THE ADJACENCY GRAPH OF A REAL ALGEBRAIC SURFACE

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Abstract. The paper deals with the question of recognizing the mutual positions of the connected components of a non-singular real projective surface S in the real projective 3-space. We present an algorithm that answers this question through the computation of the adjacency graph of the surface; it also allows to decide whether each connected component is contractible or not. The algorithm, combined with a previous one returning as an output the topology of the surface, computes a set of data invariant up to ambient-homeomorphism which, though not sufficient to determine the pair $(\mathbb{RP}^3, S)$, give information about the nature of the surface as an embedded object.

Real algebraic surfaces – adjacency graph – algorithms.

1. Introduction

Recognizing a surface up to homeomorphism means to determine a topological model for each of its connected components. In [4] and [3] this question is addressed from a constructive point of view for a non-singular real algebraic surface in the real projective space $\mathbb{RP}^3$; the authors give an algorithm to count the number of connected components of the surface and to compute the Euler characteristic of each of them, which determines them topologically.

In those papers the surface is considered as an abstract topological space, without taking into any account how it is situated in $\mathbb{RP}^3$, but a surface can be embedded in the real projective space in different ways, with possible self-knotting of a connected component and linking of distinct components. Thus it may happen that for two homeomorphic surfaces S, S' the pairs $(\mathbb{RP}^3, S)$ and $(\mathbb{RP}^3, S')$ are not homeomorphic, i.e. there exists no homeomorphism $\varphi : \mathbb{RP}^3 \to \mathbb{RP}^3$ such that $\varphi(S) = S'$. As a simple example one can take as S the torus of equation $(x_1^2 + x_2^2 + x_3^2 + 3x_0^2)^2 - 16(x_1^2 + x_3^2)x_0^2 = 0$, and as S' the one-sheeted hyperboloid $x_0^2 - x_1^2 - x_2^2 + x_3^2 = 0$. When the pairs $(\mathbb{RP}^3, S)$ and $(\mathbb{RP}^3, S')$ are homeomorphic we will say that the surfaces S and S' are ambient-homeomorphic.

One classical question concerning algebraic surfaces seen as embedded objects in $\mathbb{RP}^3$ is the celebrated Hilbert's *Ambient Topological Classification Problem*([5]): "Up to homeomorphism, what are the possible pairs $(\mathbb{RP}^3, S)$, where S is a non-singular real algebraic surface of degree d in $\mathbb{RP}^3$?"

This problem has been solved only for $d \leq 4$, and for $d \leq 3$ its solution coincides with the answer to the Topological Classification Problem, which requires to determine, up to homeomorphism, the possible distinct models for a non-singular real algebraic surface of degree d. Namely, any projective surface of degree 1 in $\mathbb{RP}^3$ is a projective plane and two planes can be transformed each into the other by means

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of a projective isomorphism of $\mathbb{RP}^3$. By the well known theorem of projective classification of quadrics, any non-singular non-empty real projective surface of degree 2 can be transformed by means of a projective isomorphism of $\mathbb{RP}^3$ either into the sphere $x_0^2-x_1^2-x_2^2-x_3^2=0$ or into the one-sheeted hyperboloid $x_0^2-x_1^2-x_2^2+x_3^2=0$, which is homeomorphic to a torus. Also for non-singular cubics the two types of classification are equivalent and classical results prove the existence of exactly five distinct models. The degree 4 is the lowest degree in which there exist pairs of homeomorphic surfaces S, S' such that $(\mathbb{RP}^3, S)$ and $(\mathbb{RP}^3, S')$ are not homeomorphic. The classification of degree 4 surfaces up to ambient-homeomorphism, begun by Hilbert, Rohn and Utkin and completed by Kharlamov, shows the existence of 66 topologically distinct models and 113 non-homeomorphic pairs $(\mathbb{RP}^3, S)$.

For a non-singular real algebraic curve C in the real projective plane the pair $(\mathbb{RP}^2, C)$ is completely determined by the knowledge of the mutual position of the ovals of the curve, which can be given either by means of the list of the nests of C or through its adjacency graph, whose vertices are the connected components of $\mathbb{RP}^2 \setminus C$ and in which two distinct vertices are joined by an edge if and only if they share a common boundary. In the case of surfaces the situation is more complicated, since the knowledge of the mutual position of its connected components is an invariant up to ambient-homeomorphism, but not sufficient to determine the pair $(\mathbb{RP}^3, S)$; an example is shown in Figure 1.

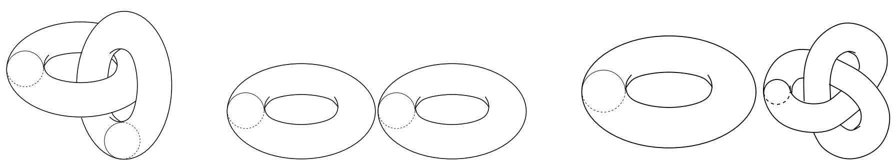


FIGURE 1. Homeomorphic, but pairwise not ambient-homeomorphic surfaces

In this paper we address the question of recognising the disposition in $\mathbb{RP}^3$ of the connected components of a non-singular real projective surface from a constructive point of view, presenting an algorithm that computes the adjacency graph of the surface, decides for each connected component whether it is contractible or not and reconstructs the inclusion partial order among the contractible components; all these notions will be defined in Section 2. We show that this result can be achieved by a more efficient handling of the techniques and computations used in [3] to determine the topological type of the surface, enriching the data computed at each iterative step with data relative to adjacencies. For this reason, even if we have made an effort to take the paper as much self-contained as possible, the knowledge of the article [3] (and also of [4], where the topology was computed for a non-singular surface disjoint from a line) will be helpful for the full comprehension of the procedures we are going to use. In any case we refer the reader to those papers both for a wider description of the theoretical background needed for the topological determination of a surface and for the algorithmic details of some constructive procedures we will use also in this paper.

The adjacency graph with each vertex and each edge marked either "contractible" or "non-contractible" and endowed with a set of roots will be called the *weighted adjacency graph* of the surface. In Section 3 we will present an algorithm that, through an iterative computation of the graphs of finitely many level curves and level surfaces, eventually computes the weighted adjacency graph of a compact non-singular affine surface in $\mathbb{R}^3$. In the case of an arbitrary surface, the basic idea is proving that the weighted adjacency graph of S can be recovered from the adjacency graph of a suitably constructed compact algebraic surface $\widehat{S}$ in $\mathbb{R}^3$; this will be the object of Section 4. In the final section we describe the global structure of the algorithm and we show on some examples the way it works.

2. The adjacency graph of curves and surfaces

Assume that S is the real projective algebraic surface in $\mathbb{RP}^3$ defined by the equation $F(x, y, z, t) = 0$, where F is a homogeneous polynomial of degree d with real coefficients; a point in $\mathbb{RP}^3$ is called a singular point of the surface S if it annihilates F and all its first partial derivatives. We will say that the surface is non-singular if it contains no real singular point, while the complex zero-set defined by F may contain non-real singular points.

Before defining the adjacency graph of curves and surfaces, we want to recall some basic classical facts about the topology of non-singular algebraic curves and surfaces, for a proof of which we refer for instance to [9], [8] and [7].

Each connected component of a non-singular real algebraic curve C in the real projective plane is homeomorphic to a circle and can be embedded in $\mathbb{RP}^2$ in two topologically distinct ways. In the first case the component does not disconnect $\mathbb{RP}^2$ and it is called a *one-sided* component. In the second case the connected component, called an *oval*, disconnects $\mathbb{RP}^2$ into two connected components: one of them is homeomorphic to a disc and is called the *interior part* of the oval, the other is homeomorphic to a Möbius band and is called the *exterior part* of the oval. A non-singular real algebraic curve contains a one-sided component, and in fact exactly one, if and only if the degree of the curve is odd.

