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Holography, $SL(2,\mathbb{R})$ symmetry, Virasoro algebra and all that in Rindler spacetime

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Abstract: It is shown that it is possible to define quantum field theory of a massless scalar free field on the event horizon of a $2D$-Rindler spacetime. Free quantum field theory on the horizon enjoys diffeomorphism invariance and turns out to be unitarily and algebraically equivalent to the analogous theory of a scalar field propagating inside Rindler spacetime, nomatter the value of the mass of the field in the bulk. More precisely, there exists a unitary transformation that realizes the bulk-boundary correspondence upon an appropriate choice for Fock representation spaces. Secondly, the found correspondence is a subcase of an analogous algebraic correspondence described by injective $*$-homomorphisms of the abstract algebras of observables generated by abstract quantum free-field operators. These field operators are smeared with suitable test functions in the bulk and exact 1-forms on the horizon. In this sense the correspondence is independent from the chosen vacua. It is proven that, under that correspondence the "hidden" $SL(2,\mathbb{R})$ quantum symmetry found in a previous work gets a clear geometric meaning, it being associated with a group of diffeomorphisms of the horizon itself. Finally it is found that there is a possible enlargement of the quantum symmetry on the horizon to a quantum Virasoro symmetry associated with vector fields on the event horizon.

1 Introduction.

Many papers have tackled the problem of the microscopic origin of black-hole entropy. In that context holographic principle [1, 2, 3] arose by the idea that gravity near the horizon should be described by a low dimensional theory with a higher dimensional group of symmetry. After that, the correspondence between quantum field theories of different dimensions was conjectured by Maldacena in his celebrated work about AdS/Cft correspondence [4]. Using the machinery of string theory, he argued that there is a correspondence between quantum field theory in a, asymptotically AdS, $d+1$ dimensional spacetime – the "bulk"– and a conformal theory in a d dimensional manifold – the (conformal) "boundary" at spacelike infinity –. Notice that Maldacena's holographic principle encompasses asymptotically-AdS black hole spacetimes.

Afterwards Witten [5] described the above correspondence in terms of relations of observables of the two theories. Results arisen by those works were proven rigorously by Rehren for free quantum fields in a AdS background, exstablishing the existence of a correspondence between bulk observables and boundary observables (usually called algebraic holography) without employing string technology [6, 7]. A crucial point to explain the correspondence in Rehren's version is that, in AdS_{d+1} space, the conformal group which acts in d dimensions can be realized as the group of the isometries of the AdS_{d+1} bulk. In this way, from a pure geometric point of view, the nature of the bulk-boundary correspondence has a straightforward explanation. Finally, Strominger [8] proposed to enlarge the extent by showing that there is an analogous correspondence between dS space and a possible conformal field theory on its timelike boundary. In another work [9], making use of the optical metric, the near horizon limit of a massless theory in Schwarzschild-like spacetime has been intepreted as a theory in a asymptotic AdS spacetime giving rise to holographic properties.

In Schwarzschild spacetime embedded in Kruskal manifold, the proper boundary (dropping the boundary at infinity) is made of the event horizon of the black hole. With that notion of boundary a natural question arises "what about a bulk-boundary correspondence in a manifold containing a Schwarzschild-like black hole?" Obviously a first and intriguing problem, tackled in section 3, is the definition of a quantum field theory on a manifold – as an event horizon – whose metric is degenerate. To approach the general issue in the simplest version, we notice that two-dimensional Rindler spacetime embedded in Minkowski spacetime approximates the nontrivial part of the spacetime structure near a bifurcate horizon as that of a Schwartzschild black hole embedded in Kruskal spacetime. The remaining transverse manifold is not so relevant in several interesting quantum effects as Hawking's radiation and it seems that it can be dropped in the simplest approximation. In that context, we have argued in a recent work [10] that free quantum field theory in two-dimensional Rindler space presents a "hidden" $SL(2,\mathbb{R})$ symmetry. In other words the theory turns out to be invariant under a unitary representation of $SL(2,\mathbb{R})$ but such a quantum symmetry cannot be induced by the geometric background which enjoys a different group of isometries. $SL(2,\mathbb{R})$ is the group of symmetry of the zero-dimensional conformal field theory in the sense of [11], so, as for the case of AdS spacetime, it suggests the existence of a possible correspondence between quantum field theory in Rindler space and a conformal field theory defined on its event horizon. In fact, as it is shown within this work, the found hidden symmetry becomes manifest when one examines, after an appropriate definition, quantum field theory on the event horizon. That theory enjoys diffeomorphism invariance and the $SL(2,\mathbb{R})$ symmetry represents, in the quantum context, the geometric invariance of the theory under a little group of diffeomorphisms of the horizon. (We stress that invariance under isometries make not sense since the metric is degenerate.) We address to section 3 for thecnical details concerning the structure of quantum field theory on the horizon that, in a sense, is the limit of the bulk theory toward the orizon. We only say here that, generalizing the symplectic approach valid in the bulk, the theory can be built up by defining a suitable quantum field operator smeared with exact 1-forms, which are defined on the event horizon, to assure the invariance under diffeomorphisms; moreover the causal propagator (which involves bosonic commutation rules) is naturally defined in spite of the absence, shared with other holographic approaches in other contexts, of any natural evolution equation. The appearance of a manifest quantum $SL(2,\mathbb{R})$ symmetry on the horizon is only a part of the results established in this paper. In fact, the manifest $SL(2,\mathbb{R})$ symmetry on the horizon is a nothing but a simple result which follows form a holographic boundary-bulk correspondence established in this paper for $2D$ Rindler spacetime either in terms of unitary equivalences and in terms of $*$-algebra homomorphisms of free field observables. Investigation of quantum field theory on the horizon also reveals the existence of a larger manifest quantum symmetry given by a Virasoro algebra without central charge. (Technical investigation of such a larger symmetry is left to a forthcoming work). This operator algebra has a clear geometric interpretation in terms of vector fields defined on the horizon and generating the group of (orientation preserving) diffeomorphisms of the horizon itself. Some overlap with our results is present in the literature. Guido, Longo, Roberts and Verch [12] discussed in some detail the extent to which an algebraic QFT on a spacetime with a bifurcate Killing horizon induces a conformal QFT on that bifurcate Killing horizon. Along a similar theme, Schroer and Wiesbrock [13] have studied the relationship between QFTs on horizons and QFTs on the ambient spacetime. They even use the term "hidden symmetry" a sense similar as we do here and we done in [10]. In related follow-up works by Schroer [14] and by Schroer and Fassarella [15] the relation to holography and diffeomorphism covariance is also discussed.

This work is organized as follows: next section is devoted to review and briefly improve a few results established in [10] concerning hidden $SL(2,\mathbb{R})$ symmetry for a free quantum scalar field propagating in $2D$ Rindler spacetime. In the third section we present the main achievement of this work: We build up a quantum field theory for a massless scalar field on the horizon which, in a sense, is the limit toward the horizon of the analogous theory developed for a (also massive) field propagating in the bulk. Moreover we show that any free quantum field theories in the bulk and on the horizon are unitarily and algebraically equivalent (nomatter the value of the mass). In other words there exists a unitary transformation that realizes the bulk-boundary correspondence upon an appropriate choice for Fock representation spaces. In particular, the vacuum expectation values of observables of the free-field theory are invariant under the unitary equivalence. Actually, as we said, the found correspondence is valid in an algebraic sense too, i.e. it is described by injective $*$-homomorphisms of the abstract algebras of observables constructed by products of free-field operators smeared by suitable test functions/1-forms. In this sense the correspondence is independent from the chosen vacua. In the forth section we show that, as we expected, hidden $SL(2,\mathbb{R})$ invariance found in [10] becomes manifast on the horizon. But that is not the whole story: In the fifty section, we show that, the horizon manifest $SL(2,\mathbb{R})$ symmetry can be enlarged to a Virasoro one (technical investigation on this topic well be developed in a forthcoming work). In the last section we make some remarks and comments on the extension of our result to more complicated spacetimes containing a bifurcate event horizon.

2 Hidden $SL(2,\mathbb{R})$ symmetry near a bifurcate Killing horizon.

2.1.*Rindler space.* In [10] we have proven that quantum mechanics in a 2D-spacetime which approximates the spacetime near a bifurcate Killing horizon enjoys *hidden* $SL(2,\mathbb{R})$ invariance. This has been done by using and technically improving some general results on $SL(2,\mathbb{R})$ invariance in quantum mechanics [11]. Let us review part of the results achieved in [10] from the point of view of quantum field theory in curved spacetime (essentially in the formulation presented in [16, 17] but using $*$-algebras instead of C^*-algebras).

Remark. We illustrate the construction of quantum field theory in the considered background in some details because the general framework will be useful later in developing quantum field theory on a horizon and holography.

Consider a Schwarzschild-like metric

$$-A(r)dt \otimes dt + A^{-1}(r)dr \otimes dr + r^2 d\Sigma^2 \,, \tag{1}$$

where Σ denotes angular coordinates. Let $R > 0$ denote the radius of the black hole. As the horizon is bifurcate, $A'(R)/2 \neq 0$ and we can use the following approximation in the limit $r \to R$

$$-\kappa^2 y^2 dt \otimes dt + dy \otimes dy + R^2 d\Sigma^2 \,, \tag{2}$$

where $\kappa = A'(R)/2$ and $r = R + A'(R)y^2/4$. Dropping the angular part $R^2 d\Sigma$, the metric becomes that of the spacetime called *(right) Rindler wedge* $\mathbf{R}$ which is part of Minkowski spacetime:

$$g_{\mathbf{R}} := -\kappa^2 y^2 dt \otimes dt + dy \otimes dy \,, \tag{3}$$

with global coordinates $t \in (-\infty, +\infty)$, $y \in (0, +\infty)$. That spacetime is static [18] with respect to the timelike Killing vector ∂_t and spacelike surfaces at constant t. Later we shall make use of *Rindler light coordinates* $u, v \in \mathbb{R}$ which cover $\mathbf{R}$ and satisfy

$$u := t - \frac{\log(\kappa y)}{\kappa} \;,\; v := t + \frac{\log(\kappa y)}{\kappa} \quad \text{where} \quad t \in \mathbb{R}\,, y \in (0, +\infty)\,. \tag{4}$$

2.2.*One-particle Hilbert space.* As $\mathbf{R}$ is globally hyperbolic [18], in particular t-constant surfaces are Cauchy surfaces, quantum field theory can be implemented without difficulties [17]. There is no guarantee for the validity of the approach to quantum field theory for static spacetimes based on the quadratic form induced by the stress energy tensor presented in [17] since $-g_{\mathbf{R}}(\partial_t, \partial_t)$ has no positive lower bound. However we build up quantum field theory of a real scalar field ϕ with mass $m \geq 0$ propagating in $\mathbf{R}$ by following a more direct stationary-mode-decomposition approach. In fact, *a posteriori* it is possible to show that our procedure produces the same quantization as that in [17]. The Klein-Gordon equation reads

$$-\partial_t^2 \phi + \kappa^2 \left(y\partial_y y \partial_y - y^2 m^2 \right) \phi = 0. \tag{5}$$

If $m > 0$, $\mathcal{S}$ denotes the vector space of *real wavefunctions*, i.e., C^∞ real solutions ψ which have Cauchy data with compact support on a Cauchy surface. If $m = 0$, (5) reduces to

$$(\partial_t + \kappa y \partial_y)(-\partial_t + \kappa y \partial_y)\phi \quad (= (-\partial_t + \kappa y \partial_y)(\partial_t + \kappa y \partial_y)\phi) \;\; = 0 \tag{6}$$

and the space of real wavefunctions we want to consider is defined as $\mathcal{S} := \mathcal{S}_{\text{out}} + \mathcal{S}_{\text{in}}$ where $\mathcal{S}_{\text{out}}$ and $\mathcal{S}_{\text{in}}$ respectively are the space of real C^∞ solutions of $(\partial_t + \kappa y \partial_y)\psi = 0$ and $(-\partial_t + \kappa y \partial_y)\psi = 0$ with compactly-supported Cauchy data. The compactness requirement does not depend on the Cauchy surface [17]. There are solutions of (5) with $m = 0$ which do not belong to $\mathcal{S}$ in spite of having compactly-supported Cauchy data[1]. Define in $\mathcal{S} \times \mathcal{S}$ the following *symplectic form* [17], which does not depend on the used spacelike Cauchy surface Λ with induced measure $d\sigma$ and unit normal vector n pointing toward the future

$$\Omega(\psi, \psi') := \int_\Lambda \left(\psi' \nabla^\mu \psi - \psi \nabla^\mu \psi'\right) n_\mu \, d\sigma \,. \tag{7}$$

The definition is well-behaved for a pair of complex-valued wavefunctions too, and also if one of these has no compactly-supported Cauchy data. Equipped with these tools we can define the one-particle Hilbert space $\mathcal{H}$ associated with the Killing field ∂_t. To this end, consider the two classes of, $C^\infty(\mathbf{R}; \mathbb{C})$, ∂_t-stationary solutions of (5) where K_a is the usual Bessel-McDonald function:

$$\Phi_E(t, y) := \sqrt{\frac{2E \sinh(\pi E/\kappa)}{\pi^2 \kappa}} K_{iE/\kappa}(my) \frac{e^{-iEt}}{\sqrt{2E}} \qquad \text{with } E \in \mathbb{R}^+, \quad \text{if } m > 0\,, \tag{8}$$

$$\Phi_E^{\binom{\text{(in)}}{\text{(out)}}}(t, y) := \frac{e^{-iE(t \pm \kappa^{-1} \ln(\kappa y))}}{\sqrt{4\pi E}} \qquad \text{with } E \in \mathbb{R}^+, \quad \text{if } m = 0\,, \tag{9}$$

where $\mathbb{R}^+ := [0, +\infty)$. Modes $\Phi_E^{(\text{in})}$ are associated with particles crossing the future horizon at $t \to +\infty$, modes $\Phi_E^{(\text{out})}$ are associated with particles crossing the past horizon at $t \to -\infty$.
We have a pair of proposition whose proof is straightforward by using properties of K_{ia}, Fourier transform and Lebedev transform [19].

Proposition 2.1. (Completeness of modes). *If $m > 0$ and $\psi \in \mathcal{S}$, the function on $\mathbb{R}^+$*

$$E \mapsto \tilde{\psi}_+(E) := -i\Omega\left(\overline{\Phi_E}, \psi\right) \,. \tag{10}$$

satisfies $\tilde{\psi}_+(E) = \sqrt{E} g(E)$, and $\overline{\tilde{\psi}_+(E)} = -\sqrt{E} g(-E)$, for some $g \in \mathsf{S}(\mathbb{R}; \mathbb{C})$ (space of complex-valuated Schwartz' functions on the whole $\mathbb{R}$). Moreover,

$$\psi(t, y) = \int_0^{+\infty} \Phi_E(t, y) \tilde{\psi}_+(E) \, dE + \int_0^{+\infty} \overline{\Phi_E(t, y) \tilde{\psi}_+(E)} \, dE \quad \textit{for } (t, y) \in \mathbb{R} \times (0, +\infty)\,. \tag{11}$$

[1] With the notation used in (7), it is sufficient to fix compact-support Cauchy data $\psi, n^\mu \partial_\mu \psi$ on a t-constant Cauchy surface Λ such that $\int_\Lambda \partial_\mu \psi \, n^\mu d\sigma \neq 0$.

If $m = 0$ and $\psi \in \mathcal{S}$, the functions on $\mathbb{R}^+$ with $\alpha =$ in, out

$$E \mapsto \tilde{\psi}_+^{(\alpha)}(E) := -i\Omega\left(\overline{\Phi_E^{(\alpha)}}, \psi\right) , \tag{12}$$

satisfy $\tilde{\psi}_+^{(\alpha)}(E) = \sqrt{E} g^{(\alpha)}(E)$, $\overline{\tilde{\psi}_+^{(\alpha)}(E)} = \sqrt{E} g^{(\alpha)}(-E)$, where $g^{(\alpha)} \in \mathsf{S}(\mathbb{R};\mathbb{C})$. Moreover, for $(t, y) \in \mathbb{R} \times (0, +\infty)$

$$\psi(t,y) = \int_0^{+\infty} \sum_\alpha \Phi_E^{(\alpha)}(t,y)\tilde{\psi}_+^{(\alpha)}(E)\, dE + \int_0^{+\infty} \sum_\alpha \overline{\Phi_E^{(\alpha)}(t,y)\tilde{\psi}_+^{(\alpha)}(E)}\, dE\,. \tag{13}$$

Proposition 2.2. (Associated Hilbert spaces). *If $\psi \in \mathcal{S}$, define the one-to-one associated* positive-frequency wavefunction *for $m > 0$ and $m = 0$ respectively*

$$\psi_+(t,y) := \int_0^{+\infty} \Phi_E(t,y)\tilde{\psi}_+(E)\, dE \ , \ \psi_+(t,y) := \int_0^{+\infty} \sum_\alpha \Phi_E^{(\alpha)}(t,y)\tilde{\psi}_+^{(\alpha)}(E)\, dE\,. \tag{14}$$

With that definition, for $\psi_1, \psi_2 \in \mathcal{S}$ it results $\Omega(\psi_{1+}, \psi_{2+}) = 0$ whereas

$$\langle \psi_{1+}, \psi_{2+}\rangle := -i\Omega(\overline{\psi_{1+}}, \psi_{2+})\,, \tag{15}$$

is well-defined[2] *(at least on t-constant surfaces) and, respectively for $m > 0$ and $m = 0$*

$$\langle \psi_{1+}, \psi_{2+}\rangle = \int_0^{+\infty} \overline{\tilde{\psi}_{1+}(E)}\tilde{\psi}_{2+}(E)\, dE \ , \ \langle \psi_{1+}, \psi_{2+}\rangle = \int_0^{+\infty} \sum_\alpha \overline{\tilde{\psi}_{1+}^{(\alpha)}(E)}\tilde{\psi}_{2+}^{(\alpha)}(E)\, dE\,. \tag{16}$$

The one-particle Hilbert *space $\mathcal{H}$ is defined as the Hilbert completion of the space of finite complex linear combinations of functions ψ_+, $\psi \in \mathcal{S}$, equipped with the extension of the scalar product (15) to complex linear combinations of arguments. It results $\mathcal{H} \cong L^2(\mathbb{R}^+, dE)$ if $m > 0$ or, if $m = 0$, $\mathcal{H} \cong \mathcal{H}_{(in)} \oplus \mathcal{H}_{(out)}$ with $\mathcal{H}_{(\alpha)} \cong L^2(\mathbb{R}^+, dE)$, $\alpha =$ in,out.*

2.3. *Quantum field operators: Symplectic approach.* As usual, the whole quantum field is represented in the symmetrized Fock space $\mathfrak{F}(\mathcal{H})$ – that is $\cong \mathfrak{F}(\mathcal{H}_{(\text{in})}) \otimes \mathfrak{F}(\mathcal{H}_{(\text{out})})$ in the massless case – and referred to a vacuum state $\Psi_{\mathbf{R}}$ – namely $\Psi_{\mathbf{R}}^{(\text{in})} \otimes \Psi_{\mathbf{R}}^{(\text{out})}$ in the massless case – said the *Rindler vacuum state.* $\Psi_{\mathbf{R}}$ does not belong Hilbert space of Minkowski particles in the sense that quantum field theory in Rindler space and Minkowski one are not unitarily equivalent [17]. The *quantum field* $\Omega(\,\cdot, \hat{\phi})$ associated with the real scalar field ϕ in (5) is the linear map [17]

$$\mathcal{S} \ni \psi \mapsto \Omega(\psi, \hat{\phi}) \quad := \quad ia(\overline{\psi_+}) - ia^\dagger(\psi_+)\,, \tag{17}$$

where $\psi \in \mathcal{S}$ and $a(\overline{\psi_+})$ and $a^\dagger(\psi_+)$ respectively denote the annihilation (the conjugation being used just the get a liner map $\psi_+ \mapsto a(\overline{\psi_+})$) and construction operator associated with the one-particle state ψ_+. The right-hand side of (17) is an essentially-self-adjoint operator defined in

[2] At least for $m = 0$, positive/negative frequency wavefunctions cannot have Cauchy data with compact support due to known analiticity properties of Fourier transform.

