REGULARITY RESULTS FOR A CLASS OF OBSTACLE PROBLEMS UNDER NON STANDARD GROWTH CONDITIONS

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Abstract. We prove regularity results for minimizers of functionals $\mathcal{F}(u,\Omega) := \int_\Omega f(x,u,Du)\,dx$ in the class $K := \{u \in W^{1,p(x)}(\Omega,\mathbb{R}) : u \geq \psi\}$, where $\psi : \Omega \to \mathbb{R}$ is a fixed function and f is quasiconvex and fulfills a growth condition of the type

$$L^{-1}|z|^{p(x)} \leq f(x,\xi,z) \leq L(1+|z|^{p(x)}),$$

with growth exponent $p : \Omega \to (1,\infty)$.

1. Introduction

The aim of this paper is to study the regularity properties for local minimizers of integral functionals of the type

$$\mathcal{F}(u,\Omega) := \int_\Omega f(x,u(x),Du(x))\,dx, \tag{1.1}$$

in the class $K := \{u \in W^{1,p(x)}(\Omega,\mathbb{R}) : u \geq \psi\}$, where ψ is a fixed obstacle function, Ω is a bounded open set in $\mathbb{R}^n$ and $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a Carathéodory function satisfying a growth condition of the type

$$L^{-1}|z|^{p(x)} \leq f(x,\xi,z) \leq L(1+|z|^{p(x)}), \tag{1.2}$$

for all $x \in \Omega$, $\xi \in \mathbb{R}$, $z \in \mathbb{R}^n$, with $p : \Omega \to (1,+\infty)$ a continuous function and $L \geq 1$. We do not assume the functional considered in (1.1) to admit an Euler-Lagrange equation, especially not the integrand to be twice differentiable. Our assumptions on the integrand f are quasiconvexity (see (H2)) and $p(x)$ growth in the sense of (H1).

Problems with non standard growth became of increasing interest in the past ten years, on one hand since they appear for example in a natural way in the modeling of non newtonian fluids (for example electrorheological fluids, see for instance [24]), on the other hand they are in particular interesting from the mathematical point of view since they represent the borderline case between standard growth and so-called (p,q) growth conditions.

It is not difficult to see that existence of local minimizers for problems of $p(x)$ type under typical structure conditions can generally be shown in the generalized Sobolev space $W^{1,p(x)}_{\mathrm{loc}}(\Omega)$ (see Definition 2.1 for more details). These spaces can be interesting by themselves. So there have been made a lot of investigations on their properties, see for example [24], [8], [9], [18], [7], [17].

Mathematical investigations of regularity for problems with $p(x)$ growth started with a first higher integrability result of Zhikov [25] for functionals of a special type. Then Acerbi & Mingione [1], [2] showed $C^{0,\alpha}$ regularity for minimizers of functionals $\int f(x,Du)\,dx$ under certain weak continuity assumptions on the exponent function p. Coscia & Mingione [6] were able to show that, in order to obtain $C^{1,\alpha}$ regularity, one needs Hölder continuity of the exponent function p itself. The authors (see [11], [19]) were able to extend results of this type to functionals $\int f(x,u,Du)\,dx$ and to higher order functionals $\int f(x,u,Du,\dots,D^m u)\,dx$ with $p(x)$ growth. All of these papers make use of the so-called "blow up technique" in their proofs. Recently, regularity results of this type were also shown by the method of $A-$harmonic approximation by Zatorska-Goldstein and one of the authors [21].

In this paper we are concerned with one sided obstacle problems with $p(x)$ growth, providing regularity results in the setting of Hölder and Morrey spaces. Obstacle problems of this type in the situation of standard growth $p = const.$ have been studied by Choe [4], where regularity in Morrey spaces was considered, and by one of the authors [12], where these results have been extended in a sharp way. It turns out that the results of [12] can be used for our purposes, providing adequate reference estimates.
To the knowledge of the authors, the present paper seems to be a first regularity result for obstacle problems with $p(x)$ growth.

The techniques in this paper are a combination of those in [12], providing the reference estimates, and suitable localization and freezing techniques to treat the non standard growth exponent. The regularity assumptions for the exponent function p enable us to establish appropriate comparison estimates between the original minimizer and the minimizer of the frozen problem.

In the first part of the paper (see Theorem 2.8) we show $C^{0,\alpha}$ regularity for minimizers of functionals of the type $\int f(x, Du)\, dx$ in the case where the exponent function satisfies a weak regularity condition in the sense of (2.7) and the obstacle lies in an appropriate Morrey space. In Theorem 2.9, we extend these results to the case of more general functionals $\int f(x, u, Du)\, dx$. Therefore we take use of the so-called Ekeland variational principle, a tool that revealed to be crucial in regularity since the paper [14]. Finally, in Theorem 2.10 we prove $C^{1,\beta}$ regularity of minimizers in the case that the function p is $C^{0,\alpha}$ and the obstacle lies in an appropriate Campanato space which is isomorphic to some Hölder space.

The results of this paper could be used to prove estimates of Calderón-Zygmund type (as done by Acerbi & Mingione for equations in [3] and extended to systems of higher order by one of the authors in [20]) also for obstacle problems with $p(x)$ growth.

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2. Notation and statements

In the sequel Ω will denote an open bounded domain in $\mathbb{R}^n$ and $B(x, R)$ the open ball $\{y \in \mathbb{R}^n : |x - y| < R\}$. If u is an integrable function defined on $B(x, R)$, we will set

$$(u)_{x,R} = \fint_{B(x,R)} u(x)dx = \frac{1}{\omega_n R^n} \int_{B(x,R)} u(x)dx,$$

where ω_n is the Lebesgue measure of $B(0, 1)$. We shall also adopt the convention of writing B_R and $(u)_R$ instead of $B(x, R)$ and $(u)_{x,R}$ respectively, when the center will not be relevant or it is clear from the context; moreover, unless otherwise stated, all balls considered will have the same center. Finally the letter c will freely denote a constant, not necessarily the same in any two occurrences, while only the relevant dependences will be highlighted.

We start with the following definition.

Definition 2.1. *A function u is said to belong to the generalized Sobolev space $W^{1,p(x)}(\Omega; \mathbb{R})$ if $u \in L^{p(x)}(\Omega; \mathbb{R})$ and the distributional gradient $Du \in L^{p(x)}(\Omega; \mathbb{R}^n)$. Here the generalized Lebesgue space $L^{p(x)}(\Omega; \mathbb{R})$ is defined as the space of measurable functions $f : \Omega \to \mathbb{R}$ such that*

$$\int_\Omega |f(x)|^{p(x)}\, dx < \infty.$$

This is a Banach space equipped with the Luxemburg norm

$$||f||_{L^{p(x)}(\Omega;\mathbb{R})} = \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{f}{\lambda} \right|^{p(x)} dx \leq 1 \right\}.$$

This definition can be extended in a straightforward way to the case of vector-valued functions.

Next, we will set

$$\mathcal{F}(u, \mathcal{A}) := \int_{\mathcal{A}} f(x, u(x), Du(x))dx$$

for all $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ and for all $\mathcal{A} \subset \Omega$.

We adopt the following notion of local minimizer and local Q-minimizer:

Definition 2.2. *We say that a function $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ is a local minimizer of the functional* (1.1) *if $|Du(x)|^{p(x)} \in L^1_{\mathrm{loc}}(\Omega)$ and*

$$\int_{\mathrm{spt}\ \varphi} f(x, u(x), Du(x))dx \leq \int_{\mathrm{spt}\ \varphi} f(x, u(x) + \varphi(x), Du(x) + D\varphi(x))dx$$

for all $\varphi \in W^{1,1}_0(\Omega)$ with compact support in Ω.

We shall consider the following growth, ellipticity and continuity conditions:

$$\text{(H1)} \qquad L^{-1}(\mu^2 + |z|^2)^{p(x)/2} \leq f(x, \xi, z) \leq L(\mu^2 + |z|^2)^{p(x)/2} \ ,$$

$$\text{(H2)} \qquad \begin{aligned} &\int_{Q_1} [f(x_0, \xi_0, z_0 + D\varphi(x)) - f(x_0, \xi_0, z_0)]dx \\ &\qquad \geq L^{-1} \int_{Q_1} (\mu^2 + |z_0|^2 + |D\varphi(x)|^2)^{\frac{p(x_0)-2}{2}} |D\varphi(x)|^2 dx \end{aligned}$$

for some $0 \leq \mu \leq 1$, for all $z_0 \in \mathbb{R}^n$, $\xi_0 \in \mathbb{R}$, $x_0 \in \Omega$, $\varphi \in \mathcal{C}^\infty_0(Q_1)$, where $Q_1 = (0,1)^n$,

$$\text{(H3)} \qquad \begin{aligned} &|f(x, \xi, z) - f(x_0, \xi, z)| \\ &\qquad \leq L\omega_1(|x - x_0|) \left[\left(\mu^2 + |z|^2\right)^{p(x)/2} + \left(\mu^2 + |z|^2\right)^{p(x_0)/2} \right] \left[1 + |\log(\mu^2 + |z|^2)|\right] \end{aligned}$$

for all $z \in \mathbb{R}^n$, $\xi \in \mathbb{R}$, x and $x_0 \in \Omega$, where $L \geq 1$. Here $\omega_1 : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing continuous function, vanishing at zero, which represents the modulus of continuity of p :

$$\text{(H4)} \qquad |p(x) - p(y)| \leq \omega_1(|x - y|).$$

We will always assume that ω_1 satisfies the following condition:

$$\limsup_{R \to 0} \omega_1(R) \log \left(\frac{1}{R} \right) < +\infty \ ; \tag{2.1}$$

thus in particular, without loss of generality, we may assume that

$$\omega_1(R) \leq L |\log R|^{-1} \tag{2.2}$$

for all $R < 1$.

We shall also consider the following continuity condition with respect to the second variable

$$\text{(H5)} \qquad |f(x, \xi, z) - f(x, \xi_0, z)| \leq L\, \omega_2(|\xi - \xi_0|)(\mu^2 + |z|^2)^{p(x)/2},$$

for any $\xi, \xi_0 \in \mathbb{R}$. As usual, without loss of generality, we shall suppose that ω_2 is a concave, bounded and, hence, subadditive function.

No differentiability is assumed on f with respect to x or with respect to z.

Since all our results are local in nature, without loss of generality we shall suppose that

$$1 < \gamma_1 \le p(x) \le \gamma_2 \qquad \forall x \in \Omega\,, \tag{2.3}$$

and

$$\int_\Omega |Du(x)|^{p(x)} dx < +\infty\,. \tag{2.4}$$

Finally we set

$$K := \{u \in W^{1,p(x)}(\Omega;\mathbb{R}) : \ u \ge \psi\},$$

where $\psi \in W^{1,p(x)}(\Omega;\mathbb{R})$ is a fixed function.

Now we recall the definition of Morrey and Campanato spaces (see for example [16]).

Definition 2.3. *(Morrey spaces).*
Let Ω be an open and bounded subset of $\mathbb{R}^n$, let $1 \le p < +\infty$ and $\lambda \ge 0$. By $L^{p,\lambda}(\Omega)$ we denote the linear space of functions $u \in L^p(\Omega)$ such that, if we set $\Omega(x_0,\rho) := \Omega \cap B(x_0,\rho)$, we get

$$||u||_{L^{p,\lambda}(\Omega)} := \left\{ \sup_{x_0\in\Omega,\ 0<\rho<\,\mathrm{diam}(\Omega)} \rho^{-\lambda} \int_{\Omega(x_0,\rho)} |u(x)|^p dx \right\}^{1/p} < +\infty.$$

It is easy to see that $||u||_{L^{p,\lambda}(\Omega)}$ is a norm respect to which $L^{p,\lambda}(\Omega)$ is a Banach space.