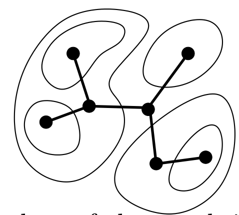
The pair $(\mathbb{RP}^2, C)$ is determined up to homeomorphism by the parity of the degree of C and by the mutual position of its ovals. Recall that two disjoint ovals either can be mutually external (i.e. each lies in the exterior part of the other) or one of them can encircle the other one. An oval that contains no other oval in its interior part is called empty. A list $[\omega_1, \dots, \omega_m]$ of ovals of a curve is called a *nest* of depth m if ω_1 is empty, ω_i is contained in the interior part of ω_{i+1} for all $i = 1, \dots m-1$ (and any other oval containing ω_i contains also ω_{i+1}) and ω_m is not contained in the interior part of any oval of the curve.

The relation "ω_i is contained in the interior part of ω_j" defines a partial order in the set of the ovals of a given curve. The set of the ovals equipped with this partial order and the marked one-sided component, if present, is called the *scheme* of the curve (alternatively the information about the presence of a one-sided component can be replaced by the parity of the degree of the curve).

An equivalent way to collect all the information concerning the mutual positions of the ovals of the curve C is through its *adjacency graph*, that we now define in a more general setting for later use:

Definition 2.1. *Let X be a topological space and Y a subspace. The adjacency graph $G(X, Y)$ is the graph whose vertices are the connected components of $X \setminus Y$ and where two distinct vertices Ω_1, Ω_2 are joined by an edge if and only if the topological closures of Ω_1 and Ω_2 are not disjoint.*

Note that the graph $G(X,Y)$ is a topological invariant of the pair (X,Y), i.e. if (X_1,Y_1) and (X_2,Y_2) are homeomorphic pairs then $G(X_1,Y_1)$ and $G(X_2,Y_2)$ are isomorphic graphs. For this reason we will think of an adjacency graph as an abstract object obtained associating a vertex to each connected component of $X \setminus Y$ and joining pairs of distinct vertices following the rule described above.



In our case, when $X = \mathbb{RP}^2$ and $Y = C$, for simplicity we will denote $G(\mathbb{RP}^2, C)$ by $G(C)$. Since the closures of two distinct regions of $\mathbb{RP}^2 \setminus C$ are either disjoint or intersect only in one oval of C, each edge of $G(C)$ corresponds exactly to one oval. Moreover any oval of C is the common boundary of two distinct regions. Thus $G(C)$ has as many edges as the number of ovals in C and the only connected component of the curve not represented among the edges of the graph is that one-sided, in the case of an odd-degree curve.

Observe that $G(C)$ is a tree. Namely, by construction it is a connected graph and, since each oval disconnects $\mathbb{RP}^2$, removing any edge from $G(C)$ disconnects the graph.

We will fix as a *root* of the tree $G(C)$ the vertex r representing the unique region external to all the ovals of C. As a consequence, the tree inherits an orientation and each oriented edge can be indicated by means of the ordered pair of its vertices. Fixing this root represents an equivalent way to express the relations of inclusion induced by the partial order on the ovals of a curve. For instance, if $l = (v_1, v_2)$ is an oriented edge of $G(C)$ corresponding to an oval ω, then the region $R(v_1)$ corresponding to the vertex v_1 is contained in the exterior part of ω, while the region $R(v_2)$ is contained in the interior one. Moreover any leaf (i.e. vertex different from the root and belonging to one single edge) corresponds to the interior part of an empty oval, and any path connecting the root to a leaf corresponds to a nest.

Because of this full parallelism between describing an embedded curve through its nests or through its associated graph, a pair $(\mathbb{RP}^2, C)$ is equivalently determined up to homeomorphism by the rooted adjacency graph $G(C)$ and the parity of the degree of C.

Turning to consider surfaces, the topological determination of a surface requires to count the number of its connected components and to characterize topologically each of them. As explained in [3], this can be done computing the Euler characteristic χ of each component. Namely, any compact connected orientable surface is homeomorphic to a torus with g holes, with $g \geq 0$ (meaning that a torus with 0 holes is a sphere), and $\chi = 2 - 2g$; instead, a non-orientable connected surface in $\mathbb{RP}^3$ is homeomorphic to the connected sum of a projective plane and a torus with g holes, so that $\chi = 1 - 2g$. We will denote by $\chi(S)$ the list of the Euler characteristics of all the components of S.

Recall that, if S is a non-singular real algebraic surface in $\mathbb{RP}^3$ of even degree, the surface and all its connected components are orientable; if the degree is odd, then S contains a non-orientable connected component, while all the other components are orientable. In particular for odd-degree surfaces the unique non-orientable component is the only one having an odd Euler characteristic.

If we now consider a surface as an embedded object, we see that a connected surface can be situated in $\mathbb{RP}^3$ in two ways: either it disconnects $\mathbb{RP}^3$ in two connected components, being the boundary for both regions, and it is then called *two-sided*, or it does not disconnect $\mathbb{RP}^3$ and it is called *one-sided*. The nature of the connected components of a non-singular real surface S with respect to the previous property is again determined by the degree of the surface: if it is even, all the components are two-sided; if it is odd, S contains exactly one component which is one-sided (the non-orientable one) and all the others are two-sided.

While for curves the property of being one-sided or two-sided fully characterizes how an oval is embedded in the projective plane, for surfaces the situation is more complicated. For instance we see that a one-sided component contains a loop (i.e. a closed path) which is not contractible (i.e. not homotopically trivial) in $\mathbb{RP}^3$. In general a subset $A \subset \mathbb{RP}^3$ is called *contractible* if any loop in A is contractible as a loop in $\mathbb{RP}^3$; otherwise it is called *non-contractible*. Equivalently A is contractible if and only if the homomorphism $\pi_1(A) \to \pi_1(\mathbb{RP}^3)$ induced by the inclusion of A in $\mathbb{RP}^3$ is trivial. As observed above, any one-sided component is non-contractible; instead a two-sided component can be either contractible or non-contractible, since for instance an affine torus is contractible while a one-sheeted hyperboloid is not.

Classical topological considerations lead to useful information, available in the literature, about the nature of components with respect to this property. One of them ensures that, if S is odd-degree, all its two-sided components are contractible. Additional results are known only for low degree surfaces; for instance, if S has degree 4, it has at most two non-contractible components: if it has two non-contractible components, then there is no other component and each of them is homeomorphic to a torus; if the number of non-contractible components is 1, then all the contractible ones are mutually external.

Looking at the complement of surface components, if S is connected and one-sided, then $\mathbb{RP}^3 \setminus S$ is connected and contractible. For two-sided components the contractibility of the complementary regions depends on the contractibility of the component. Namely, if S is connected two-sided and non-contractible, then the two connected regions of $\mathbb{RP}^3 \setminus S$ are both non-contractible (the existing non-trivial loop on S can be pushed to each region of the complement); the simplest example of this situation is a one-sheeted hyperboloid, which is homeomorphic to a torus and disconnects $\mathbb{RP}^3$ into two solid tori. If S is connected two-sided and contractible, then only one of the two regions of $\mathbb{RP}^3 \setminus S$ is contractible and is called the *interior part* of S, while the other is called the *exterior part*.

Since the interior is defined only for the contractible connected components, it is not possible to define a partial order relation in the set of the two-sided components of a surface, just repeating what we did for curves; such a partial order can be defined only in the set of the contractible components. If we choose to represent a surface through its adjacency graph and want to able to recover from it the previous partial order, it is necessary to endow the graph with weights expressing the mentioned topological features of the various components.

Again we will denote the graph $G(\mathbb{RP}^3, S)$ simply by $G(S)$. As in the case of curves, $G(S)$ has as many edges as the number of two-sided components of S and, if S is odd-degree, the only component of S not represented in the graph is the non-orientable one. Moreover $G(S)$ is a tree.

We associate to any edge e of $G(S)$ a weight, fixing $w_S(e) = c$ if the connected component of S represented by e is contractible, $w_S(e) = nc$ if it is non-contractible. By the same rule we associate to each vertex v of $G(S)$ a weight $w_S(v) = c$ or $w_S(v) = nc$.

