ON A MODEL OF PHASE RELAXATION FOR THE HYPERBOLIC STEFAN PROBLEM

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Abstract. In this paper we study a model of phase relaxation for the Stefan problem with the Cattaneo-Maxwell heat flux law. We prove an existence and uniqueness result for the resulting problem and we show that its solution converges to the solution of the Stefan problem as the two relaxation parameters go to zero, provided a relation between these parameters holds.

1. Introduction

In this paper we propose and study the nonlinear model for phase transition phenomena given by

$$\partial_t(\theta + \chi) + \operatorname{div} \mathbf{q} = f \qquad \text{in } Q := \Omega \times]0, T[, \tag{1.1}$$

$$\alpha \partial_t \mathbf{q} + \mathbf{q} = -\nabla\theta \qquad \text{in } Q, \tag{1.2}$$

$$\varepsilon \partial_t \chi + \chi \in \gamma(\theta) \qquad \text{in } Q. \tag{1.3}$$

Here Ω is a bounded domain in $\mathbb{R}^n$, $T > 0$ is a final time, and $\theta, \chi : Q \longrightarrow \mathbb{R}$, $\mathbf{q} : Q \longrightarrow \mathbb{R}^n$ are the unknown functions of the problem. The symbols ∂_t, div , and ∇ represent respectively the time derivative, the spatial divergence operator, and the gradient in space. α and ε are two (small) positive constant, $f : Q \longrightarrow \mathbb{R}$ and $\gamma : \mathbb{R} \longrightarrow \mathscr{P}(\mathbb{R})$ are two given data. Notice that γ is multivalued, so giving rise to the inclusion (1.3). We will see that a natural example of γ is given by the multivalued sign map, defined by

$$\operatorname{sign}(r) := -1 \quad \text{if } r < 0, \qquad \operatorname{sign}(r) := [-1, 1] \quad \text{if } r = 0, \qquad \operatorname{sign}(r) := 1 \quad \text{if } r > 0.$$

More generally we may consider γ as a *maximal monotone graph* in $\mathbb{R}^2$. For the theory of maximal monotone operators we refer the reader, e.g., to [2], [1], and [17].
Equation (1.1) represents the energy balance of a physical substance attaining two phases and contained in Ω: θ denotes the relative temperature, $\mathbf{q}$ the heat flux, and χ stands for the phase function: for instance if we deal with a solid-liquid system, then $(1 - \chi)/2$ represents the solid concentration, $(1 + \chi)/2$ is the liquid concentration, and $-1 \le \chi \le 1$ (cf., e.g., [21, p. 99]). Thus, if f is the external heat supply and if we make the usual assumption that the internal energy of the system is given by $e := \theta + \chi$, then we get equation (1.1). In order to describe the evolution of the system, we have to couple the energy balance with a constituve law for the heat flux and a further equation relating θ and χ: relations (1.2) and (1.3) play these roles in our model. Let us now describe such relations.

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Equation (1.2) is the well-known *Cattaneo-Maxwell heat flux law*, proposed by C. Cattaneo in [3] in order to replace the classical *Fourier law*

$$\mathbf{q} = -\nabla\theta, \tag{1.4}$$

which instead leads to the parabolic equation

$$\partial_t(\theta + \chi) - \Delta\theta = f \qquad \text{in } Q \tag{1.5}$$

(all the physical constants are normalized to 1, except of course α and ε). It is well known that equation (1.5) has the particular feature of allowing the thermal disturbances to propagate at infinite speed. Now, one can argue that heat is expected to propagate with a finite speed, so that a change of the Fourier law seems mandatory. The thermal relaxation (1.2) proposed by Cattaneo solves this problem, since in this case the energy balance (1.1) yields an equation of hyperbolic type, predicting finite speed of propagation for the temperature field. Notice that a formal integration of (1.2) gives

$$\mathbf{q}(t) = -\frac{1}{\alpha}\int_0^t \exp\left(\frac{s-t}{\alpha}\right)\nabla\theta(s)ds, \tag{1.6}$$

so that (1.2) can be considered as the starting point of the theory of materials with memory (cf. [9]). For updated reviews of Cattaneo theory we refer the reader to [13], [4], [14, Ch. 2], and [12].

Let us consider now (1.3). If $\theta = 0$ is the equilibrium temperature at which the two phases can coexist, then in order to describe the evolution of the two-phase system, it is usual to assume the classical *Stefan equilibrium condition*

$$\chi \in \operatorname{sign}(\theta) \qquad \text{in } Q. \tag{1.7}$$

This condition is fairly natural and the fact that the sign function is allowed to be setvalued is consistent with the concept of *mushy region*, a subset of Ω where very fine solid-liquid mixtures are allowed to appear at the macroscopic scale (see again [21, p. 99]). Problem (1.5), (1.7) is usually called *Stefan problem* and it is the most common model in dealing with phase transitions. See [7] and [21, Ch. II] for related existence and uniqueness results, in particular [21] contains a wide list of references about phase change problems.

It should be noted, however, that the Stefan equilibrium condition does not take into account dynamic supercooling or superheating effects, and these effects are important, since it is reasonable to assume that the phase transition is driven by a nonequilibriun condition. Accordingly, A. Visintin in [20] replaced (1.7) by the following relaxation dynamics for the phase function

$$\varepsilon\partial_t\chi + \operatorname{sign}^{-1}(\chi) \ni \theta \qquad \text{in } Q, \tag{1.8}$$

ε being a small relaxation parameter (see also [21, Sect. V.1]). Notice that the Stefan condition (1.7) can be equivalently written as

$$\operatorname{sign}^{-1}(\chi) \ni \theta \qquad \text{in } Q. \tag{1.9}$$

In this paper, instead, we use the equally natural model (1.3), by adding the term $\varepsilon\partial_t\chi$ on the left-hand side of (1.7) rather than in (1.9). Nevertheless the two models (1.8) and (1.3), are far from being equivalent. Let us also observe that, at least from a mathematical point of view, it is not so clear why one should prefer the relaxation (1.8) rather than (1.3). Surely one argument in favour of the model (1.8) is given by the fact that (1.8) can be seen as an ODE governed by an accretive operator, so that arguing in terms of the new unknown $U := (\theta + \chi, \chi)$, the system (1.5), (1.8) can be solved as an evolution equation governed by a suitable monotone operator in a suitable product space. This idea is due to Visintin, see the details in [20] and [8].

Now, we provide a further physical interpretation for the model (1.3), so that, at least from a certain point of view, it can be considered more appropriate than (1.8). The argument we are going to outline is based on a "probabilistic" interpretation of the phase transition that was given in [22]. We follow essentially [22] and [15]: in the latter paper we study model (1.3) when the Fourier heat flux law is assumed.

We postulate that our physical system is composed by several small subsystems which we call *particles*. Moreover we suppose that any of these particles can assume either the solid state or the liquid state. This is in agreement with the usual concept of mushy region. Let us call π_+ (respectively π_-) the probability of *melting a solid* (respectively *crystallizing a liquid*) particle in the unit time. Therefore we get that the melting rate per unit volume is proportional to $\pi_+(1-\chi)/2$ and the crystallizing rate per unit volume is proportional to $\pi_-(1+\chi)/2$. Hence

$$\partial_t \chi \quad \text{is proportional to} \quad \pi_+(1-\chi)/2 - \pi_-(1+\chi)/2. \tag{1.10}$$

The transition probabilities above defined depend on the temperature, i.e. there exists a function $p : \mathbb{R} \longrightarrow [-1, 1]$ such that we have two relations such $\pi_+ = p(\theta^+)$ and $\pi_- = -p(-\theta^-)$, where $\theta^+ = \max\{\theta, 0\}$ and $\theta^- = \max\{-\theta, 0\}$. Hence, the relation (1.10) means that there exists some constant $\varepsilon > 0$ such that

$$\varepsilon \partial_t \chi = \big(p(\theta^+) + p(\theta^-)\big) - \big(p(\theta^+) - p(\theta^-)\big)\chi. \tag{1.11}$$

Observe that the bigger $\partial_t \chi$ is, the smaller ε turns out to be. Equation (1.11) suggests the analysis of a relaxation dynamics like $\varepsilon \partial_t \chi = \psi(\theta, \chi)$ for a suitable class of functions $\psi : \mathbb{R}^2 \to \mathbb{R}$. This is in fact the subject of paper [22], where L^1-tecniques are used and ψ belongs to certain class of regular functions.

If instead we take p equal to the single-valued sign function and then we allow it to be multivalued, we obtain exactly our relaxation (1.3). More generally we will take p equal to a maximal monotone graph γ satisfying a growth condition at infinity and our analysis requires L^2-tecniques only.

In this paper the resulting problem (1.1)–(1.3) is coupled with suitable initial and boundary conditions: for simplicity we consider homogeneous mixed Dirichlet-Neumann boundary conditions. Precisely, letting $\{\Gamma_0, \Gamma_1\}$ be a partition of the boundary of Ω into two measurable sets, we take

$$\theta = 0 \quad \text{on } \Gamma_0 \times]0, T[, \qquad \mathbf{q} \cdot \mathbf{n} = 0 \quad \text{in } \Gamma_1 \times]0, T[, \tag{1.12}$$

$$\theta(\cdot, 0) = \theta_0, \quad \mathbf{q}(\cdot, 0) = \mathbf{q}_0 \quad \chi(\cdot, 0) = \chi_0 \qquad \text{in } \Omega, \tag{1.13}$$

where θ_0, $\mathbf{q}_0$, and χ_0 are given functions and $\mathbf{n}$ is the outward unit vector, normal to the boundary of Ω. We prove that (1.1)–(1.3), (1.12)–(1.13) admits a unique solution in the framework of Sobolev spaces.

