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IDEAL AND REAL BELIEF ABOUT BELIEF

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Abstract

The goal of this paper is to provide a formalization of monotonic belief and belief about belief in a multiagent environment. We distinguish between *ideal beliefs*, i.e., those beliefs which satisfy certain "idealized" properties which are unlikely to be possessed by real agents, and *real beliefs*. Our formalization is based on a set-theoretic specification of beliefs and, then, on the definition of the appropriate constructors which present the sets identified. This allows us to provide a uniform and taxonomic characterization of the possible ways in which ideal and real beliefs can arise. We compare our notion of ideal with the notion of logical omniscience from the modal literature, and show that the first is much weaker and more granular than the second. We provide intuitions about the conceptual importance of the cases analyzed by proving and discussing some equivalence results with some important modal systems modeling (non) logical omniscience.

1 Introduction

The specification of beliefs in a multiagent framework is a traditional topic in many research areas, including computer science. The standard approach is to start with a single propositional signature, extend it with a unary modal operator for each agent, and use possible world semantics to formally characterize what each agent believes. Traditionally, "belief" corresponds to truth in all the considered possible worlds, and a "world" is characterized by an assignment to the propositional signature. These characterizations of "belief" and "world" lead to simple and very elegant formalizations of agents' beliefs. However, the resulting agents suffer of the "logical omniscience problem" [15]: agents are idealized to the extent that they believe all the propositional consequences of their own beliefs. Though there exists a great deal of work which overcomes this problem (each approach modifying the above characterization of "belief" or "world", see [8] for an excellent overview of various approaches), all such solutions lack the simplicity and elegance of the original approach (see also [17]).

In another tradition (see e.g. [12, 16]) each agent is modeled as a set of "interacting" distinct theories, that here we call *reasoners*. The intuition is that each reasoner models a point of view that the agent has (about its own beliefs, about the beliefs of another agent, about the beliefs about the beliefs of another agent, ... and so on); while the "interaction" among reasonersmodels the fact that what is true in a view is usually compatible with what is true in another view. The major feature of this approach is that it allows for an incremental and modular specification of multiagent systems. One can specify the beliefs of an agent, the beliefs he has about other agents, ... and so on, one by one, as distinct theories, and then impose the relations which exist between these theories. A multiagent system is specified *from the inside*. This feature is crucial in the modeling of complex situations where there is no (or it is hard to define a) global scheme describing the overall application.[1]

This paper follows the second line of research and extends the current literature in several ways. First, it sets up a formal framework for the extensional definition of beliefs in a multiagent systems, i.e., a framework in which agents' beliefs are formalized as sets of sentences satisfying certain properties. Then, following what is standard practice in software specification, we define the appropriate constructors which intensionally *present* such sets of beliefs. In particular, we use the multicontext formalism originally introduced in [11] and then further developed in [12].

Second, in order to classify agents' beliefs, we introduce the notions of "ideal reasoner" (i.e. a reasoner whose beliefs are closed under propositional consequence) and of "ideal observer" (i.e. a reasoner whose beliefs about another reasoner are correct and complete). A "real reasoner" ["real observer"] is then defined as a reasoner which is believes too much or too little with respect to an ideal reasoner [observer] taken as reference. The distinction between the abilities to reason and observe, is necessary in order to take into account the rich structure (i.e., the multiple theories and their compatibility relations) of the formalism that we use. On the basis of such distinction, we are able to taxonomically characterize all the possible sources of reality by looking at the possible ways in which the constructors for ideality can "go wrong".

Third, we study the relationship between ideality/reality and (non) logical omniscience and show that, even though ideality/reality and (non) logical omniscience capture similar intuitions, there are some important differences. For instance, as discussed in detail below, reality does not coincide with not ideality, and ideality is a more granular and weaker notion than the notion of logical omniscience captured in the standard approach (e.g., by the modal system $\mathcal{K}$). In order to present ideal reasoners and observers which are also logically omniscient, we have to ensure a correspondence between the beliefs of the observer and those of the observed reasoner which is tighter than that guaranteed by the ideality in the observing capabilities. Finally, we show how it possible to capture various forms of non logical omniscience presented in the literature as particular multicontext systems.

The paper is structured as follows. In Section 2 we specify beliefs and beliefs about beliefs as certain sets of formulas constituent the notions of reasoner and observer, respectively. In this section we also show how beliefs and beliefs about beliefs can be

[1] As a matter of fact, our proposed framework formalizes ideas which have been exploited in many complex applications developed in various areas of Artificial Intelligence, e.g., computational linguistics [6, 26, 27], the formalization of opacity and transparency in belief contexts [1], the integration of information coming from heterogeneous data bases [23], planning[19], and multiagent systems [3, 4, 14, 18, 25].

presented as certain kinds of multicontext systems. In Section 3 we define the multicontext systems for ideal belief and ideal belief about belief, MBK^-. In Section 4 we characterize the possible forms of reality by analyzing how the constructors for ideal belief and ideal belief about belief can be modified to generate incompleteness, incorrectness, or a combination of them. The tricky part is in the definitions of incompleteness and incorrectness. In Section 5 we compare MBK^- and some multicontext systems for real belief with some important modal systems. The goal of this analysis is to provide intuitions about the expressive power and conceptual importance of the multicontext systems defined. Till Section 5 we limit ourselves to the case of only two reasoners. This is a very strong hypothesis which, among other things, forces us to deal only with the case of no nested beliefs. In Section 6 we show how the definitions and methodology given in the previous sections can be uniformly lifted to account for arbitrary sets of reasoners. As a particular important case, in Section 7 we show how multiple reasoners can be put together to generate nested beliefs, and discuss how the equivalence theorems presented in Section 5 can be generalized to this situation. We conclude in Section 8 with some concluding remarks and related work. In particular, we focus on the works that we consider as most close to ours, i.e., Konolige's deduction model of belief and the existing literature on formalizing belief with multicontext systems.

To make the paper easier to read, we instantiate the approach and the technical notions introduced to a running example which is developed all along the paper. The example is adapted from Chapter 10 of Raymond Smullyan's "Alice in Puzzleland" [24] and goes as follows.

Example 1 Consider the scene represented in Figure 1. Two agents –Alice (left) and Humpty Dumpty (right)– are thinking about the beliefs of a third agent –the White Knight. The White Knight is a "looking glass logician", namely he only believes false beliefs. Furthermore, he believes the negation of the formulas which follow from the negation of his (false) beliefs. In particular the White Knight believes all the contradictory sentences. Alice and Humpty Dumpty have different beliefs, different views of the beliefs of the other, different views of the beliefs of the White Knight, ... and so on. Humpty Dumpty is a "keen arguer" while Alice is not; she has limited capabilities both in reasoning and in having beliefs about beliefs. Furthermore, Humpty Dumpty knows that the White Knight is a looking glass logician while Alice does not; she ascribes to the White Knight reasoning capabilities similar to hers.

2 Reasoners and observers

As mentioned in the introduction, an agent is formalized as a set of "interacting" theories. Let us see how this vague statement can be instantiated to Example 1.

Example 2 Consider the beliefs of Alice, as shown in Figure 1. Their formalization requires only two theories, namely:

- a theory formalizing Alice's view of the world (that is, Alice's beliefs): her first and fourth belief belong to this theory; and
- a theory, formalizing Alice's view of the White Knight view of the world, which contains her second and third belief.

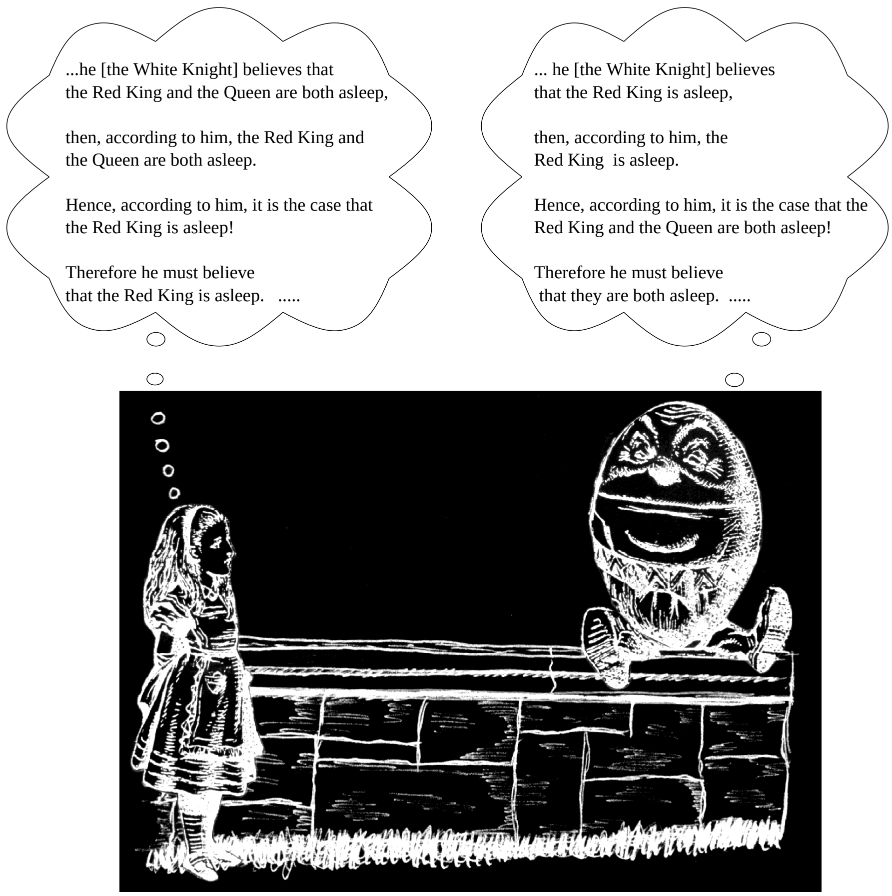

Figure 1: Alice and Humpty Dumpty.

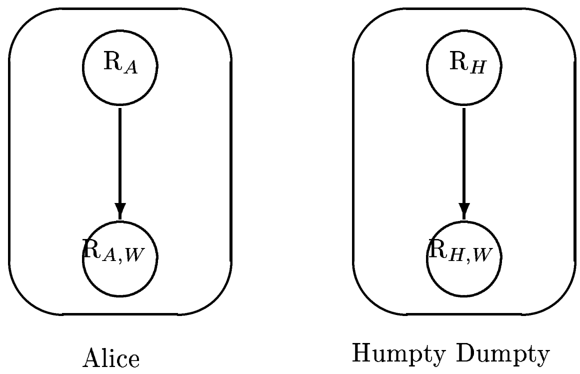

Figure 2: Alice and Humpty Dumpty's theories.

Analogously for Humpty Dumpty we have:

- a theory formalizing Humpty Dumpty's view of the world: his first and fourth belief belong to this theory; and
- a theory, formalizing Humpty Dumpty's view about the White Knight view of the world, which contains his second and third belief.

Figure 2 represents schematically the situation of Figure 1. The two boxes represent Alice and Humpty Dumpty. A circle stands for a theory formalizing a view. Sequences of indexes in the name of the theory identify the view. Thus, "A" stands for "Alice's view of the world", "A, W" stands for "Alice's view about the White Knight view of the world", and analogously for the others. Arrows intuitively represent the (compatibility) relation which exists between the connected theories. In this example the relation is that "of the source theory having beliefs about the target theory".

Notice that in Figure 2 we have only two theories per agent. Things may get more complicated and we may have many more theories and arrows connecting them. For instance, a belief that Alice has about the beliefs that the White Knight has about some other agent A_i would require a third theory. In this case, the schema of Figure 2 would have to be modified by adding a theory R_{A,W,A_i} and an arrow from $R_{A,W}$ to R_{A,W,A_i}.

Even if very simple, Example 1 and its formalization in Example 2, suggest the two basic means by which new beliefs can be inferred inside a view:

1. By reasoning inside the theory formalizing the view. Thus, for instance, in Figure 1 (left), the third belief is derived from the second by applying a conjunction elimination. To emphasize this fact we call a theory formalizing a view, a *reasoner*.
2. By inferring beliefs from the fact that another reasoner has a belief. Thus, for instance, in Figure 1 (left), the fourth belief (belonging to R_A) is derived from the fact that the third belief belongs to $R_{A,W}$. We call *observer* a reasoner which

is capable of having beliefs about the beliefs of another reasoner. In Figure 2 the observers are the reasoners which are the source of an arrow.

We abstractly represent the reasoning capabilities of a reasoner R as a pair $\langle L, T\rangle$, where L is a set of first order sentences and $T \subseteq L$. L is the *language* and T is the set of *beliefs* or *theorems* of R. (Notationally, in the following, R_i stands for the pair $\langle L_i, T_i\rangle$.) In this paper we restrict ourselves to the propositional case; formulas are thus propositional combinations of either propositional letters or expressions of the form $\mathrm{B}(\text{“}A\text{”})$, where A is a propositional formula. The latter formulas are called *belief sentences*. A in $\mathrm{B}(\text{“}A\text{”})$ is called the *argument* of B.

To represent abstractly the interaction capabilities of a reasoner we need two reasoners, one having beliefs, the other having beliefs about the first. This is captured by the notion of *belief system*.

Definition 3 (Belief System) Let B be a unary predicate symbol. A *belief system (for* B*)* is a pair of reasoners $\langle R_0, R_1\rangle_{\mathrm{B}}$. The parameter B is *the belief predicate*, R_0 is the *observer* and R_1 is the *observed reasoner* of $\langle R_0, R_1\rangle_{\mathrm{B}}$.

R_0's beliefs about R_1's beliefs are represented by the set of belief sentences which are part of the beliefs of R_0.

Example 4 Consider Example 2. Humpty Dumpty is formalized by the belief system $\langle R_H, R_{H,W}\rangle_{\mathrm{B}}$ with R_H and $R_{H,W}$ being respectively the observer and the observed reasoner. We formalize the fact that Humpty Dumpty knows that the White Knight is a looking glass logician as follows:

- the language $L_{H,W}$ of $R_{H,W}$ is a full propositional language;
- the set of beliefs $T_{H,W}$ of $R_{H,W}$ is closed under the “Rule of the Looking Glass Logician”
$$\frac{A_1, \ldots, A_k}{A}\ RLGL$$
where $RLGL$ is applicable only if $(\neg A_1 \wedge \ldots \wedge \neg A_k) \supset \neg A$ is a tautology ($k \geq 0$).

Compare this formalization with the definition of looking glass logician given in Example 1.

We formalize the fact that Humpty Dumpty is a keen arguer by imposing that he is able to derive all the possible consequences of what he knows, and that he believes all the belief sentences $\mathrm{B}(\text{“}A\text{”})$, with A a belief of his view of the White Knight. We have therefore the following:

- the language L_H of R_H is a full propositional language whose set of belief sentences is $\{\mathrm{B}(\text{“}A\text{”}) \mid A \in L_{H,W}\}$, and
- the set of beliefs T_H of R_H is closed under the “Rule of Propositional Logic”
$$\frac{A_1, \ldots, A_k}{A}\ RPL$$
where RPL is applicable only if $(A_1 \wedge \ldots \wedge A_k) \supset A$ is a tautology ($k \geq 0$). Furthermore, the set of belief sentences in T_H is
$$\{\mathrm{B}(\text{“}A\text{”}) \mid A \in T_{H,W}\}.$$

Alice is formalized by the belief system $\langle R_A, R_{A,W}\rangle_{\mathrm{B}}$. The description of Alice given in Example 1 is quite loose. We can however expect the following conditions to hold:

- the languages $L_{A,W}$ and L_A are closed under subformulas;
- the set of beliefs $T_{A,W}$ and T_A are closed under the rules

$$\frac{A_1 \wedge A_2}{A_1} \wedge E_l \qquad \frac{A_1 \wedge A_2}{A_2} \wedge E_r;$$

- the set of belief sentences in L_A corresponds to some given subset Γ of $L_{A,W}$, i.e., it is the set $\{\mathrm{B}(\text{``}A\text{''}) \mid A \in \Gamma, \Gamma \subseteq L_{A,W}\}$; and
- given the same set Γ as above, the set of belief sentences in T_A is

$$\{\mathrm{B}(\text{``}A\text{''}) \mid A \in T_{A,W} \cap \Gamma\}.$$

The first two items reflect the fact that Alice has limited reasoning capabilities, and the assumption that Alice ascribes to the White Knight reasoning capabilities similar to hers. The last two items reflect the fact that Alice has limited capabilities in having beliefs about her beliefs.