As a consequence of the previous considerations, the two vertices of an edge are non-contractible if and only if the edge is non-contractible; hence the non-contractible vertices and edges of $G(S)$ form a subgraph that we will denote $G_{nc}(S)$. We will denote by $\overline{G_c(S)}$ the subgraph formed by the contractible edges of $G(S)$ and their relative vertices. If S has an even degree, there exists at least a vertex v non-contractible; if the degree is odd, all vertices and all edges in $G(S)$ are contractible, i.e. $G_{nc}(S) = \emptyset$. While $\overline{G_c(S)}$ may be non-connected, we have that:

Proposition 2.2. *$G_{nc}(S)$ is connected.*

Proof. Let u, v be two distinct vertices of $G_{nc}(S)$ and consider the unique path $(u = u_0, u_1, \dots, u_{m-1}, u_m = v)$ joining u and v in $G(S)$ with $u_i \neq u_j$ for any $i \neq j$. We claim that u_i is non-contractible for any $i = 1, \dots, m-1$; otherwise, if r is the least integer such that u_r is contractible, then u_{r-1} is non-contractible and the edge $e = \{u_{r-1}, u_r\}$ corresponds to a contractible component of S containing u_r in its interior part. Since the u_i's are distinct, the regions $u_{r+1}, \dots, u_m = v$ are contained in the interior part of e, which contradicts the fact that v is non-contractible. □

We now fix a root in each connected component of $\overline{G_c(S)}$ in such a way that the induced order reflects the natural partial order in the set of the contractible components of the surface.

If S is even-degree, then $G_{nc}(S)$ is non-empty: any connected component of $\overline{G_c(S)}$ is a tree having a unique vertex weighted nc, that we will take as a root of the subtree. In particular, if S is even-degree and does not contain any non-contractible two-sided component, then there is a unique (non-contractible) region in $\mathbb{RP}^3 \setminus S$ external to all the two-sided components of S, the subgraph $G_{nc}(S)$ consists of a single vertex, chosen as a root of $\overline{G_c(S)}$, which is connected. If d is even but S contains non-contractible two-sided components, then $G_{nc}(S)$ is not simply a point, $\overline{G_c(S)}$ (if non-empty) may be non-connected and a root is chosen in each connected component of it. Note that for even-degree surfaces the chosen roots can be equivalently characterized as the vertices common to $\overline{G_c(S)}$ and $G_{nc}(S)$.

The previous way to fix roots in $\overline{G_c(S)}$ cannot be used when the degree of S is odd, because $G_{nc}(S) = \emptyset$. For instance if S consists of a non-orientable component and two spheres, the graph contains two edges and three vertices all marked with c; note that the information about which vertex is the root is the only one that allows to decide whether the two spheres are mutually external or if one encircles the other.

If the degree d is odd and Y_0 is the non-orientable component, then there exists a unique (contractible) region of $\mathbb{RP}^3 \setminus S$ external to all the components of S different from Y_0 (which are contractible): we choose this vertex as a root of $\overline{G_c(S)}$.

The graph $G(S)$ endowed with the function $w_S : G(S) \to \{c, nc\}$ and with the set $r(S)$ of roots chosen as above will be called the *weighted adjacency graph* of S.

The knowledge of the weighted adjacency graph allows us not only to recover the scheme of inclusions of the contractible components of S, but also to recognize whether two distinct chains of contractible components of S ordered by inclusion lie in the same or in different regions of the complement of a non-contractible component of the surface.

We can therefore collect all the information on the surface examined so far associating to S a set of data $(G(S), w_S, r(S), \chi(S))$ which is invariant up to ambient-homeomorphism. Passing from S to its graph, in principle we discard the information concerning the parity of the surface degree, since in any case the non-orientable component is not represented in the graph. As a matter of fact, this piece of information is not lost and can be recovered looking at $\chi(S)$: the surface is odd-degree if and only if $\chi(S)$ contains an odd number. As already pointed out, surfaces that are homeomorphic but not ambient-homeomorphic can share the same data $(G(S), w_S, r(S), \chi(S))$: an example was shown in Figure 1.

Even if not sufficient to determine the pair $(\mathbb{RP}^3, S)$, the ability to compute $(G(S), w_S, r(S), \chi(S))$ starting from an equation of S is very useful as a source of information about the nature of the surface as an embedded object. As announced in the introduction, the algorithm to compute all these data is an "enriched variant" of the one presented in [3] to compute $\chi(S)$ and hence the topological type of S. For this reason in the next sections we will focus on the procedure to compute the triple $(G(S), w_S, r(S))$, refering the reader to the mentioned previous papers for the details of the computation of $\chi(S)$.

3. The compact affine case

In this section we preliminarily consider the case when the surface S does not intersect in real points the plane "at infinity" $\{t = 0\}$, so that it is contained in the affine chart $\{[x, y, z, t] \in \mathbb{RP}^3 \mid t \neq 0\} \simeq \mathbb{R}^3$. We will describe an algorithmic procedure to compute its weighted adjacency graph $(G(S), w_S, r(S))$. The algorithm follows the same iterative steps as the one presented in [3] to compute the topological type of S; at each step a more efficient handling of the computations performed to study S topologically will allow us also to record the mutual dispositions of the connected components of the level surfaces and so eventually compute the weighted adjacency graph of S.

First of all let us see how we can compute the adjacency graph $G(S)$ considering S as a non-singular compact affine surface in $\mathbb{R}^3$ using the affine coordinates (x, y, z).

Up to a generic linear change of coordinates, one can assume (see for instance [1]) that the projection $p : S \to \mathbb{R}$ defined by $p(x, y, z) = z$ is a Morse function (i.e. all its real critical points are non-degenerate) and that the images of distinct critical points are distinct critical values. If $[-N, N]$ is an interval containing all the finitely many critical values of p, we subdivide it as $[-N, N] = [-N = a_0, a_1] \cup [a_1, a_2] \cup \ldots \cup [a_s, a_{s+1} = N]$ so that each a_i is non-critical for p and each interval $[a_i, a_{i+1}]$ contains only one critical value in its interior part. For each $a \in \mathbb{R}$ we will denote by C_a the level curve $p^{-1}(a) = S \cap \{z = a\}$ and by S_a the level surface $p^{-1}([-N, a]) = S \cap \{z \leq a\}$; in particular $S_{-N} = \emptyset$ and $S_N = S$, while S_a is a surface with boundary C_a.

Our goal of computing the graph $G(S)$ will be achieved by computing iteratively, for $a = a_i$, $i = 0, \ldots, s + 1$, the triple $(G(C_a), M_a, G(S_a))$, where

1. $G(C_a)$ denotes the adjacency graph $G(\{z = a\}, C_a)$
2. $G(S_a)$ denotes the adjacency graph $G(\{z \leq a\}, S_a)$
3. $M_a : G(C_a) \to G(S_a)$ is the graph morphism that associates to each vertex v of $G(C_a)$ the vertex of $G(S_a)$ representing the region of $(\mathbb{R}^3 \setminus S) \cap \{z \leq a\}$ having in its boundary the region of $\{z = a\} \setminus C_a$ represented by v. Observe that this association defines a graph morphism since the images of adjacent

vertices are adjacent vertices; in particular M_a transforms each edge e of $G(C_a)$ into the edge of $G(S_a)$ representing the connected component of S_a containing in its boundary the oval of C_a represented by e.

At the initial level both C_{-N} and S_{-N} are empty, hence both $G(C_{-N})$ and $G(S_{-N})$ consist of a single vertex and M_{-N} is defined in the natural trivial way. Recall that in [3] (see also [4]) the number of connected components of S and their topological types were computed iteratively reconstructing, for $a = a_1, \dots, a_{s+1}$, the triple $(Scheme(C_a), \mu_a, \chi(S_a))$ where:

i) $Scheme(C_a)$ is the list of all the nests of the level curve C_a (which is non-singular, compact and even-degree)
ii) μ_a is the function that associates to any oval ω of C_a the connected component of S_a containing ω in its boundary
iii) $\chi(S_a)$ is the list of the Euler characteristics of the connected components of the level surface S_a.

