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VECTOR BUNDLES ON ALGEBRAIC CURVES: HOLOMORPHIC TRIPLES IN LOW GENUS

STEFANO PASOTTI

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ADVISOR:
Prof. EDOARDO BALLICO

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Agli open spaces,
quelli di oggi e quelli di ieri.

Contents

Introduction

Let X be a smooth and projective curve over an algebraically closed field $\mathbb{K}$ of characteristic 0. A *holomorphic triple* on X is a triple (E_1, E_2, φ), where E_1 and E_2 are vector bundles on X and φ is a morphism in $\mathrm{Hom}(E_2, E_1)$.

The study of holomorphic triples has been started by Bradlow and García-Prada in [11, 7] where the authors deal with the search for solutions to some gauge theoretic equations on X obtained by dimensional reduction of the Hermitian-Einstein equation on $X \times \mathbb{P}^1$.

For those objects it is possible to introduce a notion of stability, depending on a real parameter α, and consequently to deal with the moduli spaces of α-stable triples. These spaces depends on some parameters, as the ranks and degrees of the vector bundles involved in the definition of holomorphic triple, besides of course α itself. The main properties of these triples and of their moduli spaces have been further investigated by the same authors with Gothen in [8] and, more recently, in [9] and in [12] by García-Prada, Hernández Ruipérez, Pioli and Tejero Prieto. Many different problems have been faced and solved in the papers cited above, in particular the authors have shown that some constraints on the parameter α must exist in order for a triple to be α-stable. Moreover, when for some fixed values of the parameters the moduli space is non empty, irreducibility and smoothness is proved and a computation for the dimension is provided. Several techniques are used to achieve these results, but probably one of the most valuable is that of *flips*. This method, which takes advantage of the theory of deformations and extensions for triples, takes care of how the α-stability of a holomorphic triple varies for fixed ranks and degrees as α varies in the admissible range, hence with this tool it is possible to prove non-emptiness of the moduli spaces for particular and convenient values of α (usually for "large" values of α), and then to extend this result also to the remaining cases.

However, although the general theory developed in the aforementioned papers is mainly independent of the genus g of the curve X, in fact for some particular results it is necessary to require that $g \geqslant 2$, since the cases of the projective line ($g = 0$) and of elliptic curves ($g = 1$) deserve some special treatment. This is due mainly to the fact that in these last two cases stable and semistable vector bundles are rare to appear, hence the would-be solver has to face the problem of the lack of "good" objects to use for building holomorphic triples for which it is easier to prove α-stability. Moreover in low genus the technique of flips described above, even if it is still sensible, unfortunately fails to provide useful information, thus problems like non-emptiness and irreducibility of moduli spaces in these cases are still open.

The aim of this work is then to try to answer some of the questions here mentioned precisely in the two cases of the projective line and of elliptic curves. While we emerge victorious from the challenge with elliptic curves (and in fact from the results we obtain we are also able to prove analogous results for bielliptic curves) we can obtain only partial results for the projective line, which hence deserves some more attention to be rendered in future works.

More in details in **Chapter 1** we collect the preliminary definitions and results concerning holomorphic triples and we present the main results on their moduli spaces. Here we discuss the constraints on the range of admissible α in order for α-stable triples to exist, we note that even if α is a real parameter, in fact only a finite number of different moduli spaces exists, and we introduce the general technique of flips. Also using this technique it is possible to prove several results on the moduli spaces of holomorphic triples, such as irreducibility, smoothness and dimension computation.

Chapter 2 introduces the notion of *coherent system* (namely a pair (E, V) where E is a vector bundle on X and V is a vector subspace of the vector space of global sections of E), presents the main theory of these objects and stresses the relationships between holomorphic triples and coherent systems.

The motivations for this introduction are that the situation for those new objects is analogous to that of triples: the general theory has been developed since about 1993 in several papers, but the cases $g = 0$ and $g = 1$ deserve a special treatment. For coherent systems the theory has been improved recently to cover also these last two cases by Lange and Newstead in the two papers [15, 16] (and in fact further developed by the same authors in [17, 18] in the case of curves of genus 0, since the results so far known are not yet exhaustive). The results we can obtain for holomorphic triples are analogous to the results obtained in the previously cited papers, and summarized in an appropriate section of this chapter, hence our interest in them.

Here we discuss briefly also a way of seeing both holomorphic triples and coherent systems as particular *augmented bundles*, that is as particular objects made up with one or more vector bundles on X with some extra data of some kind (prescribed sections, a map,...). A more comprehensive introduction to this point of view can be seen in [5], where also other classes of augmented bundles are taken into consideration.

Chapter 3 is devoted to study the particular case of holomorphic triples on the projective line. This is probably the most difficult case we take into consideration, an evidence of this being the fact that the results here obtained are not completely exhaustive, hence not as good as one would dream. In fact we are able to obtain some necessary conditions for non-emptiness, but in general not sufficient conditions. Some particular cases are considered (namely those corresponding to some particular values for the ranks and degrees of the two vector bundles) and more precise answers are provided with these extra hypotheses. The cases $n_2 = 1$ and $n_2 = 2$ are in fact completely solved and reveal that the necessary conditions previously proved for the projective line are in fact also sufficient, but, so far, it is not known whether this is true in general. The results presented in this Chapter have been obtained in collaboration with Francesco Prantil of the University of Trento; our main results are the existence of some stronger constraints on the values of the parameter α in order for α-stable triples to exist (see Propositions 3.2.1 and 3.2.2), the proof of some properties of the general α-stable triple (Theorems 3.3.6 and 3.3.12) and the existence results already mentioned (see Section 3.4).

Chapter 4 deals with the study of the particular case of holomorphic triples on elliptic and bielliptic curves. The results here obtained are in some sense more interesting, since for $g = 1$ it is possible to provide necessary and sufficient conditions for the moduli spaces of α-stable triples to be non empty, and in these cases some further properties of the moduli spaces, such as irreducibility, can be shown. The next natural step is to consider bielliptic curves, that is curves X which are a double covering through a map f of an elliptic curve C, to extend (hopefully!) the results on the moduli spaces also to this case. In fact it turns out that α-stability behaves well in respect with the double covering map, and hence the main properties of the moduli spaces of elliptic curves are still true for bielliptic ones. Here we consider also *elementary transformations* and investigate how these transformations make the α-stability of a triple worse. Also in this case one finds out that the elementary transformation of a sufficiently general α-stable triple is still α-stable. The results presented in this Chapter have been obtained in collaboration with Edoardo Ballico and Francesco Prantil of the University of Trento; the main results are summarized in the following Theorems.

Theorem 1 (Thms 4.4.6 and 4.4.8). *Let E_1, E_2 be polystable vector bundles with* $\operatorname{rank} E_2 < \operatorname{rank} E_1$*, and $\mu(E_2) < \mu(E_1)$. Then there exists a homomorphism $\varphi \in \operatorname{Hom}(E_2, E_1)$ such that the triple $\mathcal{T} = (E_1, E_2, \varphi)$ is α-stable for any $\alpha \in (\alpha_m, \alpha_M)$. Moreover $\mathcal{N}_\alpha(n_1, n_2, d_1, d_2)$ is irreducible, smooth of dimension $-n_1 d_2 + n_2 d_1 + 1$, for every $\alpha \in (\alpha_m, \alpha_M)$.*

Theorem 2 (Thms 4.5.4, 4.5.5 and 4.5.10). *Let* $\alpha \in \mathbb{R}$*,* C *be an elliptic curve,* $f : X \longrightarrow C$ *a double covering with* X *a smooth curve of genus* $g \geqslant 2$*,* $\sigma : X \longrightarrow X$ *the involution and* (E_1, E_2, φ) *an* α*-stable triple on* C *with* E_1 *and* E_2 *polystable vector bundles with pairwise non-isomorphic indecomposable direct factors. Then the triples* $(f^*(E_1), f^*(E_2), f^*(\varphi))$*,* $(F_1', f^*(E_2), f^*(\varphi))$ *and* (F_1', F_2', ψ') *are* 2α*-stable, where* F_1' *and* F_2' *are obtained from* $f^*(E_1)$ *and* $f^*(E_2)$ *making a general positive elementary transformation supported in a point* $p \in X$ *where* f *is not ramified.*

Chapter 5, in the end, presents some results on coherent systems, in particular on elliptic and bielliptic curves and on the projective line. Using as main tools the results of Lange and Newstead [15, 16] some spannedness-like properties are proved for sufficiently general σ-stable coherent systems. An existence result is also proved for curves of any genus g using a dimensional estimation provided by the analysis of rational curves in Grassmannians and of their Plücker embeddings performed by Ballico in [3]. This can be seen as a first step of a longer path in the direction of those values of the parameters which are not yet covered by the general theory summarized in Chapter 2.

Acknowledgements

*There are many people I would like to mention here, but the time is short and the deadline for the presentation is too close, and, you know, it is not my style to do such things (*any *thing?) hurriedly, so my choice is to thank these people directly face to face: all of you, expect to be thanked in a moment or another.*

But there are two people I absolutely want to mention. A good story needs both a good plot and a good storyteller. I do not know if I belong to this second category, and it is not my duty to determine whether the plot is good (or at least good enough), but if it is so, undoubtedly this is due also to Edoardo Ballico, who during these last years has been a constant reference point with his advice and his acute insight, and to Francesco Prantil who shares with me burdens and honours of producing mathematics in the last two years. Thank you.

Chapter 1

Holomorphic triples and their moduli space

In this Chapter we recall the definitions of holomorphic triple on a curve X and α-(semi)stability and collect some general results on α-stable triples and on their moduli spaces. The results here presented are independent of the genus g, only in the last section, where we discuss the main properties of the moduli spaces of holomorphic triples, we will require $g \geqslant 2$, since some results regarding non-emptiness and irreducibility rely on some dimensional estimations which in fact do not work for $g = 0$ or $g = 1$. For more details on the result here collected refer e.g. to [5, 7, 8].

1.1 Setting

Let X be a smooth and projective curve over an algebraically closed field $\mathbb{K}$ of characteristic 0.

Definition 1.1.1. *We call* holomorphic triple *over the curve X a triple $\mathcal{T} = (E_1, E_2, \varphi)$ where E_1 and E_2 are two vector bundles on X and $\varphi : E_2 \to E_1$ is a holomorphic map between them. If* $\mathrm{rank}(E_i) = n_i$ *and* $\deg(E_i) = d_i$, *$i = 1, 2$ we say that $\mathcal{T}$ is of* type (n_1, n_2, d_1, d_2).

If $\mathcal{T}$ and $\mathcal{T}'$ are two triples on X, then a homomorphism from $\mathcal{T}$ to $\mathcal{T}'$ is made up of two maps ψ_1 and ψ_2 such that the following diagram commutes:

$$\begin{array}{ccc} E_2 & \xrightarrow{\varphi} & E_1 \\ {\scriptstyle\psi_2}\uparrow & & \uparrow{\scriptstyle\psi_1} \\ E_2' & \xrightarrow{\varphi'} & E_1'. \end{array}$$

In order to introduce a notion of stability for holomorphic triples we need to precise what are sub-objects and to give a definition of slope which, differently from the case of vector bundles, depends on a parameter α.

Definition 1.1.2. *Let $\mathcal{T}$ be a triple on the curve X. A* subtriple *of $\mathcal{T}$ is a triple $\mathcal{T}' = (E_1', E_2', \varphi')$ such that for $i = 1, 2$, E_i' is a subbundle of E_i and the following diagram commutes:*

$$\begin{array}{ccc} E_2 & \xrightarrow{\varphi} & E_1 \\ \uparrow & & \uparrow \\ E_2' & \xrightarrow{\varphi'} & E_1'. \end{array}$$

The subtriple $(0,0,0)$ is the trivial *subtriple. A subtriple $\mathcal{T}'$ is* proper *if $\mathcal{T}' \neq (0,0,0)$ and $\mathcal{T}' \neq \mathcal{T}$.*

Note that, in general, it would be possible to let E_1' and E_2' be coherent subsheaves of E_1 and E_2, but when dealing with stability criteria it is sufficient to consider only saturated subsheaves and in our situation those are exactly the subbundles. As we said earlier, differently from the case of vector bundles, the definition of stability for holomorphic triples that we are going to introduce depends on a real parameter, thus there is (a priori) a 1-parameter family of stability criteria for triples.

Definition 1.1.3. *Let $\mathcal{T} = (E_1, E_2, \varphi)$ be a triple on X and $\alpha \in \mathbb{R}$. The* α-degree *of $\mathcal{T}$ is*

$$\deg_\alpha(\mathcal{T}) := \deg(E_1) + \deg(E_2) + \alpha \operatorname{rank}(E_2),$$

and the α-slope *of $\mathcal{T}$ is*

$$\mu_\alpha(\mathcal{T}) := \frac{\deg_\alpha(\mathcal{T})}{\operatorname{rank}(E_1) + \operatorname{rank}(E_2)} = \mu(E_1 \oplus E_2) + \frac{\operatorname{rank}(E_2)}{\operatorname{rank}(E_1) + \operatorname{rank}(E_2)}.$$

In the remainder we will always write $n_i = \operatorname{rank}(E_i)$, $d_i = \deg(E_i)$, $n_i' = \operatorname{rank}(E_i')$ and $d_i' = \deg(E_i')$, $i = 1, 2$.

Definition 1.1.4. *A holomorphic triple $\mathcal{T}$ is said to be* α-stable *(resp.* α-semistable*) if, for all proper subtriples $\mathcal{T}'$ of $\mathcal{T}$, $\mu_\alpha(\mathcal{T}') < \mu_\alpha(\mathcal{T})$ (resp. $\mu_\alpha(\mathcal{T}') \leqslant \mu_\alpha(\mathcal{T})$). $\mathcal{T}$ is said to be* α-polystable *if it is the direct sum of α-stable triples of the same α-slope.*

Remark 1.1.5. Equivalently the notion of (semi)stability for triples can be defined as follows. Let $\mathcal{T}'$ be a subtriple of $\mathcal{T}$ and $\tau \in \mathbb{R}$ and write

$$\theta_\tau(\mathcal{T}') := \mu(E_1' \oplus E_2') - \tau - \frac{n_2'(n_1 + n_2)}{n_2(n_1' + n_2')}\big(\mu(E_1 \oplus E_2) - \tau\big).$$

Then the triple $\mathcal{T}$ is *τ-(semi)stable* if, for any proper subtriple $\mathcal{T}'$, $\theta_\tau(\mathcal{T}') < 0$ (resp. $\theta_\tau(\mathcal{T}') \leqslant 0$).

It is an easy computation to show that a triple $\mathcal{T}$ is α-(semi)stable if and only if it is τ-(semi)stable and α and τ are related by

$$\alpha = \frac{n_1 + n_2}{n_2}\big(\tau - \mu(E_1 \oplus E_2)\big) \qquad \tau = \mu_\alpha(\mathcal{T}).$$

In the following we will mainly use the former definition, but of course any result we will prove can be rephrased in terms of the latter.

If $\mathcal{T}$ is a holomorphic triple, for any subtriple $\mathcal{T}'$ and for any $\alpha \in \mathbb{R}$ it is sometimes convenient to write

$$\Delta_\alpha(\mathcal{T}', \mathcal{T}) := \mu_\alpha(\mathcal{T}') - \mu_\alpha(\mathcal{T})$$

and

$$\Delta_\alpha(\mathcal{T}) := \max_{\substack{\{\mathcal{T}' \text{ proper} \\ \text{subtriple of } \mathcal{T}\}}} \Delta_\alpha(\mathcal{T}', \mathcal{T}),$$

hence $\mathcal{T}$ is α-(semi)stable if and only if $\Delta_\alpha(\mathcal{T}) < 0$ (respectively $\Delta_\alpha(\mathcal{T}) \leqslant 0$).

In some particular cases the notion of stability is particularly simple, for example when the map φ is the zero homomorphism.

Example 1.1.6 ([7, Lm. 3.4])**.** Let $\mathcal{T} = (E_1, E_2, \varphi)$ be a holomorphic triple and assume that $\varphi = 0$. Then $\mathcal{T}$ is α-semistable if and only if $\alpha = \mu(E_1) - \mu(E_2)$ and both E_1 and E_2 are semistable bundles. $\mathcal{T}$ cannot be α-stable for any α. For note that in this case the subtriples of $\mathcal{T}$ are of the form $(E_1', E_2', 0)$, where E_1' and E_2' are any subbundles of E_1 and E_2 respectively. In particular we can consider the proper subtriples $(E_1, 0, 0)$ and $(0, E_2, 0)$. The α-semistability conditions for these triples are respectively

$$\frac{d_1}{n_1} \leqslant \frac{d_1 + d_2 + \alpha n_2}{n_1 + n_2}$$
$$\frac{d_2 + \alpha n_2}{n_2} \leqslant \frac{d_1 + d_2 + \alpha n_2}{n_1 + n_2},$$

which are equivalent respectively to $\alpha \leqslant \mu(E_1) - \mu(E_2)$ and $\alpha \geqslant \mu(E_1) - \mu(E_2)$. Hence $\mathcal{T}$ cannot be α-stable for any α and it is α-semistable if and only if $\alpha = \mu(E_1) - \mu(E_2)$. In this case let E_1' be a proper subbundle of E_1. Then we can consider the proper subtriple $(E_1', 0, 0)$: the $(\mu(E_1) - \mu(E_2))$-semistability condition for this triple is

$$\frac{d_1'}{n_1'} \leqslant \frac{d_1 + d_2 + (\mu(E_1) - \mu(E_2))n_2}{n_1 + n_2}$$

which after some easy computations is equivalent to $\mu(E_1') \leqslant \mu(E_1)$, proving the semistability of E_1. The semistability of E_2 can be proved in an analogous way.

According to the Definition of α-stability 1.1.4 the moduli spaces of α-stable holomorphic triples have been built, first by Bradlow and García-Prada and then by Schmitt with the methods of geometric invariant theory of Mumford. See [20] for a survey on moduli spaces and their construction and [7, 22] for details of the two constructions of the moduli spaces above; the main properties of these moduli spaces are collected in the following Sections, and in particular in Section 1.7.

In the following we will denote by $\mathcal{N}_\alpha(X; n_1, n_2, d_1, d_2)$ the moduli space of α-stable holomorphic triples of type (n_1, n_2, d_1, d_2) on the curve X and by $\mathcal{N}_\alpha^{ss}(X; n_1, n_2, d_1, d_2)$ the moduli space of S-equivalence classes of α-semistable holomorphic triples of type (n_1, n_2, d_1, d_2). We will omit to specify the curve X whenever no ambiguities seems like to arise, hence we will write simply $\mathcal{N}_\alpha(n_1, n_2, d_1, d_2)$ and $\mathcal{N}_\alpha^{ss}(n_1, n_2, d_1, d_2)$.

If $\mathcal{T} = (E_1, E_2, \varphi)$ is a triple of type (n_1, n_2, d_1, d_2) it is always possible to consider the *dual* triple $\mathcal{T}^* = (E_1^*, E_2^*, \varphi^*)$ which is a triple of type $(n_2, n_1, -d_2, -d_1)$. It turns out that the properties of being α-(semi)stable for these two triples are strictly related.

Proposition 1.1.7 ([7, Prop. 3.12])**.** *The triple $\mathcal{T}$ is α-(semi)stable if and only if $\mathcal{T}^*$ is α-(semi)stable or, equivalently,*

$$\mathcal{N}_\alpha^{(ss)}(n_1, n_2, d_1, d_2) \cong \mathcal{N}_\alpha^{(ss)}(n_2, n_1, -d_2, -d_1).$$

As a consequence of the previous Proposition, when no additional hypothesis on the nature of the vector bundles E_1 and E_2 are assumed, the study of α-stable triples can be restrict (e.g.) to the case $n_2 \leqslant n_1$: the corresponding results for $n_2 > n_1$ can be recovered appealing to duality.

In the following we will need also some further result on vector bundles on curves; we collect them here for future reference.

Definition 1.1.8. *A vector bundle E on a curve X is said to be* generated by its global sections *if the evaluation map $H^0(X, E) \longrightarrow E_x$ is surjective for all $x \in X$, or equivalently if there exists an exact sequence*

$$0 \longrightarrow F \longrightarrow \mathcal{O} \otimes W \longrightarrow E \longrightarrow 0,$$

where W is a vector space.

Proposition 1.1.9 ([23, Prop. 6]). *Let E and F be semistable vector bundles on the curve X. Then the following facts hold.*

i) If $\mu(F) < \mu(E)$, then $\operatorname{Hom}(E, F) = \{ 0 \}$.

ii) If E and F are both stable and $\mu(E) = \mu(F)$, then either $E \cong F$ or $\operatorname{Hom}(E, F) = \{ 0 \}$.

iii) If E is stable, then it is simple, i.e. $\operatorname{End}(E) \cong \mathbb{K}$.

Proposition 1.1.10 ([23, Lm. 20]). *Let E be a semistable vector bundle of rank n and degree d such that $d > n(2g-1)$. Then E is generated by its global sections and, moreover, $H^1(X, E) = \{ 0 \}$.*

1.2 Bounds on the range of α

According to our definition of α-stability a priori α can be any real number. In fact it turns out that some constraints exist in order for α-stable triples to exist. In particular a lower bound on α always exists and, provided that $n_1 \neq n_2$, an upper bound also exists, as stated by the following Proposition.

Proposition 1.2.1 ([7, Prop. 3.13 and 3.14]). *Let $\mathcal{T}$ be an α-stable triple. Then*

$$0 < \mu(E_1) - \mu(E_2) < \alpha.$$

Moreover, if $n_1 \neq n_2$, then

$$\alpha < \left(1 + \frac{n_1 + n_2}{|n_1 - n_2|}\right)\big(\mu(E_1) - \mu(E_2)\big).$$

Proof. The proof relies on the existence of some particular subtriples of $\mathcal{T}$. This is a quite standard way to produce bounds on α (see e.g. 3.2.1 and 3.2.2). In particular in this case the lower bound comes from the α-stability of the proper subtriple $(0, E_2, 0)$, while the upper bound from the α-stability of $(0, \ker\varphi, \varphi)$ and $(\operatorname{im}\varphi, E_2, \varphi)$; see [7] for further details. □

In the following we will usually write $\mu_1 := \mu(E_1)$, $\mu_2 := \mu(E_2)$,

$$\begin{aligned} \alpha_m &= \alpha_m(n_1, n_2, d_1, d_2) := \mu_1 - \mu_2 \\ \alpha_M &= \alpha_M(n_1, n_2, d_1, d_2) := \left(1 + \frac{n_1 + n_2}{|n_1 - n_2|}\right)\big(\mu_1 - \mu_2\big). \end{aligned}$$

Note that, if $\mu_1 = \mu_2$ and $n_1 \neq n_2$, then $\alpha_m = \alpha_M = 0$, hence α-stable triples cannot exist and α-semistable triples can exist only for $\alpha = 0$.

Some easy consequences of the Proposition above are the following.

Corollary 1.2.2 ([7, Cor. 3.15]). *If $n_1 \neq n_2$, then a necessary condition for $\mathcal{T}$ to be α-stable is $\mu_1 > \mu_2$.*

Corollary 1.2.3 ([7, Cor. 3.16]). *Let $\mathcal{T}$ be an α stable triple such that $n_1 = n_2$. Then, if φ is not an isomorphism, $d_1 > d_2$. Moreover in any α-stable triple $\mathcal{T}$ φ is an isomorphism if and only if $n_1 = n_2$ and $d_1 = d_2$.*

Proposition 1.2.4 ([7, Lm. 4.5]). *For any $\alpha > 0$ the triple $\mathcal{T} = (E_1, E_1, \varphi)$ is α-stable if and only if φ is an isomorphism and E_1 is stable.*

Example 1.2.5 ([7, Prop 3.17]). When $n_1 = n_2$ the range of the possible values of α can in fact fail to be bounded: this is illustrated in this example. Let E_1 and E_2 be two stable bundles of rank n and degree d and $\varphi : E_2 \longrightarrow E_1$ be non trivial. Then for any $\alpha > \alpha_m$ the triple $\mathcal{T} = (E_1, E_2, \varphi)$ is α-stable. For note first of all that since φ is a non trivial map between two stable vector bundles of the same rank and degree, then by 1.1.9, it must be a scalar multiple of the identity, and hence, in particular, it is injective. Let now $\mathcal{T}' = (E_1', E_2', \varphi')$ be a proper subtriple of $\mathcal{T}$. Since E_1 and E_2 are stable of the same slope, we have $\mu(E_1' \oplus E_2') < \mu(E_1 \oplus E_2)$, and from the injectivity of φ it follows $n_2' \leqslant n_1'$, and hence

$$\frac{n_2'}{n_1' + n_2'} \leqslant \frac{1}{2}.$$

Therefore we have

$$\mu_\alpha(\mathcal{T}') = \mu(E_1' \oplus E_2') + \alpha \frac{n_2'}{n_1' + n_2'} < \mu(E_1 \oplus E_2) + \alpha \frac{1}{2} = \mu_\alpha(\mathcal{T}),$$

proving the α-stability of $\mathcal{T}$.

1.3 Stability, simplicity and irreducibility

As it happens for vector bundles (see e.g. Proposition 1.1.9) it turns out that the notion of α-stability is deeply bounded to those of simplicity and irreducibility. Let us start by making precise what it is meant for simplicity and irreducibility in the case of holomorphic triples.

Definition 1.3.1. *Let $\mathcal{T} = (E_1, E_2, \varphi)$ be a holomorphic triple and write*

$$\operatorname{End}(E_1, E_2, \varphi) := \{\, (u, v) \in \operatorname{End}(E_1) \oplus \operatorname{End}(E_2) \mid u \circ \varphi = \varphi \circ v \,\}.$$

$\mathcal{T}$ is said to be simple *if $\operatorname{End}(E_1, E_2, \varphi) = \mathbb{K}$, i.e. if the only endomorphisms of $\mathcal{T}$ are the scalar multiples of the pair $(\mathrm{id}_{E_1}, \mathrm{id}_{E_2})$.*

Definition 1.3.2. *A triple* $\mathcal{T} = (E_1, E_2, \varphi)$ *is said to be* reducible *if there exist direct sum decompositions* $E_i = \bigoplus_{j=1}^n E_i^{(j)}$, *and* $\varphi = \bigoplus_{j=1}^n \varphi_j$, $i = 1, 2$ *such that, for each* $1 \leqslant j \leqslant n$, $\varphi_j \in \mathrm{Hom}\left(E_2^{(j)}, E_1^{(j)}\right)$. *We adopt the convention that if either* $E_1^{(j)} = 0$ *or* $E_2^{(j)} = 0$, *then* φ_j *is the zero-map. We will write* $\mathcal{T}_j := (E_1^{(j)}, E_2^{(j)}, \varphi_j)$ *and* $\mathcal{T} := \bigoplus_{j=1}^n \mathcal{T}_j$. *If* $\mathcal{T}$ *is not reducible it is said to be* irreducible.

Proposition 1.3.3 ([7, Prop. 2.12])**.** *Let* $\mathcal{T}$ *be a simple triple. Then* $\mathcal{T}$ *is irreducible.*

Again, as it happens for vector bundles, an important consequence of α-stability is that α-stable triples are simple and, hence, irreducible. This is a Corollary of the following Proposition.

Proposition 1.3.4 ([7, Prop. 3.9])**.** *Let* $\mathcal{T} = (E_1, E_2, \varphi)$ *be an* α*-stable triple and let* $(u, v) \in \mathrm{End}(\mathcal{T})$. *Then either* $(u, v) = (\mathrm{id}_{E_1}, \mathrm{id}_{E_2})$ *or both* u *and* v *are isomorphisms.*

Corollary 1.3.5. *Let* $\mathcal{T}$ *be an* α*-stable triple for some values of* α. *Then* $\mathcal{T}$ *is simple.*

Remark 1.3.6. Note that irreducibility of α-stable holomorphic triples can be easily proved also directly. For assume by contraposition that $\mathcal{T}$ is a reducible triple and write $\mathcal{T} := \bigoplus_{j=1}^n \mathcal{T}_j$. For any $j = 1, 2, \ldots, n$ it is possible to consider the proper subtriples $\mathcal{T}' := \mathcal{T}_j$ and $\mathcal{T}'' := \mathcal{T}/\mathcal{T}_j$ which are triples of type $(n_1^j, n_2^j, d_1^j, d_2^j)$ and $(n_1 - n_1^j, n_2 - n_2^j, d_1 - d_1^j, d_2 - d_2^j)$ respectively. The α-stability conditions for these subtriples are respectively

$$\frac{d_1^j + d_2^j + \alpha n_2^j}{n_1^j + n_2^j} < \frac{d_1 + d_2 + \alpha n_2}{n_1 + n_2}$$
$$\frac{d_1 - d_1^j + d_2 - d_2^j + \alpha(n_2 - n_2^j)}{n_1 + n_2 - n_1^j - n_2^j} < \frac{d_1 + d_2 + \alpha n_2}{n_1 + n_2}$$

which are equivalent respectively to

$$\alpha(n_2 n_1^j - n_1 n_2^j) > (d_1^j + d_2^j)(n_1 + n_2) - (d_1 + d_2)(n_1^j + n_2^j)$$
$$\alpha(n_2 n_1^j - n_1 n_2^j) < (d_1^j + d_2^j)(n_1 + n_2) - (d_1 + d_2)(n_1^j + n_2^j)$$

and show that $\mathcal{T}$ is not α-stable for any α.

1.4 Relationships between α-stable triples and the stability of vector bundles

It is natural to ask whether some kind of connection between the notion of α-stability for a triple $\mathcal{T} = (E_1, E_2, \varphi)$ and the stability of the vector bundles E_1 and E_2 exists. The answer is provided by the following Propositions, at least for some special values of the parameter α, namely when α is close to α_0 and, in the case $n_2 \neq n_1$, when α is close to α_M.

Proposition 1.4.1 ([7, Prop. 3.19]). *Let $\mathcal{T} = (E_1, E_2, \varphi)$ be a holomorphic triple. There exists $\varepsilon > 0$ such that, for any $\alpha \in (\alpha_m, \alpha_m + \varepsilon)$, if $\mathcal{T}$ is α-stable, then E_1 and E_2 are both semistable vector bundles. Vice versa, if one of E_1, E_2 is stable and the other is semistable, then the triple (E_1, E_2, φ) is α-stable for any choice of $\varphi \in \mathrm{Hom}(E_2, E_1)$, $\varphi \neq 0$.*

Proof. Note that in the statement of [7] it is in fact required that both the vector bundles E_1 and E_2 are stable to prove the α-stability of $\mathcal{T}$. From the proof it turns out, however, that our assumption is enough. □

Proposition 1.4.2 ([8, Prop. 7.5]). *Assume $n_2 < n_1$ and let $\mathcal{T} = (E_1, E_2, \varphi)$ be an α-semistable triple for large enough α. Then $\mathcal{T}$ is of the form*

$$0 \longrightarrow E_2 \xrightarrow{\varphi} E_1 \longrightarrow F \longrightarrow 0,$$

where F is locally free and both E_2 and F are semistable.