It should be observed that from a macroscopic point of view it is reasonable to assume that the Fourier law is a good approximation of real phenomena, so that the heat equation is sufficient to describe the thermal evolution of the system (see however [5] for materials for which the Fourier law is not satisfactory). Therefore in most of physical applications the relaxation parameters α and ε can be supposed very small with respect to the used lenght scale. Thus it is natural to consider the Stefan problem as approximation for the relaxed system. In this paper we give a rigorous proof of this heuristic argument. More precisely we show that the solution of problem (1.1)–(1.3), (1.12)–(1.13) converges, in a suitable sense, to the solution of problem (1.5), (1.7) coupled with the following boundary-initial conditions

$$\theta = 0 \quad \text{on } \Gamma_0 \times]0,T[, \qquad -\partial_{\mathbf{n}}\theta = 0 \quad \text{on } \Gamma_1 \times]0,T[, \tag{1.14}$$

$$(\theta + \chi)(\cdot, 0) = \theta_0 + \chi_0 \qquad \text{in } \Omega, \tag{1.15}$$

provided the heat relaxation parameter α is less or equal than the kinetic parameter ε. This constraint for the relaxation parameters can be considered non restrictive, since it is realistic to suppose that the time heat relaxation is smaller than that of the phase relaxation (cf. [20, p. 229]).

Let us mention related problems. One could couple (1.1)–(1.2) with the phase relaxation (1.8). The resulting model is studied in [6] and [16], but the uniqueness of solutions of the corresponding initial-boundary value problem is an open problem, at variance with our model (1.1)–(1.3). Moreover a possible choice is given by the model (1.1)–(1.2), (1.7). This is the so-called *hyperbolic Stefan problem*. While the validity of such model could be disputed, the point is that the existence of solutions remains an open problem, in fact it is not clear how to deduce enough a priori estimates. See however [18] and [19] for partial results in this direction.

2. Main results

In this section we give the variational formulation of the problems presented in the Introduction and we state our main results. Throughout the paper we make the following assumptions.

(H1) Ω is a bounded domain in $\mathbb{R}^n$ ($n \in \mathbb{N} = \{1, 2, \ldots\}$) with Lipschitz boundary $\Gamma := \partial\Omega$. The outward normal unit vector is denoted by $\mathbf{n}$. $Q := \Omega \times]0,T[$, where T is a positive number.

(H2) Γ_0 and Γ_1 are open subsets of Γ such that $\overline{\Gamma}_0 \cup \overline{\Gamma}_1 = \Gamma$, $\Gamma_0 \cap \Gamma_1 = \varnothing$ and $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \Gamma$ is of Lipschitz class.

(H3) $\gamma : \mathbb{R} \to \mathscr{P}(\mathbb{R})$ is a maximal monotone operator such that $0 \in \gamma(0)$ and that is linearly bounded, i.e. there is a constant $C_\gamma > 0$ such that

$$|s| \le C_\gamma(1 + |r|) \qquad \forall r \in D(\gamma), \quad \forall s \in \gamma(r), \tag{2.1}$$

where $D(\gamma) = \{r \in \mathbb{R} \ : \ \gamma(r) \neq \varnothing\}$.

(H4) $f \in W^{1,1}(0,T;L^2(\Omega)) + L^1(0,T;(H^1_{\Gamma_0}(\Omega)))$, where $H^1_{\Gamma_0}(\Omega) := \{v \in H^1(\Omega) \ : \ v|_{\Gamma_0} = 0\}$.

(H5) $\theta_0 \in H^1_{\Gamma_0}(\Omega), \quad \mathbf{q}_0 \in L^2(\Omega;\mathbb{R}^n), \quad \chi_0 \in L^2(\Omega)$.

Remark 2.1. For the theory of maximal monotone operators we refer the reader to [2, Ch. II], [1, Ch. II], and [17, Ch. IV]. From condition (2.1) and from the maximal monotonicity of γ, it can be easily deduced that $D(\gamma) = \mathbb{R}$. Notice that the $(n-1)$-dimensional Hausdorff measure of Γ_D is not required to be strictly positive.

Let us now fix some notation. We set $H := L^2(\Omega)$ and $V := H^1_{\Gamma_0}(\Omega)$, endow H and V with the usual inner products, and identify H with its dual space. Then we have $V \subset H \subset V'$ with dense and compact embeddings. We also define the operator $A \in \mathscr{L}(V,V')$ by

$$
{}_{V'}\langle Av_1, v_2\rangle_V := \int_\Omega \nabla v_1 \cdot \nabla v_2, \quad v_1, v_2 \in V. \tag{2.2}
$$

Next, we consider the spaces $\mathbf{H} := L^2(\Omega;\mathbb{R}^n)$ and $L^2_{\mathrm{div}}(\Omega) := \{\mathbf{v} \in \mathbf{H} : \ \operatorname{div}\mathbf{v} \in H\}$, the latter endowed with the usual inner product defined by $(\mathbf{v}_1,\mathbf{v}_2)_{L^2_{\mathrm{div}}(\Omega)} := (\mathbf{v}_1,\mathbf{v}_2)_{\mathbf{H}} + (\operatorname{div}\mathbf{v}_1, \operatorname{div}\mathbf{v}_2)_H$ for $\mathbf{v}_1, \mathbf{v}_2 \in L^2_{\mathrm{div}}(\Omega)$. It is well-known that if $\mathbf{v} \in L^2_{\mathrm{div}}(\Omega)$, then $\mathbf{v}\cdot\mathbf{n} \in H^{-1/2}(\Gamma)$ and the restriction $\mathbf{v}\cdot\mathbf{n}|_{\Gamma_1}$ makes sense in $(H^{1/2}_{00}(\Gamma_1))'$ (see, e.g., [11]). In this functional framework we introduce the closed subspace of $L^2_{\mathrm{div}}(\Omega)$

$$
\mathbf{V} := \left\{\mathbf{v} \in L^2_{\mathrm{div}}(\Omega) : \ \mathbf{v}\cdot\mathbf{n}|_{\Gamma_1} = 0\right\}. \tag{2.3}
$$

By identifying $\mathbf{H}$ with its dual space, we get $\mathbf{V} \subset \mathbf{H} \subset \mathbf{V}'$ with dense and continuous embeddings. Moreover, we will consider the operator $B \in \mathscr{L}(\mathbf{H}, V')$ defined by

$$
{}_{V'}\langle B\mathbf{u}, v\rangle_V := -\int_\Omega \mathbf{u}\cdot\nabla v, \quad \mathbf{u} \in \mathbf{H}, \ v \in V. \tag{2.4}
$$

Now, we can present the precise formulation of problem (1.1)–(1.3), (1.12)–(1.13).

Problem $(\mathbf{P}_{\alpha\varepsilon})$. Let $\alpha, \varepsilon > 0$. Find a triplet $(\theta_{\alpha\varepsilon}, \chi_{\alpha\varepsilon}, \mathbf{q}_{\alpha\varepsilon})$ satisfying the following conditions

$$
\theta_{\alpha\varepsilon} \in L^\infty(0,T;V), \quad \theta'_{\alpha\varepsilon} \in L^2(0,T;H) + L^1(0,T;V), \tag{2.5}
$$

$$
\chi_{\alpha\varepsilon} \in H^1(0,T;H), \tag{2.6}
$$

$$
\mathbf{q}_{\alpha\varepsilon} \in L^\infty(0,T;\mathbf{V}) \cap W^{1,\infty}(0,T;\mathbf{H}), \tag{2.7}
$$

$$
(\theta_{\alpha\varepsilon} + \chi_{\alpha\varepsilon})' + \operatorname{div}\mathbf{q}_{\alpha\varepsilon} = f \qquad \text{a.e. in } Q, \tag{2.8}
$$

$$
\alpha\mathbf{q}'_{\alpha\varepsilon} + \mathbf{q}_{\alpha\varepsilon} = -\nabla\theta_{\alpha\varepsilon} \qquad \text{a.e. in } Q, \tag{2.9}
$$

$$
\varepsilon\chi'_{\alpha\varepsilon} + \chi_{\alpha\varepsilon} \in \gamma(\theta_{\alpha\varepsilon}) \qquad \text{a.e. in } Q, \tag{2.10}
$$

$$
\theta_{\alpha\varepsilon}(0) = \theta_0, \quad \chi_{\alpha\varepsilon}(0) = \chi_0, \quad \mathbf{q}_{\alpha\varepsilon}(0) = \mathbf{q}_0 \qquad \text{a.e. in } \Omega. \tag{2.11}
$$

Here and in what follows the symbol " $'$ " will denote the derivative with respect to time of vector-valued functions. We will prove the following result.

Theorem 2.1. *Problem $(\mathbf{P}_{\alpha\varepsilon})$ has a unique solution.*

Now we give the weak formulation of the classical Stefan problem.

Problem (P). Find a pair (θ, χ) satisfying the following conditions

$$\theta \in L^\infty(0,T;H) \cap L^2(0,T;V), \tag{2.12}$$

$$\chi \in L^2(Q), \tag{2.13}$$

$$(\theta + \chi)' \in L^2(0,T;V'), \tag{2.14}$$

$$(\theta + \chi)' + A\theta = f \qquad \text{in } V', \text{ a.e. in }]0,T[, \tag{2.15}$$

$$\chi \in \gamma(\theta) \qquad \text{a.e. in } Q, \tag{2.16}$$

$$(\theta + \chi)(0) = \theta_0 + \chi_0 \qquad \text{in } V'. \tag{2.17}$$

Here is the main theorem of this paper.