Since for a non-singular curve of even degree giving the list of its nests is perfectly equivalent to giving its adjacency graph, we can modify the output of the subalgorithm that studies the shape of the level curves so that it returns the scheme of C_a in the form of its adjacency graph.

The function μ_a can be seen as a function, defined on the edges of $G(C_a)$ and associating to each of them an edge of $G(S_a)$, that coincides with the action of M_a on the edges of $G(C_a)$. It is clear that the knowledge of the function M_a is essential, because it allows to iteratively recover the adjacencies among the connected components of the level surface; thus it will be computationally relevant to show that the same computations needed to reconstruct μ_a are sufficient also to reconstruct M_a.

Let us recall that, for any interval $[a, b]$, the reconstruction of μ_b from μ_a requires to investigate the surface in the strip $\{a \leq z \leq b\}$ so as to be able to decide whether an oval ω of C_a and an oval η of C_b lie in the boundary of the same connected component of $S \cap \{a \leq z \leq b\}$. If so, $\mu_b(\eta)$ is the component Y of S_b such that $Y \cap \{z \leq a\}$ contains ω in its boundary.

In the case when $[a, b]$ contains no critical value, the situation is quite simple. First of all the curves C_a and C_b have the same schemes, i.e. $G(C_a) = G(C_b)$; moreover the level surfaces S_a and S_b are homeomorphic and share the same adjacency graph $G(S_a) = G(S_b)$. The boundary of each connected component of $S \cap \{a \leq z \leq b\}$ contains exactly one oval in the plane $\{z = a\}$ and one oval in $\{z = b\}$. It is possible to detect the 1-1 correspondence between the ovals of C_a and the ovals of C_b as follows: one chooses a point P in the center of a nest n of C_a (i.e. inside the innermost oval of the nest) and computes the final point Q of a *roadmap* lifting P up to level b (following the terminology adopted in [3] by roadmap we mean a continuous semialgebraic path $\alpha : [0, 1] \to \{a \leq z \leq b\}$ not intersecting the surface and such that $\alpha(0) = P$ and $\alpha(1) = Q \in \{z = b\}$). Then the point Q lies in the center of a nest $\widetilde{n}$ having the same depth as n. Thus one recognizes that the i-th oval of $\widetilde{n}$ belongs to the boundary of the connected component of S_b that contains also the i-th oval of n; one recognizes also the correspondence among the regions comprised between two consecutive ovals of the nest n and the analogous regions determined by the ovals of the nest $\widetilde{n}$. Therefore the computation of finitely many roadmaps from points chosen in the centers of the nests of C_a up to level b allows to compute the 1-1 correspondence between the ovals and regions determined by C_a

and the ovals and regions determined by C_b. In other words, when $[a, b]$ contains no critical value, the computation of finitely many roadmaps allows to reconstruct not only μ_b from μ_a but also M_b from M_a.

The same result holds also when the interval $[a, b]$ contains exactly one critical value c, with $a < c < b$. In this case there is no 1-1 correspondence between the ovals of C_a and the ovals of C_b; according to the index of the critical point we can have different situations: an oval may appear in C_b (when the index is 0) or it may disappear (if the index is 2), while if the index is 1 it may happen either that two ovals of C_a glue into a single oval of C_b or that an oval of C_a splits into two ovals of C_b. In [4] and [3] it is explained how the previous method based on roadmaps can be used to understand what happens to the level curve and to the level surface when passing through a critical value; in this way we can reconstruct both $G(S_b)$ from $G(S_a)$ and M_b from M_a.

Case index 0. If the index is 0, we know that C_b contains one more oval than C_a and by means of roadmaps we can recognize it and also decide in which region of $\{z = b\} \setminus C_b$ it appears. This means that we are able to decide to which vertex of $G(C_a)$ it is necessary to join the new vertex in order to obtain $G(C_b)$. Similarly $G(S_b)$ contains one more vertex and one more edge than $G(S_a)$: the information on regions given by the computed roadmaps makes it possible to detect the vertex of $G(S_a)$ where the new edge has to be attached. In particular M_b coincides with M_a on the common graph $G(C_a)$ and associates to the new vertex (edge) of $G(C_b)$ the new vertex (edge) of $G(S_b)$.

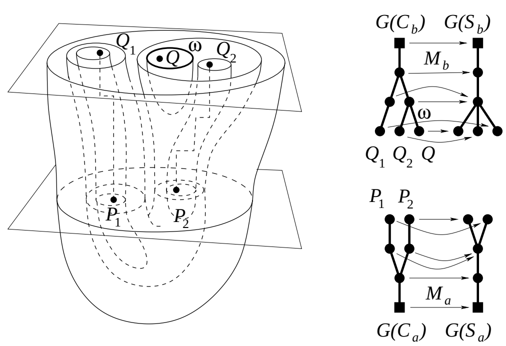


FIGURE 2. An example of how the adjacency graphs and the function M_a change passing through a critical point of index 0.

Case index 2. Passing through a critical value of index 2, a 2-cell is attached to an oval of C_a which disappears and $G(S_b) = G(S_a)$. We can reconstruct the other data in this situation just reversing the method used in the case of index 0: the computation of finitely many “reversed roadmaps” starting from points chosen in the centers of the nests of C_b and having their final points on the plane $\{z = a\}$ allows to recognize the oval that disappears and its location with respect to the ovals of C_a. Thus we can determine the edge and the vertex that disappear in $G(C_b)$ w.r.t. $G(C_a)$, i.e. to see $G(C_b)$ as a subgraph of $G(C_a)$. Then the function M_b coincides with the restriction of M_a to $G(C_b)$.

Case index 1. The case of critical points of index 1 presents several possible situations according to the way a 1-cell is attached to the boundary C_a of S_a. In

[3] one can find the list of the different situations that can arise and a procedure to determine the ovals involved in the attachment of the 1-cell and to reconstruct μ_b from μ_a.

If a 1-cell is attached to one single oval η which splits into two ovals ω_1, ω_2 of C_b, the edge η in $G(C_a)$ splits into two edges ω_1, ω_2 in $G(C_b)$ sharing a common vertex, while $G(S_b) = G(S_a)$. As for M_b, using the information given by the roadmaps about how edges and vertices of $G(C_a)$ lift to edges and vertices of $G(C_b)$, we can compute M_b on the whole graph $G(C_b)$. Two of the possible situations are presented in Figure 3.

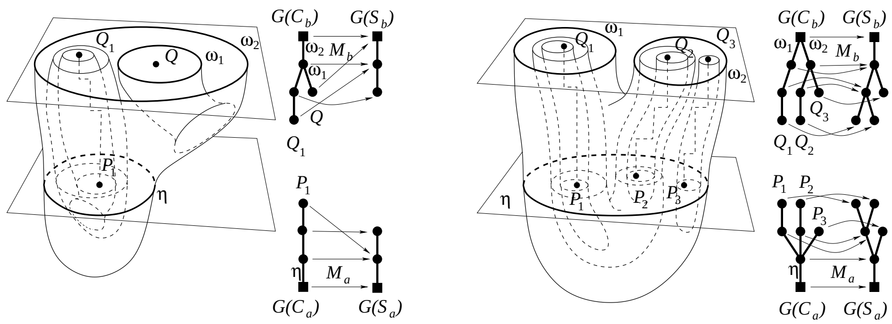


FIGURE 3. Two cases of attachment of a 1-cell to a single oval

When a 1-cell is attached to two distinct ovals ω_1, ω_2 of C_a that glue into an oval η of C_b, it may happen that two distinct connected components of S_a glue into a single component of S_b: this occurs when $\mu_a(\omega_1) \neq \mu_a(\omega_2)$ (see the right-hand surface in Figure 4). In this case the components $\mu_a(\omega_1)$ and $\mu_a(\omega_2)$ appear in $G(S_a)$ as distinct edges e_1 and e_2 having a common vertex v_0 (corresponding to the region of $\{z \leq a\} \setminus S$ containing in its boundary both $\mu_a(\omega_1)$ and $\mu_a(\omega_2)$). The glueing of the components $\mu_a(\omega_1)$ and $\mu_a(\omega_2)$ when passing through the critical value has as its counterpart in the adjacency graph the fact that the two edges e_1 and e_2 collaps into a single edge in $G(S_b)$. In other words, after recognizing via roadmaps e_1 and e_2, $G(S_b)$ can be obtained as a quotient of $G(S_a)$ identifying e_1 with e_2. The function μ_b expresses this identification associating to the edge η of $G(C_b)$ the edge obtained in $G(S_b)$ via identification; M_b can be then computed using roadmaps as in the previous case. If instead $\mu_a(\omega_1) = \mu_a(\omega_2)$ (e.g. as in the left-hand surface in Figure 4), the situation is simpler because $G(S_b) = G(S_a)$; M_b can be again computed via roadmaps.