Definition 2.4. *(Campanato spaces).*
Let Ω be an open and bounded subset of $\mathbb{R}^n$, let $p \ge 1$ and $\lambda \ge 0$. By $\mathcal{L}^{p,\lambda}(\Omega)$ we denote the linear space of functions $u \in L^p(\Omega)$ such that, if we set $\Omega(x_0,\rho) := \Omega \cap B(x_0,\rho)$, we get

$$[u]_{p,\lambda} = \left\{ \sup_{x_0\in\Omega,\ 0<\rho<\,\mathrm{diam}(\Omega)} \rho^{-\lambda} \int_{\Omega(x_0,\rho)} |u(x) - (u)_{x_0,\rho}|^p \, dx \right\}^{1/p} < +\infty,$$

where

$$(u)_{x_0,\rho} := \frac{1}{|\Omega(x_0,\rho)|} \int_{\Omega(x_0,\rho)} u(x)\, dx$$

is the average of u in $\Omega(x_0,\rho)$.

Also in this case it is not difficult to show that $\mathcal{L}^{p,\lambda}(\Omega)$ is a Banach space equipped with the norm

$$||u||_{\mathcal{L}^{p,\lambda}(\Omega)} = ||u||_{L^p(\Omega)} + [u]_{p,\lambda}.$$

Remark 2.5. The local variants $L^{p,\lambda}_{\mathrm{loc}}(\Omega)$ and $\mathcal{L}^{p,\lambda}_{\mathrm{loc}}(\Omega)$ are defined in a standard way

$$u \in L^{p,\lambda}_{\mathrm{loc}}(\Omega) \ \Leftrightarrow\ u \in L^{p,\lambda}(\Omega') \quad \forall\, \Omega' \Subset \Omega$$

$$u \in \mathcal{L}^{p,\lambda}_{\mathrm{loc}}(\Omega) \ \Leftrightarrow\ u \in \mathcal{L}^{p,\lambda}(\Omega') \quad \forall\, \Omega' \Subset \Omega.$$

The interest of Campanato's spaces lies mainly in the following result which will be used in the next sections.

Theorem 2.6. *Let Ω be a bounded open Lipschitz domain of $\mathbb{R}^n$, and let $n < \lambda < n+p$. Then the space $\mathcal{L}^{p,\lambda}(\Omega)$ is isomorphic to $\mathcal{C}^{0,\alpha}(\bar\Omega)$ with $\alpha = \frac{\lambda-n}{p}$. We also remark that, using Poincaré inequality, we have that, for a weakly differentiable function v, if $Dv \in L^{p,\lambda}(\Omega)$, then $v \in \mathcal{L}^{p,p+\lambda}(\Omega)$.*

Remark 2.7. Theorem 2.6 also holds for a larger class of domains (see [16], Sect. 2.3).

The first result we are able to obtain is for local minimizers in K of the functional

$$\mathcal{H}(u, B_R) = \int_{B_R} h(x, Du(x))\, dx \tag{2.5}$$

where $h : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is a continuous function fulfilling growth, ellipticity and continuity conditions of kind (H1), (H2) and (H3).

More precisely we have:

Theorem 2.8. *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a local minimizer of the functional* (2.5) *in K, where h is a continuous function satisfying* (H1) - (H4)*; suppose moreover that the function ψ fulfills the following assumption*

$$D\psi \in L^{q,\lambda}_{\mathrm{loc}}(\Omega), \tag{2.6}$$

for some $n - \gamma_1 < \lambda < n$, with $q = \gamma_2\, r$ for some $r > 1$, where γ_1 and γ_2 have been introduced in (2.3)*. Finally assume that the following holds*

$$\lim_{R\to 0} \omega_1(R) \log\left(\frac{1}{R}\right) = 0. \tag{2.7}$$

Then $u \in \mathcal{C}^{0,\alpha}_{\mathrm{loc}}(\Omega)$ with $\alpha = 1 - \dfrac{n-\lambda}{\gamma_1}$.

The main result is instead for local minimizers of the functional (1.1) in K.

Theorem 2.9. *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a local minimizer of the functional* (1.1) *in K, where f is a continuous function satisfying* (H1) - (H5)*; suppose moreover that the function ψ fulfills* (2.6)*, for some $n - \gamma_1 < \lambda < n$, with $q = \gamma_2\, r$ for some $r > 1$, where γ_1 and γ_2 have been introduced in* (2.3)*. Finally assume that the following holds*

$$\lim_{R\to 0} \omega_1(R) \log\left(\frac{1}{R}\right) + \omega_2(R) = 0. \tag{2.8}$$

Then $u \in \mathcal{C}^{0,\alpha}_{\mathrm{loc}}(\Omega)$ with $\alpha = 1 - \dfrac{n-\lambda}{\gamma_1}$.

Finally, if the Lagrangian f is more regular and the obstacle stays in a Campanato space, we have the following result.

Theorem 2.10. *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a local minimizer of the functional* (1.1) *in K, where f is a function of class $\mathcal{C}^2$ satisfying* (H1) - (H5) *and the function ψ fulfills the following assumption*

$$D\psi \in \mathcal{L}^{\gamma_1,\lambda}_{\mathrm{loc}}(\Omega), \tag{2.9}$$

for some $n < \lambda < n + \gamma_1$, where γ_1 has been introduced in (2.3)*. If we assume that*

$$\omega_1(R) + \omega_2(R) \leq L\, R^{\varsigma} \tag{2.10}$$

for some $0 < \varsigma \leq 1$ and all $R \leq 1$, then $Du \in \mathcal{L}^{\gamma_1,\tilde\lambda}_{\mathrm{loc}}(\Omega)$ for some suitable $n < \tilde\lambda < n + \gamma_1$ and therefore $u \in \mathcal{C}^{1,\tilde\alpha}_{\mathrm{loc}}(\Omega)$ with $\tilde\alpha = 1 - \dfrac{n-\tilde\lambda}{\gamma_1}$.

3. Preliminary results

Before proving our main theorems, we need some preliminary results and establish some basic notation.

• **A higher integrability result** We first prove a higher integrability result for functionals of type (1.1).

Lemma 3.1. *Let $\mathcal{O}$ be an open subset of Ω, let $u \in W^{1,1}_{\mathrm{loc}}(\mathcal{O})$ be a local minimizer in K of the functional* (1.1) *with $f : \mathcal{O} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ satisfying* (H1)*, with the exponent function p satisfying* (H4)*,* (2.1) *and* (2.3) *and with ψ fulfilling condition* (2.6)*. Moreover suppose that*

$$\int_{\mathcal{O}} |Du(x)|^{p(x)} dx \le M_1$$

for some constant M_1. Then, there exist two positive constants c_0, δ depending on n, r, γ_1, γ_2, L, M_1, where r is the quantity appearing in condition (2.6)*, such that, if $B_R \Subset \mathcal{O}$, then*

$$\left(\fint_{B_{R/2}} |Du(x)|^{p(x)(1+\delta)} dx\right)^{1/(1+\delta)} \le c_0 \fint_{B_R} |Du(x)|^{p(x)} dx + c_0 \left(\fint_{B_R} (|D\psi(x)|^{p(x)(1+\delta)} + 1)\, dx\right)^{1/(1+\delta)} . \tag{3.1}$$

Proof. *First step:* we set

$$p_1 := \min_{x \in \overline{B}_R} p(x), \qquad p_2 := \max_{x \in \overline{B}_R} p(x),$$

let $R/2 \le t < s \le R \le 1$, and let $\eta \in C_0^\infty(B_R)$ be a cut-off function such that $0 \le \eta \le 1$, $\eta \equiv 0$ outside B_s, $\eta \equiv 1$ on B_t, $|D\eta| \le 2(s-t)^{-1}$. Moreover we set $\varphi(x) = \eta(x)(u(x) - (u)_R)$ and let $g = u - \varphi$. We remark that $g = u$ on ∂B_s while on B_t we have $g = (u)_R$, consequently $Dg = 0$ on B_t. We would like to use the minimality of u in K. A priori g is not an element of K, so we set $\tilde g := \max\{g, \psi\}$ and $\Sigma := \{x \in \mathbb{R}^n : g(x) \ge \psi(x)\}$. This assures that $\tilde g \in K$ and so, by the minimality of u

$$\mathcal{F}(u, B_s) \le \mathcal{F}(\tilde g, B_s). \tag{3.2}$$

Therefore we estimate by (3.2) and the growth (H1)

$$\begin{aligned}
\int_{B_t} |Du(x)|^{p(x)} dx &\le L \int_{B_s} f(x, u(x), Du(x)) dx \\
&\overset{(3.2)}{\le} L \int_{B_s} f(x, \tilde g(x), D\tilde g(x)) dx \\
&= L\, \mathcal{F}(\tilde g, B_s \cap \Sigma) + L\, \mathcal{F}(\tilde g, B_s \setminus \Sigma) \\
&= L\, \mathcal{F}(g, B_s \cap \Sigma) + L\, \mathcal{F}(\psi, B_s \setminus \Sigma) \\
&\le L \int_{B_s} f(x, g(x), Dg(x))\, dx + L \int_{B_s} f(x, \psi(x), D\psi(x))\, dx \\
&\overset{(H1)}{\le} L^2 \int_{B_s} (1 + |Dg(x)|^{p(x)})\, dx + L^2 \int_{B_s} (1 + |D\psi(x)|^{p(x)})\, dx \\
&\le L^2 \int_{B_s \setminus B_t} [(1 - \eta(x))|Du(x)| + |u(x) - (u)_R||D\eta(x)|]^{p(x)} dx + \bar c \\
&\le \hat c \int_{B_s \setminus B_t} |Du(x)|^{p(x)} dx + \tilde c \int_{B_s} \left|\frac{u(x) - (u)_R}{s-t}\right|^{p(x)} dx + \bar c \\
&\le \hat c \int_{B_s \setminus B_t} |Du(x)|^{p(x)} dx + \tilde c \frac{1}{|s-t|^{p_2}} \int_{B_R} |u(x) - (u)_R|^{p(x)} dx + \bar c\, ,
\end{aligned}$$

where $\hat c = L^2 2^{\gamma_2 - 1}$, $\tilde c = L^2 2^{2\gamma_2 - 1}$, $\bar c = L^2 \int_{B_R} (1 + |D\psi(x)|^{p(x)})\, dx$.

Now we proceed in a standard way, 'filling the hole' and applying [16], Lemma 6.1 to deduce

$$\int_{B_{R/2}} |Du(x)|^{p(x)} dx \le cR^{p_1 - p_2} \int_{B_R} \left|\frac{u(x) - (u)_R}{R}\right|^{p(x)} dx + c \int_{B_R} (1 + |D\psi(x)|^{p(x)})\, dx$$

$$\leq \; cR^{-\omega_1(8R)} \int_{B_R} \left|\frac{u(x)-(u)_R}{R}\right|^{p(x)} dx + c\int_{B_R} (1+|D\psi(x)|^{p(x)})\,dx$$
$$\leq \; c\int_{B_R} \left|\frac{u(x)-(u)_R}{R}\right|^{p(x)} dx + c\int_{B_R} (1+|D\psi(x)|^{p(x)})\,dx,$$

where we used (2.2) and c is a constant depending only on γ_1, γ_2, L.

