WELL-POSEDNESS OF THE CAUCHY PROBLEM FOR p-EVOLUTION SYSTEMS OF PSEUDO-DIFFERENTIAL OPERATORS

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ABSTRACT. We study $p-$evolution pseudo-differential systems of the first order with coefficients in (t,x) and real characteristics. We find sufficient conditions for the well-posedness of the Cauchy problem in H^∞. These conditions involve the behavior as $x\to\infty$ of the coefficients, requiring some decay estimates to be satisfied.

1. Introduction and main results

We consider, in $[0,T]\times\mathbb{R}$, systems of pseudo-differential operators of the form

$$L = D_t + \begin{pmatrix} \mu_1(t,x,D_x) & & \\ & \ddots & \\ & & \mu_m(t,x,D_x) \end{pmatrix} + R(t,x,D_x), \tag{1.1}$$

where D_t stands for $D_t\cdot I$, $\mu_k(t,x,D_x)$, for $1\le k\le m$, are pseudo-differential operators with symbol in $C([0,T];S^p)$, for a given $p\ge 2$, and $R(t,x,D_x)$ is a matrix of pseudo-differential operators with symbol in $C([0,T];S^0)$. Here $D=\frac{1}{i}\partial$, and S^m is the classical class of symbols $a(x,\xi)$ defined by

$$|\partial_\xi^\alpha D_x^\beta a(x,\xi)|\le C_{\alpha,\beta,h}\langle\xi\rangle_h^{m-\alpha} \qquad \forall\alpha,\beta\in\mathbb{N},\ h\ge 1,$$

for some $C_{\alpha,\beta,h}>0$, with $\langle\xi\rangle_h:=\sqrt{h^2+\xi^2}$.

System (1.1) will be called a p-evolution system of the first order. We shall assume, in the following, that

$$\mu_k(t,x,D_x)=\mu_k^{(p)}(t,D_x)+\sum_{j=1}^{p-1}\mu_k^{(j)}(t,x,D_x) \tag{1.2}$$

with symbols $\mu_k^{(j)}\in C([0,T];S^j)$ for all $1\le k\le m$ and $1\le j\le p$.

According to the necessary condition of the Lax-Mizohata theorem for well-posedness of the Cauchy problem for scalar differential equations in Sobolev spaces, we assume that

$$\mu_k^{(p)}(t,\xi)\in\mathbb{R} \qquad \forall(t,\xi)\in[0,T]\times\mathbb{R},\ 1\le k\le m, \tag{1.3}$$

while $\mu_k^{(j)}(t,x,\xi)\in\mathbb{C}$ for $1\le j\le p-1$ and $1\le k\le m$.

When all the coefficients $\mu_k^{(j)}$ (and not only $\mu_k^{(p)}$) are real, well-posedness results for $p\ge 2$-evolution equations are known (cf., for instance, [A1], [A2], [AZ], [AC]). In the case of complex coefficients, some unavoidable decay conditions in x are needed, as shown by [I1]; this leads us to conditions (1.5)-(1.7) below. Well posedness of first order p-evolution differential equations with complex coefficients has been studied, for instance, in [I2] and [KB] for the case $p=2$, [CC] for $p=3$, [ABZ] for $p\ge 4$. Second order equations with complex coefficients have

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been considered, for example, in [CC], [ACC], [CR], for $p = 2, 3$. Higher order equations with complex coefficients have been studied, for instance, in [T] for $p = 2$ and will be studied in the forthcoming paper [AB] for $p \geq 4$.

In this paper we focus on $p \geq 2$-evolution pseudo-differential systems of the first order. The main result of this paper, Theorem 1.1, will be crucial in [AB].