the dense invariant subspace $\mathfrak{F}_0 \subset \mathfrak{F}(\mathcal{H})$ spanned by all states containing a finite arbitrarily large number of particles with states given by positive-frequency wavefunctions. (17) is formally equivalent via (11) to the non-rigorous but popular definition

$$\hat{\phi}(x) \text{ ``="} \int_0^{+\infty} \Phi_E(x) a_E + \overline{\Phi_E(x)} a_E^\dagger \, dE \,. \tag{18}$$

The given procedure can be generalized to any Klein-Gordon scalar field propagating in a (not necessarily static) globally hyperbolic spacetime provided a suitable vacuum state is given [17]. Let $\mathsf{D}(\mathbf{R};\mathbb{R})$ denote the space of real compactly-supported smooth functions in $\mathbf{R}$ and $J(A)$ the union of the *causal past* and *causal future* of a set $A \subset \mathbf{R}$. As $\mathbf{R}$ is globally hyperbolic there is a uniquely determined *causal propagator* $E : \mathsf{D}(\mathbf{R};\mathbb{R}) \to \mathcal{S}$ of the Klein-Gordon operator K of the field ϕ [17]. E enjoys the following properties. It is linear and surjective, $Ef \in J(\text{supp} f)$, $Ef = 0$ only if $f = Kg$ for some $g \in \mathsf{D}(\mathbf{R};\mathbb{R})$ and E satisfies for all $\psi \in \mathcal{S}$, $f, h \in \mathsf{D}(\mathbf{R};\mathbb{R})$

$$\int_{\mathbf{R}} \psi f \, d\mu_g = \Omega(Ef, \psi) \text{ and } \int_{\mathbf{R}} h(x)(Ef)(x) \, d\mu_g(x) = \Omega(Ef, Eh) \,, \tag{19}$$

μ_g being the measure induced by the metric in $\mathbf{R}$. (19) suggests to define [17] a quantum-field operator *smeared with functions* f *of* $\mathsf{D}(\mathbf{R};\mathbb{R})$ as the linear map

$$f \mapsto \hat{\phi}(f) := \Omega(Ef, \hat{\phi}) \,. \tag{20}$$

It is possible to smear the field operator by means of compactly-supported complex-valued functions, whose space will be denoted by $\mathsf{D}(\mathbf{R};\mathbb{C})$, simply by defining $\hat{\phi}(f + ih) := \hat{\phi}(h) + i\hat{\phi}(h)$ when $f, h \in \mathsf{D}(\mathbf{R};\mathbb{R})$. (20) entails [17]

$$[\hat{\phi}(f), \hat{\phi}(h)] = -iE(f,h) := -i \int_{\mathbf{R}} h(x)(Ef)(x) \, d\mu_g(x) \,, \tag{21}$$

that is the rigorous version of the formal identity $[\hat{\phi}(x), \hat{\phi}(x')] = -iE(x, x')$. As a further result [17] $[\hat{\phi}(f), \hat{\phi}(g)] = 0$ if the supports of f and g are *spatially separated*, that is $\text{supp} f \not\subset J(\text{supp} g)$ (which is equivalent to $\text{supp} g \not\subset J(\text{supp} f)$).
All that we said holds for $m \geq 0$. Let us specialize to the massless case giving further details. In Rindler light coordinates (4) the decomposition $\mathcal{S} = \mathcal{S}_{\text{in}} + \mathcal{S}_{\text{out}}$ (see 2.2) reads, if $\psi \in \mathcal{S}$, $\psi(u,v) = \psi(v) + \psi'(u)$ where $\psi \in \mathcal{S}_{\text{in}}$ and $\psi' \in \mathcal{S}_{\text{out}}$ are compactly supported. Trivial consequences are that ψ vanishes on the past horizon $v \to -\infty$, and ψ' vanishes on the future horizon $u \to +\infty$ (see 3.1) and $\Omega(\psi, \psi') = 0$. In the considered case

$$E = E_{\text{in}} + E_{\text{out}} \,, \tag{22}$$

where, in terms of bi-distributions interpreted as in (21),

$$E_{\text{in}}((u', v'), (u, v)) = \frac{1}{4} \text{sign}(v - v') \text{ whereas } E_{\text{out}}((u', v'), (u, v)) = \frac{1}{4} \text{sign}(u - u') \,. \tag{23}$$

The maps $f \mapsto E_{\text{in/out}}f$ from $\mathsf{D}(\mathbf{R},\mathbb{R})$ to, respectively, $\mathcal{S}_{\text{in/out}}$ are surjective and $E_{\text{in/out}}f = 0$ if and only if, respectively, $f = \partial_u g$ or $f = \partial_v g$ for some $g \in \mathsf{D}(\mathbf{R},\mathbb{R})$.
In the Fock space associated with Rindler vacuum $\Psi_{\mathbf{R}}$, we have the natural decomposition

$$\hat{\phi}(f) = \hat{\phi}_{\text{in}}(f) + \hat{\phi}_{\text{out}}(f) \quad \text{with} \quad \hat{\phi}_{\text{in/out}}(f) := \Omega(E_{\text{in/out}}f, \hat{\phi}) \tag{24}$$

and the the two kinds of field operators commute, i.e. $[\hat{\phi}_{\text{in}}(f), \hat{\phi}_{\text{out}}(g)] = i\Omega(E_{\text{in}}f, E_{\text{out}}g) = 0$.

2.4.$SL(2,\mathbb{R})$ *symmetry.* If $m > 0$ and thus $\mathcal{H} \cong L^2(\mathbb{R}^+, dE)$, the *one-particle (Rindler) Hamiltonian* H is the self-adjoint operator

$$(Hf)(E) := Ef(E) \text{ with domain } \mathcal{D}(H) = \{f \in L^2(\mathbb{R}^+, dE) \mid \int_0^{+\infty} E^2|f(E)|^2 dE < +\infty\}. \tag{25}$$

If $m = 0$ and thus $\mathcal{H} \cong L^2(\mathbb{R}^+, dE) \oplus L^2(\mathbb{R}^+, dE) \cong \mathbb{C}^2 \otimes L^2(\mathbb{R}^+, dE)$, the Hamiltonian H reads $I \otimes H'$, H' being the operator defined in the right-hand side of (25) referred to $L^2(\mathbb{R}^+, dE)$ and the identity operator I being referred to $\mathbb{C}^2$.
In [10] it was argued that the one-particle system enjoys invariance under a unitary representation of $SL(2,\mathbb{R})$ as consequence of the form of the spectrum of H which is $\sigma(H) = [0, +\infty)$ with no degeneracy for $m \neq 0$ and double degeneracy if $m = 0$. Let us state and prove rigorously some of the statements of [10] in a form which is relevant for the remaining part of this work. First of all one has to fix a real constant $\beta > 0$ [10], with the physical dimensions *energy*$^{-1}$, that is necessary for dimensional reasons in defining the relevant domain of operators as it will be clear from the proof of Theorem 2.1. *We assume to use the same value of* β *throughout this work.* Fix reals $k, m > 0$ and define the dense subspace $\mathcal{D}_k \subset \mathcal{H} \cong L^2(\mathbb{R}^+, dE)$ spanned by vectors:

$$Z_n^{(k)}(E) := \sqrt{\frac{\Gamma(n-k+1)}{E\,\Gamma(n+k)}}\, e^{-\beta E}\, (2\beta E)^k\, \mathsf{L}_{n-k}^{(2k-1)}(2\beta E)\;, \quad n = k, k+1, \ldots, \tag{26}$$

where $\mathsf{L}_p^{(\alpha)}$ are modified Laguerre's polynomials [20]. Notice that $\mathcal{D}_k \subset \mathcal{D}(H)$. Moreover, $\mathcal{D}_k$ is invariant under the linearly-independent symmetric operators defined on $\mathcal{D}_k$:

$$H_0 := H\restriction_{\mathcal{D}_k}\,, \quad D := -i\left(\frac{1}{2} + E\frac{d}{dE}\right)\,, \quad C := -\frac{d}{dE}E\frac{d}{dE} + \frac{(k-\frac{1}{2})^2}{E}\,. \tag{27}$$

which enjoy the commutation relations of the Lie algebra of $SL(2,\mathbb{R})$, $sl(2,\mathbb{R})$,

$$[iH_0, iD] = -iH_0\,, \quad [\,iC, iD] = iC\,, \quad [iH_0, iC] = -2iD\,. \tag{28}$$

iH_0, iC, iD are operatorial realizations of the basis elements of $sl(2,\mathbb{R})$

$$h = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; c = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}, \; d = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{29}$$

As a consequence, one expects that operators in (27) generate a strongly-continuous unitary representation of $SL(2,\mathbb{R})$. Let $\mathfrak{L}$ indicate the space of finite real linear combinations of operators in (27), let $\rho : sl(2,\mathbb{R}) \to i\mathfrak{L}$ be the unique Lie algebra isomorphism with $\rho(h) = iH_0$, $\rho(c) = iC$, $\rho(d) = iD$ and let $\widetilde{SL}(2,\mathbb{R})$ denote the universal covering of $SL(2,\mathbb{R})$.

Theorem 2.1. *The operators of $\mathfrak{L}$ are essentially self-adjoint, $\overline{H_0} = H$ in particular, and:*

(a) *$\mathcal{H}$ is irreducible under the unique unitary strongly-continuous representation of $\widetilde{SL}(2,\mathbb{R})$, $g \mapsto U(g) : \mathcal{H} \to \mathcal{H}$ such that $U(\exp(tx)) = e^{it\,\overline{\rho(x)}}$ for all $x \in sl(2,\mathbb{R})$, $t \in \mathbb{R}$. If (and only if) $k \in \{1/2, 1, 3/2, \ldots\}$, U is a representation of $SL(2,\mathbb{R})$ and is faithful only if $k = 1/2$. U does not depend on $\beta > 0$.*

(b) *$\{U(g)\}_{g\in\widetilde{SL}(2,\mathbb{R})}$ is a group of* symmetries *of the quantum system, that is, for every $t \in \mathbb{R}$ and $g \in \widetilde{SL}(2,\mathbb{R})$, there is $g(t) \in \widetilde{SL}(2,\mathbb{R})$ such that*

$$e^{itH}\, U(g)\, A\, U(g)^\dagger\, e^{-itH} = U(g(t))\, e^{itH}\, A\, e^{-itH}\, U(g(t))^\dagger\,, \tag{30}$$

for every observable (i.e., self-adjoint operator) A. If $g = \exp(uh + vc + wd)$, with $u, v, w \in \mathbb{R}$,

$$g(t) = \exp((u + tw + t^2 v)h + (w + 2tv)d + vc)\,. \tag{31}$$

(c) *For every $t \in \mathbb{R}$, consider the linearly independent elements of $\mathfrak{L}$*

$$H_0(t) := H_0\,, \quad D(t) := D + tH\,, \quad C(t) := C + 2tD + t^2 H\,. \tag{32}$$

The time-dependent observables generated by those operators are constants of motion, *i.e.*

$$e^{itH}\, \overline{uH_0(t) + vC(t) + wD(t)}\, e^{-itH} \;=\; \overline{uH_0 + vC + wD}\,, \quad \text{for every } t, u, v, w \in \mathbb{R}\,. \tag{33}$$

Proof. **(a)** $\{Z_n^{(k)}\}_{n=k,k+1,\ldots}$ (26) is a Hilbert base of eigenvectors of the operator $K = \frac{1}{2}(\beta H_0 + \beta^{-1}C)$ [10]. Moreover $X = \beta^2 H_0^2 + \beta^{-2}C^2 + 2D^2$ is essentially selfadjoint in $\mathcal{D}_k$ because $\{Z_n^{(k)}\}_{n=k,k+1,\ldots}$ are analytic vectors for X since $X = 4K^2 + cI$ from (28) where $c \in \mathbb{R}$. Since X is essentially self-adjoint, general results due to Nelson (Theorem 5.2, Corollary 9.1, Lemma 9.1 and Lemma 5.1 in [21]) imply that the operators in $i\mathfrak{L}$, are essentially self-adjoint on $\mathcal{D}_k$ moreover they imply the existence and uniqueness of a unitary representation on $\mathcal{H}$ of the unique simply connected group with Lie algebra given by the space generated by $\beta h, \beta^{-1}c, 2d$ (that is $\widetilde{SL}(2,\mathbb{R})$) such that $\frac{d}{dt}|_{t=0}U(\exp(tx)) = i\overline{\rho(x)}$ for all $x \in sl(2,\mathbb{R})$. The derivative in the left-hand side is evaluated in the strong operator topology sense on a suitable subspace G (Gårding space [21]) and gives a restriction of the Stone generator of the strongly continuous unitary one-parameter subgroup $\mathbb{R} \ni t \mapsto U(\exp(tx)))$. As G contains a dense set of analytical vectors for the elements of $\mathfrak{L}$ [21], $-i\frac{d}{dt}|_{t=0}U(\exp(tx))$ is essentially self-adjoint and thus its closure coincides with the usual Stone generator and $U(\exp(tx)) = e^{it\,\overline{\rho(x)}}$. As $\mathcal{D}_k \subset \mathcal{D}(H)$, the unique self-adjoint extension of H_0, $\overline{H_0}$, must coincide with H itself. Suppose that P is the orthogonal projector onto an invariant subspace for each $U(g)$. P must commute with $e^{it\overline{K}}$ in particular. Using Stone's theorem and the fact that the spectrum of K is not degenerate, one has that (in strong operator topology sense) $P = \sum_{i\in M} Z_n^{(k)} \langle Z_n^{(k)}, \cdot \rangle$ where $M \subset \mathbb{N}$. Similarly P must commute with every element of $\mathcal{L} + i\mathcal{L}$, $A_\pm := \mp iD + \frac{1}{2}(\beta H_0 - \beta^{-1}C)$ in particular. Using the fact that, for every $m, n \in \mathbb{N}$ with $m > 0$, $Z_{n+1}^{(k)} = c_n A_+ Z_n^{(k)}$ and $Z_{m-1}^{(k)} = c_m A_- Z_m^{(k)}$ for some reals $c_n, c_m > 0$ [10], one proves that $M = \mathbb{N}$, that is $P = I$ and so the representation is irreducible. The proof of the fact that the representation of $\widetilde{SL}(2,\mathbb{R})$ reduces to a representation of $SL(2,\mathbb{R})$ iff $k \in \{1/2, 1, 3/2, \dots\}$ and that the representation is faithful only for $k = 1/2$ is based on the representation of the subgroup $\{\exp t(h+c)\}_{t\in\mathbb{R}} \subset \widetilde{SL}(2,\mathbb{R})$ which is isomorphic to $SO(2)$ and is responsible for the fact that $\widetilde{SL}(2,\mathbb{R})$ is multiply connected. The proof has been sketched in section 6.2 and the footnote 4 in [10]. The self-adjoint extensions of the elements in $\mathcal{L}$ do not depend on the value of $\beta > 0$ – and thus it happens for the representation U itself since every $g \in \widetilde{SL}(2,\mathbb{R})$ is the finite product of elements of some one-parameter subgroups – because, if $\beta' \neq \beta$, there is a subspace $\mathcal{D}$ containing, with obvious notation, both $\mathcal{D}_k^{(\beta)}$ and $\mathcal{D}_k^{(\beta')}$ where each element of $\mathcal{L}$ (which is essentially self adjoint on both $\mathcal{D}_k^{(\beta)}$ and $\mathcal{D}_k^{(\beta')}$) determines the same symmetric extension nomater if one starts by $\mathcal{D}_k^{(\beta)}$ or $\mathcal{D}_k^{(\beta')}$. That extension is essentially self-adjoint since $\mathcal{D}_k^{(\beta)}$ is a dense set of analytic vectors in $\mathcal{D}$. To prove **(b)** and **(c)** notice that $e^{itH} = U(\exp(th))$. So, if $g \in \widetilde{SL}(2,\mathbb{R})$, $g(t) := \exp(th)g(\exp(th))^{-1}$ fulfils (30) by application of the representation U. Define $h(t) := \rho^{-1}(H_0(t))$, $c(t) := \rho^{-1}(C(t))$, $d(t) := \rho^{-1}(D(t))$. These matrices satisfies the commutation rules (28) for every t. Using (28) and uniqueness theorems for matrix-valued differential equations one gets, for $s, t, u, v, w \in \mathbb{R}$,

$$\exp\{th\} \exp\{s(uh(t) + vc(t) + wd(t))\}(\exp\{th\})^{-1} = exp\{s(uh + vc + wd)\}\ ,$$

which is (31) if $s = 1$. Applying U on both sides one gets $e^{itH} e^{is\,\overline{uH_0(t)+vC(t)+wD(t)}} e^{-itH} = e^{is\,\overline{uH_0+vC+wD}}$, which is equivalent to $e^{is \exp\{itH\}\overline{uH_0(t)+vC(t)+wD(t)}\exp\{-itH\}} = e^{is\,\overline{uH_0+vC+wD}}$. Stone's theorem entails (33) by strongly differentiating both sides in s. $\Box$

The generalization to the case $m = 0$ is trivial: The theorem holds true separately in each space $L^2(\mathbb{R}^+, dE)$ of $\mathcal{H} \cong L^2(\mathbb{R}^+, dE) \oplus L^2(\mathbb{R}^+, dE)$.

Remarks. **(1)** From now on we assume to work in *Heisenberg representation.* Within this picture, by (33), H, $\overline{C}$, $\overline{D}$ coincide with the Heisenberg evolution of, respectively, $\overline{H_0(t)}$, $\overline{C(t)}$, $\overline{D(t)}$ at time t. Moreover, in this picture, $e^{-i\tau H}\psi_+$ must not seen as the time evolution (at time τ) of the state ψ_+ (given at time 0), but it must considered as a different state at all. This turns out to be in accordance with the relationship between states and wavefunctions (see 2.5): ψ and $\alpha_{-\tau}^{(\partial_t)}(\psi)$ are two different wavefunctions. This point of view will be useful shortly in a context where time evolution makes no sense at all.
(2) The found $SL(2,\mathbb{R})$ symmetry is only due to the shape of spectrum of $\sigma(H)$ which is $[0, +\infty)$. The absence of degeneracy implies that the representation is irreducible. From a physical point of view, invariance under the conformal group $SL(2,\mathbb{R})$ could look very unexpected if $m > 0$ since the theory admits the scale m. However, that scale does not affect the spectrum of H.

In physical terms this is due to the gravitational energy which is encompassed by H itself since Rindler frame represents the spacetime experienced by an accelerated observer.
(3) It is clear that the found $SL(2,\mathbb{R})$ symmetry can straightforwardly be extended to the free-field quantization by defining multi-particle operators generated by H, C, D.
(4) Generators iH_0, iD, *differently* from iH_0 and iC, define a basis of the Lie algebra of a subgroup of $SL(2,\mathbb{R})$, $SL_+^{\triangle}(2,\mathbb{R})$, made of real 2×2 upper triangular matrices with unitary determinant and positive trace. (30) holds for $U(g)$, $g \in SL_+^{\triangle}(2,\mathbb{R})$, giving rise to another smaller symmetry of the system. The subgroup generated by iH_0 trivially enjoys the same fact.