As we will see in Example 7, the above conditions allow us to formalize Humpty Dumpty' and Alice's reasoning in Figure 1.

Presenting a belief system requires representing the reasoning and interaction capabilities of each reasoner in the belief system. The reasoning capabilities of a reasoner can be represented by some set of facts (its basic beliefs) together with some inference engine which allows it to derive beliefs from the beliefs it already has. For each reasoner R_i, we thus introduce a corresponding *context* C_i, defined as an axiomatic formal system, i.e., a triple $\langle L_i, \Omega_i, \Delta_i\rangle$, where L_i is the language, Ω_i is the set of axioms and Δ_i is the set of inference rules of C_i. (Notationally, in the following, a context C_i is implicitly defined as $\langle L_i, \Omega_i, \Delta_i\rangle$.) The interaction capabilities of a reasoner can be represented by some set of *bridge rules*, i.e., inference rules with premises and conclusions in different contexts. For instance, the bridge rule

$$\frac{C_1 : A_1}{C_2 : A_2}$$

allows us to derive the formula A_2 in context C_2 just because the formula A_1 has been derived in context C_1. (Notationally, we write $C : A$ to mean the formula A in the context C.) Contexts and bridge rules are the components of *multicontext systems* (*MC systems*), where an MC system is defined as a pair ⟨*family-of-contexts, set-of-bridge-rules*⟩.[2] Derivability in a MC system *MS*, in symbols $\vdash_{\mathrm{MS}}$, is defined in [11]. For sake of completeness this definition is reported in Appendix A. Appendix A contains also a detailed description of the notation used for representing contextual

[2]Multicontext systems can be thought of as particular Labelled Deductive Systems (LDS)s [9, 10]. In particular, multicontext systems are LDSs where labels are used only to keep track of the context formulas belong to, and where inference rules can be applied only to formulas belonging to the "appropriate" context.

inference rules and bridge rules. Roughly speaking, derivability in a MC system is a generalization of Prawitz' notion of deduction inside a Natural Deduction System [22] obtained by allowing multiple languages (one per context) and by indexing formulas with the context they belong to.

We are interested in interactions between R_0 and R_1 which enable R_0 to derive B("A") when R_0 derives A and/or viceversa. The particular class of MC systems presenting belief systems can therefore be characterized as follows:

Definition 5 (MR$^-$) An MC-system $\langle\{C_0, C_1\}, BR\rangle$ is an MR$^-$ system[3] if BR is

$$\frac{C_1 : A}{C_0 : \mathrm{B}(``A")}\ \mathrm{R}^{\mathrm{B}}_{up} \qquad \frac{C_0 : \mathrm{B}(``A")}{C_1 : A}\ \mathrm{R}^{\mathrm{B}}_{dn}$$

and the restrictions include:
$\mathrm{R}^{\mathrm{B}}_{up}$: $C_1 : A$ does not depend on any assumption in C_1.
$\mathrm{R}^{\mathrm{B}}_{dn}$: $C_0 : \mathrm{B}(``A")$ does not depend on any assumption in C_0.

The rule on the left is called reflection up, the one on the right, reflection down. The restrictions are such that a formula can be reflected up or reflected down only if it does not depend on any assumptions, i.e., if it is a theorem. Reflection up allows to prove B("A") in C_0 just because A has been proved in C_1, while reflection down has the dual effect, i.e., it allows to prove A in C_1 just because B("A") has been proved in C_0. The intuition is that the reasoner R_0 (whose reasoning capabilities are characterized by C_0) believes B("A"), i.e., it believes that the reasoner R_1 (whose reasoning capabilities are characterized by C_1) believes A, if R_1 actually believes A and the bridge rules allow R_0 to derive B("A").

Up to Section 6, given an MR$^-$ system $\langle\{C_\alpha, C_\beta\}, BR\rangle$ where α and β are arbitrary strings, we implicitly assume that the context of the premise of the rule of reflection down is C_α if $\alpha < \beta$ (according to the lexicographic order), and C_β otherwise.

The sense in which an MR$^-$ system presents a belief system is made precise by the following definition:

Definition 6 (Belief System presented by an MR$^-$ System) Let $\mathrm{MS} = \langle\{C_0, C_1\}, BR\rangle$ be an MR$^-$ System. MS *presents* the belief system $\langle R_0, R_1\rangle_{\mathrm{B}}$ if $T_i = \{A \mid \vdash_{\mathrm{MS}} C_i : A\}$ $(i = 0, 1)$.

Definition 6 says that the beliefs of a reasoner R_i consist of all the theorems proved by the MR$^-$ system, which belong to L_i.

Example 7 Consider the belief system $\langle R_H, R_{H,W}\rangle_{\mathrm{B}}$ formalizing Humpty Dumpty in Example 4. This belief system can be presented by an MR$^-$ system

$$\langle\{C_H, C_{H,W}\}, \{\mathrm{R}^{\mathrm{B}}_{up}, \mathrm{R}^{\mathrm{B}}_{dn}\}\rangle$$

where

- $L_{H,W}$ and L_H satisfy the corresponding conditions in Example 4,

[3] The abbreviations MR and MBK (where the letters "M", "R", "B" and "K" stand for "Multi", "Reflection", "Belief" and the modal logic $\mathcal{K}$, respectively) come from [12]. The minus in the superscript, when used, indicates that the defined system is weaker than the corresponding system in [12].

- $\Delta_{H,W}$ and Δ_H consists of *RLGL* and *RPL* respectively, and
- there are no other restrictions on the applicability of $\mathrm{R}^{\mathrm{B}}_{up}$ and $\mathrm{R}^{\mathrm{B}}_{dn}$ other than those mentioned in Definition 5.

In this class of MR$^-$ systems it is possible to formalize Humpty Dumpty's reasoning process shown in Figure 1, namely, to prove that if the White Knight believes that the Red King is Asleep (*RKA*), then he also believes that the Red King is Asleep and the Red Queen is Asleep (*RQA*). In other words, if B("*RKA*") is provable in C_H, then also B("$RKA \wedge RQA$") is provable in C_H. The proof goes as follows:

$$\cfrac{\cfrac{\cfrac{C_H : \mathrm{B}(\text{“}RKA\text{”})}{C_{H,W} : RKA}\;\mathrm{R}^{\mathrm{B}}_{dn}}{C_{H,W} : RKA \wedge RQA}\;RLGL}{C_H : \mathrm{B}(\text{“}RKA \wedge RQA\text{”})}\;\mathrm{R}^{\mathrm{B}}_{up}$$

Along the same lines, it is possible to present Alice's belief system $\langle R_A, R_{A,W}\rangle_{\mathrm{B}}$ with the corresponding MR$^-$ system

$$\langle \{C_A, C_{A,W}\}, \{\mathrm{R}^{\mathrm{B}}_{up}, \mathrm{R}^{\mathrm{B}}_{dn}\}\rangle$$

where

- $L_{A,W}$ and L_A satisfy the corresponding conditions in Example 4,
- $\Delta_{A,W}$ and Δ_A consists of $\wedge E_l$ and $\wedge E_r$, and
- the only additional restriction on the applicability of $\mathrm{R}^{\mathrm{B}}_{up}$ and $\mathrm{R}^{\mathrm{B}}_{dn}$ is that the premise of $\mathrm{R}^{\mathrm{B}}_{up}$ must belong to Γ.

In these systems, provided that *RQA* belongs to Γ, we are able to prove Alice's belief that if the White Knight believes that both the Red King and the Red Queen are asleep, then he also believes that the Red King is asleep. This corresponds to Alice's reasoning in Figure 1. The proof goes as follows:

$$\cfrac{\cfrac{\cfrac{C_A : \mathrm{B}(\text{“}RKA \wedge RQA\text{”})}{C_{A,W} : RKA \wedge RQA}\;\mathrm{R}^{\mathrm{B}}_{dn}}{C_{A,W} : RQA}\;\wedge E_r}{C_A : \mathrm{B}(\text{“}RQA\text{”})}\;\mathrm{R}^{\mathrm{B}}_{up}$$

3 Ideal reasoners and observers

Reasoners and observers can be categorized depending on their reasoning and interaction capabilities, respectively. We say that reasoners and observers are *ideal* if they satisfy certain closure properties in the sense made precise by the following definition.

Definition 8 (Ideality) Given a belief system $\langle R_0, R_1\rangle_{\mathrm{B}}$, we say that:

- R_i $(i = 0, 1)$ is an *Ideal Reasoner* if

– L_i is closed under the formation rules for propositional languages, and
– T_i is closed under tautological consequence.

- R_0 is an *Ideal Observer* if
  – $L_1 = \{A \mid \mathrm{B}(\text{“}A\text{”}) \in L_0\}$, and
  – $T_1 = \{A \mid \mathrm{B}(\text{“}A\text{”}) \in T_0\}$.

The intuition is that R_0 and R_1 are ideal reasoners if they are able to believe all and only the (tautological) consequences of what they know. R_0 is an ideal observer if it believes all and only those belief sentences whose argument is a belief of R_1. Notice that no request is made about the specific elements of the sets defining an ideal reasoner or an ideal observer. Thus, for instance, whether a formula belongs to the set of beliefs of an ideal reasoner is left undetermined; we only require that, if this is the case, so must be for all its consequences. This choice reflects how, in practice, ideality is dealt with in the literature where, for instance, $\mathcal{K}$ is the modal system for omniscience no matter what theoretic axioms are added.

Notice that the "empty" reasoner — the reasoner with empty language — is ideal. Analogously, any "absolutely contradictory" reasoner — any reasoner with a full propositional language L believing any proposition in L — is ideal. At a first sight this might go against our intuitions. However it is a fact that these reasoners satisfy all the conditions for ideality. A good way to think about this is to see these reasoners as the result of a process of limit where we progressively decrease the number of atomic formulas of a language (in the case of empty reasoner) or increase the number of theorems (in the case of absolutely contradictory reasoners).

Example 9 Consider Example 4. In general, $R_{H,W}$ is not an ideal reasoner; in fact its beliefs are not closed under tautological consequence. R_H is instead an ideal reasoner as well as an ideal observer of $R_{H,W}$. Concerning Alice, $R_{A,W}$ and R_A are again not ideal reasoners while R_A is an ideal observer of $R_{A,W}$ only if $\Gamma = L_{A,W}$.

The closure conditions for ideality can be captured by posing appropriate restrictions on MR$^-$ systems. Let us consider the following definition:

Definition 10 (MBK$^-$) An MR$^-$ system $\langle\{C_0, C_1\},\ BR\rangle$ is an MBK$^-$ system if the following conditions are satisfied:

- L_0 and L_1 contain a given set P of propositional letters, the symbol for falsity $\bot$, and are closed under implication[4];
- Δ_0 (Δ_1) includes the instances

$$\frac{A_1, \ldots, A_k}{A}\ RPL_k$$

  such that $(A_1 \wedge \ldots \wedge A_k) \supset A$ is a tautology and $k \in \{0, 1, 2\}$[5];

[4]We also use standard abbreviations from propositional logic, such as $\neg A$ for $A \supset \bot$, $A \vee B$ for $\neg A \supset B$, $A \wedge B$ for $\neg(\neg A \vee \neg B)$, $\top$ for $\bot \supset \bot$.

[5]Notice that RPL_1 is a derived inference rule of RPL_2, and thus both Δ_0 and Δ_1 are not minimal. However, the case $k = 1$ allows for a more natural formulation of Definition 28 in Section 5.2.

- $L_1 = \{A \mid \mathrm{B}(\text{"}A\text{"}) \in L_0\}$;
- the restrictions on the applicability of reflection up/down are only those listed in Definition 5.

Theorem 11 Let $\langle R_0, R_1 \rangle_{\mathrm{B}}$ be the belief system presented by an MBK$^-$ system. Then

- R_0 and R_1 are ideal reasoners, and
- R_0 is an ideal observer.

The proof is straightforward. It is sufficient to observe that the first two conditions in Definition 10 ensure that each R_i is an ideal reasoner ($i = 0, 1$); while the last two ensure that R_0 is an ideal observer.

4 Real reasoners and observers

At first, one is tempted to define reality as not ideality, in the same way as not omniscience is usually defined as absence of omniscience. However this is not what we want. Consider for instance the empty reasoner and any absolutely contradictory reasoner. These are ideal reasoners, nevertheless we would like to say that they are also real. There is in fact a sense in which the empty reasoner believes "too little" while any absolutely contradictory reasoner believes "too much" with respect to what we would consider an ideal situation. Analogously, consider a reasoner which is not aware of a proposition or does not believe a true sentence, but whose formation and inference rules are complete for propositional logic. This reasoner is an ideal reasoner, however we would like to say that it is also a real reasoner. This reasoner simply does not know all it ought to know.

Differently from what is the case for ideality, reality is a relative notion which states the absence of certain properties with respect to a specific reference. When talking of a real reasoner or a real observer we mean that such a reasoner or observer believes too little or too much (i.e., it is incomplete or incorrect) with respect to a reasoner or observer taken as reference. This intuition is already informally articulated, even if limited to beliefs and reasoners, in [13]. In particular, in that paper a reasoner is defined real relatively to another reasoner, independently of (what we have called here) the belief system of which it is part. However, the formalization of these ideas is more complex than it might seem, and, as the technical development discussed below shows, the notion of reality informally introduced in [13] is not correct. The key observation is that two reasoners or observers cannot be compared independently of the belief system of which they are part.

The proofs of the theorems in this Section are reported in Appendix B.

4.1 Realizing MR$^-$ systems

The starting point is to define when a belief system is (in)correct or (in)complete with respect to another belief system (notationally, we write $\langle R_{A_0}, R_{A_1} \rangle_{\mathrm{B}} \subseteq \langle R_{B_0}, R_{B_1} \rangle_{\mathrm{B}}$ to mean $R_{A_0} \subseteq R_{B_0}$ and $R_{A_1} \subseteq R_{B_1}$):

Definition 12 (Belief System Correctness/Completeness) A belief system $\langle R_{E_0}, R_{E_1}\rangle_{\mathrm{B}}$ is *correct [complete]* with respect to a belief system $\langle R_{I_0}, R_{I_1}\rangle_{\mathrm{B}}$ if $\langle R_{E_0}, R_{E_1}\rangle_{\mathrm{B}} \subseteq \langle R_{I_0}, R_{I_1}\rangle_{\mathrm{B}}$ $[\langle R_{I_0}, R_{I_1}\rangle_{\mathrm{B}} \subseteq \langle R_{E_0}, R_{E_1}\rangle_{\mathrm{B}}]$.

The intuition is that, for instance, in a correct belief system, each reasoner maintains a subset of the beliefs of the corresponding reasoner in the reference belief system. If the beliefs of one reasoner (for example of R_{E_0}) are strictly contained in the beliefs of the corresponding reasoner (R_{I_0}) then the belief system is incomplete. We say that $\langle R_{E_0}, R_{E_1}\rangle_{\mathrm{B}}$ is *real* with respect to $\langle R_{I_0}, R_{I_1}\rangle_{\mathrm{B}}$, to mean that $\langle R_{E_0}, R_{E_1}\rangle_{\mathrm{B}}$ is incomplete or incorrect with respect to $\langle R_{I_0}, R_{I_1}\rangle_{\mathrm{B}}$. Analogously, we say that a reasoner (observer) is *real* when it believes more or less beliefs (beliefs about beliefs) than it should, i.e., when it is *incorrect* or *incomplete* with respect to another reasoner (observer) taken as reference.

The next step is to "propagate" the notions of (in)correctness and (in)completeness from belief systems to MR^- systems. However things are complicated as a comparison between MR^- systems based simply on set inclusion of the components does not work. For instance, it is easy to think of two different sets of axioms with the same proof-theoretic power. To solve this problem we introduce a new operation $\oplus$ such that, if $C = \langle L, \Omega, \Delta\rangle$ and $C' = \langle L', \Omega', \Delta'\rangle$ are two contexts, then $C \oplus C'$ is the context $\langle L \cup L', \Omega \cup \Omega', \Delta \cup \Delta'\rangle$. This allows us to give the following definition (notationally, in the following, $\mathrm{MS}_E = \langle\{C_{E_0}, C_{E_1}\}, BR_E\rangle$ and $\mathrm{MS}_I = \langle\{C_{I_0}, C_{I_1}\}, BR_I\rangle$ are MR^- systems presenting the belief system $\langle R_{E_0}, R_{E_1}\rangle_{\mathrm{B}}$ and $\langle R_{I_0}, R_{I_1}\rangle_{\mathrm{B}}$, respectively):

Definition 13 (MR^- Correctness/Completeness) MS_E is a *correct [complete] realization* of MS_I if $\langle\{C_{E_0} \oplus C_{I_0}, C_{E_1} \oplus C_{I_1}\}, BR_E \cup BR_I\rangle$ and MS_I $[\mathrm{MS}_E]$ present the same belief system.