According to the previous facts, we find that

$$\text{(3.3)} \qquad \fint_{B_{R/2}} |Du(x)|^{p(x)}dx \leq c\fint_{B_R} \left|\frac{u(x)-(u)_R}{R}\right|^{p(x)} dx + c\fint_{B_R} (1+|D\psi(x)|^{p(x)})\,dx.$$

Second step: Fixing $\vartheta = \min\left\{\sqrt{\frac{n+1}{n}}, \gamma_1\right\}$ and taking $R < R_0/16$, where R_0 is small enough to have $\omega_1(8R_0) \leq \vartheta - 1$ and therefore $1 \leq p_2/p_1\vartheta \leq \vartheta^2 \leq (n+1)/n$, by Sobolev-Poincaré's inequality we obtain

$$\fint_{B_R} \left|\frac{u(x)-(u)_R}{R}\right|^{p(x)} dx$$
$$\leq \; 1+\fint_{B_R} \left|\frac{u(x)-(u)_R}{R}\right|^{p_2} dx$$
$$\leq \; 1+c\left(\int_{B_R}(1+|Du(x)|^{p(x)})dx\right)^{\frac{(p_2-p_1)\vartheta}{p_1}} R^{\frac{-(p_2-p_1)\vartheta n}{p_1}} \left(\fint_{B_R} |Du(x)|^{\frac{p_1}{\vartheta}}dx\right)^{\vartheta}$$
$$\leq \; c(M_1)\left(\fint_{B_R} |Du(x)|^{\frac{p_1}{\vartheta}}dx\right)^{\vartheta} + c\,,$$

where in the second inequality we use the fact that $\frac{p_1}{\vartheta} \leq \frac{p(x)}{\vartheta} \leq p(x)$ and in the last one we use again the fact that, by (2.2), $R^{\frac{-(p_2-p_1)\vartheta n}{p_1}}$ is bounded. So, by the second step

$$\text{(3.4)} \qquad \fint_{B_R} \left|\frac{u(x)-(u)_R}{R}\right|^{p(x)} dx \leq c\left(\fint_{B_R} |Du(x)|^{\frac{p(x)}{\vartheta}}dx\right)^{\vartheta} + c\,.$$

Third step: from (3.3) and (3.4) we obtain, for all $R < R_0/16$

$$\fint_{B_{R/2}} |Du(x)|^{p(x)}dx \leq c_1\left(\fint_{B_R} |Du(x)|^{\frac{p(x)}{\vartheta}}dx\right)^{\vartheta} + c_2\int_{B_R} (1+|D\psi(x)|^{p(x)})\,dx,$$

where $c_1 \equiv c_1(\gamma_1, \gamma_2, L, M_1, n)$ and $c_2 \equiv c_2(\gamma_1, \gamma_2, L)$.

We now apply Gehring's lemma (see [16], Theorem 6.6 or [15], Chapter V) and deduce that there exists $0 < \delta < r-1$ (where r appears in the higher integrability assumption (2.6) on the obstacle function ψ) such that

$$\left(\fint_{B_{R/2}} |Du(x)|^{p(x)(1+\delta)}dx\right)^{1/(1+\delta)} \leq \; c_0\fint_{B_R} |Du(x)|^{p(x)}dx + c_0\left(\fint_{B_R} (|D\psi(x)|^{p(x)(1+\delta)}+1)\,dx\right)^{1/(1+\delta)},$$

with $c_0 \equiv c_0(\gamma_1, \gamma_2, L, M_1, n, r)$. This concludes the proof. □

Proof. It follows from the first step of the previous proof, formula (3.3). □

• **A remark about local minimizers with obstacles**

Let $g : \mathbb{R}^n \to \mathbb{R}$ be a function of class $\mathcal{C}^2$ satisfying, for some $1 < \gamma_1 \le p \le \gamma_2$, the following growth and ellipticity conditions

$$L^{-1}(\mu^2 + |z|^2)^{p/2} \le g(z) \le L(\mu^2 + |z|^2)^{p/2} \,, \tag{H6}$$

$$\int_{Q_1} [g(z_0 + D\phi(x)) - g(z_0)]dx \ge L^{-1} \int_{Q_1} (\mu^2 + |z_0|^2 + |D\phi(x)|^2)^{(p-2)/2} |D\phi(x)|^2 \, dx \tag{H7}$$

for some $0 \le \mu \le 1$, for all $z_0 \in \mathbb{R}^n$, $\phi \in \mathcal{C}_0^\infty(Q_1)$, where $Q_1 = (0,1)^n$, $L \ge 1$. Moreover let v be a local minimizer in the class K of the functional

$$w \mapsto \int_{B_R} g(Dw(x))\, dx \tag{3.5}$$

with $B_R \Subset \Omega$.

Then it is possible to prove that

$$\int_\Omega \langle A(Dv(x)), D\varphi(x)\rangle \; dx \ge 0 \qquad \forall\, \varphi \in \mathcal{C}_0^\infty(\Omega) \qquad \text{such that } \varphi \ge 0, \tag{3.6}$$

where $A(z) := Dg(z)$ and $A(z)$ satisfies the following monotonicity and growth conditions

$$\langle A(z), z\rangle \ge \nu_1 |z|^p - c \tag{3.7}$$

for some $\nu_1 \equiv \nu_1(\gamma_1, \gamma_2, L)$ and $c \equiv c(\gamma_1, \gamma_2, L)$, and

$$|A(z)| \le L\,(1 + |z|^{p-1}). \tag{3.8}$$

It is also possible to show (see [13]) that g also satisfies, with $\nu_2 \equiv \nu_2(\gamma_1, \gamma_2, L) > 0$ and $0 \le \mu \le 1$

$$D^2 g(z)\lambda \otimes \lambda \ge \nu_2\, (\mu^2 + |z|^2)^{(p-2)/2}\, |\lambda|^2. \tag{3.9}$$

• **A reference estimate**

Proposition 3.2. *Let $g : \mathbb{R}^n \to \mathbb{R}$ be a function of class $\mathcal{C}^2$ satisfying* (H6) *and* (H7) *for some $1 < \gamma_1 \le p \le \gamma_2$. Moreover let v be a local minimizer in K of the functional* (3.5) *with $B_R \Subset \Omega$.*

If in addition the function ψ fulfills (2.6) *for some $n - \gamma_1 < \lambda < n$, then for all $0 < \rho < R/2$ and any $\varepsilon > 0$*

$$\int_{B_\rho} |Dv(x)|^p \, dx \le c \left[\left(\frac{\rho}{R}\right)^n + \varepsilon \right] \int_{B_R} (1 + |Dv(x)|^p)\, dx + \bar{c}\, R^\lambda,$$

where $c \equiv c(\gamma_1, \gamma_2, L)$ and $\bar{c} \equiv \bar{c}(\gamma_1, \gamma_2, L, \varepsilon)$.

Proof. The proof of this result can be carried out as in Proposition 4.1 of [12]. One indeed has to make sure that the constants involved only depend on the global bounds γ_1 and γ_2 of the exponent function p. In this respect, the key points are Theorem 2.2 of [13] and the structure conditions (3.7), (3.8) and (3.9). □

• **A up-to-the-boundary higher integrability result**

If v is a local minimizer in K of the functional (3.5), then the following up-to-the-boundary higher integrability result can be rapidly achieved.

Proposition 3.3. (see [12], Proposition 3.3) *Let $g : \mathbb{R}^n \to \mathbb{R}$ be a continuous function fulfilling* (H6) *for some $\gamma_1 \le p \le \gamma_2$. Let v be a local minimizer of the functional* (3.5) *in the Dirichlet class $\{v \in u + W_0^{1,p}(B_R) : v \in K\}$, for some $u \in W^{1,p}(B_R)$, where the function ψ fulfills the assumption* (2.6)*. If moreover $u \in W^{1,\bar{q}}(B_R)$ for a certain $p < \bar{q} < q$, then there exist $p < \bar{r} < \bar{q}$ and c depending on γ_1, γ_2, L but not on u or R such that $v \in W^{1,\bar{r}}(B_{R/2})$ and*

$$\left(\fint_{B_{R/2}} |Dv(x)|^{\bar{r}}\, dx \right)^{1/\bar{r}} \le c \left(\fint_{B_{R/2}} |Dv(x)|^p \, dx \right)^{1/p} \tag{3.10}$$

$$+ c \left[\fint_{B_R} (1 + |Du(x)|^{\bar{q}}) \, dx \right]^{1/\bar{q}} + c \left[\fint_{B_R} (1 + |D\psi(x)|^{\bar{q}}) \, dx \right]^{1/\bar{q}} .$$

Proof. The proof of this result is not very difficult. We refer the reader to [5] for the proof in the non-obstacle situation and to [12] for a short discussion on the additional difficulties due to the obstacle function. On the other hand one has to assure that the constants only depend on the global bounds γ_1 and γ_2 of the exponent function p, and not on p itself. For this discussion we refer the reader to [19]. $\square$

• Iteration lemma

We will use the following iteration lemma, which can for example be found in [16], for the proof of our results.

Lemma 3.4. *Let $\Phi(t)$ be a nonnegative and nondecreasing function. Suppose that*

$$\Phi(\rho) \leq A \left[\left(\frac{\rho}{R} \right)^{\alpha} + \varepsilon \right] \Phi(R) + BR^{\beta},$$

for all $\rho \leq R \leq R_0$, with A, B, α, β nonnegative constants, $\beta < \alpha$. Then there exists a constant $\varepsilon_0 \equiv \varepsilon_0(A, \alpha, \beta)$ such that if $\varepsilon < \varepsilon_0$, for all $\rho \leq R \leq R_0$, then

$$\Phi(\rho) \leq c \left[\left(\frac{\rho}{R} \right)^{\beta} \Phi(R) + B\rho^{\beta} \right],$$

where c is a constant depending on α, β, A, but independent of B.

• A technical lemma

Lemma 3.5. ([5], Lemma 2.2) *If $p > 1$ is such that there exist two constants γ_1, γ_2 with $\gamma_1 \leq p \leq \gamma_2$, then there exists a constant $c \equiv c(\gamma_1, \gamma_2)$ such that for any $\mu \geq 0$, $\xi, \eta \in \mathbb{R}^n$*

$$(\mu^2 + |\xi|^2)^{p/2} \leq c \, (\mu^2 + |\eta|^2)^{p/2} + c \, (\mu^2 + |\xi|^2 + |\eta|^2)^{(p-2)/2} \, |\xi - \eta|^2.$$

4. Proof of Theorem 2.8

A priori assumptions For the proof of Theorem 2.8 we will assume that the modulus of continuity of our growth exponent p satisfies the condition (2.7). Therefore, in particular we may always assume (2.2).

Step 1: Localization Let us start with Lemma 3.1 which provides a higher integrability exponent δ such that for any $\Omega' \Subset \Omega$ there holds

$$\int_{\Omega'} |Du|^{p(x)(1+\delta)} \, dx < +\infty.$$

Let us assume that the $p(x)$ energy on Ω is bounded, i.e. that there exists a constant M such that

$$\int_{\Omega} |Du|^{p(x)} \, dx \leq M < +\infty. \tag{4.1}$$

In the sequel we will explicitly point out if constants depend on this bound M.