Proposition 1.4.3 ([9, Prop. 4.13]). *Let $\mathcal{T} = (E_1, E_2, \varphi)$ be a triple of the form*

$$0 \longrightarrow E_2 \xrightarrow{\varphi} E_1 \longrightarrow F \longrightarrow 0,$$

with F locally free and such that the extension is not trivial. If E_2 and F are stable then $\mathcal{T}$ is α-stable for large enough values of α.

Proposition 1.4.4 ([8, Prop. 8.1]). *Let $\mathcal{T} = (E_1, E_2, \varphi)$ be a triple of type (n, n, d, d) and $\alpha > 0$. Then $\mathcal{T}$ is α-stable if and only if E_1 and E_2 are stable and φ is an isomorphism.*

1.5 Critical values

So far we know that some constraints on α exists, even if we have shown in Example 1.2.5 that, in general, α can fail to be bounded. In fact something stronger is true: as α varies in the admissible range described by Proposition 1.2.1 the α-stability condition does not vary continuously, but it varies only at a finite number of *critical values*, due to the fact that the numerical quantities involved in the definition of α-stability are rational numbers and the denominators are bounded by n_1 and n_2.

Fix a real number α. If a triple $\mathcal{T}$ is strictly α-semistable, then there exists a proper subtriple $\mathcal{T}'$ of $\mathcal{T}$ such that

$$\mu_\alpha(\mathcal{T}') = \frac{d_1' + d_2'}{n_1' + n_2'} + \alpha\frac{n_2'}{n_1' + n_2'} = \frac{d_1 + d_2}{n_1 + n_2} + \alpha\frac{n_2}{n_1 + n_2} = \mu_\alpha(\mathcal{T}).$$

If $n_2'/(n_1' + n_2') = n_2/(n_1 + n_2)$, then the previous condition is in fact independent of α, hence $\mathcal{T}$ can be α-stable for no values of α and it is strictly α-semistable for any values of α for which it is not α-unstable. Of course it can also happen that $n_2'/(n_1' + n_2') \neq n_2/(n_1 + n_2)$; the values of α for which this happens are precisely the critical values, as stated by the following definition.

Definition 1.5.1. *Let $\alpha \in [\alpha_m, +\infty)$. We say α is a* critical value *for $\mathcal{T}$ if there exist integers n_1', n_2', d_1', d_2' such that*

$$\frac{d_1' + d_2'}{n_1' + n_2'} + \alpha\frac{n_2'}{n_1' + n_2'} = \frac{d_1 + d_2}{n_1 + n_2} + \alpha\frac{n_2}{n_1 + n_2}$$

where $0 \leqslant n_i' \leqslant n_1$, $(n_1', n_2', d_1', d_2') \neq (n_1, n_2, d_1, d_2)$, $(n_1', n_2') \neq (0, 0)$ and $n_1' n_2 \neq n_1 n_2'$.

We say that α is generic *if it is not critical.*

It is possible to prove the following

Proposition 1.5.2 ([8, Prop. 2.6])**.** *Let $\mathcal{T}$ be a triple of type (n_1, n_2, d_1, d_2). Then the following is true.*

i) The critical values of $\mathcal{T}$ form a discrete subset of $[\alpha_m, +\infty)$. Moreover if $n_1 \neq n_2$ the number of critical values is finite and lies in the interval $[\alpha_m, \alpha_M]$.

ii) The α-stability criteria for two generic values of α lying between two consecutive critical values are equivalent, hence the corresponding moduli spaces are isomorphic.

iii) If α is generic and $\gcd(n_2, n_1 + n_2, d_1 + d_2) = 1$, then α-stability and α-semistability are equivalent.

So far we do not know of any upper bound on α in the case $n_1 = n_2$. In fact it turns out that also in this situation the number of critical values is finite and hence there exist only finitely many different moduli spaces. This is made clear by the following Stabilization Theorem.

Theorem 1.5.3 (Stabilization Theorem, [8, Thm 8.6])**.** *Assume* $n_1 = n_2 =: n$ *and* $d_1 \geqslant d_2$ *and write* $\alpha_0 = d_1 - d_2$*. Then, for any real numbers* α_1, α_2 *such that* $\alpha_0 < \alpha_1 \leqslant \alpha_2$*,*

$$\mathcal{N}_{\alpha_1}(n, n, d_1, d_2) \subseteq \mathcal{N}_{\alpha_2}(n, n, d_1, d_2).$$

Moreover there exists a real number $\alpha_L \geqslant \alpha_0$ *such that*

$$\mathcal{N}_{\alpha_1}(n, n, d_1, d_2) = \mathcal{N}_{\alpha_2}(n, n, d_1, d_2)$$

for all $\alpha_L < \alpha_1 \leqslant \alpha_2$*.*

According to the previous results it is possible to divide the range $[\alpha_m, +\infty)$ into a finite set of subintervals whose extremes are precisely the critical values:

$$0 \leqslant \alpha_0 = \alpha_m < \alpha_1 < \cdots < \alpha_L < +\infty,$$

where $\alpha_L = \alpha_M$ if $n_1 \neq n_2$.

For any critical value α_j we will write $\alpha_j^+ := \alpha_j + \varepsilon$ and $\alpha_j^- := \alpha_j - \varepsilon$, where $\varepsilon > 0$ is small enough so that the interval (α_j, α_j^+) or, respectively, (α_j^-, α_j), does not contain any critical value. Occasionally we will speak of "small" α to refer to values of α of the form α_m^+ and of "big" or "large" α to refer to values of the form α_M^- when $n_1 \neq n_2$, or of the form α_L^+ when $n_1 = n_2$.

1.6 Flip loci

We have already observed that if α_1 and α_2 are two generic values in the interval (α_i, α_{i+1}) between two consecutive critical values, then the moduli spaces $\mathcal{N}_{\alpha_1}(n_1, n_2, d_1, d_2)$ and $\mathcal{N}_{\alpha_2}(n_1, n_2, d_1, d_2)$ are isomorphic. In this section we deepen our study on how the moduli spaces $\mathcal{N}_\alpha(n_1, n_2, d_1, d_2)$ vary for fixed (n_1, n_2, d_1, d_2) as α crosses a critical value α_i. This turns out to be quite useful in proving non-emptiness results for moduli spaces, since it permits to prove this property for particular values of the parameter α, and then to extend the results to all the remaining values.

Remark 1.6.1. It is easy (but useful) to describe here how the α-stability property for a holomorphic triple $\mathcal{T} = (E_1, E_2, \varphi)$ of type (n_1, n_2, d_1, d_2) can change as α varies in the admissible range. The basic observation is that, if $\mathcal{T}' = (E_1', E_2', \varphi')$ is a proper subtriple of $\mathcal{T}$, then

$$\Delta_\alpha(\mathcal{T}', \mathcal{T}) = \frac{d_1 + d_2}{n_1 + n_2} - \frac{d_1' + d_2'}{n_1' + n_2'} + \alpha \left(\frac{n_2}{n_1 + n_2} - \frac{n_2'}{n_1' + n_2'} \right),$$

hence $\Delta(\mathcal{T}', \mathcal{T})$ is a linear function of α. Moreover, if $\frac{n_2}{n_1+n_2} - \frac{n_2'}{n_1'+n_2'} > 0$ then $\Delta(\mathcal{T}', \mathcal{T})$ is strictly increasing, if $\frac{n_2}{n_1+n_2} - \frac{n_2'}{n_1'+n_2'} < 0$ then $\Delta(\mathcal{T}', \mathcal{T})$ is strictly decreasing and if $\frac{n_2}{n_1+n_2} - \frac{n_2'}{n_1'+n_2'} = 0$ then $\Delta(\mathcal{T}', \mathcal{T})$ is constant.

Definition 1.6.2. *Let $\mathcal{T}$ be a triple and $\alpha_i \in (\alpha_m, \alpha_M)$ a critical value for $\mathcal{T}$. We write $\mathcal{S}_{\alpha_i^+}$ for the subset of $\mathcal{N}_{\alpha_i}(n_1, n_2, d_1, d_2)$ made up of triples which are α_i^+-stable but not α_i^--stable and $\mathcal{S}_{\alpha_i^-}$ for the subset of $\mathcal{N}_{\alpha_i}(n_1, n_2, d_1, d_2)$ made up of triples which are α_i^--stable but not α_i^+-stable.*

Using techniques coming from the theory of deformation of holomorphic triples (see [8, § 3]) it is possible to show that, under suitable hypotheses, the flip loci $\mathcal{S}_{\alpha_i^\pm}$ are contained in subvarieties of positive codimension in $\mathcal{N}_{\alpha_i^\pm}(n_1, n_2, d_1, d_2)$. In particular the following is true.

Proposition 1.6.3. *Let $\mathcal{T}$ be a holomorphic triple of type (n_1, n_2, d_1, d_2) and let $\alpha_i \in (\alpha_m, \alpha_M)$ be a critical value. Then the following holds.*

i) $\mathcal{N}_{\alpha_i^+}(n_1, n_2, d_1, d_2) \backslash \mathcal{S}_{\alpha_i^+} = \mathcal{N}_{\alpha_i}(n_1, n_2, d_1, d_2) = \mathcal{N}_{\alpha_i^-}(n_1, n_2, d_1, d_2) \backslash \mathcal{S}_{\alpha_i^-}$

ii) If $\alpha_i > 2g-2$, then the loci $\mathcal{S}_{\alpha_i^\pm} \subseteq \mathcal{N}_{\alpha_i^\pm}(n_1, n_2, d_1, d_2)$ are contained in subvarieties of codimension at least $g-1$. In particular they are contained in subvarieties of strictly positive codimension if $g \geqslant 2$. If $\alpha_i = 2g-2$ the same is true for $\mathcal{S}_{\alpha_i^+}$.

Remark 1.6.4. The proof of the previous Proposition relies on the fact that it is possible to give a lower bound on the codimension of the subvarieties in which the flip loci are contained. In particular the codimension is bounded from below by

$$\begin{aligned}\min \big\{ n_1'' d_1' + n_2'' d_2' + n_2' d_1'' + n_1'' d_2' - n_1' d_1'' - n_2' d_2'' + \\ + (g-1)\left(n_1' n_1'' + n_2' n_2'' - n_2' n_1''\right)\big\}\end{aligned}$$

where the minimum is to be taken over all the quadruples (n_1', n_2', d_1', d_2') and $(n_1'', n_2'', d_1'', d_2'')$ such that the following relations are fulfilled:

$$\begin{aligned}
&(n_1, n_2, d_1, d_2) = (n_1', n_2', d_1', d_2') + (n_1'', n_2'', d_1'', d_2''),\\
&\frac{d_1' + d_2'}{n_1' + n_2'} + \alpha_i \frac{n_2'}{n_1' + n_2'} = \frac{d_1'' + d_2''}{n_1'' + n_2''} + \alpha_i \frac{n_2''}{n_1'' + n_2''},\\
&\frac{n_2'}{n_1' + n_2'} < \frac{n_2''}{n_1'' + n_2''} \quad \text{in the case of } \mathcal{S}_{\alpha_i^+},\\
&\frac{n_2'}{n_1' + n_2'} > \frac{n_2''}{n_1'' + n_2''} \quad \text{in the case of } \mathcal{S}_{\alpha_i^-}.
\end{aligned}$$

Having achieved this result, the general strategy to prove non-emptiness of moduli spaces is then to prove it for a particular and convenient value of the parameter α (usually for large values of α), then to deduce the analogous result for all the other values of α noticing that crossing each critical value does not "throw away" all the stable triples from $\mathcal{N}_\alpha(n_1, n_2, d_1, d_2)$, but it "preserves" some of them. Note, however, that in the particular cases of curves of genus 0 and 1 the hypothesis of Proposition 1.6.3 are always fulfilled, but the Proposition itself does not provide any useful information, since the bound on the codimension of the flip loci is always trivial.

1.7 Moduli spaces of holomorphic triples

Here we summarize the main results on moduli spaces of α-stable holomorphic triples.

Theorem 1.7.1 ([7, Thm 6.1]). *Let X be a curve of genus g and fix n_1, n_2, d_1, d_2. Then the moduli space $\mathcal{N}_\alpha(n_1, n_2, d_1, d_2)$ of α-stable triples of type (n_1, n_2, d_1, d_2) is a complex quasi-projective variety and it is projective if $n_1 + n_2$ and $d_1 + d_2$ are coprime and α is generic. The dimension of the moduli space at a smooth point is*

$$\rho(n_1, n_2, d_1, d_2) := 1 + n_2 d_1 - n_1 d_2 + (n_1^2 + n_2^2 - n_1 n_1)(g - 1). \qquad (1.1)$$

Remark 1.7.2. Note that $\rho(n_1, n_2, d_1, d_2) \geqslant 0$ is a necessary condition for $\mathcal{N}_\alpha(n_1, n_2, d_1, d_2)$ to be non-empty. In the following we will refer to this inequality as *Brill-Noether condition.*

Proposition 1.7.3 ([7, Prop. 6.3]). *Let $\mathcal{T} = (E_1, E_2, \varphi)$ be an α-stable triple and assume that φ is either injective or surjective. Then $\mathcal{T}$ corresponds to a smooth point of $\mathcal{N}_\alpha(n_1, n_2, d_1, d_2)$.*

From now on assume that $g \geqslant 2$. The following result states some properties of moduli spaces for large values of α.

Theorem 1.7.4 ([8, Thm 7.7]). *Let $n_1 > n_2$, $\mu_1 > \mu_2$ and $\alpha = \alpha_M^-$. Then the moduli space $\mathcal{N}_\alpha(n_1, n_2, d_1, d_2)$ is smooth of dimension $\rho(n_1, n_2, d_1, d_2)$ and it is birationally equivalent to a $\mathbb{P}^N$-fibration over $\mathcal{M}(n_1 - n_2, d_1 - d_2) \times \mathcal{M}(n_2, d_2)$, where $\mathcal{M}(n, d)$ denotes the moduli space of stable vector bundles of rank n and degree d and $N = n_2 d_1 - n_1 d_2 + n_1(n_1 - n_2)(g - 1) - 1$. In particular $\mathcal{N}_\alpha(n_1, n_2, d_1, d_2)$ is non-empty and irreducible. Moreover whenever $\gcd(n_1 - n_2, d_1 - d_2) = 1$ and $\gcd(n_2, d_2) = 1$, the birational equivalence is an isomorphism.*

From the previous result and using the technique of flips presented in Section 1.6, it is possible to extend the results also to other values of α, provided that they are big enough in comparison with the genus g of the curve.

Theorem 1.7.5 ([8, Thm 7.9]). *Let α be such that $\alpha_m < 2g-2 \leqslant \alpha < \alpha_M$ and assume $n_2 \leqslant n_1$. Then $\mathcal{N}_\alpha(n_1, n_2, d_1, d_2)$ is birationally equivalent to $\mathcal{N}_{\alpha_M^-}(n_1, n_2, d_1, d_2)$, hence, in particular, it is non-empty and irreducible.*

Chapter 2

Coherent systems and their moduli spaces

In this Chapter we recall the definition of coherent system and we introduce the notion of σ-(semi)stability for those objects. We collect the main results on σ-stable coherent systems and the main properties of their moduli spaces.

Coherent systems, introduced by Le Potier in [19] and known also as *Brill-Noether pairs* (see [14]), present many analogies with holomorphic triples, and in fact the treatment here deserved to these objects is similar to that of holomorphic triples discussed in the previous Chapter. For the sake of precision one would say that the oldest notion is probably that of coherent systems, or, at least, that the study of coherent systems and their properties has started earlier, due to its relationships with the study of Brill-Noether loci (see Section 2.5). It would be more correct, hence, to say that it is the theory of triples which follows in its footsteps. Here however we choose this order of presentation since in the following holomorphic triples will be our main topic of discussion, whilst some result on coherent systems will only occasionally be proved. Both of them, moreover, can be seen in a unitary way as different examples of the so called *augmented bundles*, as (briefly) discussed at the end of Section 2.7.

For the results reported in this Chapter refer, e.g., to [6, 5].

2.1 Setting

Let X be a smooth curve on an algebraically closed field $\mathbb{K}$ of characteristic 0.

Definition 2.1.1. *A* coherent system *on the curve X is a pair (E, V) where*

E is a vector bundle on X and V is a vector subspace of the vector space $H^0(X,E)$ of the global sections of E. If $\mathrm{rank}(E) = n$, $\deg(E) = d$ *and* $\dim(V) = k$ *then we speak of coherent systems of* type (n,d,k).

Definition 2.1.2. *A* coherent subsystem *of a coherent system (E,V) is a coherent system (E',V') where E' is a subbundle of E and V is a vector subspace of $V \cap H^0(X,E')$. The subsystem $(0,0)$ is the* trivial *coherent system. A coherent subsystem is* proper *if it is not trivial and $(E',V') \neq (E,V)$.*

As we did for holomorphic triples we introduce now a notion of stability for those objects which depends on a real parameter σ.

Definition 2.1.3. *Let (E,V) be a coherent system and $\sigma \in \mathbb{R}$. The* σ-degree *of (E,V) is*

$$\deg_\sigma(E,V) = \deg(E) + \sigma \dim(V)$$

and the σ-slope *is*

$$\mu_\sigma(E,V) := \frac{\deg_\sigma(E,V)}{\mathrm{rank}(E)} = \frac{\deg(E) + \sigma \dim(V)}{\mathrm{rank}(E)}.$$

In the following we will always write $d := \deg(E)$, $n := \mathrm{rank}(E)$ and $k := \dim(V)$.

Definition 2.1.4. *A coherent system (E,V) is said to be* α-(semi)stable *if, for all proper coherent subsystems (E',V'), $\mu_\sigma(E',V') < \mu_\sigma(E,V)$ (resp. $\mu_\sigma(E',V') \leqslant \mu_\sigma(E,V)$). (E,V) is said to be* σ-polystable *if it is the direct sum of σ-stable coherent systems of the same σ-slope.*

According to Definition 2.1.4 the moduli spaces of σ-stable coherent systems and of S-equivalence classes of σ-semistable coherent systems have been built (see [22]). In the following we will denote by $\mathcal{G}_\sigma(X;n,d,k)$ (or simply by $\mathcal{G}_\sigma(n,d,k)$ if no confusion can arise on the curve X) the moduli space of σ-stable coherent systems and by $\mathcal{G}_\sigma^{ss}(X;n,d,k)$ (or simply $\mathcal{G}_\sigma^{ss}(n,d,k)$) the moduli space of S-equivalence classes of σ-semistable coherent systems.

2.1.1 Constraints on the parameter σ

In general σ cannot be any real number: some necessary conditions hold in order for σ-stable coherent systems to exist. These conditions are stated in the next Proposition.

Proposition 2.1.5 ([6, Lm. 4.2 and 4.3])**.** *Let (E,V) be a σ-stable coherent system. Then $\sigma > 0$. Moreover, if $k < n$, then $\sigma < d/(n-k)$, hence in particular we must have $d > 0$ in order for σ-stable coherent systems to exist.*

If $k \geqslant n$, then $d \geqslant 0$ is a necessary condition for σ-semistable coherent systems to exist and, moreover, we must have $d > 0$ in order for σ-stable coherent systems to exist except in the case $(n, d, k) = (1, 0, 1)$.

2.2 Critical values

As it happens for holomorphic triples the numerical quantities involved in the definition of σ-stability are rational numbers with bounded denominators, hence the σ-stability condition varies only at certain critical values.

Definition 2.2.1. *We say that $\sigma \in \mathbb{R}$ is a* critical value *for a coherent system of type (n, d, k) if either $\sigma = 0$ or there exist integers $0 < n' \leqslant n$, $0 \leqslant k' \leqslant k$ and d' such that*

$$\frac{d'}{n'} + \sigma \frac{k'}{n'} = \frac{d}{n} + \sigma \frac{k}{n},$$

and $k'/n' \neq k/n$. We say that σ is generic *if it is not critical.*

From the definition above it is clear that the critical values lie in the set

$$\left\{ \frac{nd' - n'd}{n'k - nk'} \;\middle|\; 0 \leqslant k' \leqslant k,\ 0 < n' \leqslant n,\ n'k \neq nk' \right\} \cap [0, +\infty).$$

A result analogous to Proposition 1.5.2 holds also for coherent systems. In particular it is possible to prove that if σ is generic and $\gcd(n, d, k) = 1$, then σ-stability and σ-semistability are equivalent.

Moreover, as it happens for holomorphic triples, even if in the case $k \geqslant n$ the stability condition does not provide an upper bound on σ, in fact beyond a certain value σ_L the moduli spaces do not change. This is stated in the following Proposition which plays the same role of the Stabilization Theorem 1.5.3 and is a consequence of the fact that in a σ-stable coherent system (E, V), if σ is big enough, the vector bundle E is generically generated by its global sections in V.

Proposition 2.2.2. *Let $k \geqslant n$. There exists a critical value σ_L such that $\mathcal{G}_{\sigma_1}(n, d, k) = \mathcal{G}_{\sigma_2}(n, d, k)$ for any $\sigma_L < \sigma_1 \leqslant \sigma_2$. The σ-range is then divided into a finite set of intervals bounded by critical values:*

$$0 = \sigma_0 < \sigma_1 < \cdots < \sigma_L < +\infty$$

such that

i) if σ_i and σ_{i+1} are two consecutive critical values, the moduli spaces for any two different values of σ in the interval (σ_1, σ_{i+1}) coincide;

ii) for any two different values of σ in the range $(\sigma_L, +\infty)$ the moduli spaces coincide.

For small values of σ a result analogous to Proposition 1.4.1 holds.

Proposition 2.2.3 ([6, Prop. 2.5]). *Let (E,V) be a coherent system, σ_1 be the first critical value after 0 and $0<\sigma<\sigma_1$. If (E,V) is σ-stable, then E is semistable. Conversely if E is a stable vector bundle, then (E,V) is σ-stable.*

2.3 Flip loci

As we did for holomorphic triples, if σ_i is a critical values for (E,V) we write $\sigma_i^{\pm} := \sigma_i \pm \varepsilon$, where $\varepsilon > 0$ is such that σ_i is the only critical value in the interval (σ_i^-, σ_i^+). We will speak of "small" σ to refer to values of σ of the forms $\sigma_0^+ = 0^+$ and of "large" σ to refer to values of the form σ_L^- if $k<n$, or of the form σ_L^+ if $k \geqslant n$.

Definition 2.3.1. *Let (E,V) be a coherent system of type (n,d,k) and let $\sigma_i \in (\sigma_0, \sigma_L)$ be a critical value for (E,V). We write $\mathcal{S}_{\sigma_i^+}$ for the subset of $\mathcal{G}_{\sigma_i}(n,d,k)$ made up of coherent systems which are σ_i^+-stable but not σ_i^--stable and $\mathcal{S}_{\sigma_i^-}$ for the subset of $\mathcal{G}_{\sigma_i}(n,d,k)$ made up of coherent systems which are σ_i^--stable but not σ_i^+-stable.*

The idea now is to operate as we did in the previous Chapter for holomorphic triples, that is to search for some estimates on the codimensions of the flip loci, in order to prove that crossing a critical value does not change the birational structure of a moduli space. This, however, turns out to be a more difficult effort in this situation, and in fact a result similar to Proposition 1.6.3 does not exist.

Once a critical value σ_i is fixed we will say that the flip at σ_i is *good* if $\mathcal{S}_{\sigma_i^+}$ has positive codimension in $\mathcal{G}_{\sigma_i}(n,d,k)$. Some rather technical results which show sufficient conditions for a flip to be good exist, and come from the theory of infinitesimal deformations and of extensions for coherent systems. In fact these results involve some numerical quantities, and require to show that convenient inequalities are fulfilled; see [6, Lm. 6.8, Cor. 6,9] for details.

2.4 Moduli spaces of coherent systems

Here we collect the main results on the moduli space of σ-stable coherent systems. First of all we introduce the following definition, which is related to the question of smoothness, as stated in Proposition 2.4.2.

Definition 2.4.1. *Let (E,V) be a coherent system. The* Petri map *of (E,V) is the map*

$$V \otimes H^0(E^* \otimes K) \longrightarrow H^0(\mathrm{End}(E) \otimes K)$$

given by multiplication of sections.

A curve is said to be a Petri curve *if the Petri map*

$$H^0(L) \otimes H^0(L^* \otimes K) \longrightarrow H^0(K)$$

is injective for every line bundle L over X.

Proposition 2.4.2 ([6, Prop. 3.10])**.** *Let (E, V) be an α-stable coherent system of type (n, d, k). Then the moduli space $\mathcal{G}_\sigma(n, d, k)$ is smooth of dimension*

$$\beta(n, d, k) := n^2(g-1) + 1 - k(k - d + n(g-1))$$

at the point corresponding to (E, V) if and only if the Petri map is injective.

The main results concerning the moduli space for large values of σ depend on the parameters n and k, and are stated in the following Theorems.

Theorem 2.4.3 ([6, Thm 5.4])**.** *Let $0 < k < n$, $d > 0$ and $g \geqslant 2$. Then the moduli space $\mathcal{G}_{\sigma_L^-}(n, d, k)$ is birationally equivalent to a fibration over the moduli space $\mathcal{M}(n-k, d)$ with fibre the Grassmannian $\mathrm{Gr}(k, d+(n-k)(g-1))$. In particular $\mathcal{G}_{\sigma_L^-}(n, d, k)$ is non-empty if and only if $k \leqslant d+(n-1)(g-1)$, and it is then always irreducible and smooth of dimension $\beta(n, d, k)$. If moreover $\gcd(n-k, d) = 1$ the the birational equivalence is an isomorphism.*

Note that the cases $g = 0$ and $g = 1$ deserve a particular treatment since $\mathcal{M}(n-k, d)$ may be empty. We deal with these cases in Section 2.6.

Theorem 2.4.4 ([6, Thm 5.6])**.** *Let $k = n \geqslant 2$. If $d > n$ $\widetilde{\mathcal{G}}_{\sigma_L^+}(n, d, k)$ is irreducible and $\mathcal{G}_{\sigma_L^+}(n, d, k)$ is smooth of dimension $\beta(n, d, k)$. If $d = n$, $\mathcal{G}_{\sigma_L^+}(n, d, k)$ is empty and $\widetilde{\mathcal{G}}_{\sigma_L^+}(n, d, k)$ is irreducible of dimension n. If $d = 0$ $\mathcal{G}_{\sigma_L^+}(n, d, k)$ is empty and $\widetilde{\mathcal{G}}_{\sigma_L^+}(n, d, k)$ consists of a single point.*

Theorem 2.4.5 ([6, Thm 5.11])**.** *Suppose that X is a Petri curve and that $k = n+1$. Then $\mathcal{G}_{\sigma_L^+}(n, d, k)$ is non-empty if and only if $\beta(n, d, n+1) \geqslant 0$. Moreover $\mathcal{G}_{\sigma_L^+}(n, d, k)$ has dimension $\beta(n, d, n+1)$ and it is irreducible whenever $\beta(n, d, n+1) > 0$.*

The idea is now to use again the flips technique to extend these results. The success, however, is subjected to the capability of proving that the flips are good in the sense of Section 2.3. This is known so far only for particular types of coherent systems, namely those with few sections ($k = 1, 2, 3$) or of small rank ($n = 2$); for details on these results see e.g. [6, § 7–10]. Note also that all these results have been proved under the further assumption that $g \geqslant 2$.

Some other existence results are proved in [2] without the use of flips; in particular there it is shown that σ-stable coherent systems exist for all $\sigma > 0$ provided that $g \geqslant 2$, $k > n$ and d is big enough:

Theorem 2.4.6 ([2, Thm 1])**.** *For all integers $g \geqslant 2$, $n \geqslant 2$ and $k > n$ set*

$$d(g,n,k) := \binom{\widetilde{k}}{n} + (n+1)g + \widetilde{k} - n - 1,$$

where $\widetilde{k} := \max\{\, k, 2g+2n \,\}$, and let X be a smooth curve of genus g. Then for every $d \geqslant d(g,n,k)$ there exists a coherent system (E,V) of type (n,d,k) with the following properties:

i) V spans E and E is stable;

ii) (E,V) is σ-stable for all $\sigma > 0$;

iii) the Petri map of (E,V) is injective;

iv) the natural map $\bigwedge^n(V) \longrightarrow H^0(X, \det(E))$ is injective.

2.5 Coherent systems and Brill-Noether loci

Definition 2.5.1. *Let $\mathcal{M}(n,d)$ denote the moduli space of stable vector bundles of rank n and degree d on a curve X and let $k \geqslant 0$. The* Brill-Noether loci *of stable vector bundles are defined by*

$$\mathcal{B}(n,d,k) := \left\{\, E \in \mathcal{M}(n,d) \;\middle|\; \dim H^0(E) \geqslant k \,\right\}.$$

Similarly one can define the Brill-Noether loci $\mathcal{B}^{ss}(n,d,k)$ of semistable bundles.

It is possible to prove that the Brill-Noether loci are closed subschemes of the corresponding moduli space. The aim of Brill-Noether theory is to inspect the main properties of these subschemes (non-emptiness, irreducibility, connectedness,...). The main results are summarized in the following statement.

Theorem 2.5.2 ([6, Thm 2.8])**.** *If the Brill-Noether locus $\mathcal{B}(n,d,k)$ is non-empty and $\mathcal{B}(n,d,k) \neq \mathcal{M}(n,d)$, then*

i) every irreducible component B of $\mathcal{B}(n,d,k)$ has dimension

$$\dim B \geqslant \beta(n,d,k),$$

where $\beta(n,d,k) := n^2(g-1) + 1 - k(k-d+n(g-1))$ is called the Brill-Noether number*.*

ii) $\mathcal{B}(n,d,k+1) \subseteq \operatorname{Sing}\mathcal{B}(n,d,k)$.

iii) *The tangent space of* $\mathcal{B}(n,d,k)$ *at a point* E *with* $\dim H^0(E) = k$ *can be identified with the dual of the cokernel of the Petri map*

$$H^0(E) \otimes H^0(E^* \otimes K) \longrightarrow H^0(\operatorname{End} E \otimes K).$$

iv) $\mathcal{B}(n,d,k)$ *is smooth of dimension* $\beta(n,d,k)$ *at* E *if and only if the Petri map is injective.*

The Brill-Noether loci are deeply bounded with the moduli space of σ-stable coherent systems: every vector bundle which occurs as part of a coherent system must have at least a prescribed number of linearly independent sections and, vice versa, a vector bundles E in $\mathcal{B}(n,d,k)$ determines naturally a coherent system of type (n,d,k).

