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SIDEPATH RESULTS ON PACKING $\VEC{P}_1$'S AND $\VEC{P}_2$'S

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Abstract

We provide proofs of some results from our companion paper.

Key words: $\{\vec{P}_1, \vec{P}_2\}$-packing, directed path packing, circuit packing.

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1 Introduction

This report provides proofs and refinements of some results from our companion paper [2]. We refer the reader to [2] for a more detailed introduction to notation, background, and motivation.

Let $\mathcal{G}$ be a fixed set of digraphs. Given a digraph H, a $\mathcal{G}$-packing in H is a collection $\mathcal{P}$ of vertex disjoint subgraphs of H, not necessarily induced, each isomorphic to a member of $\mathcal{G}$. A $\mathcal{G}$-packing $\mathcal{P}$ is *maximum* if the number of vertices belonging to members of $\mathcal{P}$ is maximum, over all $\mathcal{G}$-packings. The analogous problem for undirected graphs has been extensively studied in the literature. In a companion paper we initiate the study of digraph packing problems, focusing on the case when $\mathcal{G}$ is a family of directed paths. We showed that unless $\mathcal{G}$ is (essentially) either $\{\vec{P_1}\}$, or $\{\vec{P_1}, \vec{P_2}\}$, the $\mathcal{G}$-packing problem is NP-complete. We use the notation $\vec{P_k}$ for the *directed path of length* k, i.e., the path $u_0, u_1, \ldots, u_k$ in which all arcs are oriented from u_{i-1} to u_i for $i = 1, 2, \ldots k$.

When $\mathcal{G} = \{\vec{P_1}\}$, the $\mathcal{G}$-packing problem is simply the matching problem. In [2], we treat in detail the one remaining case, $\mathcal{G} = \{\vec{P_1}, \vec{P_2}\}$. We give in this case a polynomial time algorithm based on augmenting configurations, and a corresponding Berge-type and Tutte-type theorems. We also give a reduction to bipartite matching. In this report, we give a direct combinatorial algorithm based on augmentations, explore weighted variants of the problem, and give a polyhedral analysis.

2 Min-max characterization

In this report, the term *packing* without further specification always refers to a $\{\vec{P_1}, \vec{P_2}\}$-packing. Let $H = (V, A)$ be a digraph and let $\mathcal{P}$ be a packing in H. Then $exp_H(\mathcal{P})$ denotes the number of vertices of H left exposed by $\mathcal{P}$. (In this report, subscripts are omitted when convenient).

Given a vertex v, the *in-neighbourhood* of v, denoted $N^+(v)$, respectively *out-neighbourhood* of v, denoted $N^-(v)$, is the set $\{u | (u, v) \text{ is an arc of } H\}$, respectively $\{u | (v, u) \text{ is an arc of } H\}$. Given a set of vertices S, $N^+(S)$ is the union of $N^+(v)$ taken over all vertices v in S. The set $N^-(S)$ is analogously defined using $N^-(v)$. For any set S of vertices from H, the *deficiency* of S in H is defined as

$$def_H(S) := |S| - |N^+(S)| - |N^-(S)|.$$

Clearly, $exp(\mathcal{P}) \geq def(S)$ for every set S and every packing $\mathcal{P}$. The following min-max characterization is the main result of our analysis.

Theorem 2.1 *In every digraph H there exists a set of nodes S and a packing $\mathcal{P}$ such that $exp(\mathcal{P}) = def(S)$.*

In the next section, we provide an algorithmic proof of Theorem 2.1.