2.5.*Hidden and manifest symmetries.* A differentiable group of *geometric* symmetries of a classical Klein-Gordon field in $\mathbf{R}$ (however everything we say can be extended to any globally hyperbolic spacetime along the procedures presented in [17]) is defined as follows. Take a differentiable, locally-bijective, representation, $G \ni g \mapsto d_g$, of a connected Lie group, G, where $d_g : \mathbf{R} \to \mathbf{R}$ are isometries. The representation automatically induces a group of transformations $\{\alpha_g\}_{g\in G}$ of scalar fields $f : \mathbf{R} \to \mathbb{R}$ (or $\mathbb{C}$), with $(\alpha_g(f))(x) := f\left(d_{g^{-1}}(x)\right)$. As d_g are (orientation-preserving) isometries, α_g define *geometric symmetries* of the field in the sense that they transform elements of $\mathcal{S}$ into elements of $\mathcal{S}$ not affecting the symplectic form. If quantization is implemented, solutions of the equation of motion in $\mathcal{S}$ are associated with one-particle quantum states through the decompositions (11),(13). Consider a group of quantum symmetries in the sense of (30), described by a strongly-continuous representation of a Lie group G in terms of unitary operators $\{U(g)\}_{g\in G}$. In this picture, the one-parameter unitary group generated by the Hamiltonian is assumed to be a subgroup of $\{U(g)\}_{g\in G}$. If the symmetry "does not depend on time", i.e., $g(t) = g$ in (30), that assumption can be dropped or, equivalently, the subgroup generated by the Hamiltonian can be considered in the center of $\{U(g)\}_{g\in G}$ (i.e. it commutes with the other elements of the group). If $\{U(g)\}_{g\in G}$ is related, by means of (11), (13) and (14), to a group of geometric symmetries $\{\alpha_g\}_{g\in G}$, that is $\widetilde{(\alpha_g(\psi))}_+ = U(g)\tilde{\psi}_+$ for all $g \in G$ and $\psi \in \mathcal{S}$, we say that the symmetry is *manifest*. Otherwise we say that the symmetry is *hidden*. In Rindler space, the quantum symmetry group $\{e^{i\tau H}\}_{\tau\in\mathbb{R}}$ give rise to manifest symmetry because it is associated with the geometric group of symmetries $\{\alpha_\tau\}_{\tau\in\mathbb{R}}$, induced by the one-parameter group of isometries generated by the Killing vector ∂_t. The extent changes dramatically if considering the whole $\widetilde{SL}(2,\mathbb{R})$ symmetry. The space of Killing fields of $\mathbf{R}$ has a basis $\partial_t, \partial_X, \partial_T$ with $\{\partial_T, \partial_X\} = 0$, $\{\partial_T, \partial_t\} = \partial_X$ and $\{\partial_X, \partial_t\} = \partial_T$. ($X$ and T are the spatial and temporal coordinate of a Minkowski frame with $\partial_t = X\partial_T + T\partial_X$.) It is trivially proven that no Killing field $a\partial_t + b\partial_X + c\partial_T$ enjoys, with respect to ∂_t, the commutation rule that D enjoys with respect to H_0 in (28) so that no Killing field corresponding to C makes sense. Summarizing, $\mathbf{R}$ cannot support isometry representations of $SL(2,\mathbb{R})$ (or $\widetilde{SL}(2,\mathbb{R})$) or the subgroup $SL_+^{\triangle}(2,\mathbb{R})$ generated by H_0, D. Hence the whole $SL(2,\mathbb{R})$ symmetry and that associated with D are *hidden*.

3 Conformal field on the horizon.

3.1.*Restriction to horizons.* In [10], a similar analysis is performed for $\mathbf{AdS}_2$ spacetime since

the spectrum of the Hamiltonian of a particle has the same structure as that in Rindler space. However as a remarkable difference $SL(2,\mathbb{R})$ is a *manifest* symmetry of a quantum particle moving in $\mathbf{AdS}_2$ because $SL(2,\mathbb{R})$ admits a representation in terms of $\mathbf{AdS}_2$ isometries.
Coming back to Rindler space viewed as a (open) submanifold of Minkowski spacetime, a natural question arises: "*Regardless the found $SL(2,\mathbb{R})$ symmetry is hidden, does it become manifest if one considers quantum field theory in an appropriate subregion of* $\mathbf{R}\cup\partial\mathbf{R}$*?*"
We shall see that investigation on this natural question has several implications with holography because it naturally leads to the formulation of a quantum field theory on the horizon which is algebraically and unitarily related with that formulated in the bulk, but also it suggests that the symmetry of the theory is greater than $SL(2,\mathbb{R})$ it being described by a *Virasoro algebra*.
It is clear from 2.5 that the only region which could give a positive answer to the question is the boundary $\partial\mathbf{R}$ of Rindler wedge, i.e. a *bifurcate event horizon* made of three disjoint parts $\mathbf{F}\cup\mathbf{P}\cup\mathbf{S}$. $\mathbf{S}$ (a point in our $2D$ case) is the spacelike submanifold of Minkowski spacetime where the limit of the Killing field ∂_t vanishes whereas the lightlike submanifold of Minkowski spacetime $\mathbf{F}$ and $\mathbf{P}$ (the former in the causal future of the latter) are the *future* and the *past* horizon respectively, where the limit of ∂_t becomes lightlike but *not* vanishes. Since the induced metric on $\mathbf{F}\cup\mathbf{P}$ is degenerate, the diffeomorphisms of $\mathbf{F}\cup\mathbf{P}$ can be viewed as isometries and the question about a possible *manifest* $SL(2,\mathbb{R})$ symmetry on the horizon must be interpreted in that sense: The unitary representation has to be associated with a group of diffeomorphisms induced by a Lie algebra of vector fields defined on the horizon.
To go on, let us investigate the limit of wavefunctions when the horizon is approached by Rindler-time evolution. To this end, consider the Rindler light coordinates (4). $\mathbf{F}$ is represented by $u\to+\infty$, $v\in\mathbb{R}$ whereas $\mathbf{P}$ is given by $v\to-\infty$, $u\in\mathbb{R}$. Coordinates u,v actually cover the Rindler space only, but, separately, the limit of v is well defined on the lightlike submanifold $\mathbf{F}$ and the limit of u is well defined on the submanifold $\mathbf{P}$ and they define well-behaved global coordinate frames on these submanifolds[3]. This can be proven by passing to Minkowski light coordinates $U:=T-X, V:=T+X$ which satisfy $V=e^{\kappa v}$, $U=e^{-\kappa u}$ in $\mathbf{R}$. So, from now v and u are also interpreted as coordinates on $\mathbf{F}$ and $\mathbf{P}$ respectively. We have the following remarkable technical result (where, if $a\in\mathbb{C}$, "$a+c.c.$" means "$a+$ *complex conjugation of* a")

Proposition 3.1. *Take $\psi\in\mathcal{S}$, with associated (Rindler) positive frequency parts $\tilde{\psi}_+$ or $\tilde{\psi}_+^{(\alpha)}$ as in (11) and (13) and consider the evolution of ψ in the whole Minkowski spacetime. In coordinate $v\in\mathbb{R}$, the restriction of ψ to $\mathbf{F}$ reads respectively for $m>0$ and $m=0$,*

$$\psi(v)=\int_{\mathbb{R}^+}\frac{e^{-iEv}}{\sqrt{4\pi E}}N_{m,\kappa}(E)\tilde{\psi}_+(E)\,dE+c.c.\;,\;\;\psi(v)=\int_{\mathbb{R}^+}\frac{e^{-iEv}}{\sqrt{4\pi E}}\tilde{\psi}_+^{(\mathrm{in})}(E)\,dE+c.c. \qquad (34)$$

In coordinate $u\in\mathbb{R}$, the restriction of ψ to $\mathbf{P}$ reads respectively for $m>0$ and $m=0$,

$$\psi(u)=\int_{\mathbb{R}^+}\frac{e^{-iEu}}{\sqrt{4\pi E}}\overline{N_{m,\kappa}(E)}\tilde{\psi}_+(E)\,dE+c.c.\;,\;\;\psi(u)=\int_{\mathbb{R}^+}\frac{e^{-iEu}}{\sqrt{4\pi E}}\tilde{\psi}_+^{(\mathrm{out})}(E)\,dE+c.c.\,. \qquad (35)$$

[3]It holds in the $2D$ case. For greater dimension, v (resp. u) together with other "transverse" coordinates defines global coordinates on $\mathbf{F}$ (resp. $\mathbf{P}$) as well.

The function $N_{m,\kappa}$ (that is restricted to $\mathbb{R}^+$ in (34) and (35)) can be defined on the whole $\mathbb{R}$ as

$$N_{m,\kappa}(E) := e^{-i\frac{E}{\kappa}\log\frac{m}{2\kappa}}\,\mathrm{sign}(E)\,\Gamma\left(\frac{iE}{\kappa}\right)\sqrt{\frac{E}{\kappa\pi}\sinh\frac{\pi E}{\kappa}}\,. \tag{36}$$

It belongs to $C^\infty(\mathbb{R})$ and satisfies $|N_{m,\kappa}(E)| = 1$ and $\overline{N_{m,\kappa}(E)} = -N_{m,\kappa}(-E)$ for all $E \in \mathbb{R}$.

Proof. As $t = 0$ is part of a Cauchy surface in Minkowski spacetime, ψ can uniquely be extended into a smooth solution of Klein Gordon equation in Minkowski spacetime, therefore it makes sense to consider its restriction to $\mathbf{P}$ or $\mathbf{F}$. As ψ is smooth, those restrictions can be computed by taking the limit of the function represented in light Rindler coordinates. First consider the case $m = 0$ and $u \to \infty$. One has $\Phi^{(\mathrm{out})}_E(t(u,v), y(u,v)) = \frac{e^{-iEu}}{\sqrt{4\pi E}}$ and $\Phi^{(\mathrm{in})}_E(t(u,v), y(u,v)) = \frac{e^{-iEv}}{\sqrt{4\pi E}}$. Insert these functions in (13) and extend each integrations on the whole $\mathbb{R}$ axis by defining $\tilde{\psi}^{(\alpha)}_+(E) = 0$ for $E \leq 0$. Using the properties of $\tilde{\psi}^{(\alpha)}_+$ stated in Proposition 2.1 before (13) one sees that $\psi(u,v)$ can be decomposed as a sum of two functions (one in the variable u and the other in the variable v) which are the real part of the Fourier transform of a couple of $L^1(\mathbb{R})$ functions. Taking the limit $u \to +\infty$ the function containing only modes *out* vanishes as a consequence of Rieman-Lebesgue lemma and (34) with $m = 0$ arises. The case $m = 0$ and $v \to -\infty$ is strongly analogous. The case $m > 0$ is based on the following expansion [20] at $x \to 0$ with ω fixed in $\mathbb{R}$

$$K_{i\omega}(x) = \frac{i\pi e^{\pi\omega/2}}{2}\left(\frac{ix}{2}\right)^{i\omega}\frac{1 + O_\omega(x^2)}{\Gamma(1+i\omega)\sinh(\pi\omega)} - \frac{i\pi e^{-\pi\omega/2}}{2}\left(\frac{ix}{2}\right)^{-i\omega}\frac{1 + O'_\omega(x^2)}{\Gamma(1-i\omega)\sinh(\pi\omega)}\,, \tag{37}$$

where, for ω fixed, $|O_\omega(x^2)| \leq C_\omega|x|^2$ and $|O'_\omega(x^2)| \leq C'_\omega|x|^2$ for some real finite C_ω, C'_ω, whereas for x fixed in $\mathbb{R}$, $\omega \mapsto |O_\omega(x^2)|$ and $\omega \mapsto |O'_\omega(x^2)|$ are bounded. Inserting the expansion above in the expression (8) and taking the limit as $u \to \infty$ in (11), Riemann-Lebesgue's lemma together with some trivial properties of Γ function ([20]) produces (34) for $m > 0$. The function (36) is nothing but $\mathrm{sign}(E)\ e^{-iE(\log(m/2\kappa))/\kappa}\Gamma\left(\frac{iE}{\kappa}\right)\left|\Gamma\left(\frac{iE}{\kappa}\right)\right|^{-1}$ [20] and so $|N_{m,\kappa}(E)| = 1$ for $E \neq 0$ is trivially true. $\Gamma(ix)$ is smooth along the real axis with a simple pole in $x = 0$ that is canceled out by the zero of $\mathrm{sign}(x)\sqrt{x\sinh x}$ that is smooth in the whole $\mathbb{R}$. Thus $N_{m,\kappa} \in C^\infty((-\infty, +\infty))$. $|N_{m,\kappa}(E)| = 1$ for $E = 0$ is trivially valid by continuity. $\overline{N_{m,\kappa}(E)} = -N_{m,\kappa}(-E)$ is a straightforward consequence of $\overline{\Gamma(ix)} = \Gamma(-ix)$ for $x \in \mathbb{R}$. The case $v \to -\infty$ can be proven similarly. □

From a pure mathematical point of view perhaps straightforwardly extending known results (see Chap. 5 of [17]), Proposition 3.1 shows that a solution in $\mathcal{S}$ of the massive Klein-Gordon equation in $2D$ Rindler space are completely determined by its values on *either* the future *or* the past horizon. Whereas, in the massless case, a solution $\mathcal{S}$ is completely determined by its values on *both* the future *and* the past horizon.
As $|N_{m,\kappa}(E)| = 1$ we can write $N_{m,\kappa}(E) = e^{i\rho_{m,\kappa}(E)}$ where the phase $\rho_{m,\kappa}(E)$ is real-valued. The restriction of ψ to the horizon $\mathbf{F}$ (the other case is analogous) depends from the mass of the field through the phase $\rho_{m,\kappa}$ only. As a trivial result we see that two (free scalar QFT) theories

in $\mathbf{R}$ with different strictly-positive masses $m \neq m'$ and Rindler vacua $\Psi_m, \Psi_{m'}$ turn out to be unitarily equivalent. This is due to the unitary operator $U : \mathfrak{F}(\mathcal{H}_m) \to \mathfrak{F}(\mathcal{H}_{m'})$ naturally defined by the requirement $U\Psi_m = \Psi_{m'}$ and induced by the scalar-product-preserving map between positive frequency wavefunctions

$$\psi_+ \mapsto \psi'_+ \ , \ \text{with} \ \ \psi'_+(E) := e^{+i(\rho_{m,\kappa}(E)-\rho_{m',\kappa}(E))}\psi_+(E) \quad \text{for all } E \geq 0$$

where $\psi_+ \in \mathcal{H}_m$ and $\psi'_+ \in \mathcal{H}_{m'}$. Similarly, each theory is unitarily equivalent to the massless theory built up using only *in* modes. Avoiding any choice for the mass, one is naturally lead to consider the class of the "fields defined on the horizon"

$$\psi(v) = \int_{\mathbb{R}^+} \frac{e^{-iEv}}{\sqrt{4\pi E}} \tilde{\psi}_+(E)\, dE + \int_{\mathbb{R}^+} \frac{e^{+iEv}}{\sqrt{4\pi E}} \overline{\tilde{\psi}_+(E)}\, dE \tag{38}$$

as the object which makes possibles all those crossed unitary identifications and exists independently from the quantum fields defined in the bulk $\mathbf{R}$ with their own masses. We want to try to consider this object as a *quantum* field in a sense we go to specify.

3.2. *Quantum field theory on* $\mathbf{F}$ *and* $\mathbf{P}$. Following the procedure presented in 2.2 and 2.3 we want to show that it is possible to define a quantum field theory on $\mathbf{F}$ which matches with that defined in the bulk. (That idea is not new in the literature and it has been used in important works as [22]). First of all define the space of "wavefunctions" $\mathcal{S}_{\mathbf{F}}$. A suitable definition which will be useful later, is the following: $\mathcal{S}_{\mathbf{F}}$ is the space $\mathsf{S}(\mathbb{R};\mathbb{R})$ of the real-valued Schwartz' functions on $\mathbb{R}$ where $\mathbb{R}$ is identified with $\mathbf{F}$ itself by means of the coordinate v. Actually the name "wavefunction" is not appropriate because there is no wave equation to fulfill in our context. As a consequence the correct point of view to interpret the formalism is the Heisenberg's picture. $\mathcal{S}_{\mathbf{F}}$ has a natural nondegenerate symplectic form which is *invariant under the diffeomorphisms of* $\mathbf{F}$ *which preserve its orientation*:

$$\Omega_{\mathbf{F}}(\varphi, \varphi') := \int_{\mathbf{F}} \varphi' d\varphi - \varphi d\varphi' \ . \tag{39}$$

To define the one-particle Hilbert space, consider the modes

$$F_E(v) := \frac{e^{-iEv}}{\sqrt{4\pi E}} \quad \text{with } E \in \mathbb{R}^+ \ . \tag{40}$$

Analogous definitions can be given with analogous notations for the past event horizon $\mathbf{P}$ using modes as in (40) with $-iEv$ replaced for $-iEu$. The following pair of propositions can be simply proven using Fourier transform theory for Schwartz' functions.

Proposition 3.2. (Completeness of modes). *If φ belongs to $\mathcal{S}_{\mathbf{F}}$, the function*

$$\mathbb{R}^+ \ni E \mapsto \tilde{\varphi}_+(E) := -i\Omega_{\mathbf{F}}\left(\overline{F_E}, \psi\right) \ . \tag{41}$$

satisfies $\tilde{\varphi}_+(E) = \sqrt{E}g(E)$, $\overline{\tilde{\varphi}_+(E)} = \sqrt{E}g(-E)$, *where* $g \in \mathsf{S}(\mathbb{R}, \mathbb{R})$. *Moreover, for* $v \in \mathbb{R}$ *(38),*

$$\varphi(v) = \int_0^{+\infty} F_E(v)\tilde{\varphi}_+(E)\, dE + \int_0^{+\infty} \overline{F_E(v)\tilde{\varphi}_+(E)}\, dE\,. \tag{42}$$

Similar results hold replacing **F** *for* **P** *everywhere.*

Proposition 3.3. (Associated Hilbert spaces). *If* φ *belongs to either* $\mathcal{S}_\mathbf{F}$, *define the one-to-one associated* positive-frequency wavefunction

$$\varphi_+(v) := \int_0^{+\infty} F_E(v)\tilde{\varphi}_+(E)\, dE\,. \tag{43}$$

With that definition and for φ_1, φ_2 *in* $\mathcal{S}_\mathbf{F}$, *it results* $\Omega_\mathbf{F}(\varphi_{1+}, \varphi_{2+}) = 0$ *whereas*

$$\langle \varphi_{1+}, \varphi_{2+} \rangle_\mathbf{F} := -i\Omega_\mathbf{F}(\overline{\varphi_{1+}}, \varphi_{2+})\,, \tag{44}$$

is well-defined and

$$\langle \varphi_{1+}, \varphi_{2+} \rangle_\mathbf{F} = \int_0^{+\infty} \overline{\tilde{\varphi}_{1+}(E)}\tilde{\varphi}_{2+}(E)\, dE\,. \tag{45}$$

The one-particle Hilbert *space* $\mathcal{H}_\mathbf{F}$ *is defined as the Hilbert completion of the space of finite complex linear combinations of functions* φ_+, *for all* φ *in* $\mathcal{S}_\mathbf{F}$, *equipped with the extension of the scalar product (44) to complex linear combinations of arguments. It results* $\mathcal{H}_\mathbf{F} \cong L^2(\mathbb{R}^+, dE)$. *Similar results and definitions hold replacing* **F** *for* **P** *everywhere.*

Definition 3.1. (Quantum field operators on horizons). *Consider the symmetrized Fock space* $\mathfrak{F}_\mathbf{F}(\mathcal{H}_\mathbf{F})$ *with vacuum state* $\Psi_\mathbf{F}$ *and scalar product* $\langle\, ,\rangle_\mathbf{F}$. *The* quantum field operator on **F**, $\Omega_\mathbf{F}(\,.\,, \hat{\phi}_\mathbf{F})$ *is the symmetric-operator valued function*

$$\varphi \mapsto \Omega(\varphi, \hat{\phi}_\mathbf{F}) \quad := \quad ia_\mathbf{F}(\overline{\varphi_+}) - ia^\dagger_\mathbf{F}(\varphi_+)\,, \tag{46}$$

where $\varphi \in \mathcal{S}_\mathbf{F}$. $a_\mathbf{F}(\overline{\varphi_+})$ *and* $a^\dagger_\mathbf{F}(\varphi_+)$ *respectively denote the annihilation and construction operator associated with the one-particle state* φ_+ *which are defined in the dense invariant subspace* $\mathfrak{F}_{0\mathbf{F}}$ *spanned by all states containing a finite, arbitrarily large, number of particles with states given by positive-frequency wavefunctions. An analogous definition is given replacing* **F** *for* **P** *everywhere.*

Operators $\Omega_\mathbf{F}(\varphi, \hat{\phi}_\mathbf{F})$ and $\Omega_\mathbf{P}(\varphi, \hat{\phi}_\mathbf{P})$ are essentially self-adjoint on $\mathfrak{F}_{0\mathbf{F}}$ and $\mathfrak{F}_{0\mathbf{P}}$ respectively since their elements are analytic vectors.
We want to define an analogous procedure to that in the bulk (see (20)) for smearing field operators by means of "functions" instead of "wavefunctions". The issue is however relevant because it permits to introduce the analogue $E_\mathbf{F}$ of the causal propagator E in spite of having no

equation of motion in $\mathbf{F}$. The idea is that something like (19) should work also in our context. A clear difficulty is that there is no a diffeomorphism invariant measure which can be used in the analogue of (19) in place of $d\mu_g$. On the other hand integration of k-forms is diffeomorphism invariant on (oriented manifolds). Therefore we aspect that the space of "functions" used to smear quantum fields should properly be a space of 1-forms rather than functions. To go on we notice that *a posteriori* $E_{\mathbf{F}}$ has to fulfill something like $[\hat{\phi}_{\mathbf{F}}(v), \hat{\phi}_{\mathbf{F}}(v')] = -iE_{\mathbf{F}}(v, v')$. By a formal but straightforward computation which uses $[a_E, a_E^\dagger] = \delta(E-E')$ and the analogue of (18) with Φ_E replaced for F_E, one finds that $i[\hat{\phi}_{\mathbf{F}}(v), \hat{\phi}_{\mathbf{F}}(v')] = \frac{1}{4}\text{sign}\,(v-v')$. This v-parametrized distribution actually defines a well-behaved transformation from the space of exact (smooth) 1-forms in $\mathbf{F}$ with compact support to the space of smooth functions on $\mathbf{F}$. As the functions $f \in \mathcal{S}_{\mathbf{F}}$ vanish (with all of their derivatives) as $v \to \infty$, if $\eta = df$

$$\int_{v' \in \mathbb{R}} \text{sign}\,(v-v')\eta(v') = f(v) - (-f(v)) = 2f(v)\,.$$

In the following, $\mathsf{D}(\mathbf{F}; \mathbb{R})$ is the real space of the 1-forms $\eta = d\varphi$ such that $\varphi \in \mathcal{S}(\mathbf{F}; \mathbb{R})$.