It is evident that in order to achieve a direct correspondence between Brill-Noether loci and coherent systems a precise relationship between vector bundles stability and coherent systems σ-stability is needed. In this case the key step is provided by Proposition 2.2.3, which permits to define a map

$$\Psi : \begin{cases} \mathcal{G}_{\sigma_0}(n,d,k) & \longrightarrow & \mathcal{B}^{ss}(n,d,k) \\ (E,V) & \longmapsto & E \end{cases}$$

whose image contains $\mathcal{B}(n,d,k)$. In general, hence, it is possible to obtain information on $\mathcal{B}(n,d,k)$ from properties of $\mathcal{G}_{\sigma_0}(n,d,k)$ (as an example if $\gcd(n,d) = 1$, then $\mathcal{B}(n,d,k)$ and $\mathcal{B}^{ss}(n,d,k)$ coincide and the map above is also surjective), and this motivates the interest in studying the properties of the moduli space $\mathcal{G}_{\sigma_0}(n,d,k)$. We have the following results.

Theorem 2.5.3 ([6, § 11])**.** *Assume* $g \geqslant 2$, $\beta(n,d,k) \leqslant n^2(g-1)$, $\mathcal{G}_{\sigma_0}(n,d,k)$ *is irreducible and* $\mathcal{B}(n,d,k) \neq \emptyset$. *Then the following facts hold.*

i) $\mathcal{B}(n,d,k)$ *is irreducible.*

ii) Ψ *is one-to-one over* $\mathcal{B}(n,d,k) \setminus \mathcal{B}(n,d,k+1)$.

iii) $\dim \mathcal{B}(n,d,k) = \dim \mathcal{G}_{\sigma_0}(n,d,k)$.

iv) *For any* $E \in \mathcal{B}(n,d,k) \setminus \mathcal{B}(n,d,k+1)$ *the linear map*

$$\mathrm{d}\Psi : T_{(E,H^0(E))}\mathcal{G}_{\sigma_0}(n,d,k) \longrightarrow T_E\mathcal{B}(n,d,k)$$

of Zariski tangent spaces is an isomorphism.

v) *Assume* $\mathcal{G}_{\sigma_0}(n,d,k)$ *is smooth. Then* Ψ *is an isomorphism and moreover, if* $\gcd(n,d,k) = 1$, *then* $\mathcal{G}_{\sigma_0}(n,d,k)$ *is a desingularization of the closure of* $\mathcal{B}(n,d,k)$ *in the projective variety* $\mathcal{M}^{ss}(n,d)$.

Applying these techniques provides information on the irreducibility, the dimension and the birational structure of $\mathcal{B}(n,d,k)$. Of course in order to apply this method we need a good understanding of the moduli space $\mathcal{G}_{\sigma_0}(n,d,k)$, which we can obtain, as already observed, by combining the study of the moduli space of σ-stable holomorphic triples for large values of σ, and the use of flips to "move" the information to small values of σ. The known results, hence, are mainly in the cases presented in the previous Sections and in which we have good estimates for the codimensions of the flips (see [6, § 11.2] for details on the results).

2.6 Coherent systems in low genus

The general theory of coherent systems is true for vector bundles on curves of any genus, but some of the results presented therein, in particular those regarding the moduli spaces and their properties, requires $g \geqslant 2$. In fact coherent systems on the projective line and on elliptic curves deserve a particular treatment. This is mainly due to the particularity of these two cases in respect with the stability of vector bundles. In fact in both these cases it is true that we have a more or less complete classification of stable and semistable vector bundles, but it is also true that, from this classification, it turns out that quite often stable (or even semistable) object are rare to appear.

The study of coherent systems in genus 0 and 1 has been carried on by Lange and Newstead in [15, 16] (and it is still in active development for the case of the projective line, see [17, 18]). Here we summarize their main results, since we will use them later to prove analogous results for holomorphic triples on curves of low genus.

Before starting recall that in [13] it has been proved that every vector bundle E on the projective line $\mathbb{P}^1$ can be written uniquely as

$$E \cong \bigoplus_{i=1}^{n} \mathcal{O}(a_i),$$

with $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_n$. Such a bundle is stable if and only if $n = 1$ and is semistable if and only if $a_1 = a_2 = \ldots = a_n$. A vector bundle E is said to be of *generic splitting type* if it can be written in the form

$$E \cong \mathcal{O}(a)^{n-t} \oplus \mathcal{O}(a-1)^t,$$

where a and t are defined by $d = an - t$ with $0 \leqslant t < n$. It can be shown that a bundle E is of generic splitting type if and only if $h^1(\mathbb{P}^1, \mathrm{End}(E)) = 0$.

Dealing with vector bundles on elliptic curves recall that in [1] a complete classification of indecomposable vector bundles on elliptic curves is given, in particular it is proved that the indecomposable bundles of any fixed rank and degree form a family parametrized by X. An indecomposable bundle is always semistable and it is stable if and only if its rank and degree are coprime. Moreover in [24] it is shown that the moduli space of S-equivalence classes of semistable vector bundles of rank n and degree d is isomorphic to the hth symmetric product S^hX of X, where $h = \gcd(n, d)$. In particular every point of S^hX is represented by a polystable vector bundle

$$E = E_1 \oplus \cdots \oplus E_h \tag{2.1}$$

where each direct summand is stable of rank n/h and degree d/h.

2.6.1 Coherent systems on the projective line

The main results for coherent systems on the projective line are here summarized. Note that they are still partial results: necessary conditions for non-emptiness are provided and it is shown that the moduli space, whenever non-empty, is smooth and irreducible of the expected dimension. A complete classification is provided when the dimension of V is 1 or 2 and sufficient conditions are provided in some other particular cases. In fact it turns out that the case $g = 0$ is the most difficult to deal with, a situation that will still be true also for holomorphic triples, as we will see in Chapter 3.

One of the more useful tools is provided by the following Lemma.

Lemma 2.6.1 ([15, Lm. 3.1]). *Let (E, V) be a σ-stable coherent system of type (n, d, k) for some $\sigma > 0$ and assume $k > 0$. Then $E \cong \bigoplus_{i=1}^{n} \mathcal{O}(a_i)$, where for all $i = 1, 2, \ldots, n$ $a_i > 0$, hence E is generated by its global sections.*

Moving from this result it is possible to prove the following.

Theorem 2.6.2 ([15, Thm 3.2]). *Suppose $k > 0$ and $\mathcal{G}_\sigma(n, d, k)$ non-empty. Then $\mathcal{G}_\sigma(n, d, k)$ is smooth and irreducible and has dimension $\beta(n, d, k)$. Moreover, for a general $(E, V) \in \mathcal{G}_\sigma(n, d, k)$, E is of generic splitting type.*

It is known that when σ is close to 0 the σ-stability of a coherent system (E, V) implies the semistability of E, but semistable vector bundles on the projective line can exist if and only if $n|d$, hence one could expect that quite often the moduli space for small values of σ would be empty and thus, hopefully, it would be possible to provide a better lower bound on σ. In fact this is true and, whenever $k < n$, also a better upper bound exists. Note that in the following Proposition we recover the standard lower bound precisely when $t = 0$, that is when $n|d$ and hence semistable vector bundles of type (n, d) exist.

Proposition 2.6.3 ([15, Props 4.1 and 4.2]). *Suppose $\mathcal{G}_\sigma(n,d,k)$ is non-empty and let the numbers a and t be defined by $d = na - t$ with $0 \leqslant t < n$. Then*

$$\sigma > \frac{t}{k}.$$

If, moreover, $0 < k < n$, and l, m are defined by $ka - t = l(n-k) + m$ with $0 \leqslant m < n - k$, then

$$\sigma < \frac{d}{n-k} - \frac{mn}{k(n-k)}.$$

As previously stated in some particular cases more precise results can be proved; they are summarized in the following Theorem.

Theorem 2.6.4 ([15, 5.1, 5.4, 6.1, 6.3, 6.4]). *The following facts hold.*

i) If $k = 1$ and $n \geqslant 2$, then $\mathcal{G}_\sigma(n,d,1)$ is non-empty if and only if

$$t < \sigma < \frac{d}{n-k} - \frac{mn}{n-1},$$

hence in this case the constraints on σ expressed by Proposition 2.6.3 are optimal.

ii) If $k = 2$ and $n \geqslant 3$, then $\mathcal{G}_\sigma(n,d,2)$ is non-empty if and only if $(n,d) \neq (4,6)$,

$$\frac{t}{2} < \sigma < \frac{d}{n-2} - \frac{mn}{2(n-2)}$$

and

$$\beta(n,d,2) \geqslant 0,$$

hence in this case the constraints on σ expressed by Proposition 2.6.3 are optimal.

iii) If $k = n-1$, then $\mathcal{G}_\sigma(n,d,n-1)$ is non-empty for some σ if and only if $d \geqslant n$ and in this case an upper bound for σ is precisely d.

iv) If $k = n$, then $\mathcal{G}_\sigma(n,d,n)$ is non-empty for some σ if and only if $d > n$ and in this case there is no upper bound on σ.

v) If $k = n+1$, then $\mathcal{G}_\sigma(n,d,n+1)$ is non-empty for some σ if and only if $d \geqslant n$ and in this case if we write $d = na - t$ with $0 \leqslant t < n$, then $\mathcal{G}_\sigma(n,d,n+1)$ is always non-empty if $\sigma > t$.

2.6.2 Coherent systems on elliptic curves

The results for elliptic curves are definitively more exhaustive, since in this case it is possible to prove that the moduli space of σ-stable coherent systems of type (n, d, k), if non-empty, is smooth and irreducible of the expected dimension and, moreover, precise conditions for non-emptiness can be shown.

Similarly to what happens for the projective line, the following can be proved.

Lemma 2.6.5 ([16, Lm. 4.1])**.** *Suppose $n \geqslant 2$ and $k > 0$ and let (E, V) be a σ-stable coherent system of type (n, d, k). Then every indecomposable direct summand of E is of positive degree.*

Theorem 2.6.6 ([16, Thm 4.3])**.** *Suppose $n \geqslant 2$, $k > 0$ and $\mathcal{G}_\sigma(n, d, k)$ non-empty. Then $\mathcal{G}_\sigma(n, d, k)$ is smooth and irreducible of dimension $\beta(n, d, k)$. Moreover, for a general $(E, V) \in \mathcal{G}_\sigma(n, d, k)$,*

$$E \cong E_1 \oplus \cdots \oplus E_h,$$

where $h = \gcd(n, d)$ and E_i are stable and pairwise non-isomorphic vector bundles of the same slope.

Theorem 2.6.7 ([16, 3.1, 3.2, 4.5, 5.1, 5.2, 5.4])**.** *The following facts hold.*

i) If $k = 0$, then for all σ, $\mathcal{G}_\sigma(n, d, 0) \cong \mathcal{M}(n, d)$, where we write $\mathcal{M}(n, d)$ for the moduli space of stable vector bundles of rank n and degree d; in particular it is non-empty if and only if $\gcd(n, d) = 1$.

ii) If $k = d$ and $\gcd(n, d) > 1$, then $\mathcal{G}_\sigma(n, d, d)$ is empty for all σ.

iii) If $0 < k < n$ and either $k < d$ or $k = d$ and $\gcd(n, d) = 1$, then $\mathcal{G}_\sigma(n, d, k)$ is non-empty if and only if

$$0 < \sigma < \frac{d}{n-k}.$$

iv) If $k \geqslant n$ and either $k < d$ or $k = d$ and $\gcd(n, d) = 1$, then $\mathcal{G}_\sigma(n, d, k)$ is non-empty for all $\sigma > 0$.

Note in particular that, in fact, $\mathcal{G}_\sigma(n, d, k)$ is non-empty for the full range of admissible values of σ.

In the following we will need also this Lemma from [16]; we collect it here for future reference.

Lemma 2.6.8 ([16, Lm 2.2])**.** *Suppose that $E = E_1 \oplus \cdots \oplus E_h$ with all E_i indecomposable. Then $\dim \mathrm{Aut}(E) \geqslant h$. Moreover equality holds if and only if E is polystable and the E_i are pairwise non-isomorphic.*

2.7 Holomorphic triples and coherent systems as augmented bundles

It is interesting and useful to note that coherent systems can be seen as "specialized" holomorphic triples. The two notions of σ-stability and α-stability, however, are not exactly equivalent even if, of course, they are deeply related.

If (E, V) is a coherent system of type (n_1, d_1, n_2) and

$$V := \operatorname{span}\{\, \varphi_1, \ldots, \varphi_{n_2} \,\},$$

then it is possible to consider the evaluation map $\varphi : V \otimes \mathcal{O} \longrightarrow E$, where $V \otimes \mathcal{O}$ is a trivial vector bundle of rank n_2, and hence to recover a holomorphic triple $(E, \mathcal{O}^{n_2}, \varphi)$. Moreover the map $H^0(X, \varphi)$ induced by φ on the global sections of $\mathcal{O}^{n_2}$ is injective. Vice versa, if $(E_1, \mathcal{O}^{n_2}, \varphi)$ is a holomorphic triple such that $H^0(X, \varphi) : H^0(X, \mathcal{O}^{n_2}) \longrightarrow H^0(X, E_1)$ is injective, then by setting $V := H^0(X, \varphi)(H^0(X, \mathcal{O}^{n_2}))$ it is possible to recover a coherent system (E_1, V) of type (n_1, d_1, n_2).

The following can be easily proved (see also [5, Prop. 1.14]).

Lemma 2.7.1. *Let $\mathcal{O}^{n_2} \xrightarrow{\varphi} E_1$ be a triple such that $H^0(X, \mathcal{O}^{n_2})$ maps injectively into $H^0(X, E_1)$. Then the corresponding coherent system is σ-(semi)stable if and only if the triple is α-(semi)stable in the restricted sense that the stability condition is fulfilled by those subtriples which have a trivial second component, where the numbers σ and α are related by*

$$\alpha = \mu(E_1) + \sigma \frac{n_2 + n_1}{n_1} \qquad \sigma = -\mu(\mathcal{O}^{n_2} \oplus E_1) + \alpha \frac{n_1}{n_2 + n_1}.$$

Remark 2.7.2. Note that, as explicitly stated in the previous Lemma, the two notions of α-stability and σ-stability are not equivalent. In particular it is obvious that α-stability is a stronger condition, in the sense that a σ-stable coherent system can fail to be α-stable as a holomorphic triple since, in general, a proper subtriple with a non trivial second component and which violates the α-stability condition can exist, but such a triple does not correspond to a coherent subsystem. Some further discussions on the connections between the two definitions of stability are carried on in Section 4.2.

In fact the hypothesis on the injectivity of $H^0(X, \varphi)$ can be dropped as far as we are interested in α-stable holomorphic triples, since the following is true.

Lemma 2.7.3. *Let $\mathcal{T} = (E_1, \mathcal{O}^{n_2}, \varphi)$ be α-stable for some α. Then the map $H^0(X, \varphi)$ induced on the global sections is injective. The same holds for α-semistable triples provided that $\alpha \neq \alpha_m$.*

Proof. Assume by contraposition that $H^0(X, \varphi)$ is not injective and write $\emptyset \neq N := \ker(H^0(X, \varphi))$ and $t := \dim N > 0$. Then $N \otimes \mathcal{O}$ is a trivial subbundle of $\mathcal{O}^{n_2}$ of rank t, hence the triple $\mathcal{T}' = (0, \mathcal{O}^t, \varphi_{|\mathcal{O}^t})$ is a proper subtriple of $\mathcal{T}$. The α-semistability condition for $\mathcal{T}'$ is

$$\mu_\alpha(0, \mathcal{O}^t, \varphi_{|\mathcal{O}^t}) = \alpha \leqslant \frac{d_1 + \alpha n_2}{n_1 + n_2} = \mu_\alpha(\mathcal{T})$$

which is equivalent to $\alpha \leqslant \alpha_m(n_1, n_2, d_1, 0)$, and hence $\mathcal{T}$ is not α-stable for any α and can be α-semistable only if $\alpha = \alpha_m$. □

According to these Lemmas the study of α-(semi)stable holomorphic triples with a trivial second component can always be related to the study of σ-(semi)stable coherent systems. In fact in the following we will often start from results that are known to be true for coherent systems and deduce from them the corresponding results for holomorphic triples by keeping somehow into consideration also the subtriples with a non trivial second component (see e.g. Remark 3.4.9).

More in general, note that both coherent systems and holomorphic triples are part of the so called *augmented bundles* (also known as *decorated bundles* in [22]). This term denotes in general an object which consists of one or more vector bundles on a curve X together with certain extra data, which are in our cases some prescribed global sections of the vector bundle for coherent systems, or a map between the vector bundles for holomorphic triples. Even if, of course, every different class of augmented bundles has special and particular properties, there are important aspects common to all of these kinds of objects, as suggested by the two particular cases we considered in these first two Chapters. In particular those properties related to the notion of stability and to the construction of moduli spaces are rather similar for all the augmented bundles. A treatment of augmented bundles from a sort of "unified" point of view is carried on extensively in [5], to which the Reader is referred for more details.

Chapter 3

Holomorphic triples on the projective line

In the previous Chapters we have introduced the definition of holomorphic triple and the main properties of these objects. As we observed many times the general theory of triples is independent of the genus g of the curve on which the vector bundles are considered, however some results concerning moduli spaces and their properties need the further assumption that $g \geqslant 2$. In particular the problems of non-emptiness and of irreducibility deserve a particular treatment on the projective line and on elliptic curves. An analogous problem arises in the study of Coherent Systems and has been faced by Lange and Newstead in two papers ([15, 16]). In this Chapter and in the following one we extend (mutatis mutandis) the results already known for holomorphic triples to cover also the case of triples on curves of low genus.

In particular in this Chapter we consider a curve X of genus 0, hence isomorphic to the projective line $\mathbb{P}^1$. This is doubtless the hardest case we consider, mainly because the situation is complicated by the fact that on the projective line stable and semistable vector bundles are rare to appear. We can provide only partial results: necessary conditions for non-emptiness which show that the range of admissible values of α is in fact smaller than the usual one, but sufficient conditions only in some particular cases and quite often only for a subset of the whole admissible range of values of α. Note that also the case of coherent systems of genus 0 is still under development since it presents the same difficulties. In fact our results parallelize the results of [15] (even if it is worth to observe that in the case of coherent systems the study of the 0 genus case has been further improved in the two recent preprint [17, 18]).

All the results proved in this Chapter have been obtained in collaboration with Francesco Prantil of the University of Trento.

3.1 Setting

Before starting note that if $g = 0$, then α is always bigger that $2g-2$, hence the hypothesis of Theorem 1.7.5 are always fulfilled. Moreover in this particular case the Brill-Noether condition expressed by Theorem 1.7.1 can be restated in the following way.

Corollary 3.1.1. *Assume that $\mathcal{T} \in \mathcal{N}_\alpha(n_1, n_2, d_1, d_2) \neq \emptyset$; then $\mathcal{T}$ is smooth and*

$$d_1 \geqslant \frac{n_1^2 + n_2^2 - n_1 n_2 + n_1 d_2 - 1}{n_2}.$$

In the following we will need also the following Lemma. Note that it is independent of the genus of the curve X, hence we will make use of it freely also for elliptic and bielliptic curves in Chapter 4. If E is a vector bundle on a curve X and $\mathcal{L}$ is a line bundle, then the α-(semi)stability for E is equivalent to the α-(semi)stability for $E \otimes \mathcal{L}$. The Lemma shows that the same holds also for holomorphic triples.

Lemma 3.1.2. *Let X be a curve of any genus g and $\mathcal{L}$ be a line bundle over X. Then, for any $\alpha \in \mathbb{R}$, the triple $\mathcal{T} = (E_1, E_2, \varphi)$ is α-(semi)stable if and only if the triple $\mathcal{T} \otimes \mathcal{L} = (E_1 \otimes \mathcal{L}, E_2 \otimes \mathcal{L}, \varphi \otimes \mathrm{id}_{\mathcal{L}})$ is α-(semi)stable.*

Proof. The proof follows easily from the fact that

$$\deg(E_i \otimes \mathcal{L}) = \deg(E_i) + \mathrm{rank}(E_i) \deg(L).$$

Note that the requirement that $\mathcal{L}$ is a line bundle is necessary to have a 1-1 correspondence between the subtriples of $\mathcal{T}$ and those of $\mathcal{T} \otimes \mathcal{L}$. □

As a consequence of this Lemma we can always assume that $\deg E_i \geqslant 0$, $i = 1, 2$.

3.2 Constraints on α

As it happens for coherent systems, in the particular case of curves of genus 0 both the upper and lower bound of Proposition 1.2.1 can be improved, as shown by the following results.

Proposition 3.2.1. *Let $\bigoplus_{i=1}^{n_2} \mathcal{O}(b_i) \longrightarrow \bigoplus_{i=1}^{n_1} \mathcal{O}(a_i)$ be an α-stable holomorphic triple for some α. Then the following inequalities hold:*

$$\alpha > a_1 + \frac{na_1 - a - b}{m}, \tag{3.1}$$

$$\alpha > \frac{b + a - (m+n)b_m}{n}. \tag{3.2}$$

where $a = \deg\left(\bigoplus_{i=1}^{n_1} \mathcal{O}(a_i)\right)$ *and* $b = \deg\left(\bigoplus_{i=1}^{n_2} \mathcal{O}(b_i)\right)$.

Proof. The triple $(\mathcal{O}(a_1), 0, 0)$ is a proper subtriple and it is a straightforward computation to see that the α-stability condition for it gives the first inequality.

Consider now the proper subtriple $\left(\bigoplus_{i=1}^{n_1} \mathcal{O}(a_i), E_2', \varphi_{|E_2'}\right)$, where E_2' is the direct sum of all the $\mathcal{O}(b_i)$ except $\mathcal{O}(b_m)$. The α-stability condition for this subtriple is

$$\frac{a + b - b_{n_2} + (n_2 - 1)\alpha}{n_1 + n_2 - 1} < \frac{a + b + n_2\alpha}{n_1 + n_2}.$$

and it is equivalent to inequality (3.2). □

Proposition 3.2.2. *Let* $\bigoplus_{i=1}^{n_2} \mathcal{O}(b_i) \longrightarrow \bigoplus_{i=1}^{n_1} \mathcal{O}(a_i)$ *be an* α*-stable triple for some* α *and assume that* $n_2 < n_1$*. Then*

$$\alpha < 2\frac{a + b - (n_1 + n_2)b_1}{n_1 - n_2}. \tag{3.3}$$

Proof. Consider the subtriple $\left(\operatorname{im}\left(\varphi_{|\mathcal{O}(b_1)}\right), \mathcal{O}(b_1), \varphi_{|\mathcal{O}(b_1)}\right)$. The inequality follows from the α-stability condition for this triple. □

Remark 3.2.3. Note that $n_1 a_1 \geqslant a$, so the lower bound in (3.1) is always better or equal than the standard α_m. Moreover this bound is equivalent to $\alpha > \mu_1 - \mu_2$ if and only if, for all $i = 1, \dots, n_1$, $a_i = a/n_1$, that is if and only if $\bigoplus_{i=1}^{n_1} \mathcal{O}(a_i)$ is semistable. In the same way the lower bound in (3.2) is always better or equal to the standard one, equality holding if and only if $\bigoplus_{i=1}^{n_2} \mathcal{O}(b_i)$ is semistable. Note also the similarity to what happens for coherent systems (see Proposition 2.6.3 and the considerations immediately before it).

In the same way the upper bound in (3.3) is always stricter than that given by α_M and equality holds if and only if $\bigoplus_{i=1}^{n_2} \mathcal{O}(b_i)$ is semistable and $\operatorname{im}\left(\varphi_{|\mathcal{O}(b_1)}\right)$ is saturated (under the assumption that $\operatorname{im}\left(\varphi_{|\mathcal{O}(b_1)}\right)$ is not saturated, the upper bound above can be further improved).

In the following we will denote the bounds above by

$$\overline{\alpha}_m = a_1 + \frac{n_1 a_1 - a - b}{n_2}$$
$$\overline{\overline{\alpha}}_m = \frac{a + b - (n_1 + n_2)b_{n_2}}{n_1}$$
$$\overline{\alpha}_M = 2\frac{a + b - (n_1 + n_2)b_1}{n_1 - n_2}.$$

Corollary 3.2.4. *Let $\bigoplus_{i=1}^{n_2}\mathcal{O}(b_i) \longrightarrow \bigoplus_{i=1}^{n_1}\mathcal{O}(a_i)$ be an α-stable triple for some α and assume that $a_1 \geqslant 0$. Then, for all $i = 1, 2, \ldots, n_2$, $b_i > -\alpha$, and thus $b > -n_2\alpha$.*

Proof. Our assumption on the existence of at least one $a_i \geqslant 0$ is equivalent to require that $a + b + n_2\alpha \geqslant 0$. In fact an easy computation shows that, if $a_1 \geqslant 0$, then

$$\mu_1 - \mu_2 - a\frac{n_1 + n_2}{n_1 n_2} \leqslant a_1 + \frac{n_1 a_1 - a - b}{n_2} = \overline{\alpha}_m$$

so, in particular, α is greater than or equal to the first term in the above inequality, which is equivalent to our claim.

It follows therefore from the previous Proposition that

$$\frac{a + b + n_2\alpha}{n_1 + n_2} \leqslant \frac{a + b + n_2\alpha}{n_1 + n_2 - 1} < \frac{a + b + n_2\alpha}{n_1 + n_2} + \frac{b_{n_2} + \alpha}{n_1 + n_2 - 1}$$

and so $b_{n_2} > -\alpha$. □

3.3 Triples with E_2 semistable

In this section we prove some results for holomorphic triples on the projective line under the further assumption that one of the vector bundles is semistable. In particular in the remainder we will assume that E_2 is semistable, since the case in which E_1 is semistable can be dealt with by duality.

If E_2 is a semistable vector bundle, then we have the following Corollary of Lemma 3.1.2.

Corollary 3.3.1. *The triple $\mathcal{O}(b)^{n_2} \xrightarrow{\varphi} \bigoplus_{i=1}^{n_1}\mathcal{O}(a_i)$ is α-(semi)stable if and only if the triple $\mathcal{O}^{n_2} \xrightarrow{\varphi} \bigoplus_{i=1}^{n_1}\mathcal{O}(c_i)$ is α-(semi)stable, where for all $i = 1, 2, \ldots n_1$, $c_i := a_i - b$.*

According to this Corollary in the following, if $\mathcal{O}(b)^{n_2} \xrightarrow{\varphi} \bigoplus_{i=1}^{n_1}\mathcal{O}(a_i)$ is an α-stable triple, we can assume without loss of generality that $\mathcal{O}(b)^{n_2} \cong \mathcal{O}^{n_2}$. First of all we can prove an analogous of Lemma 2.6.1.

Proposition 3.3.2. *Let $\mathcal{O}^{n_2} \xrightarrow{\varphi} \bigoplus_{i=1}^{n_1}\mathcal{O}(c_i)$ be an α-stable triple for some $\alpha \in \mathbb{R}$. Then, for all $i = 1, 2, \ldots n_1$, $c_i \geqslant 1$.*

Proof. Assume that there exists $c_j < 0$ and consider the composition map

$$\mathcal{O}^{n_2} \xrightarrow{\varphi} \bigoplus_{i=1}^{n_1}\mathcal{O}(c_i) \xrightarrow{\pi} \mathcal{O}(c_j). \tag{3.4}$$

It is a classical result that a necessary condition for this map to be different from zero is $c_i \geqslant 0$, so the map is the zero map, that is $\mathcal{O}(c_j) \cap \mathrm{im}(\varphi) = \emptyset$. The subtriples $(\mathcal{O}(c_j), 0, 0)$ and $(E_1', \mathcal{O}^{n_2}, \varphi)$, where E_1' is the direct sum of all $\mathcal{O}(c_i)$ except $\mathcal{O}(c_j)$, are thus proper subtriples which contradict α-stability for every α.

Assume now that $c_n = 0$. Then the subtriple $\left(\bigoplus_{i=1}^{n-1} \mathcal{O}(c_i), \mathcal{O}^{m-1}, \varphi|_{\mathcal{O}^{m-1}}\right)$ is a proper subtriple and contradicts the α-stability condition. □

The previous result is in fact a particular case of a more general Theorem, in fact its proof relies on the fact that the composition map (3.4) is zero, and hence the holomorphic triple $(\bigoplus_{i=1}^{n_1} \mathcal{O}(c_i), \mathcal{O}^{n_2}, \varphi)$ splits into the direct sum of two non trivial holomorphic triples, contradicting the α-stability. In the same way as above but dropping the hypothesis on E_2, it is thus possible to prove also the following.

Proposition 3.3.3. *Let* $\mathcal{T} = (\bigoplus_{i=1}^{n_1} \mathcal{O}(a_i), \bigoplus_{i=1}^{n_2} \mathcal{O}(b_i), \varphi)$ *be an* α*-stable triple for some* α*. Then* $a_i > b_{n_2}$ *for all* $i = 1, 2, \ldots, n_1$*.*

In the next Theorem we obtain a characterization of the general α-stable triples analogous to that obtained on the projective line for σ-stable coherent systems in Theorem 2.6.2.