FURTHER LOCALIZATION. Thus we end up with a maximal radius of size R_M, such that there holds $\omega_1(8R_M) \leq \delta/4$. Let $\mathcal{O} \Subset \Omega$ be a set whose diameter does not exceed R_M. We denote

$$p_2 := \max\{p(x) : x \in \overline{\mathcal{O}}\} = p(x_0), \qquad p_1 := \min\{p(x) : x \in \overline{\mathcal{O}}\}. \tag{4.2}$$

Then there holds

$$p_2 - p_1 \leq \omega_1(8R_M) \leq \delta/4; \tag{4.3}$$

$$p_2(1 + \delta/4) \leq p(x)(1 + \delta/4 + \omega_1(R)) \leq p(x)(1 + \delta). \tag{4.4}$$

Furthermore we note that the localization together with the bound (2.2) for the modulus of continuity provides for any $R \le 8R_M \le 1$:

$$R^{-n\omega_1(R)} \le \exp(nL) = c(n,L), \qquad R^{-\frac{n\omega_1(R)}{1+\omega_1(R)}} \le c(n,L). \tag{4.5}$$

HIGHER INTEGRABILITY. By our higher integrability result and the localization, we immediately obtain

$$\fint_{B_R} |Du|^{p_2}\,dx \le c\left[\left(\fint_{B_{2R}} |Du|^{p(x)}\,dx\right)^{1+\delta/4} + \fint_{B_{2R}} |D\psi|^{p(x)(1+\delta/4)}\,dx + 1\right] \tag{4.6}$$

Step 2: Freezing

Let B_R be a ball in $\mathcal{O}$. We define $v \in u + W_0^{1,p_2}(B_R)$ as the unique minimum of the functional

$$\mathcal{G}(v) := \int_{B_R} h(x_0, Dv)\,dx =: \int_{B_R} g(Dv)\,dx$$

in the class K. Since the functional $\mathcal{G}$ is frozen in the point x_0, it satisfies the growth and ellipticity conditions (H6) and (H7) with the maximal exponent $p = p_2$. For our proof we will assume that $g \in \mathcal{C}^2$. Removing the $\mathcal{C}^2$ regularity of g can then be done by a suitable approximation, arguing exactly as in [2].

Note that by the minimizing property of v, we obtain the following bound for the p_2 energy of v (since $u \in K$):

$$\int_{B_R} |Dv|^{p_2}\,dx \le L^2 \int_{B_R} (1 + |Du|^{p_2})\,dx \le c(M), \tag{4.7}$$

where in the last inequality we used (4.6), (4.1) and (2.6).

REFERENCE ESTIMATE. v is a K-minimizer of the frozen functional with constant p_2 growth. Therefore it satisfies the assumptions of Proposition 3.2 with $1 < \gamma_1 \le p \equiv p_2 \le \gamma_2$. Thus there holds for any $\varepsilon > 0$ and any ρ with $2\rho < R$:

$$\int_{B_\rho} |Dv|^{p_2}\,dx \le c\left[\left(\frac{\rho}{R}\right)^n + \varepsilon\right] \int_{B_R} (1 + |Dv|^{p_2})\,dx + \bar{c}R^\lambda, \tag{4.8}$$

with $c \equiv c(\gamma_1, \gamma_2, L)$ and $\bar{c} \equiv \bar{c}(\gamma_1, \gamma_2, L, \varepsilon)$.

COMPARISON ESTIMATE. We prove the following comparison estimate

$$\begin{aligned} &\int_{B_R} (\mu^2 + |Du|^2 + |Dv|^2)^{\frac{p_2-2}{2}} |Du - Dv|^2\,dx \\ &\qquad \le c\left(\omega_1(R)\log\left(\tfrac{1}{R}\right)\right)\left[\int_{B_{4R}} (1 + |Du|^{p_2})\,dx + R^\lambda\right]. \end{aligned} \tag{4.9}$$

Using the differentiability of g and the ellipticity (3.9) with $p = p_2$, we estimate

$$\begin{aligned} \mathcal{G}(u) - \mathcal{G}(v) &= \int_{B_R} [g(Du) - g(Dv)]\,dx \\ &= \int_{B_R} \langle Dg(Dv), Du - Dv\rangle dx \qquad [= 0] \\ &\quad + \int_{B_R} dx \int_0^1 (1-t) D^2 g(tDu + (1-t)Dv)(Du - Dv) \otimes (Du - Dv)dt \\ &\ge \nu_2 \int_{B_R} dx \int_0^1 (1-t)(\mu^2 + |tDu + (1-t)Dv|^2)^{(p_2-2)/2} |Du - Dv|^2 dt \end{aligned} \tag{4.10}$$

$$\geq \quad c^{-1}\int_{B_R}(\mu^2+|Du|^2+|Dv|^2)^{(p_2-2)/2}|Du-Dv|^2dx\ ,$$

with $c \equiv c(\gamma_1,\gamma_2,L)$.

On the other hand we have

$$\begin{aligned}\int_{B_R}[g(Du)-g(Dv)]\,dx &= \int_{B_R}[h(x_0,Du)-h(x,Du)]\,dx\\ &\quad+\int_{B_R}[h(x,Du)-h(x,Dv)]\,dx\\ &\quad+\int_{B_R}[h(x,Dv)-h(x_0,Dv)]\,dx\\ &= I^{(1)}+I^{(2)}+I^{(3)}.\end{aligned}$$

Obviously we have $I^{(2)}\leq 0$, since u is a minimizer of the functional (2.5) in the class K and v belongs to K. We estimate $I^{(1)}$, using the continuity of the integrand with respect to the variable x, then splitting as follows:

$$\begin{aligned}I^{(1)} &\leq c\int_{B_R}\omega_1(|x-x_0|)((\mu^2+|Du|^2)^{p(x)/2}+(\mu^2+|Du|^2)^{p_2/2})(1+\log(\mu^2+|Du|^2))\,dx\\ &\leq c\,\omega_1(R)\int_{B_R\cap\{|Du|\geq e\}}|Du|^{p_2}\log|Du|^{p_2}\,dx+c\,\omega_1(R)R^n\\ &\leq c\,\omega_1(R)R^n \fint_{B_R}|Du|^{p_2}\log\left(e+|||Du|^{p_2}||_{L^1(B_R)}\right)\,dx\\ &\quad+c\,\omega_1(R)\int_{B_R}|Du|^{p_2}\log\left(e+\frac{|Du|^{p_2}}{|||Du|^{p_2}||_{L^1(B_R)}}\right)\,dx+c\,\omega_1(R)R^n\\ &= I_1^{(1)}+I_2^{(1)}+I_3^{(1)}.\end{aligned}$$

We estimate $I_2^{(1)}$, using higher integrability (3.1), the localization (4.4), the bound M for the $p(x)$ energy and some basic facts from the theory of Orlicz spaces (for some details we refer to [22], see also [2], [3])

$$\begin{aligned}I_2^{(1)} &\leq c\,\omega_1(R)R^n\left(\fint_{B_R}|Du|^{p_2(1+\delta/4)}\,dx\right)^{1/(1+\delta/4)}\\ &\leq c\,\omega_1(R)R^n+c\,\omega_1(R)R^n\left(\fint_{B_R}|Du|^{p(x)(1+\delta/4+\omega_1(R))}\,dx\right)^{1/(1+\delta/4)}\\ &\leq c\,\omega_1(R)R^n+c(\delta)\omega_1(R)R^n\Bigg[\left(\fint_{B_{2R}}|Du|^{p(x)}\,dx\right)^{\frac{1+\delta/4+\omega_1(R)}{1+\delta/4}}\\ &\qquad\qquad+\left(\fint_{B_{2R}}(|D\psi|^{p(x)(1+\delta)}+1)\,dx\right)^{1/(1+\delta/4)}\Bigg]\\ &\leq c\,\omega_1(R)R^n+c\,\omega_1(R)R^nR^{-n\frac{\omega_1(R)}{1+\delta/4}}\left(\fint_{B_{2R}}|Du|^{p(x)}\,dx\right)\left(\int_{B_{2R}}|Du|^{p(x)}\,dx\right)^{\frac{\omega_1(R)}{1+\delta/4}}\\ &\quad+c\,\omega_1(R)R^n\,R^{\frac{(\lambda-n)}{1+\delta/4}}\\ &\overset{(4.5)}{\leq} c\,\omega_1(R)R^n+c\,\omega_1(R)R^n\left(\fint_{B_{2R}}(1+|Du|^{p_2})\,dx\right)\left(\int_{B_{2R}}|Du|^{p(x)}\,dx\right)^{\frac{\omega_1(R)}{1+\delta/4}}\\ &\quad+c\,\omega_1(R)R^\lambda\,R^{(n-\lambda)\,\frac{\delta/4}{1+\delta/4}}\end{aligned}$$

$$\leq \quad c\,\omega_1(R)\cdot M\cdot \int_{B_{2R}} (1+|Du|^{p_2})\,dx + c\,\omega_1(R)\,R^{\lambda}.$$

In the last step we used the fact that $R\leq 1$. We estimate $I_1^{(1)}$, using estimates for the $L\log L$-norm of $|Du|^{p_2}$, which can for example be found in [3]:

$$\begin{aligned}
I_1^{(1)} \quad &\leq \quad c\,\omega_1(R)\log\left(R^{-n}e + R^{-n}\int_{B_R}|Du|^{p_2}\,dx\right)\int_{B_R}|Du|^{p_2}\,dx \\
&\leq \quad c\,\omega_1(R)\int_{B_R}|Du|^{p_2}\,dx\cdot\log\left(e+\int_{B_R}|Du|^{p_2}\,dx\right) + c\,\omega_1(R)\log\left(\tfrac{1}{R}\right)\int_{B_R}|Du|^{p_2}\,dx \\
&\leq \quad c(\delta)\,\omega_1(R)\left(1+\int_{B_R}|Du|^{p_2}\,dx\right)^{\delta/4}\int_{B_R}|Du|^{p_2}\,dx + c\,\omega_1(R)\log\left(\tfrac{1}{R}\right)\int_{B_R}|Du|^{p_2}\,dx \\
&\leq \quad c(M,n,\delta)\left(\omega_1(R)\log\left(\tfrac{1}{R}\right)\right)\int_{B_R}(1+|Du|^{p_2})\,dx.
\end{aligned}$$

Thus, alltogether we obtain

$$I^{(1)} \leq c\left(\omega_1(R)\log\left(\tfrac{1}{R}\right)\right)\left[\int_{B_{2R}}(1+|Du|^{p_2})\,dx + R^{\lambda}\right].$$