Definition 3.2. **(Causal propagator and associated quantum field on horizons)**. *With the given notations, the* causal propagator *in* $\mathbf{F}$*, is the mapping* $E_{\mathbf{F}} : \mathsf{D}(\mathbf{F}; \mathbb{R}) \to \mathcal{S}(\mathbf{F}; \mathbb{R})$ *with*

$$(E_{\mathbf{F}}\eta)(v) := \frac{1}{4}\int_{v' \in \mathbb{R}} \textit{sign}\,(v-v')\eta(v')\,, \tag{47}$$

and the quantum-field operator on $\mathbf{F}$ smeared with forms η of $\mathsf{D}(\mathbf{F}; \mathbb{R})$ *is the mapping*

$$\eta \mapsto \hat{\phi}_{\mathbf{F}}(\eta) := \Omega_{\mathbf{F}}(E_{\mathbf{F}}\eta, \hat{\phi}_{\mathbf{F}})\,. \tag{48}$$

Analogous definitions are given replacing $\mathbf{F}$ *for* $\mathbf{P}$ *and* v *for* u *everywhere.*

The given definitions are good generalizations of the analogous tools in usual quantum field theory (see (19) and (21) in particular) as stated in the following pair of propositions whose proof is trivial.

Proposition 3.4. *If* $\varphi \in \mathcal{S}_{\mathbf{F}}$*,* $\omega = 2d\varphi$ *is the unique element of* $\mathsf{D}(\mathbf{F}; \mathbb{R})$ *such that* $\varphi = E_{\mathbf{F}}(\omega)$*. Moreover, if* $\eta, \omega \in \mathsf{D}(\mathbf{F}; \mathbb{R})$

$$\int_{\mathbf{F}} \varphi\eta = \Omega_{\mathbf{F}}(E_{\mathbf{F}}\eta, \varphi) \;\; \textit{and} \;\; \int_{\mathbf{F}} (E_{\mathbf{F}}\omega)\eta = \Omega_{\mathbf{F}}(E_{\mathbf{F}}\eta, E_{\mathbf{F}}\omega)\,. \tag{49}$$

An analogous result holds replacing $\mathbf{F}$ *for* $\mathbf{P}$ *everywhere.*

Proposition 3.5. *If* $\varphi \in \mathcal{S}(\mathbf{F}; \mathbb{R})$ *and* $\eta, \omega \in \mathsf{D}(\mathbf{F}; \mathbb{R})$

$$[\hat{\phi}_{\mathbf{F}}(\eta), \hat{\phi}_{\mathbf{F}}(\omega)] \;\; = \;\; -iE(\eta, \omega) := -i\int_{\mathbf{F}} (E_{\mathbf{F}}\eta)\omega\,. \tag{50}$$

In particular $[\hat{\phi}_{\mathbf{F}}(\eta), \hat{\phi}_{\mathbf{F}}(\omega)] = 0$ *if* $supp\, \eta \cap supp\, \omega = \emptyset$. *An analogous result holds replacing* $\mathbf{F}$ *for* $\mathbf{P}$ *everywhere.*

3.3. *The algebraic approach.* To state holographic theorems it is necessary to re-formulate quantum field theory in an algebraic approach either in the bulk and on the horizon. In globally hyperbolic spacetimes, linear QFT can be formulated independently from a preferred vacuum state and Fock representation. It is worthwhile stressing that [17] physics implies the existence of meaningful quantum states which cannot be represented in the same Hilbert (Fock) representation of the algebra of observables. In this sense the algebraic approach is more fundamental than the usual Fock approach in QFT in curved spacetime. Let us summarize the procedure in $\mathbf{R}$ which, at least for $m > 0$, could be replaced by any globally hyperbolic spacetime. The basic tool is an abstract $*$-algebra, $\mathcal{A}_{\mathbf{R}}$, made of the linear combinations of products of formal field operators $\phi(f), \phi(f)^*$ ($f \in \mathsf{D}(\mathbf{R};\mathbb{C})$) and the unit I, which enjoy the same properties of operators $\hat{\phi}(f), \hat{\phi}(f)^\dagger$ (and the identity operator I). From a physical point of view, the Hermitian elements of $\mathcal{A}_{\mathbf{R}}$ represent the *local observables* of the free-field theory. For $m > 0$, the required properties are:
(1) $\phi(f)^* = \phi(\overline{f})$ for all $f \in \mathsf{D}(\mathbf{R};\mathbb{C})$,
(2) $\phi(af + bg) = a\phi(f) + b\phi(g)$ for all $f, g \in \mathsf{D}(\mathbf{R};\mathbb{C})$, $a, b \in \mathbb{C}$,
(3) $[\phi(f), \phi(g)] = -iE(f,g)I$ for all $f, g \in \mathsf{D}(\mathbf{R};\mathbb{C})$, and
(4) $\phi(f) = 0$ if (and only if) $f = Kh$ for some $h \in \mathsf{D}(\mathbf{R};\mathbb{C})$.
$\mathcal{A}_{\mathbf{R}}$ is rigorously realized as follows. Consider the complex unital algebra $\mathcal{A}_{0\mathbf{R}}$, freely generated by the unit I, and abstract objects $\phi(f)$ and $\phi(f)^*$ for all $f \in \mathsf{D}(\mathbf{R};\mathbb{C})$. The involution * on $\mathcal{A}_{0\mathbf{R}}$ is the unique antilinear involutive function $^* : \mathcal{A}_{0\mathbf{R}} \to \mathcal{A}_{0\mathbf{R}}$ such that $I^* = I$, $(\phi(f))^* = \phi(f)^*$. Let $\mathcal{I} \subset \mathcal{A}_{\mathbf{R}}$ be the double-side ideal whose elements are linear combinations of products containing at least one of the following factors $\phi(f)^* - \phi(\overline{f})$, $\phi(af+bg) - a\phi(f) - b\phi(g)$, $[\phi(f), \phi(g)] + iE(f,g)I$, and $\phi(Kg)$ for $f, g \in \mathsf{D}(\mathbf{R};\mathbb{C})$, $a, b \in \mathbb{C}$. $\mathcal{A}_{\mathbf{R}}$ is defined as the space of equivalence classes with respect to the equivalence relation in $\mathcal{A}_{0\mathbf{R}}$, $A \sim B$ iff $A - B \in \mathcal{I}$ and $\mathcal{A}_{\mathbf{R}}$ is equipped with the $*$-algebra structure induced by $\mathcal{A}_{0\mathbf{R}}$ through $\sim$. If, with little misuse of notation, $\phi(f)$ and I respectively denote the classes $[\phi(f)]$ and $[I] \in \mathcal{A}_{\mathbf{R}}$, the properties (1),(2),(3), (4) are fulfilled.
If $m = 0$, there are two relevant algebras $\mathcal{A}^{(in)}_{\mathbf{R}}$ and $\mathcal{A}^{(out)}_{\mathbf{R}}$. $\mathcal{A}^{(in)}_{\mathbf{R}}$ is the unital $*$-algebra generated by I, $\phi_{\mathrm{in}}(f)$ and $\phi_{\mathrm{in}}(f)^*$ for every $f \in \mathsf{D}(\mathbf{R}, \mathbb{C})$ whereas $\mathcal{A}^{(out)}_{\mathbf{R}}$ is the unital $*$-algebra generated by I, $\phi_{\mathrm{out}}(f)$ and $\phi_{\mathrm{out}}(f)^*$ for every $f \in \mathsf{D}(\mathbf{R}, \mathbb{C})$. By definition these algebras satisfy the constraints (1),(2),(3) and (4) with the difference that, in (3), E must be replaced for E_{in} or E_{out} respectively and, in (4), K must be replaced by ∂_u or ∂_v respectively. The rigorous definitions can be given similarly to the case $m > 0$, by starting from freely generated algebras and passing to quotient algebras. We recall that if $\mathcal{A}$, $\mathcal{B}$ are $*$-algebras with field $\mathbb{C}$ and units $I_{\mathcal{A}}$, $I_{\mathcal{B}}$, $\mathcal{A} \otimes \mathcal{B}$ denotes (see p.143 of [23]) the $*$-algebra whose associated vector-space structure is the tensor product $\mathcal{A} \otimes \mathcal{B}$, the unit is $I := I_{\mathcal{A}} \otimes I_{\mathcal{B}}$, the involution and the algebra product are respectively given by $(\sum_k A_k \otimes B_k)^* := \sum_k A_k^* \otimes B_k^*$ and $(\sum_k A_k \otimes B_k)(\sum_i A_i' \otimes B_i') := \sum_{ki} A_k A_i' \otimes B_k B_i'$ with obvious notation. Assuming (24) as the definition of $\phi(f)$, the whole field algebra $\mathcal{A}_{\mathbf{R}}$ is *defined* as $\mathcal{A}_{\mathbf{R}} := \mathcal{A}^{(in)}_{\mathbf{R}} \otimes \mathcal{A}^{(out)}_{\mathbf{R}}$. That unital $*$-algebra satisfies (1),(2),(3) and (4).

An *algebraic state* on a $*$-algebra $\mathcal{A}$ with unit I, is a linear map $\mu : \mathcal{A} \to \mathbb{C}$ that is normalized (i.e. $\mu(I) = 1$) and positive (i.e. $\mu(A^*A) \geq 0$ for $A \in \mathcal{A}$). The celebrated GNS theorem (e.g., see [17]) states that for every algebraic state μ on $\mathcal{A}$ there is a triple $(\mathfrak{H}_\mu, \Pi_\mu, \Omega_\mu)$ such that the following facts hold. $\mathfrak{H}_\mu$ is a Hilbert space, Π_μ is a $*$-algebra representation of $\mathcal{A}$ in terms of operators on $\mathfrak{H}_\mu$ which are defined on a dense invariant subspace $\mathfrak{D}_\mu \subset \mathfrak{H}_\mu$ and such that $\Pi_\mu(A^*) = (\Pi_\mu(A))^\dagger \restriction_{\mathfrak{D}_\mu}$. Finally $\mathfrak{D}_\mu$ is spanned by all the vectors $\Pi_\mu(A)\Omega_\mu$, $A \in \mathcal{A}$, and $\mu(A) = \langle \Omega_\mu, \Pi_\mu(A)\Omega_\mu \rangle$ for all $A \in \mathcal{A}$, $\langle\, ,\, \rangle$ denoting the scalar product in $\mathfrak{H}_\mu$. If $(\mathfrak{H}'_\mu, \Pi'_\mu, \Omega'_\mu)$ is another similar triple associated with the same μ, there is a unitary operator $U : \mathfrak{H}_\mu \to \mathfrak{H}'_\mu$ such that $\Omega'_\mu = U\Omega_\mu$ and $\Pi'_\mu = U\Pi_\mu$. If $\mathcal{A} = \mathcal{A}_{\mathbf{F}}$, by direct inspection one finds that quantum field theory in $\mathbf{R}$ in the Fock-space $\mathfrak{F}(\mathcal{H})$ with $\Psi_{\mathbf{R}}$ as vacuum state coincides with that in a GNS representation of $\mathcal{A}_{\mathbf{R}}$ associated with the (*quasifree* [17]) algebraic state $\mu_{\mathbf{R}}$ completely determined by $\mu_{\mathbf{R}}(\phi(f)\phi(g)) := \left\langle \Psi_{\mathbf{R}}, \hat{\phi}(f)\hat{\phi}(g)\Psi_{\mathbf{R}} \right\rangle$ via Wick expansion of symmetrized n-point functions. Moreover it results $\mathfrak{D}_\mu = \mathfrak{F}_0$.

All the procedure can be used to give an algebraic approach for QFT on $\mathbf{F}$ (or $\mathbf{P}$): Define $\mathsf{D}(\mathbf{F};\mathbb{C}) := \mathsf{D}(\mathbf{F};\mathbb{R}) + i\mathsf{D}(\mathbf{F};\mathbb{R})$ and define $\hat{\phi}_{\mathbf{F}}(\omega + i\eta) := \hat{\phi}_{\mathbf{F}}(\omega) + i\hat{\phi}_{\mathbf{F}}(\eta)$ when $\omega, \eta \in \mathsf{D}(\mathbf{F};\mathbb{R})$. Finally, consider the abstract $*$-algebra $\mathcal{A}_{\mathbf{F}}$ with unit I, generated by I, $\phi_{\mathbf{F}}(\omega)$, $\phi_{\mathbf{F}}(\omega)^*$ for all $\omega \in \mathsf{D}(\mathbf{F};\mathbb{C})$, such that, for all $a, b \in \mathbb{C}$ and $\omega, \eta \in \mathsf{D}(\mathbf{F};\mathbb{C})$:

(1) $\phi_{\mathbf{F}}(\omega)^* = \phi_{\mathbf{F}}(\overline{\omega})$,

(2) $\phi_{\mathbf{F}}(a\omega + b\eta) = a\phi_{\mathbf{F}}(\omega) + b\phi_{\mathbf{F}}(\eta)$ and

(3) $[\phi_{\mathbf{F}}(\omega), \phi_{\mathbf{F}}(\eta)] = -iE_{\mathbf{F}}(\omega, \eta)I$.

(The rigorous definition is given in terms of quotient algebras as usual.) From a physical point of view the (Hermitean) elements of $\mathcal{A}_{\mathbf{F}}$ represents the *(quasi) local observables* of the free-field theory on the future horizon. By direct inspection one finds that quantum field theory in $\mathbf{F}$ in the Fock-space referred to the vacuum state $\Psi_{\mathbf{F}}$ coincides with that in a GNS representation of $*$-algebra $\mathcal{A}_{\mathbf{F}}$ associated with a the (quasifree) algebraic state $\mu_{\mathbf{F}}$ completely determined, via Wick expansion, by $\mu_{\mathbf{F}}(\phi_{\mathbf{F}}(\eta)\phi_{\mathbf{F}}(\omega)) := \left\langle \Psi_{\mathbf{F}}, \hat{\phi}_{\mathbf{F}}(\eta)\hat{\phi}_{\mathbf{F}}(\omega)\Psi_{\mathbf{F}} \right\rangle_{\mathbf{F}}$ and $\mathfrak{D}_\mu = \mathfrak{F}_{0\mathbf{F}}$.

Everything can similarly be stated for quantum filed theory on $\mathbf{P}$ with trivial changes in notations.

3.4. *Two holographic theorems.* Here we prove two *holographic* theorems for the observables of free fields, one in the algebraic approach and the latter in the Hilbert-space formulation under the choice of suitable vacuum states. The former theorem says that, in the massive case, there is a one-to-one transformation from the algebra of the fields in the bulk $\mathcal{A}_{\mathbf{R}}$ – that is the local observables of the free field in the bulk – to a subalgebra of fields on the future horizon $\mathcal{A}_{\mathbf{F}}$ – that is the observables of the free field in the future horizon –. The mapping preserves the structure of $*$-algebra and thus the two classes of obsevables can be identified completely nomatter the value of the mass of the field in the bulk and the fact that there is no mass associated with the field on the horizon. Remarkably, this identification does not requires any choice for reference vacuum states since it is given at a pure algebraic level. To build up the said mapping, take any compactly supported function f in the bulk, consider the generated wavefunction $\psi_f = E(f)$ (that is assumed to be defined in the whole Minkowski space), restrict

ψ_f on $\mathbf{F}$ obtaining a horizon wavefunction φ_f and associate with that function the unique form ω_f with $\varphi_f = E_{\mathbf{F}}\omega_f$. Finally define $\chi_{\mathbf{F}}(\phi(f)) := \phi_{\mathbf{F}}(\omega_f)$. Next step is to extend $\chi_{\mathbf{F}}$ to the whole algebra $\mathcal{A}_{\mathbf{R}}$ by requiring that the $*$-algebra structure is preserved that is, I is mapped in I, $\phi(f)^*$ is mapped into $\chi_{\mathbf{F}}(\phi(f))^*$, products of fields $\phi(f)\phi(g)$ are mapped into $\chi_{\mathbf{F}}(\phi(f))\chi_{\mathbf{F}}(\phi(g))$ and so on. In the massless case, the procedure is similar but one has to consider also the past evolution of wavefunctions toward the past event horizon $\mathbf{P}$. The theorem says that the required extensions into algebra homomorphisms actually exists, are uniquely determined and injective so that the observable algebra in the bulk can be seen as a observable subalgebra on the horizon.