Theorem 3.3.4. *Fix* $\alpha \in \mathbb{R}$ *and let* $\mathcal{O}^{n_2} \xrightarrow{\varphi} \bigoplus_{i=1}^{n_1} \mathcal{O}(c_i)$ *be a general* α*-stable triple. Then the vector bundle* $\bigoplus_{i=1}^{n_1} \mathcal{O}(c_i)$ *is of generic splitting type, i.e.*

$$\bigoplus_{i=1}^{n_1} \mathcal{O}(c_i) \cong \mathcal{O}(q)^{n_1 - t} \oplus \mathcal{O}(q-1)^t$$

where the integers q *and* t *are defined by*

$$c = n_1 q - t, \ \text{with } 0 \leqslant t < n_1.$$

Proof. If $n_2 \leqslant n_1$, then the result follows from the results on coherent systems in Theorem 2.6.2. Assume now that $n_2 > n_1$ and write $F_c := \bigoplus_{i=1}^{n_1} \mathcal{O}(c_i)$ and $c := \deg F_c = \sum_{i=1}^{n_1} c_i$. It is a standard result (see [21, Thm 2.1] for a recent formulation) that there always exists a surjective map between $\mathcal{O}^{n_2}$ and F_c, so we can assume without loss of generality that the map φ is surjective. Therefore we have the exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{O}^{n_2} \xrightarrow{\varphi} F_c \longrightarrow 0$$

where $\mathcal{K} := \ker(\varphi)$, which shows that $\mathrm{rank}(\mathcal{K}) = n_2 - n_1$ and $k := \deg(\mathcal{K}) = -c$. Let us now consider the subscheme U of $\mathcal{N}_\alpha(n_1, n_2, c, 0)$ defined by the exact sequences

$$0 \longrightarrow \mathcal{K} \longrightarrow E' \longrightarrow F_c \longrightarrow 0.$$

Thus we have

$$\begin{aligned}\dim(U) =&\dim \operatorname{Ext}^1(F_c,\mathcal{K}) - \dim \operatorname{Aut}(\mathcal{K}) - \dim \operatorname{Aut}(F_c) + 1 =\\ =&h^1(\mathbb{P}^1,\mathcal{K}\otimes F_c^*) - (n_2-n_1)^2 - h^1(\mathbb{P}^1,\operatorname{End}(\mathcal{K}))+\\ &- n_1^2 - h^1(\mathbb{P}^1,\operatorname{End}(F_c)) + 1 =\\ =&h^0(\mathbb{P}^1,\mathcal{K}\otimes F_c^*) - n_1(n_2-n_1) - n_1k + (n_2-n_1)c - (n_2-n_1)^2+\\ &- n_1^2 + 1 - h^1(\mathbb{P}^1,\operatorname{End}(\mathcal{K})) - h^1(\mathbb{P}^1,\operatorname{End}(F_c)) =\\ =&\rho(n_1,n_2,c,0) - h^1(\mathbb{P}^1,\operatorname{End}(\mathcal{K})) - h^1(\mathbb{P}^1,\operatorname{End}(F_c)),\end{aligned}$$

where $h^0(\mathbb{P}^1,\mathcal{K}\otimes F_c^*) = 0$ because for every i,j, $k_j - c_i < 0$. Hence we have

$$h^1(\mathbb{P}^1,\operatorname{End}(\mathcal{K})) = h^1(\mathbb{P}^1,\operatorname{End}(F_c)) = 0. \qquad \square$$

Corollary 3.3.5. *Let $\mathcal{O}(b)^{n_2} \xrightarrow{\varphi} \bigoplus_{i=1}^{n_1}\mathcal{O}(a_i)$ be a general α-stable triple. Then the vector bundle $\bigoplus_{i=1}^{n_1}\mathcal{O}(a_i)$ is of the form*

$$\bigoplus_{i=1}^{n_1}\mathcal{O}(a_i) \cong \mathcal{O}(q)^{n_1-t}\oplus\mathcal{O}(q-1)^t$$

where the integers q and t are defined by

$$a = n_1q - t, \text{ with } 0\leqslant t < n_1.$$

In the general case, with the same technique, and observing that by tensorization with a suitable line bundle one can always obtain that $b_m > 0$, it is also possible to prove the following.

Theorem 3.3.6. *Let $\bigoplus_{i=1}^{n_2}\mathcal{O}(b_i) \xrightarrow{\varphi} \bigoplus_{i=1}^{n_1}\mathcal{O}(a_i)$ be a general α-stable triple such that $n_2 < n_1$. Then $\bigoplus_{i=1}^{n_2}\mathcal{O}(b_i)$ is of generic splitting type.*

Proof. Write $F_a = \bigoplus_{i=1}^{n_1}\mathcal{O}(a_i)$, $F_b = \bigoplus_{i=1}^{n_2}\mathcal{O}(b_i)$, $a = \deg F_a$ and $b = \deg F_b$. Without loss of generality we can assume that φ is injective, therefore we have the exact sequence

$$0 \longrightarrow F_b \xrightarrow{\varphi} F_a \longrightarrow C \longrightarrow 0$$

where $C = \operatorname{coker}(\varphi)$, thus we have $\operatorname{rank}(C) = n_1 - n_2$ and $\deg(C) = a - b$. Let us now consider the subscheme U of $\mathcal{N}_\alpha(n_1,n_2,a,b)$ define by the exact sequences

$$0 \longrightarrow F_b \xrightarrow{\varphi} E' \longrightarrow C \longrightarrow 0.$$

In an analogous way as in the proof of the previous result we can compute

$$\dim(U) = \rho(n_1,n_2,a,b) - h^1(\mathbb{P}^1,\operatorname{End}(F_b)) - h^1(\mathbb{P}^1,\operatorname{End}(C))$$

and conclude $h^1(\mathbb{P}^1,\operatorname{End}(F_b)) = 0$. $\square$

The following Proposition shows that in an α-stable triple $\mathcal{T} = (E_1, E_2, \varphi)$ the image of the map φ in some sense reaches all the direct summands of E_1.

Proposition 3.3.7. *Let $\mathcal{T} : \bigoplus_{i=1}^{n_2} \mathcal{O}(b_i) \xrightarrow{\varphi} \mathcal{O}(a)^{n_1}$ be an α-stable triple. Then, for any direct summand $\mathcal{O}(a)$, $\operatorname{im}(\varphi) \cap \mathcal{O}(a) \neq \emptyset$.*

Proof. By contraposition let $\mathcal{O}(a)$ be such that $\operatorname{im}(\varphi) \cap \mathcal{O}(a) = \emptyset$. Then the subtriple $\mathcal{O}^{n_2} \xrightarrow{\varphi} \mathcal{O}(a)^{n_1-1}$ is a proper subtriple and the α-stability condition for it is

$$\frac{(n_1-1)a + n_2\alpha}{n_1+n_2-1} < \frac{n_1 a + n_2\alpha}{n_1+n_2},$$

which leads to $\alpha < a = \mu_1 - \mu_2 = \alpha_m$. Thus the triple $\mathcal{T}$ is not α-stable for any α. □

Fix now n_1, n_2 and a triple $\mathcal{T}$ of the form $\mathcal{O}^{n_2} \xrightarrow{\varphi} \bigoplus_{i=1}^{n_1} \mathcal{O}(c_i)$, and write $F_c := \bigoplus_{i=1}^{n_1} \mathcal{O}(c_i)$, $c := \deg F_c$. It is an easy computation using Riemann-Roch formula to see that a necessary condition for $\mathcal{T}$ to be related to a coherent system is $n_2 \leqslant n_1 + c$. For the general α-stable triple this is, in fact, also a sufficient condition as follows from 2.7.3. This can be deduced also directly from the exact sequences that are proved to exist in the following Lemmas.

Lemma 3.3.8. *Let $\mathcal{O}^{n_2} \xrightarrow{\varphi} \bigoplus_{i=1}^{n_1} \mathcal{O}(c_i)$ be a general α-stable triple such that $n_2 < n_1$. Then there exists an exact sequence*

$$0 \longrightarrow \mathcal{O}^{n_2} \longrightarrow \bigoplus_{i=1}^{n_1} \mathcal{O}(c_i) \longrightarrow G \longrightarrow 0. \tag{3.5}$$

In particular $H^0(\mathbb{P}^1, \mathcal{O}^{n_2})$ maps injectively into $H^0(\mathbb{P}^1, \bigoplus_{i=1}^{n_1} \mathcal{O}(c_i))$.

Proof. By Proposition 3.3.2, $\bigoplus_{i=1}^{n_1} \mathcal{O}(c_i)$ is generated by its global sections. So, by a classical result of Atiyah (see [1, Thm. 2]), we obtain the exact sequence of the statement. The last claim is immediate. □

Lemma 3.3.9. *Let $\mathcal{O}^{n_2} \xrightarrow{\varphi} F_c$ be a general α-stable triple such that $n_1 = n_2$. Then there exists an exact sequence*

$$0 \longrightarrow \mathcal{O}^{n_2} \longrightarrow F_c \longrightarrow T \longrightarrow 0, \tag{3.6}$$

where T is a torsion sheaf. In particular $H^0(\mathbb{P}^1, \mathcal{O}^{n_2})$ maps injectively into $H^0(\mathbb{P}^1, F_c)$.

Proof. Again, by Proposition 3.3.2, F_c is generated by its global sections. Fixing a point x of $\mathbb{P}^1$ and a basis for $\mathcal{O}_x^{n_2}$ defines an injective map $\mathcal{O}_x^{n_2} \longrightarrow F_{cx}$, and therefore an exact sequence (3.6). □

Lemma 3.3.10. *Let $\mathcal{O}^{n_2} \xrightarrow{\varphi} F_c$ be a general α-stable triple such that $n_2 > n_1$. Then there exists an exact sequence*

$$0 \longrightarrow G \longrightarrow \mathcal{O}^{n_2} \longrightarrow F_a \longrightarrow 0. \tag{3.7}$$

Moreover, if $n_1 < n_2 \leqslant n_1 + c$, $H^0(\mathbb{P}^1, \mathcal{O}^{n_2})$ maps injectively into $H^0(\mathbb{P}^1, F_c)$.

Proof. In general, by Proposition 3.3.2, the existence of an exact sequence (3.7) is a standard result (again see [21, Thm 2.1] for a recent proof). Assume now that $n_1 < n_2 \leqslant n_1+c$ and let V be a dimension n_2 subspace of $H^0(\mathbb{P}^1, F_c)$. Then the sheaf $V \otimes \mathcal{O}$ defines a map from the global sections of $\mathcal{O}^{n_2}$ to that of F_c which is injective (in fact an isomorphism) into V. □

Remark 3.3.11. Note that from Lemma 2.7.3 it follows that, for any $\alpha > \alpha_m$, α-stable holomorphic triples of type $(n_1, n_2, c, 0)$ with a trivial second component cannot exist unless $n_2 \leqslant n_1 + c$. Consequently α-stable holomorphic triples of type $(n_1, n_2, a, n_2 b)$ cannot exist unless $n_2 + n_1(b-1) \leqslant a$.

In the case in which $n_2 \leqslant n_1 + c$ we can inherit some properties for the triple $\mathcal{T}$ by the case of coherent systems (see Theorem 2.6.4).

Theorem 3.3.12. *Let $\mathcal{T} = (\bigoplus_{i=1}^{n_1} \mathcal{O}(c_i), \mathcal{O}^{n_2}, \varphi)$ be an α-stable triple and write $c := \sum_{i=1}^{n_1} c_i$. Then the following is true:*

1. *for all $i = 1, 2, \ldots, n_1$, $c_i \geqslant 1$;*
2. *if $\mathcal{T}$ is general, then $\bigoplus_{i=1}^{n_1} \mathcal{O}(c_i)$ is of generic splitting type:*
$$\bigoplus_{i=1}^{n_1} \mathcal{O}(c_i) \cong \mathcal{O}(q)^{n_1-t} \oplus \mathcal{O}(q-1)^t;$$
3. *if $\mathcal{T}$ is general and $n_2 < n_1$, then the vector bundle G of (3.5) is of the form*
$$G \cong \mathcal{O}(q+l+1)^u \oplus \mathcal{O}(q+l)^{n_1-n_2-u}.$$
4. *$\alpha > \mu_1 + \frac{n_1+n_2}{n_1 n_2} t$;*
5. *if $n_2 < n_1$ then $\alpha < 2\frac{c}{n_1-n_2} - \frac{u(n_1+n_2)}{n_2(n_1-n_2)} = \alpha_M - \frac{u(n_1+n_2)}{n_2(n_1-n_2)}$;*
6. *the bounds at the previous items are sharp in the case $n_2 = 1$,*

where t, q, l and u are defined by

$$c = n_1 q - t \text{ with } 0 \leqslant t < n_1, \text{ and } n_2 q - t = l(n_1 - n_2) + u \text{ with } 0 \leqslant u < n_1 - n_2.$$

Proof. All the results above can be easily deduced from the respective results in 2.6.4 using Lemma 2.7.1. □

Corollary 3.3.13. *Let $\mathcal{T} = (\bigoplus_{i=1}^{n_1} \mathcal{O}(a_i), \mathcal{O}(b)^{n_2}, \varphi)$ be an α-stable triple and write $a := \sum_{i=1}^{n_1} a_i$. Then the following is true:*

1. *for all $i = 1, 2, \ldots, n_1$, $a_i \geqslant b + 1$;*

2. *if $\mathcal{T}$ is general $\bigoplus_{i=1}^{n_1} \mathcal{O}(a_i)$ is of generic splitting type:*

$$\bigoplus_{i=1}^{n_1} \mathcal{O}(a_i) \cong \mathcal{O}(q)^{n_1-t} \oplus \mathcal{O}(q-1)^t;$$

3. *if $\mathcal{T}$ is general and $n_2 < n_1$, then the vector bundle G of (3.5) is of the form*

$$G \cong \mathcal{O}(q+l+1)^u \oplus \mathcal{O}(q+l)^{n_1-n_2-u}.$$

4. *$\alpha > \mu_1 - \mu_2 + \frac{n_1+n_2}{n_1 n_2} t$;*

5. *if $n_2 < n_1$, then $\alpha < 2\frac{a-n_1 b}{n_1-n_2} - \frac{u(n_2+n_1)}{n_2(n_1-n_2)}$;*

6. *the bounds at the previous items are sharp in the case $n_2 = 1$,*

where t, q, l and u are defined by

$$a = n_1 q - t \text{ with } 0 \leqslant t < n_1, \; a - n_2 b = (l+q)(n_1-n_2) + u \text{ with } 0 \leqslant u < n_1 - n_2.$$

Remark 3.3.14. Note that if $\bigoplus_{i=1}^{n_1} \mathcal{O}(a_i)$ is of generic splitting type, then the lower bound above coincides with the lower bound of Proposition 3.2.1.

3.4 Some special cases

In the final part of this Chapter we deal with the analysis of some special cases, with the goal of proving, at least in these particular situations, some sufficient conditions for non-emptiness, and hence obtaining a deeper understanding of the moduli spaces. In particular we take into consideration many of the cases Lange and Newstead investigated in [15] for coherent systems, mainly for two reasons. First of all we can make comparisons between the results we prove for holomorphic triples and those that have been proved for coherent systems. Second and more important, our strategy is mainly to refer back to coherent systems to deal with subtriples with a trivial component and then to arrange in some way the remaining cases (see also Remark 3.4.9).

In the following we will write $F_a := \bigoplus_{i=1}^{n_1} \mathcal{O}(a_i)$, $F_b := \bigoplus_{i=1}^{n_2} \mathcal{O}(b_i)$, $F_c := \bigoplus_{i=1}^{n_1} \mathcal{O}(c_i)$, $a := \deg F_a$, $b := \deg F_b$ and $c := \deg F_c$. Note that, as usual, the cases with n_1 and n_2 interchanged can be dealt with appealing to duality.

The case $n_2 = 1$

The triple $(\bigoplus_{i=1}^{n_1} \mathcal{O}(a_i), \mathcal{O}(b), \varphi)$ can always be thought as a coherent system (E, V) with $\dim(V) = 1$ and the two notions of stability for the two different structures are in this case exactly the same, so the results from [15] apply:

Theorem 3.4.1. *Suppose $n \geqslant 2$. Then the moduli space of α-stable triples with parameters $(n_1, 1, a, b)$ is non-empty if and only if*

$$\mu_1 - \mu_2 + \frac{n_1+1}{n_1} t < \alpha < 2\frac{a - n_1 b}{n_1 - 1} - \frac{(n_1+1)}{n_1 - 1} u.$$

where t, q, l and u are defined by

$a = n_1 q - t$ *with* $0 \leqslant t < n_1$, *and* $a - b = (l+q)(n_1 - 1) + u$ *with* $0 \leqslant u < n_1 - 1$.

The case $n_2 = 2$

Note that, again, if the vector bundle E_2 is semistable and $n_1 \geqslant 3$, then the triples $(\bigoplus_{i=1}^{n_1} \mathcal{O}(a_i), \mathcal{O}(b)^2, \varphi)$ can be related to coherent systems of type $(n_1, a - n_1 b, 2)$ but the two notions of α-stability and σ-stability are not, a priori, equivalent. We would like to prove that the bounds of Theorem 3.3.12 are sharp also in this case, that is

Conjecture 3.4.2. Suppose $n_1 \geqslant 3$. Then the moduli space of α-stable triples with parameters $(n_1, 2, a, 2b)$ is non-empty if and only if α satisfies

$$\alpha_m + \frac{n_1+2}{2n_1} t < \alpha < \alpha_M - \frac{u(n_1+2)}{2(n_1-2)},$$

a, b and n_1 satisfy the Brill-Noether condition

$$a - n_1 b \geqslant \frac{1}{2} n_1 (n_1 - 2) + \frac{3}{2}$$

and $(n_1, a - n_1 b) \neq (4, 6)$.

Unfortunately so far we do not have a complete proof of this fact (see Remark 3.4.5 for details and comments). A useful tool for the proof could be the following Lemma, which is a generalization of [15, Lm. 5.3].

Definition 3.4.3. *Let $\mathcal{T} = (F_c, \mathcal{O}^2, \varphi)$ be a general triple as in 3.3.12, that is $F_c \cong \mathcal{O}(q)^{n_1 - t} \oplus \mathcal{O}(q-1)^t$. For $t \geqslant 1$ and $d \leqslant 0$ we define the d-*invariant $\delta_d(\mathcal{T})$ *of the triple as the minimal rank of a direct factor of $\mathcal{O}(q-1)^t$ containing the image of some $\mathcal{O}(d) \subseteq \mathcal{O}^2$ under the composed map*

$$\mathcal{O}(d) \longrightarrow \mathcal{O}^2 \xrightarrow{\varphi} F_c \longrightarrow \mathcal{O}(q-1)^t.$$

Lemma 3.4.4. *Let $q \geqslant 1$ and $t \geqslant 1$. Then the general triple $\mathcal{T} = (F_c, \mathcal{O}^2, \varphi)$ satisfies*

$$\delta_d(\mathcal{T}) = \begin{cases} t, & q \geqslant t - 2d + 1, \\ t-1, & q = t - 2d, \\ q + 2d, & 1 \leqslant q \leqslant t - 2d - 1. \end{cases}$$

Proof. The composition map $\mathcal{O}(d) \longrightarrow F_c$ is of the form

$$\begin{pmatrix} f_1 & \dots & f_{n_1-t} & g_1 & \dots & g_t \\ f'_1 & \dots & f'_{n_1-t} & g'_1 & \dots & g'_t \end{pmatrix}^T \cdot \begin{pmatrix} h \\ h' \end{pmatrix}$$

where f_i, f'_i are binary forms of degree q, g_i, g'_i are binary forms of degree $q-1$ and h, h' are binary forms of degree $-d$. The pair h, h' corresponds naturally to the matrix

$$H = \begin{pmatrix} h_1 & \dots & h_{-d+1} \\ h'_1 & \dots & h'_{-d+1} \end{pmatrix}$$

whose elements are complex numbers such that every 2×2 minor has full rank. Thus the composition map is given by the vector

$$(hg_1 + h'g'_1, \dots, hg_t + h'g'_t)^T.$$

The proof now is a slight variation of [15, Lm. 5.3]; for the sake of completeness we report here the details. Note that, by definition of $\delta_d(\mathcal{T})$ we have

$$t - \delta_d(\mathcal{T}) = \max_{A \in GL(t,\mathbb{K}),\ H \text{ as above}} \{ \text{ number of 0 entries in } Av \},$$

which is equivalent to the maximum number of linearly independent vectors $(\lambda_1, \dots, \lambda_t) \in \mathbb{K}^t$ such that

$$(\lambda_1 hg_1 + \lambda_1 h'g'_1) + \ldots + (\lambda_t hg_t + \lambda_t h'g'_t) = 0, \tag{3.8}$$

the maximum to be taken over all matrices H as above. Consider the Segre embedding

$$\iota : \mathbb{P}^{-2d+1} \times \mathbb{P}^{t-1} \longrightarrow \mathbb{P}^{-2dt+2t-1}$$

and note that the conditions on the minors of H define open subsets of the projective space $\mathbb{P}^{-2d+1}$. Consider now the map $\psi : \mathbb{K}^{-2dt+2t} \longrightarrow H^0(\mathbb{P}^1, \mathcal{O}(a-1))$ defined by

$$(x_1, y_1, \dots, x_t, y_t) \mapsto x_1 g_1 + y_1 g'_1 + \ldots x_t g_t + y_t g'_t,$$

where $x_i, y_i \in \mathbb{K}^{-d+1}$, and write W for its kernel. We have

$$t - \delta_d(\mathcal{T}) - 1 \leqslant \dim \left(\mathbb{P}(W) \cap \iota(\mathbb{P}^{-2d+1} \times \mathbb{P}^{t-1}) \right). \tag{3.9}$$

By varying $\mathcal{T}$ generically the vector space W can be chosen arbitrarily, and for a general choice of g_i and g'_i we have $\dim W = \max\{\, 2t - 2dt - q, 0 \,\}$.

If $q \geqslant t - 2d + 1$ then

$$(2t - 2dt - q) + (-2d + 1 + t - 1) < -2dt + 2t - 1,$$

hence W can be chosen so that $\mathbb{P}(W) \cap \iota(\mathbb{P}^{-2d+1} \times \mathbb{P}^{t-1}) = \emptyset$ and thus, by (3.9), $\delta_d(\mathcal{T}) = t$.

If $q = t-2d$, then we can choose W so that $\mathbb{P}(W)\cap\iota(\mathbb{P}^{-2d+1}\times\mathbb{P}^{t-1})$ is finite and non-empty, hence by (3.9) $\delta_d(\mathcal{T}) \geqslant t - 1$. Any point in the intersection $\mathbb{P}(W) \cap \iota(\mathbb{P}^{-2d+1} \times \mathbb{P}^{t-1})$ is in fact a solution of (3.8), thus $t - \delta_d(\mathcal{T}) \geqslant 1$.

If $1 \leqslant q \leqslant t - 2d - 1$ we can choose W so that

$$\dim \left(\mathbb{P}(W) \cap \iota(\mathbb{P}^{-2d+1} \times \mathbb{P}^{t-1})\right) = t - q - 2d,$$

and this intersection is irreducible, hence the maximal dimension of a linear space contained in it is exactly $t - q - 2d - 1$, which concludes the proof. □

Remark 3.4.5. Note that, according to Theorem 2.6.4, all the conditions we assume in the statement of Conjecture 3.4.2 are necessary conditions for the existence of a σ-stable coherent system, and so, by Remark 2.7.2, are necessary conditions also in the case of triples. Moreover these conditions are also sufficient for a coherent system to exist, so to show sufficiency of the conditions we have only to show that the corresponding triple is α-stable for a proper subtriple $\mathcal{T}' = (F', E', \varphi')$ where E' is a rank 1 non trivial subbundle of $\mathcal{O}^2$, and so $E' \cong \mathcal{O}(d)$ for some $d < 0$. Denote $h = \operatorname{rank}(F')$ and $f = \deg(F')$. Let us compute

$$\begin{aligned}\Delta_\alpha(\mathcal{T}', \mathcal{T}) =&\mu_\alpha(\mathcal{T}') - \mu_\alpha(\mathcal{T}) = \frac{f + d + \alpha}{h + 1} - \frac{n_1 q - t + 2\alpha}{n_1 + 2} = \\ =&\alpha\frac{n_1 - 2h}{(h + 1)(n_1 + 2)} + \frac{f + d}{h + 1} - \frac{n_1 q - t}{n_1 + 2}.\end{aligned} \tag{3.10}$$

Note that if $t > 0$ and we write S for the image of F in $\mathcal{O}(q-1)$ and $s = \operatorname{rank}(S)$, then, by the definition of the invariant $\delta_d(\mathcal{T})$, $h^0(S^*(q-1)) \geqslant \delta_d(\mathcal{T})$ and, by Riemann-Roch, $h^0(S^*(q-1)) = -\deg(S) + sq$, hence

$$f \leqslant (h - s)q + sq - \delta_d(\mathcal{T}) = hq - \delta_d(\mathcal{T}).$$

Moreover from $2q - t = l(n_1 - 2) + u \geqslant n_1 - 2 \geqslant 0$ it follows that $2q \geqslant t$.

So far we are able to prove the Conjecture only under the further assumption that $n_1 \leqslant 2h$, while the remaining case needs a deeper analysis.

Case $n_1 < 2h$. The coefficient of α is strictly negative, so we obtain a strict upper bound for (3.10) by considering $\alpha = \overline{\alpha}_m = \frac{2q+t}{2}$, therefore we can prove

$$\frac{(2q+t)(n_1-2h)}{2(h+1)(n_1+2)} + \frac{f+d}{h+1} - \frac{n_1q-t}{n_1+2} \leqslant 0,$$

or equivalently,

$$t - 2qh + 2(f+d) \leqslant 0. \tag{3.11}$$

If $t = 0$ the α-stability condition for the subtriples is always fulfilled as $\alpha > q = \alpha_m$, so we can assume $t \geqslant 1$. In this case we have

$$t - 2qh + 2(f+d) \leqslant t - 2\delta_d + 2d$$

and we are now in position for applying the previous Lemma.

If $q \geqslant t-d+1$ then $\delta_d = t$, so condition (3.11) is $\delta \geqslant 2d$ which is always true for $d < 0$. If $q = t-d$ then condition (3.11) is $2d \leqslant t-2$ which is true because $2d \leqslant -2$ and $t-2 \geqslant -1$. Finally if $q \leqslant t-d-1$, condition (3.11) is $2q - t \geqslant 0$ which is always fulfilled as already observed.

Case $n_1 = 2h$. In this case (3.10) is

$$\frac{f+d}{h+1} - \frac{2hq-t}{2(h+1)}.$$

Assume first that $t \neq 0$, so we have to prove

$$2d - 2\delta + t < 0$$

which follows in the same way of the previous case.

Let now $t = 0$, so that $F_c \cong \mathcal{O}(q)^{n_1}$. In this case the α-stability for $\mathcal{T}'$ is

$$\frac{d + rq + \alpha}{r+1} < \frac{rq + \alpha}{r+1},$$

which is equivalent to $d < 0$, always true under our assumptions.

The case $n_2 = n_1 - 1$

In this case we have results only for the moduli space for large α. We will show that for large enough α the moduli space is always non-empty.

In particular we can show the following.

Proposition 3.4.6. *There exists α for which $\mathcal{N}_\alpha\big(n_1, n_1-1, a, (n_1-1)b\big)$ is non-empty if and only if $a \geqslant n_1(b+1)$*

Proof. In Theorem 2.6.4 it has been proved that there exists an α-stable coherent system (F_c, V) of type (n_1, c, n_1-1) provided that α is sufficiently large and $c \geqslant n_1$. Let us consider the exact sequence

$$0 \longrightarrow \mathcal{O}^{n_1-1} \xrightarrow{\varphi} F_c \longrightarrow G \longrightarrow 0$$

associated to such a coherent system.

We claim that the triple $\mathcal{T} = (F_c, \mathcal{O}^{n_1-1}, \varphi)$ is α-stable for large alpha. If it is so, then the triple $\mathcal{O}(b)^{n_1-1} \xrightarrow{\varphi} F_a$, where as usual $c_i = a_i - b$, is α-stable for large α, proving the Proposition.

Let us prove our claim. From the stability of the corresponding coherent system we already know that α-stability condition is fulfilled by all subtriples (F', E', φ') where E' is a trivial subbundle of $\mathcal{O}^{n_1-1}$, so consider a proper subtriple

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathcal{O}^{n_1-1} & \xrightarrow{\varphi} & F_c \\ & & \uparrow & & \uparrow \\ & & E' & \xrightarrow{\varphi'} & F' \end{array}$$

where E' is non-trivial, denote it by $\mathcal{T}'$ and write $l = \operatorname{rank}(E')$, $e = \deg(E')$, $h = \operatorname{rank}(F')$ and $f = \deg(F')$. For large enough α, there exists $\varepsilon > 0$ such that $\alpha = \alpha_M - \varepsilon = 2c - \varepsilon$. We therefore have

$$\begin{aligned} \Delta_\alpha(\mathcal{T}', \mathcal{T}) =& \frac{f + e + l\alpha}{l + h} - \frac{c + (n_1 - 1)\alpha}{2n_1 - 1} = \\ =& \varepsilon \frac{h(n_1 - 1) - n_1 l}{(l + h)(2n_1 - 1)} + \frac{c(l - h) + f + e}{l + h}. \end{aligned} \tag{3.12}$$

Provided that $e \leqslant 0$ and $f \leqslant c$, we have

$$c(l - h) + e + f \leqslant c(l - h) + c = c(l - h + 1),$$

so, whenever $l < h - 1$, the second term in (3.12) is strictly negative, hence for small enough ε α-stability condition is fulfilled for $\mathcal{T}'$. Note that $l \leqslant h$, so we have to analyse only two more cases to complete the proof.

Case $l = h - 1$. In this case we have for the second term in (3.12)

$$\frac{-c + f + e}{2h - 1} \leqslant \frac{e}{2h - 1} \leqslant 0$$

and equality holds if and only if $c = f + e$, from which follows $c = f$ and $e = 0$. In this case $E' \cong \mathcal{O}^{h-1}$, hence α-stability is guaranteed by the result on coherent systems.

Case $l = h$. We have for the second term in (3.12)

$$\frac{f + e}{2h} = \frac{e}{h} \leqslant 0.$$

Moreover the first term is

$$\varepsilon \frac{-1}{2h(2n - 1)} < 0,$$

thus we are done. ∎

Remark 3.4.7. Note that if E_2 is a non-semistable vector bundle, then the previous argument does not apply, in fact the bound $\overline{\alpha}_M$ is always better than α_M, hence $\varepsilon > 0$ cannot be too small, or, in other words, we cannot be sure to reach a value of α sufficiently close to α_M, as it is required by the previous proof.

The case $n_2 = n_1$

Also in this case we can obtain a partial result: an α-stable triple exists provided that α is sufficiently large and a is sufficiently large in respect to b. Write $n := n_1 = n_2$.

Theorem 3.4.8. *The moduli space $\mathcal{N}_\alpha\big(n, n, a, nb\big)$ is non-empty for sufficiently large α if and only if $a > n(b+1)$, and in this case there is no upper bound on α.*

Proof. Let

$$\begin{array}{ccc} \mathcal{O}^n & \xrightarrow{\varphi} & F_c \\ \uparrow & & \uparrow \\ E' & \xrightarrow{\varphi'} & F' \end{array}$$

be a proper subtriple of $\mathcal{T}$, and denote it by $\mathcal{T}'$. Write $F_c = \mathcal{O}(q)^{n-t} \oplus \mathcal{O}(q-1)^t$ and, as usual, write $r = \operatorname{rank}(E')$, $h = \operatorname{rank}(F')$, $e = \deg(E')$, $f = \deg(F')$ and compute

$$\begin{aligned} \Delta_\alpha(\mathcal{T}', \mathcal{T}) =& \frac{f + e + r\alpha}{r+h} - \frac{nq - t + n\alpha}{2h} = \\ =& \alpha \frac{n(r-h)}{2n(r+h)} - \frac{q}{2} + \frac{2n(f+e) + t(r+h)}{2n(r+h)}. \end{aligned} \tag{3.13}$$

If $r - h < 0$ then the coefficient of α in (3.13) is strictly negative, so (3.13) is itself strictly negative, provided that α is sufficiently large.