MS_E is *equivalent* to MS_I if it is correct and complete with respect to MS_I. We talk of *realization* to emphasize the process by which the constructors of a real belief system [real reasoner, real observer] are defined starting from those of a reference belief system [reasoner, observer]. Consider for instance the notion of correct realization. MS_E is a correct realization of MS_I if adding its proof-theoretic power to that of MS_I results into a system which still has the same proof-theoretic power as MS_I. From the above definition, it trivially follows that MS_E is a correct realization of MS_I if and only if MS_I is a complete realization of MS_E. As trivial examples, the empty system is a correct realization of any reference system MS_I. Any absolutely contradictory system MS is complete with respect to any reference system whose two languages stand in a subset relation with the corresponding languages of MS.

Consider the following two examples in which Γ is a given, finite set of propositional formulas.

Example 14 Let MS_I be the smallest MBK^- system. Let MS_E be defined as MS_I except that Δ_{E_1} consists of the instances of RPL_0 whose conclusions belongs to the set of formulas Γ. Then, MS_E is a correct but incomplete realization of MS_I.

Example 15 Let MS_I be the smallest MBK^- system. Let MS_E be defined as MS_I except that $\mathrm{R}^{\mathrm{B}}_{up}$ has the additional restriction that the premise belongs to the set of formulas Γ. Then, MS_E is a correct but incomplete realization of MS_I.

In the above two examples, the observer of the belief system presented by MS_E is the same. However, Example 14 and Example 15 model two very different situations. In the first example, we have an ideal reasoner R_{E_0} ideally interacting with a non ideal reasoner R_{E_1}. In the second example, both R_{E_0} and R_{E_1} are ideal reasoners, but the interaction between the two is not ideal. If we assume that R_{E_1} formalizes an agent a's beliefs about the world while R_{E_0} formalize a's beliefs about his beliefs about the world, then in the first case we think that a is not capable of ideal reasoning about the world, while in the second he is assumed to be not capable of ideally observing its own beliefs.

To save space, from now on, we consider incompleteness only. With some provisos, all the results presented below can be replicated for incorrectness.

The link between MS_E being an incomplete realization of MS_I and the incompleteness of $\langle R_{E_0}, R_{E_1}\rangle_{\mathrm{B}}$ with respect to $\langle R_{I_0}, R_{I_1}\rangle_{\mathrm{B}}$ is established by the following theorem.

Theorem 16 Let MS_E be a correct realization of MS_I. Then

- $\langle R_{E_0}, R_{E_1}\rangle_{\mathrm{B}}$ is correct with respect to $\langle R_{I_0}, R_{I_1}\rangle_{\mathrm{B}}$, and
- $\langle R_{E_0}, R_{E_1}\rangle_{\mathrm{B}}$ is incomplete with respect to $\langle R_{I_0}, R_{I_1}\rangle_{\mathrm{B}}$ if and only if MS_E is an incomplete realization of MS_I.

The proof is a consequence of the fact that $\langle\{C_{E_0} \oplus C_{I_0}, C_{E_1} \oplus C_{I_1}\}, BR_E \cup BR_I\rangle$ and MS_I present the same belief system. The second item of Theorem 16 states that we have achieved what we wanted, i.e., that incompleteness between two MR^- systems corresponds to incompleteness in the belief systems presented, and viceversa. Notice however that this result holds under the hypothesis that MS_E is a correct realization of MS_I. This hypothesis is necessary in order to guarantee that $\langle R_{E_0}, R_{E_1}\rangle_{\mathrm{B}}$ is correct with respect to $\langle R_{I_0}, R_{I_1}\rangle_{\mathrm{B}}$. In fact, the viceversa of the first item of Theorem 16 does not hold. That is, MS_E can be an incorrect realization of MS_I and $\langle R_{E_0}, R_{E_1}\rangle_{\mathrm{B}}$ be correct with respect to $\langle R_{I_0}, R_{I_1}\rangle_{\mathrm{B}}$. Consider the following example.

Example 17 Let MS_I be the smallest MBK^- system. Suppose MS_E is defined as MS_I except that Δ_{E_1} is empty. Then MS_E is a correct but incomplete realization of MS_I. From Theorem 16, we have that $\langle R_{E_0}, R_{E_1}\rangle_{\mathrm{B}}$ is correct and incomplete with respect to $\langle R_{I_0}, R_{I_1}\rangle_{\mathrm{B}}$, i.e.,

$$\langle R_{E_0}, R_{E_1}\rangle_{\mathrm{B}} \subset \langle R_{I_0}, R_{I_1}\rangle_{\mathrm{B}}.$$

However, consider the system MS'_E obtained from MS_E by adding to Δ_{E_1} the inference rule

$$\frac{A \vee \neg A}{A}\ \rho$$

for any propositional letter A in L_{E_1}. Then MS'_E is an incorrect realization of MS_I even though the belief systems presented by MS_E and MS'_E are the same (In fact ρ can never be applied in MS'_E, as an empty Δ_{E_1} implies that there is now way to derive $A \vee \neg A$).

From Definition 13, correct realizations have the property that the result of adding the components of MS_I to MS_E defines an MR^- system which still presents a correct belief system. Intuitively, this property guarantees that correct realizations generate theorems in a way which is consistent with how theorems are generated by the reference MR^- system.

4.2 Realizing contexts and bridge rules

The next step is to find necessary and sufficient conditions for having realizations of MR^- systems. Via Theorem 16 this provides necessary and sufficient conditions on the presented belief systems.

Definition 18 (Context Incompleteness) We say that

- C_{E_0} is an *incomplete* realization of C_{I_0} if MS_E is an incomplete realization of $\langle\{C_{E_0} \oplus C_{I_0}, C_{E_1}\},\ BR_E\rangle$;
- C_{E_1} is an *incomplete* realization of C_{I_1} if MS_E is an incomplete realization of $\langle\{C_{E_0}, C_{E_1} \oplus C_{I_1}\},\ BR_E\rangle$;
- BR_E is an *incomplete* realization of BR_I if MS_E is an incomplete realization of $\langle\{C_{E_0}, C_{E_1}\}, BR_E \cup BR_I\rangle$.

Example 19 Consider Example 14. In this case, C_{E_0} is a complete realization of C_{I_0}, C_{E_1} is an incomplete realization of C_{I_1} and BR_E is a complete realization of BR_I.

Example 20 Consider Example 15. In this case, C_{E_0} is a complete realization of C_{I_0}, C_{E_1} is a complete realization of C_{I_1} and BR_E is an incomplete realization of BR_I.

Notice that in both examples C_{E_0} is a complete realization of C_{I_0} even though the beliefs of R_{E_0} are strictly contained in the beliefs of R_{I_0}, i.e., $T_{E_0} \subset T_{I_0}$. This is exactly what one would expect since the reasoning capabilities of R_{I_0} and R_{E_0} (modeled by C_{E_0} and C_{I_0} respectively) are the same. Section 5.2 provides two substantial examples of incomplete realizations of contexts and bridge rules.

Theorem 21 MS_E is an incomplete realization of MS_I if and only if at least one of the following three conditions is satisfied:

- C_{E_0} is an incomplete realization of C_{I_0};
- C_{E_1} is an incomplete realization of C_{I_1};
- BR_E is an incomplete realization of BR_I.

4.3 Realizing the components of contexts

The next and final step is to iterate what done in Section 4.2 to the components of contexts. Via Theorem 16 and Theorem 21, this provides necessary and sufficient conditions on the presented belief systems.

(Notationally, in the following, if C_i is a context, P_i and W_i are the set of atomic formulas and the set of construction rules for L_i, respectively. L_i is therefore defined as the smallest set generated from P_i and closed under W_i, in symbols $L_i = Cl(P_i, W_i)$.)

Definition 22 (Context's Component Incompleteness) Let A be one of the letters in $\{P, W, \Omega, \Delta\}$. For $i \in \{0, 1\}$, we say that A_{E_i} is an *incomplete realization of* A_{I_i} if MS_E is an incomplete realization of $\langle\{C_0, C_1\},\ BR_E\rangle$, where[6]:

$$C_j = \begin{cases} \langle Cl(P_{E_j} \cup P_{I_j}, W_{E_j}), \Omega_{E_j}, \Delta_{E_j}\rangle & \text{if } j = i \text{ and } A = P; \\ \langle Cl(P_{E_j}, W_{E_j} \cup W_{I_j}), \Omega_{E_j}, \Delta_{E_j}\rangle & \text{if } j = i \text{ and } A = W; \\ \langle L_{E_j}, W_{E_j}, \Omega_{E_j} \cup \Omega_{I_j}, \Delta_{E_j}\rangle & \text{if } j = i \text{ and } A = \Omega; \\ \langle L_{E_j}, W_{E_j}, \Omega_{E_j}, \Delta_{E_j} \cup \Delta_{I_j}\rangle & \text{if } j = i \text{ and } A = \Delta; \\ C_{E_j} & \text{otherwise.} \end{cases}$$

Example 23 Consider Example 14. In this case, Δ_{E_1} is an incomplete realization of Δ_{I_1}.

Definition 22 fixes the intuitively correct but formally wrong classification provided in [13]. That paper discusses in detail the intuitions underlying this classification and provides various examples. As already discussed in [13], the various forms of incompleteness (in the signature, formation rules, axioms, inference rules) model very different intuitions. For instance, the incompleteness in the signature models the case in which a reasoner is not aware of some primitive propositions. This is the case, for example, of the Bantu tribesman in [7] who is not aware that personal computer prices are going down. A "more civilized" tribesman might be aware of computers and their prices, but he might not believe that their prices are decreasing. The latter situation is modeled with a reasoner incomplete in the axioms. Incompleteness in the formation rules and/or inference rules are best suited for modeling the limitation of resources that real reasoners have both in constructing sentences and in proving theorems.

Theorem 24 C_{E_i} is an incomplete realization of C_{I_i} $(i = 0, 1)$ if and only if at least one of the following four conditions is satisfied:

- P_{E_i} is an incomplete realization of P_{I_i};
- W_{E_i} is an incomplete realization of W_{I_i};
- Ω_{E_i} is an incomplete realization of Ω_{I_i};
- Δ_{E_i} is an incomplete realization of Δ_{I_i}.

[6]Strictly speaking, $\langle L_{E_j}, W_{E_j}, \Omega_{E_j} \cup \Omega_{I_j}, \Delta_{E_j}\rangle$ is not assured to be a context unless $\Omega_{I_j} \subseteq L_{E_j}$. More carefully, we should write $\langle L_{E_j}, W_{E_j}, \Omega_{E_j} \cup (\Omega_{I_j} \cap L_{E_j}), \Delta_{E_j}\rangle$. Analogously for $\langle L_{E_j}, W_{E_j}, \Omega_{E_j}, \Delta_{E_j} \cup \Delta_{I_j}\rangle$.

It is important to notice that the classification provided by Definitions 18 and 22 is exhaustive in the sense that it considers all the constructors on MR^- systems. This, together with Theorems 21 and 24, achieves the goal set up in Section 1, that is, these definitions provide an exhaustive classification of all the possible forms and sources of reality.

5 A comparison with modal systems

The most common approach to the formalization of the beliefs of an agent a is to take an axiomatic formal system, extend its language with a modal operator $\mathcal{B}$,[7] and (under the *objective interpretation* of belief [21]) take $\mathcal{B}A$ as representing the fact that a believes A. The desired properties of a's beliefs are obtained by considering some set of axioms (each set characterizing a particular modal system Σ, see for example [2, 8]). The beliefs ascribed to the agent are represented by the formulas A such that $\mathcal{B}A$ is provable in Σ. More formally, a *modal system* Σ is a pair $\langle L, T\rangle$ such that

- L contains the set $P \cup \{\bot\}$ of propositional letters (with P as in Definition 10), is closed under implication and the modal operator $\mathcal{B}$; and
- $T \subseteq L$ and T is closed under tautological consequence.

A formula A is a *theorem* of Σ ($\vdash_\Sigma A$) if $A \in T$. A is *derivable* from a set Γ of formulas ($\Gamma \vdash_\Sigma A$) if $(A_1 \wedge \ldots \wedge A_n) \supset A \in T$ and $\{A_1, \ldots, A_n\} \subseteq \Gamma$. We also say that Σ is $\mathcal{K}_n$*-classical* if any formula of the form $(\mathcal{B}A_1 \wedge \ldots \wedge \mathcal{B}A_n) \supset \mathcal{B}A$ belongs to T whenever $(A_1 \wedge \ldots \wedge A_n) \supset A$ belongs to T, $(n = 0, 1, 2)$. $\mathcal{K}_1$-classical modal systems are said to be *monotone*, $\mathcal{K}_2$-classical modal systems are said to be *regular*, and $\{\mathcal{K}_0, \mathcal{K}_2\}$-classical modal systems are said to be *normal.*

An MC system $\text{MS} = \langle\{C_0, C_1\}, BR\rangle$ and a modal system $\Sigma = \langle L, T\rangle$ are said to be *equivalent* if for any formula A in L_0,

$$\vdash_{\text{MS}} C_0 : A \quad \Longleftrightarrow \quad \vdash_\Sigma A^+, \tag{1}$$

where A^+ is the modal counterpart of A, i.e., it is obtained replacing any monadic atomic formula M(“B”) with $\mathcal{M}B$ in A.

The proofs of the theorems in this Section are in Appendix C. Notice that in this section, coherently with the analysis given above, we restrict ourselves to the case of no nesting of modal operators (no nested beliefs). This assumption is lifted in Section 7.

5.1 Normal modal systems

We start by studying whether the smallest normal modal system $\mathcal{K}$ and the smallest MBK^- system are equivalent. This is motivated by the fact that the observer presented by an MBK^- system is both an ideal reasoner and an ideal observer; in other words, it is saturated with respect to the properties that we have considered (of reasoning, of observing). Analogously, $\mathcal{K}$ is the smallest normal system which is meant

[7] Notationally, we write modal operators using calligraphic style.

to model omniscient agents (see [8]). MBK$^-$ and $\mathcal{K}$ turn out to be not equivalent. In fact, for any set of propositional formulas $\Gamma \cup \{A\}$, we have

$$\Gamma \vdash_{\mathcal{K}} A \Longrightarrow \{\mathcal{B}A : A \in \Gamma\} \vdash_{\mathcal{K}} \mathcal{B}A.$$

This property gives $\mathcal{K}$ a form of ideality with respect to derivations which is much stronger than the form of ideality possessed by R_0, which is only with respect to theorems. One way to fill the gap between $\mathcal{K}$ and R_0 is to add to the set of axioms of the context C_0 the corresponding of the $\mathcal{K}$ axiom $\mathcal{B}A \supset (\mathcal{B}(A \supset B) \supset \mathcal{B}B)$. Namely, for any propositional formulas A and B,

$$\mathrm{B}(\text{“}A\text{”}) \supset (\mathrm{B}(\text{“}A \supset B\text{”}) \supset \mathrm{B}(\text{“}B\text{”}))$$

should be an element of Ω_0. The resulting system can easily be proved equivalent to $\mathcal{K}$. Another possibility is to drop the restriction on reflection down and take Δ_0 to be the set of inference rules for Classical Natural Deduction Systems.

Definition 25 (MBK) An MC system $\langle\{C_0, C_1\}, BR\rangle$ is an MBK system if it satisfies all the conditions of Definition 10 except that

- Δ_0 includes the set of Classical Natural Deduction Rules,
- $\mathrm{R}^{\mathrm{B}}_{dn}$ has no restrictions.

Theorem 26 Let $\mathcal{K}$ be the smallest normal modal system and let MBK be the smallest MBK system. $\mathcal{K}$ and MBK are equivalent.

The above theorem (similar to Theorem 5.1 in [12]) is proved as a corollary of Theorem 29 (see Section 5.2). The effect of unrestricting reflection down is to allow a very interesting form of reasoning called *simulative reasoning* in [6], and also implemented in many applications (see, e.g., [6, 14]). Intuitively, in any MBK system it is possible to

1. make assumptions in C_0 about C_1's beliefs (e.g., assume $C_0 : \mathrm{B}(\text{“}A_1\text{”}), \ldots, C_0 : \mathrm{B}(\text{“}A_n\text{”})$),
2. evaluate in C_1 the consequences of these assumptions (e.g., infer by reflection down $C_1 : A_1, \ldots, C_1 : A_n$ and perform deduction in C_1 deriving $C_1 : B$), and
3. reflect back the result in C_0 (e.g., infer by reflection up $C_0 : \mathrm{B}(\text{“}B\text{”})$ depending on $C_0 : \mathrm{B}(\text{“}A_1\text{”}), \ldots, C_0 : \mathrm{B}(\text{“}A_n\text{”})$).