After finitely many calls of the iterative step, eventually we get the adjacency graph $G(S) = G(S_N)$.

In order to know the weighted adjacency graph, we still have to compute w_S and $r(S)$. Since S is contained in the affine chart $\{t \neq 0\}$ of $\mathbb{RP}^3$, necessarily it has an even degree, all its components are two-sided contractible and the regions of $\mathbb{RP}^3 \setminus S$ are contractible except the only one external to all the components of S, which is non-contractible. We can easily detect this region Σ by means of our algorithm: $\Sigma \cap \{z \leq -N\}$ corresponds to the only point in $G(S_{-N})$, thus it is sufficient to mark with nc this unique vertex $v(\Sigma)$ at the initial step and, after completing the computation of $G(S) = G(S_N)$, to mark with c all the edges in $G(S)$ and all the vertices different from $v(\Sigma)$, and to choose $v(\Sigma)$ as the only root in the graph.

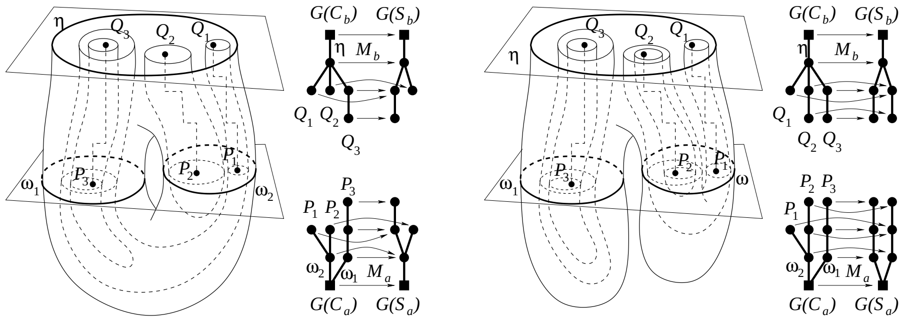


FIGURE 4. Two cases of attachment of a 1-cell to distinct ovals

4. The general case algorithm

In this section we will describe a constructive procedure to compute the triple $(G(S), w_S, r(S))$ for an arbitrary non-singular real algebraic surface $S \subset \mathbb{RP}^3$. If S does not intersect in real points the plane at infinity $\{t = 0\} \subset \mathbb{RP}^3$, then we can apply the algorithm described in the previous section. If S is not affine, in [3] the authors proposed a method to reduce the computation of $\chi(S)$ to the computation of $\chi(\widehat{S})$ for a compact affine surface $\widehat{S} \subset \mathbb{R}^3$ suitably constructed. In this paper we will exploit the same reduction process, that we now briefly recall, explaining how $G(S)$, w_S and $r(S)$ can be recovered from the computation of the adjacency graph $G(\widehat{S})$.

Let S be a non-singular real projective surface in $\mathbb{RP}^3$ defined by the equation $F(x, y, z, t) = 0$, where F is a homogeneous polynomial of degree d in $\mathbb{Q}[x, y, z, t]$. Denote by $\pi : S^3 \to \mathbb{RP}^3$ the map that associates to any point (x, y, z, t) of the 3-sphere S^3 the point of homogeneous coordinates $[x, y, z, t]$ in $\mathbb{RP}^3$, so that each fiber contains two antipodal points on the sphere. In this way S^3 is a 2-sheeted covering space of $\mathbb{RP}^3$ on which we can lift the surface S through π considering the non-singular surface

$$\widetilde{S} = \pi^{-1}(S) = \{(x, y, z, t) \in \mathbb{R}^4 \mid F(x, y, z, t) = 0\} \cap S^3.$$

The antipodal map $ap : S^3 \to S^3$ defined by $ap(v) = -v$ induces an involution on the set $\mathcal{F}$ of the connected components of $\widetilde{S}$, so that we can split $\mathcal{F}$ as the union of $\mathcal{F}_1$ and $\mathcal{F}_2$, where

$$\mathcal{F}_1 = \{\widetilde{T} \mid \widetilde{T} \in \mathcal{F},\ ap(\widetilde{T}) = \widetilde{T}\} \quad \text{and} \quad \mathcal{F}_2 = \mathcal{F} \setminus \mathcal{F}_1.$$

The map ap acts as an involution also on the set $\mathcal{R}$ of the connected components of $S^3 \setminus \widetilde{S}$, thus we can similarily split $\mathcal{R}$ as $\mathcal{R}_1 \cup \mathcal{R}_2$ where

$$\mathcal{R}_1 = \{\widetilde{\Sigma} \mid \widetilde{\Sigma} \in \mathcal{R},\ ap(\widetilde{\Sigma}) = \widetilde{\Sigma}\} \quad \text{and} \quad \mathcal{R}_2 = \mathcal{R} \setminus \mathcal{R}_1.$$

The preimage $\widetilde{Y} = \pi^{-1}(Y)$ of a connected component Y of S can be either connected (so that $\widetilde{Y} \in \mathcal{F}_1$) or the union of two distinct connected components of $\widetilde{S}$ corresponding each to the other through the pairing induced by ap in $\mathcal{F}_2$. Using this fact, in [3] it is proved that the knowledge of the topology of $\widetilde{S}$ and of the two sets $\mathcal{F}_1$ and $\mathcal{F}_2$ makes it possible to compute the Euler characteristic of all the components of S, that is the list $\chi(S)$, and hence to determine the topological type of the surface. We will see that the computation of the sets $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{R}_1$, $\mathcal{R}_2$ is sufficient both to recover the graph $G(S)$ from $G(\widetilde{S})$ and to compute w_S and the set of roots $r(S)$. This will be a consequence of the following

Lemma 4.1. *Let A be a connected subset of $\mathbb{RP}^3$. Then A is non-contractible if and only if $\pi^{-1}(A)$ is connected.*

Proof. Assume that A is non-contractible and, by contradiction, that $\pi^{-1}(A)$ is not connected. Let C_1, C_2 be two connected components of $\pi^{-1}(A)$ such that $C_2 = ap(C_1)$. If $x_1 \in C_1$ and $x_2 = ap(x_1) \in C_2$, then $\pi(x_1) = \pi(x_2) = y \in A$. By hypothesis there exists a non-contractible closed path $\gamma : [0,1] \to A$ such that $\gamma(0) = \gamma(1) = y$. If we lift γ to a path $\widetilde{\gamma}$ in $\pi^{-1}(A)$ such that $\widetilde{\gamma}(0) = x_1$, then necessarily $\widetilde{\gamma}(1) = x_2$: if otherwise $\widetilde{\gamma}(1) = x_1$, then $\widetilde{\gamma}$ would be contractible in S^3, while γ is not homotopically trivial. Then the points x_1, x_2 lying in different connected components of $\pi^{-1}(A)$ could be joined by means of a continuous path, which is impossible. Conversely, since $\pi^{-1}(A)$ is ap-invariant, if it is connected then we can find a path σ in $\pi^{-1}(A)$ joining two antipodal points. Then $\pi \circ \sigma$ is a non-trivial loop in A. $\square$

Corollary 4.2. *A connected component Y of S is non-contractible if and only if $\pi^{-1}(Y) \in \mathcal{F}_1$; a region Σ of $\mathbb{RP}^3 \setminus S$ is non-contractible if and only if $\pi^{-1}(\Sigma) \in \mathcal{R}_1$.*

The previous corollary explains how vertices and edges in $G(S)$ lift in the adjacency graph $G(\widetilde{S}) = G(S^3, \widetilde{S})$ of the surface $\widetilde{S}$ according to their contractibility. Namely, for every edge e of $G(S)$, if e is non-contractible then $\pi^{-1}(e)$ is an edge of $G(\widetilde{S})$, while if e is contractible then $\pi^{-1}(e)$ consists of two edges e_1, e_2 of $G(\widetilde{S})$ with $e_2 = ap(e_1)$. Similarly, each non-contractible vertex of $G(S)$ lifts to a single vertex in $G(\widetilde{S})$, while the preimage of each contractible vertex of $G(S)$ consists of a pair of vertices v_1, v_2 such that $v_2 = ap(v_1)$.