Theorem 3.1. **(Algebraic holography)** *In a 2D-Rindler space $\mathbf{R}$ viewed as immersed in a corresponding 2D Minkowski spacetime, consider quantum field theory of a scalar field with mass $m \geq 0$ satisfying Klein-Gordon equation (5). Consider the algebra $\mathcal{A}_{\mathbf{R}}$ (including $\mathcal{A}^{(out)}_{\mathbf{R}}$, $\mathcal{A}^{(in)}_{\mathbf{R}}$ if $m = 0$) of local observables in the bulk and the algebras $\mathcal{A}_{\mathbf{F}}$ and $\mathcal{A}_{\mathbf{P}}$ of the observables on the horizons $\mathbf{F}$ and $\mathbf{P}$. The following statements hold.*
(a) *If $m > 0$, there is a unital-$*$-algebra homomorphism $\chi_{\mathbf{F}} : \mathcal{A}_{\mathbf{R}} \to \mathcal{A}_{\mathbf{F}}$ uniquely determined by*

$$\chi_{\mathbf{F}} : \phi(f) \mapsto \phi_{\mathbf{F}}(\omega_f) \quad \textit{with } \omega_f := 2d((Ef)\restriction_{\mathbf{F}}) \textit{ for all } f \in \mathsf{D}(\mathbf{R}, \mathbb{R}) \,, \tag{51}$$

$(Ef)\restriction_{\mathbf{F}}$ denoting the limit of $E(f)$ on $\mathbf{F}$. $\chi_{\mathbf{F}}$ turns out to be injective.
An analogous statement holds replacing $\mathbf{F}$ for $\mathbf{P}$.
(b) *If $m = 0$, there are two unital-$*$-algebra homomorphisms $\pi_{\mathbf{P}} : \mathcal{A}^{(out)}_{\mathbf{R}} \to \mathcal{A}_{\mathbf{P}}$ and $\pi_{\mathbf{F}} : \mathcal{A}^{(in)}_{\mathbf{R}} \to \mathcal{A}_{\mathbf{F}}$ uniquely determined by*

$$\pi_{\mathbf{F}} : \phi_{in}(f) \mapsto \phi_{\mathbf{F}}(\eta_f) \qquad \textit{with } \omega_f := 2d(E(f)\restriction_{\mathbf{F}}) \textit{ for all } f \in \mathsf{D}(\mathbf{R}, \mathbb{R}) \,, \tag{52}$$

$$\pi_{\mathbf{P}} : \phi_{out}(f) \mapsto \phi_{\mathbf{P}}(\omega_f) \qquad \textit{with } \eta_f := 2d(E(f)\restriction_{\mathbf{P}}) \textit{ for all } f \in \mathsf{D}(\mathbf{R}, \mathbb{R}) \,. \tag{53}$$

$\pi_{\mathbf{F}}$ and $\pi_{\mathbf{P}}$ turn out to be injective.
(c) *$\pi_{\mathbf{F}}(\mathcal{A}^{(in)}_{\mathbf{R}}) \subset \mathcal{A}_{\mathbf{F}}$ is the subalgebra generated by I and abstracts field operators smeared by the elements of $\mathsf{D}(\mathbf{F}, \mathbb{C})$ with compact support. The analogous statement holds for $\pi_{\mathbf{P}}(\mathcal{A}^{(out)}_{\mathbf{R}})$.*

Proof. **(a)** The uniqueness of the homomorphism is trivially proven by noticing that the elements of $\mathcal{A}_{\mathbf{R}}$ are of the form $A = aI + \sum_k b_k\phi(f_k) + \sum_h c_h\phi(g_h)^* + \sum_{ls} d_{ls}\phi(h_l)\phi(p_s) + ...$ where the overall summation as well as every partial summation is finite. As $\chi_{\mathbf{F}}$ is a homomorphism, $\chi_{\mathbf{F}}(A) = aI + \sum_k b_k\chi_{\mathbf{F}}(\phi(f_k)) + \sum_h c_h\chi_{\mathbf{F}}(\phi(g_h))^* + \sum_{ls} d_{ls}\chi_{\mathbf{F}}(\phi(h_l))\chi_{\mathbf{F}}(\phi(p_s)) + ...$ Moreover $\chi_{\mathbf{F}}(\phi(h)) = \chi_{\mathbf{F}}(\phi(Re\, h)) + i\chi_{\mathbf{F}}(\phi(Im\, h))$ and thus the values $\chi_{\mathbf{F}}(\phi(f))$ with h real determine the homomorphism provided it exists. Let us prove the existence of the homomorphism. Take $f \in \mathsf{D}(\mathbf{R}, \mathbb{R})$ and consider $\psi_f = Ef$ and the associated function $\tilde{\psi}_{f+} = \tilde{\psi}_{f+}(E)$. It holds $\tilde{\psi}_{f+}(E) = \sqrt{E}f(E)$ with $f \in \mathsf{S}(\mathbb{R}, \mathbb{C})$ such that $\overline{f(E)} = -f(-E)$ as stated in Proposition 2.1 and $N_{m,\kappa} \in C^\infty(\mathbb{R})$ (with $|N_{m,\kappa}(E)| = 1$) and $\overline{N_{m,\kappa}(E)} = -N_{m,\kappa}(-E)$ as stated in Proposition 3.1. As a consequence $N_{m,\kappa}(E)\tilde{\psi}_+(E) = \sqrt{E}h(E)$ where $\overline{h(E)} = h(-E)$ and $h \in \mathsf{S}(\mathbb{R}, \mathbb{C})$. Passing to the function $v \mapsto \psi_f(v)$ in Proposition 3.1 and using these results one gets

$$\psi_f(v) = \text{const.} \int_0^{+\infty} e^{-iEv}h(E)dE + \text{c.c.} = \text{const.} \int_{-\infty}^{+\infty} e^{-iEv}h(E)dE \,.$$

As h belongs to Schwartz' space, ψ_f belongs to the same space because Fourier transform preserves Schwartz' space. Moreover ψ_f is real since $h(E) = \overline{h(-E)}$. We have found that $\psi_f \in \mathcal{S}_{\mathbf{F}}$ and thus $\omega_f := 2d\psi_f = 2d[(Ef)\restriction_{\mathbf{F}}]$ is an element of $\mathsf{D}(\mathbf{F}, \mathbb{R})$. Using $f \in \mathsf{D}(\mathbf{R}, \mathbb{C})$ the result is preserved trivially by the linear decomposition in real and imaginary part. Assume once again that $f \in \mathsf{D}(\mathbf{R}, \mathbb{R})$. Notice that ω_f contains the same information as ψ_f because $\psi_f(v) = 2\int_{-\infty}^{v} \omega_f$. In turn ψ_f determines the function $\tilde{\psi}_{f_+}$ which determines Ef. As we said in 2.3, Ef determines f up to a term Kh with $h \in \mathsf{D}(\mathbf{R}, \mathbb{R})$. We conclude that $\omega_f = \omega_g$ if and only if $f = g + Kh$ with $h \in \mathsf{D}(\mathbf{R}, \mathbb{R})$. The same result arises for functions $f, g \in \mathsf{D}(\mathbf{R}, \mathbb{C})$ by linearity and from the fact that E transforms real functions into real functions. We have found that there is a well-defined linear map $\mathsf{D}(\mathbf{R}, \mathbb{C}) \ni f \mapsto \omega_f \in \mathsf{D}(\mathbf{F}, \mathbb{C})$ that transforms real functions in real forms and such that $\omega_f = \omega_g$ if and only if $f = g + Kh$. Now we define $\chi_{0\mathbf{F}}(\phi(f)) = \phi_{\mathbf{F}}(\omega_f)$ and $\chi_{\mathbf{F}}(I) = I$ and $\chi_{\mathbf{F}}(\phi(f)^*) = \phi_{\mathbf{F}}(\omega_f)^*$. That map extends straightforwardly from the $*$-algebra $\mathcal{A}_{0\mathbf{R}}$ freely generated by I, $\phi(f)$, $\phi(f)^*$ (with involution uniquely determined as said in 2.4) to the analogous free $*$-algebra on $\mathbf{F}$ giving rise to a $*$-algebra homomorphism $\chi_{0\mathbf{F}}$. However it is not injective since it results $\chi_{0\mathbf{F}}(\phi(f)) = \chi_{0\mathbf{F}}(\phi(g))$ whenever $f = g + Kh$, and more generally injectivity failure arises for any pair of elements of the algebra which are different to each other because of the presence of factors $\phi(f)$ and $\phi(g)$ with $f - g = Kh$. The injectivity is however restored if we take the quotient $*$-algebra $\mathcal{A}_{1\mathbf{R}}$ in $\mathcal{A}_{0\mathbf{R}}$ with respect to the both-side ideal containing linear combinations of products with at least one factor $\phi(Kf)$ or $\phi(Kf)^*$ for any $f \in \mathsf{D}(\mathbf{F}, \mathbb{C})$ and redefine the injective map $\chi_{1\mathbf{F}} : \mathcal{A}_{1\mathbf{R}} \to \mathcal{A}_{0\mathbf{F}}$ as the map induced by $\chi_{0\mathbf{F}}$ through the canonical projection of $\mathcal{A}_{0\mathbf{R}}$ onto $\mathcal{A}_{1\mathbf{R}}$. By construction $\chi_{1\mathbf{F}}$ is an injective $*$-algebra isomorphism. In this context and from now on, it is convenient to think the objects $\phi(f)$ as smeared by the equivalence class $[f]$ rather than f itself, where $[f]$ belong to the complex vector space obtained by taking the quotient of $\mathsf{D}(\mathbf{R}, \mathbb{C})$ with respect to the subspace $K\mathsf{D}(\mathbf{R}, \mathbb{C})$. The map $[f] \mapsto \omega_f$ is a well-defined injective vector space isomorphism that preserves the complex conjugation. To conclude we have to extract the algebras $\mathcal{A}_{\mathbf{R}}$ and $\mathcal{A}_{\mathbf{F}}$ by the procedure outlined in 2.4, based on the projection on suitable quotient spaces, and prove that the $*$-homeomorphism $\chi_{1\mathbf{F}}$ induces a $*$-homeomorphism $\chi_{\mathbf{F}} : \mathcal{A}_{\mathbf{R}} \to \mathcal{A}_{\mathbf{F}}$. To this end we have to consider the double-side ideal $\mathfrak{I} \subset \mathcal{A}_{1\mathbf{R}}$ whose elements are linear combinations of products containing at least one of the following factors $\phi(f)^* - \phi(\overline{f})$, $\phi(af + bg) - a\phi(f) - b\phi(g)$, $[\phi(f), \phi(g)] + iE(f, g)I$, for $f, g \in \mathsf{D}(\mathbf{R}; \mathbb{C})$, $a, b \in \mathbb{C}$. $\mathcal{A}_{\mathbf{R}}$ is defined as the space of equivalence classes with respect to the equivalence relation in $\mathcal{A}_{1\mathbf{R}}$, $A \sim_{\mathfrak{I}} B$ iff $A - B \in \mathfrak{I}$ and $\mathcal{A}_{\mathbf{R}}$ is equipped with the $*$-algebra structure induced by $\mathcal{A}_{1\mathbf{R}}$ through the canonical projection. The analogous procedure must be used for $\mathcal{A}_{1\mathbf{F}}$ with respect to an analogous ideal $\mathfrak{I}_{\mathbf{F}} \subset \mathcal{A}_{1\mathbf{F}}$ in order to produce $\mathcal{A}_{\mathbf{F}}$. Then the injective $*$-homomorphism $\chi_{1\mathbf{F}}$ induces a injective $*$-homomorphism $\chi_{\mathbf{F}} : \mathcal{A}_{\mathbf{R}} \to \mathcal{A}_{\mathbf{F}}$ if the equivalence relations induced by $\mathfrak{I}$ and $\mathfrak{I}_{\mathbf{F}}$ are preserved by $\chi_{1\mathbf{F}}$ itself, i.e. $A \sim_{\mathfrak{I}} B$ if and only if $\chi_{1\mathbf{F}}(A) \sim_{\mathfrak{I}_{\mathbf{F}}} \chi_{1\mathbf{F}}(B)$. We leave the simple but tedious proof of this fact to the reader, proving the only non trivial point which concerns factors $[\phi(f), \phi(g)] + iE(f, g)I$. It is simply found that, among other trivially fulfilled conditions, the preservation of the equivalence relation arises if $\chi_{1\mathbf{F}}([\phi(f), \phi(g)] + iE(f, g)I) = [\phi_{\mathbf{F}}(\omega_f), \phi_{\mathbf{F}}(\omega_g)] + iE_{\mathbf{F}}(\omega_f, \omega_g)I$. Which is equivalent to $E(f, g) = E_{\mathbf{F}}(\omega_f, \omega_g)$. (Notice that, by the known properties of the causal propagator both sides are invariant under the addition of a term Kh to f or g.) $E(f,g) = E_{\mathbf{F}}(\omega_f, \omega_g)$ is equivalent to, with obvious notations, $\Omega(\psi_f, \psi_g) = \Omega_{\mathbf{F}}(\varphi_f, \varphi_g)$. It is sufficient to prove that identity for real f, g. By Propositions 2.2 and 3.3 one finds $-i\Omega(\psi_f, \psi_g) = \langle \psi_{f+}, \psi_{g+} \rangle - \overline{\langle \psi_{f+}, \psi_{g+} \rangle}$ and $-i\Omega_{\mathbf{F}}(\varphi_f, \varphi_g) = \langle \varphi_{f+}, \varphi_{g+} \rangle_{\mathbf{F}} - \overline{\langle \varphi_{f+}, \varphi_{g+} \rangle_{\mathbf{F}}}$. Passing in energy representation, where the scalar product is simply that of $L^2(\mathbb{R}^+, dE)$ in both spaces, ψ_{f+} and ψ_{g+} are represented by some $E \mapsto \tilde{\psi}_{f+}(E)$ and $E \mapsto \tilde{\psi}_{g+}(E)$ respectively whereas, by Proposition 3.1, φ_{f+} and φ_{g+} are represented by $E \mapsto N_{m,\kappa}(E)\tilde{\psi}_{f+}(E)$ and $E \mapsto N_{m,\kappa}(E)\tilde{\psi}_{g+}(E)$ respectively. Since $|N_{m,\kappa}(E)| = 1$ it results $\langle \psi_{f+}, \psi_{g+} \rangle = \langle \varphi_{f+}, \varphi_{g+} \rangle_{\mathbf{F}}$ that entails $\Omega(\psi_f, \psi_g) = \Omega_{\mathbf{F}}(\varphi_f, \varphi_g)$ and concludes the proof. **(b)** Following a proof very similar to that as in the case (b) (but simpler since the phases $N_{m,\kappa}$ disappear when one uses Proposition 3.1) one sees that $\mathcal{A}^{(\text{in})}_{\mathbf{R}}$ is isomorphic to $\mathcal{A}_{\mathbf{F}}$ under the unique extension, into a injective $*$-algebra-with-unit homomorphism, of the map $\pi_{\mathbf{F}} : \phi_{\text{in}}(f) \mapsto \phi_{\mathbf{F}}(\omega_f)$ with $\omega_f := 2d(E_{\text{in}}(f))$ and this is equivalent to the thesis because $E_{\text{in}} f = E(f) \restriction_{\mathbf{F}}$ since $E_{\text{out}}(f) \restriction_{\mathbf{F}} = 0$ and $E_{\text{in}}(f) \restriction_{\mathbf{F}} = E_{\text{in}} f$, for smooth compactly supported f defined in $\mathbf{R}$ (these facts are consequences of Proposition 3.1). The case of $\mathcal{A}^{(\text{out})}_{\mathbf{R}}$ is strongly analogous. **(c)** Consider the case of $\pi_{\mathbf{F}}$ the other being analogous. If f is smooth and compactly supported in $\mathbf{R}$, $E_{\text{in}} f$ is a compactly-supported function of v and thus $\omega_f = 2d(E(f)\restriction_{\mathbf{F}}) = 2d(E_{\text{in}} f)$ is compactly supported on $\mathbf{F}$. Conversely if $\omega = d\varphi \in \mathsf{D}(\mathbf{F}, \mathbb{C})$ is compactly supported on $\mathbf{F}$, φ must be compactly supported and $f(u,v) := 2\varphi(u)h(u)$ is smooth, compactly supported in $\mathbf{R}$ for every smooth compactly supported function $h : \mathbb{R} \to \mathbb{R}$ and $\omega = 2d(E_{\text{in}}(f))$ if $\int_{\mathbb{R}} h(u)du = 1$. $\Box$

Remarks. **(1)** We stress that QFT on the horizon is the same nomatter the value of the mass of the filed in the bulk: Different choices of the mass determine different injective $*$-algebra homomorphisms from the algebra in the bulk to the *same* algebra of observables on the horizon. **(2)** There are strong differences between the cases $m > 0$ and $m = 0$. If f is compactly supported in the bulk, the horizon restriction of Ef is compactly supported if $m = 0$ but that is not the case when $m > 0$. For that reason we have defined $\mathcal{S}(\mathbf{F})$ (and $\mathsf{D}(\mathbf{F}, \mathbb{R})$) as a space of rapidly decreasing functions (1-forms) rather than a space of compactly supported functions (1-forms). Moreover, if $m > 0$, $\mathcal{A}_{\mathbf{R}}$ is isomorphic to a subalgebra of $\mathcal{A}_{\mathbf{F}}$ (or equivalently $\mathcal{A}_{\mathbf{P}}$). Conversely, if $m = 0$, $\mathcal{A}_{\mathbf{R}} (= \mathcal{A}^{(\text{in})}_{\mathbf{R}} \otimes \mathcal{A}^{(\text{out})}_{\mathbf{R}})$ is isomorphic to a subalgebra of $\mathcal{A}_{\mathbf{F}} \otimes \mathcal{A}_{\mathbf{P}}$ by means of the injective unital-$*$-algebra homomorphism $\pi_{\mathbf{P}} \otimes \pi_{\mathbf{F}} : \mathcal{A}_{\mathbf{R}} \to \mathcal{A}_{\mathbf{P}} \otimes \mathcal{A}_{\mathbf{F}}$.
(3) The existence of the $*$-homorphisms $\chi_{\mathbf{F}}$ and $\pi_{\mathbf{F}/\mathbf{P}}$ implies that, for all $f, g \in \mathsf{D}(\mathbf{F}, \mathbb{C})$ or $\mathsf{D}(\mathbf{F}, \mathbb{C})$ and respectively for $m > 0$ or $m = 0$,

$$[\phi(f), \phi(g)] = [\phi_{\mathbf{F}}(\omega_f), \phi_{\mathbf{F}}(\omega_g)] \text{ or } [\phi_{\text{in/out}}(f), \phi_{\text{in/out}}(g)] = [\phi_{\mathbf{F}/\mathbf{P}}(\omega_f), \phi_{\mathbf{F}/\mathbf{P}}(\omega_g)] , \qquad (54)$$

as a consequence the *causal propagator and the symplectic forms are preserved too.*

The second theorem concerns the unitary implementation of the $*$-homomorphism given in Theorem 3.1. This theorem states that, if one realizes the algebras of observables $\mathcal{A}_{\mathbf{R}}$ and $\mathcal{A}_{\mathbf{P}}$, $\mathcal{A}_{\mathbf{F}}$ in terms of proper field operators in the Fock spaces constructed over respectively, the Rindler vacuum $\Psi_{\mathbf{R}}$ and $\Psi_{\mathbf{P}}$, $\Psi_{\mathbf{F}}$, then the injective homomorphisms presented in Theorem 3.1 are implemented by unitary operators which preserve the vacuum states. In other words, *with the said choice of the vacuum states and Fock representation of the algebras of observables*, the theory in the bulk and that on the horizon are *unitarily equivalent*. As an immediate consequence, it arises that the *vacuum expectation values* are preserved passing from the theory in the bulk $\mathbf{R}$ to the theory on the horizon $\mathbf{F}$ (or $\mathbf{P}$).

Theorem 3.2. (**Unitary holography**). *In the same hypotheses as in Theorem 3.1, consider the realization of the algebra of the local observables of the bulk $\mathcal{A}_{\mathbf{R}}$, in the Fock space $\mathfrak{F}(\mathcal{H})$ with Rindler vacuum $\Psi_{\mathbf{R}}$ ($= \Psi_{\mathbf{R}}^{(out)} \otimes \Psi_{\mathbf{R}}^{(in)}$ if $m = 0$) and the realizations of the algebras of observables of the horizons $\mathcal{A}_{\mathbf{P}}$, $\mathcal{A}_{\mathbf{F}}$ in the Fock spaces $\mathfrak{F}(\mathcal{H}_{\mathbf{P}})$, $\mathfrak{F}(\mathcal{H}_{\mathbf{F}})$ of Definition 3.1 with horizon vacua $\Psi_{\mathbf{P}}$,$\Psi_{\mathbf{F}}$. With these realizations, the homomorphisms $\chi_{\mathbf{P}/\mathbf{F}}$ and $\pi_{\mathbf{P}/\mathbf{F}}$ can be implemented by unitary transformations which preserves the vacuum states. More precisely:*
(a) *If $m > 0$, the map that associates a positive frequency wavefunction ψ_+ in Rindler space with the element of $\mathcal{H}_{\mathbf{F}} \cong L^2(\mathbb{R}^+, dE)$, $\phi : E \mapsto M_{m,\kappa}(E)\tilde{\psi}_+(E)$ extends into the unitary operator $U_{\mathbf{F}} : \mathfrak{F}(\mathcal{H}) \to \mathfrak{F}(\mathcal{H}_{\mathbf{F}})$ such that*

$$U_{\mathbf{F}}\Psi_{\mathbf{R}} = \Psi_{\mathbf{F}} \quad (55)$$

$$\chi_{\mathbf{F}}(\hat{A}) = U_{\mathbf{F}}\hat{A}U_{\mathbf{F}}^{-1} \qquad \text{for all } \hat{A} \in \mathcal{A}_{\mathbf{R}}, \quad (56)$$

The analogous statement holds replacing $\mathbf{F}$ for $\mathbf{P}$.
(b) *If $m = 0$, the maps which associate positive frequency wavefunctions $\psi_+^{(in)}$ and $\psi_+^{(out)}$ in Rindler space with respectively elements of $\mathcal{H}_{\mathbf{F}} \cong L^2(\mathbb{R}^+, dE)$ and $\mathcal{H}_{\mathbf{P}} \cong L^2(\mathbb{R}^+, dE)$, $\phi^{(in)} : E \mapsto \tilde{\psi}_+^{(in)}(E)$ and $\phi^{(out)} : E \mapsto \tilde{\psi}_+^{(out)}(E)$, extend into unitary operators $V_{\mathbf{F}} : \mathfrak{F}(\mathcal{H}_{(in)}) \to \mathfrak{F}(\mathcal{H}_{\mathbf{F}})$ and $V_{\mathbf{P}} : \mathfrak{F}(\mathcal{H}_{(out)}) \to \mathfrak{F}(\mathcal{H}_{\mathbf{P}})$, such that*

$$V_{\mathbf{F}}\Psi_{\mathbf{R}}^{(in)} = \Psi_{\mathbf{F}} \,, \qquad V_{\mathbf{P}}\Psi_{\mathbf{R}}^{(out)} = \Psi_{\mathbf{P}} \,, \quad (57)$$

$$\pi_{\mathbf{F}}(\hat{A}) = V_{\mathbf{F}}\, \hat{A}\, V_{\mathbf{F}}^{-1} \qquad \text{for all } \hat{A} \in \mathcal{A}_{\mathbf{R}}^{(in)}, \quad (58)$$

$$\pi_{\mathbf{P}}(\hat{A}) = V_{\mathbf{P}}\, \hat{A}\, V_{\mathbf{P}}^{-1} \qquad \text{for all } \hat{A} \in \mathcal{A}_{\mathbf{R}}^{(out)}, \quad (59)$$