Consider for instance the following deduction of (the translation of) the $\mathcal{K}$ axiom:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{C_0 : \mathrm{B}(\text{“}A\text{”})}{C_1 : A}\,\mathrm{R}^{\mathrm{B}}_{dn} \quad \dfrac{C_0 : \mathrm{B}(\text{“}A \supset B\text{”})}{C_1 : A \supset B}\,\mathrm{R}^{\mathrm{B}}_{dn}}{C_1 : B}\,\supset\!\mathrm{E}_1}{C_0 : \mathrm{B}(\text{“}B\text{”})}\,\mathrm{R}^{\mathrm{B}}_{up}}{C_0 : \mathrm{B}(\text{“}A \supset B\text{”}) \supset \mathrm{B}(\text{“}B\text{”})}\,\supset\!\mathrm{I}_0}{C_0 : \mathrm{B}(\text{“}A\text{”}) \supset (\mathrm{B}(\text{“}A \supset B\text{”}) \supset \mathrm{B}(\text{“}B\text{”}))}\,\supset\!\mathrm{I}_0}{}$$

C_1's reasoning capabilities are used by C_0 to infer modal formulas (e.g., the $\mathcal{K}$ axiom) which would otherwise have to be asserted as axioms, or derived from these axioms by modal reasoning (in $\mathcal{K}$, by using the $\mathcal{K}$ axiom). In this perspective the choice (in Definition 25) of Δ_0 as a set of Classical Natural Deduction Rules is crucial in order to guarantee that

$$\Gamma, C_0 : A \vdash_{\text{MS}} C_0 : B \Longrightarrow \Gamma \vdash_{\text{MS}} C_0 : A \supset B$$

holds, also when $\Gamma, C_0 : A \vdash_{\text{MS}} C_0 : B$ is proved by a deduction across languages (notice that this property is exploited twice — in the last two steps — in the deduction above). Using RPL_k as a rule for propositional logic (see Definition 10) the above deduction would not be possible as there would be no way to make an assumption in C_0, make inferences in C_1 with an open assumption in C_0, and then close this assumption once the appropriate theorems have been reflected up from C_1 to C_0.

As originally shown in [12], MBK can be extended to obtain MC systems equivalent to stronger normal modal logics. This can be done simply by adding further appropriate bridge rules. Technically, the idea is to keep reflection down unrestricted and use the new bridge rules to increase and control the propagation of information between C_0 and C_1. Intuitively this shows how the idea of simulative reasoning can be extended beyond what originally described in [6]. More importantly, it provides an effective tool for formally specifying the agents' simulative capabilities in the current implementations of multiagent systems. Consider for instance an agent a (modeled by C_0) which is able to export assumptions about its own beliefs in the mental image that it has of itself (modeled by C_1). Technically, this can be modeled via the bridge rule:

$$\frac{C_0 : \text{B}(\text{“}A\text{”})}{C_1 : \text{B}(\text{“}A\text{”})}\ 4_{br}.$$

Adding 4_{br} to the set of bridge rules of an MBK system, makes it possible to prove $C_0 : \text{B}(\text{“}A\text{”}) \supset \text{B}(\text{“B(“}A\text{”)”})$, i.e., the first order translation of the *positive introspective axiom* $\mathcal{B}A \supset \mathcal{B}\mathcal{B}A$ (if a believes A, then it believes that it believes A). A proof is the following:

$$\cfrac{\cfrac{\cfrac{C_0 : \text{B}(\text{“}A\text{”})}{C_1 : \text{B}(\text{“}A\text{”})}\ 4_{br}}{C_0 : \text{B}(\text{“B(“}A\text{”)”})}\ \text{R}^{\text{B}}_{up}}{C_0 : \text{B}(\text{“}A\text{”}) \supset \text{B}(\text{“B(“}A\text{”)”})}\ \supset\!\text{I}_0. \tag{2}$$

We may also assume that a is able to export its own assumptions about what it does not believe in the mental image that it has of itself. This situation can be modeled by the following bridge rule:

$$\frac{C_0 : \neg\text{B}(\text{“}A\text{”})}{C_1 : \neg\text{B}(\text{“}A\text{”})}\ 5_{br}$$

which allows to prove the corresponding of the *negative introspective axiom* $\neg\mathcal{B}A \supset \mathcal{B}\neg\mathcal{B}A$ (if the a does not believe A, then it believes that it does not believe A) along the same lines as the proof (2).

5.2 Classical modal systems

Let us consider $\mathcal{K}_{p,q}$ modal systems, namely systems which are $\mathcal{K}_p$-classical and $\mathcal{K}_q$-classical. Consider the smallest $\mathcal{K}_{p,q}$ system. Then,

- if $p = 0$ and $q = 1$ we have Fagin' and Halpern's system of local reasoning [7, 8];
- if $p = 0$ and $q = 2$ we have the smallest normal modal system $\mathcal{K}$;
- if $p = q = 1$ we have the smallest monotone modal system $\mathcal{M}$ [2];
- if $p = q = 2$ we have the smallest regular modal system $\mathcal{R}$ [2].

Starting from an MBK system $\langle\{C_0, C_1\}, BR\rangle$ —presenting the belief system $\langle R_0, R_1\rangle_{\text{B}}$— we can obtain equivalent MC systems by weakening the reflection principles or the deductive machinery of C_1. In the first case R_0 is no longer guaranteed to be an ideal observer; in the second R_1 is no longer guaranteed to be an ideal reasoner. Notice that to obtain an equivalence result we have to start from MBK and not MBK$^-$. In fact the modal systems mentioned above still possess a (limited) form of ideality with respect to derivations. That is, in any $\mathcal{K}_{p,q}$ modal system Σ if $k \in \{p, q\}$ then

$$A_1, \ldots, A_k \vdash_\Sigma A \Longrightarrow \mathcal{B}A_1, \ldots, \mathcal{B}A_k \vdash_\Sigma \mathcal{B}A.$$

Definition 27 (MBK$_{p,q}$) An MC system $\langle\{C_0, C_1\}, BR\rangle$ is an MBK$_{p,q}$ system if it satisfies all the conditions of Definition 25 except that R^{B}_{up} has the further restriction that the premise depends on exactly p or q occurrences of formulas in L_0.

Definition 28 (MBK$'_{p,q}$) An MC system $\langle\{C_0, C_1\}, BR\rangle$ is an MBK$'_{p,q}$ system if it satisfies all the conditions of Definition 25 except that RPL_k in Δ_1 has the further restriction that $k \in \{p, q\}$.

Theorem 29 Let $\mathcal{K}_{p,q}$ be the smallest $\mathcal{K}_{p,q}$-classical system and let MBK$_{p,q}$, MBK$'_{p,q}$ be the smallest MBK$_{p,q}$ and MBK$'_{p,q}$ systems respectively. $\mathcal{K}_{p,q}$ is equivalent to MBK$_{p,q}$ and MBK$'_{p,q}$.

Similar to what happens in Examples 14 and 15, MBK$_{p,q}$ and MBK$'_{p,q}$ present the same observer. However in MBK$_{p,q}$ the source of the (eventual) reality is in the bridge rules (modeling the reasoners' interaction capabilities), while in MBK$'_{p,q}$ is in one of the contexts (modeling the reasoners' reasoning capabilities).

Finally, it is easy to prove that MBK, as defined in Theorem 26, and MBK$'_{0,2}$, as defined in Theorem 29, are equivalent. The two systems differ only for the set of inference rules in C_1. However, these two sets are equivalent: any inference rule in one set is a derived inference rule in the other.

5.3 The Logic of General Awareness

Let us now consider Fagin' and Halpern's logic of general awareness [7, 8]. The idea of this logic is to distinguish between what an agent *explicitly* believes (modeled via a modal operator $\mathcal{X}$) and what an agent *implicitly* believes (modeled via the modal operator $\mathcal{B}$). Intuitively, an agent's explicit beliefs represent its actual beliefs, while its implicit beliefs model how the world would be if the agent explicit beliefs were true [7, 20]. In order to characterize what an agent explicitly believes, Fagin and Halpern introduce another modal operator $\mathcal{A}$: the formula $\mathcal{A}A$ means that the agent is "aware" of A. An agent explicitly believes A if and only if it implicitly believes A and is aware of A. More precisely, a normal modal system $\mathcal{AW} = \langle L_{\mathcal{AW}}, T_{\mathcal{AW}} \rangle$ is a *system of general awareness* if

- $L_{\mathcal{AW}}$ is closed under the modal operators $\mathcal{B}$, $\mathcal{X}$ and $\mathcal{A}$[8]; and
- any formula of the form $\mathcal{X}A \leftrightarrow (\mathcal{B}A \wedge \mathcal{A}A)$ *(explicit belief definition axiom)* belongs to $T_{\mathcal{AW}}$.

Given a set of formulas Γ, an agent is aware of Γ in a system of general awareness $\mathcal{AW} = \langle L_{\mathcal{AW}}, T_{\mathcal{AW}} \rangle$, if for any formula $A \in \Gamma$, the *awareness axiom* $\mathcal{A}A$ belongs to $T_{\mathcal{AW}}$. If this is the case, we say that $\mathcal{AW}$ is an $\mathcal{AW}_\Gamma$ *system.*

To capture $\mathcal{AW}_\Gamma$ systems, we introduce a new predicate X corresponding to the modal operator $\mathcal{X}$. The set of formulas Γ of which an agent is aware determines the set of R_1's beliefs the reasoner R_0 can explicitly observe. In other words, if A has been derived in the context C_1, then X("A") can be derived in C_0 by reflection up only if $A \in \Gamma$.

Definition 30 (MAW$_\Gamma$) Let Γ be a set of formulas. An MC system $\mathrm{MS} = \langle \{C_0, C_1\}, BR \rangle$ is an MAW$_\Gamma$ system if the following conditions are satisfied:

- L_0 and L_1 are full propositional languages,
- Δ_0 includes the set of Classical Natural Deduction rules,
- Δ_1 includes

$$\frac{A_1, \ldots, A_k}{A}\ RPL_k$$

  with the restriction that $(A_1 \wedge \ldots \wedge A_k) \supset A$ be a tautology and $k \in \{0, 2\}$.
- $L_1 = \{A \mid \mathrm{B}(\text{"}A\text{"}) \in L_0\} = \{A \mid \mathrm{X}(\text{"}A\text{"}) \in L_0\}$,
- BR consists of the following bridge rules:

$$\frac{C_1 : A}{C_0 : \mathrm{B}(\text{"}A\text{"})}\ \mathrm{R}^{\mathrm{B}}_{up} \qquad \frac{C_0 : \mathrm{B}(\text{"}A\text{"})}{C_1 : A}\ \mathrm{R}^{\mathrm{B}}_{dn}$$

$$\frac{C_1 : A}{C_0 : \mathrm{X}(\text{"}A\text{"})}\ \mathrm{R}^{\mathrm{X}}_{up} \qquad \frac{C_0 : \mathrm{X}(\text{"}A\text{"})}{C_1 : A}\ \mathrm{R}^{\mathrm{X}}_{dn}$$

  and the restrictions for the applicability are:

  $\mathrm{R}^{\mathrm{B}}_{up}$: $C_1 : A$ does not depend on any assumption in C_1.
  $\mathrm{R}^{\mathrm{X}}_{up}$: $C_1 : A$ does not depend on any assumption in C_1 and $A \in \Gamma$.

Theorem 31 Let Γ be a set of propositional formulas. Let $\mathcal{AW}_\Gamma$ be the smallest $\mathcal{AW}_\Gamma$ system, and let MAW$_\Gamma$ be the smallest MAW$_\Gamma$ system. MAW$_\Gamma$ and $\mathcal{AW}_\Gamma$ are equivalent.

[8] As before, we do not allow for nesting of modal operators.

Notice that in a MAW_Γ system there is no explicit notion of awareness in the observer language. Instead, this notion is built in the structure of the system. Of course, analogously to what happens for modal $\mathcal{K}$ (and all the other modal logics), we can construct an MBK system equivalent to MAW_Γ simply by adding to the observer's set of axioms the (corresponding of the) awareness and explicit belief definition axioms.

6 Complex belief systems

So far we have concentrated on the basic configuration of one reasoner having beliefs about another reasoner. However, we may have more complex configurations with multiple reasoners organized in multiple belief systems possibly sharing one or both reasoners, or their belief predicate. For example, we may have an "isolated" reasoner (a reasoner which is neither being observed by another reasoner nor an observer of other reasoners) as well as a reasoner observing another reasoner which, in turn, is (possibly) observing other reasoners and so on. These arbitrary configurations of reasoners are formally described as *complex belief systems.*

Definition 32 (Complex belief system) Let I be a set of indexes. Let $\mathbf{B}$ be a tuple of binary relations over I. Then a *complex belief system* is a pair $\langle\{R_i\}_{i\in I},\, \mathbf{B}\rangle$, where $\{R_i\}_{i\in I}$ is a family of reasoners. The k-th element of $\mathbf{B}$ has an associated belief predicate B^k.

The k-th binary relation in $\mathbf{B}$ describes all the pairs of reasoners (belief systems) whose belief predicate is B^k. In the simplest case, a complex belief system consists of a set of belief systems, each two belief systems with distinct reasoners and distinct belief predicate. In this case, the cardinality n_i of I is twice the cardinality n_b of $\mathbf{B}$. As the simplest example of this situation, the complex belief system $\langle\{R_0, R_1\},\ \langle\{\langle 0,\ 1\rangle\}\rangle\rangle$ corresponds to the belief system $\langle R_0, R_1\rangle_{\text{B}}$. n_i may be also smaller than $2\times n_b$, in which case there is at least one reasoner which is the constituent of more than one belief system. Finally, n_i may be greater than $2\times n_b$, in which case either there are isolated reasoners or there is at least one belief predicate which is shared by more than one belief system.

We use MC systems to present complex belief systems generalizing in the obvious way the notion of MR^- system and of MR^- system presenting a belief system.

Definition 33 ($\mathbf{MR}_I^{\text{B}-}$) Let I be a set of indexes. Let $\mathbf{B}$ be a tuple of binary relations over I. An MC system $\langle\{C_i\}_{i\in I}, BR\rangle$ is an $\text{MR}_I^{\text{B}-}$ system if BR consists of the bridge rules

$$\frac{C_j : A}{C_i : \text{B}^k(\text{“}A\text{”})}\ \text{R}_{up}^{\text{B}^k} \qquad\qquad \frac{C_i : \text{B}^k(\text{“}A\text{”})}{C_j : A}\ \text{R}_{dn}^{\text{B}^k} \tag{3}$$

with $\langle i,\ j\rangle$ belonging to the k-th relation in $\mathbf{B}$, and the restrictions include that the premise of each bridge rule does not depend on assumptions in the context it belongs to.

Definition 34 (Complex Belief System presented by an $\mathbf{MR}_I^{\text{B}-}$ System) An $\text{MR}_I^{\text{B}-}$ system MS *presents* the complex belief system $\langle\{R_i\}_{i\in I},\ \mathbf{B}\rangle$ if $T_i = \{A \mid\ \vdash_{\text{MS}} C_i : A\}$ $(i \in I)$.

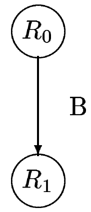

Figure 3: A belief system.

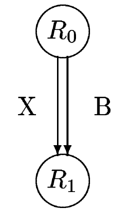

Figure 4: A complex belief system.

It is possible to represent the structure of a complex belief system as a direct graph, whose nodes represent reasoners and whose edges represent relations among reasoners. Each edge is labeled with the unary predicate corresponding to the relation. Thus, for example, a belief system $\langle\{R_0, R_1\},\ \langle\{\langle 0,\ 1\rangle\}\rangle\rangle$ —as, e.g., presented by an MR^- system (see Definition 5)— corresponds to Figure 3. The complex belief system $\langle\{R_0, R_1\},\ \langle \mathrm{B}, \mathrm{X}\rangle\rangle$ (where both B and X are the binary relation $\{\langle 0,\ 1\rangle\}$) —as, e.g., presented by an MAW_Γ system (see Definition 30)— corresponds to Figure 4.