Theorem 2.2. *Assume that $f \in W^{1,1}(0,T;H)$. Then Problem (**P**) admits a unique solution (θ, χ). Moreover, for any pair $\alpha, \varepsilon > 0$ such that $\alpha \le \varepsilon$, let $(\theta_{\alpha\varepsilon}, \mathbf{q}_{\alpha\varepsilon}, \chi_{\alpha\varepsilon})$ be the solution of Problem ($\mathbf{P}_{\alpha\varepsilon}$), we have*

$$\theta_{\alpha\varepsilon} \rightharpoonup \theta \qquad \text{in } L^2(0,T;H), \tag{2.18}$$

$$\chi_{\alpha\varepsilon} \rightharpoonup \chi \qquad \text{in } L^2(0,T;H), \tag{2.19}$$

$$\mathbf{q}_{\alpha\varepsilon} \rightharpoonup -\nabla\theta \qquad \text{in } L^2(0,T;\mathbf{H}), \tag{2.20}$$

as $\alpha, \varepsilon \searrow 0$.

Remark 2.2. Existence and uniqueness results for Problem (**P**) are well known, see e.g. [21] and the references therein. However our proof of the existence/uniqueness part of Theorem 2.2 is independent and uses a rather different method, since we recover the solution of (**P**) as the limit of the sequence $(\theta_{\alpha\varepsilon}, \mathbf{q}_{\alpha\varepsilon}, \chi_{\alpha\varepsilon})$. Of course our convergence result is completely new.

Let us introduce a general notation which will hold throughout the sequel. For a map $\phi \in L^1(0,T;X)$, where X is a Banach space, we define $I_0\phi : [0,T] \longrightarrow X$ by

$$(I_0\phi)(t) := \int_0^t \phi(s)ds, \qquad t \in [0,T]. \tag{2.21}$$

Finally, we will use the symbol C to denote a positive constant which depends only on the data and may vary from line to line.

3. Analysis of the relaxed problem

This section is devoted to the proof of Theorem 2.1. For simplicity, when writing a solution of Problem ($\mathbf{P}_{\alpha\varepsilon}$), we omit the subscript $\alpha\varepsilon$. The first result we prove concerns uniqueness.

Lemma 3.1. *Problem ($\mathbf{P}_{\alpha\varepsilon}$) has at most one solution.*

Proof. Let $(\theta_i, \mathbf{q}_i, \chi_i)$, $i = 1, 2$, be two solutions to Problem ($\mathbf{P}_{\alpha\varepsilon}$). Then, for $i = 1, 2$, let $\xi_i \in \gamma(\theta_i)$ a.e. in Q such that

$$\varepsilon\chi_i' + \chi = \xi_i \qquad \text{a.e. in } Q, \quad i = 1, 2. \tag{3.1}$$

Set $\widetilde{\theta} := \theta_1 - \theta_2$, $\widetilde{\mathbf{q}} := \mathbf{q}_1 - \mathbf{q}_2$, $\widetilde{\chi} := \chi_1 - \chi_2$, and $\widetilde{\xi} := \xi_1 - \xi_2$. Fix $t \in [0,T]$. At first let us subtract the respective equations (2.8) for $(\theta_i, \mathbf{q}_i, \chi_i)$, $i = 1,2$, from each other and test the result by $\varepsilon\widetilde{\theta}$. Integrating over $\Omega\times]0,t[$, thanks to the Green formula and to (2.11) we have that

$$\frac{\varepsilon}{2}\|\widetilde{\theta}(t)\|_H^2 + \varepsilon\int_0^t\int_\Omega \widetilde{\chi}'(s)\widetilde{\theta}(s)ds - \varepsilon\int_0^t\int_\Omega \widetilde{\mathbf{q}}(s)\cdot\nabla\widetilde{\theta}(s)ds = 0. \tag{3.2}$$

Now subtract the respective equations (2.8) for (θ_i, χ_i), $i = 1,2$, from each other and integrate the difference from 0 to $s \in]0,t[$, $t \in [0,T]$. Then we test the result by $\widetilde{\theta}(s)$ and finally integrate over $]0,t[$. We get that

$$\|\widetilde{\theta}\|^2_{L^2(0,t;H)} + \int_0^t\int_\Omega \widetilde{\chi}(s)\widetilde{\theta}(s)ds - \int_0^t\int_\Omega (I_0\widetilde{\mathbf{q}})(s)\cdot\nabla\widetilde{\theta}(s)ds = 0. \tag{3.3}$$

Then let us multiply the difference of equations (2.10) by $\widetilde{\theta}$ and integrate over $\Omega\times]0,t[$. Thanks to the monotonicity of γ we find

$$0 \le \varepsilon\int_0^t\int_\Omega \widetilde{\chi}'(s)\widetilde{\theta}(s)ds + \int_0^t\int_\Omega \widetilde{\chi}(s)\widetilde{\theta}(s)ds. \tag{3.4}$$

Now observe that using equation (2.9) written for the two solutions, it is readily seen that

$$\begin{aligned} &-\varepsilon\int_0^t\int_\Omega \widetilde{\mathbf{q}}(s)\cdot\nabla\widetilde{\theta}(s)ds - \int_0^t\int_\Omega (I_0\widetilde{\mathbf{q}})(s)\cdot\nabla\widetilde{\theta}(s)ds \\ &= \frac{\alpha\varepsilon}{2}\|\widetilde{\mathbf{q}}(t)\|^2_{\mathbf{H}} + \varepsilon\|\widetilde{\mathbf{q}}\|^2_{L^2(0,t;\mathbf{H})} + \alpha\int_0^t\int_\Omega (I_0\widetilde{\mathbf{q}})(s)\cdot\widetilde{\mathbf{q}}'(s)ds + \frac{1}{2}\|(I_0\widetilde{\mathbf{q}})(t)\|^2_{\mathbf{H}}. \end{aligned} \tag{3.5}$$

Let us rewrite the integral appearing at the right hand side of (3.5). For all $\delta > 0$ we have

$$\begin{aligned} &-\alpha\int_0^t\int_\Omega (I_0\widetilde{\mathbf{q}})(s)\cdot\widetilde{\mathbf{q}}'(s)ds \\ &= -\alpha((I_0\widetilde{\mathbf{q}})(t),\widetilde{\mathbf{q}}(t))_{\mathbf{H}} + \alpha\int_0^t \|\widetilde{\mathbf{q}}(s)\|^2_{\mathbf{H}}ds \\ &\le \alpha\|\widetilde{\mathbf{q}}(t)\|_{\mathbf{H}}\|(I_0\widetilde{\mathbf{q}})(t)\|_{\mathbf{H}} + \alpha\int_0^t \|\widetilde{\mathbf{q}}(s)\|^2_{\mathbf{H}}ds \\ &\le \delta\alpha\|\widetilde{\mathbf{q}}(t)\|^2_{\mathbf{H}} + \frac{\alpha}{4\delta}\left\|\int_0^t \widetilde{\mathbf{q}}(s)ds\right\|^2_{\mathbf{H}} + \alpha\int_0^t \|\widetilde{\mathbf{q}}(s)\|^2_{\mathbf{H}}ds \\ &\le \delta\alpha\|\widetilde{\mathbf{q}}(t)\|^2_{\mathbf{H}} + \frac{\alpha}{4\delta}\left\|\left(t\int_0^t |\widetilde{\mathbf{q}}(s)|^2ds\right)^{1/2}\right\|^2_{\mathbf{H}} + \alpha\int_0^t \|\widetilde{\mathbf{q}}(s)\|^2_{\mathbf{H}}ds \\ &= \delta\alpha\|\widetilde{\mathbf{q}}(t)\|^2_{\mathbf{H}} + \frac{\alpha}{4\delta}t\int_0^t \|\widetilde{\mathbf{q}}(s)\|^2_{\mathbf{H}}ds + \alpha\int_0^t \|\widetilde{\mathbf{q}}(s)\|^2_{\mathbf{H}}ds \end{aligned} \tag{3.6}$$

Now, adding (3.2), (3.3), and (3.4), and taking into account (3.5) and (3.6) with $\delta = \varepsilon/4$, we finally get

$$\|\widetilde{\theta}(t)\|^2_H + \|\widetilde{\theta}\|^2_{L^2(0,t;H)} + \|\widetilde{\mathbf{q}}(t)\|^2_{\mathbf{H}} + \|\widetilde{\mathbf{q}}\|^2_{L^2(0,t;\mathbf{H})} + \|(I_0\widetilde{\mathbf{q}})(t)\|^2_{\mathbf{H}} \le C\int_0^t \|\widetilde{\mathbf{q}}(s)\|^2_{\mathbf{H}}ds, \tag{3.7}$$

where C is a positive constant depending only on α, ε, and T. At this point it suffices to apply Gronwall lemma to deduce that $\widetilde{\theta} = 0$ and $\widetilde{\mathbf{q}} = 0$ a.e. in Q. Then by a comparison in (2.8) it follows that $\widetilde{\chi}$ is zero almost everywhere and the lemma is proved. $\square$

Now we address our attention to the existence of solutions for Problem $(\mathbf{P}_{\alpha\varepsilon})$. In order to prove that such solutions exist, we regularize the problem adding the term $-\mu\Delta\theta$ to the left hand side of (2.8) and by replacing the multivalued function γ by its Yosida approximation $\gamma_\mu := (\gamma^{-1} + \mu I_{\mathbb{R}})^{-1}$, $I_{\mathbb{R}}$ being the identity in $\mathbb{R}$ and μ being a positive parameter. Then we establish suitable estimates on the solutions and pass to limits as $\mu \searrow 0$. This procedure also provides a larger class of test functions to get the a priori estimates needed for the asymptotic analysis performed in sections 4 and 5. We recall that $\gamma_\mu : \mathbb{R} \longrightarrow \mathbb{R}$ is a Lipschitz continuous function with Lipschitz constant $1/\mu$ and that γ_μ also satisfies condition (2.1) with the same constant C_γ (cf. [2, Ch. II]). For the sake of clarity we start by stating a regularity result for the heat equation in its weak formulation. Its proof is standard and can be obtained, e.g., by using Galerkin method with a basis made of eigenfunctions of A.