3 An algorithm

We only describe a procedure which, given a digraph $H = (V, A)$ and a packing $\mathcal{P}$, returns either an *augmentation*, that is a packing $\mathcal{P}'$ with $exp(\mathcal{P}') < exp(\mathcal{P})$, or a set S with $exp(\mathcal{P}) = def(S)$. At every step, the procedure will perform a move out of a finite set of possible moves. At every step, three subsets S, R_+ and R_- of V will be updated (only by adding nodes to them). We denote by R the set $R_+ \cup R_-$ and by T the set $S \cup R$. There are two kinds of moves. A first kind of move leads to an augmentation, hence to the termination of the procedure. The second kind of move increases both $|S|$ and $|R_+| + |R_-|$. This is more than enough to show that at most $O(|V|)$ labelings can occur between two consecutive augmentations in the main algorithm. If at a certain step no move out of the finite set can be applied, then the procedure halts and S is returned. This procedure leads to an $O(mn)$ algorithm for finding a maximum $\{\vec{P}_1, \vec{P}_2\}$-packing.

The correctness of the procedure rests on the following invariants.

- Let v be any node in S. Then in $H \setminus v$ we can produce a packing $\mathcal{P}'$ such that $exp_{H\setminus v}(\mathcal{P}') < exp(\mathcal{P})$ by just altering $\mathcal{P}$ within $H[T] \setminus v$;
- let u and v be any two nodes in S. Let H' be the digraph obtained from H by adding arc (u, v). Then in H' we can produce a packing $\mathcal{P}'$ such that $exp_{H'}(\mathcal{P}') < exp(\mathcal{P})$ by just altering $\mathcal{P}$ within $H'[T]$;
- S and R are disjoint;
- the two endnodes of a $\vec{P}_1$ in $\mathcal{P}$ are either both in T or both outside T. In the first case, one of them is in S and the other is either in R_+ or in R_-;

- for $\vec{P}_2$'s in $\mathcal{P}$ we have three possibilities: either all three nodes are outside T, or the two endpoints are in S and the middle node is both in R_+ and in R_-, or one endpoint is in S, the other is not in T and the middle node is either in R_+ or in R_-.

Initialization. Set $R_+ := R_- := S := \emptyset$. The invariants trivially hold.

3.1 Augmenting moves

By the invariants, we have an augmentation whenever one of the following situations occur.

1 two nodes in S are adjacent;

2 a node s in S is adjacent to a node in $V \setminus T$ which is the endpoint of a $\vec{P}_1$ in $\mathcal{P}$;

3 a node s in S is adjacent to a node in $V \setminus T$ which is the endpoint of a $\vec{P}_2$ in $\mathcal{P}$;

4.1 there exists an arc (s, b) where $s \in S$ and $b \in R_-$ is the middle node of a $\vec{P}_2$ in $\mathcal{P}$;

4.2 there exists an arc (b, s) where $s \in S$ and $b \in R_+$ is the middle node of a $\vec{P}_2$ in $\mathcal{P}$.

The rationale behind 3 and 4 is the only which needs some word of explanation, to be read after the whole picture of the algorithm has been gathered. Actually, the rationale behind 3 and 4 is essentially the same, hence we choose to expose it for 3. Let $(a, b)(b, c)$ be the $\vec{P}_2$ and assume a is the node outside T and adjacent to s. If b and c are also outside T, then no further explanation should be needed. Assume therefore b in R and $c \in S$. Again, if among the moves which led to put s in S no move based on $c \in S$ was needed, then we can actually pretend that the move which put b and c in T has never occurred, and the argument above applies. Otherwise we have found a circuit going through b and disjoint from what $H[T']$ was at the step which put b and c in T because of an arc (b, s') with $s' \in S$. Even if the $\vec{P}_2$'s involved into this circuit can partially fall outside of it (but always remain within $V \setminus T$), since the circuit has been obtained through a sequence of labeling moves and in view of the remark concerning labeling moves given in the next section, we conclude that it is always possible to rearrange things within this circuit in such a way to leave exposed only node b or only nodes a and b of the circuit, leaving precisely the same status (exposed/not exposed) for nodes outside this circuit. But then s', after this rearrangement outside $H[T']$, could have been covered with the extra $\vec{P}_1$ (b, s') or with the extra $\vec{P}_2$ $(a, b)(b, s')$.