Proof. **(a)** We consider the case of $\mathbf{F}$, the case of $\mathbf{P}$ being similar. Under the identifications $\mathcal{H} \cong L^2(\mathbb{R}^+, dE)$ (Proposition 2.2) and $\mathcal{H}_{\mathbf{F}} \cong L^2(\mathbb{R}^+, dE)$ (Proposition 3.3), consider the map $V : \mathcal{H} \ni \psi \mapsto \phi \in \mathcal{H}_{\mathbf{F}}$ where we have defined $\phi(E) := N_{m,\kappa}(E)\psi(E)$ for all $\psi \in \mathcal{H}$. V is a unitary transformation by construction since $N_{m,\kappa}$ is a smooth function with $|N_{m,\kappa}(E)| = 1$ for all E as stated in Proposition 3.1. That unitary transformation can be extended into a unitary transformation $U_{\mathbf{F}} : \mathfrak{F}(\mathcal{H}) \to \mathfrak{F}(\mathcal{H}_{\mathbf{F}})$ by defining $U_{\mathbf{F}}\Psi_{\mathbf{R}} := \Psi_{\mathbf{F}}$ and $U_{\mathbf{F}}\restriction_{\mathcal{H}_s^{\otimes n}} := U_1 \otimes \cdots \otimes U_n$ for all $n = 1, 2, 3, \ldots$, where $\mathcal{H}_s^{\otimes n}$ indicates the symmetrized tensor product of n copies of $\mathcal{H}$ and $U_k = V$ for $k = 1, 2, \ldots, n$. $U_{\mathbf{F}}$ preserves the vacuum states by construction and induces a unital-$*$-algebra homomorphism $\rho : \mathcal{A}_{\mathbf{R}} \to \mathcal{A}_{\mathbf{F}}$ such that $\rho(A) = U_{\mathbf{F}}AU_{\mathbf{F}}^{-1}$ for every $A \in \mathcal{A}_{\mathbf{R}}$. To conclude the proof, by the uniqueness of $\chi_{\mathbf{F}}$ proven in Theorem 3.1, it is sufficient to show that $\rho(\hat{\phi}(f)) = \chi_{\mathbf{F}}(\hat{\phi}(f))$ for every $f \in \mathsf{D}(\mathbf{R}, \mathbb{R})$. To this end, take $f \in \mathsf{D}(\mathbf{R}, \mathbb{R})$ and consider the positive-frequency part of $\psi := Ef$, ψ_+. The construction used to define $U_{\mathbf{F}}$ implies that $U_{\mathbf{F}} a^\dagger(\psi_+) U_{\mathbf{F}}^{-1} = a^\dagger_{\mathbf{F}}(V\psi_+)$ and $U_{\mathbf{F}} a(\overline{\psi_+}) U_{\mathbf{F}}^{-1} = a_{\mathbf{F}}(\overline{V\psi_+})$ and thus, by Definitions 3.1 and 3.2, $U_{\mathbf{F}}\hat{\phi}(f)U_{\mathbf{F}}^{-1} = \hat{\phi}_{\mathbf{F}}(\omega_f)$ where $\omega_f = 2d\varphi_f$ with $\varphi_f(v) = \int_{\mathbb{R}^+} \frac{e^{-iEv}}{\sqrt{4\pi E}} N_{m,\kappa}(E)\tilde{\psi}_+(E)\, dE + \text{c.c.}$. By (a) of Proposition 3.1, $\varphi_f = (Ef)\restriction_{\mathbf{F}}$ and thus it holds $\rho(\hat{\phi}(f)) = U_{\mathbf{F}}\hat{\phi}(f)U_{\mathbf{F}}^{-1} = \chi_{\mathbf{F}}(\hat{\phi}(f))$ that concludes the proof. **(b)** The proof is strongly analogous to that in the massive case with obvious changes. □

Remark. Once again, the crucial difference between the massive and the massless case is that the Hilbert space of the bulk field is isomorphic to either the Fock spaces $\mathfrak{F}(\mathcal{H}_{\mathbf{F}})$ and $\mathfrak{F}(\mathcal{H}_{\mathbf{P}})$ if $m > 0$, whereas it is isomorphic to $\mathfrak{F}(\mathcal{H}_{\mathbf{F}}) \otimes \mathfrak{F}(\mathcal{H}_{\mathbf{P}})$ if $m = 0$. In the latter case the unitary transformation $V_{\mathbf{F}} \otimes V_{\mathbf{P}} : \mathfrak{F}(\mathcal{H}) \to \mathfrak{F}(\mathcal{H}_{\mathbf{F}}) \otimes \mathfrak{F}(\mathcal{H}_{\mathbf{P}})$, satisfies $(V_{\mathbf{F}} \otimes V_{\mathbf{P}})\Psi_{\mathbf{R}} = \Psi_{\mathbf{P}} \otimes \Psi_{\mathbf{F}}$ and $(\pi_{\mathbf{F}} \otimes \pi_{\mathbf{P}})(\hat{B}) = (V_{\mathbf{F}} \otimes V_{\mathbf{P}})\, \hat{B}\, (V_{\mathbf{F}}^{-1} \otimes V_{\mathbf{P}}^{-1})$ for all $\hat{B} \in \mathcal{A}_{\mathbf{R}}$.

4 Horizon manifest symmetry.

4.1. *$SL(2,\mathbb{R})$ unitary representations on the horizon.* Consider QFT on the future event horizon $\mathbf{F}$ in the Fock representation of the algebra $\mathcal{A}_{\mathbf{F}}$ referred to the vacuum state $\Psi_{\mathbf{F}}$. The one-particle space $\mathcal{H}_{\mathbf{F}}$ is isomorphic to $L^2(\mathbb{R}^+, dE)$. An irreducible unitary representation $\widetilde{SL}(2,\mathbb{R})$, $g \mapsto \mathcal{U}_{\mathbf{F}}(g)$, generated by the operators (27) $\overline{H_{\mathbf{F}0}}$, $\overline{C_{\mathbf{F}}}$ and $\overline{D_{\mathbf{F}}}$ with

$$H_{\mathbf{F}0} := E\,, \quad D_{\mathbf{F}} := -i\left(\frac{1}{2} + E\frac{d}{dE}\right)\,, \quad C_{\mathbf{F}} := -\frac{d}{dE}E\frac{d}{dE} + \frac{(k-\frac{1}{2})^2}{E}\,, \tag{60}$$

can uniquely be defined in $\mathcal{H}_{\mathbf{F}}$ as proven in Theorem 2.1. The operators (60) are defined on the dense invariant subspace $\mathcal{D}_{\mathbf{F}k} \subset L^2(\mathbb{R}^+, dE) \cong \mathcal{H}_{\mathbf{F}}$ which has the same definition as $\mathcal{D}_k$. If $m > 0$, that representation induces an analogous representation in the one-bulk-particle space $\mathcal{H}$ through unitary holography. That is $SL(2,\mathbb{R}) \ni g \mapsto \mathcal{U}(g) := U_{\mathbf{F}}^{-1}\mathcal{U}_{\mathbf{F}}(g)U_{\mathbf{F}}$ whose generators are $U_{\mathbf{F}}^{-1}\overline{H_{\mathbf{F}0}}U_{\mathbf{F}}$, $U_{\mathbf{F}}^{-1}\overline{D_{\mathbf{F}}}U_{\mathbf{F}}$ and $U_{\mathbf{F}}^{-1}\overline{C_{\mathbf{F}}}U_{\mathbf{F}}$. We stress that $g \mapsto \mathcal{U}(g)$ does *not* coincides with the representation $g \mapsto U(g)$ given in Theorem 2.1 but it is unitarily equivalent to that and thus (a) of Theorem 2.1 can be restated with trivial changes. Moreover (see below) $U_{\mathbf{F}}^{-1}\overline{H_{\mathbf{F}0}}U_{\mathbf{F}}$ still coincides with the Hamiltonian H of the bulk theory. As a consequence also the analogues of points (b) and (c) in Theorem 2.1 can be re-stated for the representation $g \mapsto \mathcal{U}(g)$ which, in turn, defines a $\widetilde{SL}(2,\mathbb{R})$-*symmetry* of the system in the bulk. We are interested on *that* $\widetilde{SL}(2,\mathbb{R})$-*symmetry* which is induced by the $\widetilde{SL}(2,\mathbb{R})$ unitary representation on the horizon QFT via (unitary) holography nomatter the mass of the field in the bulk. We stress that this $\widetilde{SL}(2,\mathbb{R})$-symmetry is *hidden* in the bulk because the same argument used in 2.5 applies to this case too, however it could be manifest, in the sense of 2.5, when examined on the horizon. That is the issue we want to discuss in the following.
Everything we have said for $\mathbf{F}$ can be re-stated for $\mathbf{P}$ with obvious changes. If $m = 0$ and using (b) of Theorem 3.2, everything we said above concerning the representations of $\widetilde{SL}(2,\mathbb{R})$ in $\mathcal{H}_{\mathbf{F}}$ and those induced on $\mathcal{H}$ by means of $U_{\mathbf{F}}$ can be restated concerning the triples $\mathcal{H}_{\mathbf{F}}$, $\mathcal{H}_{(\text{in})}$, $V_{\mathbf{F}}$

and $\mathcal{H}_{\mathbf{P}}$, $\mathcal{H}_{(\mathrm{out})}$, $V_{\mathbf{P}}$ separately. Moreover by the comment after Theorem 3.2, one sees that a pair of $SL(2,\mathbb{R})$ representations in $\mathcal{H}_{\mathbf{F}}$ and $\mathcal{H}_{\mathbf{P}}$ naturally induces a *reducible* $SL(2,\mathbb{R})$ on $\mathcal{H}$ by means of $V_{\mathbf{F}} \otimes V_{\mathbf{P}}$.

4.2. *Horizon analysis of the bulk symmetry associated with* $H_{\mathbf{F}0}$. Let us focus attention on the first generator $H_{\mathbf{F}0}$ in the case $m > 0$. Concerning QFT on $\mathbf{P}$ and the case $m = 0$, there are completely analogous results. From now on we use the following conventions referring to a representation of an algebra of observables $\mathcal{A}$ in a symmetrized Fock space $\mathfrak{F}(\mathcal{H})$. If X is a self-adjoint operator in the one-particle Hilbert space $\mathcal{H}$ and $\hat{A} \in \mathcal{A}$, $\hat{A}^{(X)}_\tau := e^{i\tau\mathbf{X}}\hat{A}e^{-i\tau\mathbf{X}}$ where $\mathbf{X} := 0 \oplus X \oplus (X \otimes I + I \otimes X) \oplus \cdots$ is the operator naturally associated with X in the Fock space $\mathfrak{F}(\mathcal{H}) = \mathbb{C} \oplus \mathcal{H} \oplus (\mathcal{H} \otimes \mathcal{H})_s \oplus \cdots$. In other words, $\hat{A}^{(X)}_\tau$ is the *Heisenberg evolution* of $\hat{A}$ at time τ with respect to the noninteracting multiparticle Hamiltonian $\mathbf{X}$ induced by the one-particle Hamiltonian X. We have the following theorems.

Theorem 4.1. *Unitary holography associates the self-adjoint operator* $\overline{H_{\mathbf{F}0}}$ *with the one-particle Hamiltonian in the bulk* H *(25), i.e.,*

$$U_{\mathbf{F}}^{-1}\overline{H_{\mathbf{F}0}}U_{\mathbf{F}} = H\,. \tag{61}$$

Defining $H_{\mathbf{F}} := \overline{H_{\mathbf{F}0}}$, *the following further statements hold.*
(a) *Referring to Fock representations of algebras of observables* $\mathcal{A}_{\mathbf{R}}$ *and* $\mathcal{A}_{\mathbf{F}}$ *on vacuum states* $\Psi_{\mathbf{R}}$ *and* $\Psi_{\mathbf{F}}$, *Heisenberg-like evolution is preserved by unitary holography:*

$$U_{\mathbf{F}}\hat{A}^{(H)}_\tau U_{\mathbf{F}}^{-1} = (U_{\mathbf{F}}\hat{A}U_{\mathbf{F}}^{-1})^{(H_{\mathbf{F}})}_\tau\,, \tag{62}$$

(b) $\{e^{i\tau H_{\mathbf{F}}}\}_{\tau\in\mathbb{R}}$ *induces, via (42) a group of transformations* $\{\alpha^{(\partial_v)}_\tau\}_{\tau\in\mathbb{R}}$ *of horizon wavefunctions* φ *such that*

$$\left(\alpha^{(\partial_v)}_\tau(\varphi)\right)(v) := \varphi(v-\tau) \quad \textit{for all } \varphi \in \mathcal{S}_{\mathbf{F}} \textit{ and } v \in \mathbb{R}\,. \tag{63}$$

That is the same group of transformations of functions induced by the group of diffeomorphisms of $\mathbf{F}$ *generated by the vector field* ∂_v.
(c) *If* $\{\alpha^{(\partial_t)}_\tau\}_{\tau\in\mathbb{R}}$ *denotes the one-parameter group of Rindler-time displacements of Rindler wavefunctions (see 2.4),*

$$\alpha^{(\partial_v)}_\tau(\psi\restriction_{\mathbf{F}}) = \left(\alpha^{(\partial_t)}_\tau(\psi)\right)\restriction_{\mathbf{F}} \quad \textit{for all } \psi \in \mathcal{S}\,. \tag{64}$$

Proof. Consider the self-adjoint operator on $\mathcal{H}_{\mathbf{F}} \cong L^2(\mathbb{R}^+, dE)$:

$$(H_{\mathbf{F}}f)(E) := Ef(E) \text{ for } f \in \mathcal{D}(H_{\mathbf{F}}) = \{h \in L^2(\mathbb{R}^+, dE) \mid \textstyle\int_0^{+\infty} E^2|h(E)|^2 dE < +\infty\}\,. \tag{65}$$

Since $\mathcal{D}_{\mathbf{F}k} \subset \mathcal{D}(H_{\mathbf{F}})$ and $H_{\mathbf{F}0} = H$ in $\mathcal{D}_{\mathbf{F}k}$ where $H_{\mathbf{F}0}$ is essentially self-adjoint, it must hold $H_{\mathbf{F}} = \overline{H_{\mathbf{F}0}}$. The definition of $U_{\mathbf{F}}$ (its restriction to $\mathcal{H}$ is sufficient) given in (a) in Theorem 3.2,

(25) and (65) entail (61). **(a)** is an immediate consequence of (61). **(b)** By Proposition 3.2 and (42), $\varphi \in \mathcal{S}_{\mathbf{F}}$ is the Fourier (anti)transform of a Schwartz' function f with $\tilde{\varphi}_+(E) = \sqrt{E}f(E)$ if $E \geq 0$ and the application of $e^{i\tau H_{\mathbf{F}}}$ on $\tilde{\varphi}_+$ changes f into $\mathbb{R} \ni E \mapsto e^{iE\tau}f(E)$ which still is a Schwartz' function. Hence, $\alpha^{(\partial_t)}_\tau(\varphi)$ is constructed by: (1) Fourier transforming φ into f, (2) replacing $f(E)$ by $e^{iE\tau}f(E)$ and (3) transforming back that function into $\alpha^{(\partial_t)}_\tau(\varphi)$ via Fourier transformation. By direct inspection one finds $(\alpha^{(\partial_t)}_\tau(\varphi))(v) = \varphi(v-\tau)$ trivially. **(c)** In $\mathcal{H} \cong L^2(\mathbb{R}^+, dE)$ and $\mathcal{H}_{\mathbf{F}} \cong L^2(\mathbb{R}^+, dE)$, (61) states that both $e^{i\tau H}$ and $e^{i\tau H_{\mathbf{F}}}$ are represented by the same multiplicative operator $e^{i\tau E}$ in the respective spaces. Then (14) and (34) imply (64). □

Remark. Since the one-parameter unitary group generated by $H_{\mathbf{F}}$ turns out to be associated with a vector field of $\mathbf{F}$, ∂_v, which induces a group of (orientation-preserving) diffeomorphisms, the bulk-symmetry generated by $H_{\mathbf{F}}$ via unitary holography is *manifest* also on the horizon.

The machinery can be implemented at algebraic level. To this end, using the relation (see Proposition 3.4) $\omega = 2dE_{\mathbf{F}}\omega$, define the one-parameter group of transformations of forms $\omega \in \mathsf{D}(\mathbf{F},\mathbb{R})$ $\{\beta^{(\partial_v)}_\tau\}_{\tau\in\mathbb{R}}$, where $(\beta^{(\partial_v)}_\tau(\omega))(v) := 2d(\alpha^{(\partial_v)}_\tau(E_{\mathbf{F}}\omega))$. Finally, define the action of $\beta^{(\partial_v)}_\tau$ on quantum fields as $\gamma^{(\partial_v)}_\tau(\phi_{\mathbf{F}}(\omega)) := \phi_{\mathbf{F}}(\beta^{(\partial_v)}_{-\tau}(\omega))$, for $\omega \in \mathsf{D}(\mathbf{F},\mathbb{C})$. One has the following result.

Theorem 4.2. *The transformations $\gamma^{(\partial_v)}_\tau$, $\tau \in \mathbb{R}$ uniquely extended into a group of automorphisms of $\mathcal{A}_{\mathbf{F}}$, $\{\gamma^{(\partial_v)}_\tau\}_{\tau\in\mathbb{R}}$ such that:*
(a) *if $\{\gamma^{(\partial_t)}_\tau\}_{\tau\in\mathbb{R}}$ denotes the analogous group of automorphisms of the bulk algebra $\mathcal{A}_{\mathbf{R}}$ generated by Rindler time-displacements,*

$$\left(\chi_{\mathbf{F}} \circ \gamma^{(\partial_t)}_\tau\right)(A) = \left(\gamma^{(\partial_v)}_\tau \circ \chi_{\mathbf{F}}\right)(A) \quad \textit{for all } A \in \mathcal{A}_{\mathbf{F}} \textit{ and } \tau \in \mathbb{R}. \tag{66}$$

(b) *In the Fock space realization of $\mathcal{A}_{\mathbf{F}}$ referred to $\Psi_{\mathbf{F}}$,*

$$(\hat{B})^{(H_{\mathbf{F}})}_\tau = \gamma^{(\partial_v)}_\tau(\hat{B}) \quad \textit{for all } \hat{B} \in \mathcal{A}_{\mathbf{F}} \textit{ and } \tau \in \mathbb{R}. \tag{67}$$

Sketch of proof. $\alpha^{(\partial_v)}_\tau(E_{\mathbf{F}}\omega) = E_{\mathbf{F}}\beta^{(\partial_v)}_\tau(\omega)$, the preservation of the symplectic form under the action of $\alpha^{(\partial_v)}_\tau$ and Proposition 3.4 entail $E_{\mathbf{F}}(\beta^{(\partial_v)}_\tau(\omega), \beta^{(\partial_v)}_\tau(\omega')) = E_{\mathbf{F}}(\omega,\omega')$. This property trivially extended to complex valued forms. $\gamma^{(\partial_v)}_\tau$ must be extended on the whole algebra $\mathcal{A}_{\mathbf{F}}$ requiring the preservation of the unital $*$-algebra structure. The proof of the existence of such an extension is based on the preservation of the causal propagator established above. If $A = \phi(f)$, (66) an immediate consequence of (64) and the definition of $\chi_{\mathbf{F}}$ in Theorem 3.2. Then (66) extends to the whole algebra since $\gamma^{(\partial_v)}_\tau$, $\gamma^{(\partial_t)}_\tau$ and $\chi_{\mathbf{F}}$ are homomorphisms. (67) is an immediate consequence of the fact that $\gamma^{(\partial_v)}_\tau(\hat{\phi}_{\mathbf{F}}(\omega))$ is the Heisenberg-like evolution of $\hat{\phi}_{\mathbf{F}}(\omega)$ induced by the "Hamiltonian" $H_{\mathbf{F}}$ and evaluated at "time" τ. □

4.3. *Horizon analysis of the bulk symmetry associated with* $D_{\mathbf{F}}$. Let us examine the properties of the unitary one-parameter group, $\{e^{i\,\overline{\mu D_{\mathbf{F}}}}\}_{\mu\in\mathbb{R}}$.