Lemma 3.2. *Let $\mu \in]0,1[$ and let $F \in L^2(0,T;H) + L^1(0,T;V)$ and $u_0 \in V$. Then there exists a unique function u such that*

$$u \in L^\infty(0,T;V), \quad u' \in L^2(0,T;H) + L^1(0,T;V), \tag{3.8}$$

$$\nabla u \in L^2(0,T;\mathbf{V}), \quad Au = -\Delta u \in L^2(0,T;H), \tag{3.9}$$

$$u' - \mu\Delta u = F \qquad a.e. \ in \ Q, \tag{3.10}$$

$$u(0) = u_0 \qquad a.e. \ in \ \Omega. \tag{3.11}$$

The regularized problem will be solved by means of the Banach shrinking theorem. We need the following lemma.

Lemma 3.3. *Let $\mu \in]0,1[$, $\mathbf{p} \in L^2(0,T;\mathbf{V})$, and $X \in H^1(0,T;H)$. Then there exists a unique triplet $(\theta_\mu, \mathbf{q}_\mu, \chi_\mu)$ such that*

$$\theta_\mu \in L^\infty(0,T;V), \quad \theta_\mu' \in L^2(0,T;H) + L^1(0,T;V), \tag{3.12}$$

$$\nabla\theta_\mu \in L^2(0,T;\mathbf{V}), \quad A\theta_\mu = -\Delta\theta_\mu \in L^2(0,T;H), \tag{3.13}$$

$$\mathbf{q}_\mu \in H^1(0,T;\mathbf{V}), \tag{3.14}$$

$$\chi_\mu \in H^1(0,T;H^1(\Omega)), \tag{3.15}$$

$$\theta_\mu' - \mu\Delta\theta_\mu = f - X' - \operatorname{div}\mathbf{p} \qquad a.e. \ in \ Q, \tag{3.16}$$

$$\alpha\mathbf{q}_\mu' + \mathbf{q}_\mu = -\nabla\theta_\mu \qquad a.e. \ in \ Q, \tag{3.17}$$

$$\varepsilon\chi_\mu' + \chi_\mu = \gamma_\mu(\theta_\mu) \qquad a.e. \ in \ Q, \tag{3.18}$$

$$\theta_\mu(0) = \theta_0, \quad \mathbf{q}_\mu(0) = \mathbf{q}_0, \quad \chi_\mu(0) = \chi_0 \qquad a.e. \ in \ \Omega. \tag{3.19}$$

Proof. For convenience we omit the subscript μ. Since $X \in H^1(0,T;H)$ and $\mathbf{p} \in L^2(0,T;\mathbf{V})$, we have that $f - X' - \operatorname{div}\mathbf{p} \in L^2(0,T;H) + L^1(0,T;V)$, therefore thanks to (H6) and to Lemma 3.2 there exists a unique θ_μ satisfying (3.12)–(3.13), (3.16), and the first condition in (3.19). Now, since (H6) holds and γ_μ is a Lipschitz function, we have that $\gamma_\mu(\theta_\mu) \in$

$L^2(0,T;H^1(\Omega))$. Therefore thanks to (3.13) and (H6), we find $\mathbf{q}_\mu$ and χ_μ satisfying (3.14), (3.15), the ODEs (3.17)–(3.18), and the initial conditions in (3.19). $\square$

Now we can state and prove the result concerning the approximated problem.

Lemma 3.4. *Let* $\mu \in]0,1[$. *Then there exists a unique triplet* $(\theta_\mu, \mathbf{q}_\mu, \chi_\mu)$ *satisfying the following conditions.*

$$\theta_\mu \in L^\infty(0,T;V), \quad \theta_\mu' \in L^2(0,T;H) + L^1(0,T;V), \tag{3.20}$$

$$\nabla\theta_\mu \in L^2(0,T;\mathbf{V}), \quad \Delta\theta_\mu = -A\theta_\mu \in L^2(0,T;H), \tag{3.21}$$

$$\mathbf{q}_\mu \in H^1(0,T;\mathbf{V}), \tag{3.22}$$

$$\chi_\mu \in H^1(0,T;H^1(\Omega)), \tag{3.23}$$

$$(\theta_\mu + \chi_\mu)' - \mu\Delta\theta_\mu + \operatorname{div}\mathbf{q}_\mu = f \qquad a.e.\ in\ Q, \tag{3.24}$$

$$\alpha\mathbf{q}_\mu' + \mathbf{q}_\mu = -\nabla\theta_\mu \qquad a.e.\ in\ Q, \tag{3.25}$$

$$\varepsilon\chi_\mu' + \chi_\mu = \gamma_\mu(\theta_\mu) \qquad a.e.\ in\ Q, \tag{3.26}$$

$$\theta_\mu(0) = \theta_0, \quad \mathbf{q}_\mu(0) = \mathbf{q}_0, \quad \chi_\mu(0) = \chi_0 \qquad a.e.\ in\ \Omega. \tag{3.27}$$

Proof. Thanks to Lemma 3.3 we can define a nonlinear mapping

$$\mathcal{S} : L^2(0,T;\mathbf{V}) \times H^1(0,T;H) \longrightarrow L^2(0,T;\mathbf{V}) \times H^1(0,T;H)$$

that assignes to $(\mathbf{p}, X)$ the unique pair $(\mathbf{q}_\mu, \chi_\mu)$ satisfying (3.12)–(3.19). It is clear that a triplet $(\theta_\mu, \mathbf{q}_\mu, \chi_\mu)$ is a solution to problem (3.20)–(3.27) if and only if $(\mathbf{q}_\mu, \chi_\mu)$ is a fixed point of $\mathcal{S}$ and θ_μ satisfies (3.12)–(3.13) and (3.16)–(3.19). Here we endow $L^2(0,T;\mathbf{V})\times H^1(0,T;H)$ with the norm defined by

$$\|(\mathbf{r},\zeta)\|^2_{L^2(0,T;\mathbf{V})\times H^1(0,T;H)} := \|\mathbf{r}\|^2_{L^2(0,T;\mathbf{V})} + \|\zeta\|^2_{H^1(0,T;H)}$$

for $(\mathbf{r},\zeta) \in L^2(0,T;\mathbf{V}) \times H^1(0,T;H)$. We want to prove a contracting property of $\mathcal{S}$. For convenience we omit the subscript μ. Let $(\mathbf{p}_i, X_i) \in L^2(0,T;\mathbf{V}) \times H^1(0,T;H)$, $i = 1,2$, and let $(\theta_i, \mathbf{q}_i, \chi_i)$, $i = 1,2$, satisfying (3.12)–(3.19) with $\mathbf{p}$, X, θ_μ, $\mathbf{q}_\mu$, and χ_μ replaced respectively by $\mathbf{p}_i$, X_i, θ_i, $\mathbf{q}_i$, and χ_i, $i = 1,2$. Set $\widetilde{\mathbf{p}} := \mathbf{p}_1 - \mathbf{p}_2$, $\widetilde{X} := X_1 - X_2$, $\widetilde{\theta} := \theta_1 - \theta_2$, $\widetilde{\chi} := \chi_1 - \chi_2$, $\widetilde{\mathbf{q}} := \mathbf{q}_1 - \mathbf{q}_2$. Let us multiply the difference of equations (3.17) by $\widetilde{\mathbf{q}}$, and integrate over $\Omega\times]0,s[$, where $s \in [0,T]$. We find, thanks to (3.19) and to an application of Young inequality,

$$\frac{\alpha}{2}\|\widetilde{\mathbf{q}}(s)\|^2_{\mathbf{H}} + \frac{1}{2}\|\widetilde{\mathbf{q}}\|^2_{L^2(0,s;\mathbf{H})} \le \frac{1}{2}\|\nabla\widetilde{\theta}\|^2_{L^2(0,s;\mathbf{H})}. \tag{3.28}$$

Now let us take the divergence of equations (3.17), in order to obtain

$$\alpha\operatorname{div}\widetilde{\mathbf{q}}' + \operatorname{div}\widetilde{\mathbf{q}} = -\Delta\widetilde{\theta} \qquad \text{a.e. in } Q.$$

Multiplying this equation by $\operatorname{div}\widetilde{\mathbf{q}}$, integrating in time and space, and applying the Young inequality, we can infer that

$$\frac{\alpha}{2}\|\operatorname{div}\widetilde{\mathbf{q}}(s)\|^2_H + \frac{1}{2}\|\operatorname{div}\widetilde{\mathbf{q}}\|^2_{L^2(0,s;H)} \le \frac{1}{2}\|\Delta\widetilde{\theta}\|^2_{L^2(0,s;H)}. \tag{3.29}$$

Let us add inequalities (3.28) and (3.29) and integrate the resulting inequality over $]0,t[$, where $t \in [0,T]$. We infer that there exists a constant $C > 0$, depending only on α, such that

$$\|\widetilde{\mathbf{q}}\|^2_{L^2(0,t;\mathbf{V})} \le C \int_0^t (\|\nabla\widetilde{\theta}\|^2_{L^2(0,s;\mathbf{H})} + \|\Delta\widetilde{\theta}\|^2_{L^2(0,s;H)})ds. \tag{3.30}$$

Now multiply the difference of equations (3.16) by $-\Delta\widetilde{\theta}$ to obtain, after an integration in time and space,

$$\frac{1}{2}\|\nabla\widetilde{\theta}(s)\|^2_{\mathbf{H}} + \frac{\mu}{2}\|\Delta\widetilde{\theta}\|^2_{L^2(0,s;H)} \le \frac{1}{\mu}\|\widetilde{X}'\|^2_{L^2(0,s;H)} + \frac{1}{\mu}\|\mathrm{div}\,\widetilde{\mathbf{p}}\|^2_{L^2(0,s;\mathbf{H})}. \tag{3.31}$$