3.2 Labeling moves and termination

Labeling moves come about when a node s in S is adjacent to a node x with certain properties. These properties are the preconditions for the move. We assume (s, x) to be the arc which triggered the move, since the arguments remain the same if one reverses the direction of all arcs involved and swaps between R_+ and R_-. Essentially, there is only one possible labeling move.

Labeling Move.

Precondition: $x \notin R$ and (a, x) is a $\vec{P}_1$ in $\mathcal{P}$ or $x \notin R_-$ and $(a, x)(x, b)$ is a $\vec{P}_2$ in $\mathcal{P}$.

Action: a is put into S and x is put into R_-.

Comments: It is easy to check that the first invariant is maintained. The second invariant for the two nodes s and a follows directly from the first invariant on s. (In the case when (a, x) is a $\vec{P}_1$ in $\mathcal{P}$, consider that in every orientation of K_3 there is a $\vec{P}_2$.) The second invariant for a and another node $t \in S$ follows from the second invariant itself for s and t, just exploiting (s, x) and the arc added between a and t but discarding (a, x) to emulate the effect of an arc between s and t. All other invariants are trivially maintained.

Remark: in the case when x is put into R_-, then arcs (s, x) and (a, x) are *both* directed towards x. In the case when x is put into R_+, then arcs (x, s) and (x, a) are *both* exiting x.

Termination.

At termination, if no augmentation has occurred, then the following properties do hold.

- S is an independent set of nodes (otherwise *Augment Move 1*);
- $N^+(S)$ and $N^-(S)$ are both contained into $R = T \setminus S$ (otherwise *Augmenting Move 2* or 3 or 4 or *Labeling Move*).

Consider to associate to every non-exposed node s in S that node which was put in R_+ or in R_- at the same step when s was put into S. The min-max characterization follows by observing that this function is onto R, is not defined on the exposed nodes, and for every node v in $R_+ \cap R_-$ there are two distinct nodes into S which are mapped into v.

3.3 A reduction to bipartite matching

In this section, we propose a shorter solution through a reduction to the bipartite matching problem. This also allows for a $O(m\sqrt{n})$ algorithm and for more convenient derivations of positive results for weighted versions of our packing problem.

Let H be a digraph with vertex set V and arc set A. We define the *bipartite graph G associated with H* (sometimes denoted by $G = G(H)$), as follows: Let V^+, V^* and V^- be three distinct copies of V, with $u^+ \in V^+$, $u^* \in V^*$, and $u^- \in V^-$ denoting the vertices corresponding to $u \in V$ respectively. Let $E^+ = \{u^+v^* \ : uv \in A\}$ and $E^- = \{\ u^*v^- \ : uv \in A\}$. The graph G has the vertex set $W = V^+ \cup V^* \cup V^-$ and the edge set $E = E^+ \cup E^-$. Note that G is indeed a bipartite graph with bipartition $V^*, V^+ \cup V^-$. We shall describe a correspondence between sets of vertices in H that can be covered by packings of H, and sets of vertices in V^* that can be covered by matchings of G.

Lemma 3.1 *Let G be the bipartite graph associated with a digraph H. For every packing $\mathcal{P}$ of H, there exists a matching M of G such that $u \in V$ is covered by $\mathcal{P}$ if and only if u^* is covered by M.*

Proof: For each arc uv which forms a $\vec{P}_1$ in $\mathcal{P}$, we put in M the edges u^+v^* and u^*v^-. For each pair of arcs uv, vw which form a $\vec{P}_2$ in $\mathcal{P}$, we put in M the edges u^+v^*, u^*v^-, and v^+w^*. ∎

Lemma 3.2 *Let G be the bipartite graph associated with a digraph H. For every matching M of G, there exists a packing $\mathcal{P}$ of H such that $u \in V$ is covered by $\mathcal{P}$ whenever u^* is covered by M.*

Proof: Every nontrivial directed path or cycle admits a perfect $\{\vec{P}_1, \vec{P}_2\}$-packing; thus, it suffices to find a packing of H by directed paths and cycles covering the appropriate vertices. However, the subgraph of H which naturally corresponds to M may contain vertices incident with three arcs (one arc for each of u^-, u^*, u^+). Thus, we shall process M to form a new matching M' covering the same vertices of V^* as M, but which corresponds to a collection of directed paths and cycles in H.