Trivially, all the results presented in Sections 2, 3, 4 can be applied to each belief system (MR^- system) inside a complex belief system ($\mathrm{MR}_I^{\mathtt{B}-}$ system). An open question remains of which are the interesting structures that complex belief systems may have. If we had such a characterization, we could (try to) give notions of ideality and reality for complex belief systems and (try to) see how the results presented in Sections 2, 3, 4 can be generalized. However, as far as we know, such a characterization does not exist. We hope that our current research on the formal specification of multiagent systems will provide us with some ideas in this direction.

7 A comparison with modal systems – part II

We do not have any uniform way to translate MC systems presenting complex belief systems into equivalent modal systems. This does not seem possible in general, mainly because modal logics do not seem to have the modularity and flexibility provided by complex belief systems. Technically, a result proved in [5] shows that there are (complex) belief systems such that it is not possible to construct an equivalent modal system with a finite schematic set of axioms.

We can provide equivalence results with modal logics in the case of nested beliefs possibly in presence of multiple agents. This can be done by generalizing the equivalence theorems presented in Section 5. In our framework the situation of nested beliefs and multiple agents can be represented by a complex belief system with a tree structure like that in Figure 5. In Figure 5 there is an external observer, represented by R_ϵ, which observes n agents, each agent a_i represented by a reasoner R_i. Each reasoner R_i (with $i \in [1, n]$) can be taken as a model of a_i's beliefs as seen by the external observer. Analogously, each reasoner $R_{i_1 \ldots i_k}$ (with $i_1, \ldots, i_k$ in $[1, n]$) can be

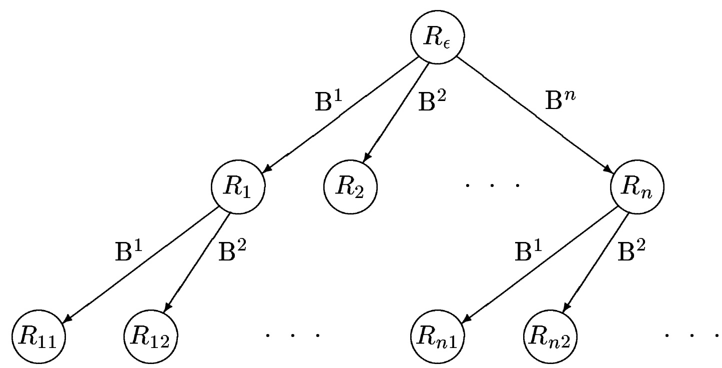

Figure 5: A tree of reasoners.

taken as a model of agent a_{i_k} as seen by agent $a_{i_{k-1}}$, as seen by ..., as seen by the external observer (see [13] for a longer discussion about this system).[9]

Example 35 Consider Example 1. Let us now suppose that we want to formalize the situation where there is only one agent, e.g. a reader of Example 1, who has his own views of the situation described in Figure 1. We have only one agent, the reader, that we formalize as an external observer R_ϵ having beliefs about Alice' and Humpty Dumpty's beliefs. The complex belief system formalizing this situation is

$$\langle\{R_\epsilon, R_A, R_H, R_{A,W}, R_{H,W}\},\ \langle\{\langle\epsilon,\ A\rangle\}, \{\langle\epsilon,\ H\rangle\}, \{\langle A,\ \ A, W\rangle, \langle H,\ \ H, W\rangle\}\rangle\rangle.$$

This complex belief system is depicted inside the box of Figure 6. The surrounding box represents the external observer.

Notice that introducing an external observer allows us to represent Alice and Humpty Dumpty's beliefs in a single theory. Thus, if

$$\mathrm{B}^A(\text{“}\mathrm{B}^W(\text{“}RQA\text{”})\text{”}) \vee \mathrm{B}^H(\text{“}\mathrm{B}^W(\text{“}RKA \wedge RQA\text{”})\text{”})$$

belongs to T_ϵ this intuitively means that the external observer believes that Alice believes that the White Knight believes that the Red Queen is Asleep, or that Humpty Dumpty believes that the White Knight believes that the Red King and the Red Queen are both asleep.

For the sake of simplicity, let us concentrate on the case of a single agent a corresponding, in modal logics, to the nesting of a single modal operator. The tree structure reduces to a chain of reasoners $\langle\{R_i\}_{0\leq i<n},\ \langle\{\langle i,\ i+1\rangle \mid 0 \leq i < n-1\}\rangle\rangle$ in which R_0 observes R_1, R_1 observes R_2, and so on. Assuming that this chain is finite and that the language of the bottom reasoner is propositional, we are able to model agents whose beliefs are nested up to the height of the chain.

The limit case of an infinite chain allows for the representation of agents with arbitrary level of nesting in their own beliefs. In order to generalize the equivalence

[9]The notion of external observer was first introduced by Konolige [16] and it is used here with the same intuitive meaning.

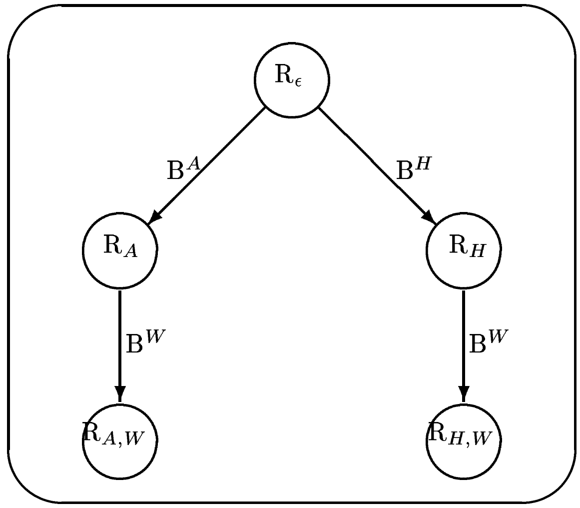

Figure 6: The complex belief system formalizing the external observer.

results in Section 5 to the nested case, we consider MC systems corresponding to an infinite chain of reasoners. More precisely, we consider MC systems $\langle\{C_i\}_{i\in\omega}, BR\rangle$, where BR contains for all $i \in \omega$, a pair of bridge rules like those given in (3) (page 21) with $j = i+1$. An MC system MS $= \langle\{C_i\}_{i\in\omega}, BR\rangle$ and a modal system $\Sigma = \langle L, T\rangle$ are said to be *equivalent* if for any formula A in L_0, Equation (1) holds.

Notationally from now on we will write MBK, $\text{MBK}_{p,q}$, $\text{MBK}'_{p,q}$, MAW_Γ, $\mathcal{K}$, $\mathcal{K}_{p,q}$, AW_Γ meaning the systems defined in the previous sections extended to allow for arbitrary nesting (these systems will be formally defined below). Furthermore, if MS $= \langle\{C_i\}_{i\in\omega}, BR\rangle$ is an MC system, then $\text{MS}^{\downarrow i}$ is the MC system $\langle\{C_i, C_{i+1}\}, BR^{\downarrow i}\rangle$, where $BR^{\downarrow i}$ is the set of bridge rules in BR of the form

$$\frac{C_{i+1}:A}{C_i:\text{B}(\text{“}A\text{”})}\ \text{R}^{\text{B}}_{up} \qquad \frac{C_i:\text{B}(\text{“}A\text{”})}{C_{i+1}:A}\ \text{R}^{\text{B}}_{dn}$$

Notice that $BR^{\downarrow i}$ is the set of the bridge rules given in (3) with $j = i+1$.

We are now ready to see how the results in Section 5 generalize. The following definition and theorem generalize Definition 25 and Theorem 26.

Definition 36 (MBK) An MC system $\langle\{C_i\}_{i\in\omega}, BR\rangle$ is an MBK system if for each $i \in \omega$, $\text{MBK}^{\downarrow i}$ satisfies the conditions in Definition 25.

Theorem 37 Let $\mathcal{K}$ be the smallest normal modal system and let MBK be the smallest MBK system. $\mathcal{K}$ and MBK are equivalent.

Even more, consider the smallest MBK system MS such that for each $i \in \omega$,

$$\frac{C_i:\text{B}(\text{“}A\text{”})}{C_{i+1}:\text{B}(\text{“}A\text{”})}\ 4^i_{br},$$

or

$$\frac{C_i : \neg \mathrm{B}(\text{“}A\text{”})}{C_{i+1} : \neg \mathrm{B}(\text{“}A\text{”})}\ 5^i_{br},$$

or both, belong to the set of bridge rules. Then, the positive introspective axiom or/and the negative introspective axiom become provable in any context C_i (e.g., for the positive introspective axiom, substitute C_i for C_0 and C_{i+1} for C_1 in the proof (2)). Indeed, MS turns out to be provably equivalent to the smallest normal modal system containing the positive introspective axiom, or the negative introspective axiom, or both of them; depending on the set of bridge rules that are added (see [12] for formal proofs of these facts).

The following definition generalizes Definition 27.

Definition 38 ($\mathbf{MBK}_{p,q}$) An MC system $\langle \{C_i\}_{i\in\omega}, BR\rangle$ is an $\mathrm{MBK}_{p,q}$ system if for each $i \in \omega$, $\mathrm{MBK}^{\downarrow i}_{p,q}$ satisfies the conditions in Definition 27.

Theorem 39 Let $\mathcal{K}_{p,q}$ be the smallest normal modal system and let $\mathrm{MBK}_{p,q}$ be the smallest $\mathrm{MBK}_{p,q}$ system. $\mathcal{K}_{p,q}$ and $\mathrm{MBK}_{p,q}$ are equivalent.

Notice that in Section 5 we have two MC systems equivalent to $\mathcal{K}_{p,q}$ in the case of no nested beliefs (see Definitions 27, 28 and Theorem 29). Indeed, the above theorem only partially generalizes Theorem 29. In fact, though the MC systems in Theorem 29 present the same observer, they have different properties and generalize in different ways. From Theorem 39, $\mathrm{MBK}_{p,q}$ turns out to be equivalent to $\mathcal{K}_{p,q}$. Consider instead $\mathrm{MBK}'_{p,q} = \langle \{C_i\}_{i\in\omega}, BR\rangle$, defined as the smallest MC system such that for each $i \in \omega$, $\mathrm{MBK}'^{\downarrow i}_{p,q}$ satisfies the conditions in Definition 28. $\mathrm{MBK}'_{p,q}$ is not equivalent to $\mathcal{K}_{p,q}$. This is because a belief system with an ideal reasoner ideally observing a (possibly) real reasoner (as in Definition 28) cannot be iterated without making each observed reasoner ideal. More precisely, for each $i \in \omega$, Δ_i has to be complete for propositional logic (since —considering $\mathrm{MBK}'^{\downarrow i}_{p,q}$ and Definition 28— Δ_i includes the set of Classical Natural Deduction Rules). As a consequence, $\mathrm{MBK}'_{p,q}$ is not equivalent to $\mathcal{K}_{p,q}$ unless $\{p, q\} = \{0, 2\}$.

Finally, the following definition and theorem generalize Definition 30 and Theorem 31 respectively.

Definition 40 ($\mathbf{MAW}_\Gamma$) Let Γ be a set of formulas. An MC system $\langle \{C_i\}_{i\in\omega}, BR\rangle$ is an MAW_Γ system if for each $i \in \omega$, $\mathrm{MAW}^{\downarrow i}_\Gamma$ satisfies the conditions in Definition 30.

Theorem 41 Let Γ be a set of propositional formulas. Let $\mathcal{AW}_\Gamma$ be the smallest $\mathcal{AW}_\Gamma$ system and let MAW_Γ be the smallest MAW_Γ system. $\mathcal{AW}_\Gamma$ and MAW_Γ are equivalent.

In Appendix D we hint the proofs of the theorems in this section.

8 Conclusions

In this paper we have provided a taxonomic analysis of ideal and real belief. As far as we know such an analysis has never been done before. The main novelties are:

- Our analysis is based on a set-theoretic (instead of model-theoretic) characterization of belief and belief about belief.
- We distinguish between the abilities to reason and observe. We introduce the notions of ideality and reality. In particular, for us reality does not mean not ideality. Ideality characterizes the presence of certain idealized properties; reality the fact that a real reasoner (observer) computes too little or too much with respect to another reasoner (observer). We have also characterized how each constructor in an MR^- system affects reality.
- We compare our notion of ideality with the notion of logical omniscience [15], equivalent to Konolige's notion of "saturation" [16]. We see that ideality is a more granular and weaker notion than the notion of logical omniscience. We show how it is possible to define multicontext systems presenting logically omniscient agent and capture also various forms of non logical omniscience presented in the literature.

Considering the other approaches, we see the following technical differences. The main difference with modal approaches is our use of multiple theories. We have argued that our approach allows for a modular and incremental specification of beliefs in a multiagent framework. As shown by the equivalence result in Theorem 29, various multicontext systems —each modeling a different situation— can correspond to a single modal logic. We believe that it is possible to define modal systems which capture such different situations. However, the question is on how natural and simple the resulting approach would be.

If we consider Konolige's deduction model of belief [16], our approach shares the intuitions. There are however some important technical differences. First, in his belief systems, the deduction rules associated to each theory must be sound (with respect to classical logic). This is not the case in our approach. For example, the "rule of the looking glass logician" $RLGL$ in Example 4 is not sound. Second, Konolige's "attachment rule" (relating what an agent believes to what is true in the views of the agent) is much stronger than the reflection principles that we use to ensure the correspondence between the observer and the observed reasoner. This for example is pointed out by the fact that Konolige's "saturated reasoners" do not necessarily correspond to ours reasoners with ideal reasoning and observing capabilities. Third, Konolige considers only configurations of reasoners corresponding to trees. Our framework is more flexible. For example, in MAW_Γ systems, different views are implemented with a single reasoners accessed in different ways. Furthermore, we can easily extend our framework to allow also for bridge rules modeling various relations among reasoners. For example, we can impose that the set of belief of one reasoner R_1 is a subset of the set of belief of another reasoner R_2 simply by adding the bridge rule

$$\frac{C_1 : A}{C_2 : A} \, .$$

This rule is for example useful if R_1 [R_2] represents the "explicit" ["implicit"] belief of an agent (see [13, 20]).

Finally, this work improves on [1, 3, 4, 12, 13] for the novelties introduced above. All these works are about the formalization of belief with MR systems, i.e., MR^-

systems in which the restriction on reflection down is dropped. In particular, [1] is about the formalization of opacity and transparency within the belief predicate. [3, 4] show how multicontext systems allows for "elaboration tolerant" formalizations of the Three Wise Men puzzle. [12] is the first paper on the subject: it defines the underlying intuitions and focuses on defining various MR systems provably equivalent to the most popular normal modal logics. [13] defines some MR systems formalizing various kind of non logical omniscience, and proves equivalence results with corresponding modal logics. In this paper, we introduce: the notions of reasoner, observer, belief system (Section 2); the notion of ideality (Section 3); the notion of reality, the taxonomy showing how multicontext systems can present reality (Section 4); the comparison with logical omniscience and also equivalence results with various modal systems modeling non logical omniscience (Section 5). Notice that in most of the paper we consider MR^- systems, and introduce MR systems only in Section 5. As we have seen, the apparently minor detail of dropping the restriction on reflection down has a dramatic impact on the properties of the presented reasoners. Finally, it is worth remarking that this work can be seen as providing the foundations to the work in formalizing belief with multicontext systems.

Acknowledgements

The authors would like to thank Massimo Benerecetti, Alessandro Cimatti and Luciano Serafini for many fruitful discussions on issues related to the subject of this paper. Some parts of this article appear also in a paper with the same title presented at the International Conference on Formal and Applied Practical Reasoning (FAPR), Bonn (Germany), 3-7 June 1996.

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A Derivability in a MC system

We follow Prawitz [22] in the notation and terminology.

Let $C = \langle L, \Omega, \Delta \rangle$ be a context (presented as an axiomatic formal system). We represent an inference rule $\iota \in \Delta$ as follows $(0 \leq n \leq m)$:

$$\frac{A_1 \dots A_n \quad \begin{matrix}[B_{n+1}]\\ A_{n+1}\end{matrix} \quad \dots \quad \begin{matrix}[B_m]\\ A_m\end{matrix}}{A}\ \iota. \tag{4}$$

with the understanding that $A_1 \dots A_n, A_{n+1} \dots A_m, A, B_{n+1} \dots B_m \in L$. (4) represents a rule ι discharging the assumptions $B_{n+1}, \dots, B_m$.