Therefore from (3.30) we get

$$\|\widetilde{\mathbf{q}}\|^2_{L^2(0,t;\mathbf{V})} \le C \int_0^t (\|\widetilde{X}'\|^2_{L^2(0,s;H)} + \|\mathrm{div}\,\widetilde{\mathbf{p}}\|^2_{L^2(0,s;\mathbf{H})})ds, \tag{3.32}$$

for some positive constant C which depends only on α and μ. Let us multiply the difference of equations (3.18) by $\widetilde{\chi}'$. Exploiting the Lipschitz continuity of γ_μ we deduce that

$$\begin{aligned}\frac{\varepsilon}{2}\|\widetilde{\chi}'\|^2_{L^2(0,t;H)} + \frac{1}{2}\|\widetilde{\chi}(t)\|^2_H &\le \frac{1}{2\varepsilon}\int_0^t\int_\Omega |\gamma_\mu(\theta_1(s)) - \gamma_\mu(\theta_2(s))|^2 ds \\ &\le \frac{1}{2\varepsilon}\int_0^t\int_\Omega \frac{1}{\mu^2}|\theta_1(s) - \theta_2(s)|^2 ds = \frac{1}{2\varepsilon\mu^2}\int_0^t \|\widetilde{\theta}(s)\|^2_H ds.\end{aligned} \tag{3.33}$$

Finally test the difference of equations (3.16) by the function $\widetilde{\theta}$. Integrating in time and applying Hölder and Young inequalities yields

$$\begin{aligned}&\frac{1}{2}\|\widetilde{\theta}(s)\|^2_H + \frac{\mu}{2}\|\nabla\widetilde{\theta}\|^2_{L^2(0,s;\mathbf{H})} \\ &\le \frac{1}{2}\|\widetilde{X}'\|^2_{L^2(0,s;H)} + \frac{1}{2\mu}\|\widetilde{\mathbf{p}}\|^2_{L^2(0,s;\mathbf{H})} + \frac{1}{2}\int_0^s \|\widetilde{\theta}(\tau)\|^2_H d\tau,\end{aligned} \tag{3.34}$$

therefore the Gronwall lemma yields

$$\frac{1}{2}\|\widetilde{\theta}(s)\|^2_H + \frac{\mu}{2}\|\nabla\widetilde{\theta}\|^2_{L^2(0,s;\mathbf{H})} \le C(\|\widetilde{X}'\|^2_{L^2(0,t;H)} + \|\widetilde{\mathbf{p}}\|^2_{L^2(0,t;\mathbf{H})}), \tag{3.35}$$

C being positive and depending only on μ. Hence we deduce from (3.33) and (3.35) that

$$\|\widetilde{\chi}\|^2_{H^1(0,t;H)} \le C \int_0^t (\|\widetilde{X}'\|^2_{L^2(0,s;H)} + \|\widetilde{\mathbf{p}}\|^2_{L^2(0,s;\mathbf{H})})ds \tag{3.36}$$

for some constant C that depends only on ε and μ. Collecting (3.32) and (3.36) we get that

$$\|(\widetilde{\mathbf{q}}, \widetilde{\chi})\|^2_{L^2(0,t;\mathbf{V})\times H^1(0,t;H)} \le C \int_0^t \|(\widetilde{\mathbf{p}}, \widetilde{X})\|^2_{L^2(0,s;\mathbf{V})\times H^1(0,s;H)} ds \tag{3.37}$$

where the constant C dependes only on α, ε, and μ. From estimate (3.37), arguing inductively, it is easy to infer that there exists a positive integer m such that $\mathcal{S}^m$ is a strict contraction. Therefore by Banach fixed point theorem $\mathcal{S}$ has a unique fixed point and the lemma is proved. ☐

In the following lemmas we collect some inequalities that we will use to get the a priori bounds needed to take the limit in problem (3.24)–(3.27) as $\mu \searrow 0$. We state them in a form that will also be useful for the subsequent section.

Lemma 3.5. *Let $(\theta_\mu, \mathbf{q}_\mu, \chi_\mu)$ satisfy (3.20)–(3.27). Then for every $t \in [0,T]$ we have*

$$\frac{1}{2}\|\theta_\mu(t)\|_H^2 + \frac{\mu}{2}\|\nabla\theta\|^2_{L^2(0,t;\mathbf{H})} + \frac{\alpha}{2}\|\mathbf{q}_\mu(t)\|^2_{\mathbf{H}} + \|\mathbf{q}_\mu\|^2_{L^2(0,t;\mathbf{H})}$$
$$\le \frac{\varepsilon}{2}\|\theta_0\|_H^2 + \frac{\varepsilon}{2}\|\mathbf{q}_0\|^2_{\mathbf{H}} + \int_0^t \|f(s)\|_H\|\theta_\mu(s)\|_H ds - \int_0^t\int_\Omega \chi_\mu'(s)\theta_\mu(s)ds. \tag{3.38}$$

Proof. It suffices to multiply equation (3.24) by θ and to test equation (3.25) by $\mathbf{q}$. Then add the resulting equations and integrate over $\Omega\times]0,t[$, $t \in [0,T]$. The lemma follows thanks to Hölder inequality, to (H6), and to a cancellation. ☐

Lemma 3.6. *Let $(\theta_\mu, \mathbf{q}_\mu, \chi_\mu)$ satisfy (3.20)–(3.27) and let $f_H \in W^{1,1}(0,T;H)$ and $f_V \in L^1(0,T;V)$ such that $f = f_H + f_V$. Then for all $t \in [0,T]$*

$$\frac{\varepsilon}{2}\|\nabla\theta_\mu(t)\|_H^2 + \mu\varepsilon\|\Delta\theta_\mu\|^2_{L^2(0,t;\mathbf{H})} - \varepsilon\int_0^t\int_\Omega \operatorname{div}\mathbf{q}_\mu(s)\Delta\theta_\mu(s)ds$$
$$\le \frac{\varepsilon}{2}\|\nabla\theta_0\|^2_{\mathbf{H}} + \varepsilon\|f_H(t)\|_H\|\Delta(I_0\theta_\mu(t))\|_H + \varepsilon\int_0^t \|f_H'(s)\|_H\|\Delta(I_0\theta_\mu(s))\|_H ds$$
$$+ \varepsilon\int_0^t \|\nabla f_V(s)\|_{\mathbf{H}}\|\nabla\theta_\mu(s)\|_{\mathbf{H}} ds - \int_0^t\int_\Omega \chi_\mu(s)\Delta\theta_\mu(s)ds \tag{3.39}$$

Proof. For simplicity we omit the subscript μ. First we multiply equation (3.20) by $-\varepsilon\Delta\theta$ and integrate by parts, respectively in time and space, the terms $-\varepsilon\int_0^t\int_\Omega f_H(s)\Delta\theta(s)ds$ and $-\varepsilon\int_0^t\int_\Omega f_V(s)\Delta\theta(s)ds$. Then let us multiply (3.26) by $-\Delta\theta$. This yields

$$-\varepsilon\int_0^t\int_\Omega \chi'(s)\Delta\theta(s)ds - \int_0^t\int_\Omega \chi(s)\Delta\theta(s)ds$$
$$= -\int_0^t\int_\Omega \gamma_\lambda(\theta)\Delta\theta = \int_0^t\int_\Omega \gamma_\mu'(\theta(s))|\nabla\theta(s)|^2 ds \ge 0, \tag{3.40}$$

due to the monotonicity of γ_μ. Finally add the two relations obtained and observe that there is a cancellation. Several applications of Hölder inequality and (H6) yield (3.39). ☐

Lemma 3.7. *Let $(\theta_\mu, \mathbf{q}_\mu, \chi_\mu)$ satisfy (3.20)–(3.27). Then, if $t \in [0,T]$ we have*

$$\|\nabla\theta_\mu\|^2_{L^2(0,t;\mathbf{H})} + \frac{\mu}{2}\|\Delta(I_0\theta_\mu)(t)\|_H^2 + \frac{1}{2}\|\Delta(I_0\theta_\mu)(t)\|_H^2$$
$$\le \|\theta_0 + \chi_0 + (I_0 f)(t)\|_H\|\Delta(I_0\theta_\mu)(t)\|_H + \int_0^t \|f(s)\|_H\|\Delta(I_0\theta_\mu)(s)\|_H ds$$
$$+ \int_0^t\int_\Omega \chi_\mu(s)\Delta\theta_\mu(s)ds - \alpha\int_0^t\int_\Omega \operatorname{div}\mathbf{q}_\mu(s)\Delta\theta_\mu(s)ds. \tag{3.41}$$

Proof. Let us integrate equation (3.24) in time from 0 to $s \in [0,t]$ and multiply the result by $-\Delta\theta$. Then integrate again in time over $]0,t[$ the equality obtained and estimate the term

$-\int_0^t\int_\Omega(\theta_0+\chi_0+(I_0f)(s))\Delta\theta(s)ds$ by means of an integration by parts. Now let us take the divergence of equation (3.25) we obtain (cf. (3.21))

$$\alpha\text{div}\,\mathbf{q}'+\text{div}\,\mathbf{q}=-\Delta\theta \qquad \text{a.e. in } Q. \tag{3.42}$$

Integrate in time (3.42) and multiply the resulting equality by $-\Delta\theta$. The lemma follows if we add the two relations we have obtained. □