If $r = h$, we have to show that

$$-nhq + n(f+e) + th < 0. \tag{3.14}$$

We distinguish two cases.

Case $f + e > 0$. In this case the first part of the inequality (3.14) is big for small h, so let us assume $h = 1$ This leads to show that $n(f+e) - nq + t < 0$. We have

$$n(f+e) - nq + t \leqslant nhq - nq + t + ne = t + ne,$$

and this, provided that $t < n$, is always true if $e \leqslant -1$. If $e = 0$ then the α-stability follows from the σ-stability of the corresponding coherent system.

Case $f + e \leqslant 0$. In this case we have

$$-nhq + n(f+e) + th \leqslant t - nq,$$

so it suffices to show that $t - nq < 0$. This is always true if $\deg(F_c) \geqslant 1$. If $\deg(F_c) = 0$, then $f = 0$, and α-stability for $\mathcal{T}'$ follows from the σ-stability of the corresponding coherent system. □

Remark 3.4.9. Assume that F_b is semistable, and so, according to our convention, $F_b \cong \mathcal{O}^{n_2}$. In all the cases we have analyzed the strategy was the same: to refer back to the coherent system related to the triple to guarantee the α-stability condition for all those subtriples with a trivial second component, and then to check in some way all the remaining subtriples. The result is that in all the cases we took into consideration the α-stability condition for the subtriples of the second kind above does not ever provide a stronger condition than the α-stability for coherent system does, so it is natural to ask whether this is always true for $n_2 \leqslant n_1 + a$ and for all α in the admissible range. A good question but, unfortunately, so far we do not have a good answer for it.

Conjecture 3.4.10. Let $g = 0$ and $n_2 \leqslant n_1 + a$. Then α-stable triples $(E_1, \mathcal{O}^{n_2}, \varphi)$ with E_2 semistable are equivalent to σ-stable coherent systems, where α and σ are related by Lemma 2.7.1.

In the next Chapter a positive answer to the question above is provided for curves of any genus g, but only when $\alpha = \alpha_m^+$ (see Theorem 4.2.4).

Chapter 4

Holomorphic triples on elliptic and bielliptic curves

In this Chapter we study the α-stability for holomorphic triples over curves of genus $g = 1$ and on bielliptic curves, hence in the remainder X will always denote an elliptic curve. We are able to prove necessary and sufficient conditions for the moduli space of α-stable triples on elliptic curves to be non-empty and, in these cases, we can show that it is smooth and irreducible. We deepen also the analysis of the relationships that exist between the two stability conditions for holomorphic triples and coherent systems, at least in some particular cases. This is interesting by its own, but here will be used as a tool to prove the existence results for elliptic triples.

From these results we move to deduce properties of the moduli spaces on bielliptic curves, studying in particular pullbacks of holomorphic triples and elementary transformations.

The results on elliptic curves are due to a collaboration with Francesco Prantil of the University of Trento, while the results on bielliptic curves have been proved in a joint work with Francesco and Edoardo Ballico of the University of Trento.

4.1 First results

First of all we collect some general results concerning α-stability for holomorphic triples on elliptic curves.

Remark 4.1.1. Note that, by tensorization for a suitable line bundle according to Lemma 3.1.2, it is possible to operate as we did for triples on the projective line and to reduce to the case in which $\deg(E_2) = 0$ if and only if $n_2 | d_2$

i.e. if and only if $\mu(E_2) \in \mathbb{Z}$. If we assume that $E_2 = \bigoplus_{j=1}^{k} E_2^j$ is polystable with all the direct summands E_2^j with the same slope, then $\gcd(d_2^j, n_2^j) = 1$, so we can reduce to the degree 0 case if and only if $n_2^j = 1$.

In any case, again according to Lemma 3.1.2, we can always assume without loss of generality that $d_1 > 0$ and $d_2 > 0$. Moreover if we write E_1 and E_2 as direct sums of indecomposable vector bundles, we can always assume that every direct summand which appears in the decomposition has positive degree, and so $H^1(X, E_1) = H^1(X, E_2) = 0$.

Remark 4.1.2. Note that, for any integer l, $\gcd(n_1, d_1) = \gcd(n_1, d_1 + ln_1)$, so the number of direct summands in the "canonical decomposition" of a semistable vector bundle E_1 is the same as in the decomposition of $E_1 \otimes \mathcal{L}$ for any line bundle $\mathcal{L}$.

Lemma 4.1.3. *Suppose that the triple $\mathcal{T} = (E_1, E_2, \varphi)$ is α-stable for some α. Then the set*

$$I(\mathcal{T}) := \{\alpha \in \mathbb{R} \mid \mathcal{T} \text{ is } \alpha\text{-stable}\}$$

is an open, non-empty interval in $(\alpha_m, +\infty)$; if, moreover, $n_1 \neq n_2$, then $I(\mathcal{T}) \subseteq (\alpha_m, \alpha_M)$.

Proof. $I(\mathcal{T})$ is obviously non-empty. Fix a subtriple $\mathcal{T}'$ of $\mathcal{T}$. The stability condition for $\mathcal{T}'$ gives a condition on α, which can be either $\alpha > a$ or $\alpha < b$, where a, b belong to the discrete set of the critical values of α. The set $I(\mathcal{T})$ is thus the intersection of semi-infinite open intervals, hence it is itself open. □

4.2 Links between triples and coherent systems

This section is devoted to deepen the relationship between holomorphic triples and coherent systems. Note that the results here proved are independent of the genus g of X, hence they remain true also for non-elliptic curves.

Recall also that in Lemma 2.7.3 we have proved that for a holomorphic triple $\mathcal{T} = (E_1, E_2, \varphi)$ the condition that the map induced by φ on the global sections is injective is a necessary condition for α-stability, and that in Section 1.5 we agree to denote by α_m^+ a value of α which lies in the interval between $\alpha_m = \mu_1 - \mu_2$ and the first critical value for $\mathcal{T}$ and by α_M^- a value of α in the interval between the last critical value and α_M. The next results are based on the fundamental observation stated in the following Lemma.

Lemma 4.2.1. *Let $\mathcal{T} = (E_1, E_2, \varphi)$ be a holomorphic triple such that E_1 and E_2 are semistable vector bundles and assume that $\alpha = \alpha_m^+$. Then in a subtriple $\mathcal{T}' = (E_1', E_2', \varphi')$ that contradicts α_m^+-stability either $E_1' = 0$ and E_2' has maximum slope, or both E_1' and E_2' have maximum slope.*

Proof. Write as usual $d_1' = \deg(E_i')$ and $n_i' = \operatorname{rank}(E_i')$, $i = 1, 2$. The α-stability condition for $\mathcal{T}'$ is

$$\frac{d_1' + d_2' + \alpha n_2'}{n_1' + n_2'} < \frac{d_1 + d_2 + \alpha n_2}{n_1 + n_2},$$

which is equivalent to

$$(d_1' + d_2')(n_1 + n_2) - (d_1 + d_2)(n_1' + n_2') < \alpha(n_2 n_1' - n_1 n_2'). \tag{4.1}$$

Write $\overline{\alpha} := \frac{(d_1'+d_2')(n_1+n_2)-(d_1+d_2)(n_1'+n_2')}{n_2 n_1' - n_1 n_2'}$. We distinguish three cases.

Case $\boldsymbol{n_2 n_1' - n_1 n_2' > 0}$. The equation (4.1) is

$$\overline{\alpha} < \alpha,$$

so it is enough to show that $\overline{\alpha} \leqslant \mu_1 - \mu_2$. This is equivalent to

$$d_1' + d_2' \leqslant d_1 \frac{n_1'}{n_1} + d_2 \frac{n_2'}{n_2}$$

which is always true by the semistability of E_1 and E_2.

Case $\boldsymbol{n_2 n_1' - n_1 n_2' = 0}$. In this case the equation (4.1) is

$$(d_1' + d_2')(n_1 + n_2) - (d_1 + d_2)(n_1' + n_2') < 0.$$

By the semistability of E_1 and E_2 $d_1' \leqslant d_1 \frac{n_1'}{n_1}$ and $d_2' \leqslant d_2 \frac{n_2'}{n_2}$, hence if either E_1 or E_2 are not of maximum slope, then it is enough to prove that

$$\left(d_1 \frac{n_1'}{n_1} + d_2 \frac{n_2'}{n_2} \right) (n_1 + n_2) - (d_1 + d_2)(n_1' + n_2') \leqslant 0,$$

which is always true as a straightforward computation shows.

Case $\boldsymbol{n_2 n_1' - n_1 n_2' < 0}$. The equation (4.1) is $\overline{\alpha} > \alpha$, so it is enough to show that $\overline{\alpha} > \mu_1 - \mu_2$. This is equivalent to

$$d_1' + d_2' < d_1 \frac{n_1'}{n_1} + d_2 \frac{n_2'}{n_2}$$

which is true whenever at least one between E_1' and E_2' is not of maximum slope or, in the case $E_1' = 0$, E_2' is not of maximum slope. □

Remark 4.2.2. Note that from the first case of the previous proof it follows that if (E_1, E_2, φ) is a triple such that E_1 and E_2 are semistable vector bundles and there exists $\widehat{\alpha}$ such that $\mathcal{T}$ is $\widehat{\alpha}$-stable, then $\mathcal{T}$ is α-stable for all α in the open interval $(\alpha_m, \widehat{\alpha})$. More in detail it is possible to prove the following Proposition, which in fact is independent of the genus g of the curve involved and provides sufficient conditions for extending the α-stability of a triple $\mathcal{T}$ for which α-stability is known for a fixed value of α, to a whole subinterval of (α_m, α_M).

Proposition 4.2.3. *Let X be a curve of any genus g and $\mathcal{T} = (E_1, E_2, \varphi)$ be a $\widehat{\alpha}$-stable triple for $\widehat{\alpha} \in (\alpha_m, \alpha_M)$ such that φ is injective. Write $E_3 := \operatorname{coker}(\varphi)$, so to have an exact sequence*

$$0 \to E_2 \to E_1 \to E_3 \to 0.$$

The following facts hold:

i) *if E_1 and E_2 are semistable, then $\mathcal{T}$ is α-stable for all $\alpha \in (\alpha_m, \widehat{\alpha}]$;*

ii) *if E_2 and E_3 are semistable, then $\mathcal{T}$ is α-stable for all $\alpha \in [\widehat{\alpha}, \alpha_M)$;*

iii) *if E_1, E_2, E_3 are semistable, then $\mathcal{T}$ is α-stable for all $\alpha \in (\alpha_m, \alpha_M)$.*

Proof.

i) The result follows from 4.2.1.

ii) Consider the subtriple $\widetilde{\mathcal{T}} = (\varphi(E_2), E_2, \varphi)$, and the quotient triple $\widehat{\mathcal{T}} = (E_3, 0, 0)$, hence we have an exact sequence

$$0 \to \widetilde{\mathcal{T}} \to \mathcal{T} \to \widehat{\mathcal{T}} \to 0.$$

It is easy to see that $\widetilde{\mathcal{T}}$ and $\widehat{\mathcal{T}}$ are α-semistable for any $\alpha > 0$ and that $\mu_{\alpha_M}(\widetilde{\mathcal{T}}) = \mu_{\alpha_M}(\widehat{\mathcal{T}})$; this implies the α_M-semistability of $\mathcal{T}$. Hence we cannot have a subtriple of $\mathcal{T}$ giving a condition like $\alpha < a$ for $a < \alpha_M$, proving *ii)*.

iii) It follows from *i)* and *ii)*. □

As a consequence of Lemma 4.2.1 above we can prove the following result.

Theorem 4.2.4. *Let (E, V) be a coherent system of type (n, d, k) and let $\mathcal{T} = (E, \mathcal{O}^k, \varphi)$ be the corresponding holomorphic triple. Then (E, V) is 0^+-stable if and only if $\mathcal{T}$ is α_m^+-stable.*

Proof. One implication is obvious. Assume now that (E, V) is 0^+-stable. By Proposition 2.2.3 the vector bundle E is semistable. If, by the sake of contradiction, (E', F', φ') is a proper subtriple of $\mathcal{T}$ that violates α_m^+-stability, then by the previous Lemma both E' and F' must have maximum slope, hence, in particular $\mu(F') = 0$, which shows that $F' \cong \mathcal{O}^{k'}$ for a suitable $k' \leqslant k$. This contradicts 0^+-stability for (E, V), concluding the proof. □

It is possible to prove a similar result also for α_M^--stability, but so far only in a very particular case.

Proposition 4.2.5. *Let (E_1, V) be a coherent system of type (n_1, d_1, n_2), and $\mathcal{T} = (E_1, \mathcal{O}^{n_2}, \varphi)$ the corresponding holomorphic triple. Assume moreover that $n_2 = n_1 - 1$, φ is injective, E_1 is semistable and σ is close enough to the last critical value $\sigma_M = d_1/(n_1 - n_2)$. If (E_1, V) is σ_M^--stable, then $\mathcal{T}$ is α_M^--stable.*

Proof. Let (E_1', E_2', φ') be a proper subtriple of $\mathcal{T}$ that violates α_M^--stability. If $\deg E_1' \leqslant 0$, then the proper subtriple $\left(\varphi(\mathcal{O}^{n_2'}), \mathcal{O}^{n_2'}, \varphi\right)$ corresponds to a coherent subsystem which violates the σ_M^--stability, so we can assume that $\deg E_1' > 0$. Moreover if $n_1' = n_2'$, then $E_1' \cong E_2'$, hence $\deg E_1' = \deg E_2' \leqslant 0$, so we can assume also $n_2' < n_1'$. As in the proof of Lemma 4.2.1, the condition $\mu_\alpha(\mathcal{T}') < \mu_\alpha(\mathcal{T})$ is equivalent to the condition

$$d_1' + d_2' < d_1 \frac{n_1' - n_2'}{n_1 - n_2}.$$

Note that

$$n_2 n_1' - n_1 n_2' = (n_1 - 1)n_1' - n_1 n_2' = n_1(n_1' - n_2') - n_1' \geqslant n_1 - n_1' \geqslant 0$$

which shows that $\frac{n_1'}{n_1} \leqslant \frac{n_1' - n_2'}{n_1 - n_2}$, hence it is enough to prove

$$d_1' + d_2' < d_1 \frac{n_1'}{n_1}.$$

By the semistability of E_1, this is true whenever $\mu(E_2') < 0$ and $\mu(E_1') < \mu(E_1)$, hence in the triple $\mathcal{T}'$ both E_1' and E_2' must have maximum slope. In particular $\mu(E_2') = 0$, hence $E_2' \cong \mathcal{O}^{n_2'}$ for a suitable $n_2' \leqslant n_2$, a contradiction with the σ_M^--stability of (E_1, V). □

4.3 Holomorphic triples with a trivial second component

Throughout this section we assume that $E_2 \cong \mathcal{O}^{n_2}$, the general case will be considered in 4.4.

Remark 4.3.1. Note that, according to Lemma 2.7.3, any α-stable holomorphic triple with trivial second component is linked with a σ-stable coherent system, hence, by Lemma 2.6.5, every indecomposable direct summand of E_1 has positive degree.

An immediate consequence of the Remark above is the following Corollary.

Corollary 4.3.2. *If $(E_1, \mathcal{O}^{n_2}, \varphi)$ is an α-stable triple, then $n_2 \leqslant d_1$.*

Proof. According to Lemma 2.7.3 the map on the global sections is injective, hence, by Riemann-Roch and Remark 4.3.1,

$$n_2 = h^0(X, \mathcal{O}^{n_2}) \leqslant h^0(X, E_1) = d_1. \qquad \square$$

Let now $\mathcal{T} = (E_1, \mathcal{O}^{n_2}, \varphi)$ be a holomorphic triple and assume $n_2 < n_1$. Then recall that by [16, Thm 3.3 and Prop 3.6] for a general σ-stable coherent system (and hence for a general α-stable triple with trivial second component) we have an exact sequence

$$0 \to \mathcal{O}^{n_2} \xrightarrow{\varphi} E_1 \to F \to 0,$$

where E_1 and F are polystable vector bundles with pairwise non-isomorphic indecomposable direct summands of the same slope.

Theorem 4.3.3. *The general triple $\mathcal{T} = (E_1, \mathcal{O}^{n_2}, \varphi)$ is α-stable for every α in the admissible range (α_m, α_M).*

Proof. In this situation $\alpha_m = \mu(E_1)$ and $\alpha_M = \left(\frac{2n_1}{n_1 - n_2}\right)\mu(E_1)$.

The α_m^+-stability for $\mathcal{T}$ follows from Proposition 4.2.4 and Theorem 2.6.7.

We now follow the idea of [9, Prop. 4.13] and consider the subtriple $\mathcal{T}' = (\varphi(\mathcal{O}^{n_2}), \mathcal{O}^{n_2}, \varphi)$ (which determines the upper bound on α), which gives rise to the following commutative diagram:

$$\begin{array}{ccc} \mathcal{O}^{n_2} & \xrightarrow{\varphi} & \varphi(\mathcal{O}^{n_2}) \\ \| & & \downarrow \\ \mathcal{O}^{n_2} & \xrightarrow{\varphi} & E_1 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & F \end{array}$$

If we write $\mathcal{T}''$ for the quotient triple given by the bottom line of the diagram above we can consider the triple $\mathcal{T}$ as an extension

$$0 \to \mathcal{T}' \to \mathcal{T} \to \mathcal{T}'' \to 0.$$

The triples $\mathcal{T}'$ and $\mathcal{T}''$ are α-semistable for every $\alpha > 0$ (the α-semistability of the first one is a consequence of [8, Prop 8.1]), in particular they are α_M-semistable. Moreover an easy calculation shows that $\mu_{\alpha_M}(\mathcal{T}') = \mu_{\alpha_M}(\mathcal{T}'')$. It is a general fact that an extension of α-semistable triples of the same α-slope is itself α-semistable. Thus $\mathcal{T}$ is α_M-semistable. Let $\widetilde{\mathcal{T}}$ be a subtriple of $\mathcal{T}$, so we have a commutative diagram:

$$\begin{array}{ccc} \mathcal{O}^{n_2} & \xrightarrow{\varphi} & E_1 \\ \uparrow & & \uparrow \\ \widetilde{E}_2 & \xrightarrow{\widetilde{\varphi}} & \widetilde{E}_1 \end{array}$$

We write $\widetilde{n}_2 = \operatorname{rank} \widetilde{E}_2$ and $\widetilde{n}_1 = \operatorname{rank} \widetilde{E}_1$. We have to show that the subtriple is not α-destabilizing for any $\alpha_m < \alpha < \alpha_M$. We divide this check in three cases. If $\frac{\widetilde{n}_2}{\widetilde{n}_1} < \frac{n_2}{n_1}$, then the stability condition of $\widetilde{\mathcal{T}}$ gives a condition $\alpha > a$ where $a \leq \alpha_m$ because of the α_m^+-stability of $\mathcal{T}$. If $\frac{\widetilde{n}_2}{\widetilde{n}_1} = \frac{n_2}{n_1}$ the subtriple gives a condition independent of α which must be always true for the same reason of the previous case. If $\frac{\widetilde{n}_2}{\widetilde{n}_1} > \frac{n_2}{n_1}$ we have a condition $\alpha < b$ and we note that $b \geq \alpha_M$ because of the α_M-semistability of $\mathcal{T}$. □

4.4 The general case

In this section we drop the hypothesis that E_2 is semistable and prove the existence of α-stable holomorphic triples for any α in the admissible range (α_m, α_M).

Remark 4.4.1. Note that, by duality, we can assume that $n_2 \leqslant n_1$. Moreover, unless $n_1 = n_2$, to have α-stability for some α a necessary condition for E_1 and E_2 is $\mu(E_2) < \mu(E_1)$. From this follows that a necessary condition for the non-emptiness of $\mathcal{N}_\alpha(n_1, n_2, d_1, d_2)$ is $d_1 > d_2$.

Remark 4.4.2. Assume E_1 and E_2 polystable with pairwise non isomorphic direct summands. Note that for a map from E_2 to E_1 to exist, the two vector bundles have to fulfill $\mu(E_2) \leqslant \mu(E_1)$.

If, in particular, $\mu(E_2) = \mu(E_1)$, then the triple $\mathcal{T}$ is not α-stable for any α unless E_1 and E_2 are stable and $n_1 = n_2$, $d_1 = d_2$. For, if $E_1 = \bigoplus_{i=1}^h E_1^i$ and $E_2 = \bigoplus_{j=1}^k E_2^j$, write for any $1 \leqslant j \leqslant k$, $\varphi_j := \varphi_{|E_2^j} : E_2^j \longrightarrow E_1$. By [23, Prop. 8], $\operatorname{coker}(\varphi_j)$ has no torsion and $\mu(\ker(\varphi_j)) = \mu(E_2^j) = \mu(E_2)$. By the semistability of E_2 we deduce that either $\ker(\varphi_j) = E_2^j$ or $\ker(\varphi_j) = 0$. In the first case $\varphi_j = 0$, hence the triple $\mathcal{T}$ admits the subtriples $(0, E_2^j, \varphi_j)$ and $(E_1, E_2/E_2^j, \varphi)$ which contradict α-stability for any α. Assume now that $\ker(\varphi_j) = 0$, that is φ_j is injective. From the fact that $\varphi_j(E_2^j)$ is a stable vector sub-bundle of E_1 of maximum slope, we deduce that $E_2^j \cong E_1^i$ for some $1 \leqslant i \leqslant h$. If we now fix i and denote by E_2' the direct sum of all those E_2^j such that $\varphi_j(E_2^j) = E_1^i$, the triple $\mathcal{T}$ admits the two subtriples (E_1^i, E_2', φ') and $(E_1/E_1^i, E_2/E_2', \varphi'')$ which contradict α-stability.

From now on and until different stated we will always assume that E_1 and E_2 are both polystable vector bundles with pairwise non-isomorphic direct summands of the same slope and $n_2 < n_1$.

Remark 4.4.3. Note that, if n'_1, n'_2, n''_1, n''_2 are integers such that

$$n_2 n'_1 - n_1 n'_2 > 0$$
$$n_2 n''_1 - n_1 n''_2 > 0$$

then, adding member to member, $n_2(n'_1 + n''_1) - n_1(n'_2 + n''_2) > 0$.

Proposition 4.4.4. *Let $\mathcal{T} = (E_1, E_2, \varphi)$ be a holomorphic triple. Write $h = \gcd(n_1, d_1)$, $k = \gcd(n_2, d_2)$ and assume that $\alpha = \alpha_m^+$, $h < k$, both E_1 and E_2 are polystable vector bundles of the form (2.1) with non-isomorphic direct summands and $\varphi_j \neq 0$ for any $1 \leqslant j \leqslant k$, where $\varphi_j := \varphi_{|E_2^j}$. Then the triple $\mathcal{T}$ is α_m^+-stable.*

Proof. Let $\mathcal{T}' = (E'_1, E'_2, \varphi')$ be a subtriple of $\mathcal{T}$ and write $\alpha = \alpha_m + \varepsilon$, $\varepsilon > 0$. By Lemma 4.2.1 we can consider only those subtriples in which both the vector bundles have maximum slope, that is those subtriples in which E'_1 and, respectively, E'_2 are direct sums of some of the E_1^i or, respectively, E_2^j. Note that, in this case,

$$\Delta_\alpha(E'_1, E'_2, \varphi') := \mu_\alpha(E'_1, E'_2, \varphi') - \mu_\alpha(E_1, E_2, \varphi) = \varepsilon \frac{n_1 n'_2 - n'_1 n_2}{(n_1 + n_2)(n'_1 + n'_2)}$$

so we have to show that subtriples that verify condition $n_1 n'_2 - n'_1 n_2 \geqslant 0$ do not exist.

Moreover, according to Remark 4.4.3, we can assume without loss of generality that $E'_1 = E_1^i$ and $E'_2 = E_2^j$ for fixed i, j. It is now enough to observe that $\operatorname{rank}(E_1^i) = n_1/h$ and $\operatorname{rank}(E_2^j) = n_2/k$, and hence, under our assumptions,

$$n_1 n'_2 - n_2 n'_1 = n_1 \frac{n_2}{k} - n_2 \frac{n_1}{h} = n_1 n_2 \left(\frac{1}{k} - \frac{1}{h} \right) < 0$$

to conclude. □

If we agree to renounce to prove the result for any triple which fulfills the hypothesis of the previous Proposition, but we consider only the triples where φ is a general map of $\operatorname{Hom}(E_2, E_1)$, then we can avoid the limitation of assuming that the number of direct summands of E_1 is smaller than that of E_2, as shown by the following results.

Lemma 4.4.5. *Let $E_1 = \bigoplus_{i=1}^h E_1^i$ and $E_2 = \bigoplus_{j=1}^k E_2^j$ be polystable vector bundles with pairwise non-isomorphic direct summands, let φ be a general map of $\operatorname{Hom}(E_2, E_1)$ and write $\varphi_j := \varphi_{|E_2^j}$. Then, for any $1 \leqslant j \leqslant k$, $\operatorname{im}(\varphi_j) \cap E_1^i \neq \emptyset$.*

Proof. We first claim that the set of the homomorphisms $\varphi \in \mathrm{Hom}(E_2, E_1)$ such that every restriction map φ_j is general in $\mathrm{Hom}(E_2^j, E_1)$ is an open dense subset of $\mathrm{Hom}(E_2, E_1)$. For, let $A = \bigoplus_{j \in J_1} E_2^j$ and $B = \bigoplus_{j \in J_2} E_2^j$ such that $J_1, J_2 \subset \{1, \ldots, k\}$ and $J_1 \cap J_2 = \emptyset$. An easy calculation shows that $h^0(X, \mathrm{Hom}(A, B)) = h^1(X, \mathrm{Hom}(A, B)) = 0$.

Consider a generic $f_A \in \mathrm{Hom}(A, E_1)$, hence an exact sequence

$$0 \to A \xrightarrow{f_A} E_1 \to \underbrace{E_1/f_A(A)}_{\text{semistable}} \to 0.$$

A homomorphism $\widetilde{f_B} : B \to E_1/f_A(A)$ induces an exact sequence

$$0 \to \mathrm{Hom}(B, A) \to \mathrm{Hom}(B, E_1) \to \mathrm{Hom}(B, E_1/f_A(A)) \to 0$$

which gives the following cohomology sequence:

$$\underbrace{H^0(X, \mathrm{Hom}(B, A))}_{0} \to H^0(X, \mathrm{Hom}(B, E_1)) \to$$
$$\to H^0(X, \mathrm{Hom}(B, E_1/f_A(A))) \to \underbrace{H^1(X, \mathrm{Hom}(B, A))}_{0}.$$

Hence if we have two generic maps $f_A : A \to E_1$, $f_B : B \to E_1$, we obtain an injective map $f_A \oplus f_B : A \oplus B \to E_1$. If we write U_j for the open subset of the generic homomorphism in $\mathrm{Hom}(E_2^j, E_1)$, then we have a map:

$$\psi : U_1 \times \cdots \times U_h \to \mathrm{Hom}(E_2, E_1).$$

Clearly this map is injective and its image has the right dimension and this proves the claim.

Fix now $1 \leqslant j \leqslant k$ and assume the existence of a subtriple of the form $E_2^j \longrightarrow \oplus_{i \in I} E_1^i$, where I is a proper subset of $\{1, \ldots, h\}$. We have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E_2^j & \xrightarrow{\varphi_j} & E_1 & \longrightarrow & G & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \gamma\uparrow & & \\
0 & \longrightarrow & E_2^j & \xrightarrow{\varphi_j} & \bigoplus_{i \in I} E_1^i & \longrightarrow & G' & \longrightarrow & 0
\end{array}$$

where G is semistable for the assumption of φ_j to be generic. An easy calculation shows that $\mu(G') > \mu(G)$, and this together with the injectivity of the map γ (by the Snake Lemma) gives a contradiction, hence such a subtriple cannot exist. $\square$

Theorem 4.4.6. *Let E_1, E_2 be two polystable vector bundles with $n_2 < n_1$, and $\mu(E_2) < \mu(E_1)$. Then there exists a homomorphism $\varphi \in \mathrm{Hom}(E_2, E_1)$ such that the triple $\mathcal{T} = (E_1, E_2, \varphi)$ is α-stable for any $\alpha \in (\alpha_m, \alpha_M)$.*

Proof. Let φ be a general map of $\mathrm{Hom}(E_2, E_1)$. By Lemma 4.2.1 a subtriple (E_1', E_2', φ) which violates α_m^+-stability is such that E_2' has maximum slope and either $E_1' = 0$ or it has maximum slope too. By Lemma 4.4.5 both these cases cannot occur, thus $\mathcal{T}$ is α_m^+-stable.

The stability for the other values of α now follows as in the proof of Theorem 4.3.3. □

Let us now consider the moduli space $\mathcal{N}_\alpha = \mathcal{N}_\alpha(n_1, n_2, d_1, d_2)$ of α-stable triple $E_2 \xrightarrow{\varphi} E_1$. Recall that in [4] the following Proposition is proved.

Proposition 4.4.7 ([4, Prop 2.3])**.** *Let F and G be polystable vector bundles on an elliptic curve X, with $\mathrm{rank}(F) \geqslant \mathrm{rank}(G)$ and $\mu(F) < \mu(G)$. Assume that no two among the indecomposable factors of F (respectively of G) are isomorphic. Then we have the following.*

- *i) If $\mathrm{rank}(F) \leqslant \mathrm{rank}(G)$, then a general $f \in \mathrm{Hom}(F, G)$ is injective.*
- *ii) If $\mathrm{rank}(F) < \mathrm{rank}(G)$, then a general $f \in \mathrm{Hom}(F, G)$ is injective, and $\mathrm{coker}(f)$ is torsion-free.*

We can now prove the following result.

Theorem 4.4.8. *If $n_2 < n_1$ and $\mu(E_2) < \mu(E_1)$, then $\mathcal{N}_\alpha(n_1, n_2, d_1, d_2)$ is non-empty, irreducible, smooth of dimension*

$$\rho(n_1, n_2, d_1, d_2) = -n_1 d_2 + n_2 d_1 + 1$$

for every $\alpha \in (\alpha_m, \alpha_M)$. Moreover the general element of $\mathcal{N}_\alpha(n_1, n_2, d_1, d_2)$ gives rise to an exact sequence

$$0 \longrightarrow E_2 \xrightarrow{\varphi} E_1 \longrightarrow G \longrightarrow 0$$

where E_1, E_2 and G are polystable vector bundles with non-isomorphic direct summands of the same slope.