We estimate $I^{(3)}$ in an analogue way to $I^{(1)}$. Doing the splitting into $I_1^{(3)}$ to $I_3^{(3)}$ in the same way as for $I^{(1)}$, we use higher integrability up to the boundary for v (3.10) (where we set $\bar r = p_2(1+\tilde\varepsilon/4)$ with $\tilde\varepsilon\in(0,\delta)$ being the up-to-the-boundary higher integrability exponent) and the estimate (4.7) for the p_2 energy of v:

$$\begin{aligned}
I_2^{(3)} \quad &\leq \quad c\,\omega_1(R)R^n\left(\fint_{B_R}|Dv|^{p_2(1+\tilde\varepsilon/4)}\,dx\right)^{1/(1+\tilde\varepsilon/4)} \\
&\leq \quad c\,\omega_1(R)R^n\Bigg[\fint_{B_R}|Dv|^{p_2}\,dx + \left(\fint_{B_{2R}}(|Du|^{p_2(1+\delta/4)}+1)\,dx\right)^{\frac{1}{1+\delta/4}} \\
&\qquad\qquad + \left(\fint_{B_{2R}}(|D\psi|^{p_2(1+\delta/4)}+1)\,dx\right)^{\frac{1}{1+\delta/4}}\Bigg] \\
&\leq \quad c\,\omega_1(R)R^n\int_{B_R}(1+|Du|^{p_2})\,dx + c\,\omega_1(R)R^n\left(\fint_{B_{2R}}|Du|^{p_2(1+\delta/4)}\,dx\right)^{\frac{1}{1+\delta/4}} \\
&\qquad + c\,\omega_1(R)\,R^{\lambda}\,R^{(n-\lambda)\,\frac{\delta/4}{1+\delta/4}} \\
&\leq \quad c(M)\,\omega_1(R)\int_{B_{4R}}(1+|Du|^{p_2})\,dx + c\,\omega_1(R)\,R^{\lambda}.
\end{aligned}$$

In the last step we used again the fact that $R\leq 1$ and also an estimate analogue to the one of the term $I_2^{(1)}$; that's why the radius of the ball doubles once more. $I_1^{(3)}$ is estimated in an analogue way to $I_1^{(1)}$, additionally using (4.7) for passing over from the p_2 energy of v to the energy of u. Alltogether we end up with

$$I^{(3)} \leq c\left(\omega_1(R)\log\left(\tfrac{1}{R}\right)\right)\left[\int_{B_{4R}}(1+|Du|^{p_2})\,dx + R^{\lambda}\right].$$

Taking the estimates for $I^{(1)}$ to $I^{(3)}$ together, we end up with the desired comparison estimate (4.9).

CONCLUSION. Now we put together our reference estimate and the comparison estimate to deduce a decay estimate for the p_2 energy of u.

By the technical Lemma 3.5 we now split as follows

$$\begin{aligned}\int_{B_\rho} |Du|^{p_2}\,dx &\le \int_{B_\rho} (\mu^2+|Du|^2)^{p_2/2}\,dx\\ &\le c\int_{B_\rho}(\mu^2+|Dv|^2)^{p_2/2}\,dx\\ &\qquad\qquad + c\int_{B_\rho}(\mu^2+|Du^2|+|Dv|^2)^{\frac{p_2-2}{2}}|Du-Dv|^2\,dx\\ &=: [A]+[B].\end{aligned}$$

For $[A]$ we use the reference estimate and estimate (4.7) to deduce (note that $\rho\le 1$)

$$\begin{aligned}[A] &\le c\rho^n+\int_{B_\rho}|Dv|^{p_2}\,dx\\ &\le c\left[\left(\frac{\rho}{R}\right)^n+\varepsilon\right]\int_{B_R}(1+|Dv|^{p_2})\,dx + cR^\lambda\\ &\le c\left[\left(\frac{\rho}{R}\right)^n+\varepsilon\right]\int_{B_R}(1+|Du|^{p_2})\,dx + cR^\lambda.\end{aligned}$$

For the term $[B]$ we use the comparison estimate (4.9)

$$[B]\le c\left(\omega_1(R)\log\left(\tfrac{1}{R}\right)\right)\left[\int_{B_{4R}}(1+|Du|^{p_2})\,dx+R^\lambda\right]. \tag{4.11}$$

Thus, alltogether we end up with

$$\int_{B_\rho}|Du|^{p_2}\,dx\le c\left[\left(\frac{\rho}{R}\right)^n+\varepsilon+\omega_1(R)\log\left(\tfrac{1}{R}\right)\right]\int_{B_{4R}}(1+|Du|^{p_2})\,dx+\bar{c}R^\lambda, \tag{4.12}$$

where $c\equiv c(n,L,M,\gamma_1,\gamma_2)$ and $\bar{c}\equiv\bar{c}(n,M,L,\gamma_1,\gamma_2,\varepsilon)$.

Step 3: Proof of the Theorem

Let B_{R_0} be a ball whose radius is small enough to satisfy $R_0\le R_M$. Then estimate (4.12) holds for any radii $0<\rho\le R\le R_0$. Let $\varepsilon_0\equiv\varepsilon_0(n,M,L,\gamma_1,\gamma_2,\lambda)$ be the quantity provided by Lemma 3.4. We choose $\varepsilon\equiv\varepsilon_0/2$. This fixes the dependencies of the constant in (4.12) $\bar{c}\equiv\bar{c}(n,L,M,\gamma_1,\gamma_2,\lambda)$. Then by our assumption (2.7) we can find a radius $R_1>0$ so small that $\omega_1(R_1)\log(1/R_1)<\varepsilon_0/2$ and therefore

$$\omega_1(R)\log\left(\tfrac{1}{R}\right)+\varepsilon<\varepsilon_0,$$

for any $0<R\le R_1$ and thus we have $R_1\equiv R_1(n,\gamma_1,\gamma_2,L,M,\omega_1,\lambda)$. Lemma 3.4 yields

$$\int_{B_\rho}|Du(x)|^{p_2}\,dx\le c\rho^\lambda,$$

with $c\equiv c(n,M,L,\gamma_1,\gamma_2,\lambda)$, whenever $0<\rho<R_1$. Since we have $\gamma_1\le p_2\le\gamma_2$, we deduce by a standard covering argument that

$$Du\in L^{\gamma_1,\lambda}_{\mathrm{loc}}(\Omega),$$

and thus by Poincaré's inequality we conclude $u\in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ with $\alpha=1-\frac{n-\lambda}{\gamma_1}$. □

5. Proof of Theorem 2.9

We start with a technical lemma which we will need later in the proof. The proof of a slightly modified lemma can be found in [12].

Proposition 5.1. *Let $g : \mathbb{R}^n \to \mathbb{R}$ be a continuous function satisfying* (H6) *and* (H7) *with* $1 < \gamma_1 \le p \le \gamma_2 < \infty$. *Let* $u \in K$, $B_R \Subset \Omega$ *and let* $v_0 \in W^{1,p}(\Omega)$ *be a minimizer of the functional*

$$\mathcal{H}(w, B_R) := \int_{B_R} g(Dw(x))\,dx + \theta_0 \left(\int_{B_R} |Dw - Dv_0|^p\,dx\right)^{1/p}$$

in the Dirichlet class

$$\overline{D} := \{w \in K : w = u \text{ on } \partial B_R\},$$

where $\theta_0 \ge 0$. *Then, for all* $\beta > 0$, *for all* $A_0 > 0$ *and for any* $\varepsilon > 0$ *we have*

$$\begin{aligned}\int_{B_\rho} |Dv_0(x)|^p\,dx \quad &\le \quad c\left[\left(\frac{\rho}{R}\right)^n + \varepsilon\right]\int_{B_R}(1 + |Dv_0(x)|^p)\,dx + \bar{c}\,R^\lambda \\ &\quad + c\,\theta_0\left(\int_{B_R}|Du(x) - Dv_0(x)|^p\,dx\right)^{1/p} + c\,\theta_0^{\frac{p}{p-1}}\left[\frac{1}{A_0}\right]^{\frac{p\beta}{p-1}} \\ &\quad + c\,[A_0]^{p\beta}\int_{B_R}(1 + |Du(x)|^p)\,dx,\end{aligned}$$

for any $0 < \rho < R/2$, *where the constants* c *depend only on* L, γ_1, γ_2 *while the constant* $\bar{c}$ *depends also on* ε.

Let B_R be a ball in $\mathcal{O}$, where $\mathcal{O}$ has been defined in (4.2). We define $v \in u + W_0^{1,p_2}(B_R)$ as the unique minimum in the class K of the following functional

$$\mathcal{G}_0(v, B_R) := \int_{B_R} f(x_0, (u)_R, Dv(x))\,dx. \tag{5.1}$$

Since the functional $\mathcal{G}_0$ is frozen in the point $(x_0, (u)_R)$, it satisfies the growth and ellipticity conditions (H6) and (H7) with maximal exponent $p = p_2$. From now on, since we are going to prove local regularity results, we shall assume that, due to Theorem 2.8 in [12] , our minimizer v in K is globally Hölder continuous, that is there exists $0 < \gamma < 1$ and a constant $[v]_\gamma$ such that

$$|v(x) - v(y)| \le [v]_\gamma\,|x - y|^\gamma \tag{5.2}$$

for all $x, y \in \Omega$.

Remark 5.2. We use Theorem 2.8 in [12] with the choice $p = p_2$; in this respect we have to make sure that $\lambda > n - p_2$, but this is satisfied as we are assuming that $\lambda > n - \gamma_1$.

We start applying Lemma 3.1 in order to get a higher integrability exponent for the gradient Du, $\delta > 0$. Obviously, we can replace at will the exponent δ with smaller constants, so we choose

$$\delta < \min\left\{\frac{\gamma}{1-\gamma}, \frac{p_2 + \lambda - n}{n - \lambda}\right\}.$$

We moreover set

$$\tilde{m} := \min\left\{\frac{\lambda - n + p_2}{p_2}, \gamma + \gamma\,\delta - \delta, \frac{p_2 + (1+\delta)\,(\lambda - n)}{p_2}\right\}$$

and due to our assumptions it turns out that $0 < \tilde{m} < 1$. We will show that

$$\mathcal{G}_0(u) - \mathcal{G}_0(v) \le c\,\left(\omega_1(R)\log\left(\tfrac{1}{R}\right) + \omega_2^\sigma(R^{\tilde{m}})\right)\left[\int_{B_{4R}}(1 + |Du|^{p_2})\,dx + R^\lambda\right], \tag{5.3}$$

for some suitable $\sigma > 0$ we will specify later and with a constant $c \equiv c(n, \gamma_1, \gamma_2, M, L, \gamma, \lambda)$. Note here that M is the bound on the $p(x)$ energy which has been introduced in (4.1).

Since u is a local minimizer in K of the functional (1.1), we obtain

$$\mathcal{G}_0(u) \quad \le \quad \mathcal{G}_0(v) + \int_{B_R}[f(x_0, (u)_R, Du(x)) - f(x, u(x), Du(x))]\,dx$$

$$
\begin{aligned}
&\quad + \int_{B_R} [f(x, v(x), Dv(x)) - f(x_0, (u)_R, Dv(x))]\, dx \\
&\leq \mathcal{G}_0(v) + \int_{B_R} [f(x_0, (u)_R, Du(x)) - f(x, (u)_R, Du(x))]\, dx \\
&\quad + \int_{B_R} [f(x, (u)_R, Du(x)) - f(x, (v)_R, Du(x))]\, dx \\
&\quad + \int_{B_R} [f(x, (v)_R, Du(x)) - f(x, v(x), Du(x))]\, dx \\
&\quad + \int_{B_R} [f(x, v(x), Du(x)) - f(x, u(x), Du(x))\, dx \\
&\quad + \int_{B_R} [f(x, v(x), Dv(x)) - f(x, (v)_R, Dv(x))]\, dx \\
&\quad + \int_{B_R} [f(x, (v)_R, Dv(x)) - f(x, (u)_R, Dv(x))]\, dx \\
&\quad + \int_{B_R} [f(x, (u)_R, Dv(x)) - f(x_0, (u)_R, Dv(x))]\, dx \\
&\leq \mathcal{G}_0(v) + I^{(4)} + I^{(5)} + I^{(6)} + I^{(7)} + I^{(8)} + I^{(9)} + I^{(10)},
\end{aligned}
$$

with the obvious labelling. Note that we do not have a condition similar to (5.2) for the function u, which forces us to split in a bit more complicated way.