Let MS= $\langle \{C_i\}_{i \in I},\ BR \rangle$ be an MC system (where $\{C_i\}_{i \in I}$ is a family of contexts and BR is a set of bridge rules). We represent a bridge rule $\delta \in BR$ as follows $(0 \leq n \leq m)$:

$$\frac{C_1 : A_1 \dots C_n : A_n \quad \begin{matrix}[C'_{n+1} : B_{n+1}]\\ C_{n+1} : A_{n+1}\end{matrix} \quad \dots \quad \begin{matrix}[C'_m : B_m]\\ C_m : A_m\end{matrix}}{C : A}\ \delta. \tag{5}$$

(5) must be read similarly to (4). Notice that in (4) we drop C in $C : A$ leaving this information implicit. This trick allows us to use directly and without rephrasing the notation and terminology from [22]. This cannot be done for bridge rules (nor in the notion of deduction given below) as bridge rules (and deductions) involve formulas belonging to different contexts.

We now define what we mean by Π being a *deduction in* MS of a formula $C : A$ *depending on* a set of formulas Γ.

1. A formula $C_i : A$ is a deduction in MS of $C_i : A$ depending on (i) the empty set if $A \in \Omega_i$; (ii) $\{C_i : A\}$, otherwise.

2. If for each $i \in \{1, \ldots, m\}$ Π_i is a deduction in MS of $C_k : A_i$ depending on Γ_i and ι as represented in (4) belongs to Δ_k, then

$$\frac{\Pi_1 \ldots \Pi_m}{C_k : A}\ \iota$$

is a deduction of $C_k : A$ depending on

$$\bigcup_{1 \le i \le m} \Gamma_i \setminus \{C_k : B_{n+1}, \ldots, C_k : B_m\}.$$

3. If for each $i \in \{1, \ldots, m\}$ Π_i is a deduction in MS of $C_i : A_i$ depending on Γ_i and δ as represented in (5) belongs to BR, then

$$\frac{\Pi_1 \ldots \Pi_m}{C : A}\ \delta$$

is a deduction of $C : A$ depending on

$$\bigcup_{1 \le i \le m} \Gamma_i \setminus \{C'_{n+1} : B_{n+1}, \ldots, C'_m : B_m\}.$$

$C : A$ is *derivable from* a set of formulas Γ in MS ($\Gamma \vdash_{\text{MS}} C_i : A$) if there exists a deduction of $C_i : A$ depending on Γ' and $\Gamma' \subseteq \Gamma$. $C_i : A$ is a *theorem* in MS ($\vdash_{\text{MS}} C_i : A$) if it is derivable from the empty set.

B Proofs of the Theorems in Section 4

In this section

- C_i is an abbreviation for $C_{E_i} \oplus C_{I_i}$,
- MS is the MC system $\langle \{C_0, C_1\}, BR_E \cup BR_I \rangle$, and
- $\langle R_0, R_1 \rangle_{\text{B}}$ is the belief system presented by MS.

If MS′ and MS″ are two MR$^-$ systems, we also write MS′ $\preceq$ MS″, MS′ $\asymp$ MS″ and MS′ $\prec$ MS″ as abbreviations for

- MS′ is a correct realization of MS″,
- MS′ is equivalent to MS″ (or, MS′ $\preceq$ MS″ and MS″ $\preceq$ MS′),
- MS′ is a correct and incomplete realization of MS″ (or, MS′ $\preceq$ MS″ and MS′ $\not\asymp$ MS″),

respectively.

B.1 Proof of Theorem 16

Theorem 16 *Let MS_E be a correct realization of MS_I. Then*

- $\langle R_{E_0}, R_{E_1}\rangle_{\mathrm{B}}$ *is correct with respect to* $\langle R_{I_0}, R_{I_1}\rangle_{\mathrm{B}}$, *and*
- $\langle R_{E_0}, R_{E_1}\rangle_{\mathrm{B}}$ *is incomplete with respect to* $\langle R_{I_0}, R_{I_1}\rangle_{\mathrm{B}}$ *if and only if MS_E is an incomplete realization of MS_I.*

By hypothesis, we have that MS_E is a correct realization of MS_I, *i.e.*

$$\langle R_{I_0}, R_{I_1}\rangle_{\mathrm{B}} = \langle R_0, R_1\rangle_{\mathrm{B}}. \tag{6}$$

Considering the first item, we have to prove that

$$\langle R_{E_0}, R_{E_1}\rangle_{\mathrm{B}} \subseteq \langle R_{I_0}, R_{I_1}\rangle_{\mathrm{B}}. \tag{7}$$

Clearly, $\langle R_{E_0}, R_{E_1}\rangle_{\mathrm{B}} \subseteq \langle R_0, R_1\rangle_{\mathrm{B}}$. The thesis follows from equation (6).

For the second item, we prove that $\langle R_{E_0}, R_{E_1}\rangle_{\mathrm{B}}$ is complete with respect to $\langle R_{I_0}, R_{I_1}\rangle_{\mathrm{B}}$ if and only if MS_E is a complete realization of MS_I.

$$\begin{array}{rcl} \langle R_{E_0}, R_{E_1}\rangle_{\mathrm{B}} \text{ is complete with respect to } \langle R_{I_0}, R_{I_1}\rangle_{\mathrm{B}} & \Longleftrightarrow & \text{(Definition 12)} \\ \langle R_{I_0}, R_{I_1}\rangle_{\mathrm{B}} \subseteq \langle R_{E_0}, R_{E_1}\rangle_{\mathrm{B}} & \Longleftrightarrow & \text{(eq. (7))} \\ \langle R_{I_0}, R_{I_1}\rangle_{\mathrm{B}} = \langle R_{E_0}, R_{E_1}\rangle_{\mathrm{B}} & \Longleftrightarrow & \text{(eq. (6))} \\ \langle R_0, R_1\rangle_{\mathrm{B}} = \langle R_{E_0}, R_{E_1}\rangle_{\mathrm{B}} & \Longleftrightarrow & \text{(Definition 13)} \\ \mathrm{MS}_E \text{ is a complete realization of } \mathrm{MS}_I. & & \end{array}$$

B.2 Proof of Theorem 21

Theorem 21 *MS_E is an incomplete realization of MS_I if and only if at least one of the following three conditions is satisfied:*

- *C_{E_0} is an incomplete realization of C_{I_0};*
- *C_{E_1} is an incomplete realization of C_{I_1};*
- *BR_E is an incomplete realization of BR_I.*

We prove the two directions of the equivalence separately.

$\Longrightarrow$) We prove the counter-positive. By hypothesis, we have that

$$MS_E \asymp \langle \{C_{E_0} \oplus C_{I_0}, C_{E_1}\},\ BR_E\rangle, \tag{8}$$

and

$$MS_E \asymp \langle \{C_{E_0}, C_{E_1} \oplus C_{I_1}\},\ BR_E\rangle, \tag{9}$$

and

$$MS_E \asymp \langle \{C_{E_0}, C_{E_1}\},\ BR_E \cup BR_I\rangle. \tag{10}$$

We have to prove that $\langle R_{E_0}, R_{E_1}\rangle_{\mathrm{B}} = \langle R_0, R_1\rangle_{\mathrm{B}}$.

From equations (8) and (9) we have $L_{I_0} \subseteq L_{E_0}$ and $L_{I_1} \subseteq L_{E_1}$ respectively. Since $L_i = L_{I_i} \cup L_{E_i}$, we also have $L_i = L_{E_i}$.

We now show that $T_i = T_{E_i}$. Let Π be a proof of $C_i : A$ in MS. We show that $C_{E_i} : A$ is provable in MS_E, by induction on the number n of applications of bridge rules in Π.

$\underline{n = 0}$ If $i = 0$ the thesis follows from equation (8). If $i = 1$ the thesis follows from equation (9).

$\underline{n = m + 1}$ For simplicity, we assume $i = 1$. The case for $i = 0$ is analogous. The proof Π has the form

$$\dfrac{\dfrac{\dfrac{\Sigma_1}{C_0 : \mathrm{B}(\text{“}A_1\text{”})}}{C_1 : A_1}\,\mathrm{R}^{\mathrm{B}}_{dn} \quad \ldots \quad \dfrac{\dfrac{\Sigma_n}{C_0 : \mathrm{B}(\text{“}A_n\text{”})}}{C_1 : A_n}\,\mathrm{R}^{\mathrm{B}}_{dn}}{\dfrac{\Sigma}{C_1 : A}}$$

where $C_0 : \mathrm{B}(\text{“}A_1\text{”}), \ldots, C_0 : \mathrm{B}(\text{“}A_n\text{”})$ are the first occurrences of formulas in C_0 met on the thread from $C_1 : A$ to the leaves of Π.

Let $k \in \{1, \ldots, n\}$. Given the restrictions on the bridge rules, we have that

$$\dfrac{\Sigma_k}{C_0 : \mathrm{B}(\text{“}A_k\text{”})}$$

is a proof in MS of $C_0 : \mathrm{B}(\text{“}A_k\text{”})$. Hence — Σ_k contains less than n applications of bridge rules — $C_{E_0} : \mathrm{B}(\text{“}A_k\text{”})$ is provable in MS_E, and — from equation (10) — there exists a proof Π_k of $C_{E_1} : A_k$ in MS_E. From this it follows that

$$\dfrac{\Pi_1 \quad \ldots \quad \Pi_n}{\dfrac{\Sigma}{C_1 : A}}$$

is a proof of $C_1 : A$ in $\langle \{C_{E_0}, C_{E_1} \oplus C_{I_1}\},\ BR_E \rangle$. The thesis then follows from equation (9).

$\Longleftarrow$) We consider only the incompleteness of C_{E_0}. The other two cases are analogous. Suppose the first condition is satisfied. Hence:

$$MS_E \prec \langle \{C_{E_0} \oplus C_{I_0}, C_{E_1}\},\ BR_E \rangle$$

and then

$$MS_E \prec \langle \{C_{E_0} \oplus C_{I_0}, C_{E_1} \oplus C_{I_1}\},\ BR_E \cup BR_I \rangle.$$

B.3 Proof of Theorem 24

Theorem 24 *C_{E_i} is an incomplete realization of C_{I_i} $(i = 0, 1)$ if and only if at least one of the following four conditions is satisfied:*

- *P_{E_i} is an incomplete realization of P_{I_i};*
- *W_{E_i} is an incomplete realization of W_{I_i};*
- *Ω_{E_i} is an incomplete realization of Ω_{I_i};*
- *Δ_{E_i} is an incomplete realization of Δ_{I_i}.*

We consider the case $i = 0$ (the case $i = 1$ is analogous).

$\Longrightarrow$) Suppose none of the conditions is satisfied. From

$$MS_E \asymp \langle\{\langle Cl(P_{E_0} \cup P_{I_0}, W_{E_0}), \Omega_{E_0}, \Delta_{E_0}\rangle, C_{E_1}\},\ BR_E\rangle$$

we have $P_{I_0} \subseteq L_{E_0}$. Hence, from

$$MS_E \asymp \langle\{\langle Cl(P_{E_0}, W_{E_0} \cup W_{I_0}), \Omega_{E_0}, \Delta_{E_0}\rangle, C_{E_1}\},\ BR_E\rangle$$

we also have $L_{I_0} \subseteq L_{E_0}$. Finally, given

$$MS_E \asymp \langle\{\langle L_{E_0}, W_{E_0}, \Omega_{E_0} \cup \Omega_{I_0}, \Delta_{E_0}\rangle, C_{E_1}\},\ BR_E\rangle$$

and

$$MS_E \asymp \langle\{\langle L_{E_0}, W_{E_0}, \Omega_{E_0}, \Delta_{E_0} \cup \Delta_{I_0}\rangle, C_{E_1}\},\ BR_E\rangle$$

it is easy to show that for any proof in $\langle\{C_{E_0} \oplus C_{I_0}, C_{E_1}\},\ BR\rangle$, there exists a proof with the same conclusion in MS_E.

$\Longleftarrow$) We consider only the incompleteness of P_{E_0}. The other two cases are analogous. Suppose the first condition is satisfied. Hence:

$$MS_E \prec \langle\{\langle Cl(P_{E_0} \cup P_{I_0}, W_{E_0}), \Omega_{E_0}, \Delta_{E_0}\rangle, C_{E_1}\},\ BR_E\rangle$$

and then

$$MS_E \prec \langle\{C_{E_0} \oplus C_{I_0}, C_{E_1}\},\ BR_E\rangle.$$

C Proofs of the Theorems in Section 5

From here on, without loss of generality, we assume that the set of inference rules Δ_0 of the context C_0 be

$$\frac{A \quad A \supset B}{B} \supset\!E_0 \qquad \text{(modus ponens)},$$

$$\frac{\begin{array}{c}[A]\\ B\end{array}}{A \supset B} \supset\!I_0 \qquad \text{(imply introduction)},$$

$$\frac{\begin{array}{c}[\neg A]\\ \bot\end{array}}{A} \bot_0 \qquad \text{(reductio ad absurdum)}.$$

Furthermore, for any modal operator $\mathcal{B}$, by $\mathcal{B}^n A$ we mean $\mathcal{B}A$ if $n = 1$, A otherwise. Analogously, for any belief predicate B, by $\mathrm{B}^n(\text{“}A\text{”})$ we mean $\mathrm{B}(\text{“}A\text{”})$ if $n = 1$, A otherwise.

Consider either $\mathrm{MBK}_{p,q}$ or $\mathrm{MBK}'_{p,q}$ or MAW_Γ. It can be easily shown that any deduction Π of $C_0 : A$ depending on $C_{i_1} : A_1, \ldots, C_{i_n} : A_n$ is such that $i_1 = \ldots = i_n = 0$, (the bridge rules with conclusions in C_0 have the restriction that the premise does not depend on any assumption in C_1). In the following, we will make extensive and tacit use of this fact.

The following lemma will be useful later.

Lemma 42 Let Σ be either $\mathcal{K}_{p,q}$ or $\mathcal{AW}_\Gamma$. If A is a propositional formula and $\vdash_\Sigma A$ then A is a tautology.

All the theorems in Section 5 are about the equivalence between an MC system $\mathrm{MS} = \langle \{C_0, C_1\}, BR\rangle$ and a modal system $\Sigma = \langle L, T\rangle$. We recall that MS and Σ are *equivalent* if for any formula A in L_0,

$$\vdash_{\mathrm{MS}} C_0 : A \Longleftrightarrow \vdash_\Sigma A^+,$$

where A^+ is obtained replacing any monadic atomic formula $\mathrm{M}(\text{“}B\text{”})$ with $\mathcal{M}B$ in A.

For convenience, we define $(.)^*$ as the inverse function of $(.)^+$, i.e., such that if A is a formula in a given modal language, A^* is the expression obtained replacing any modal atomic formula $\mathcal{M}B$ with $\mathrm{M}(\text{“}B\text{”})$ in A. Then the above definition of equivalence can be restated as follows. An MC system $\mathrm{MS} = \langle \{C_0, C_1\}, BR\rangle$ and a modal system $\Sigma = \langle L, T\rangle$ are *equivalent* if for any formula A in L such that A^* belongs to L_0,

$$\vdash_{\mathrm{MS}} C_0 : A^* \Longleftrightarrow \vdash_\Sigma A.$$

C.1 Proof of Theorem 29

Theorem 29 *Let $\mathcal{K}_{p,q}$ be the smallest $\mathcal{K}_{p,q}$-classical system and let $MBK_{p,q}$, $MBK'_{p,q}$ be the smallest $MBK_{p,q}$ and $MBK'_{p,q}$ systems respectively. $\mathcal{K}_{p,q}$ is equivalent to $MBK_{p,q}$ and $MBK'_{p,q}$.*

If $\mathcal{K}_{p,q} = \langle L, T \rangle$, T is the smallest set closed under the inference rules

$$\frac{(A_1 \wedge \ldots \wedge A_k) \supset A_{k+1}}{(\mathcal{B}A_1 \wedge \ldots \wedge \mathcal{B}A_k) \supset \mathcal{B}A_{k+1}}\ RK_k$$

if $(A_1 \wedge \ldots \wedge A_k) \supset A_{k+1}$ is a propositional formula and $k \in \{p, q\}$, and

$$\frac{A_1 \ldots A_k}{A_{k+1}}\ RPL_k$$

if $(A_1 \wedge \ldots \wedge A_k) \supset A_{k+1}$ is a tautology and $k \in \{0, 2\}$.