Theorem 4.3. *The unitary one-parameter group,* $\{e^{i\,\overline{\mu D_{\mathbf{F}}}}\}_{\mu\in\mathbb{R}}$ *enjoys the following properties.*
(a) *If* $\tilde{\varphi}\in L^2(\mathbb{R}^+,dE)\cong\mathcal{H}_{\mathbf{F}}$, *for all* $\mu\in\mathbb{R}$ *and* $E\in\mathbb{R}^+$,

$$(e^{i\mu\,\overline{D_{\mathbf{F}}}}\tilde{\varphi})(E) \;=\; e^{\mu/2}\tilde{\varphi}(e^{\mu}E)\,. \tag{68}$$

(b) *By means of (42),* $\{e^{i\,\overline{\mu D_{\mathbf{F}}}}\}_{\mu\in\mathbb{R}}$ *induces a group* $\{\alpha^{(v\partial_v)}_\mu\}_{\mu\in\mathbb{R}}$ *of transformations of horizon wavefunctions* φ *with*

$$\left(\alpha^{(v\partial_v)}_\mu(\varphi)\right)(v) \;:=\; \varphi\left(e^{-\mu}v\right)\,. \tag{69}$$

for all $\varphi\in\mathcal{S}_{\mathbf{F}}$ *and* $\mu\in\mathbb{R}$. $\{\alpha^{(v\partial_v)}_\mu\}_{\mu\in\mathbb{R}}$ *is the same group of transformations of functions associated with the group of diffeomorphisms of* $\mathbf{F}$ *induced by the vector field* $v\partial_v$.

Sketch of proof. **(a)** Consider the one-parameter group of unitary operators $\{V_\mu\}_{\mu\in\mathbb{R}}$ with $V_\mu(\tilde{\varphi})(E)=e^{\mu/2}\tilde{\varphi}(e^{\mu}E)$, for $\tilde{\varphi}\in L^2(\mathbb{R}^+,dE)$. For every $f\in\mathcal{D}_{\mathbf{F}k}$, $\langle f,V_\mu\tilde{\varphi}\rangle=\langle V_{-\mu}f,\tilde{\varphi}\rangle$. On the other hand, using the definition of Schwartz space and Lebesgue's dominated-convergence theorem, it is simply proven that $V_{-\mu}f\to f$ as $\mu\to 0$ and so $\langle f,V_\mu\tilde{\varphi}\rangle\to\langle f,\tilde{\varphi}\rangle$ as $\mu\to 0$ for every $f\in\mathcal{D}_{\mathbf{F}k}$ which is dense in $L^2(\mathbb{R}^+,dE)$. As a consequence $\{V_\mu\}_{\mu\in\mathbb{R}}$ is weakly continuous and thus strongly continuous it being made of unitary operators and Stone's theorem can be used. With a similar procedure (also using Lagrange's theorem to estimate an incremental ratio) one gets that, if $\tilde{\varphi}\in\mathcal{D}_{\mathbf{F}k}$ and interpreting the derivative in the topology of $L^2(\mathbb{R}^+,dE)$, $\frac{d}{d\mu}|_{\mu=0}(V_\mu\tilde{\varphi})$ can be computed pointwisely. A straightforward calculation of the pointwise derivative gives $\frac{d}{d\mu}|_{\mu=0}(V_\mu\tilde{\varphi})=i(D_{\mathbf{F}})\tilde{\varphi}$. Stone's theorem implies that generator G of $V_\mu=e^{i\mu G}$ is well-defined on $\mathcal{D}_{\mathbf{F}k}$ and coincides with $D_{\mathbf{F}}$ therein. Since $D_{\mathbf{F}}$ is essentially self-adjoint on that domain it must be $G=\overline{D_{\mathbf{F}}}$ and this proves (a). **(b)** Take $\varphi\in\mathcal{S}_{\mathbf{F}}$, use the decomposition (42) as in the proof of Theorem 4.1, and transform $\tilde{\varphi}_+\in L^2(\mathbb{R},dE)$ under the action of $e^{i\,\overline{\mu D_{\mathbf{F}}}}$ taking (68) into account. With a trivial change of variables in the decomposition (42) one sees that, if φ belongs to Schwartz' space, the obtained transformed wavefunction is just $\varphi\left(e^{-\mu}v\right)$ which still is in $\mathcal{S}_{\mathbf{F}}$. □

Remark. Since the one-parameter unitary group generated by $D_{\mathbf{F}}$ turns out to be associated with the vector field of $\mathbf{F}$, $v\partial_v$, which induces a group of (orientation-preserving) diffeomorphisms, the bulk-symmetry generated by $D_{\mathbf{F}}$ via unitary holography is *manifest* on the horizon.

Once again the machinery can be implemented at algebraic level. We consider the group associated with $v\partial_v$ only. Define the one-parameter group of transformations of forms $\omega\in\mathsf{D}(\mathbf{F},\mathbb{R})$, $\{\beta^{(v\partial_v)}_\tau\}_{\tau\in\mathbb{R}}$, with $(\beta^{(v\partial_v)}_\tau(\omega))(v):=2d(\alpha^{(v\partial_v)}_\tau(E_{\mathbf{F}}\omega))$. Finally, extend the action of $\beta^{(v\partial_v)}_\tau$ on quantum fields as $\gamma^{(v\partial_v)}_\tau(\phi_{\mathbf{F}}(\omega)):=\phi_{\mathbf{F}}(\beta^{(v\partial_v)}_{-\tau}(\omega))$, for $\omega\in\mathsf{D}(\mathbf{F},\mathbb{C})$. The following result, whose proof is essentially the same as that of the relevant part of Theorem 4.2, holds.

Theorem 4.4. *Transformations* $\gamma_\tau^{(v\partial_v)}$ *uniquely extended into a one-parameter group of automorphisms of* $\mathcal{A}_{\mathbf{F}}$, $\{\gamma_\tau^{(v\partial_v)}\}_{\tau\in\mathbb{R}}$ *such that in the Fock space realization of* $\mathcal{A}_{\mathbf{F}}$ *referred to* $\Psi_{\mathbf{F}}$,

$$(\hat{B})_\tau^{(D_{\mathbf{F}})} = \gamma_\tau^{(v\partial_v)}(\hat{B}) \quad \textit{for all } \hat{B} \in \mathcal{A}_{\mathbf{F}} \textit{ and } \tau \in \mathbb{R}. \tag{70}$$

4.4. *Horizon analysis of the unitary group generated by* $C_{\mathbf{F}}$. The analysis of the action of the group generated by $C_{\mathbf{F}}$ is much more complicated than the other considered cases. The point is the following. A necessary condition to associate with a transformed state $e^{it\overline{C_{\mathbf{F}}}}\psi$ ($\psi \in \mathcal{H}_{\mathbf{F}}$) a wavefunction of $\mathcal{S}_{\mathbf{F}}$ by (43) (with $\tilde{\varphi}_+ = e^{it\overline{C_{\mathbf{F}}}}\psi$ and taking the real part of the right-hand side) is that $e^{it\overline{C_{\mathbf{F}}}}\psi$ belong to the domain of $H_{\mathbf{F}}^{-1/2}$. Indeed in the general case (43) must be interpreted as the Fourier-Plancherel transform[4] of the $L^2(\mathbb{R}, dE)$ function given by 0 if $E < 0$ and $\left((4\pi H_{\mathbf{F}})^{-1/2} e^{it\overline{C_{\mathbf{F}}}}\psi\right)(E)$ if $E \geq 0$. That requirement is, in fact, fulfilled concerning $e^{i\overline{uH_{\mathbf{F}}+vD_{\mathbf{F}}}}\psi$ if $\psi \in \mathcal{S}_{\mathbf{F}}$ because $e^{i\overline{uH_{\mathbf{F}}+vD_{\mathbf{F}}}}\psi \in \mathcal{S}_{\mathbf{F}}$ and so the usual Fourier transformation is sufficient to interpret the formalism. Concerning $C_{\mathbf{F}}$ the extent needs a careful treatment and the space $\mathcal{S}_{\mathbf{F}}$ must, in fact, changed in order to assure that $e^{it\overline{C_{\mathbf{F}}}}\psi$ belongs to the domain of $H_{\mathbf{F}}^{-1/2}$. There are several possibilities to do it at least in the case $k = 1$ in the definition of $C_{\mathbf{F}}$. To go on we need some preliminary results. If $k = 1$, focus attention on the operator analogous to K in the proof of Theorem 2.1, $K_{\mathbf{F}} := \beta H_{\mathbf{F}0} + \beta^{-1} C_{\mathbf{F}}$. It is known [10] that $\sigma\left(\overline{K_{\mathbf{F}}}\right) = \{1, 2, \ldots\}$ (nomatter the value of $\beta > 0$) with corresponding eigenvectors $Z_1^{(1)}, Z_2^{(1)}, \ldots$ (which do depend on β) given in (26). Thus defining $\Theta := e^{i\pi\overline{K_{\mathbf{F}}}}$ one also gets $\Theta = \Theta^\dagger = \Theta^{-1}$. $\{\Theta, I\}$ is the image under $\mathcal{U}$ of the discrete subgroup $\{\vartheta, \vartheta^2 = -I, \vartheta^3 = -\vartheta, \vartheta^4 = I\} \subset SL(2, \mathbb{R})$ with

$$\vartheta = \begin{bmatrix} 0 & \beta \\ -\beta^{-1} & 0 \end{bmatrix} = e^{\pi(\beta h + \beta^{-1} c)/2}\,, \tag{71}$$

Proposition 4.1. *Fix* $k = 1$ *in the definition (60) so that the representation of* $\widetilde{SL}(2, \mathbb{R})$ *is in fact a representation of* $SL(2, \mathbb{R})$. *For every* $\beta > 0$,

$$\Theta\, \beta H_{\mathbf{F}}\, \Theta = \frac{1}{\beta}\overline{C_{\mathbf{F}}} \quad , \quad \Theta\, \overline{D_{\mathbf{F}}}\, \Theta = -\overline{D_{\mathbf{F}}}\,, \tag{72}$$

$-\Theta$ *is nothing but the* J_1*-Hankel unitary transform:*

$$(-\Theta\psi)(E) := \beta \lim_{M\to+\infty} \int_0^M J_1\left(\beta\sqrt{4EE'}\right)\psi(E')dE' \ , \quad \textit{for all } \psi \in L^2(\mathbb{R}^+, dE), \tag{73}$$

where the limit is computed in the norm of $L^2(\mathbb{R}^+, dE)$ *and coincides with the* L^1 *integral over* $\mathbb{R}^+$ *if* $E \mapsto E^{-1/4}\psi(E)$ *belongs to* $L^1(\mathbb{R}^+, dE)$ *and* $E \mapsto \sqrt{E}\psi(E)$ *belongs to* $L^1([0,1], dE)$.

[4]That is the unique unitary extension of the Fourier transform defined on $L^2(\mathbb{R}, dE)$.

Sketch of proof. By Stone's theorem, identities in (72) are equivalent to analogue identities with self-adjoint operators $H_{\mathbf{F}}$, $\overline{C_{\mathbf{F}}}$ and $\overline{D_{\mathbf{F}}}$ replaced by the respectively generated one-parameter unitary groups. In that form, the thesis can be proven, first for the corresponding one-parameter groups in $SL(2,\mathbb{R})$, using simple analytic procedures based on the uniqueness theorem of the matrix-valued solutions of differential equations, and then the result can be extended to unitary operators using the representation introduced in Theorem 2.1. The second part arises straightforwardly from chapter 9 in [24] with trivial adaptations of the definitions. □

Proposition 4.2. *Take $\varphi \in \mathcal{S}_{\mathbf{F}}$ using notation as in (42), define $\tilde{\varphi}_{\beta+} := \Theta\tilde{\varphi}_+$ and*

$$\varphi_\beta(v) = \varphi\left(-\frac{\beta^2}{v}\right) - \varphi(0) \quad \textit{for all } v \in \mathbf{F}. \tag{74}$$

(a) *$\varphi \mapsto \varphi_\beta$ is the transformation induced by Θ on wavefunctions, i.e. (42) holds by replacing φ for φ_β and $\tilde{\varphi}_+$ for $\tilde{\varphi}_{\beta+}$.*
(b) *If $X := \frac{\beta}{2}\partial_v + \frac{1}{2\beta}v^2\partial_v$ and $\alpha_\epsilon^{(X)}(\varphi)$ denotes the natural action of the local one-parameter group of diffeomorphisms generated by X on φ, the first term in the right hand side of (74) is*

$$\lim_{\epsilon\to\pi}\left(\alpha_\epsilon^{(X)}(\varphi)\right)(v)\,, \quad \textit{for all } v \in \mathbf{F}. \tag{75}$$

Sketch of proof. By hypotheses $\tilde{\varphi}_+$ satisfies the conditions which enables us to represent $\Theta\tilde{\varphi}_+$ as in (73). In that case, by the expansion of $J_1(x)$ at $x=0$, one sees that the L^2, and continuous on $(0,+\infty)$, function $E \mapsto (\Theta\tilde{\varphi}_+)(E)$ is $O(E^{1/2})$ as $E \to 0^+$ and thus it belongs to the domain of $H_{\mathbf{F}}^{-1/2}$. Using Fubini-Tonelli's and dominated convergence theorems we have that $\varphi_\beta(v)$ reads, (where the limit in the left-hand side is in the L^2-convergence sense),

$$\lim_{\epsilon\to 0^+}\int_0^\infty e^{-iE(v-i\epsilon)}\frac{(\Theta\tilde{\varphi}_+)(E)}{\sqrt{4\pi E}}dE = -\beta\int_0^\infty\lim_{\epsilon\to 0^+}\left(\int_0^\infty e^{-iE(v-i\epsilon)}\frac{J_1(\beta\sqrt{4EE'})}{\sqrt{4\pi E}}dE\right)\tilde{\varphi}_+(E')dE'\,.$$

The limit in right-hand side can explicitly be computed by using known results [20] obtaining that it is $(e^{iE'\beta^2/v}-1)/\sqrt{4\pi E'}$. This result produces $\varphi_\beta(v) = \varphi(-\beta^2/v) - \varphi(0)$. Concerning the second statement, it is simply proven that, for $\epsilon \in (-\pi,\pi)$,

$$\left(\alpha_\epsilon^{(X)}(\varphi)\right)(v) = \varphi\left(\frac{-\beta^2\tan(\epsilon/2) + \beta v}{\beta + v\tan(\epsilon/2)}\right)\,.$$

With our hypotheses for φ, the limit as $\epsilon \to \pi$ is well defined for every $v \in \mathbb{R}$ and proves the statemet in (b).□

By direct inspection and using (74) one sees that, if $\varphi \in \mathcal{S}_{\mathbf{F}}$, usually $\varphi_\beta \notin \mathcal{S}_{\mathbf{F}}$, but $\varphi_\beta \in \mathsf{W}_\infty(\mathbb{R})$ in any cases, the latter being the Sobolev space of the C^∞ complex-valued functions which are $L^2(\mathbb{R}, dv)$ with all of derivatives of every order.

Now, using (72), the geometric action of $e^{i\lambda \overline{C_{\mathbf{F}}}} = \Theta e^{i\lambda\beta^2 H_{\mathbf{F}}}\Theta$ can easily be computed for wavefunctions φ of $\mathcal{S}_{\mathbf{F}}$ such that $\varphi(0) = 0$ and $v \mapsto \varphi(-1/v)$ still belongs to $\mathcal{S}_{\mathbf{F}}$. Take such a φ, extract $\tilde{\varphi}_+$ and apply Θ. The resulting wavefunction is an element of $\mathcal{S}_{\mathbf{F}}$ by Proposition 4.2. The application of the one-parameter group generated by $\beta^2 H_{\mathbf{F}}$, $e^{i\lambda\beta^2 H_{\mathbf{F}}}$, gives rise to wavefunctions (see Theorem 4.1) $v \mapsto \varphi(-\beta^2/(v-\beta^2\lambda))$ which still belongs to $\mathcal{S}_{\mathbf{F}}$. Finally, since it is possible, apply Θ once again. All that procedure is equivalent to apply the group $e^{i\lambda \overline{C_{\mathbf{F}}}} = \Theta e^{i\lambda\beta^2 H_{\mathbf{F}}}\Theta$, on the initial $\tilde{\varphi}_+$. By this way one gets that the following theorem.

Theorem 4.5. *Consider the horizon wavefunctions $\varphi \in \mathcal{S}_{\mathbf{F}}$ such that such that $\varphi(0) = 0$ and $v \mapsto \varphi(-1/v)$ still belongs to $\mathcal{S}_{\mathbf{F}}$. The unitary group, $\{e^{i\lambda \overline{C_{\mathbf{F}}}}\}_{\lambda\in\mathbb{R}}$ induces a class $\{\alpha^{(v^2\partial_v)}_\lambda\}_{\lambda\in\mathbb{R}}$ of transformations of the said wavefunctions by means of (42), with*

$$\left(\alpha^{(v^2\partial_v)}_\lambda(\varphi)\right)(v) \quad := \quad \varphi\left(\frac{v}{1+\lambda v}\right) - \varphi\left(\frac{1}{\lambda}\right), \quad \textit{for all } \lambda \in \mathbb{R}. \tag{76}$$

The transformation of wavefunctions defined by the first term in the right-hand side of (76) is that generated by the local *group of diffeomorphisms of* $\mathbf{F}$ *associated with the field $v^2\partial_v$.*

Remarks. **(1)** In our hypotheses, $\alpha^{(v^2\partial_v)}_\lambda(\varphi) \in \mathsf{W}_\infty(\mathbb{R})$, but in general $\alpha^{(v^2\partial_v)}_\lambda(\varphi) \notin \mathcal{S}_{\mathbf{F}}$ so that the class of transformations does not define a group of transformations of wavefunctions in $\mathcal{S}_{\mathbf{F}}$. It is worthwhile stressing that these transformations define a group when working on the space $\mathcal{E}_{\mathbf{F}}$ of complex wavefunctions $\psi = \psi(v)$ whose positive-frequency and negative-frequency parts of Fourier transform are linear combinations of functions $E \mapsto Z^{(1)}_n(|E|)/\sqrt{4\pi|E|}$. In fact $\mathcal{E}_{\mathbf{F}}$ is invariant under (76). On the other hand $\mathcal{E}_{\mathbf{F}} \cap (\mathcal{S}_{\mathbf{F}} + i\mathcal{S}_{\mathbf{F}}) = \emptyset$.
(2) The integral curves of the field $v^2\partial_v$, $v(t) = v(0)/(1-tv(0))$, have domain which *depends on the initial condition*: That is $\mathbb{R}\setminus\{1/v(0)\}$, and $v(t)$ diverges if t approaches the singular point (barring the initial condition $v(0) = 0$ that produces a constant orbit). Thus the one-parameter group of (orientation-preserving) diffeomorphisms generated by $v^2\partial_v$ is only *local*. However, as the functions in $\mathcal{S}_{\mathbf{F}}$ vanish at infinity with all their derivatives, the singular point of the domain is harmless in (76).
(3) It makes sense to extend the definition of symplectically-smeared field operator when $\varphi \in \mathsf{W}_\infty(\mathbb{R})$ by means of Definition 3.1. Indeed the Fourier-Plancherel transform of φ, f satisfies $\int_{\mathbb{R}^+}(1+|E|^k)^2|f(E)|^2 dE < \infty$, for $k = 0, 1, 2, \ldots$ and so $\mathbb{R}^+ \ni E \to \tilde{\varphi}_+(E) := \sqrt{4\pi E}f(E)$ is a one-particle quantum state of $L^2(\mathbb{R}^+, dE)$. With the same hypotheses $E_{\mathbf{F}}d\varphi$ is well defined[5] and enjoys the relevant properties stated in Proposition 3.4 and 3.5. Then enlarging $\mathsf{D}(\mathbf{F},\mathbb{C})$ to include elements $\omega = d\varphi$ where φ is real and belongs to $\mathsf{W}_\infty(\mathbb{R})$, one can define $\hat{\phi}(\omega)$ as in Definition 3.2 non affecting the relevant properties stated in Proposition 3.4 and 3.5. By this way, the algebraic approach can be implemented in terms of formal quantum fields smeared by

[5]In particular $d^k\varphi(v)/dv^k \to 0$ as $v \to \pm\infty$ for $k = 0, 1, 2, \ldots$: by elementary calculus and Cauchy-Schwartz inequality, every $d^k\varphi(v)/dv^k$ is uniformly continuous. If $d^k\varphi(v)/dv^k \not\to 0$ as $v \to \pm\infty$ for some k, there are $\epsilon > 0$ and a sequence of intervals I_n with $\int_{I_n} dv = l > 0$ and $|d^k\varphi(v)/dv^k\restriction_{I_n}| > \epsilon$. Thus $\int_{\mathbb{R}}|d^k\varphi(v)/dv^k|^2 dv = \infty$ which is impossible.

functions of $\mathsf{W}_\infty(\mathbb{R})$.