At this point, the term $I^{(4)}$ can be estimated as $I^{(1)}$, giving

$$
I^{(4)} \leq c\left(\omega_1(R)\log\left(\tfrac{1}{R}\right)\right)\left[\int_{B_{2R}} (1 + |Du|^{p_2})\, dx + R^\lambda\right].
$$

In a similar way, the term $I^{(10)}$ can be estimated as the term $I^{(3)}$, giving

$$
I^{(10)} \leq c\left(\omega_1(R)\log\left(\tfrac{1}{R}\right)\right)\left[\int_{B_{4R}} (1 + |Du|^{p_2})\, dx + R^\lambda\right].
$$

The terms $I^{(6)}$ and $I^{(8)}$ can be estimated using (5.2) as follows

$$
\begin{aligned}
I^{(6)} &\leq L \int_{B_R} \omega_2(|v - (v)_R|)\,(\mu^2 + |Du(x)|^2)^{p(x)/2}\, dx \\
&\leq c\,\omega_2(R^\gamma) \int_{B_R} (1 + |Du(x)|^{p_2})\, dx
\end{aligned}
$$

and, using also (4.7)

$$
\begin{aligned}
I^{(8)} &\leq L \int_{B_R} \omega_2(|v - (v)_R|)\,(\mu^2 + |Dv(x)|^2)^{p(x)/2}\, dx \\
&\leq c\,\omega_2(R^\gamma) \int_{B_R} (1 + |Du(x)|^{p_2})\, dx.
\end{aligned}
$$

The main difficulty arises when we try to estimate the remaining terms. First of all, using Poincaré inequality, the minimality in K of u and v respectively for the functionals (1.1) and (5.1) and also the Caccioppoli inequality for v (which can be obtained working in a similar way as in Theorem 3.2. in [12]), we obtain

$$
\begin{aligned}
\fint_{B_R} |v(x) - u(x)|\, dx &\leq c\,R \fint_{B_R} |Dv(x) - Du(x)|\, dx \\
&\leq c\left(R^{p_2} \fint_{B_R} |Dv(x) - Du(x)|^{p_2}\, dx\right)^{1/p_2}
\end{aligned}
$$

$$
\begin{aligned}
&\le \quad c\left(R^{p_2} \fint_{B_R}(1+|Du|^{p_2})\,dx\right)^{1/p_2}\\
&\overset{(4.6)}{\le} \quad c\,R + c\,R\left(\fint_{B_{2R}}|Du|^{p(x)}\,dx\right)^{(1+\delta)/p_2} + c\,R^{\frac{p_2+\lambda-n}{p_2}}\\
&\overset{(H1)}{\le} \quad c\,R^{\tilde m} + c\,R\left[\fint_{B_{2R}}(1+|Dv|^{p(x)})\,dx\right]^{\frac{(1+\delta)}{p_2}}\\
&\le \quad c\,R^{\tilde m} + c\,R\left[\fint_{B_{4R}}\left(1+\left|\frac{v(x)-(v)_{4R}}{R}\right|^{p_2}\right)dx + c\,R^{\lambda-n}\right]^{\frac{(1+\delta)}{p_2}}\\
&\le \quad c\,R^{\tilde m} + c\,R\left[\fint_{B_{4R}}\left(\frac{[v]_\gamma^{p_2}\,R^{p_2\,\gamma}}{R^{p_2}}+1\right)dx\right]^{\frac{(1+\delta)}{p_2}}\\
&\le \quad c\,R^{\tilde m},
\end{aligned}
$$

and thus by the monotonicity of ω_2

$$
\omega_2\left(\fint_{B_R}|v(x)-u(x)|\,dx\right) \le c\,\omega_2(R^{\tilde m}). \tag{5.4}
$$

At this point we have

$$
\begin{aligned}
I^{(5)} &\le L\int_{B_R}\omega_2\left(\fint_{B_R}|u(x)-v(x)|\,dx\right)\left(\mu^2+|Du|^2\right)^{p(x)/2}\,dx\\
&\le c\,\omega_2(R^{\tilde m})\int_{B_R}(1+|Du|^{p_2})\,dx
\end{aligned}
$$

and in a similar way, using (4.7)

$$
\begin{aligned}
I^{(9)} &\le L\int_{B_R}\omega_2\left(\fint_{B_R}|v(x)-u(x)|\,dx\right)\left(\mu^2+|Dv|^2\right)^{p(x)/2}\,dx\\
&\le c\,\omega_2(R^{\tilde m})\int_{B_R}(1+|Du|^{p_2})\,dx.
\end{aligned}
$$

The term $I^{(7)}$ has to be treated in a slightly different way, i.e. we have

$$
\begin{aligned}
I^{(7)} &\le L\int_{B_R}\omega_2(|v(x)-u(x)|)\,(\mu^2+|Du(x)|^2)^{p(x)/2}\,dx\\
&\le L\int_{B_R}\omega_2(|v(x)-u(x)|)\,(\mu^2+|Du(x)|^2)^{p_2/2}\,dx + L\int_{B_R}\omega_2(|v(x)-u(x)|)\,dx\\
&\le I_1^{(7)} + I_2^{(7)}.
\end{aligned}
$$

Let us set $\tilde r = p_2(1+\delta/4)$ and $\sigma := \frac{\tilde r - p_2}{\tilde r}$. Using the concavity of ω_2 we deduce

$$
\begin{aligned}
I_1^{(7)} &\le c\left[\int_{B_R}(\mu^2+|Du|^2)^{\tilde r/2}\,dx\right]^{p_2/\tilde r}\left[\int_{B_R}\omega_2^{\frac{\tilde r}{\tilde r-p_2}}(|v(x)-u(x)|)\,dx\right]^{\frac{\tilde r-p_2}{\tilde r}}\\
&\le c\,R^n\left[\fint_{B_R}\omega_2(|v(x)-u(x)|)\,dx\right]^{\frac{\tilde r-p_2}{\tilde r}}\left(\fint_{B_R}(1+|Du|^{p_2(1+\delta/4)})\,dx\right)^{\frac{1}{1+\delta/4}}\\
&\le c\,R^n\,\omega_2^\sigma\left(\fint_{B_R}(|v(x)-u(x)|)\,dx\right)\left(\fint_{B_R}(1+|Du|^{p(x)(1+\delta/4+\omega_1(R))})\,dx\right)^{\frac{1}{1+\delta/4}}
\end{aligned}
$$

$$
\begin{aligned}
&\overset{(5.4)}{\leq} \quad c\,\omega_2^{\sigma}(R^{\tilde m})\,R^n\,\left[\left(\fint_{B_{2R}}|Du|^{p(x)}\,dx\right)^{\frac{1+\delta/4+\omega_1(R)}{1+\delta/4}}+\left(\fint_{B_{2R}}(|D\psi|^{p(x)(1+\delta)}+1)\,dx\right)^{\frac{1}{1+\delta/4}}\right]\\
&\leq \quad c\,\omega_2^{\sigma}(R^{\tilde m})\,R^n\,\left[R^{-n\,\frac{\omega_1(R)}{1+\delta/4}}\left(\fint_{B_{2R}}|Du|^{p(x)}\,dx\right)\left(\int_{B_{2R}}|Du|^{p(x)}\,dx\right)^{\frac{\omega_1(R)}{1+\delta/4}}+R^{\frac{\lambda-n}{1+\delta/4}}\right]\\
&\leq \quad c(M)\,\omega_2^{\sigma}(R^{\tilde m})\,\left[\int_{B_{2R}}(1+|Du|^{p_2})\,dx+R^{\lambda}\right].
\end{aligned}
$$

On the other hand

$$
I_2^{(7)}\leq\,c\,R^n\,\omega_2\left(\fint_{B_R}|v(x)-u(x)|\,dx\right)\overset{(5.4)}{\leq}\,c\,\omega_2(R^{\tilde m})\,R^n.
$$

Collecting the previous bounds and summing up we obtain

$$
I^{(4)}+\dots I^{(10)}\leq\,c\,\left(\omega_1(R)\log\left(\tfrac{1}{R}\right)+\omega_2^{\sigma}(R^{\tilde m})\right)\left[\int_{B_{4R}}(1+|Du|^{p_2})\,dx+R^{\lambda}\right],
$$

with $c\equiv c(n,\gamma_1,\gamma_2,M,L,\gamma,\lambda)$, which provides the desired estimate (5.3).

We set for simplicity

$$
F(R):=\omega_1(R)\log\left(\frac{1}{R}\right)+\omega_2^{\sigma}(R^{\tilde m})\,.
$$

The assumption (2.8) allows us to say that

$$
\lim_{R\to 0}F(R)=0\,.
$$

Now, by the minimality of v, we obtain

$$
\mathcal{G}_0(u)\leq\inf_V\mathcal{G}_0+H(R)\,,
$$

where we set

$$
H(R):=c\,F(R)\left[\int_{B_{4R}}(1+|Du(x)|^{p_2})dx+R^{\lambda}\right]
$$

and

$$
V=\{v\in u+W_0^{1,1}(B_R):\,v\in K\}.
$$

Let V be equipped with the distance

$$
d(w_1,w_2):=H(R)^{-\frac{1}{p_2}}\left(\int_{B_R}|Dw_1(x)-Dw_2(x)|^{p_2}\,dx\right)^{1/p_2}. \tag{5.5}
$$

It is easy to see that the functional $\mathcal{G}_0$ is lower semicontinuous with respect to the topology induced by the distance d. Then by [10], Theorem 1 ("Ekeland variational principle") there exists $v_0\in V$ such that

(i) $\displaystyle\int_{B_R}|Du(x)-Dv_0(x)|^{p_2}\,dx\leq\,H(R),$

(ii) $\mathcal{G}_0(v_0)\leq\mathcal{G}_0(u),$

(iii) v_0 is a local minimizer in V of the functional

$$
w\mapsto\mathcal{H}(w):=\mathcal{G}_0(w)+[H(R)]^{\frac{p_2-1}{p_2}}\left(\int_{B_R}|Dw-Dv_0|^{p_2}\,dx\right)^{1/p_2}.
$$

Remark 5.3. We choose to apply the Ekeland variational principle with the distance (5.5) which derives from a suitable weighted $L^{p_2}-$norm instead of the corresponding L^1-norm; the same trick has been successfully applied in the paper [21]. The advantage of this choice is that we can directly estimate the term

$$\int_{B_R} |Du(x) - Dv_0(x)|^{p_2}\, dx$$

by means of (i) given by the Ekeland lemma without needing any further interpolation argument (which has been instead employed for example in [5] or in [12]).