C.1.1 From $\mathcal{K}_{p,q}$ to $\mathbf{MBK}_{p,q}$ and $\mathbf{MBK}'_{p,q}$

Lemma 43 If $\vdash_{\mathcal{K}_{p,q}} A$ then $\vdash_{\mathrm{MBK}_{p,q}} C_0 : A^*$ and $\vdash_{\mathrm{MBK}'_{p,q}} C_0 : A^*$.

Proof By induction on the structure of the proof Π of A.

Base case (RPL_0): A is a tautology (and hence also A^* is a tautology). Since Δ_0 is complete for propositional logic, A^* is provable in C_0.

Step case (RPL_2): If A is the consequence of an application of RPL_2 then the thesis follows from the induction hypothesis and the fact that Δ_0 is complete for tautological consequence.

Step case (RK_k ($k \in \{p, q\}$)): A has the form $(\mathcal{B}A_1 \wedge \ldots \wedge \mathcal{B}A_k) \supset \mathcal{B}A_{k+1}$ and is the conclusion of an application of RK_k. Then $(A_1 \wedge \ldots \wedge A_k) \supset A_{k+1}$ is a propositional formula provable in $\mathcal{K}_{p,q}$ and hence (Lemma 42) a tautology. From this, it follows that

$$\cfrac{\cfrac{C_0 : \mathrm{B}(\text{``}A_1\text{''})}{C_1 : A_1}\ \mathrm{R}^{\mathrm{B}}_{dn} \quad \ldots \quad \cfrac{C_0 : \mathrm{B}(\text{``}A_k\text{''})}{C_1 : A_k}\ \mathrm{R}^{\mathrm{B}}_{dn}}{\cfrac{C_1 : A_{k+1}}{C_0 : \mathrm{B}(\text{``}A_{k+1}\text{''})}\ \mathrm{R}^{\mathrm{B}}_{up}}\ RPL_k$$

is a deduction in $\mathrm{MBK}_{p,q}$ and $\mathrm{MBK}'_{p,q}$ of $C_0 : \mathrm{B}(\text{``}A_{k+1}\text{''})$ depending on $C_0 : \mathrm{B}(\text{``}A_1\text{''}), \ldots, C_0 : \mathrm{B}(\text{``}A_k\text{''})$. Hence $C_0 : (\mathrm{B}(\text{``}A_1\text{''}) \wedge \ldots \wedge \mathrm{B}(\text{``}A_k\text{''})) \supset \mathrm{B}(\text{``}A_{k+1}\text{''})$ is provable in $\mathrm{MBK}_{p,q}$ and $\mathrm{MBK}'_{p,q}$.

C.1.2 From $\mathbf{MBK}_{p,q}$ to $\mathcal{K}_{p,q}$

Consider a deduction Π in $\mathrm{MBK}_{p,q}$. If $assume(C_i : A, \Pi)$ is the total number of times the formula $C_i : A$ is assumed in Π, we say that Π is *stratified* if each occurrence in Π of the conclusion of RPL_2 depends on a set of formulas Γ such that

- $\Gamma \subseteq L_1$, or
- $\Gamma \subseteq L_0$ and $(\sum_{C_0 : A \in \Gamma} assume(C_0 : A, \Pi)) \leq max(p, q)$.

Lemma 44 If Π is a deduction of $C_i : A$ depending on $C_i : A_1, \ldots, C_i : A_n$ then Π is stratified.

Proof : Assume there exists a conclusion $C_1 : B$ of an application of RPL_2 not satisfying the two conditions. Hence, $C_1 : B$ depends on a formula $C_0 : D$. Even more, given the restriction on $\mathrm{R}^{\mathrm{B}}_{up}$, all the formulas in the thread from $C_i : A$ to $C_1 : B$ are the conclusion of an application of RPL_2 and thus

- belong to L_1, and
- depend on $C_0 : D$.

In particular, for the conclusion $C_i : A$ of the whole deduction, we have that $i = 1$ and that $C_i : A$ depends on $C_0 : D$, contradicting the hypothesis that all the formulas $C_i : A$ depends on belong to L_i.

Lemma 45 If Π is a stratified deduction of $C_i : A^*$ depending on the formula occurrences $C_{i_1} : A_1^*, \ldots, C_{i_n} : A_n^*$ and $r = min(i, i_1, \ldots, i_n)$ then

$$\mathcal{B}^{i_1 - r} A_1, \ldots, \mathcal{B}^{i_n - r} A_n \vdash_{\mathcal{K}_{p,q}} \mathcal{B}^{i-r} A.$$

Proof By induction on the structure of the deduction Π.

Base case (assumption): $C_i : A^*$ is an assumption. Trivially, $A \vdash_{\mathcal{K}_{p,q}} A$.

Base case (RPL_0): $C_1 : A$ is a tautology. A is provable in $\mathcal{K}_{p,q}$ by one application of RPL_0.

Step case (RPL_2): The deduction Π has the form $(0 \leq s \leq n)$

$$\cfrac{\cfrac{\begin{matrix} C_j : A_1^* \ldots C_j : A_s^* \\ \Sigma_1 \end{matrix}}{C_1 : B_1} \qquad \cfrac{\begin{matrix} C_j : A_{s+1}^* \ldots C_j : A_n^* \\ \Sigma_2 \end{matrix}}{C_1 : B_2}}{C_1 : A} \; RPL_2$$

If $j = 1$ or $n = 0$ the thesis trivially follows from the induction hypothesis.

If $j = 0$ and $0 < n \leq max(p, q)$, there are two cases:

- $0 < s < n = max(p, q) = 2$: by induction hypothesis we have $A_1 \vdash_{\mathcal{K}_{p,q}} \mathcal{B}B_1$ and $A_2 \vdash_{\mathcal{K}_{p,q}} \mathcal{B}B_2$. The thesis follows from the fact that $(B_1 \wedge B_2 \supset A)$ is a tautology and hence $\mathcal{B}B_1, \mathcal{B}B_2 \vdash_{\mathcal{K}_{p,q}} \mathcal{B}A$.
- $s = 0$ $[s = n]$: in this case

$$\cfrac{\Sigma_1}{C_1 : B_1} \qquad \left[\cfrac{\Sigma_2}{C_1 : B_2} \right]$$

is a proof of $C_1 : B_1$ $[C_1 : B_2]$. Hence

$$\cfrac{\cfrac{\Sigma_1}{C_1 : B_1} \quad C_1 : B_2}{C_1 : A} \; RPL_2 \qquad \left[\cfrac{C_1 : B_1 \quad \cfrac{\Sigma_2}{C_1 : B_2}}{C_1 : A} \; RPL_2 \right]$$

is a deduction of $C_1:A$ depending on $C_1:B_2$ $[C_1:B_1]$ with length strictly less than the length of the deduction Π. Hence, by induction hypothesis, we have $B_2 \vdash_{\mathcal{K}_{p,q}} A$ $[B_1 \vdash_{\mathcal{K}_{p,q}} A]$ and then (since $max(p,q) > 0$) $\mathcal{B}B_2 \vdash_{\mathcal{K}_{p,q}} \mathcal{B}A$ $[\mathcal{B}B_1 \vdash_{\mathcal{K}_{p,q}} \mathcal{B}A]$. The thesis trivially follows.

Step case ($\supset$E$_0$): The deduction Π has the form $(0 \leq s \leq n)$

$$\dfrac{\dfrac{\begin{array}{c}C_0:A_1^* \ldots C_0:A_s^*\\ \Sigma_1\end{array}}{C_0:B^*} \qquad \dfrac{\begin{array}{c}C_0:A_{s+1}^* \ldots C_0:A_n^*\\ \Sigma_2\end{array}}{C_0:B^* \supset A^*}}{C_0:A^*} \supset\mathrm{E}_0$$

The thesis trivially follows from the induction hypothesis.

Step case ($\supset$I$_0$ [$\bot_0$]): The deduction Π has the form

$$\dfrac{\dfrac{\begin{array}{c}C_0:A_1^* \ldots C_0:A_n^*\\ \Sigma\end{array}}{C_0:C^*}}{C_0:B^* \supset C^*} \supset\mathrm{I}_0 \qquad \left[\dfrac{\dfrac{\begin{array}{c}C_0:A_1^* \ldots C_0:A_n^*\\ \Sigma\end{array}}{C_0:\bot}}{C_0:A^*} \bot_0 \right]$$

Two cases:

- $C_0:C^*$ $[C_0:\bot]$ depends on $C_0:B^*$ $[C_0:\neg A^*]$: by induction hypothesis we have $A_1,\ldots,A_n,B \vdash_{\mathcal{K}_{p,q}} C$ $[A_1,\ldots,A_n,\neg A \vdash_{\mathcal{K}_{p,q}} \bot]$.
- $C_0 : C^*$ $[C_0 : \bot]$ does not depend on $C_0 : B^*$ $[C_0 : \neg A^*]$: by induction hypothesis we have $A_1,\ldots,A_n \vdash_{\mathcal{K}_{p,q}} C$ $[A_1,\ldots,A_n \vdash_{\mathcal{K}_{p,q}} \bot]$.

In both cases, the thesis $A_1,\ldots,A_n \vdash_{\mathcal{K}_{p,q}} B \supset C$ $[A_1,\ldots,A_n \vdash_{\mathcal{K}_{p,q}} A]$ trivially follows.

Step case ($\mathrm{R}_{up}^{\mathrm{B}}$): The deduction Π has the form

$$\dfrac{\dfrac{\begin{array}{c}C_0:A_1^* \ldots C_0:A_n^*\\ \Sigma\end{array}}{C_1:B}}{C_0:\mathrm{B}(\text{“}B\text{”})} \mathrm{R}_{up}^{\mathrm{B}}$$

(notice that in this case $n \in \{p,q\}$).

Two cases:

- $n = 0$: By induction hypothesis $\vdash_{\mathcal{K}_{p,q}} B$. Since $0 \in \{p,q\}$, we also have $\vdash_{\mathcal{K}_{p,q}} \mathcal{B}B$.
- $n > 0$: The thesis trivially follows from the induction hypothesis.

Step case ($\mathrm{R}^{\mathrm{B}}_{dn}$): The deduction Π has the form

$$\dfrac{\dfrac{\begin{array}{c}C_0 : A_1^* \ldots C_0 : A_n^* \\ \Sigma\end{array}}{C_0 : \mathrm{B}(\text{“}A\text{”})}}{C_1 : A}\ \mathrm{R}^{\mathrm{B}}_{dn}$$

Two cases:

- $n = 0$: By induction hypothesis, $\vdash_{\mathcal{K}_{p,q}} \mathcal{B}A$. From this it follows that $0 \in \{p, q\}$ and hence the thesis $\vdash_{\mathcal{K}_{p,q}} A$.
- $n > 0$: The thesis trivially follows from the induction hypothesis.

Lemma 46 If $\vdash_{\mathrm{MBK}_{p,q}} C_0 : A^*$ then $\vdash_{\mathcal{K}_{p,q}} A$.

Proof Trivial consequence of Lemma 44 and Lemma 45.

C.1.3 From $\mathbf{MBK}'_{p,q}$ to $\mathcal{K}_{p,q}$

Lemma 47 If Π is a deduction of $C_i : A^*$ depending on $C_{i_1} : A_1^*, \ldots, C_{i_n} : A_n^*$ then $\mathcal{B}^{i_1} A_1, \ldots, \mathcal{B}^{i_n} A_n \vdash_{\mathcal{K}_{p,q}} \mathcal{B}^i A$.

Proof The proof is by induction on the length of the deduction Π. All the cases almost trivially follow from the induction hypothesis. The only non trivial case is when we consider the inference rule RPL_k in Δ_1 ($k \in \{p, q\}$). In this case, the deduction Π has the form

$$\dfrac{\dfrac{\Sigma_1}{C_1 : A_1} \quad \ldots \quad \dfrac{\Sigma_k}{C_1 : A_k}}{C_1 : A_{k+1}}\ RPL_k$$

If each premise $C_1 : A_j$, with $1 \leq j \leq k$, depends on $\{C_{j_1} : A_{j_1}^*, \ldots, C_{j_n} : A_{j_n}^*\}$, then by induction hypothesis $\mathcal{B}^{j_1} A_{j_1}, \ldots, \mathcal{B}^{j_n} A_{j_n} \vdash_{\mathcal{K}_{p,q}} \mathcal{B}A_j$. The thesis follows from the fact that $(A_1 \wedge \ldots \wedge A_k) \supset A_{k+1}$ is a tautology and hence $\mathcal{B}A_1, \ldots, \mathcal{B}A_k \vdash_{\mathcal{K}_{p,q}} \mathcal{B}A_{k+1}$.

C.2 Proof of Theorem 31

Theorem 31 *Let Γ be a set of propositional formulas. Let $\mathcal{AW}_\Gamma$ be the smallest $\mathcal{AW}_\Gamma$ system, and let MAW_Γ be the smallest MAW_Γ system. MAW_Γ and $\mathcal{AW}_\Gamma$ are equivalent.*

Without loss of generality, we assume $\Gamma \subseteq L_{\mathcal{AW}}$.

C.2.1 From $\mathcal{AW}_\Gamma$ to $\mathbf{MAW}_\Gamma$

If $A \in L_{\mathcal{AW}}$, define $A^\dagger$ as follows:

- $A^\dagger = (B^\dagger \supset C^\dagger)$, if $A = (B \supset C)$;

- $A^\dagger = \mathrm{B}(\text{“}B^\dagger\text{”})$, if $A = \mathcal{B}B$;
- $A^\dagger = \mathrm{X}(\text{“}B^\dagger\text{”})$, if $A = \mathcal{X}B$;
- $A^\dagger = \begin{cases} \top, \text{ if } A = \mathcal{A}B \text{ and } B \in \Gamma, \\ \mathrm{X}(\text{“}B^\dagger\text{”}), \text{ if } A = \mathcal{A}B \text{ and } B \notin \Gamma; \end{cases}$
- $A^\dagger = A$, otherwise.

Notice that for any formula A not containing the modal operator $\mathcal{A}$, $A^\dagger = A^*$.

Lemma 48 If $\vdash_{\mathcal{AW}_\Gamma} A$ then $\vdash_{\mathrm{MAW}_\Gamma} C_0 : A^\dagger$.

Proof The proof is by induction on the structure of the proof of A.

Base case (RPL_0): A is a tautology (and hence also $A^\dagger$ is a tautology). Since Δ_0 is complete for propositional logic, $C_i : A^\dagger$ is provable.

Base case (awareness axiom): $A = \mathcal{A}B$ ($B \in \Gamma$). $A^\dagger = \top$, which is provable in C_0.

Base case (explicit belief definition): $A = \mathcal{X}B \leftrightarrow (\mathcal{A}B \wedge \mathcal{B}B)$. Two cases:

- If $B \notin \Gamma$, $A^\dagger = \mathrm{X}(\text{“}B\text{”}) \leftrightarrow (\mathrm{X}(\text{“}B\text{”}) \wedge \mathrm{B}(\text{“}B\text{”}))$. $(\mathrm{X}(\text{“}B\text{”}) \wedge \mathrm{B}(\text{“}B\text{”})) \supset \mathrm{X}(\text{“}B\text{”})$ is a tautology and hence provable in C_0. $\mathrm{X}(\text{“}B\text{”}) \supset (\mathrm{X}(\text{“}B\text{”}) \wedge \mathrm{B}(\text{“}B\text{”}))$ is logically equivalent to $\mathrm{X}(\text{“}B\text{”}) \supset \mathrm{B}(\text{“}B\text{”})$, which is provable in C_0 as a consequence of the following deduction:

$$\dfrac{\dfrac{C_0 : \mathrm{X}(\text{“}B\text{”})}{C_1 : B}\,\mathrm{R}^{\mathrm{X}}_{dn}}{C_0 : \mathrm{B}(\text{“}B\text{”})}\,\mathrm{R}^{\mathrm{B}}_{up}$$

- If $B \in \Gamma$, $A^\dagger = \mathrm{X}(\text{“}B\text{”}) \leftrightarrow (\top \wedge \mathrm{B}(\text{“}B\text{”}))$, which is logically equivalent to $\mathrm{X}(\text{“}B\text{”}) \leftrightarrow \mathrm{B}(\text{“}B\text{”})$. We have already established that $\mathrm{X}(\text{“}B\text{”}) \supset \mathrm{B}(\text{“}B\text{”})$ is provable in C_0. $\mathrm{B}(\text{“}B\text{”}) \supset \mathrm{X}(\text{“}B\text{”})$ is also provable in C_0 as a consequence of the following deduction:

$$\dfrac{\dfrac{C_0 : \mathrm{B}(\text{“}B\text{”})}{C_1 : B}\,\mathrm{R}^{\mathrm{B}}_{dn}}{C_0 : \mathrm{X}(\text{“}B\text{”})}\,\mathrm{R}^{\mathrm{X}}_{up}$$

Step case (RPL_2): If A is the consequence of an application of RPL_2, then the thesis follows from the induction hypothesis and the fact that Δ_0 is complete for tautological consequence.