Proof. The non-emptiness of $\mathcal{N}_\alpha$ follows from Theorem 4.4.6 together with the existence of polystable vector bundles of any degree and rank. The smoothness and the dimension $\rho(n_1, n_2, d_1, d_2)$ follow from the general theory [8, Thm 3.8].

Note that by [7, Prop 4.4] a necessary condition for the existence of an α-stable triple $\mathcal{T} = (E_1, E_2, \varphi)$ is $h^0(X, \mathrm{Hom}(E_1, E_2)) = 0$, which is equivalent to $h^1(X, \mathrm{Hom}(E_2, E_1)) = 0$ by the Serre duality (now we are not assuming

E_1, E_2 semistable). Write as usual $E_1 = \oplus_{i=1}^h E_1^i$ and $E_2 = \oplus_{j=1}^k E_2^j$; set $n_i^j = \operatorname{rank} E_i^j$ and $d_i^j = \deg E_i^j$, $i = 1, 2$. By Lemma 3.1.2 and the previous discussion we can always suppose $d_i^j \geqslant 1$. Fix n_i^j and d_i^j such that $\sum d_i^j = d_i$ and $\sum n_i^j = n_i$. Then the sequences $(E_i^1, \dots, E_i^h)$ of indecomposable vector bundles of the fixed degree and rank form a family parametrized by X^h. Write Z for the projective bundle of homomorphism over $X^h \times X^k$, and U for the open subset of α-stable objects. If we denote by

$$\Psi : U \to N_\alpha(n_1, n_2, d_1, d_2)$$

the canonical morphism, it follows that

$$\dim \Psi^{-1}(\mathcal{T}) = \dim \operatorname{Aut}(E_1) + \dim \operatorname{Aut}(E_2) - 1$$

for every $\mathcal{T} = (E_2, E_1, \varphi) \in \operatorname{im} \Psi$. Hence

$$\begin{aligned} \dim \operatorname{im} \Psi &= h^0(X, \operatorname{Hom}(E_2, E_1)) + h + k - \\ &- \min \dim \operatorname{Aut}(E_2) - \min \dim \operatorname{Aut}(E_1) + 1 \\ &= -n_1 d_2 + n_2 d_1 + h + k - \\ &- \min \dim \operatorname{Aut}(E_2) - \min \dim \operatorname{Aut}(E_1) + 1 \\ &= \beta + h + k - \min \dim \operatorname{Aut}(E_2) - \min \dim \operatorname{Aut}(E_1). \end{aligned}$$

It is clear that if $\min \dim \operatorname{Aut}(E_2) + \min \dim \operatorname{Aut}(E_1) > h + k$ we cannot have an open set of a component of $\mathcal{N}_\alpha(n_1, n_2, d_1, d_2)$. By Lemma 2.6.8 we note that E_1 and E_2 have to be polystable with factors pairwise non-isomorphic if the closure of $\operatorname{im} \Psi$ is an irreducible component of $\mathcal{N}_\alpha(n_1, n_2, d_1, d_2)$. This together with the property (2.1) of page 23 determine uniquely U, thus giving the irreducibility of the moduli space.

Now, for the general α-stable triple, it is clear that E_1 and E_2 are semistable vector bundles. Moreover we have $h^1(X, \operatorname{Hom}(E_2, E_1)) = 0$, hence the dimension of $H^0(X, \operatorname{Hom}(E_2, E_1))$ is independent of the vector bundles E_1 and E_2 and, by semicontinuity, the same holds in a neighborhood of E_1 and E_2. If we fix $\varphi \in \operatorname{Hom}(E_2, E_1)$ we can deform the vector bundles E_1 and E_2 into two vector bundles with the required properties. From Proposition 4.4.7 it follows that φ is injective and G is semistable. Now the same argument of above applies for G. □

Assume now that the triple (E_1, E_2, φ) is such that $n_1 = n_2$ and write n for this common value.

Remark 4.4.9. Note that, in general when $\operatorname{rank} E_1 = \operatorname{rank} E_2$, by [7, Prop. 3.18], we cannot have α-stable triples with $d_2 > d_1$, and if $d_2 = d_1$, then a necessary condition for the α-stability is that φ is injective.

Proposition 4.4.10. *Let $\mathcal{T} = (E_1, E_2, \varphi)$ be a triple with $d_1 = d_2$. Then $\mathcal{T}$ is α-stable for any $\alpha > 0$ if and only if E_1 and E_2 are stable vector bundles and φ is an isomorphism; $\mathcal{T}$ is α-semistable for all $\alpha \geqslant 0$ if and only if E_1, E_2 are polystable and φ is an isomorphism.*

Proof. The first part follows from Remark 4.4.2. For the second part the proof is analogous to that of the case of curves of genus $g \geqslant 2$: see [8, Prop 8.1]. □

Theorem 4.4.11. *Let $\mathcal{T} = (E_1, E_2, \varphi)$ be a triple such that $d_2 < d_1$. Then the moduli space $\mathcal{N}_\alpha(n, n, d_1, d_2)$ is non-empty and irreducible for every $\alpha > \alpha_m$.*

Proof. Take $E_1 = \bigoplus_{i=1}^{h}$ and $E_2 = \bigoplus_{j=1}^{k}$ two rank n polystable vector bundles of degree d_1 and d_2 respectively. By Proposition 4.4.7 and the construction made in the case $\operatorname{rank} E_2 < \operatorname{rank} E_1$, the general $\varphi \in \operatorname{Hom}(E_2, E_1)$ is injective and such that every restriction of φ to E_i is general. So the triple $\mathcal{T} = (E_1, E_2, \varphi)$ is α_m^+-stable. Let $\mathcal{T}' = (E_1', E_2', \varphi')$ be a subtriple; the injective condition on φ implies $\operatorname{rank} E_2' \leqslant \operatorname{rank} E_1'$. Such a triple cannot give an upper bound condition on α, so $\mathcal{T}$ is α-stable for every $\alpha \in (\alpha_m, \infty)$. The proof of the irreducibility follows as in the proof of Theorem 4.4.8. □

4.5 Holomorphic triples on bielliptic curves

In this section we study the α-stability for holomorphic triples on bielliptic curves, deducing, in particular, some existence theorems for α-stable triples from the results proved in Section 4.4 for holomorphic triples on elliptic curves. To pursue this task we investigate whether for a triple the property of being α-stable is preserved through pullbacks. Note that in this way we can provide results for moduli spaces of triples of type $(n_1, n_2, 2d_1, 2d_2)$ since the pullback of a vector bundle through a double covering map has even degree. In order to extend the results to the remaining cases we use elementary transformations and take into consideration how α-stability gets worse when we apply such a transformation on some of the vector bundles involved. This is done in the particular case of bielliptic curves in the proofs of Theorems 4.5.5 and 4.5.10, and in the last section some results are proved for double coverings of curves of any genus g.

Recall that a *(positive) elementary transformation* E' of a vector bundle E on a curve X supported at a point $p \in X$ is defined by an exact sequence

$$0 \longrightarrow E \xrightarrow{j} E' \longrightarrow \mathbb{K}(p) \longrightarrow 0$$

where $\mathbb{K}(p)$ is the skyscraper sheaf with support in p and fibre $\mathbb{K}$. The set of elementary transformations of E is parametrized by pairs (p, j), where $p \in X$

and $j : E_p \longrightarrow \mathbb{K}$ is a linear form of the vector space E_p, and hence is a vector bundle on the curve X of rank rank(E). If E' is an elementary transformation of E, then $\mathrm{rank}(E') = \mathrm{rank}(E)$ and $\deg(E') = \deg(E) + 1$.

Recall also that in [4] the following has been proved.

Lemma 4.5.1 ([4, Lm 3.2])**.** *Let X be a curve of genus g and let $f : X \to C$ be a double covering on an elliptic curve C. Let M, N be two semistable vector bundles on C, with $\mathrm{rank}(M) = \mathrm{rank}(N)$, $\deg(M) = \deg(N) + 1$ and $N \hookrightarrow M$. Then every vector bundles E on X with $f^*(N) \hookrightarrow E \hookrightarrow f^*(M)$ and $\mathrm{length}(E/f^*(N)) = 1$ is semistable.*

From now on let C be an elliptic curve, $f : X \longrightarrow C$ be a double covering of C with X a smooth curve of genus $g \geqslant 2$ and $\sigma : X \longrightarrow X$ the bielliptic involution. In Section 4.4 it has been shown that, if $\mathcal{T} = (E_1, E_2, \varphi)$ is the general α-stable holomorphic triple on C, then E_1 and E_2 are polystable vector bundles with pairwise non isomorphic direct summands. The next Lemmas show how these properties behave through pullback, and will be used later in the proofs of the existence results.

Lemma 4.5.2. *Let C be an elliptic curve, $f : X \longrightarrow C$ a double covering with X a smooth curve of genus $g \geqslant 2$ and E a stable vector bundle on C. Assume $f_*(\mathcal{O}_X) \cong \mathcal{O}_C \oplus M$ with $M \in \mathrm{Pic}^{-g+1}(C)$. If $h^0(C, \mathrm{End}(E)) = 1$, then $h^0(X, \mathrm{End}(f^*(E))) = 1$.*

Proof. We have

$$\begin{aligned} h^0(X, \mathrm{End}(f^*(E))) &= h^0(C, f_*(\mathrm{End}(f^*(E)))) = \\ &= h^0(X, \mathrm{End}(E)) + h^0(C, \mathrm{End}(E)(-M)). \end{aligned}$$

Since E is a stable vector bundle $\mathrm{End}(E)$ is semistable of degree 0, and since $\deg(M) = -g + 1 < 0$, $h^0(C, \mathrm{End}(E)(-M)) = 0$, proving the Lemma. □

Lemma 4.5.3. *Let C be an elliptic curve, $f : X \longrightarrow C$ a double covering with X a smooth curve of genus $g \geqslant 2$ and E a polystable vector bundle on C with pairwise non-isomorphic direct summands. Then $f^*(E)$ is polystable with the same property.*

Proof. Note that, by [4, Rmk 2.6], $f^*(E)$ is a polystable vector bundles. Moreover, if A and B are non-isomorphic indecomposable factors of E, then the vector bundles $f^*(A)$ and $f^*(B)$ are polystable and, by Lemma 4.5.2, simple, and hence stable. Moreover $\mathrm{Hom}(f^*(A), f^*(B)) = f^*(\mathrm{Hom}(A, B))$ and $\mathrm{Hom}(A, B)$ is semistable of degree 0 on C by [1], hence by the proof of lemma

4.5.2, since $A \neq B$,

$$H^0(X, \mathrm{Hom}(f^*(A), f^*(B))) = 0,$$

and so $f^*(A)$ and $f^*(B)$ are non-isomorphic. $\square$

The next Theorem shows that α-stability is preserved through pullbacks provided that the vector bundles are polystable with pairwise non isomorphic direct summands.

Theorem 4.5.4. *Let $\alpha \in \mathbb{R}$, C be an elliptic curve, $f : X \longrightarrow C$ a double covering with X a smooth curve of genus $g \geqslant 2$ and $\sigma : X \longrightarrow X$ the involution. If (E_1, E_2, φ) is an α-stable triple on C with E_1 and E_2 polystable vector bundles with pairwise non-isomorphic indecomposable direct factors, then $(f^*(E_1), f^*(E_2), f^*(\varphi))$ is 2α-stable on X.*

Proof. Assume by the sake of contradiction that $(f^*(E_1), f^*(E_2), f^*(\varphi))$ is not 2α-stable and let $\overline{\mathcal{T}} = (\overline{F}_1, \overline{F}_2, \overline{\varphi} = f^*(\varphi)|_{\overline{F}_2})$ be a proper subtriple with maximal 2α-slope and such that $\mathrm{rank}(\overline{F}_1)$ is minimal. Note that, since $(f^*(E_1), f^*(E_2), f^*(\varphi))$ is σ-invariant, the subtriple

$$\sigma^*\left(\overline{\mathcal{T}}\right) := \left(\sigma^*(\overline{F}_1), \sigma^*(\overline{F}_2), \sigma^*(\overline{\varphi})\right)$$

has the same property. Consider now the exact sequence

$$0 \longrightarrow \overline{\mathcal{T}} \cap \sigma^*\left(\overline{\mathcal{T}}\right) \longrightarrow \overline{\mathcal{T}} \oplus \sigma^*\left(\overline{\mathcal{T}}\right) \longrightarrow \overline{\mathcal{T}} + \sigma^*\left(\overline{\mathcal{T}}\right) \longrightarrow 0. \qquad (4.2)$$

By the minimality of $\overline{F}_1$, both $\overline{\mathcal{T}}$ and $\sigma^*\left(\overline{\mathcal{T}}\right)$ are 2α-stable and, moreover,

$$\mu_{2\alpha}\left(\overline{\mathcal{T}} + \sigma^*\left(\overline{\mathcal{T}}\right)\right) \leqslant \mu_{2\alpha}\left(\overline{\mathcal{T}} \oplus \sigma^*\left(\overline{\mathcal{T}}\right)\right).$$

Assume first that $\overline{\mathcal{T}} \cap \sigma^*\left(\overline{\mathcal{T}}\right) \neq (0,0)$ and $\overline{\mathcal{T}} \neq \sigma^*(\overline{\mathcal{T}})$. Then, by the maximality of $\mu_{2\alpha}(\overline{\mathcal{T}})$, $\mu_{2\alpha}(\overline{\mathcal{T}} \cap \sigma^*(\overline{\mathcal{T}})) < \mu_{2\alpha}(\overline{\mathcal{T}})$ and by the exact sequence (4.2), $\mu_{2\alpha}(\overline{\mathcal{T}} + \sigma^*(\overline{\mathcal{T}})) > \mu_{2\alpha}(\overline{\mathcal{T}})$, which contradicts the maximality of $\mu_{2\alpha}(\overline{\mathcal{T}})$.

Assume now $\overline{\mathcal{T}} = \sigma^*(\overline{\mathcal{T}})$. Then, by [4, Rmk 3.1], there exists a triple $\mathcal{T}'$ on C such that $\overline{\mathcal{T}} = f^*(\mathcal{T}')$, but this contradicts the α-stability of (E_1, E_2, φ).

Assume in the end that $\overline{\mathcal{T}} \neq \sigma^*(\overline{\mathcal{T}})$ and $\overline{\mathcal{T}} \cap \sigma^*(\overline{\mathcal{T}}) = (0,0)$. Again there exists a triple $\mathcal{T}''$ on C such that $\overline{\mathcal{T}} + \sigma^*(\overline{\mathcal{T}}) = f^*(\mathcal{T}'')$, and $\mathcal{T}''$ contradicts the α-stability of (E_1, E_2, φ) whenever

$$\overline{\mathcal{T}} \oplus \sigma^*(\overline{\mathcal{T}}) \cong \overline{\mathcal{T}} + \sigma^*(\overline{\mathcal{T}}) \neq (f^*(E_1), f^*(E_2), f^*(\varphi)),$$

but this is always true by the decomposition of $f^*(E_1)$ and $f^*(E_2)$ and Lemma 4.5.3. $\square$

Our next effort now is to understand how the 2α-stability of a holomorphic triple behaves when an elementary transformation is performed on one of the vector bundles involved. The answer is provided by the following Theorem and by Corollary 4.5.6.

Theorem 4.5.5. *Let C be an elliptic curve, $f : X \longrightarrow C$ a double covering with X a smooth curve of genus $g \geqslant 2$ and $\sigma : X \longrightarrow X$ the involution. Fix integers d_1, d_2, n_1 and n_2, write $\widetilde{\alpha}_m = 2\alpha_m + \frac{2}{n_1}$, and fix $\alpha \in \mathbb{R}$ such that $\widetilde{\alpha}_m/2 < \alpha$ and, if $n_1 \neq n_2$, $\alpha < \alpha_M$. Then $\mathcal{N}_{2\alpha}(X; n_1, n_2, 2d_1+1, 2d_2)$ is non-empty.*

Proof. By Theorem 4.4.6 let $\mathcal{T} = (E_1, E_2, \varphi)$ be in $\mathcal{N}_\alpha(C; n_1, n_2, d_1, d_2)$. Let $p \in X$ be a point such that $p \neq \sigma(p)$, that is p is not a ramification point for f, and let F_1' be the general vector bundle obtained from $f^*(E_1)$ making a positive elementary transformation supported in p. We have a natural inclusion $\iota_1 : f^*(E_1) \longrightarrow F_1'$ and $\deg(F_1') = \deg(f^*(E_1)) + 1$. In a natural way we can define a map $\psi' := \iota_1 \circ \psi$ and we can consider the triple $\mathcal{T}' := (F_1', f^*(E_2), \psi')$. We claim that this triple is 2α-stable.

First of all note that, since p is not a ramification point for f, we can identify the fibers of $f^*(E_1)$ over p and of E_1 over $f(p)$, hence there exists a vector bundle E_1'' on C obtained from E_1 making a positive elementary transformation supported at $f(p)$ such that F_1' is a subsheaf of $f^*(E_1'')$ and

$$\operatorname{length}(f^*(E_1'')/F_1') = \operatorname{length}(F_1'/f^*(E_1)) = 1.$$

By the generality of the elementary transformation we can assume that E_1'' is polystable with pairwise non-isomorphic direct summands, and hence the same is true also for $f^*(E_1'')$ by Lemma 4.5.3. Note also that if we write $j : E_1 \longrightarrow E_1''$ for the natural inclusion and $\varphi'' := j \circ \varphi$, then the triple $\mathcal{T}'' := (E_1'', E_2, \varphi'')$ is α-stable since E_1'' is semistable and $\mathcal{T}$ is α-stable, and hence, by Theorem 4.5.4, the triple $(f^*(E_1''), f^*(E_2), f^*(\varphi''))$ is 2α-stable. The situation is sketched by the following diagrams:

$$\begin{array}{ccc} f^*(E_2) & \xrightarrow{f^*(\varphi'')} & f^*(E_1'') \\ \| & & \uparrow \iota_2 \\ f^*(E_2) & \xrightarrow{\psi'} & F_1' \\ \| & & \uparrow \iota_1 \\ f^*(E_2) & \xrightarrow{f^*(\varphi)} & f^*(E_1) \end{array} \qquad \begin{array}{ccc} E_2 & \xrightarrow{\varphi''} & E_1'' \\ \| & & \uparrow j \\ E_2 & \xrightarrow{\varphi} & E_1 \end{array}$$

Assume now, by the sake of contradiction, that the tripe $\mathcal{T}'$ is not 2α-stable and let $\overline{\mathcal{T}}$ be a proper subtriple with maximum 2α-slope. Again note that the same holds also for the triple $\sigma^*(\overline{T})$ as a subtriple of $(\sigma^*(F_1'), f^*(E_2), \sigma^*(\varphi'))$. If $\overline{F}_1 \cap f^*(E_1) = \overline{F}_1$, then $\overline{T}$ is a proper subtriple of $(f^*(E_1), f^*(E_2), f^*(\varphi))$ and

$$\begin{aligned}\mu_{2\alpha}(\overline{T}) \geqslant \mu_{2\alpha}(F_1', f^*(E_2), \psi') =\\ = \mu_{2\alpha}(f^*(E_1), f^*(E_2), f^*(\varphi)) + \frac{1}{n_1+n_2} >\\ > \mu_{2\alpha}(f^*(E_1), f^*(E_2), f^*(\varphi)),\end{aligned}$$

which contradicts the 2α-(semi)stability of $(f^*(E_1), f^*(E_2), f^*(\varphi))$. Hence we can assume that $\overline{F}_1 \cap f^*(E_1) \neq \overline{F}_1$, and $\deg(\overline{F}_1) = \deg(\overline{F}_1 \cap f^*(E_1)) + 1$. Write $\overline{n}_1 := \operatorname{rank}(\overline{F}_1)$, $\overline{n}_2 := \operatorname{rank}(\overline{F}_2)$ and let F be the saturation of $\overline{F_1}$ in $f^*(E_1'')$. Since $\overline{F_1} \cap f^*(E_1) \neq \overline{F_1}$, $f^*(E_1) = F_1' \cap \sigma^*(F_1')$ and $F_1' + \sigma^*(F_1') = f^*(E_1'')$, then $\sigma^*(\overline{F_1})$ is not saturated in $f^*(E_1'')$ at p, and hence $\overline{F_1}$ is not saturated at $\sigma(p)$, thus $F \neq \overline{F_1}$ and $\deg(F) = \deg(\overline{F_1}) + 1$. Therefore we have

$$\begin{aligned}\mu_{2\alpha}(F, \overline{F_2}, \overline{\psi}) =& \mu_{2\alpha}(\overline{F_1}, \overline{F_2}, \overline{\psi}) + \frac{1}{\overline{n}_1 + \overline{n}_2} \geqslant\\ \geqslant& \mu_{2\alpha}(F_1', f^*(E_2), \psi') + \frac{1}{\overline{n}_1 + \overline{n}_2} =\\ =& \mu_{2\alpha}(f^*(E_1''), f^*(E_2), f^*(\varphi'')) - \frac{1}{n_2+n_2} + \frac{1}{\overline{n}_1 + \overline{n}_2} >\\ >& \mu_{2\alpha}(f^*(E_1''), f^*(E_2), f^*(\varphi'')),\end{aligned}$$

and this contradicts the 2α-(semi)stability of $(f^*(E_1''), f^*(E_2), f^*(\varphi''))$, concluding the proof. □

Corollary 4.5.6. *Let C be an elliptic curve, $f : X \longrightarrow C$ a double covering with X a smooth curve of genus $g \geqslant 2$ and $\sigma : X \longrightarrow X$ the involution. Fix integers d_1, d_2, n_1 and n_2, write $\widetilde{\alpha}_m = 2\alpha_m + \frac{2}{n_2}$, and fix $\alpha \in \mathbb{R}$ such that $\widetilde{\alpha}_m/2 < \alpha$ and, if $n_1 \neq n_2$, $\alpha < \alpha_M$. Then $\mathcal{N}_{2\alpha}(X; n_1, n_2, 2d_1, 2d_2 - 1)$ is non-empty.*

Proof. It follows from Theorem 4.5.5 by duality. □

Remark 4.5.7. Here we want to give a motivation for the bounds on α that appear in the statement of the previous Theorem. Note that for a triple of type $(n_1, n_2, 2d_1 + 1, 2d_2)$ the classical bounds on α are

$$\overline{\alpha}_m := 2\alpha_m + \frac{1}{n_1}$$

$$\overline{\alpha}_M := \begin{cases} 2\alpha_M + \dfrac{2}{n_1 - n_2} & \text{if} \quad n_1 > n_2, \\ \\ 2\alpha_M + \dfrac{2n_2}{n_1(n_2 - n_1)} & \text{if} \quad n_1 < n_2, \end{cases}$$

hence the bounds on α of Theorem 4.5.5 are not the best possible since $\widetilde{\alpha}_m > \overline{\alpha}_m$ and $\overline{\alpha}_M > 2\alpha_M$. However we need to assume these bounds because in the proof we can obtain contradictions by violating the 2α-stability of triples either of type $(n_1, n_2, 2d_1, 2d_2)$ or of type $(n_1, n_2, 2d_1 + 2, 2d_2)$ and

$$\widetilde{\alpha}_m = \alpha_m(n_1, n_2, 2d_1 + 2, 2d_2), \qquad 2\alpha_M = \alpha_M(n_1, n_2, 2d_1, 2d_2).$$

Note that we have inequalities $\alpha_m < \overline{\alpha}_m/2 < \widetilde{\alpha}_m/2$ and $\alpha_M < \overline{\alpha}_M/2 < \widetilde{\alpha}_M/2$.

Remark 4.5.8. Here we want to give a bound on the degree of α-instability of a triple that fails to be α-stable in the whole range of admissible values of α. In the particular case of bielliptic curves these results can be improved, as shown in Theorem 4.5.9 which proves that the triple $\mathcal{T}$ of Theorem 4.5.5 is in fact α-stable for any $\alpha \in (\alpha_m, \alpha_M)$, but the calculations here carried on do not rely on the fact that C is an elliptic curve, and hence they remain true for any curve Y and for any double covering $f : X \xrightarrow{2:1} Y$ of Y with a smooth curve X.

To fix the notation assume now $n_2 < n_1$. Note that in this case

$$\overline{\alpha}_M = 2\alpha_M + \frac{2}{n_1 - n_2}.$$

With the notation of the previous proof, if $(\overline{F}_1, \overline{F}_2, \overline{\psi})$ is a proper subtriple of $(F_1', f^*(E_2), \psi')$ that fails the 2α-stability test for $2\alpha_M < 2\alpha < \overline{\alpha}_M$, then

$$\mu_{2\alpha_M}(\overline{F}_1, \overline{F}_2, \overline{\psi}) = \mu_{2\alpha_M}(f^*(E_1), f^*(E_2), f^*(\varphi)).$$

Moreover

$$\mu_{\overline{\alpha}_M}(\overline{F}_1, \overline{F}_2, \overline{\psi}) = \mu_{2\alpha_M}(\overline{F}_1, \overline{F}_2, \overline{\psi}) + \frac{2\overline{n}_2}{(n_1 - n_2)(\overline{n}_1 + \overline{n}_2)}$$

and

$$\begin{aligned} &\mu_{\overline{\alpha}_M}(f^*(E_1), f^*(E_2), f^*(\varphi)) = \\ &\qquad = \mu_{2\alpha_M}(f^*(E_1), f^*(E_2), f^*(\varphi)) + \frac{2n_2}{(n_1 - n_2)(n_1 + n_2)}. \end{aligned}$$

So we can exhibit an estimate on how badly the stability can fail outside the α-range of the statement of Theorem 4.5.5:

$$\begin{aligned} \Delta_{\overline{\alpha}_M}(\overline{F}_1, \overline{F}_2, \overline{\psi}) &:= \mu_{\overline{\alpha}_M}(\overline{F}_1, \overline{F}_2, \overline{\psi}) - \mu_{\overline{\alpha}_M}(F_1', f^*(E_2), \psi') = \\ &= \mu_{\overline{\alpha}_M}(\overline{F}_1, \overline{F}_2, \overline{\psi}) - \mu_{\overline{\alpha}_M}(f^*(E_1), f^*(E_2), f^*(\varphi)) - \frac{1}{n_1 + n_2} = \\ &= \frac{2\overline{n}_2}{(n_1 - n_2)(\overline{n}_1 + \overline{n}_2)} - \frac{2n_2}{(n_1 - n_2)(n_1 + n_2)} - \frac{1}{n_1 + n_2} = \\ &= \frac{1}{n_1 - n_2} \cdot \frac{\overline{n}_2 - \overline{n}_1}{\overline{n}_1 + \overline{n}_2} \leqslant \frac{1}{n_1 - n_2}. \end{aligned}$$

In the same way if, instead, $n_2 > n_1$, then

$$\overline{\alpha}_M = 2\alpha_M + \frac{2n_2}{n_1(n_2-n_1)},$$

and hence, from

$$\begin{aligned}\mu_{\overline{\alpha}_M}(\overline{F}_1,\overline{F}_2,\overline{\psi}) &= \mu_{2\alpha_M}(\overline{F}_1,\overline{F}_2,\overline{\psi}) + \frac{2n_2\overline{n}_2}{n_1(n_2-n_1)(\overline{n}_1+\overline{n}_2)}\\ \mu_{\overline{\alpha}_M}(f^*(E_1),f^*(E_2),f^*(\varphi)) &=\\ &= \mu_{2\alpha_M}(f^*(E_1),f^*(E_2),f^*(\varphi)) + \frac{2n_2^2}{n_1(n_2-n_1)(n_2+n_1)}\end{aligned}$$

we can recover:

$$\begin{aligned}&\Delta_{\overline{\alpha}_M}(\overline{F}_1,\overline{F}_2,\overline{\psi}) =\\ &\quad= \mu_{\overline{\alpha}_M}(\overline{F}_1,\overline{F}_2,\overline{\psi}) - \mu_{\overline{\alpha}_M}(f^*(E_1),f^*(E_2),f^*(\varphi)) - \frac{1}{n_1+n_2} =\\ &\quad= \frac{2n_2\overline{n}_2}{n_1(n_2-n_1)(\overline{n}_1+\overline{n}_2)} - \frac{2n_2^2+n_1n_2-n_1^2}{n_1(n_2-n_1)(n_2+n_1)}.\end{aligned}$$

In general, thus,

$$\begin{aligned}\Delta_{\overline{\alpha}_M}(\overline{F}_1,\overline{F}_2,\overline{\psi}) \leqslant \max\Bigg\{&\frac{1}{n_1-n_2},\\ &\frac{2n_2\overline{n}_2}{n_1(n_2-n_1)(\overline{n}_1+\overline{n}_2)} - \frac{2n_2^2+n_1n_2-n_1^2}{n_1(n_2-n_1)(n_2+n_1)}\Bigg\}.\end{aligned}$$

For small values of α, the comparison is made for $\widetilde{\alpha}_m = \alpha_m(n_1,n_2,2d_1{+}2,2d_2)$. We have

$$\widetilde{\alpha}_m = \overline{\alpha}_m + \frac{1}{n_1},$$

hence from

$$\begin{aligned}\mu_{\widetilde{\alpha}_m}(F,\overline{F}_2,\overline{\psi}) &= \mu_{\widetilde{\alpha}_M}(f^*(E_1''),f^*(E_2),f^*(\varphi'')),\\ \mu_{\overline{\alpha}_m}(\overline{F}_1,\overline{F}_2,\overline{\psi}) &= \mu_{\widetilde{\alpha}_m}(F,\overline{F}_2,\overline{\psi}) - \frac{1}{\overline{n}_1+\overline{n}_2} =\\ &= \mu_{\widetilde{\alpha}_m}(F,\overline{F}_2,\overline{\psi}) - \frac{\overline{n}_2+n_1}{n_1(\overline{n}_1+\overline{n}_2)},\\ \mu_{\overline{\alpha}_m}(f^*(E_1''),f^*(E_2),f^*(\varphi'')) &=\\ &= \mu_{\widetilde{\alpha}_m}(f^*(E_1''),f^*(E_2),f^*(\varphi'')) - \frac{2n_2}{2n_1(n_1+n_2)}.\end{aligned}$$

we can obtain

$$\begin{aligned}\Delta_{\overline{\alpha}_m}(\overline{F}_1,\overline{F}_2,\overline{\psi}) &:= \mu_{\overline{\alpha}_m}(\overline{F}_1,\overline{F}_2,\overline{\psi}) - \mu_{\overline{\alpha}_m}(F_1',f^*(E_2),\psi') =\\ &= \frac{\overline{n}_1-n_1}{n_1(\overline{n}_1+\overline{n}_2)}.\end{aligned}$$

Note in particular that this last case is independent of what is the biggest between n_1 and n_2, and, moreover, $\Delta_{\overline{\alpha}_m}(\overline{F}_1, \overline{F}_2, \overline{\psi}) \leqslant 0$, hence a subtriple which contradicts 2α-stability for $\overline{\alpha}_m < 2\alpha < \widetilde{\alpha}_m$ cannot exist (as already observed this can be deduced also from the proof of Lemma 4.2.1, as shown in Proposition 4.2.3).