First of all, from the growth assumption (H6) with exponent $p = p_2$ and from property (ii), as $u \in K$, we have

$$L^{-1}\int_{B_R} |Dv_0(x)|^{p_2} dx \leq \mathcal{G}_0(v_0) \leq \mathcal{G}_0(u) \leq L\int_{B_R} (1 + |Du(x)|^{p_2})\, dx, \tag{5.6}$$

Now, we apply Proposition 5.1 with the following choices: $h(z) := f(x_0, (u)_R, z)$, $p = p_2$, $A_0 = F(R)$ and $\vartheta_0 = [H(R)]^{\frac{p_2-1}{p_2}}$ Then, by property (i) and using (5.6), we have for every $\beta > 0$

$$\begin{aligned}
\int_{B_\rho} |Dv_0(x)|^{p_2} dx \quad &\leq \quad c\left[\left(\frac{\rho}{R}\right)^n + \varepsilon\right]\int_{B_R} (1 + |Dv_0(x)|^{p_2})\, dx + \bar{c}\, R^\lambda \\
&\quad + c\,[H(R)]^{\frac{p_2-1}{p_2}}\left(\int_{B_R} |Du(x) - Dv_0(x)|^{p_2}\, dx\right)^{1/p_2} + c\, H(R)\, F(R)^{\frac{p_2\,\beta}{1-p_2}} \\
&\quad + c\,[F(R)]^{p_2\beta}\int_{B_R} (1 + |Du(x)|^{p_2}) dx \\
&\leq \quad c\left[\left(\frac{\rho}{R}\right)^n + \varepsilon\right]\int_{B_R} (1 + |Du(x)|^{p_2})\, dx + \bar{c}\, R^\lambda + c\, H(R) \\
&\quad + c\, H(R)\,[F(R)]^{\frac{p_2\,\beta}{1-p_2}} + c\,[F(R)]^{p_2\,\beta}\int_{B_R} (1 + |Du(x)|^{p_2})\, dx,
\end{aligned}$$

for any $0 < \rho < R$, where $c \equiv c(n, \gamma_1, \gamma_2, L, M, \lambda, \gamma)$ and $\bar{c} \equiv \bar{c}(n, \gamma_1, \gamma_2, L, M, \varepsilon, \lambda, \gamma)$. Now we choose $\beta \equiv \beta(\gamma_1, \gamma_2) > 0$ such that

$$\beta < \frac{p_2 - 1}{p_2^2} < \frac{\gamma_2 - 1}{\gamma_1^2}.$$

With this choice of β deduce

$$H(R)[F(R)]^{\frac{p_2\beta}{1-p_2}} \leq c\, F(R)^{p_2\beta}\left[\int_{B_{4R}} (1 + |Du(x)|^{p_2}\, dx + R^\lambda\right].$$

Therefore, combining the previous facts, we easily get

$$\begin{aligned}
\int_{B_\rho} |Dv_0(x)|^{p_2} dx \quad &\leq \quad c\left[\left(\frac{\rho}{R}\right)^n + \varepsilon\right]\int_{B_R} (1 + |Du(x)|^{p_2})\, dx \\
&\quad + c\,[F(R)]^{p_2\beta}\int_{B_{4R}} (1 + |Du(x)|^{p_2}) dx + \bar{c}\, R^\lambda.
\end{aligned} \tag{5.7}$$

Using once more (i), we end up with

$$\begin{aligned}
\int_{B_\rho} |Du(x)|^{p_2} dx \quad &\leq \quad c\int_{B_\rho} |Dv_0(x)|^{p_2} dx + c\int_{B_\rho} |Du(x) - Dv_0(x)|^{p_2} dx \\
&\leq \quad c\left[\left(\frac{\rho}{R}\right)^n + [F(R)]^{p_2\beta} + \varepsilon\right]\int_{B_{4R}} (1 + |Du(x)|^{p_2})\, dx + \bar{c}\, R^\lambda,
\end{aligned} \tag{5.8}$$

for any $0 < \rho < R$, where $c \equiv c(n, \gamma_1, \gamma_2, L, M, \lambda, \gamma)$ and $\bar{c} \equiv \bar{c}(n, \gamma_1, \gamma_2, L, M, \varepsilon, \lambda, \gamma)$.

Let B_{R_0} be a ball such that $B_{R_0} \subset \mathcal{O}$. Then estimate (5.8) holds for any radii $0 < \rho \le R \le R_0$. Let $\varepsilon_0 \equiv \varepsilon_0(n, M, L, \gamma_1, \gamma_2, \lambda, \gamma)$ be the quantity provided by Lemma 3.4. We choose $\varepsilon = \varepsilon_0/2$ and this fixes the dependencies in (5.8) of the constant $\bar{c} \equiv \bar{c}(n, \gamma_1, \gamma_2, L, M, \lambda, \gamma)$. Then by our assumption (2.8) we can find a radius $R_1 > 0$ so small that $[F(R)]^{p_2\,\beta} < \varepsilon_0/2$ and therefore $[F(R)]^{p_2\,\beta} + \varepsilon < \varepsilon_0$ for any $0 < R \le R_1$ and thus we have $R_1 \equiv R_1(n, \gamma_1, \gamma_2, L, M, \omega_1, \omega_2, \lambda, \gamma)$. Now Lemma 3.4 yields

$$\int_{B_\rho} |Du(x)|^{p_2}\, dx \le c\, \rho^\lambda,$$

with $c \equiv c(n, M, L, \gamma_1, \gamma_2, \lambda, \gamma)$, whenever $0 < \rho < R_1$. Since we have $\gamma_1 \le p_2 \le \gamma_2$, we deduce by standard covering argument that

$$Du \in L^{\gamma_1, \lambda}_{\mathrm{loc}}(\Omega),$$

and thus by Poincaré's inequality we conclude that $u \in \mathcal{C}^{0,\alpha}_{\mathrm{loc}}(\Omega)$ with $\alpha = 1 - \frac{n-\lambda}{\gamma_1}$. This finishes the proof. □

6. Proof of Theorem 2.10

Le $\mathcal{O}$ be defined as in (4.2), B_R be a ball in $\mathcal{O}$ and let u be a local minimizer of the functional (1.1) in K. As in the proof of Theorem 2.8, we define

$$\mathcal{G}_0(v, B_R) := \int_{B_R} f(x_0, (u)_R, Dv(x))\, dx =: \int_{B_R} \bar{g}(Dv(x))\, dx, \tag{6.1}$$

and let $v \in u + W^{1,p_2}_0(B_R)$ be the unique solution of the problem

$$\min \left\{ \mathcal{G}_0(w, B_R) :\ w \in K \cap u + W^{1,p_2}_0(B_R) \right\}. \tag{6.2}$$

We set $A(\eta) := D\bar{g}(\eta)$. As $f \in \mathcal{C}^2$, then $\bar{g} \in \mathcal{C}^2$ and it satisfies the conditions (H6), (H7) and (3.9) with exponent $p = p_2$ while the linear and continuous operator A fulfills (3.7) and (3.8) with exponent $p = p_2$. We also introduce $w \in v + W^{1,p_2}_0(B_R)$ to be the solution of the following equation:

$$\int_{B_R} \langle A(Dw(x)), D\varphi(x)\rangle\, dx = \int_{B_R} \langle A(D\psi(x)), D\varphi(x)\rangle\, dx \qquad \forall\, \varphi \in W^{1,p_2}_0(B_R). \tag{6.3}$$

Then, by the maximum principle, we get that $w \ge \psi$ in B_R, since $v \ge \psi$ on ∂B_R. We also have

$$\int_{B_R} \langle A(Dv(x)), Dv(x) - Dw(x)\rangle\, dx \le 0, \tag{6.4}$$

since $v - w \in W^{1,p_2}_0(B_R)$ and $w \ge v$ in B_R.

At this point let z be the solution of the following minimum problem

$$\min \left\{ \mathcal{G}_0(z, B_R) : z \in u + W^{1,p_2}_0(B_R) \right\}, \tag{6.5}$$

where $\mathcal{G}_0$ has been introduced in (6.1). It is clear that z satisfies

$$\int_{B_R} \langle A(Dz(x)), D\varphi(x)\rangle\, dx = 0 \qquad \forall\, \varphi \in W^{1,p_2}_0(B_R);$$

moreover $z = w$ on ∂B_R, so for example

$$\int_{B_R} \langle A(Dz(x)), Dw(x) - Dz(x)\rangle\, dx = 0. \tag{6.6}$$

We prove Theorem 2.10 by comparison to the minimizer z of the frozen problem in the whole class $u + W^{1,p_2}_0(B_R)$ which can be shown to fulfill a nice estimate. Additionally we need a suitable comparison estimate between z and the original minimizer u which is established via some comparison steps between. First we start with a reference estimate for z, then we compare z and w, after that w and v. Finally we compare v and u. Note that all comparisons between the functions v, w and z can be cited from [12], since these functions are solutions or minimizers, respectively, of suitable frozen problems with constant exponent p_2. Therefore we shorten these steps, only citing the results and the structure conditions needed, referring the reader to [12] for a more detailed discussion.

Using the estimates (2.4) and (2.5) in [23] we deduce the reference estimate

$$\fint_{B_\rho} |Dz(x) - (Dz)_\rho|^{p_2}\, dx \le c\, \left(\frac{\rho}{R}\right)^{\beta\, p_2} \fint_{B_R} (1 + |Dz(x)|^{p_2})\, dx, \tag{6.7}$$

where $c > 0,\ 0 < \beta < 1$ and both c and β depend only on γ_1, γ_2, L.

Using the fact that by Theorem 2.6 we have

$$D\psi \in \mathcal{L}^{\gamma_1,\lambda}(\Omega) \Rightarrow D\psi \in \mathcal{C}^{0,\alpha}(\bar{\Omega})$$

where $\alpha = \frac{\lambda - n}{\gamma_1}$, we obtain for any $p_2 > 1$

$$|A(D\psi(x)) - A(D\psi(y))| \le c\, |x - y|^{\alpha(p_2 - 1)}. \tag{6.8}$$

This allows us to deduce

$$\int_{B_R} |Dw(x) - Dz(x)|^{p_2}\, dx \le c(L)\, R^{\frac{\alpha(p_2-1)}{2}} \int_{B_R} (1 + |Dw(x)|^{p_2})\, dx. \tag{6.9}$$

On the other hand by the minimality of z we immediately deduce

$$\int_{B_R} |Dz(x)|^{p_2}\, dx \le\ c(L) \int_{B_R} (1 + |Dw(x)|^{p_2})\, dx. \tag{6.10}$$

Moreover, using (3.7), (3.8) and (6.3), we deduce the following estimate

$$\int_{B_R} |Dw(x)|^{p_2}\, dx \le\ c \int_{B_R} (|Dv(x)|^{p_2} + 1)\, dx, \tag{6.11}$$

with $c \equiv c(L, \gamma_1, \gamma_2, \alpha)$.

This, together with (6.10) and the minimality of v in K, yields

$$\int_{B_R} |Dz(x)|^{p_2}\, dx \le\ c \int_{B_R} (1 + |Du(x)|^{p_2})\, dx, \tag{6.12}$$

with $c \equiv c(L, \gamma_1, \gamma_2, \alpha)$.

The comparison between v and w can be established in an analogue way, obtaining

$$\int_{B_R} |Dv(x) - Dw(x)|^{p_2}\, dx \le\ c\, R^{\frac{\alpha(p_2-1)}{2}} \int_{B_R} (|Dv(x)|^{p_2} + 1)\, dx. \tag{6.13}$$

Now we compare u and v. First of all we show that under the new assumption (2.9) on the obstacle function we get

$$\int_{B_R} |D\psi(x)|^{p_2}\, dx \le\ c\, R^n. \tag{6.14}$$

In fact, using (6.8), which holds for all $p_2 \ge 1$, and (3.7), we have

$$\begin{aligned}
\int_{B_R} |D\psi(x)|^{p_2}\, dx \ &\le\ \frac{1}{\nu_1} \int_{B_R} \langle\, A(D\psi(x)), D\psi(x)\rangle\, dx +\ c\, R^n \\
&\le\ \frac{1}{\nu_1} \int_{B_R} \langle\, A(D\psi(x)) - (A(D\psi))_R, D\psi(x)\rangle\, dx +\ c\, R^n \\
&\le\ \frac{R^{\alpha(p_2-1)}}{\nu_1} \int_{B_R} (|D\psi(x)|^{p_2} + 1)\, dx +\ c\, R^n \\
&\le\ \frac{1}{2} \int_{B_R} (|D\psi(x)|^{p_2} + 1)\, dx +\ c\, R^n,
\end{aligned}$$

where we also used the fact that $\frac{R^{\alpha\,(p_2-1)}}{\nu_1} \leq \frac{1}{2}$, assumption which is not restrictive as we are proving local regularity results.