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. Let $(\theta_\mu,\mathbf{q}_\mu,\chi_\mu)$ satisfy (3.20)–(3.27). Let us multiply the phase equation (3.26) by χ'. Using Hölder and Young inequalities and taking advantage of (2.1) (holding also for $\gamma_m u$) we find

$$\begin{aligned}\frac{\varepsilon}{2}\|\chi_\mu'\|^2_{L^2(0,t;H)}+\frac{1}{2}\|\chi_\mu(t)\|^2_H &\le \frac{1}{2}\|\chi_0\|^2_H+\frac{1}{2\varepsilon}\int_0^t\int_\Omega|\gamma_\lambda(\theta_\mu(s))|^2ds\\ &\le \frac{1}{2}\|\chi_0\|^2_H+\frac{C_\gamma^2}{\varepsilon}t|\Omega|+\frac{C_\gamma^2}{\varepsilon}\int_0^t\|\theta_\mu(s)\|^2_Hds.\end{aligned}\tag{3.43}$$

for every $t\in[0,T]$. Now multiply (3.38) by ε and add the resulting inequality to (3.43). Since

$$-\int_0^t\int_\Omega\chi_\mu'(s)\theta_\mu(s)ds\le\frac{\varepsilon}{4}\|\chi_\mu'\|^2_{L^2(0,t;H)}+\frac{1}{\varepsilon}\int_0^t\|\theta_\mu(s)\|^2_Hds,$$

we find, thanks to an application of a generalized version of Gronwall lemma (cf. [2, Lemma A.4, Lemma A.5, pp. 156-157]) a positive constant $C>0$, depending only on ε, α, C_γ, T, $|\Omega|$, θ_0, $\mathbf{q}_0$, χ_0, and f, such that

$$\|\theta_\mu\|^2_{L^\infty(0,t;H)}+\mu\|\nabla\theta_\mu\|^2_{L^2(0,t;\mathbf{H})}+\|\mathbf{q}_\mu\|^2_{L^\infty(0,t;\mathbf{H})}+\|\chi_\mu\|^2_{H^1(0,t;H)}\le C. \tag{3.44}$$

Now let us consider the three following inequalities:

$$\|f_H(t)\|_H\|\Delta(I_0\theta_\mu)(t)\|_H\le\sigma_1\|\Delta(I_0\theta_\mu)(t)\|^2_H+\frac{1}{4\sigma_1}\|f_H(t)\|^2_H, \tag{3.45}$$

$$\begin{aligned}\int_0^t\int_\Omega\chi_\mu(s)\Delta\theta_\mu(s)ds&=\int_\Omega\chi_\mu(t)\Delta(I_0\theta_\mu)(t)-\int_0^t\int_\Omega\chi_\mu'(s)\Delta(I_0\theta_\mu)(s)ds\\ \le\sigma_2\|\Delta(I_0\theta_\mu)(t)\|^2_H&+\frac{1}{4\sigma_2}\|\chi_\mu(t)\|^2_H+\|\chi_\mu'(s)\|^2_{L^2(0,t;H)}+\int_0^t\|\Delta(I_0\theta_\mu)(s)\|^2_Hds,\end{aligned}\tag{3.46}$$

(3.47)
$$\|\theta_0+\chi_0+(I_0f)(t)\|_H\|\Delta(I_0\theta_\mu)(t)\|_H\le\sigma_3\|\Delta(I_0\theta_\mu)(t)\|^2_H+\frac{1}{4\sigma_3}\|\theta_0+\chi_0+(I_0f)(t)\|^2_H.$$

Let us multiply (3.41) by ε/α and add it to (3.39). Taking into account of (3.45)–(3.47) with suitable values of σ_i, $i=1,2,3$ (which will depend on ε and α, but not on μ), and utilizing again the generalized Gronwall lemma, it is easy to infer that

$$\|\nabla\theta_\mu\|^2_{L^\infty(0,t;\mathbf{H})}+\mu\|\Delta\theta_\mu\|^2_{L^2(0,t;H)}+\mu\|\Delta(I_0\theta_\mu)\|^2_{L^\infty(0,t;H)}+\|\Delta(I_0\theta_\mu)\|^2_{L^\infty(0,t;H)}\le C, \tag{3.48}$$

where C is a positive constant independent of μ, but depending only on α, ε, T, Ω, and f. Finally if we integrate in time equation (3.42) and multiply it by $-\Delta\theta_\mu$ we find

$$\frac{\alpha}{2}\|\mathrm{div}\,\mathbf{q}_\mu\|^2_{L^2(0,t;\mathbf{H})} + \frac{1}{2}\|\mathrm{div}\,\mathbf{q}_\mu(t)\|^2_{\mathbf{H}} \le \frac{1}{2\alpha}\|\Delta(I_0\theta_\mu)\|^2_{L^2(0,t;H)}. \tag{3.49}$$

From estimate (3.44), (3.48), and (3.49), it follows that there exist θ, $\mathbf{q}$, χ, and ξ such that, letting $f_H \in W^{1,1}(0,T;H)$ and $f_V \in L^1(0,T;V)$ such that $f = f_H + f_V$, we have the following convergences, at least for some subsequence.

$$\theta_\mu \stackrel{*}{\rightharpoonup} \theta \qquad \text{in } L^\infty(0,T;V), \tag{3.50}$$

$$\mu\Delta\theta_\mu \to 0 \qquad \text{in } L^2(0,T;H) \tag{3.51}$$

$$\Delta(I_0\theta_{\alpha\varepsilon}) \stackrel{*}{\rightharpoonup} \Delta(I_0\theta) \qquad \text{in } L^\infty(0,T;H) \tag{3.52}$$

$$\chi_\mu \stackrel{*}{\rightharpoonup} \chi \qquad \text{in } H^1(0,T;H), \tag{3.53}$$

$$\mathbf{q}_\mu \stackrel{*}{\rightharpoonup} \mathbf{q} \qquad \text{in } L^\infty(0,T;\mathbf{V}) \cap W^{1,\infty}(0,T;\mathbf{H}), \tag{3.54}$$

$$\gamma_\mu(\theta_\mu) \stackrel{*}{\rightharpoonup} \xi \qquad \text{in } L^2(0,T;H), \tag{3.55}$$

$$\theta'_\mu - f_V \rightharpoonup \theta' - f_V \qquad \text{in } L^2(0,T;H), \tag{3.56}$$

as $\mu \searrow 0$. Therefore, taking the limit in (2.8)–(2.11) as $\mu \searrow 0$ we find that (2.8), (2.9), and (2.11) hold, and that

$$\varepsilon\chi' + \chi = \xi \qquad \text{a.e. in } Q. \tag{3.57}$$

It remain to prove (2.10), i.e. that $\xi \in \gamma(\theta)$ a.e. in Q. To verify this inclusion it suffices to prove that (see e.g. [1, Prop. 1.1, Ch. II])

$$\limsup_{\mu\searrow 0} \int_Q \gamma_\mu(\theta_\mu)\theta_\mu \le \int_Q \xi\theta. \tag{3.58}$$

By (3.50), (3.56), and by the Ascoli compactness theorem we have that

$$\theta_\mu - I_0 f_V \to \theta - I_0 f_V \qquad \text{in } C([0,T];H). \tag{3.59}$$

Thanks to (3.26) we can write

$$\int_Q \gamma_\mu(\theta_\mu)\theta_\mu = \int_0^t \Big(\gamma_\mu(\theta_\mu(s)), \theta_\mu(s) - (I_0 f_V)(s)\Big)_H ds + \int_0^t \Big(\gamma_\mu(\theta_\mu(s)), (I_0 f_V)(s)\Big)_H ds \tag{3.60}$$

therefore, convergences (3.59), (3.55), and equality (3.57) yield

$$\lim_{\mu\searrow 0} \int_Q \gamma_\mu(\theta_\mu)\theta_\mu = \int_0^t \Big(\xi(s), \theta(s) - (I_0 f_V)(s)\Big)_H ds + \int_0^t \Big(\xi(s), (I_0 f_V)(s)\Big)_H ds = \int_Q \xi\theta, \tag{3.61}$$

and (3.58) is proved. □

4. Uniform estimates

Within this section, taking advantage of the preparatory lemmas proved in the previous number, we establish the necessary a priori estimates which will allow us to perform the asymptotic analysis of our problem when the relaxation parameters α and ε tend to zero. For convenience we omit the subscript $\alpha\varepsilon$. We will assume that α is less or equal than ε. We use again the approximation used to prove the existence result for the relaxed problem. Of course it is not restrictive to assume $\alpha, \varepsilon < 1$.