Therefore the map ap induces an automorphism , that we denote again by ap, of the adjacency graph $G(\widetilde{S})$, with $ap^2 = id$, which fixes the vertices corresponding to regions in $\mathcal{R}_1$ and the edges corresponding to components in $\mathcal{F}_1$, while it induces a pairing in the set of vertices in $\mathcal{R}_2$ and in the set of edges in $\mathcal{F}_2$.

We want now to relate the way ap acts on $G(\widetilde{S})$ with the structure of the graph $G(S)$ and with π. To do that, denote by $\{v_1, \ldots, v_s\}$ the set of the vertices of $\overline{G_c(S)} \cap G_{nc}(S)$. For each $i = 1, \ldots, s$, the vertex v_i represents a non-contractible region, so that $\pi^{-1}(v_i)$ consists of only one vertex in $G(\widetilde{S})$ and represents a connected region Σ_i of $S^3 \setminus \widetilde{S}$ invariant with respect to ap. For each i, denote by $A(v_i)$ the connected component of $\overline{G_c(S)}$ containing v_i. As a consequence of the previous considerations, $\pi^{-1}(A(v_i))$ is a subgraph of $G(\widetilde{S})$ on which ap acts as an involution having as its fixed locus only the vertex $\pi^{-1}(v_i)$.

If the degree d is even, these observations are sufficient to reconstruct $G(S)$, w_S and $r(S)$ from the knowledge of $G(\widetilde{S})$ and of the sets $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{R}_1$, $\mathcal{R}_2$ according to the following rules:

(a) leave unchanged the edges of $G(\widetilde{S})$ corresponding to components in $\mathcal{F}_1$ and the vertices corresponding to regions in $\mathcal{R}_1$ and denote by G' this subgraph. Note that G' is non-empty because there exists at least a non-contractible region Σ in $\mathbb{RP}^3 \setminus S$, hence $\pi^{-1}(\Sigma) \in \mathcal{R}_1$ and it corresponds to a vertex of the subgraph G';

(b) for each vertex $v \in G'$ which is the vertex of at least an edge in $G(\widetilde{S}) \setminus G'$, consider the set $\widetilde{A}(v)$ constructed as above starting from the graph $G(\widetilde{S})$ and its subgraph G'. Using the natural pairing existing in $\widetilde{A}(v)$, delete one copy in each pair of isomorphic elements in $\widetilde{A}(v)$; denote by G the graph so obtained from $G(\widetilde{S})$;
(c) mark with nc all the edges and vertices in G' and mark with c all the other elements in the constructed graph;
(d) choose as roots the vertices in G' which are vertices also for at least one edge in $G \setminus G'$.

The resulting marked graph equipped with the chosen roots is isomorphic to the weighted adjacency graph $(G(S), w_S, r(S))$ of S.

When the degree d is odd, we must be more careful because S contains a non-orientable and non-contractible component Y_0 not represented in $G(S)$ while the component $\pi^{-1}(Y_0)$, lying in $\mathcal{F}_1$, is represented in $G(\widetilde{S})$ as an edge e_0. Since in this case all the other components of S are contractible, $\mathcal{F}_1$ contains only one element, and each component of S different from Y_0 has a preimage in S^3 formed by two components lying in $\mathcal{F}_2$ and exchanged by ap. Moreover the preimage of each region in $\mathbb{RP}^3 \setminus S$ consists of two connected regions of $S^3 \setminus \widetilde{S}$ exchanged by ap (in other words $\mathcal{R}_1 = \emptyset$); in particular, if Σ_0 is the region in $\mathbb{RP}^3 \setminus Y_0$ external to all the two-sided components of S, then the two regions in $\pi^{-1}(\Sigma_0)$ correspond to the vertices v_1, v_2 of e_0.

Thus the graph $G(\widetilde{S})$, apart the "special" edge e_0, detectable as the unique element in $\mathcal{F}_1$, having no counterpart in $G(S)$, is such that $G(\widetilde{S}) \setminus \{e_0\}$ consists of two connected and isomorphic subgraphs $\widetilde{G_1}$ and $\widetilde{G_2}$.

This explains why, when the degree d is odd, we can reconstruct $G(S)$, w_S and $r(S)$ from $G(\widetilde{S})$, $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{R}_1$, $\mathcal{R}_2$ as follows:

(a) we detect in $G(\widetilde{S})$ the edge e_0 as the unique element in $\mathcal{F}_1$;
(b) we choose one of the two connected components of $G(\widetilde{S}) \setminus \{e_0\}$, say for instance $\widetilde{G_1}$;
(c) we mark with c all the edges and vertices in $\widetilde{G_1}$;
(d) we choose v_1 as a root.

Again the marked graph so constructed with the chosen root is isomorphic to the weighted adjacency graph of S. Note that in this case only the knowledge of $\mathcal{F}_1$ and $\mathcal{F}_2$ is nedeed; the splitting of $\mathcal{R}$ is useless since $\mathcal{R}_1 = \emptyset$ and $\mathcal{R}_2 = \mathcal{R}$.

Thus our original question has been reduced to the computation of $G(\widetilde{S})$ and of the sets $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{R}_1$, $\mathcal{R}_2$. We will accomplish all these tasks working on a compact affine surface $\widehat{S} \subset \mathbb{R}^3$ homeomorphic to $\widetilde{S}$. Namely we can assume, up to an affine translation, that $[0,0,0,1] \notin S$ so that $N = (0,0,0,1) \notin \widetilde{S}$. Then, projecting $S^3 \setminus \{N\}$ to $\mathbb{R}^3$ via the stereographic projection φ, the image $\widehat{S} = \varphi(\widetilde{S})$ is a compact non-singular surface in $\mathbb{R}^3$, homeomorphic to $\widetilde{S}$ and given by the polynomial equation $f(X) = F(2X, \|X\|^2 - 1) = 0$, where $X = (x, y, z)$ and $\|X\|^2 = x^2 + y^2 + z^2$.

We will denote by $inv = \varphi \circ ap \circ \varphi^{-1} : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}^3 \setminus \{0\}$ the involution, given by $inv(X) = -\frac{X}{\|X\|^2}$, corresponding to ap through the stereographic projection.

Since $G(\widetilde{S}) = G(\widehat{S})$, in order to compute $G(\widetilde{S})$ it is sufficient to apply to $\widehat{S}$ the algorithm to compute the adjacency graph of a compact affine surface in $\mathbb{R}^3$ described in the previous section.

Also for the computation of the splittings of $\mathcal{F}$ and $\mathcal{R}$, we will equivalently determine the sets of components $\widehat{\mathcal{F}}, \widehat{\mathcal{F}}_1, \widehat{\mathcal{F}}_2$ of $\widehat{S}$ and the sets of regions $\widehat{\mathcal{R}}, \widehat{\mathcal{R}}_1, \widehat{\mathcal{R}}_2$ of $\mathbb{R}^3 \setminus \widehat{S}$ corresponding through φ respectively to $\mathcal{F}$, $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{R}$, $\mathcal{R}_1$, $\mathcal{R}_2$.