Consider the bipartite graph F on the vertex set $V^+ \cup V^-$, with the edges u^+v^- for all arcs uv of H. Consider the matching $M_1 := M \cap E^+$: by replacing each asterisk with a plus (in the superscripts), we can view M_1 to be a matching of F. Similarly, we can view $M_2 := M \cap E^-$ as a matching of F by replacing all asterisks with a minus sign. Note that v^* is covered by M if and only if v^- is covered by M_1 or v^+ is covered by M_2. By the Dulmage-Mendelsohn theorem [9], we can find (in linear time) a matching M' in F which covers all vertices of V^+ covered by M_1 and also all vertices of V^- covered by M_2. To this matching M' in F there corresponds in H a set of disjoint directed paths and cycles, covering a vertex v whenever v^* is covered by M. $\blacksquare$

Recall that the size of a packing of H is the number of covered vertices. While the size of a matching of G is formally the number of *edges* in the matching, we note that this equals the number of vertices in V^* covered by the matching.

Theorem 3.3 *Let G be the bipartite graph associated with a digraph H. Then the size of a maximum packing of H equals the size of a maximum matching of G.* $\diamond$

These results (together with the bipartite matching algorithm and a linear-time algorithm inherent in the Dulmage-Mendelsohn theorem) yield a polynomial time algorithm to find a maximum packing.

We remark the above theorem also gives an alternative derivation of our min-max formula. Trivially, for $S \subseteq V^*$, M will leave at least $|S| - |N^+(S)| - |N^-(S)|$ vertices exposed in V^*. Note that this fully corresponds to deficiency in the digraph H. Moreover, by Hall's theorem for bipartite graphs, we know that when M is a maximum matching, there exists a set of vertices S with exactly $def(S)$ exposed vertices.

3.4 Polyhedral considerations

Note that the convex hull of those node subsets which are covered by a $\{\vec{P}_1, \vec{P}_2\}$-packing is precisely the same polytope as the convex hull of those node subsets which are covered by a packing of directed paths. The convex hull, and indeed the very family of coverable node subsets, would still be the same even if we considered packings of directed paths and cycles. The equivalence among the three polytopes still holds when we are interested in maximum size node subsets which can be covered by a packing.

If we are interested in these two polytopes, then the reduction to bipartite matching already provides a solution. It follows indeed from the reduction to bipartite matching that the maximal sets of nodes which are covered by a packing (we call such a packing *maximal covering*) in the original digraph are precisely the maximal subsets of V^* which can be covered by a matching in the bipartite graph considered by the reduction. Moreover, given a bipartite graph on color classes U and V, the subsets of nodes in U which can be matched to nodes in B are known to be the independent sets of a matroid. Hence, we could at worst rely on the exponential but algorithmically well behaved polyhedral description for the basis of a matroid given by Edmonds. As a consequence, we can find maximum cost packings (whether $\{\vec{P}_1, \vec{P}_2\}$-packings, or packings of directed paths, or packings of directed paths and directed cycles). When the costs are non negative, then the maximum cost packings obtained will also be *maximum covering* in the sense that they will cover maximum sets of nodes, since the basis of a matroid all have the same cardinality. In practice, to find maximum cost packings we can exploit our reduction to bipartite matching and just solve a maximum weight bipartite matching where the weight of the edges is inherited from the costs on the nodes. Namely, all edges incident with $v^* \in V^*$ receive the cost of v^* as their weight. By adding a suitably big constant to the cost of every node, we can also find minimum cost maximum covering packings. That is, we have the polyhedral description of the convex hull of maximum covering packings.

What however if we wanted to have costs on the arcs? Here the three problems are not any longer the same problem and we must distinguish between maximum $\{\vec{P}_1, \vec{P}_2\}$-packings, maximum packing of directed paths, and maximum packings of directed paths and cycles.