The action of the one-group generated by $C_{\mathbf{F}}$ can be implemented at algebraic level. If $\omega \in \mathsf{D}(\mathbf{F},\mathbb{R})$ (without the enlargement said in the remark (3) above), one can define $(\beta^{(v^2\partial_v)}_\tau(\omega)) := 2d(\alpha^{(v^2\partial_v)}_\tau(E_{\mathbf{F}}\omega))$. By direct inspection one sees that each $\alpha^{(v^2\partial_v)}_\tau$ preserves the symplectic form $\Omega_{\mathbf{F}}$ and each $\beta^{(v^2\partial_v)}_\tau$ preserves the causal propagator $E_{\mathbf{F}}$. Notice that these results are not evident *a priori* since the action of $\alpha^{(v^2\partial_v)}_\tau$ (76) is not that canonically induced by a vector field. Finally, extend the action of $\beta^{(v^2\partial_v)}_\tau$ on quantum fields as $\gamma^{(v^2\partial_v)}_\tau(\phi_{\mathbf{F}}(\omega)) := \phi_{\mathbf{F}}(\beta^{(v\partial_v)}_{-\tau}(\omega))$, for $\omega \in \mathsf{D}(\mathbf{F},\mathbb{C})$. The following result, whose proof is essentially the same as that of the relevant part of Theorem 4.2, holds.

Theorem 4.6. *Transformations $\gamma^{(v^2\partial_v)}_\tau$ uniquely extended into a one-parameter class of automorphisms of $\mathcal{A}_{\mathbf{F}}$, $\{\gamma^{(v^2\partial_v)}_\tau\}_{\tau\in\mathbb{R}}$ such that in the Fock space realization of $\mathcal{A}_{\mathbf{F}}$ referred to $\Psi_{\mathbf{F}}$,*

$$(\hat{B})^{(\overline{C_{\mathbf{F}}})}_\tau = \gamma^{(v^2\partial_v)}_\tau(\hat{B}) \quad \textit{for all } \hat{B} \in \mathcal{A}_{\mathbf{F}} \textit{ and } \tau \in \mathbb{R}. \tag{77}$$

4.5. *The full $SL(2,\mathbb{R})$ action.* To conclude we show the general action of $\mathfrak{U}(SL(2,\mathbb{R}))$ on horizon wavefunctions. With a strightforward generalization of the notion of manifest symmetry due to the appearance of the addend in the right-hand side of (79) below, the symmetry associated with the whole group $SL(2,\mathbb{R})$ can be considered as *manifest*. We leave possible comments on the field algebra extension to the reader. Remind that $\vartheta := e^{\pi(\beta h+\beta^{-1}c)/2} \in SL(2,\mathbb{R})$ and consider

$$A = \left[\begin{array}{cc} a & b \\ c & d \end{array}\right] \in SL(2,\mathbb{R})\ . \tag{78}$$

Referring to (71) and generators (29), only one of the following facts hold for suitable λ,μ,τ uniquely determined by a,b,c,d in the examined cases: If $a>0$, $A = e^{\lambda c}e^{\mu d}e^{\tau h}$ or, if $a<0$, $A = \vartheta e^{\lambda c}e^{\mu d}e^{\tau h}$, or, if $a=0$ and $b>0$, $A = \vartheta e^{\mu d}e^{\tau h}$, or, if $a=0$ and $b<0$, $A = \vartheta^3 e^{\mu d}e^{\tau h}$. Using these decompositions, part of Theorems 4.1, 4.3, 4.5 and Proposition 4.2 the following final theorem can simply be proven.

Theorem 4.6. *Take $\varphi \in \mathcal{S}_{\mathbf{F}}$ such that such that $\varphi(0)=0$ and $v \mapsto \varphi(-1/v)$ still belongs to $\mathcal{S}_{\mathbf{F}}$. If $A \in SL(2,\mathbb{R})$ has the form (78), let $\alpha^{(A)}(\varphi)$ denote the right-hand side of (42) with $\tilde{\varphi}_+$ replaced for $\mathfrak{U}(A)\tilde{\varphi}_+$ where $\tilde{\varphi}_+$ is defined as in (41). For $v\in\mathbb{R}$ it holds*

$$\left(\alpha^{(A)}(\varphi)\right)(v) = \varphi\left(\frac{dv-b}{a-cv}\right) - \varphi\left(-\frac{d}{c}\right)\ . \tag{79}$$

The second term in the right-hand side disappears if either $d=0$ or $c=0$. Finally, the transformation of wavefunctions defined by the first term in the right-hand side of (79) is that generated by the local *group of diffeomorphisms of* $\mathbf{F}$ *generated by the basis of fields $\partial_v, v\partial_v, v^2\partial_v$.*

5 Appearance of Horizon Virasoro symmetry.

5.1. *A more general symmetry.* To conclude the work we show that, at least formally, the found $SL(2,\mathbb{R})$ is embedded into a more general symmetry described by a Virasoro algebra (without central charge) which has a clear geometric meaning on the event horizon. First of all we notice that, from a pure geometric point of view, the $SL(2,\mathbb{R})$ symmetry is associated to the Lie algebra of fields ∂_v, $v\partial_v$, $v^2\partial_v$. This suggests to focus on the set of fields defined on $\mathbf{F}$, $\{\mathcal{L}_n\}_{n\in\mathbb{Z}}$ with

$$\mathcal{L}_n := v^{n+1}\partial_v\,, \quad n\in\mathbb{Z}\,. \tag{80}$$

By direct inspection one gets that, if $\{\ ,\ \}$ denotes the Lie bracket of vector fields,

$$\{\mathcal{L}_n, \mathcal{L}_m\} = (n-m)\,\mathcal{L}_{n+m}\,, \tag{81}$$

that is, the generators $\mathcal{L}_n$ span a Virasoro algebra without central charge. We remark that, in fact, the fields L_n with $n<0$ are not smooth since a singularity arises at $v=0$. It is anyway interesting to investigate the issue of the quantum representation of that Lie algebra in terms of one-particle operators of a quantum field defined on the event horizon. (These operators, if they exists, can straightforwardly extended to multi-particle operators in the Fock space built up over the vacuum $\Psi_{\mathbf{F}}$).

5.2. *Quantum Virasoro algebra.* We expect that the *infinitesimal* action of the transformation $\alpha^{(\mathcal{L}_n)}_\lambda$ induced by $\mathcal{L}_n$ on $\varphi\in\mathcal{S}_{\mathbf{F}}$ reads $\delta\varphi = \alpha^{(\mathcal{L}_n)}_\lambda(\varphi)-\varphi = -\lambda\mathcal{L}_n(\varphi)+O(\lambda^2)$ (nomatter the presence of further terms in the expression of the *finite* action as those in the right-hand side of (76)). First we consider the case $n\geq -1$. Take $\varphi\in\mathcal{S}_{\mathbf{F}}$, expand it as in (42) and use the properties of the positive frequency part: $\tilde{\varphi}_+(E)=\sqrt{E}g(E)$, $\overline{\tilde{\varphi}_+(E)}=\sqrt{E}g(-E)$, where $g\in\mathsf{S}(\mathbb{R},\mathbb{R})$. In this way, one sees that $-\mathcal{L}_n(\varphi)$, with $n\geq -1$, can be decomposed as in (42) provided its positive frequency part reads $L_n\tilde{\varphi}_+$ with

$$(L_n\tilde{\varphi}_+)(E) := i^{n+2}\sqrt{E}\frac{d^{n+1}}{dE^{n+1}}\sqrt{E}\tilde{\varphi}_+(E)\,, \quad n\geq -1\,. \tag{82}$$

Notice that formally, $L_{-1}=iH_{\mathbf{F}0}$, $L_0=iD_{\mathbf{F}}$ and $L_1=iC_{\mathbf{F}}$.
Any L_n is at least symmetric in the dense subspace of $L^2(\mathbb{R}^+,dE)$ containing functions of the form $\psi(E)=\sqrt{E}f(E)$ where f is the restriction to $\mathbb{R}^+$ of a Schwartz' function and $f(E)=O(E^k)$ as $E\to 0^+$ with $k>(n-1)/2$.
Then consider the case $n<-1$. Take $\varphi\in\mathcal{S}_{\mathbf{F}}$, with Fourier transform $\tilde{\varphi}$, such that $E\mapsto\tilde{\varphi}_1(E):=\int_0^E E_1\tilde{\varphi}(E_1)dE_1$ is in $\mathcal{S}_{\mathbf{F}}$, $E\mapsto\tilde{\varphi}_2(E):=\int_0^E\tilde{\varphi}_1(E_2)dE_2$, $E\mapsto\tilde{\varphi}_3(E):=\int_0^E\tilde{\varphi}_2(E_2)dE_2$ and so on, up to $\tilde{\varphi}_{-n-1}$, are in $\mathcal{S}_{\mathbf{F}}$. With the same procedure as above, one sees that $-\mathcal{L}_n(\varphi)$, with $n<-1$, can be decomposed as in (42) provided its positive frequency part reads $L_n\tilde{\varphi}_+$ with

$$(L_n\tilde{\varphi}_+)(E) := -i^{-(n+2)}\sqrt{E}\int_0^E dE_1\int_0^{E_1}dE_2\cdots\int_0^{E_{-(n+2)}}dE_{-(n+1)}\sqrt{E_{-(n+1)}}\tilde{\varphi}_+(E_{-(n+1)})\,, \quad n<-1. \tag{83}$$

Any L_n is at least Hermitean in the subspace of $L^2(\mathbb{R}^+, dE)$ containing the functions ψ such that all the functions $E \mapsto \psi_k(E) := \int_0^E \sqrt{E'}\psi_{k-1}(E')dE'$, with $\psi_0 := \psi$, for all integers $k = 0, 1, 2, \ldots - (n+1)$ are restrictions to $\mathbb{R}^+$ of Schwartz functions.
Self-adjoint extensions of operators $-iL_n$ if any, are the generators of one-parameters unitary groups associated with $\alpha_\lambda^{(\mathcal{L}_n)}$. It is not very difficult to show that, from a very formal point of view, the operators above can equivalently be built up from L_{-1}, L_0, L_1 following the procedure suggested by J.Kumar in [25] by interpreting $\int_0^E dE' g(E')$ as $(\frac{d}{dE})^{-1} g(E)$ on a suitable space of functions. For $n < -1$, the action of L_n can alternatively be written down as

$$(L_n\psi)(E) := \frac{i^{(n+2)}}{2} \int_0^{+\infty} \sqrt{EE'}\, \mathrm{sign}(E' - E)\, (E' - E)^{-(n+2)} \psi(E')\,, \quad n < -1, \tag{84}$$

Finally, by direct inspection one sees that Virasoro's commutation relations

$$[L_n, L_m] = (n - m)\, L_{n+m}\,, \tag{85}$$

are satisfied wherever (in the said domains) both sides of the identity make sense. A rigorous investigation of the properties of the found Virasoro algebra (e.g. essentially self-adjointness of the found operators and the possible relevance, at quantum level, of the singularity of the fileds L_n with $n < 0$) will be presented in a forthcoming paper. Here we remark only that, from a pure physical point of view and barring perhaps complicated mathematical problems related with the infinite dimension of the algebra, operators iL_n (or better their self-adjoint extensions) describes a horizon manifest *symmetry* of the free quantum field in the sense of 2.5. This is because Virasoro algebra is a Lie algebra which contains the Hamiltonian iL_{-1} as a generator.

6 Discussion, overview and open problems.

In this paper we have rigorously proven that it is possible to define a diffeomorphism invariant quantum field theory for a massless free scalar field defined on the event horizon of a Rindler spacetime. Actually all the procedure could be implemented in a manifold diffeomorphic to $\mathbb{R}$ without fixing any metric structure. The diffeomorphism invariance is a consequence of the fact that the field operators and the symplectic form act on exact 1-forms instead of smooth smearing functions and thus they do not need a metric invariant measure. Moreover, when the theory is realized on the (future and/or past) event horizon in Rindler spacetime, there is a natural injective $*$-algebra homomorphism from any quantum field theory of a (generally massive) scalar field propagating in the bulk. This holographic identification can be implemented in terms of unitary equivalences if the algebras of the fields are represented in suitable Fock spaces. In this case the vacuum state in the bulk is that associated to Fulling-Unruh quantization. In a approximated picture where Rindler space corresponds to the spacetime near the horizon of a Schwartzschild black hole, Fulling-Unruh particles are just the Poincaré-invariant particles we experience everydays. Conversely, if the Rindler background is taken seriously as part of the actual spacetime (Minkowski spacetime) without approximation, Rindler vacuum has to be

thought as the vacuum state of an acceleratd observer in Minkowski spacetime and Rindler particles have nothing to do with ordinary Poicaré invariant particles.
Another achieved result is that the hidden $SL(2,\mathbb{R})$ symmetry of the bulk theory corresponds to an analogous symmetry for the horizon theory and this horizon symmetry has a clear geometric interpretation in terms of invariance under diffeomorphisms. Barring technical open issues to be fixed in a future work, we have also found strong evidence of the fact that the $SL(2,\mathbb{R})$ quantum symmetry can be enlarged to include a full Virasoro algebra of self-adjoint generators. The geometric meaning of that enlarged symmetry is anyhow evident in terms of generators of one-parameters group of local diffeomorphisms of the horizon. In fact, we have found the expression of the quantum analogues of these generators in terms of differential operators without giving a rigorous investigation on their self-adjoint extensions. That will be the subject of a future work.

All the work has been developed in the case of a two-diemensional spacetime. Nevertheless we expect that the result obtained for this simple case can be generalized to encompass some four-dimensional cases. Considering a four-dimensional Schwartzschild black hole manifold within the near horizon approximation, angular degrees of freedom are embodied in the solutions of Klein-Gordon equation by multiplication of a two-dimensional solutions and a spherical harmonic $Y^l_m(\theta,\phi)$. All field states are elements of an appropriate tensor product of Hilbert spaces. For instance, in the massive case, the final space is the direct sum of spaces $\mathbb{C}^{2l+1}\otimes L^2(\mathbb{R}^+,dE)$ with $l=0,1,\ldots$. (The "square angular momentum" eigenvalue l defines an effective mass of the field when considered at fixed value of l. In this way the massless theory behaves as the massive one when $l\neq 0$.) With simple adaptations (e.g., the appropriate causal propagator on $\mathbf{F}$ reads

$$E_{\mathbf{F}}(x,x') = (1/4)\text{sign}\,(v-v')\delta(\theta-\theta')\delta(\phi-\phi')\sqrt{g_{\mathbb{S}^2}(\theta,\phi)}$$

and the horizon field operator $\hat{\phi}_{\mathbf{F}}$ has to be smeared with 3-forms as $df(v,\theta,\phi)\wedge d\theta\wedge d\phi$) all the results found in this paper can be re-stated for that apparently more general case. The same conclusion can be achieved when considering a four dimensional Rindler spacetime.
Some comments can be supplied for the case of the exact Schwartzschild spacetime dropping the near-horixon approximation in spite of the absence of exact solutions of the Klein Gordon equation. By the analysis of the effective potential – which depends on the angular momentum – of a either massive or massless particle propagating in the external region of the black hole spacetime, one sees that the energy spectrum is $\sigma(H)=[0,+\infty)$ once again for any values of the angular momentum. If the particle is massive no degeneracy affects a value E of the energy if the mass is greater than E, otherwise twice degeneracy arises. That is the only possible case for a massless particle. Therefore we expect that our results, with appropriate adaptations, may hold for the massless case but they could need some substantial change dealing with the massive case.
Another interesting topic that deserves investigation is if, and how, the holographic procedure can be extended in order to encompass a larger algebra of fields containing Wick monomials "ϕ^n" which naturally arise dealing with perturbative interacting quantum field theory.

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References

[1] G. 't Hooft, *Dimensional Reduction in Quantum Gravity*, preprint: **gr-qc/9310026** (1993).

[2] G. 't Hooft, *The scattering matrix approach for the quantum black hole, an overview*, Int. J. Mod. Phys. **A11**, 4623-4688 (1996).

[3] L. Susskind, *The World as a Hologram*, J. Math. Phys. **36**, 6377-6396 (1995).

[4] J. Maldacena, *The Large N Limit of Superconformal Field Theories and Supergravity*, Adv. Theor Math. Phys. **2**, 231 (1998).

[5] E. Witten, *Anti-de Sitter space and holography*, Adv. Theor. Math. Phys. **2**, 253-291 (1998).

[6] K. H. Rehren, *Algebraic Holography*, Annales Henri Poincare **1** 607-623 (2000).

[7] K. H. Rehren *Local Quantum Observables in the Anti-deSitter - Conformal QFT Correspondence*, Phys. Lett. **B493** 383-388 (2000).

[8] A. Strominger, *The dS/CFT Correspondence*, JHEP **0110** 034 (2001).

[9] I. Sachs, S. N. Solodukhin, *Horizon Holography*, Phys. Rev. **D64** 124023 (2001).

[10] V. Moretti, N. Pinamonti, *Aspects of hidden and manifest* $SL(2,\mathbb{R})$*symmetry in 2d near-horizon black-hole backgrounds*, Nuc. Phys. **B647** 131 (2002).

[11] V. de Alfaro, S. Fubini and G. Furlan, *Conformal Invariance In Quantum Mechanics*, Nuovo Cim. **A34** 569 (1976).

[12] D. Guido, R. Longo, J. E. Robertz, R.Verch *Charged sectors, spin and statistics in Quantum Field Theory on Curved Spacetimes* Rev.Math Phys. **13** 1203 (2001)

[13] B.Schroer and H-W. Wiesbrock *Looking beyond the thermal horizon: Hidden symmetries in chiral modes* Rev.Math. Phys. **12** 461 (2000)

[14] B. Schroer *Lightfront Formalism versus Holography&Chiral Scanning* hep-th/0108203

[15] B. Schroer and L. Fassarella *Wigner particle theory and Local Quantum Physics* hep-th/0106064

[16] B. S. Kay and R. M. Wald, *Theorems on the uniqueness and thermal properties of stationary, nonsingular, quasifree states on spacetimes with a bifurcate Killing horizon*, Phys.Rep. **207**, 49, (1991)

[17] R. M. Wald, *Quantum field theory in curved spacetime and black hole thermodynamics*, Chicago University Press, Chicago (1994).

[18] S.A. Fulling, *Aspects of Quantum Field Theory in Curved Spacetime*, (Cambridge, Cambridge University Press, 1989).

[19] N.N. Lebedev, *Special functions and their applications*, Dover Publications, Inc, New York, 1972.

[20] I. S. Gradshteyn, I. M. Ryzhik, *Table of Integrals, Series, and Products*, Academic Press, New York, 1995.

[21] E. Nelson, *Analytic vectors*, Ann. Math. **70** 572 (1959).

[22] G.L. Sewell, *Quantum fields on manifolds: PCT and gravitationally induced thermal states* Ann. Phys. (NY) **141** 201 (1982).

[23] N. Jacobson, *Basic Algebra II* (second edition), Freeman and Co., New York, 1989.

[24] N. I. Akhiezer, *Lectures on Integral Transforms*, Translations of Mathematical Monographs, Vol 70., American Mathematical Society, Providence, 1988.

[25] J. Kumar, *Riders of the loest AdS*, JHEP05 (2000), 035.