Step cases (RK_k ($k \in \{0, 2\}$)): A has the form $((\mathcal{B}A_1 \wedge \ldots \wedge \mathcal{B}A_k) \supset \mathcal{B}A_{k+1})$. By hypothesis, the propositional formula $(A_1 \wedge \ldots \wedge A_k) \supset A_{k+1}$ is provable in $\mathcal{AW}_\Gamma$ and hence (Lemma 42) is a tautology. $A^\dagger = ((\mathrm{B}(\text{“}A_1\text{”}) \wedge \ldots \wedge \mathrm{B}(\text{“}A_k\text{”})) \supset \mathrm{B}(\text{“}A_{k+1}\text{”}))$ is provable in C_0 as a trivial consequence of the following deduction:

$$\dfrac{\dfrac{\dfrac{C_0 : \mathrm{B}(\text{“}A_1\text{”})}{C_1 : A_1}\,\mathrm{R}^{\mathrm{B}}_{dn} \quad \ldots \quad \dfrac{C_0 : \mathrm{B}(\text{“}A_k\text{”})}{C_1 : A_k}\,\mathrm{R}^{\mathrm{B}}_{dn}}{C_1 : A_{k+1}}\,RPL_k}{C_0 : \mathrm{B}(\text{“}A_{k+1}\text{”})}\,\mathrm{R}^{\mathrm{B}}_{up}$$

Lemma 49 Let A a formula not containing the awareness operator. If $\vdash_{\mathcal{AW}_\Gamma} A$ then $\vdash_{\mathrm{MAW}_\Gamma} C_0 : A^*$.

Proof By Lemma 48, if $\vdash_{\mathcal{AW}_\Gamma} A$ then $\vdash_{\mathrm{MAW}_\Gamma} C_0 : A^\dagger$. Since A does not contain the awareness operator $\mathcal{A}$, $A^\dagger = A^*$.

C.2.2 From MAW$_\Gamma$ to $\mathcal{AW}_\Gamma$

Lemma 50 If Π is a deduction of $C_i : A^*$ depending on $C_{i_1} : A_1^*, \ldots, C_{i_n} : A_n^*$ then $\mathcal{B}^{i_1} A_1, \ldots, \mathcal{B}^{i_n} A_n \vdash_{\mathcal{AW}_\Gamma} \mathcal{B}^i A$.

Proof We prove the Lemma by induction on the structure of Π.

Base case (assumption): $C_i : A^*$ is an assumption. Trivially $\mathcal{B}^i A \vdash_{\mathcal{AW}_\Gamma} \mathcal{B}^i A$.

Base case (RPL_0): $C_1 : A$ is a tautology. Trivially $\vdash_{\mathcal{AW}_\Gamma} A$ and hence (by RK_0) $\vdash_{\mathcal{AW}_\Gamma} \mathcal{B}A$.

Step case (RPL_2): The deduction Π has the form ($0 \le j \le m \le n$)

$$\dfrac{\dfrac{\begin{array}{c} C_{i_1} : A_1^* \ldots C_{i_m} : A_m^* \\ \Sigma_1 \end{array}}{C_1 : B_1} \qquad \dfrac{\begin{array}{c} C_{i_{j+1}} : A_{j+1}^* \ldots C_{i_n} : A_n^* \\ \Sigma_2 \end{array}}{C_1 : B_2}}{C_1 : A} \; RPL_2$$

By induction hypothesis, we have both

$$\mathcal{B}^{i_1} A_1, \ldots, \mathcal{B}^{i_m} A_m \vdash_{\mathcal{AW}_\Gamma} \mathcal{B}B_1$$

and

$$\mathcal{B}^{i_{j+1}} A_{j+1}, \ldots, \mathcal{B}^{i_n} A_n \vdash_{\mathcal{AW}_\Gamma} \mathcal{B}B_2.$$

The thesis follows from the fact $\mathcal{B}B_1, \mathcal{B}B_2 \vdash_{\mathcal{AW}_\Gamma} \mathcal{B}A$.

Step cases ($\supset E_0, \supset I_0, \bot_0$): The thesis easily follows from the induction hypothesis.

Step cases ($\mathrm{R}^{\mathrm{B}}_{up}, \mathrm{R}^{\mathrm{B}}_{dn}, \mathrm{R}^{\mathrm{X}}_{up}, \mathrm{R}^{\mathrm{X}}_{dn}$): Let $\rho \in \{\mathrm{R}^{\mathrm{B}}_{up}, \mathrm{R}^{\mathrm{B}}_{dn}, \mathrm{R}^{\mathrm{X}}_{up}, \mathrm{R}^{\mathrm{X}}_{dn}\}$. The deduction Π has the form

$$\dfrac{\dfrac{\begin{array}{c} C_{i_1} : A_1^*, \ldots, C_{i_n} : A_n^* \\ \Sigma \end{array}}{C_j : B^*}}{C_i : A^*} \; \rho$$

By induction hypothesis we have $\mathcal{B}^{i_1} A_1, \ldots, \mathcal{B}^{i_n} A_n \vdash_{\mathcal{AW}_\Gamma} \mathcal{B}^j B$.

Various cases:

- $\rho = \mathrm{R}^{\mathrm{B}}_{up}$: $A^* = \mathrm{B}(\text{“}B\text{”})$, $j = i + 1$. The thesis is straightforward noticing that $\mathcal{B}^j B = \mathcal{B}^{i+1} B = \mathcal{B}^i \mathcal{B} B = \mathcal{B}^i A$.
- $\rho = \mathrm{R}^{\mathrm{B}}_{dn}$: $B^* = \mathrm{B}(\text{“}A\text{”})$, $i = j + 1$. The thesis is straightforward noticing that $\mathcal{B}^j B = \mathcal{B}^{i-1} B = \mathcal{B}^{i-1} \mathcal{B} A = \mathcal{B}^i A$.

- $\rho = \mathrm{R}^{\mathrm{X}}_{up}$: $A^* = \mathrm{X}(\text{“}B\text{”})$, $j = i + 1$. $\mathcal{B}^j B = \mathcal{B}B$ and $\mathcal{B}^i A = \mathcal{X}B$. By construction, $\mathrm{R}^{\mathrm{X}}_{up}$ is applicable only if $B \in \Gamma$. Hence $\vdash_{\mathcal{AW}_\Gamma} \mathcal{A}B$. Since $\vdash_{\mathcal{AW}_\Gamma} (\mathcal{A}B \wedge \mathcal{B}B) \supset \mathcal{X}B$, the thesis follows.
- $\rho = \mathrm{R}^{\mathrm{X}}_{dn}$: $B^* = \mathrm{X}(\text{“}A^*\text{”})$, $i = j + 1$. $\mathcal{B}^j B = \mathcal{X}A$ and $\mathcal{B}^i A = \mathcal{B}A$. But $\vdash_{\mathcal{AW}_\Gamma} \mathcal{X}A \supset \mathcal{B}A$, whence the thesis.

Lemma 51 If $\vdash_{\mathrm{MAW}_\Gamma} C_0 : A^*$ then $\vdash_{\mathcal{AW}_\Gamma} A$.

Proof Trivial consequence of Lemma 50.

D Proofs of the Theorems in Section 7

Without loss of generality, we assume that the set of inference rules of each context C_i be $\{\supset\mathrm{E}_i, \supset\mathrm{I}_i, \bot_i\}$ (we index the name of the inference rule with the index of the corresponding context), each defined analogously to the corresponding inference rule given at the beginning of Appendix C.

For the proof of Theorem 37 see [12].

D.1 Proof of Theorem 39

Theorem 39 *Let $\mathcal{K}_{p,q}$ be the smallest normal modal system and let $MBK_{p,q}$ be the smallest $MBK_{p,q}$ system. $\mathcal{K}_{p,q}$ and $MBK_{p,q}$ are equivalent.*

The direction from $\mathcal{K}_{p,q}$ to $\mathrm{MBK}_{p,q}$ can be easily established as a consequence of the following lemma (partially generalizing Lemma 43).

Lemma 52 If $\vdash_{\mathcal{K}_{p,q}} A$ then $\vdash_{\mathrm{MBK}_{p,q}} C_i : A^*$, $(i \in \omega)$.

Proof The proof of the above lemma is analogous to the proof of Lemma 43. The only different step is when we consider an application of RK_k. In this case, by induction hypothesis we have that for each $j \in \omega$

$$\vdash_{\mathrm{MBK}_{p,q}} C_j : (A_1^* \wedge \ldots \wedge A_k^*) \supset A_{k+1}^*.$$

In particular, the above equation holds for $j = i + 1$. The thesis trivially follows from the following deduction.

$$\dfrac{\dfrac{C_i : \mathrm{B}(\text{“}A_1^*\text{”})}{C_{i+1} : A_1^*}\,\mathrm{R}^{\mathrm{B}}_{dn} \quad \ldots \quad \dfrac{C_i : \mathrm{B}(\text{“}A_k^*\text{”})}{C_{i+1} : A_k^*}\,\mathrm{R}^{\mathrm{B}}_{dn} \quad C_{i+1} : (A_1^* \wedge \ldots \wedge A_k^*) \supset A_{k+1}^*}{\dfrac{C_{i+1} : A_{k+1}^*}{C_i : \mathrm{B}(\text{“}A_{k+1}^*\text{”})}\,\mathrm{R}^{\mathrm{B}}_{up}}\,\supset\mathrm{E}_{i+1}$$

For the direction from $\mathrm{MBK}_{p,q}$ to $\mathcal{K}_{p,q}$ it is important Lemma 53 —similar to Lemma 4.2 in [12]— given below.

We generalize the notion of “stratified deduction” given in Appendix C. We now say that a deduction Π is *stratified* if

- each occurrence in Π of the conclusion of either $\supset\mathrm{I}_i$ or $\bot_i$ depends on formulas belonging to L_i, and

- each occurrence in Π of the conclusion of $\supset\mathrm{E}_i$ depends on a set of formulas Γ such that
  - $\Gamma \subseteq L_i$, or
  - $\Gamma \subseteq L_{i-1}$ and $\sum_{C_{i-1}:A \in \Gamma} \mathit{assume}(C_{i-1}:A, \Pi) \leq \mathit{max}(p,q)$.

Lemma 53 If $C_i:A_1, \ldots, C_i:A_n \vdash_{\mathrm{MBK}_{p,q}} C_i:A$ then there is a stratified deduction in $\mathrm{MBK}_{p,q}$ of $C_i:A$ from $C_i:A_1, \ldots, C_i:A_n$.

Proof The proof is in two steps. We first show that for any deduction Π of $C_i:A$ from a set Γ of formulas, there is a deduction Π' of $C_i:A$ from Γ such that the conclusion $C_k:B$ of an $\supset\mathrm{I}_k$ or $\bot_k$ depends on $\Gamma' \subseteq L_k$. For such a Π', we then show that if Π' is a deduction of $C_i:A$ from $C_i:A_1, \ldots, C_i:A_n$ then Π' is stratified. The proof of the first step is an easy generalization of Lemma 4.2 in [12].

If the conclusion $C_k:B$ of an $\supset\mathrm{I}_k$ depends on $C_j:C$ with $k > j$, let $C_{k-1}:\mathrm{B}(\text{“}D\text{”})$ be the first occurrence at level $k-1$ met on the thread from $C_k:B$ to (an occurrence of the assumption) $C_j:C$. The sub-deduction of $C_{k-1}:\mathrm{B}(\text{“}D\text{”})$, containing $C_j:C$, is moved aside in the following way (the same argument applies to $\bot_k$ as well):

$$\begin{array}{c} \Pi_1 \\ \dfrac{C_{k-1}:\mathrm{B}(\text{“}D\text{”})}{C_k:D} \\ \Pi_2 \\ \dfrac{C_k:B}{C_k:E \supset B} \\ \Pi_3 \end{array} \quad \Longrightarrow \quad \begin{array}{c} \dfrac{\dfrac{\begin{array}{c} C_k:D \\ \Pi_2 \\ \dfrac{C_k:B}{C_k:E \supset B} \end{array}}{C_k:D \supset (E \supset B)} \quad \dfrac{\begin{array}{c}\Pi_1 \\ C_{k-1}:\mathrm{B}(\text{“}D\text{”})\end{array}}{C_k:D}}{C_k:E \supset B} \\ \Pi_3 \end{array} \tag{11}$$

$$\begin{array}{c} \Pi_1 \\ \dfrac{C_{k-1}:\mathrm{B}(\text{“}D\text{”})}{C_k:D} \\ \Pi_2 \\ \dfrac{C_k:\bot}{C_k:B} \\ \Pi_3 \end{array} \quad \Longrightarrow \quad \begin{array}{c} \dfrac{\dfrac{\begin{array}{c} C_k:D \\ \Pi_2 \\ \dfrac{C_k:\bot}{C_k:B} \end{array}}{C_k:D \supset B} \quad \dfrac{\begin{array}{c}\Pi_1 \\ C_{k-1}:\mathrm{B}(\text{“}D\text{”})\end{array}}{C_k:D}}{C_k:B} \\ \Pi_3 \end{array} \tag{12}$$

Note that the result of the transformation is still an $\mathrm{MBK}_{p,q}$ deduction. In fact no application of $\mathrm{R}^{\mathrm{B}}_{up}$ or $\mathrm{R}^{\mathrm{B}}_{dn}$ is performed in Π_2 from $C_k:D$ to $C_k:B$ (or $C_k:\bot$) which means that $C_k:D$ can be an assumption of Π_2. Furthermore the assumptions and the conclusion of the starting deduction and the target deduction are the same.

The second part of the proof is analogous to the proof of Lemma 44.

The thesis is an easy consequence of Lemma 53 and of the following lemma.

Lemma 54 If Π is a stratified deduction of $C_i:A^*$ depending on the formula occurrences $C_{i_1}:A_1^*, \ldots, C_{i_n}:A_n^*$ and $r = \mathit{min}(i, i_1, \ldots, i_n)$ then

$$\mathcal{B}^{i_1-r}A_1, \ldots, \mathcal{B}^{i_n-r}A_n \vdash_{\mathrm{K}_{p,q}} \mathcal{B}^{i-r}A.$$

Proof The proof is analogous to the proof of Lemma 45 (with the step $\supset\mathrm{E}_i$ corresponding to the step RPL_2 in the proof of Lemma 45).

D.2 Proof of Theorem 41

Theorem 41 *Let Γ be a set of propositional formulas. Let $\mathcal{AW}_\Gamma$ be the smallest $\mathcal{AW}_\Gamma$ system and let MAW_Γ be the smallest MAW_Γ system. $\mathcal{AW}_\Gamma$ and MAW_Γ are equivalent.*

The above theorem is an easy consequence of the following lemmas.

Lemma 55 If $\vdash_{\mathcal{AW}_\Gamma} A$ then $\vdash_{\mathrm{MAW}_\Gamma} C_i : A^\dagger$, $(i \in \omega)$.

Proof The proof of the above lemma is analogous to the proof of Lemma 48. The only different step is when we consider an application of RK_k. In this case, we can reason similarly to what done in the proof of Lemma 52.

Lemma 56 Let A a formula not containing the awareness operator. If $\vdash_{\mathcal{AW}_\Gamma} A$ then $\vdash_{\mathrm{MAW}_\Gamma} C_0 : A^*$.

Proof The proof is analogous to the proof of Lemma 49.

Lemma 57 If Π is a deduction of $C_i : A^*$ depending on $C_{i_1} : A_1^*, \ldots, C_{i_n} : A_n^*$ then $\mathcal{B}^{i_1} A_1, \ldots, \mathcal{B}^{i_n} A_n \vdash_{\mathcal{AW}_\Gamma} \mathcal{B}^i A$.

Proof First, we assume that each occurrence in Π of the conclusion of either $\supset\mathrm{I}_i$ or $\bot_i$ depends on formulas belonging to L_i. This does not cause any loss of generality because we can "move aside" deductions as in Equations 11 and 12 also in MAW_Γ. Then, the proof is analogous to the proof for the not nested case (with $\supset\mathrm{E}_i$ corresponding to the step RPL_2).