In the following subsection, we show that finding a maximum covering $\{\vec{P}_1, \vec{P}_2\}$-packing of maximum cost (cost on the arcs) is NP-hard. This makes it unlikely to find a polyhedral description for the polytope of maximum $\{\vec{P}_1, \vec{P}_2\}$-packing. This also indicates why post-processing (Mendelsohn-Dulmage) was actually needed in our reductions. In Subsection 3.6, we will give some positive polyhedral results for packings of directed paths and circuits.

3.5 Negative polyhedral results

In this subsection, we will show the NP-completeness of the following decision problems.

Problem 3.4 *Given a planar digraph D with 0/1 costs on the arcs and an integer k, does there exists a maximum packing $\mathcal{P}$ in D such that the sum of the costs on the arcs in $\mathcal{P}$ is at least k?*

Let B be a boolean formula in conjunctive normal form. Let $X = \{x_1, \ldots, x_n\}$ be the set of variables and $C = \{c_1, \ldots, c_m\}$ be the set of clauses in B. Consider the bipartite graph $G_B = (X, C; E_B)$, with color classes X and C, and with edge set $E_B = \{xc\ :$ variable x occurs in clause $c\}$. The boolean formula B is called *planar* when G_B is planar. In [8], Lichtenstein showed that Problem 1 is NP-complete.

Problem 1 (PLANAR 3SAT) *Given a planar boolean formula B, is B satisfiable?*

Moreover, the obvious reduction shows that PLANAR 3SAT remains NP-complete even when restricted to boolean formulas with precisely three literals per clause (even though this is not really needed into our two reductions). The reduction we propose will encode the given boolean formula B into a digraph $D(B)$ with $0, 1$ weights on the arcs. The digraph $D(B)$ that we will associate to B will always have a packing covering all nodes. The costs placed on the arcs play therefore a crucial role. To construct $D(B)$, take a node c_j for each clause c_j of B. Moreover, to each variable x_i associate a *truth setting component* as displayed in Figure 1. The component consists of a cycle on $5 \cdot 3 = 15$ nodes and is made of an alternating sequence of $\vec{P}_2$'s and $\vec{P}_3$'s. The arcs of the $\vec{P}_2$'s have weight 0 and are all oriented into a same direction, say anti-clockwise, whereas the arcs of the $\vec{P}_3$'s have weight 1 and are all oriented clockwise. For $j = 1, \ldots, m$, the j-th $\vec{P}_2$ is associated to the j-th occurrence of x_i into B, its tail node $\overline{x}_i{}^j$ is meant to offer a negated literal, whereas its head node $x_i{}^j$ is meant to offer a positive literal.

Consider the p-th occurrence of variable x_i into B and assume this occurrence is in clause c_j. Now, if the occurrence is positive, then we add an arc with tail in $x_i{}^p$ and head in c_j, otherwise, if the occurrence is negative, then we add an arc with tail in c_j and head in $\overline{x}_i{}^j$. These arcs weight 0.

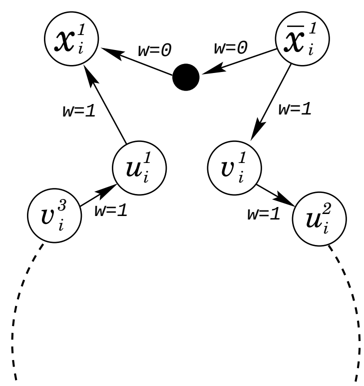

Figure 1: The truth setting component associated to variable x_i.

Note that the truth-setting components are planar. Moreover, since the connections among components and clause nodes are dictated by G_B, and by suitably numbering the occurrences of the variables into the truth-setting components, the resulting graph will also be planar. Finally, in- and out-degrees are bounded by 3.