Going through the proof of Lemma 3.1 and using estimate (6.14) one can easily see that we have higher integrability for u in the following sense

$$\left(\fint_{B_{R/2}} |Du(x)|^{p(x)(1+\delta)} dx\right)^{1/(1+\delta)} \leq c_0 \fint_{B_R} (|Du(x)|^{p(x)} + 1)\, dx. \tag{6.15}$$

A similar argument together with Proposition 3.3 allows us to conclude

$$\left(\fint_{B_{R/2}} |Dv(x)|^{\bar r}\, dx\right)^{1/\bar r} \leq c \left(\fint_{B_{R/2}} |Dv(x)|^{p}\, dx\right)^{1/p} + c \left[\fint_{B_R} (1 + |Du(x)|^{\bar q})\, dx\right]^{1/\bar q}. \tag{6.16}$$

At this point it is clear that, working as in the proof of Theorem 2.9 but using this time (6.15) and (6.16) instead of (3.1) and (3.10) respectively, we obtain

$$\mathcal{G}_0(u) - \mathcal{G}_0(v) \leq c\,(\omega_1(R)\log\left(\tfrac{1}{R}\right) + \omega_2^{\sigma}(R^{\tilde m}))\left[\int_{B_{4R}} (|Du(x)|^{p_2} + 1)\, dx + R^n\right]$$

which in turn entails, using assumption (2.10) and recalling the definition of $\tilde m$ and σ given in Section 5

$$\mathcal{G}_0(u) - \mathcal{G}_0(v) \leq c\, R^{\zeta} \left[\int_{B_{4R}} (|Du(x)|^{p_2} + 1)\, dx\right]$$

where $\zeta \equiv \zeta(n, \gamma_1, \gamma_2, \delta, \gamma, \lambda, \varsigma)$. Arguing in a standard way, as we did in (4.10), and distinguishing the cases $p_2 \geq 2$ and $1 < p_2 < 2$, we end up with

$$\int_{B_R} |Du(x) - Dv(x)|^{p_2}\, dx \leq c\, R^{\zeta/2} \left[\int_{B_{4R}} (|Du(x)|^{p_2} + 1)\, dx\right]. \tag{6.17}$$

Thus summing up, taking together the estimates (6.9), (6.11), (6.13) and (6.17), additionally setting

$$\mathcal{M} := \min\left\{\frac{\alpha(p_2-1)}{2}, \frac{\zeta}{2}\right\}$$

we conclude, using the minimality of v in K

$$\begin{aligned}
&\int_{B_R} |Dz(x) - Du(x)|^{p_2}\, dx \\
&\quad\leq \int_{B_R} |Dz(x) - Dw(x)|^{p_2}\, dx + \int_{B_R} |Dw(x) - Dv(x)|^{p_2}\, dx + \int_{B_R} |Dv(x) - Du(x)|^{p_2}\, dx \\
&\quad\leq c\, R^{\mathcal{M}} \int_{B_{4R}} (1 + |Du(x)|^{p_2})\, dx. \qquad (6.18)
\end{aligned}$$

Combining this comparison estimate with the reference estimate (6.7) and using (6.12) we deduce for any $0 < \rho < R/2$

$$\begin{aligned}
&\int_{B_\rho} |Du(x) - (Du)_\rho|^{p_2}\, dx \\
&\quad\leq \int_{B_\rho} |Dz(x) - (Dz)_\rho|^{p_2}\, dx + \int_{B_\rho} |Dz(x) - Du(x)|^{p_2}\, dx \\
&\quad\leq c\left(\frac{\rho}{R}\right)^{\beta p_2 + n} \int_{B_R} (1 + |Dz(x)|^{p_2})\, dx + cR^{\mathcal{M}} \int_{B_{4R}} (1 + |Du(x)|^{p_2})\, dx \\
&\quad\leq c\left[\left(\frac{\rho}{R}\right)^{\beta\, p_2 + n} + R^{\mathcal{M}}\right] \int_{B_{4R}} (|Du(x)|^{p_2} + 1)\, dx.
\end{aligned}$$

On the other hand, using Theorem 2.2 of [13], (6.18) and (6.12), we get

$$\begin{aligned}\int_{B_\rho} |Du(x)|^{p_2}\,dx &\leq \int_{B_\rho} |Dz(x)|^{p_2}\,dx + \int_{B_R} |Dz(x)-Du(x)|^{p_2}\,dx \\ &\leq c\left(\frac{\rho}{R}\right)^n \int_{B_R} (1+|Dz(x)|^{p_2})\,dx + c\,R^{\mathcal{M}} \int_{B_{4R}} (|Du(x)|^{p_2}+1)\,dx \\ &\leq c\left[\left(\frac{\rho}{R}\right)^n + R^{\mathcal{M}}\right] \int_{B_{4R}} |Du(x)|^{p_2}\,dx + c\,R^n.\end{aligned}$$

Now, by a standard iteration lemma, we are able to deduce the existence of a radius R_0 such that for all $R \leq R_0$

$$\int_{B_R} |Du(x)|^{p_2}\,dx \leq \; c\,R^{n-\tau}$$

for all $0 < \tau < 1$. For our purposes, we can choose any $\tau < \dfrac{p_2\beta\mathcal{M}}{n+p_2\beta}$, for example $\tau := \dfrac{1}{2}\,\dfrac{p_2\mathcal{M}\beta}{n+p_2\beta}$. At this point we choose ρ such that $\rho = \dfrac{1}{2}R^{1+\theta}$ where $\theta := \dfrac{\mathcal{M}}{n+\beta\,p_2}$. With such a choice of ρ, θ and τ, we have that

$$\int_{B_\rho} |Du(x)-(Du)_\rho|^{p_2}\,dx \leq \; c(L,\gamma_1,\gamma_2,\alpha)\,\rho^{\tilde{\lambda}} \tag{6.19}$$

where

$$\tilde{\lambda} := n + \frac{p_2\,\beta\,\mathcal{M}}{2(n+p_2\,\beta+\mathcal{M})}.$$

But the choice of R was arbitrary, so without loss of generality we may assume that (6.19) holds for all $0 < \rho \leq R_0$. Now we would like to conclude by Theorem 2.6; thus we have to make sure that $\tilde{\lambda} < n+\gamma_1$. If $\beta < (2\gamma_1)/\gamma_2$, this is true, otherwise we can choose $\mathcal{M}$ sufficiently small (this is not restrictive). Thus Theorem 2.6 gives that $Du \in \mathcal{C}^{0,\tilde{\alpha}}_{\mathrm{loc}}(\Omega)$ with $\tilde{\alpha} = 1 - \dfrac{n-\tilde{\lambda}}{\gamma_1}$. This yields the thesis.

□

References

[1] E. Acerbi, G. Mingione: Regularity results for a class of quasiconvex functionals with nonstandard growth, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **30** (2001), 311-339.

[2] E. Acerbi, G. Mingione: Regularity results for a class of functionals with nonstandard growth, *Arch. Ration. Mech. Anal.,* **156**, 121 (2001), 121-140.

[3] E. Acerbi, G. Mingione: Gradient estimates for the $p(x)$-laplacean system, *J. Reine Angew. Math.,* **584** (2005), 117-148.

[4] H. J. Choe A regularity theory for a general class of quasilinear elliptic partial differential equations and obstacle problems, *Arch. Ration. Mech. Anal.,* **114** (1991), 383-394.

[5] G. Cupini, N. Fusco, R. Petti: Hölder continuity of local minimizers, *J. Math. Anal. Appl.,* **235** (1999), 578-597.

[6] A. Coscia, G. Mingione: *Hölder continuity of the gradient pf $p(x)$- harmonic mappings*, C. R. Acad. Sci. Paris Ser. I, **328** (1999), 363-368.

[7] L. Diening: *Riesz potentials and Sobolev embeddings on generalized Lebesgue and Sobolev spaces $L^{p(\cdot)}$ and $W^{k,p(\cdot)}$*, Math. Nachr., **268** (2004), 31-43.

[8] D. Edmunds, J. Rákosník: *Sobolev embeddings with variable exponent*, Stud. Math., **143**, No. 3 (2000), 267-293.

[9] D. Edmunds, J. Rákosník: *Sobolev embeddings with variable exponent II*, Math. Nachr., **246-27** (2002), 53-67.

[10] I. Ekeland: Nonconvex minimization problems, *Bull. Am. Math. Soc.,* **(3) 1** (1979), 443-474.

[11] M. Eleuteri: Hölder continuity results for a class of functionals with non standard growth, *Boll. Unione Mat. Ital.,* **8**, 7 - B (2004), 129-157.

[12] M. Eleuteri: Regularity results for a class of obstacle problems, *Appl. Math.,* **52**, No. 2 (2007), 137-169.

[13] I. Fonseca, N. Fusco: Regularity results for anisotropic image segmentation models, *Ann. Scuola Norm. Sup. Pisa. Cl. Sci.,* **24** (1997), 463-499.

[14] N. Fusco, J. Hutchinson: $C^{1,\alpha}$ *regularity of functions minimising quasiconvex integrals*, Manuscripta Math., **54**, No. 1-2 (1985), 121-143.
[15] M. Giaquinta: *Multiples integrals in the calculus of variations and nonlinear elliptic systems*, Princeton University Press, Princeton, New Jersey, (1983).
[16] E. Giusti: *Direct methods in the calculus of variations*, Singapore: World Scientific, vii, (2003).
[17] P. Hästö: *Counter examples of regularity in variable exponent Sobolev spaces*, Contemp. Math., **370** (2005), 133-143.
[18] P. Harjulehto, P. Hästö, O. Martio: *Fuglede's theorem in variable exponent Sobolev spaces*, Collet. Math., **55**, No. 3 (2004), 315-324.
[19] J. Habermann: *Partial regularity for minima of higher order functionals with $p(x)$ growth*, to appear in Manuscripta Math.
[20] J. Habermann: *Calderón-Zygmund estimates for higher order systems with $p(x)$ growth*, Math. Zeitschrift, online DOI 10.1007/s00209-007-0180-x (2007)
[21] J. Habermann, A. Zatorska-Goldstein: *Regularity for minimizers of functionals with nonstandard growth by A-harmonic approximation*, to appear in NoDEA.
[22] T. Iwaniec, A. Verde: *On the operator $\mathcal{L}(f) = f \log|f|$*, J. Funct. Anal., **169** (1999), 391-420.
[23] J.J. Manfredi: Regularity for minima of functionals with p-growth, *J. Differential Equations,* **76** (1988).
[24] M. Růžička: *Electrorheological Fluids: Modeling and mathematical theory*, Springer, Heidelberg (2000).
[25] V. V. Zhikov: On some variational problems, *Russian J. Math. Phys.,* **5** (1997), 105-116.

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