In [3] the authors described a procedure to split $\widehat{\mathcal{F}}$ as $\widehat{\mathcal{F}}_1 \cup \widehat{\mathcal{F}}_2$ based on an investigation of the plane level curve $\widehat{C}_0 = \widehat{S} \cap \widehat{W}$ where $\widehat{W} = \{z = 0\} \subset \mathbb{R}^3$, provided that $\widehat{C}_0$ is non-singular; they proved that a component $\widehat{T}$ of $\widehat{S}$ belongs to $\widehat{\mathcal{F}}_1$ if and only if there exist two ovals ω_1, ω_2 of $\widehat{S} \cap \widehat{W}$ both contained in $\widehat{T}$ and such that $inv(\omega_1) = \omega_2$.

Using the same argument it is easily seen that a region $\widehat{\Sigma}$ of $\mathbb{R}^3 \setminus \widehat{S}$ belongs to $\widehat{\mathcal{R}}_1$ if and only if there exist two regions σ_1, σ_2 of $\widehat{W} \setminus (\widehat{S} \cap \widehat{W})$ both contained in $\widehat{\Sigma}$ and such that $inv(\sigma_1) = \sigma_2$.

All these conditions can be constructively tested. First of all one can assume that 0 is not a critical value for the projection p and choose it as one of the levels a_i to be studied in the iterative procedure, so that $\widehat{C}_0$ is automatically studied by the algorithm that computes the graph. Thus the components of $\widehat{S}$ containing ω_1 and ω_2 can be detected taking into account the function μ_0 and recording the glueing of distinct components of $\widehat{S}$ through critical values > 0. Similarily the fact that σ_1 and σ_2 are both contained in $\widehat{\Sigma}$ can be tested using the function M_0 in the place of μ_0.

The second condition $inv(\omega_1) = \omega_2$ was tested in [3] by means of a procedure that recognizes the images through inv of the ovals of $\widehat{S} \cap \widehat{W}$ and also shows how the regions of $\widehat{W} \setminus (\widehat{S} \cap \widehat{W})$ are transformed by the involution inv.

Therefore not only the splitting $\widehat{\mathcal{R}}_1 \cup \widehat{\mathcal{R}}_2$ can be easily computed, but it is in fact obtained by performing the same computations needed to split $\widehat{\mathcal{F}}$ into $\widehat{\mathcal{F}}_1 \cup \widehat{\mathcal{F}}_2$ with no additional computational cost.

5. Computational remarks and examples

Before presenting some examples, let us recall the main steps in which our algorithm is organized. Assume that the projective surface S to be studied is given by means of a defining equation $F(x, y, z, t) = 0$, with F is a square–free homogeneous polynomial of degree d with rational coefficients. We assume also that the surface is non-singular, that is it contains no real singular point while complex singularities are allowed; one of the possible methods to test that can be found in [4].

As explained in the previous section, the algorithm preliminarily checks whether S intersects in real points the plane $\{t = 0\}$: if S has no real point "at infinity", it is defined as an affine surface in $\mathbb{R}^3$ by the equation $f(x, y, z) = F(x, y, z, 1)$; otherwise the algorithm computes the equation $f(X) = F(2X, \|X\|^2 - 1) = 0$ of the affine surface $\widehat{S} \subset \mathbb{R}^3$. Note that if S is non-singular, $\widehat{S}$ is non-singular too.

The part of the algorithm that studies this affine surface requires to work in a good system of coordinates:

Definition 5.1. *The system of affine coordinates (x, y, z) is called a "good frame" for the non-singular surface $\{f(x, y, z) = 0\}$ if*

(1) *the projection $p(x, y, z) = z$ is a Morse function on the surface*

(2) *the images of distinct real critical points are distinct critical values.*

Up to a generic projective change of the original homogeneous coordinates, we can assume without loss of generality that the previous conditions are satisfied.

The real critical locus of p on the surface is the real zero-set of the ideal $K = (f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$. The projection $p(x, y, z) = z$ is a Morse function if no real critical point annihilates the determinant $H(x, y, z)$ of the matrix $\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$, i.e. if the real zero-set defined by the ideal $(K, H(x, y, z))$ is empty. A method to test that can be found in [4]. In that paper it is also showed that, if p is Morse and K is 1-dimensional, it is possible to compute a zero-dimensional ideal, say I, such that the real zero-set $V_{\mathbb{R}}(I)$ coincides with $V_{\mathbb{R}}(K)$. Of course, without changing its zero-set, we can also assume that I is radical.

Another datum necessary for the algorithm is the list of the indexes. A partial information on the index of a real critical point P is obtained by evaluating at P the determinant $H(x, y, z)$ of the Hessian matrix: if $H(P)$ is negative, then P has index 1, otherwise it has index either 0 or 2 (recall that $H(P)$ cannot vanish since P is non-degenerate). This partial information, combined with other data already computed in the iterative steps, are sufficient to determine the index: if P is a critical point contained in the strip $a < z < b$ and $H(P) > 0$, we can decide whether the index of P is 0 or 2 simply checking whether C_b has more or fewer ovals than C_a. Therefore, in order to compute the indexes it is sufficient to estimate the sign of the function $H(x, y, z)$ at the points of the variety $V_{\mathbb{R}}(I)$.

Note that the algorithm does not really require the knowledge of the real critical points, but only of the corresponding critical values and of their indexes: if these data are known, the remaining procedure in the iterative step to lift the needed data is of a pure combinatorial nature and based only on roadmaps.

Actually it is possible to compute both the real critical values and their indexes, avoiding the whole computation of the real critical points. First of all, since the real critical values of p on the surface are the images through p of the real points in $V(I)$, they can be computed as the real roots of the minimal degree univariate polynomial $m(z)$ in $I \cap \mathbb{Q}[z]$. If the degree of $m(z)$ coincides with the dimension of the finite-dimensional $\mathbb{Q}$-vector space $V = \mathbb{Q}[x, y, z]/I$, then I is in general position (which we can always assume) and the fiber over any critical value contains exactly one critical point.

Moreover recall that (see for instance [2]), for any polynomial $h \in \mathbb{C}[x, y, z]$ and any zero-dimensional ideal $J \subset \mathbb{C}[x, y, z]$, the values of h at the points of $V(J)$ coincide with the eigenvalues of the multiplication matrix M_h associated, with respect to a monomial basis, to the linear map from $\mathbb{C}[x, y, z]/J$ to itself that transforms $[g]$ into $[h] \cdot [g]$.

Therefore in our situation the values of the function $H(x, y, z)$ at the points of the variety $V(I)$ are the eigenvalues of the multiplication matrix M_H. Also the eigenvalues of the multiplication matrix M_z coincide with the values of the function z on $V(I)$, i.e. with the critical values of the projection p, which we already know to be distinct. Hence the commuting matrices M_z and M_H are simultaneously diagonalizable by means of an invertible matrix L and therefore, if a real critical value $c = p(P)$ appears in the position(i, i) of the diagonal matrix $L^{-1}M_zL$, then the value of $H(P)$ appears in the position(i, i) of the diagonal matrix $L^{-1}M_HL$.

Thus by means of linear algebra techniques we can avoid the numerical difficulties due to the explicit computation of the critical points.

Once computed the real critical values and relative indexes, the algorithm computes the topology and the weighted adjacency graph of S, if it is affine, or otherwise of the doubled surface $\widehat{S}$. In the latter case, the procedure described in Section 4 allows us to compute the splittings $\widehat{\mathcal{F}} = \widehat{\mathcal{F}}_1 \cup \widehat{\mathcal{F}}_2$ and $\widehat{\mathcal{R}} = \widehat{\mathcal{R}}_1 \cup \widehat{\mathcal{R}}_2$ and thus to recover the topology and the weighted adjacency graph of S.

We conclude the paper with some examples intended as a means to exemplify the way our algorithm works, not to show the full potentiality of the procedure. All the computations have been performed using a preliminary implementation of the algorithm we have produced in Axiom ([6]).