Lemma 3.5 *The digraph $D(B)$ admits a packing covering all its nodes. Moreover, $D(B)$ admits such a packing of cost $6m$ if and only if B is satisfiable.*

Proof: To indicate why the first statement is true, we restate it in a stronger form: the digraph $D(B)$ admits a packing covering all its nodes and where all arcs $v_i^1 u_i^2$, $v_i^2 u_i^3$ and $v_i^3 u_i^1$ belong to the packing as $\vec{P}_1$'s, for all clauses i.

For the second statement, note that any packing can not collect more than 6 arcs of weight 1 from a given truth setting component, since no packing can collect more than 2 arcs from a $\vec{P}_3$. Furthermore, if a packing collects 6 arcs of weight 1 from a component, then these arcs have been collected by 3 $\vec{P}_2$'s, otherwise one of the three nodes of the component which are incident with two weight 0 arcs of the component would not be covered. For the same reason, the 3 $\vec{P}_2$'s which collect 6 units of weight from the clause either cover nodes x_i^1, x_i^2 and x_i^3 or cover nodes $\overline{x}_i^1$, $\overline{x}_i^2$ and $\overline{x}_i^3$. In the first case, variable x_i is true; in the second case, variable x_i is false. Now the reader should be in condition to ascertain that $D(B)$ admits a packing covering all its nodes and of cost $6m$ if and only if B is satisfiable. ∎

3.6 Positive polyhedral results

The system here below describes the convex hull of characteristic vectors of packings of directed paths and cycles. Hence we can find packings of directed paths and cycles of maximum cost.

$$\max cx \\ \begin{cases} x(\delta^+(v)) \leq & 1 \quad \forall v \in V \\ x(\delta^-(v)) \leq & 1 \quad \forall v \in V \\ x \geq & 0 \end{cases} \tag{1}$$

Theorem 3.6 *The vertices of the above polytope are precisely the characteristic vectors of packings of directed paths and cycles.*

Proof: It suffices to show that the above polytope is integral.

This can be done by standard methods from polyhedral combinatorics, and along the same lines of existing proofs for the bipartite matching polytope. Namely, one has to assume the existence of a fractional vertex and then get a contradiction deriving two feasible points having the given vertex in their convex hull. In doing this, one concentrates on the digraph of only those arcs associated to fractional components of the assumed fractional vertex. The key observation is that in any finite non-empty digraph there exists either an *alternating circuit*, i.e. a circuit where the directions of the arcs along the circuit alternate, or a maximal alternating path. ∎

Also the dual, here below, has integral solutions.

$$\min 1y^+ + 1y^- \\ \begin{cases} y^+(u) + y^-(v) \geq & c_{(u,v)} \quad \forall (u,v) \in A \\ y \geq & 0 \end{cases} \tag{2}$$

Theorem 3.7 *The above polytope is integral.*

Proof: Again it is a matter of adapting standard arguments which assume a fractional vertex and then get a contradiction deriving two feasible points having the given vertex in their convex hull. The fact that such a proof can be derived along the lines of classical results for bipartite matching is better explained by the reduction given here below. ∎

In any case, we can also propose a purely combinatorial algorithm not based on Linear Programming to find packings of directed paths and cycles of maximum cost. Indeed, once again, the following reduction to bipartite matching does the job.

Given an input digraph $D = (V, A)$, construct a bipartite graph on color classes $V^+ = \{v^+ : v \in V\}$ and $V^- = \{v^- : v \in V\}$ and having an edge u^+v^- if and only if $(u, v) \in A$. Note that there is a natural correspondence between the matchings of G and the packings of directed circuits and paths in D.

Packing of directed circuits

System 1 here above, after the addition of the $|V|$ equations $x(\delta^+(v)) = x(\delta^-(v))$, describes an integral polytope (and hence the convex hull of packings of directed circuits), as one can see by essentially the same reduction to bipartite matching: take two disjoint copies V^+ and V^- of V and put one edge u^+v^- whenever $(u, v) \in A$. This time, also add the edges v^+v^- for every $v \in V$.

Note that there is a bijection between the *perfect* matchings of G and the packings of directed circuits and paths in D.

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