Lemma 4.1. *Let* $(\theta_\mu, \mathbf{q}_\mu, \chi_\mu)$ *satisfy* (3.20)–(3.27) *with* $\alpha \le \varepsilon$. *Then there exists a constant* $C > 0$, *independent of* μ, α, *and* ε, *such that for all* $t \in [0, T]$

$$\begin{aligned} &\|\nabla(I_0\theta_\mu)\|^2_{L^\infty(0,t;\mathbf{H})} + \|\theta_\mu\|^2_{L^2(0,t;H)} + \mu\|\nabla(I_0\theta_\mu)\|^2_{L^\infty(0,t;\mathbf{H})} \\ &\quad + \varepsilon\|\theta_\mu\|^2_{L^\infty(0,t;H)} + \varepsilon\mu\|\nabla\theta_\mu\|^2_{L^2(0,t;\mathbf{H})} \le C. \end{aligned} \tag{4.1}$$

Proof. We will take advantage of Lemma 3.5. Let $t \in [0, T]$. In order to control the last term in (3.38), we test (3.26) by θ and integrate over $\Omega\times]0, t[$. We get

$$0 \le \varepsilon \int_0^t\int_\Omega \chi'_\mu(s)\theta_\mu(s)ds + \int_0^t\int_\Omega \chi_\mu(s)\theta_\mu(s)ds. \tag{4.2}$$

Now let us integrate in time (3.24) and multiply the result by θ. An integration over $\Omega\times]0, t[$ and some application of Hölder and Young inequalities yield

$$\begin{aligned} &\frac{1}{2}\|\theta_\mu\|^2_{L^2(0,t;H)} + \frac{\mu}{2}\|\nabla(I_0\theta_\mu)(t)\|^2_{\mathbf{H}} - \int_0^t\int_\Omega (I_0\mathbf{q}_\mu)(s) \cdot \nabla\theta_\mu(s)ds \\ &\le C + \int_0^t\int_\Omega \chi_\mu(s)\theta_\mu(s)ds, \end{aligned} \tag{4.3}$$

where C is a positive constant depending only on θ_0, χ_0, f, and T. At this point we integrate in time equation (3.25) and multiply the resulting equation by $\nabla\theta$. We infer, after an integration in time and space, the following identity:

$$\alpha\int_0^t\int_\Omega \mathbf{q}_\mu(s) \cdot \nabla\theta_\mu(s)ds + \int_0^t\int_\Omega (I_0\mathbf{q}_\mu)(s) \cdot \nabla\theta_\mu(s)ds = -\frac{1}{2}\|\nabla(I_0\theta_\mu)(t)\|^2_{\mathbf{H}}. \tag{4.4}$$

Multiplying inequality (3.38) by ε and add it to (4.2), (4.3), and (4.4), we get, observing that there are two cancellations,

$$\begin{aligned} &\frac{1}{2}\|\nabla(I_0\theta_\mu)(t)\|^2_{\mathbf{H}} + \frac{1}{2}\|\theta_\mu\|^2_{L^2(0,t;H)} + \frac{\mu}{2}\|\nabla(I_0\theta_\mu)(t)\|^2_{\mathbf{H}} + \frac{\varepsilon}{2}\|\theta_\mu(t)\|^2_H \\ &\quad + \frac{\varepsilon\mu}{2}\|\nabla\theta_\mu\|^2_{L^2(0,t;\mathbf{H})} + \frac{\varepsilon\alpha}{2}\|\mathbf{q}_\mu(t)\|^2_{\mathbf{H}} + \varepsilon\|\mathbf{q}_\mu\|^2_{L^2(0,t;\mathbf{H})} \\ &\le C + \varepsilon\int_0^t \|f(s)\|_H\|\theta_\mu(s)\|_H ds - \alpha\int_0^t\int_\Omega \mathbf{q}_\mu(s) \cdot \nabla\theta_\mu(s)ds, \end{aligned} \tag{4.5}$$

with C depending only on θ_0, $\mathbf{q}_0$, χ_0, f, and T. Now, exploiting (3.25), we have that

$$
\begin{aligned}
-\alpha\int_0^t\int_\Omega \mathbf{q}_\mu(s)\cdot\nabla\theta_\mu(s)ds &= \alpha\int_0^t\int_\Omega \mathbf{q}_\mu(s)\cdot\left(\alpha\mathbf{q}_\mu'(s)+\mathbf{q}_\mu(s)\right)ds \\
&= \frac{\alpha^2}{2}\|\mathbf{q}_\mu(t)\|^2_{\mathbf{H}} - \frac{\alpha^2}{2}\|\mathbf{q}_0\|^2_{\mathbf{H}} + \alpha\|\mathbf{q}_\mu\|^2_{L^2(0,t:\mathbf{H})} \\
&\leq \frac{\varepsilon\alpha}{2}\|\mathbf{q}_\mu(t)\|^2_{\mathbf{H}} - \frac{\alpha^2}{2}\|\mathbf{q}_0\|^2_{\mathbf{H}} + \varepsilon\|\mathbf{q}_\mu\|^2_{L^2(0,t:\mathbf{H})},
\end{aligned}
\tag{4.6}
$$

therefore (4.5), (4.6), and an application of Gronwall lemma yield (4.1). □

Lemma 4.2. *Let $(\theta_\mu, \mathbf{q}_\mu, \chi_\mu)$ satisfy (3.20)–(3.27) with $\alpha \leq \varepsilon$. Then there exists a constant $C > 0$, independent of μ, α, and ε, such that if $t \in [0,T]$, then*

$$
\begin{aligned}
&\varepsilon\|\nabla\theta_\mu\|^2_{L^\infty(0,t;\mathbf{H})} + \mu\varepsilon\|\Delta\theta_\mu\|^2_{L^2(0,t;H)} + \|\nabla\theta_\mu\|^2_{L^2(0,t;\mathbf{H})} \\
&+ \mu\|\Delta(I_0\theta_\mu)\|^2_{L^\infty(0,t;H)} + \|\Delta(I_0\theta_\mu)\|^2_{L^\infty(0,t;H)} \leq C.
\end{aligned}
\tag{4.7}
$$

Proof. We will prove (4.7) by using Lemma 3.6 and Lemma 3.7. Let us start by observing that, due to (3.25),

$$
\begin{aligned}
&(\varepsilon-\alpha)\int_0^t\int_\Omega \operatorname{div}\mathbf{q}_\mu(s)\Delta\theta_\mu(s)ds \\
&= (\alpha-\varepsilon)\int_0^t\int_\Omega \operatorname{div}\mathbf{q}_\mu\left(\alpha\operatorname{div}\mathbf{q}_\mu'(s)+\operatorname{div}\mathbf{q}_\mu(s)\right)ds \\
&= (\alpha-\varepsilon)\left[\frac{\alpha}{2}\|\operatorname{div}\mathbf{q}_\mu(t)\|^2_{\mathbf{H}} - \frac{\alpha}{2}\|\operatorname{div}\mathbf{q}_0\|^2_{\mathbf{H}} + \|\operatorname{div}\mathbf{q}_\mu\|^2_{L^2(0,t:\mathbf{H})}\right] \\
&\leq \frac{\alpha(\varepsilon-\alpha)}{2}\|\operatorname{div}\mathbf{q}_0\|^2_{\mathbf{H}} \leq \frac{1}{8}\|\operatorname{div}\mathbf{q}_0\|^2_{\mathbf{H}},
\end{aligned}
\tag{4.8}
$$

as $\alpha \leq \varepsilon$. Therefore adding (3.39) and (3.41) and taking into account (4.8) we find (4.7). □

Lemma 4.3. *Let $(\theta, \mathbf{q}, \chi)$ be a solution of Problem $(\mathbf{P}_{\alpha\varepsilon})$, with $\alpha \leq \varepsilon$. Then there exists a positive constant C, independent of μ, α, and ε, such that for all $t \in [0,T]$*

$$
\begin{aligned}
&\|\nabla(I_0\theta)\|^2_{L^\infty(0,t;\mathbf{H})} + \|\theta\|^2_{L^2(0,t;H)} + \varepsilon\|\theta\|^2_{L^\infty(0,t;H)} \\
&+ \varepsilon\|\nabla\theta\|^2_{L^\infty(0,t;\mathbf{H})} + \|\nabla\theta\|^2_{L^2(0,t;\mathbf{H})} + \|\Delta(I_0\theta)\|^2_{L^\infty(0,t;H)} \\
&+ \alpha\|\mathbf{q}\|^2_{L^\infty(0,t;\mathbf{H})} + \|\mathbf{q}\|^2_{L^2(0,t;\mathbf{H})} + \|\operatorname{div}(I_0\mathbf{q})\|^2_{L^2(0,t;H)} \\
&+ \|\chi\|^2_{L^2(0,t;H)} + \varepsilon\|\chi\|^2_{L^\infty(0,t;H)} \leq C.
\end{aligned}
\tag{4.9}
$$

Proof. Taking the limit as $\mu \searrow 0$ in (4.1) and in (4.7), by (3.50)–(3.52) and by the lower semicontinuity of the norm we get

$$
\begin{aligned}
&\|\nabla(I_0\theta)\|^2_{L^\infty(0,t;\mathbf{H})} + \|\theta\|^2_{L^2(0,t;H)} + \varepsilon\|\theta\|^2_{L^\infty(0,t;H)} \\
&+ \varepsilon\|\nabla\theta\|^2_{L^\infty(0,t;\mathbf{H})} + \|\nabla\theta\|^2_{L^2(0,t;\mathbf{H})} + \|\Delta(I_0\theta)\|^2_{L^\infty(0,t;H)} \leq C.
\end{aligned}
\tag{4.10}
$$

Then multiply (2.9) by $\mathbf{q}$ we find

$$
\frac{\alpha}{2}\|\mathbf{q}(t)\|^2_{\mathbf{H}} + \frac{1}{2}\|\mathbf{q}\|^2_{L^2(0,t;\mathbf{H})} \leq \frac{\alpha}{2}\|\mathbf{q}_0\|^2_{\mathbf{H}} + \frac{1}{2}\|\nabla\theta\|^2_{L^2(0,t;\mathbf{H})}.
\tag{4.11}
$$

Now multiply (2.10) by χ. We get, thanks to the sublinearity of γ,

$$\frac{\varepsilon}{2}\|\chi(t)\|_H^2 + \frac{1}{2}\|\chi\|_{L^2(0,t;H)}^2 \le \frac{\varepsilon}{2}\|\chi_0\|_H^2 + \frac{1}{2}\|\xi_\mu\|_{L^2(0,t;\mathbf{H})}^2 \le C(1 + \|\theta\|_{L^2(0,t;H)}^2). \tag{4.12}$$

Finally multiply equation (2.9) by $\mathrm{div}\,(I_0\mathbf{q})$ and get

$$\frac{\alpha}{2}\|\mathrm{div}\,(I_0\mathbf{q})(t)\|_H^2 + \frac{1}{2}\|\mathrm{div}\,(I_0\mathbf{q})\|_{L^2(0,t;H)}^2 \le \frac{\alpha}{2}\|\mathrm{div}\,(I_0\mathbf{q}_0)\|_H^2 + \frac{1}{2}\|\Delta(I_0\theta)\|_{L^2(0,t;H)}^2. \tag{4.13}$$