Example 1. The projective closure of the affine surface S defined by $f(x,y,z) = [(z^4 + (2y^2 + 2x^2 - 10)z^2 + y^4 + (2x^2 + 6)y^2 + x^4 - 10x^2 + 9) \cdot ((x + \frac{1}{2}z)^2 + (y-1)^2 + 90(z - \frac{1}{2}x)^2 - 80) + 200] \cdot (x^2 + y^2 + (z-2)^2 - \frac{1}{4}) = 0$ does not contain any real point in the plane at infinity $\{t = 0\}$. Inspecting the defining equation, it is easily seen that it contains a sphere, while the other factor of f defines a perturbation of the union of a torus and an ellypsoid. The Figure 5 shows the 12 steps through which the algorithm recognizes that S consists of a torus with 2 holes and three spheres, two of which are nested, lying in the exterior part of the torus.

Example 2. Consider the projective surface S defined by the homogeneous equation $F(x,y,z,t) = (x^2 + y^2 + z^2 - \frac{1}{36}t^2) \cdot ((x - \frac{3}{2}t)^2 + y^2 + z^2 - \frac{1}{4}t^2) \cdot (x^2 + z^2 - y^2 - \frac{1}{16}t^2) = 0$. The real singular locus of S is empty, and the surface intersects the plane $\{t = 0\}$ in real points, so the algorithm preliminarily investigates the doubled surface $\widetilde{S} \subset S^3$ studying the affine surface $\widehat{S} \subset \mathbb{R}^3$ of degree 12 which is homeomorphic to $\widetilde{S}$. At the end of the iterative steps corresponding to the 12 distinct real critical points on $\widehat{S}$ we get the output showed in Figure 6.

Since $\widehat{\mathcal{F}}_1$ contains only the edge $\{1,2\}$, the algorithm recognizes that S contains only one non-contractible two-sided component, which splits $\mathbb{RP}^3$ into two regions corresponding to the vertices 1 and 2. Removing this edge from $G(\widehat{S})$ we get two connected graphs $\widehat{A}(1)$ and $\widehat{A}(2)$; in each of them the two edges correspond each to the other through the natural pairing, so we have to delete one edge in each graph. Thus we get the graph $G(S)$ with vertices $1,2,3,5$ joined by the edges $\{1,2\}, \{2,3\}, \{1,5\}$, where $\{1,2\}$ is non-contractible and $\{2,3\}, \{1,5\}$ are contractible. The topological output of the algorithm yields that $\chi(\{1,2\}) = 0$, $\chi(\{2,3\}) = 2$ and $\chi(\{1,5\}) = 2$. Hence we recognize that the non-contractible two-sided component of S is a torus, which splits $\mathbb{RP}^3$ into two regions each containing a sphere.

Example 3. The non-singular surface S defined by the homogeneous equation $F(x,y,z,t) = ((x-2t)^2 + y^2 + z^2 - 5t^2) \cdot [(x+3t)z^2 - (4x+12t)zt + (x+3t)y^2 + x^3 - 3x^2t - 27xt^2 - 28t^3] = 0$ has an odd degree, hence it necessarily intersects the plane at infinity $\{t = 0\}$ and contains one non-orientable one-sided component. We already know that all its other components are two-sided and contractible, but we can also detect their mutual disposition studying the doubled 10-degree affine surface $\widehat{S}$.

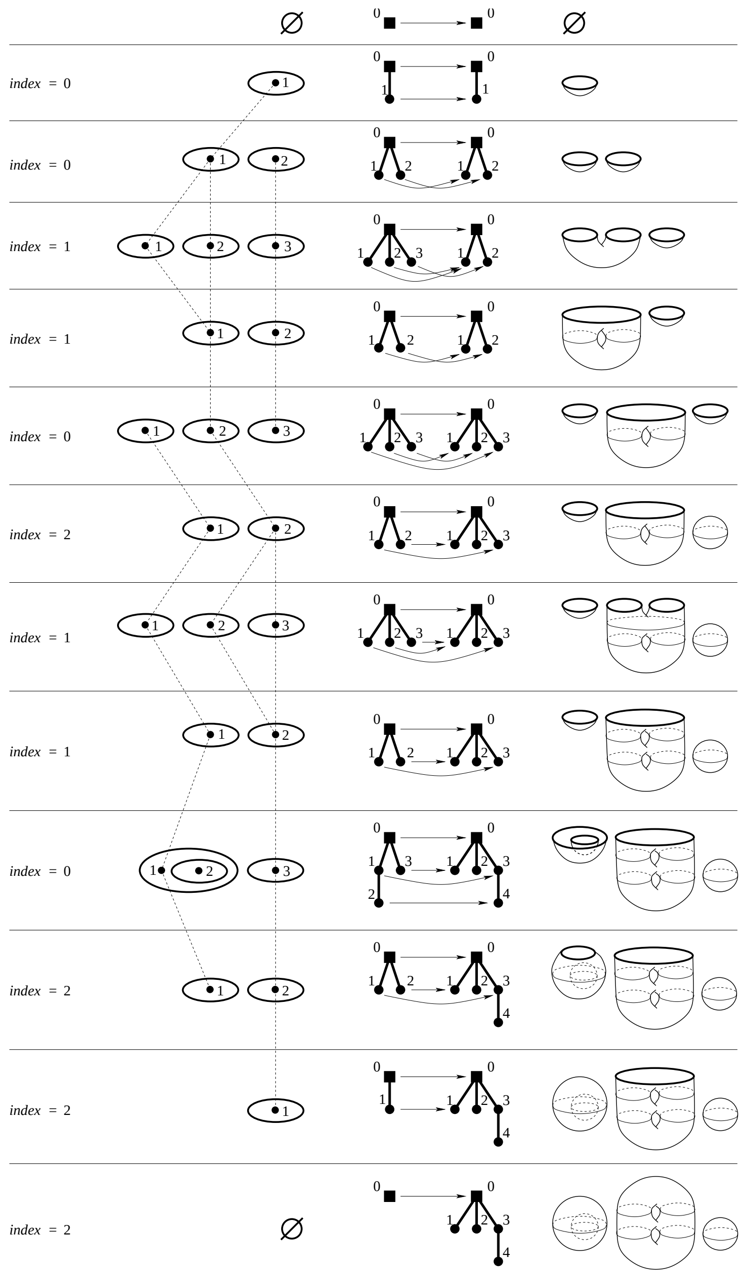


FIGURE 5. Reconstruction of the topology and the adjacency graph of the surface of Example 1. The horizontal strips (to be examined in a downward order) correspond to the 12 iterative steps of the algorithm. Each strip schematically reproduces the data computed in that step; from left to right: the index of the critical point, the scheme of the level curve, the adjacency graphs of the level curve and of the level surface with arrows representing the action of the function M, and finally a visual image of the level surface reconstructed on the basis of the computed data. Vertical dotted segments represent the roadmaps computed by the algorithm.

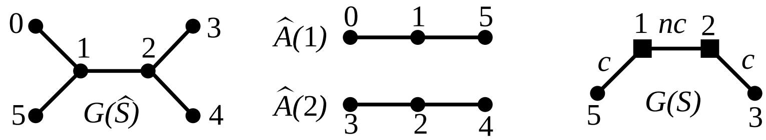


FIGURE 6. Graphical representation of the algorithm output relative to the surface of Example 2.

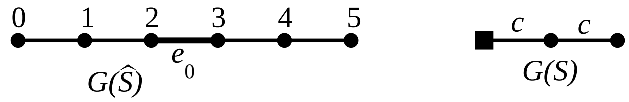


FIGURE 7. Graphical representation of the algorithm output relative to the surface of Example 3.

In the output represented in Figure 7 we see that, as expected, $\widehat{\mathcal{F}}_1$ contains only one element, i.e. the edge $\{2, 3\}$ corresponding to the preimage in $\widehat{S}$ of the non-orientable component in S. Since the algorithm computations yield that $\chi(\{2, 3\}) = 1$, we realize that the non-orientable component is a projective plane. Moreover, following the rules described in Section 4, we discard the edge $\{2, 3\}$ and choose one of the two connected graphs so obtained, say for instance the graph having $3, 4, 5$ as vertices joined by the edges $\{3, 4\}$ and $\{4, 5\}$ and we choose 3 as its root. Since $\chi(\{3, 4\}) = 2$ and $\chi(\{4, 5\}) = 2$, we get that the two contractible two-sided components of S are two nested spheres.

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