Appealing to Proposition 4.2.3 the result of Theorem 4.5.5 can be extended to the whole range of admissible α, as shown in Theorem 4.5.9 below.

Theorem 4.5.9. *Let X be as in Theorem 4.5.5. Then the moduli space $\mathcal{N}_{2\alpha}(X; n_1, n_2, 2d_1+1, 2d_2)$ is non-empty for all possible α.*

Proof. With the same notations of the proof of Theorem 4.5.5, by Proposition 4.2.3 it is sufficient to prove the semistability of $f^*(E_2)$, F_1' and $\operatorname{coker}(\psi')$. $f^*(E_2)$ is semistable by Lemma 4.5.3. The semistability of F_1' follows from Lemma 4.5.1 since by construction we have inclusions $f^*(E_1) \hookrightarrow F_1' \hookrightarrow f^*(E_1'')$ and $\operatorname{length}(F_1'/f^*(E_1)) = 1$. Now write $E_3 := \operatorname{coker}(\varphi)$ and $E_3'' := \operatorname{coker}(\varphi'')$. E_3 is semistable by construction and E_3'' is semistable by [4] and the generality of the elementary transformation ι_1, hence also $f^*(E_3)$ and $f^*(E_3'')$ are semistable. It is now easy to check that we have a chain of inclusions $f^*(E_3) \hookrightarrow \operatorname{coker}(\psi') \hookrightarrow f^*(E_3'')$, which give the semistability of $\operatorname{coker}(\psi')$, concluding the proof. □

Now we deal with the case an elementary transformation is performed on both the vector bundles of a 2α-stable holomorphic triple obtained as pullback of an α-stable triple on C. The result is analogous to the one we achieved before: a priori the α-stability is guaranteed in a subinterval of (α_m, α_M), but with a little effort it is possible to extend the result to the whole admissible α-range.

Theorem 4.5.10. *Let C be an elliptic curve, $f : X \longrightarrow C$ a double covering with X a smooth curve of genus $g \geqslant 2$ and $\sigma : X \longrightarrow X$ the involution. Fix integers d_1, d_2, n_1 and n_2, write $\widetilde{\alpha}_m = 2\alpha_m - 2\frac{n_1-n_2}{n_1 n_2}$, $\widetilde{\alpha}_M = 2\alpha_M - \frac{4}{n_2}$, and fix $\alpha \in \mathbb{R}$ such that $\max\{\widetilde{\alpha}_m/2,\ \alpha_m\} < \alpha$ and, if $n_1 \neq n_2$, $\alpha < \max\{\widetilde{\alpha}_M/2,\ \alpha_M\}$. Then $\mathcal{N}_{2\alpha}(X; n_1, n_2, 2d_1+1, 2d_2+1)$ is non-empty.*

Proof. The proof is similar to the proof of Theorem 4.5.5. Let (E_1, E_2, φ) be an α-stable triple of type (n_1, n_2, d_1, d_2) by Theorem 4.4.6 and $p \in X$ be a point such that $p \neq \sigma(p)$. Let F_1' and F_2' be the general vector bundles obtained from $f^*(E_1)$ and $f^*(E_2)$ respectively making a positive elementary transformation supported in p and consider the triple $\mathcal{T}' = (F_1', F_2', \psi')$. We claim that $\mathcal{T}'$ is 2α-stable.

In the same way of the proof of Theorem 4.5.5 we have two polystable vector bundles E_1'' and E_2'' on C with pairwise non-isomorphic direct summands such that the following diagrams commute:

$$\begin{array}{ccc} f^*(E_2'') & \xrightarrow{f^*(\varphi'')} & f^*(E_1'') \\ \uparrow & & \uparrow \\ F_2' & \xrightarrow{\psi'} & F_1' \\ \uparrow & & \uparrow \\ f^*(E_2) & \xrightarrow{f^*(\varphi)} & f^*(E_1) \end{array} \qquad \begin{array}{ccc} E_2'' & \xrightarrow{\varphi''} & E_1'' \\ \uparrow & & \uparrow \\ E_2 & \xrightarrow{\varphi} & E_1 \end{array}$$

and the triple $(f^*(E_1''), f^*(E_2''), f^*(\varphi''))$ is 2α-stable by Theorem 4.5.4.

Assume now that the triple $\mathcal{T}' = (F_1', F_2', \psi')$ is not 2α-stable, let $\overline{\mathcal{T}}$ be a proper subtriple with maximum 2α-slope and consider the triple $\sigma^*(\overline{\mathcal{T}})$ which is a proper subtriple of $(\sigma^*(F_1'), \sigma^*(F_2'), \sigma^*(\psi'))$ with maximum 2α-slope.

If $\overline{F}_1 \cap f^*(E_1) = \overline{F}_1$ and $\overline{F}_2 \cap f^*(E_2) = \overline{F}_2$ then $\overline{\mathcal{T}}$ is a subtriple also of $(f^*(E_1), f^*(E_2), f^*(\varphi))$ which contradicts the α-stability as happens in the proof of Theorem 4.5.5.

If $\overline{F}_1 \cap f^*(E_1) \neq \overline{F}_1$ and $\overline{F}_2 \cap f^*(E_2) \neq \overline{F}_2$ then a contradiction can be recovered considering the saturation of $\overline{\mathcal{T}}$ in $(f^*(E_1''), f^*(E_2''), f^*(\varphi''))$ as in the proof of Theorem 4.5.5.

Assume now $\overline{F}_1 \cap f^*(E_1) \neq \overline{F}_1$ and $\overline{F}_2 \cap f^*(E_2) = \overline{F}_2$. Write $\overline{n}_1 = \mathrm{rank}(\overline{F}_1)$, $\overline{n}_2 = \mathrm{rank}(\overline{F}_2)$ and let F be the saturation of $\overline{F}_1$ in $f^*(E_2'')$. Note that, since $\overline{F_1} \cap f^*(E_1) \neq \overline{F_1}$, $f^*(E_1) = F_1' \cap \sigma^*(F_1')$ and $F_1' + \sigma^*(F_1') = f^*(E_1'')$, $\overline{F_1}$ is not saturated in $f^*(E_1'')$ at $\sigma(p)$, hence $\deg(F) = \deg(\overline{F_1}) + 1$. We distinguish now two cases.

Case $\boldsymbol{n_1 + n_2 > 2(\overline{n}_1 + \overline{n}_2)}$. We have

$$\begin{aligned} \mu_{2\alpha}(F, \overline{F_2}, \overline{\psi}) &= \mu_{2\alpha}(\overline{F}_1, \overline{F}_2, \overline{\psi}) + \frac{1}{\overline{n}_1 + \overline{n}_2} \geqslant \\ &\geqslant \mu_{2\alpha}(F_1', F_2', \psi') + \frac{1}{\overline{n}_1 + \overline{n}_2} = \\ &= \mu_{2\alpha}(f^*(E_1''), f^*(E_2''), f^*(\varphi'')) - \frac{2}{n_1 + n_2} + \frac{1}{\overline{n}_1 + \overline{n}_2} > \\ &> \mu_{2\alpha}(f^*(E_1''), f^*(E_2''), f^*(\varphi'')), \end{aligned}$$

which contradicts the 2α-(semi)stability of $(f^*(E_1''), f^*(E_2''), f^*(\varphi''))$.

Case $n_1 + n_2 \leqslant 2(\overline{n}_1 + \overline{n}_2)$. We have

$$\begin{aligned}
\mu_{2\alpha}(\overline{F}_1 \cap f^*(E_1), \overline{F}_2, \overline{\psi}) &= \mu_{2\alpha}(\overline{F_1}, \overline{F}_2, \overline{\psi}) - \frac{1}{\overline{n}_1 + \overline{n}_2} \geqslant \\
&\geqslant \mu_{2\alpha}(F_1', F_2', \psi') - \frac{1}{\overline{n}_1 + \overline{n}_2} = \\
&= \mu_{2\alpha}(f^*(E_1), f^*(E_2), f^*(\varphi)) + \frac{2}{n_1 + n_2} - \frac{1}{\overline{n}_1 + \overline{n}_2} \geqslant \\
&\geqslant \mu_{2\alpha}(f^*(E_1), f^*(E_2), f^*(\varphi)),
\end{aligned}$$

which contradicts the 2α-stability of $(f^*(E_1), f^*(E_2), f^*(\varphi))$.

In the end the situation $\overline{F}_1 \cap f^*(E_1) = \overline{F}_1$ and $\overline{F}_2 \cap f^*(E_2) \neq \overline{F}_2$ can be dealt with in an analogous way. □

Remark 4.5.11. Again we want to motivate the bounds in the statement of Theorem 4.5.10. For a triple of type $(n_1, n_2, 2d_1 + 1, 2d_2 + 1)$ the classical bounds on α are

$$\overline{\alpha}_m := 2\alpha_m + \frac{n_2 - n_1}{n_1 n_2}$$

$$\overline{\alpha}_M := \begin{cases} 2\alpha_M - \dfrac{2}{n_2} & \text{if} \quad n_1 > n_2, \\ \\ 2\alpha_M + \dfrac{2}{n_1} & \text{if} \quad n_1 < n_2. \end{cases}$$

If we write

$$\begin{aligned}
\widetilde{\alpha}_m &= \alpha_m(n_1, n_2, 2d_1 + 2, 2d_2 + 2) \\
\widetilde{\alpha}_M &= \alpha_M(n_1, n_2, 2d_1 + 2, 2d_2 + 2),
\end{aligned}$$

then, depending on n_1 and n_2, we have the following inequalities:

$$\widetilde{\alpha}_m/2 < \overline{\alpha}_m/2 < \alpha_m \text{ and } \widetilde{\alpha}_M/2 < \overline{\alpha}_M/2 < \alpha_M \qquad \text{if } n_1 > n_2 \qquad (4.3)$$

$$\alpha_m < \overline{\alpha}_m/2 < \widetilde{\alpha}_m/2 \text{ and } \alpha_M < \overline{\alpha}_M/2 < \widetilde{\alpha}_M/2 \qquad \text{if } n_1 < n_2 \qquad (4.4)$$

which give rise to the bounds on α.

Remark 4.5.12. With the notation of the previous proof, let $(\overline{F}_1, \overline{F}_2, \overline{\psi})$ be a proper subtriple of (F_1', F_2', ψ') that fails the 2α-stability test. To compute an estimate on $\Delta_{\overline{\alpha}_m}(\overline{F}_1, \overline{F}_2, \overline{\psi})$ and $\Delta_{\overline{\alpha}_M}(\overline{F}_1, \overline{F}_2, \overline{\psi})$ is a little bit more difficult in this situation rather than in the situation of Remark 4.5.8 because we have more cases to take into consideration depending on inequalities (4.3) and (4.4). Note again that these calculations are true for any curve Y and any double covering $f : X \xrightarrow{2:1} Y$.

Assume first $n_2 < n_1$. In this situation

$$\begin{aligned} \overline{\alpha}_m &= 2\alpha_m + \frac{n_2 - n_1}{n_1 n_2}, \qquad \widetilde{\alpha}_m = 2\alpha_m + 2\frac{n_2 - n_1}{n_1 n_2}, \\ \overline{\alpha}_M &= 2\alpha_M - \frac{2}{n_2}, \qquad \widetilde{\alpha}_M = 2\alpha_M - \frac{4}{n_2}. \end{aligned} \tag{4.5}$$

Case 1. Assume first $\overline{F}_1 \cap f^*(E_1) = \overline{F}_1$, $\overline{F}_2 \cap f^*(E_2) = \overline{F}_2$ and α small. In this case in the proof of Theorem 4.5.10 we compare the triple $(\overline{F}_1, \overline{F}_2, \overline{\psi})$ with $(f^*(E_1), f^*(E_2), f^*(\varphi))$, hence we can assume

$$\mu_{2\alpha_m}(\overline{F}_1, \overline{F}_2, \overline{\psi}) = \mu_{2\alpha_m}(f^*(E_1), f^*(E_2), f^*(\varphi)).$$

We compute

$$\begin{aligned} \mu_{\overline{\alpha}_m}(\overline{F}_1, \overline{F}_2, \overline{\psi}) &= \mu_{2\alpha_m}(\overline{F}_1, \overline{F}_2, \overline{\psi}) + \frac{\overline{n}_2(n_2 - n_1)}{n_1 n_2(\overline{n}_1 + \overline{n}_2)}, \\ \mu_{\overline{\alpha}_m}(f^*(E_1), f^*(E_2), f^*(\varphi)) &= \\ &= \mu_{2\alpha_m}(f^*(E_1), f^*(E_2), f^*(\varphi)) + \frac{n_2 - n_1}{n_1(n_1 + n_2)}, \end{aligned} \tag{4.6}$$

and thus

$$\begin{aligned} \Delta_{\overline{\alpha}_m}(\overline{F}_1, \overline{F}_2, \overline{\psi}) &:= \mu_{\overline{\alpha}_m}(\overline{F}_1, \overline{F}_2, \overline{\psi}) - \mu_{\overline{\alpha}_m}(F_1', F_2', \psi') = \\ &= \mu_{\overline{\alpha}_m}(\overline{F}_1, \overline{F}_2, \overline{\psi}) - \mu_{\overline{\alpha}_m}(f^*(E_1), f^*(E_2), f^*(\varphi)) - \frac{2}{n_1 + n_2} = \\ &= \frac{\overline{n}_2(n_2 - n_1)}{n_1 n_2(\overline{n}_1 + \overline{n}_2)} - \frac{n_2 - n_1}{n_1(n_1 + n_2)} - \frac{2}{n_1 + n_2} = \\ &= \frac{\overline{n}_2(n_2 - n_1)}{n_1 n_2(\overline{n}_1 + \overline{n}_2)} - \frac{1}{n_1} < 0. \end{aligned}$$

Case 2. Assume $\overline{F}_1 \cap f^*(E_1) \neq \overline{F}_1$, $\overline{F}_2 \cap f^*(E_2) \neq \overline{F}_2$ and α big. We have:

$$\mu_{\widetilde{\alpha}_M}(F_1, F_2, \psi) = \mu_{\widetilde{\alpha}_M}(f^*(E_1''), f^*(E_2''), f^*(\varphi'')),$$

and

$$\begin{aligned} \mu_{\overline{\alpha}_M}(\overline{F}_1, \overline{F}_2, \overline{\psi}) &= \mu_{\widetilde{\alpha}_M}(F_1, F_2, \psi) - \frac{2}{\overline{n}_1 + \overline{n}_2} = \\ &= \mu_{\widetilde{\alpha}_M}(F_1, F_2, \psi) + 2\frac{\overline{n}_2 - n_2}{n_2(\overline{n}_1 + \overline{n}_2)}, \\ \mu_{\overline{\alpha}_M}(f^*(E_1''), f^*(E_2''), f^*(\varphi'')) &= \\ &= \mu_{\widetilde{\alpha}_M}(f^*(E_1''), f^*(E_2''), f^*(\varphi'')) + \frac{2}{n_1 + n_2}, \end{aligned} \tag{4.7}$$

hence

$$\begin{aligned}
\Delta_{\overline{\alpha}_M}(\overline{F}_1,\overline{F}_2,\overline{\psi}) &:= \mu_{\overline{\alpha}_M}(\overline{F}_1,\overline{F}_2,\overline{\psi}) - \mu_{\overline{\alpha}_M}(F_1',F_2',\psi') = \\
&= \mu_{\overline{\alpha}_M}(\overline{F}_1,\overline{F}_2,\overline{\psi}) - \mu_{\overline{\alpha}_M}(f^*(E_1''),f^*(E_2''),f^*(\varphi'')) + \frac{2}{n_1+n_2} = \\
&= 2\frac{\overline{n}_2-n_2}{n_2(\overline{n}_1+\overline{n}_2)} - \frac{2}{n_1+n_2} + \frac{2}{n_1+n_2} = \\
&= 2\frac{\overline{n}_2-n_2}{n_2(\overline{n}_1+\overline{n}_2)} \leqslant 0.
\end{aligned}$$

Case 3. Assume $\overline{F}_1 \cap f^*(E_1) \neq \overline{F}_1$, $\overline{F}_2 \cap f^*(E_2) = \overline{F}_2$ and α big. We have:

$$\mu_{\widetilde{\alpha}_M}(F_1,\overline{F}_2,\overline{\psi}) = \mu_{\widetilde{\alpha}_M}(f^*(E_1''),f^*(E_2''),f^*(\varphi'')),$$

and

$$\begin{aligned}
\mu_{\overline{\alpha}_M}(\overline{F}_1,\overline{F}_2,\overline{\psi}) &= \mu_{\overline{\alpha}_M}(F_1,\overline{F}_2,\overline{\psi}) - \frac{1}{\overline{n}_1+\overline{n}_2} = \\
&= \mu_{\widetilde{\alpha}_M}(F_1,\overline{F}_2,\overline{\psi}) + \frac{2\overline{n}_2-n_2}{n_2(\overline{n}_1+\overline{n}_2)},
\end{aligned}$$

hence, considering (4.7),

$$\begin{aligned}
\Delta_{\overline{\alpha}_M}(\overline{F}_1,\overline{F}_2,\overline{\psi}) &= \\
&= \mu_{\overline{\alpha}_M}(\overline{F}_1,\overline{F}_2,\overline{\psi}) - \mu_{\overline{\alpha}_M}(f^*(E_1''),f^*(E_2''),f^*(\varphi'')) + \frac{2}{n_1+n_2} = \\
&= \frac{2\overline{n}_2-n_2}{n_2(\overline{n}_1+\overline{n}_2)} - \frac{2}{n_1+n_2} + \frac{2}{n_1+n_2} = \\
&= \frac{2\overline{n}_2-n_2}{n_2(\overline{n}_1+\overline{n}_2)} \leqslant 0.
\end{aligned}$$

Case 4. Assume $\overline{F}_1 \cap f^*(E_1) \neq \overline{F}_1$, $\overline{F}_2 \cap f^*(E_2) = \overline{F}_2$ and α small. We have:

$$\mu_{2\alpha_m}(\overline{F}_1 \cap f^*(E_1),\overline{F}_2,\overline{\psi}) = \mu_{2\alpha_m}(f^*(E_1),f^*(E_2),f^*(\varphi)),$$

and

$$\begin{aligned}
\mu_{\overline{\alpha}_m}(\overline{F}_1,\overline{F}_2,\overline{\psi}) &= \mu_{\overline{\alpha}_m}(\overline{F}_1 \cap f^*(E_1),\overline{F}_2,\overline{\psi}) + \frac{1}{\overline{n}_1+\overline{n}_2} = \\
&= \mu_{2\alpha_m}(\overline{F}_1 \cap f^*(E_1),\overline{F}_2,\overline{\psi}) + \frac{(n_2-n_1)\overline{n}_2+n_1n_2}{n_1n_2(\overline{n}_1+\overline{n}_2)},
\end{aligned}$$

hence, considering (4.6),

$$\begin{aligned}
\Delta_{\overline{\alpha}_m}(\overline{F}_1,\overline{F}_2,\overline{\psi}) =&\\
&= \mu_{\overline{\alpha}_m}(\overline{F}_1,\overline{F}_2,\overline{\psi}) - \mu_{\overline{\alpha}_m}(f^*(E_1),f^*(E_2),f^*(\varphi)) - \frac{2}{n_1+n_2} =\\
&= \frac{(n_2-n_1)\overline{n}_2+n_1n_2}{n_1n_2(\overline{n}_1+\overline{n}_2)} - \frac{n_2-n_1}{n_1(n_1+n_2)} - \frac{2}{n_1+n_2} =\\
&= \frac{-n_1\overline{n}_2+n_1n_2-n_2\overline{n}_1}{n_1n_2(\overline{n}_1+\overline{n}_2)}.
\end{aligned}$$

Assume now $n_2 > n_1$. In this situation $\overline{\alpha}_m$ and $\widetilde{\alpha}_m$ are the same as in (4.5) and

$$\overline{\alpha}_M = 2\alpha_M + \frac{2}{n_1}, \qquad \widetilde{\alpha}_M = 2\alpha_M + \frac{4}{n_1}.$$

Case 5. Assume $\overline{F}_1 \cap f^*(E_1) = \overline{F}_1$, $\overline{F}_2 \cap f^*(E_2) = \overline{F}_2$ α big. We have:

$$\mu_{2\alpha_M}(\overline{F}_1,\overline{F}_2,\overline{\psi}) = \mu_{2\alpha_M}(f^*(E_1),f^*(E_2),f^*(\varphi)),$$

and

$$\begin{aligned}
\mu_{\overline{\alpha}_M}(\overline{F}_1,\overline{F}_2,\overline{\psi}) &= \mu_{2\alpha_M}(\overline{F}_1,\overline{F}_2,\overline{\psi}) + \frac{2\overline{n}_2}{n_1(\overline{n}_1+\overline{n}_2)},\\
\mu_{\overline{\alpha}_M}(f^*(E_1),f^*(E_2),f^*(\varphi)) &=\\
&=\mu_{2\alpha_M}(f^*(E_1),f^*(E_2),f^*(\varphi)) + \frac{2n_2}{n_1(n_1+n_2)},
\end{aligned} \tag{4.8}$$

hence

$$\begin{aligned}
\Delta_{\overline{\alpha}_M}(\overline{F}_1,\overline{F}_2,\overline{\psi}) &:= \mu_{\overline{\alpha}_M}(\overline{F}_1,\overline{F}_2,\overline{\psi}) - \mu_{\overline{\alpha}_M}(F_1',F_2',\psi') =\\
&= \mu_{\overline{\alpha}_M}(\overline{F}_1,\overline{F}_2,\overline{\psi}) - \mu_{\overline{\alpha}_M}(f^*(E_1),f^*(E_2),f^*(\varphi)) - \frac{2}{n_1+n_2} =\\
&= \frac{2\overline{n}_2}{n_1(\overline{n}_1+\overline{n}_2)} - \frac{2n_2}{n_1(n_1+n_2)} - \frac{2}{n_1+n_2} =\\
&= \frac{-2\overline{n}_1}{n_1(\overline{n}_1+\overline{n}_2)} \leqslant 0.
\end{aligned}$$

Case 6. Assume $\overline{F}_1 \cap f^*(E_1) \neq \overline{F}_1$, $\overline{F}_2 \cap f^*(E_2) \neq \overline{F}_2$ and α small. We have:

$$\mu_{\widetilde{\alpha}_m}(F_1,F_2,\psi) = \mu_{\widetilde{\alpha}_m}(f^*(E_1''),f^*(E_2''),f^*(\varphi'')),$$

and

$$\begin{aligned}
\mu_{\overline{\alpha}_m}(\overline{F}_1,\overline{F}_2,\overline{\psi}) &= \mu_{\overline{\alpha}_m}(F_1,F_2,\psi) - \frac{2}{\overline{n}_1+\overline{n}_2} =\\
&= \mu_{\widetilde{\alpha}_m}(F_1,F_2,\psi) - \frac{\overline{n}_2(n_2-n_1)}{n_1n_2(\overline{n}_1+\overline{n}_2)} - \frac{2}{\overline{n}_1+\overline{n}_2},\\
\mu_{\overline{\alpha}_m}(f^*(E_1''),f^*(E_2''),f^*(\varphi'')) &=\\
&=\mu_{\widetilde{\alpha}_m}(f^*(E_1''),f^*(E_2''),f^*(\varphi'')) - \frac{n_2-n_1}{n_1(n_1+n_2)},
\end{aligned} \tag{4.9}$$

hence

$$\begin{aligned}
\Delta_{\overline{\alpha}_m}(\overline{F}_1,\overline{F}_2,\overline{\psi}) &:= \mu_{\overline{\alpha}_m}(\overline{F}_1,\overline{F}_2,\overline{\psi}) - \mu_{\overline{\alpha}_m}(F_1',F_2',\psi') = \\
&= \mu_{\overline{\alpha}_m}(\overline{F}_1,\overline{F}_2,\overline{\psi}) - \mu_{\overline{\alpha}_m}(f^*(E_1''),f^*(E_2''),f^*(\varphi'')) + \frac{2}{n_1+n_2} = \\
&= -\frac{\overline{n}_2(n_2-n_1)}{n_1n_2(\overline{n}_1+\overline{n}_2)} - \frac{2}{\overline{n}_1+\overline{n}_2} + \frac{n_2-n_1}{n_1(n_1+n_2)} + \frac{2}{n_1+n_2} = \\
&= \frac{n_1\overline{n}_2+n_2\overline{n}_1-2n_1n_2}{n_1n_2(\overline{n}_1+\overline{n}_2)} \leqslant 0.
\end{aligned}$$

Case 7. Assume $\overline{F}_1\cap f^*(E_1)\neq\overline{F}_1$, $\overline{F}_2\cap f^*(E_2)=\overline{F}_2$ and α big. We have:

$$\mu_{2\alpha_M}(\overline{F}_1\cap f^*(E_1),\overline{F}_2,\overline{\psi}) = \mu_{2\alpha_M}(f^*(E_1),f^*(E_2),f^*(\varphi)),$$

and

$$\begin{aligned}
\mu_{\overline{\alpha}_M}(\overline{F}_1,\overline{F}_2,\overline{\psi}) &= \mu_{\overline{\alpha}_M}(\overline{F}_1\cap f^*(E_1),\overline{F}_2,\overline{\psi}) + \frac{1}{\overline{n}_1+\overline{n}_2} = \\
&= \mu_{2\alpha_M}(\overline{F}_1\cap f^*(E_1),\overline{F}_2,\overline{\psi}) + \frac{2\overline{n}_2+n_1}{n_1(\overline{n}_1+\overline{n}_2)},
\end{aligned}$$

hence, considering (4.8),

$$\begin{aligned}
&\Delta_{\overline{\alpha}_M}(\overline{F}_1,\overline{F}_2,\overline{\psi}) = \\
&\quad= \mu_{\overline{\alpha}_M}(\overline{F}_1,\overline{F}_2,\overline{\psi}) - \mu_{\overline{\alpha}_M}(f^*(E_1),f^*(E_2),f^*(\varphi)) - \frac{2}{n_1+n_2} = \\
&\quad= \frac{2\overline{n}_2+n_1}{n_1(\overline{n}_1+\overline{n}_2)} - \frac{2n_2}{n_1(n_1+n_2)} - \frac{2}{n_1+n_2} = \\
&\quad= \frac{n_1-2\overline{n}_1}{n_1(\overline{n}_1+\overline{n}_2)}.
\end{aligned}$$

Case 8. Assume in the end $\overline{F}_1\cap f^*(E_1)\neq\overline{F}_1$, $\overline{F}_2\cap f^*(E_2)=\overline{F}_2$ and α small. We have:

$$\mu_{\widetilde{\alpha}_m}(F_1,\overline{F}_2,\overline{\psi}) = \mu_{\widetilde{\alpha}_m}(f^*(E_1''),f^*(E_2''),f^*(\varphi'')),$$

and

$$\begin{aligned}
\mu_{\overline{\alpha}_m}(\overline{F}_1,\overline{F}_2,\overline{\psi}) &= \mu_{\overline{\alpha}_m}(F_1,\overline{F}_2,\overline{\psi}) - \frac{1}{\overline{n}_1+\overline{n}_2} = \\
&= \mu_{\widetilde{\alpha}_m}(F_1,\overline{F}_2,\overline{\psi}) - \frac{(n_2-n_1)\overline{n}_2+n_1n_2}{n_1n_2(\overline{n}_1+\overline{n}_2)},
\end{aligned}$$

hence, considering (4.9),

$$\begin{aligned}
&\Delta_{\overline{\alpha}_m}(\overline{F}_1,\overline{F}_2,\overline{\psi}) = \\
&\quad = \mu_{\overline{\alpha}_m}(\overline{F}_1,\overline{F}_2,\overline{\psi}) - \mu_{\overline{\alpha}_m}(f^*(E_1''), f^*(E_2''), f^*(\varphi'')) + \frac{2}{n_1+n_2} = \\
&\quad = -\frac{(n_2-n_1)\overline{n}_2 + n_1 n_2}{n_1 n_2(\overline{n}_1+\overline{n}_2)} + \frac{n_2-n_1}{n_1(n_1+n_2)} + \frac{2}{n_1+n_2} = \\
&\quad = \frac{n_2\overline{n}_1 + \overline{n}_2 n_1 - n_1 n_2}{n_1 n_2(\overline{n}_1+\overline{n}_2)} \leqslant 0.
\end{aligned}$$

Summarizing we have obtained the following bounds:

$$\begin{aligned}
\Delta_{\overline{\alpha}_m}(\overline{F}_1,\overline{F}_2,\overline{\psi}) &\leqslant \max\left\{\frac{2\overline{n}_2 - n_2}{n_2(\overline{n}_1+\overline{n}_2)}, \ \frac{n_2\overline{n}_1 + n_1\overline{n}_2 - n_1 n_2}{n_1 n_2(\overline{n}_1+\overline{n}_2)}\right\}, \\
\Delta_{\overline{\alpha}_M}(\overline{F}_1,\overline{F}_2,\overline{\psi}) &\leqslant \max\left\{\frac{n_1 n_2 - n_1\overline{n}_2 - n_2\overline{n}_1}{n_1 n_2(\overline{n}_1+\overline{n}_2)}, \ \frac{n_1 - 2\overline{n}_1}{n_1(\overline{n}_1+\overline{n}_2)}\right\}.
\end{aligned}$$

Again the result of Theorem 4.5.10 can be further improved to cover all the possible α in the admissible range (α_m, α_M).

Theorem 4.5.13. *Let X be as in Theorem 4.5.10. Then the moduli space $\mathcal{N}_{2\alpha}(X; n_1, n_2, 2d_1+1, 2d_2+1)$ is non-empty for all possible $\alpha \in (\alpha_m, \alpha_M)$.*

Proof. This proof is analogous to that of Theorem 4.5.9. □

4.6 Stability and elementary transformations

In this section, as anticipated, we face the general problem to give an estimate on the behaviour of α-stable holomorphic triples when subjected to an elementary transformation. This will be done for curves of any genus g. The results here presented are in some sense inspired by the problems considered in [10] for vector bundles by Brambila-Paz and Lange; they are still in progress and are the aim of further future investigations.