We can conclude collecting (4.10)–(4.13). □

5. Convergence to the Stefan problem

In this section we finally prove Theorem 2.2 about the asymptotic behaviour of the solution of $(\mathbf{P}_{\alpha\varepsilon})$ as $\alpha, \varepsilon \searrow 0$. The uniqueness of the solution of Problem $(\mathbf{P})$ follows a standard argument based on the monotonicity of γ, we refer, e.g., to [21, Ch. II]. Consider now the convergence result. Let $\xi_{\alpha\varepsilon} \in L^2(Q)$ such that $\xi_{\alpha\varepsilon} \in \gamma(\theta_{\alpha\varepsilon})$ almost everywhere in Q and

$$\varepsilon\chi_{\alpha\varepsilon}' + \chi_{\alpha\varepsilon} = \xi_{\alpha\varepsilon} \qquad \text{a.e. in } Q. \tag{5.1}$$

Thanks to Lemma 4.3 and to (2.1), we deduce that there exist four functions θ, $\mathbf{q}$, χ, and ξ such that, possibly for a subsequence, we have

$$\theta_{\alpha\varepsilon} \rightharpoonup \theta \qquad \text{in } L^2(0,T;V), \tag{5.2}$$

$$\Delta(I_0\theta_{\alpha\varepsilon}) \overset{*}{\rightharpoonup} \Delta(I_0\theta) \qquad \text{in } L^\infty(0,T;H) \tag{5.3}$$

$$\mathbf{q}_{\alpha\varepsilon} \rightharpoonup \mathbf{q} \qquad \text{in } L^2(0,T;\mathbf{H}), \tag{5.4}$$

$$I_0\mathbf{q}_{\alpha\varepsilon} \rightharpoonup I_0\mathbf{q} \qquad \text{in } L^2(0,T;\mathbf{V}), \tag{5.5}$$

$$\alpha I_0\mathbf{q}_{\alpha\varepsilon} \overset{*}{\rightharpoonup} 0 \qquad \text{in } L^\infty(0,T;\mathbf{V}), \tag{5.6}$$

$$\chi_{\alpha\varepsilon} \rightharpoonup \chi \qquad \text{in } L^2(0,T;H), \tag{5.7}$$

$$\xi_{\alpha\varepsilon} \rightharpoonup \xi \qquad \text{in } L^2(0,T;H), \tag{5.8}$$

$$(\theta_\mu + \chi_{\alpha\varepsilon})' \overset{*}{\rightharpoonup} (\theta + \chi)' \qquad \text{in } L^2(0,T;V'). \tag{5.9}$$

as α and ε goes to zero, with $\alpha \le \varepsilon$. Let us note that thanks to (4.9), the sequence $\varepsilon\chi_{\alpha\varepsilon}'$ is bounded in $L^2(Q)$. Therefore there is a function $\zeta \in L^2(Q)$ such that, at least for a subsequence, $\varepsilon\chi_{\alpha\varepsilon}' \rightharpoonup \zeta$ in $L^2(Q)$. On the other hand from (5.7) we infer that $\chi_{\alpha\varepsilon} \to \chi$ and $\chi_{\alpha\varepsilon}' \to \chi'$ in $\mathscr{D}'(Q)$. Thus $\varepsilon\chi_{\alpha\varepsilon}' \to 0$ in $\mathscr{D}'(Q)$. Hence by the uniqueness of the limit we deduce that $\zeta = 0$. The same argument applies to the sequence $\alpha\mathbf{q}_{\alpha\varepsilon}$, therefore we have that

$$\varepsilon\chi_{\alpha\varepsilon}' \rightharpoonup 0 \qquad \text{in } L^2(Q), \tag{5.10}$$

$$\alpha\mathbf{q}_{\alpha\varepsilon}' \rightharpoonup \mathbf{0} \qquad \text{in } L^2(0,T;\mathbf{H}). \tag{5.11}$$

Taking the limit in (2.8)–(2.11) we get

$$(\theta + \chi)' + B\theta = f \qquad \text{in } V', \text{ a.e. in }]0,T[, \tag{5.12}$$

$$\mathbf{q} = -\nabla\theta \qquad \text{a.e. in } Q, \tag{5.13}$$

$$\chi = \xi \qquad \text{a.e. in } Q, \tag{5.14}$$

and we recover the initial condition (2.17). From (5.12) and (5.13) we then deduce (2.15). It remains to show the Stefan condition (2.16). Let us note that, thanks to (5.1), we have

$$\int_Q \xi_{\alpha\varepsilon}\theta_{\alpha\varepsilon} = \int_Q \varepsilon\chi'_{\alpha\varepsilon}\theta_{\alpha\varepsilon} + \int_Q \chi_{\alpha\varepsilon}\theta_{\alpha\varepsilon}. \tag{5.15}$$

Now, using equation (2.8), we can write

$$\begin{aligned}\int_Q \varepsilon\chi'_{\alpha\varepsilon}\theta_{\alpha\varepsilon} &= \int_Q \varepsilon\left(f - \theta'_{\alpha\varepsilon} - \operatorname{div}\mathbf{q}_{\alpha\varepsilon}\right)\theta_{\alpha\varepsilon} \\ &= \int_Q \varepsilon f\theta_{\alpha\varepsilon} - \varepsilon\frac{1}{2}\|\theta_{\alpha\varepsilon}(t)\|_H^2 + \varepsilon\frac{1}{2}\|\theta_0\|_H^2 - \int_0^t {}_{\mathbf{V}'}\langle\varepsilon B\mathbf{q}_{\alpha\varepsilon}(s), \theta_{\alpha\varepsilon}(s)\rangle_{\mathbf{V}} ds \\ &\le \int_Q \varepsilon f\theta_{\alpha\varepsilon} + \varepsilon\frac{1}{2}\|\theta_0\|_H^2 - \int_0^t {}_{\mathbf{V}'}\langle\varepsilon B\mathbf{q}_{\alpha\varepsilon}(s), \theta_{\alpha\varepsilon}(s)\rangle_{\mathbf{V}} ds.\end{aligned} \tag{5.16}$$

Now, since (5.4) holds, we have that $B\mathbf{q}_{\alpha\varepsilon} \rightharpoonup B\mathbf{q}$ in $L^2(0,T;V')$ and therefore $\varepsilon B\mathbf{q}_{\alpha\varepsilon} \to 0$ in $L^2(0,T;V')$. Thus, thanks to (5.2),

$$\limsup_{\substack{\alpha,\varepsilon\searrow 0\\ \alpha\le\varepsilon}} \int_Q \varepsilon\chi'_{\alpha\varepsilon}\theta_{\alpha\varepsilon} \le 0. \tag{5.17}$$

On the other hand, using equation (2.8) integrated in time, we have that

$$\begin{aligned}\int_Q \chi_{\alpha\varepsilon}\theta_{\alpha\varepsilon} &= \int_Q \left(\theta_0 + \chi_0 + I_0 f - \theta_{\alpha\varepsilon} - \operatorname{div}\left(I_0\mathbf{q}_{\alpha\varepsilon}\right)\right)\theta_{\alpha\varepsilon} \\ &= \int_Q \left(\theta_0 + \chi_0 + I_0 f\right)\theta_{\alpha\varepsilon} - \|\theta_{\alpha\varepsilon}\|_{L^2(0,T;H)}^2 - \int_0^t {}_{V'}\langle B(I_0\mathbf{q}_{\alpha\varepsilon})(s), \theta_{\alpha\varepsilon}(s)\rangle_V ds.\end{aligned} \tag{5.18}$$

Now observe that from (5.5) and (5.4) it follows that $B(I_0\mathbf{q}_{\alpha\varepsilon}) \rightharpoonup B(I_0\mathbf{q})$ in $L^2(0,T;H)$ and that $B\mathbf{q}_{\alpha\varepsilon} \rightharpoonup B\mathbf{q}$ in $L^2(0,T;V')$, therefore by the Aubin-Lions compactness lemma (cf. [10, p. 58]) we find that $B(I_0\mathbf{q}_{\alpha\varepsilon}) \to B(I_0\mathbf{q})$ in $L^2(0,T;V')$. Hence, by (5.2) and (5.12), we find

$$\begin{aligned}\lim_{\substack{\alpha,\varepsilon\searrow 0\\ \alpha\le\varepsilon}} \int_Q \chi_{\alpha\varepsilon}\theta_{\alpha\varepsilon} &= \int_Q \left(\theta_0 + \chi_0 + I_0 f\right)\theta - \|\theta\|_{L^2(0,T;H)}^2 - \int_0^t {}_{V'}\langle B(I_0\mathbf{q})(s), \theta(s)\rangle_V ds \\ &= \int_0^t {}_{V'}\langle \chi(s), \theta(s)\rangle_V ds = \int_Q \chi\theta.\end{aligned} \tag{5.19}$$

Now collecting (5.18), (5.19), and (5.14), we find

$$\lim_{\substack{\alpha,\varepsilon\searrow 0\\ \alpha\le\varepsilon}} \int_Q \xi_{\alpha\varepsilon}\theta_{\alpha\varepsilon} \le \int_Q \xi\theta = \int_Q \chi\theta,$$

that gives (2.16) (cf. [2, Prop. 2.5, Ch. 2]). By the uniqueness of the solution to Problem (**P**), we infer that the entire sequences $(\theta_{\alpha\varepsilon}, \mathbf{q}_{\alpha\varepsilon}, \chi_{\alpha\varepsilon})$ converges, therefore Theorem 2.2 is proved.

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