B,$C \in \mathcal{P}(A)$*

$$B \neq C \Rightarrow \sum_{x \in B} x \neq \sum_{y \in C} y$$

then

$$\sum_{a \in A} \frac{1}{a} < 2$$

where $\mathcal{P}(A)$ denotes the set of all subsets of a given set.

Loosely speaking one can see a "competition " between the number and the size of elements of the set A and the fact that all the possible subsets must have different sums. This theorem can be proved,in various ways. One of them, as one can find in [1], uses High School Algebra only, and so it can be regarded as fully elementary.

4. The elementary proof of Theorem 1

Proof. One starts with the a well known result related with the series of the reciprocals of primes. This series as Euler proved is a divergent series. That means

$$\sum_{p \in P} \frac{1}{p} = +\infty$$

so we can find a first prime p_1 such that

$$\sum_{\substack{p \in P \\ 2 \leq p \leq p_1}} \frac{1}{p} \geq 2 \tag{1}$$

Let

$$S_1 = \{2, 3 \cdots p_1\}$$

If we consider all the subsets of S_1 we notice that it's not possible for them to have all different sums because if so, by Lemma (1)

$$\sum_{\substack{p \in P \\ 2 \leq p \leq p_1}} \frac{1}{p} < 2$$

against (1). So there are two different subsets of S_1, say A_1 and B_1 such that

$$\sum A_1 = \sum B_1$$

Let

$$a_1 = \sum A_1 = \sum B_1$$

Now, we choose a prime q_1 such that $q_1 > a_1$ and then the first prime p_2 such that

$$\sum_{\substack{p \in P \\ q_1 \le p \le p_2}} \frac{1}{p} \ge 2$$

With the same argument as before, we find two different subsets A_2 ,B_2 of

$$S_2 = \{q_1 \cdots p_2\}$$

such that

$$\sum A_2 = \sum B_2$$

So we can choose

$$a_2 = \sum A_2 = \sum B_2$$

and so on.

If one define $A = \{a_1, a_2 \cdots\}$ one has that A is a set with the property of Theorem 1 and by construction it is plain that the sets $D_n = A_n \cup B_n$ are pairwise disjoint. If a_n is expressed in two different ways as sum of primes, obviously, all the common summands can be deleted, obtaining a new a'_n witch has two different representations without any prime in common. In other words it is always possible think that A_n e B_n are disjoint sets.

$\square$

Note 1. *We can formulate Theorem 1 using odd primes only.In this case for every n we have that the numbers of elements of A_n and of B_n have the same parity.*

Figure 1 illustrates the situation. Here

$$a_n = \begin{cases} p_{r_1} + p_{r_2} + p_{r_3} \\ q_{r_1} + q_{r_2} + q_{r_3} \end{cases}$$

and

$$a_{n+1} = \begin{cases} p_{s_1} + p_{s_2} + p_{s_3} \\ q_{s_1} + q_{s_2} + q_{s_3} \end{cases}$$

Of course the draw is not in scale, but only suggests the meaning of the main result.

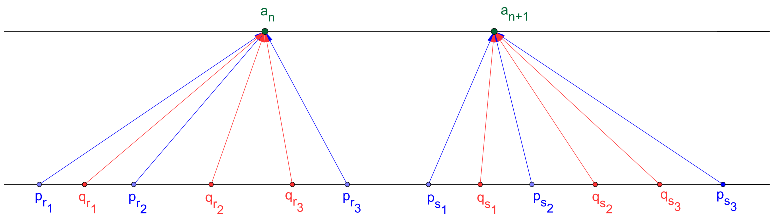

Figure 1

References

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Università di Trento, Dipartimento di Matematica, v. Sommarive 14, 56100 Trento, Italy
E-mail address: `goldoni@science.unitn.it`