Let X be a smooth curve of any genus g, $x \in X$, $\alpha \in \mathbb{R}$ and $\mathcal{T} = (E_1, E_2, \varphi)$ a holomorphic triple on X. Let $\mathcal{T}' = (E_1', E_2, \varphi')$ be the holomorphic triple obtained from $\mathcal{T}$ making a positive elementary transformation supported in x on E_1. Write $S(\mathcal{T}; \overline{n}_1, \overline{n}_2)$ for the set of all the proper subtriples $\overline{\mathcal{T}} = (\overline{E}_1, \overline{E}_2, \overline{\varphi})$ of $\mathcal{T}$ such that $\operatorname{rank}(\overline{E}_1) = \overline{n}_1$ and $\operatorname{rank}(\overline{E}_2) = \overline{n}_2$, consider the map

$$\Phi : \begin{cases} S(\mathcal{T}'; \overline{n}_1, \overline{n}_2) & \longrightarrow \quad S(\mathcal{T}; \overline{n}_1, \overline{n}_2) \\ \qquad \overline{\mathcal{T}} & \longmapsto \quad \overline{\mathcal{T}} \cap \mathcal{T} \end{cases}$$

and write $\Delta_\alpha(\overline{\mathcal{T}}, \mathcal{T}) := \mu_\alpha(\overline{\mathcal{T}}) - \mu_\alpha(\mathcal{T})$ and $\Delta_\alpha(\mathcal{T}) := \max \Delta_\alpha(\overline{\mathcal{T}}, \mathcal{T})$, where the maximum is taken among all the proper subtriples $\overline{\mathcal{T}}$ of $\mathcal{T}$.

Lemma 4.6.1. *The map Φ defined above is a bijection.*

Proof. The proof is a slight variation of [10, Lm. 1.3]. It is enough to observe that the inverse map is the map that associates to any $\overline{\mathcal{T}} \in S(\mathcal{T};\overline{n}_1,\overline{n}_2)$ the triple generated by $\overline{\mathcal{T}}$ in $\mathcal{T}'$. □

Note that, if $\Phi\left(\overline{\mathcal{T}}\right) = \overline{\mathcal{T}}$, then

$$\Delta_\alpha(\overline{\mathcal{T}},\mathcal{T}') = \Delta_\alpha(\overline{\mathcal{T}},\mathcal{T}) - \frac{1}{n_1+n_2} < \Delta_\alpha(\overline{\mathcal{T}},\mathcal{T}),$$

while, if $\Phi\left(\overline{\mathcal{T}}\right) \neq \overline{\mathcal{T}}$, then

$$\Delta_\alpha(\overline{\mathcal{T}},\mathcal{T}') = \Delta_\alpha\left(\Phi\left(\overline{\mathcal{T}}\right),\mathcal{T}\right) - \frac{1}{n_1+n_2} + \frac{1}{\overline{n}_1+\overline{n}_2} > \Delta_\alpha\left(\Phi\left(\overline{\mathcal{T}}\right),\mathcal{T}\right).$$

Lemma 4.6.2. *Let X be a smooth curve of any genus g, $x \in X$, $\alpha \in \mathbb{R}$ and $\mathcal{T} = (E_1, E_2, \varphi)$ a holomorphic triple on X. Let $\mathcal{T}' = (E_1', E_2, \varphi')$ be the holomorphic triple obtained from $\mathcal{T}$ making a positive elementary transformation supported in x on E_1. Fix integers $\overline{n}_1$ and $\overline{n}_2$ and let $\overline{\mathcal{T}}$ be a proper subtriple of $\mathcal{T}$ with maximal α-degree among all the triples of $S(\mathcal{T},\overline{n}_1,\overline{n}_2)$. Then either $\Phi^{-1}\left(\overline{\mathcal{T}}\right)$ has maximal α-degree in $S(\mathcal{T}';\overline{n}_1,\overline{n}_2)$ or one less than the maximal degree of a triple of $S(\mathcal{T}',\overline{n}_1,\overline{n}_2)$.*

Proof. By contraposition let $\overline{\mathcal{T}}'$ be a proper subtriple of $\mathcal{T}'$ of maximum α-degree in $S(\mathcal{T}';\overline{n}_1,\overline{n}_2)$ and assume that $\deg_\alpha \Phi^{-1}(\overline{\mathcal{T}}) < \deg_\alpha \overline{\mathcal{T}}' - 1$. Then

$$\deg_\alpha \overline{\mathcal{T}} \leqslant \deg_\alpha \Phi^{-1}(\overline{\mathcal{T}}) < \deg_\alpha \overline{\mathcal{T}}' - 1 \leqslant \deg_\alpha \Phi(\overline{\mathcal{T}}').$$ □

Lemma 4.6.3. *Let X be a smooth curve of any genus g, $x \in X$, $\alpha \in \mathbb{R}$ and $\mathcal{T} = (E_1, E_2, \varphi)$ a holomorphic triple on X. Let $\mathcal{T}' = (E_1', E_2, \varphi')$ be the holomorphic triple obtained from $\mathcal{T}$ making a positive elementary transformation supported in x either on E_1 or on E_2 and $\overline{\mathcal{T}}$ be a subtriple of $\mathcal{T}'$ such that $\Phi(\overline{\mathcal{T}}) = \overline{\mathcal{T}}$. If $\overline{\mathcal{T}}$ is a proper subtriple with maximum α-slope, then $\Phi(\overline{\mathcal{T}})$ is a proper subtriple of $\mathcal{T}$ with maximum α-slope, and vice versa.*

Proof. By the sake of contradiction assume that there exists a proper subtriple $\widetilde{\mathcal{T}}$ of $\mathcal{T}$ such that $\Delta_\alpha(\widetilde{\mathcal{T}},\mathcal{T}) > \Delta_\alpha(\overline{\mathcal{T}},\mathcal{T})$. Then

$$\Delta_\alpha(\mathcal{T}') \geqslant \Delta_\alpha(\Phi^{-1}(\widetilde{\mathcal{T}}),\mathcal{T}') \geqslant \Delta_\alpha(\widetilde{\mathcal{T}},\mathcal{T}') = \Delta_\alpha(\widetilde{\mathcal{T}},\mathcal{T}) - \frac{1}{n_1+n_2} >$$
$$> \Delta_\alpha(\overline{\mathcal{T}},\mathcal{T}) - \frac{1}{n_1+n_2} = \Delta_\alpha(\overline{\mathcal{T}},\mathcal{T}') = \Delta_\alpha(\mathcal{T}'),$$

a contradiction. The vice versa is analogous. □

Proposition 4.6.4. *Let X be a smooth curve of any genus g, $x \in X$, $\alpha \in \mathbb{R}$ and $\mathcal{T} = (E_1, E_2, \varphi)$ a holomorphic triple on X. Let $\mathcal{T}' = (E_1', E_2, \varphi')$ be the holomorphic triple obtained from $\mathcal{T}$ making a positive elementary transformation supported in x either on E_1 or on E_2. If there exists a proper subtriple $\overline{\mathcal{T}}$ of $\mathcal{T}'$ with maximal α-slope such that $\overline{\mathcal{T}} \cap \mathcal{T} = \overline{\mathcal{T}}$ (or, equivalently, a proper subtriple $\widetilde{\mathcal{T}}$ of $\mathcal{T}$ with maximum α-slope such that $\Phi^{-1}(\widetilde{\mathcal{T}}) = \widetilde{\mathcal{T}}$), then*

$$\Delta_\alpha(\mathcal{T}') = \Delta_\alpha(\mathcal{T}) - \frac{1}{n_1 + n_2}.$$

Otherwise, let $\overline{\mathcal{T}}$ be a proper subtriple of $\mathcal{T}'$ with maximal α-slope and maximum $\overline{n}_1 + \overline{n}_2$, and $\widetilde{\mathcal{T}}$ be a proper subtriple of $\mathcal{T}$ with maximum α-slope and minimum $\widetilde{n}_1 + \widetilde{n}_2$. Then

$$\Delta_\alpha(\mathcal{T}) - \frac{1}{n_1 + n_2} + \frac{1}{\widetilde{n}_1 + \widetilde{n}_2} \leqslant \Delta_\alpha(\mathcal{T}') \leqslant \Delta_\alpha(\mathcal{T}) - \frac{1}{n_1 + n_2} + \frac{1}{\overline{n}_1 + \overline{n}_2}.$$

Proof. Assume first that there exists a proper subtriple $\overline{\mathcal{T}}$ of $\mathcal{T}$ of maximum α-slope such that $\overline{\mathcal{T}} \cap \mathcal{T} = \overline{\mathcal{T}}$, i.e. $\Phi(\overline{\mathcal{T}}) = \overline{\mathcal{T}}$. Then, by Lemma 4.6.3, $\overline{\mathcal{T}}$ is a maximal subtriple of $\mathcal{T}$, and hence

$$\Delta_\alpha(\mathcal{T}) = \Delta_\alpha(\overline{\mathcal{T}}, \mathcal{T}) = \Delta_\alpha(\overline{\mathcal{T}}, \mathcal{T}') + \frac{1}{n_1 + n_2} = \Delta_\alpha(\mathcal{T}') + \frac{1}{n_1 + n_2}.$$

Assume now that none of the maximal subtriples of $\mathcal{T}'$ is also a subtriple of $\mathcal{T}$. Then

$$\begin{aligned} \Delta_\alpha(\mathcal{T}) \geqslant \Delta_\alpha(\Phi(\overline{\mathcal{T}}), \mathcal{T}) &= \Delta_\alpha(\overline{\mathcal{T}}, \mathcal{T}') + \frac{1}{n_1 + n_2} - \frac{1}{\overline{n}_1 + \overline{n}_2} = \\ &= \Delta_\alpha(\mathcal{T}') + \frac{1}{n_1 + n_2} - \frac{1}{\overline{n}_1 + \overline{n}_2}. \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} \Delta_\alpha(\mathcal{T}) = \Delta_\alpha(\widetilde{\mathcal{T}}, \mathcal{T}) &= \Delta_\alpha(\Phi^{-1}(\widetilde{\mathcal{T}}), \mathcal{T}') + \frac{1}{n_1 + n_2} - \frac{1}{\widetilde{n}_1 + \widetilde{n}_2} \leqslant \\ &\leqslant \Delta_\alpha(\mathcal{T}') + \frac{1}{n_1 + n_2} - \frac{1}{\widetilde{n}_1 + \widetilde{n}_2}. \end{aligned}$$

which concludes the proof. □

Remark 4.6.5. Note that if $\overline{\mathcal{T}}$ is a proper maximal subtriple of $\mathcal{T}'$ such that $\Phi(\overline{\mathcal{T}}) \neq \overline{\mathcal{T}}$ and $\Phi(\overline{\mathcal{T}})$ is a maximal subtriple of $\mathcal{T}$ (i.e. an analogous of Lemma 4.6.3 is true also in the second situation of the Proposition above), then the inequalities in the statement of the previous Proposition are in fact equalities since

$$\begin{aligned} \Delta_\alpha(\mathcal{T}') = \Delta_\alpha(\overline{\mathcal{T}}, \mathcal{T}') &= \Delta_\alpha(\Phi(\overline{\mathcal{T}}), \mathcal{T}) - \frac{1}{n_1 + n_2} + \frac{1}{\overline{n}_1 + \overline{n}_2} = \\ &= \Delta_\alpha(\mathcal{T}) - \frac{1}{n_1 + n_2} + \frac{1}{\overline{n}_1 + \overline{n}_2}. \end{aligned}$$

We are able to exclude that a maximal proper subtriple $\widetilde{\mathcal{T}}$ of $\mathcal{T}$ such that $\Delta_\alpha(\widetilde{\mathcal{T}},\mathcal{T}) > \Delta_\alpha(\Phi(\overline{\mathcal{T}}),\mathcal{T})$ exists whenever $\widetilde{n}_1+\widetilde{n}_2 \leqslant n_1+n_2$, but so far not in the remaining case.

For, if $\widetilde{n}_1+\widetilde{n}_2 = \overline{n}_1+\overline{n}_2$ then $\deg_\alpha(\widetilde{\mathcal{T}}) > \deg_\alpha(\Phi(\overline{\mathcal{T}})) = \deg_\alpha(\overline{\mathcal{T}})-1$, thus $\Delta_\alpha(\widetilde{\mathcal{T}},\mathcal{T}') \geqslant \Delta_\alpha(\overline{\mathcal{T}},\mathcal{T}') = \Delta_\alpha(\mathcal{T}')$, hence $\widetilde{\mathcal{T}}$ is maximal in $\mathcal{T}'$, a contradiction since $\Phi\left(\widetilde{\mathcal{T}}\right) = \widetilde{\mathcal{T}}$.

If now $\widetilde{n}_1+\widetilde{n}_2 < \overline{n}_1+\overline{n}_2$, then by Lemma 4.6.2, $\deg_\alpha(\widetilde{\mathcal{T}}) = \deg_\alpha(\underline{\mathcal{T}})-1$ where $\underline{\mathcal{T}}$ is a subtriple with maximum α-degree in $S(\mathcal{T}',\widetilde{n}_1,\widetilde{n}_2)$, hence

$$\begin{aligned}\Delta_\alpha(\mathcal{T}) = \Delta_\alpha(\widetilde{\mathcal{T}},\mathcal{T}) = \Delta_\alpha(\widetilde{\mathcal{T}},\mathcal{T}') + \frac{1}{n_1+n_2} &= \\ = \Delta_\alpha(\underline{\mathcal{T}},\mathcal{T}') + \frac{1}{n_1+n_2} - \frac{1}{\widetilde{n}_1+\widetilde{n}_2} &\leqslant \\ \leqslant \Delta_\alpha(\mathcal{T}') + \frac{1}{n_1+n_2} - \frac{1}{\widetilde{n}_1+\widetilde{n}_2} &< \\ < \Delta_\alpha(\mathcal{T}') + \frac{1}{n_1+n_2} - \frac{1}{\overline{n}_1+\overline{n}_2}&,\end{aligned}$$

which contradicts (4.10).

If $\widetilde{n}_1+\widetilde{n}_2 > \overline{n}_1+\overline{n}_2$ then we cannot exclude that

$$\begin{cases}\Delta_\alpha(\Phi^{-1}(\widetilde{\mathcal{T}}),\mathcal{T}') < \Delta_\alpha(\overline{\mathcal{T}},\mathcal{T}') \\ \\ \Delta_\alpha(\widetilde{\mathcal{T}},\mathcal{T}) > \Delta_\alpha(\Phi(\overline{\mathcal{T}}),\mathcal{T}).\end{cases}$$

Chapter 5

Some results on coherent systems

This final Chapter presents some few more results concerning the moduli space of coherent systems, mainly on curves of low genus and on bielliptic curves. These are in some sense only preliminary results: we plan for the future further investigations of the cases therein presented.

5.1 Geometric properties (strong t-spannedness) of generic σ-stable coherent systems

In this section we study the geometric properties of general σ-stable coherent systems (E, V) on curves of genus 0 and 1 using as a main tool the results on coherent systems proved by Lange and Newstead in [15, 16] and collected in Section 2.6. We prove also some results for coherent systems on curves of genus $g \geqslant 2$ provided that E is stable and general in its moduli space.

Definition 5.1.1. *Let X be a smooth and connected projective curve and (E, V) a coherent system on X such that $V \neq \{0\}$. Fix an integer $t \geqslant 0$. We will say that (E, V) is* generically strongly t-spanned *if*

$$\dim(V \cap H^0(X, E(-(t+1)P))) = \dim(V) - (t+1)\operatorname{rank}(E)$$

for a general $P \in X$. We will say that (E, V) is strongly t-spanned *if*

$$\dim(V \cap H^0(X, E(-Z))) = \dim(V) - (t+1)\operatorname{rank}(E)$$

for every effective divisor Z of X such that length$(Z) = t+1$. We will say that E is generically strongly t-spread *if*

$$h^0(X, E(-P_0 - \cdots - P_t)) = h^0(X, E) - (t+1)\operatorname{rank}(E)$$

for a general $(P_0, \ldots, P_t) \in X^{t+1}$.

Obviously, without changing the definition of strongly t-spannedness we could write length$(Z) \leqslant t+1$ instead of length$(Z) = t+1$ in the previous definition. If $X \cong \mathbb{P}^1$, then strongly t-spanned is equivalent to generically strongly t-spanned.

In the following, for any finite-dimensional vector space W and any integer k such that $0 \leqslant k \leqslant \dim(W)$, let Grass$(k, W)$ denote the Grassmannian of all k-dimensional linear subspaces of W. If E is a vector bundle on X and $V \subseteq H^0(X, E)$ a linear subspace we will denote by $e_{E,V} : V \otimes \mathcal{O}_X \longrightarrow E$ the evaluation map. We will often write e_E instead of $e_{E,V}$ when $V = H^0(X, E)$.

Remark 5.1.2. Assume that X has genus $g \geqslant 2$. For all integers r, d such that $r > 0$ let $M(X; r, d)$ denote the moduli space of all stable vector bundles on X with rank r and degree d. The scheme $M(X; r, d)$ is non-empty, irreducible and $\dim(M(X; r, d)) = r^2(g-1)+1$. Fix a general $E \in M(X; r, d)$. We have $h^0(X, E) = 0$ and $h^1(X, E) = r(g-1) - d$ if $d \leqslant r(g-1)$. We have $h^0(X, E) = d + r(1-g)$ and $h^1(C, E) = 0$ if $d \geqslant r(g-1)$. If $r(g-1)+1 \leqslant d \leqslant r(g-1)+r-1$, then the evaluation map e_E is injective and with a locally free cokernel. If $d = rg$, then e_E is injective and hence E is generically strongly 0-spanned. If $d \geqslant rg+1$, then e_E is surjective. Notice that for general $P \in X$ when E moves in $M(X; r, d)$ the vector bundle $E(-P)$ may be considered as a general element of $M(X; r, d)$. Hence if $d \geqslant r(g-1+t)$ for some integer $t \geqslant 0$, we see that E is generically strongly t-spanned, while if $d \geqslant r(g-1+t)-1$, then E is strongly t-spanned.

Remark 5.1.3. Let X be an elliptic curve and E a semistable vector bundle on X with rank r and degree d. By Atiyah's classification of vector bundles on an elliptic curve [1, Part II] we have $h^0(X, E) = 0$ and $h^1(X, E) = -d$ if $d < 0$, $0 \leqslant h^0(X, E) = h^1(X, E) \leqslant r$ if $d = 0$, $h^0(X, E) = d$ and $h^1(X, E) = 0$ if $d > 0$. Furthermore, if $d = 0$, then $h^0(X, E \otimes M) = h^1(X, E \otimes M) = 0$ for a general $M \in \mathrm{Pic}^0(X)$. Hence we immediately obtain that if $d \geqslant r(t+1)$ for some integer $t \geqslant 0$, then E is generically strongly t-spanned (and hence generically strongly t-spread), while if $d \geqslant r(t+1)+1$, then E is strongly t-spanned.

Remark 5.1.4. Let E a rigid vector bundle on $\mathbb{P}^1$ with rank r and degree d. Obviously, E is strongly t-spanned for some integer $t \geqslant 0$ if and only if $d \geqslant (t+1)r$. Fix integers n, d, k and $\sigma \in \mathbb{R}$ such that $\mathcal{G}_\sigma(\mathbb{P}^1; n, d, k) \neq \emptyset$. By [15, Thm 3.2] the moduli space $\mathcal{G}_\sigma(\mathbb{P}^1; n, d, k)$ is irreducible and for a general $(E, V) \in \mathcal{G}_\sigma(\mathbb{P}^1; n, d, k)$ the vector bundle E is rigid. Hence we may apply to E the first part of this remark.

Remark 5.1.5. Fix $\sigma \in \mathbb{R}$ such that $\sigma > 0$ and an σ-stable (resp. σ-

semistable) coherent system (E, V) on X of type (n, d, k). By the openness of σ-stability (resp. σ-semistability) the coherent system (E, W) is σ-stable (resp. σ-semistable) for a general k-dimensional linear subspace W of $H^0(X, E)$.

Lemma 5.1.6. *Let t be a non-negative integer, X a smooth and connected projective curve and (E, V) a generically t-spanned coherent system of type (n, d, k). Then the coherent system (E, W) is generically t-spanned for a general k-dimensional linear subspace W of $H^0(X, E)$.*

Proof. Fix any k-dimensional linear subspace W of $H^0(X, E)$. Note that (E, W) is generically t-spanned if and only if for a general $P \in X$ the restriction map

$$\rho_{(t+1)P,W} : W \longrightarrow H^0((t+1)P, E|(t+1)P) \cong \mathbb{K}^{n(t+1)}$$

is surjective. This is obviously an open condition on W. □

Lemma 5.1.7. *Let t be a non-negative integer, X a smooth and connected projective curve and (E, V) a generically t-spread coherent system of type (n, d, k). Then the coherent system (E, W) is generically t-spanned for a general k-dimensional linear subspace W of $H^0(X, E)$.*

Proof. The proof is analogous to that of Lemma 5.1.6. □

Lemma 5.1.8. *Fix integers $n > 0$ and $t \geqslant 0$. Let X be a smooth and connected projective curve and E a rank n strongly t-spanned vector bundle on X such that $h^0(X, E) > (t+1)n$. Fix any integer k such that*

$$n(t+1) + 1 \leqslant k \leqslant h^0(X, E)$$

and take a general $V \in \mathrm{Grass}(k, H^0(X, E))$. Then (E, V) is strongly t-spanned.

Proof. If $k = h^0(X, E)$, then the Lemma is obviously true, hence we may assume $k < h^0(X, E)$.

Fix $P \in X$ and let $\mathcal{A}(P)$ be the set of all $W \in \mathrm{Grass}(k, H^0(X, E))$ such that the restriction map

$$W \longrightarrow H^0((t+1)P, E|(t+1)P) \cong \mathbb{K}^{n(t+1)}$$

is not surjective. See $H^0(X, E(-(t+1)P))$ as a linear subspace of $H^0(X, E)$. Since E is strongly t-spanned, $H^0(X, E(-(t+1)P))$ has codimension $n(t+1)$ in $H^0(X, E)$. Hence

$$\mathcal{A}(P) = \{W \in \mathrm{Grass}(k, H^0(X, E)) : \dim(W \cap H^0(X, E(-(t+1)P)) > k - t(n+1))\}.$$

Since $k > n(t+1)$ the Schubert cell $\mathcal{A}(P)$ has codimension at least two in $\mathrm{Grass}(k, H^0(X,E))$. Since $\dim(X) = 1$, a general $V \in \mathrm{Grass}(k, H^0(X,E))$ is not contained in $\bigcup_{P\in X} \mathcal{A}(P)$, hence we are done. □

Theorem 5.1.9. *Fix integers $d \geqslant k > n > 0$, $t \geqslant 0$, $\sigma \in \mathbb{R}$, $\sigma \geqslant 0$, and a smooth and connected elliptic curve X. Assume $G_\sigma(X; n, d, k) \neq \emptyset$ and take a general $(E, V) \in G_\sigma(X; n, d, k)$. Then (E, V) is generically strongly t-spanned if and only if $k \geqslant (t+1)n$. If $d > k \geqslant n(t+1)$, then (E, V) is strongly t-spanned.*

Proof. By [16, Thm 3.3] $G_\sigma(n, d, k)$ is irreducible, hence it makes sense to consider its general element. Moreover E is polystable and thus in particular it is semistable. By Remark 5.1.5 we may assume that V is a general k-dimensional linear subspace of $H^0(X,E)$. Since $d \geqslant k \geqslant (t+1)n$, the vector bundle E is generically strongly t-spanned (Remark 5.1.3). Since $k \geqslant t+1$, (E, V) is generically strongly t-spanned by Lemma 5.1.6, proving the first part. The last assertion follows from Lemma 5.1.8. □

Remark 5.1.10. By Remark 5.1.4 both Theorem 5.1.9 and its proof are true with only trivial modification if we take $\mathbb{P}^1$ instead of an elliptic curve. The only missing tool to extend it to the case of curves with genus $g \geqslant 2$ is a stability theorem for coherent systems (E, V) with E general in some $\mathcal{M}(n, d)$.

Theorem 5.1.11. *Fix integers $n > \rho > 0$, $t \geqslant 0$, $k \geqslant (t+1)n$. Let X be a smooth and connected curve, E a rank n generically strongly t-spread vector bundle on X such that $h^0(X,E) \geqslant k$ and V a general element of* $\mathrm{Grass}(k, H^0(X,E))$. *Then for all rank ρ subsheaves F of E we have*

$$\dim(H^0(X,F) \cap V) \leqslant k - (t+1)(n-\rho).$$

Proof. Fix any rank ρ subsheaf F of E and take a general $(P_0, \ldots, P_t) \in X^{t+1}$. Since (E, V) is generically strongly t-spread (Lemma 5.1.7), we have

$$\dim(V \cap H^0(X, E(-P_0 - \cdots - P_t))) = k - (t+1)n.$$

Since $\mathrm{rank}(F) = \rho$, the vector space $H^0(X, F(-P_0 - \cdots - P_t))$ has codimension at most $(t+1)\rho$ in $H^0(X,F)$. Hence

$$\dim(V \cap H^0(X, F(-P_0 - \cdots - P_t))) \geqslant \dim(V \cap H^0(X,F)) - (t+1)\rho.$$

Since

$$\dim(V \cap H^0(X, F(-P_0 - \cdots - P_t))) \leqslant \dim(V \cap H^0(X, E(-P_0 - \cdots - P_t))),$$

we are done. □

Theorem 5.1.12. *Fix integers $n > \rho > 0$, $t \geqslant 0$, $k < (t+1)n$. Let X be a smooth and connected curve, E a rank n generically strongly t-spread vector bundle on X such that $h^0(X, E) \geqslant k$ and V a general element of* $\mathrm{Grass}(k, H^0(X, E))$. *Then for all rank ρ subsheaves F of E we have*

$$\dim(H^0(X, F) \cap V) \leqslant (t+1)\rho.$$

Proof. Fix any rank ρ subsheaf F of E and take a general $(P_0, \dots, P_t) \in X^{t+1}$. By assumption $H^0(X, E(-P_0 - \cdots - P_t))$ has codimension $(t+1)n$ in $H^0(X, E)$. Since $k < (t+1)n$ and V is general in $\mathrm{Grass}(k, H^0(X, E))$, we have

$$V \cap H^0(X, E(-P_0 - \cdots - P_t)) = \{0\}.$$

Hence $V \cap H^0(X, F(-P_0 - \cdots - P_t)) = \{0\}$. Since $\mathrm{rank}(F) = \rho$, the vector space $H^0(X, F(-P_0 - \cdots - P_t))$ has codimension at most $(t+1)\rho$ in $H^0(X, F)$. Hence $\dim(H^0(X, F) \cap V) \leqslant (t+1)\rho$. □

Theorem 5.1.13. *Let X be a smooth and projective curve of genus $g \geqslant 2$. Fix integers $k > n \geqslant 2$, $t \geqslant 0$, $d \geqslant k + n(g-1)$ and take a general $E \in \mathcal{M}(X; n, d)$. Fix a general $V \in$* $\mathrm{Grass}(k, H^0(X, E))$. *Let F be a subsheaf of E such that $1 \leqslant \rho := \mathrm{rank}(F) \leqslant n - 1$. Then*

$$\dim(V \cap H^0(X, F)) \leqslant \max\{(t+1)\rho, k - (t+1)(n-\rho)\}.$$

Proof. Note that t is a non-negative integer. By Remark 5.1.2 the vector bundle E is generically t-spread. Apply Theorems 5.1.11 and 5.1.12. □

Theorem 5.1.14. *Let X be a smooth and projective curve of genus $g \geqslant 2$. Fix integers $n \geqslant 2$, $t \geqslant 0$, $d \geqslant n(g+t)$ and take a general $E \in \mathcal{M}(X; n, d)$. Fix a general $V \in$* $\mathrm{Grass}(n(t+1), H^0(X, E))$. *Then (E, V) is σ-stable for all $\sigma \geqslant 0$.*

Proof. Fix a real number $\sigma \geqslant 0$, an integer ρ such that $1 \leqslant \rho \leqslant n-1$ and a rank ρ subsheaf F of E. Since E is stable, we have $\mu(F) < \mu(E)$. By Theorem 5.1.13 we have $\dim(V \cap H^0(X, F)) \leqslant (t+1)\rho$. Hence

$$\mu_\sigma(F, V \cap H^0(X, F)) \leqslant \sigma(t+1) + \mu(F) < \sigma(t+1) + \mu(E) = \mu_\sigma(E, V),$$

concluding the proof. □

5.2 Rational curves in Grassmannians and their Plücker embeddings

In [3] the following Theorem is proved.

Theorem 5.2.1 ([3, Thm 1]). *Fix integers $k > n \geqslant 2$ and $a_1 \geqslant \cdots \geqslant a_n$ such that $a_n \geqslant \left\lfloor \left(\binom{k}{n} - 1\right)/n \right\rfloor$ and $a_1 + \cdots + a_n + 1 \geqslant \binom{k}{n}$. Set $E := \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^1}(a_i)$. Let V be a general k-dimensional linear subspace of $H^0(\mathbb{P}^1, E)$. Then for all n-dimensional linear subspaces W of V the evaluation map $W \otimes \mathcal{O}_{\mathbb{P}^1} \longrightarrow E$ is an injection of sheaves. Equivalent, the natural map $\bigwedge^k(V) \longrightarrow H^0(\mathbb{P}^1, \det(E))$ is injective.*

Here we prove the following Corollary of the previous Theorem, which provides sufficient conditions for the existence of σ-stable coherent systems of type (n, d, k) for some $k > n$.

Proposition 5.2.2. *Fix $\sigma \in \mathbb{R}$ and integers integers $n \geqslant 2$, $a_1 \geqslant \cdots \geqslant a_n > 0$ and k such that $\binom{k}{n} \leqslant 1 + na_n$ and $\sigma > (na_1 - \sum_{i=1}^{n} a_i)/(k - n)$. Set $E := \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^1}(a_i)$ and take a general k-dimensional linear subspace V of $H^0(\mathbb{P}^1, E)$. Then the coherent system (E, V) is σ-stable. Furthermore, for all coherent subsystems (F, W) of (E, V) such that $1 \leqslant \mathrm{rank}(F) < n$ we have $\mu_\sigma(E, V) - \mu_\sigma(F, W) \geqslant (\sum_{i=1}^{n} a_i)/n + (k - n)\sigma/n - a_n$.*

Proof. By the previous Theorem for all integers r such that $1 \leqslant r < n$ and all rank r subsheaves F of E we have $\dim(V) \cap H^0(\mathbb{P}^1, F)) \leqslant r$. Since $\mu_+(E) = a_1$, we have $\mu(F) \leqslant a_1$. Thus $\mu_\sigma(F, W) \leqslant a_1 + \sigma < (\sum_{i=1}^{n} a_i)/n + (k/n)\sigma$, concluding the proof. □

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