We thus consider the operator (1.1)-(1.3) and assume that

$$\partial_\xi \mu_k^{(p)}(t,\xi) \geq C_p \langle \xi \rangle_h^{p-1} \qquad \forall (t,\xi) \in [0,T] \times \mathbb{R},\ 1 \leq k \leq m \tag{1.4}$$

for some $C_p > 0$, and moreover that for all $(t,x,\xi) \in [0,T] \times \mathbb{R}^2$, $1 \leq k \leq m$ and $\alpha \in \mathbb{N}$:

$$|\operatorname{Im} \partial_\xi^\alpha \mu_k^{(j)}(t,x,\xi)| \leq C_\alpha \langle x \rangle^{-\frac{j}{p-1}} \langle \xi \rangle_h^{j-\alpha},\ j = 1, \dots, p-1 \tag{1.5}$$

$$|\operatorname{Im} \partial_\xi^\alpha D_x \mu_k^{(j)}(t,x,\xi)| \leq C_\alpha \langle x \rangle^{-\frac{j-1}{p-1}} \langle \xi \rangle_h^{j-\alpha},\ j = 2, \dots, p-1 \tag{1.6}$$

$$|\operatorname{Im} \partial_\xi^\alpha D_x^\beta \mu_k^{(j)}(t,x,\xi)| \leq C_\alpha \langle x \rangle^{-\frac{j-[\beta/2]}{p-1}} \langle \xi \rangle_h^{j-\alpha},\ \left[\frac{\beta}{2}\right] = 1, \dots, j-1,\ j = 3, \dots, p-1 \tag{1.7}$$

for some $C_\alpha > 0$, where $[\beta/2]$ denotes the integer part of $\beta/2$ and $\langle \cdot \rangle := \langle \cdot \rangle_1$.

Under the above assumptions, we prove the following

Theorem 1.1. *Let L be a system of the form (1.1) satisfying (1.2)-(1.7). Then there exists a constant $\sigma > 0$ such that for every $U \in C([0,T]; H^{s+p}) \cap C^1([0,T]; H^s)$ the following estimate holds:*

$$|||U(t,\cdot)|||_{s-\sigma}^2 \leq C_s \Big(|||U(0,\cdot)|||_s^2 + \int_0^t |||LU(\tau,\cdot)|||_s^2 d\tau \Big), \qquad \forall t \in [0,T], \tag{1.8}$$

for some $C_s > 0$, where for a given vector $V = (V_1, \cdots, V_m)$ we denote $|||V|||_s^2 := \sum_{j=1}^m \|V_j\|_s^2$.

The energy estimate (1.8) leads to H^∞ well-posedness of the Cauchy problem

$$\begin{cases} LU(t,x) = F(t,x) & (t,x) \in [0,T] \times \mathbb{R} \\ U(0,x) = G(x) & x \in \mathbb{R} \end{cases} \tag{1.9}$$

with loss of σ derivatives.

In order to prove Theorem 1.1 we have to consider first the scalar case, for a pseudo-differential operator P of the form

$$P(t,x,D_t,D_x) = D_t + a_p(t,D_x) + \sum_{j=0}^{p-1} a_j(t,x,D_x) \tag{1.10}$$

with $a_j \in C([0,T]; S^j)$, $0 \leq j \leq p$,

$$a_p(t,\xi) \in \mathbb{R} \quad \forall (t,\xi) \in [0,T] \times \mathbb{R} \tag{1.11}$$

and $a_j(t,x,\xi) \in \mathbb{C}\ \forall (t,x,\xi) \in [0,T] \times \mathbb{R}^2,\ 0 \leq j \leq p-1$. For the scalar operator (1.10) we prove the following:

Theorem 1.2. *Let us consider an operator of the form (1.10) satisfying (1.11) and*

$$\partial_\xi a_p(t,\xi) \geq C_p \langle \xi \rangle_h^{p-1} \tag{1.12}$$

for some $C_p > 0$. *Assume that*

$$|\operatorname{Im}\partial_\xi^\alpha a_j(t,x,\xi)| \leq C_\alpha \langle x\rangle^{-\frac{j}{p-1}}\langle\xi\rangle_h^{j-\alpha},\ 1\leq j\leq p-1 \tag{1.13}$$

$$|\operatorname{Im}\partial_\xi^\alpha D_x a_j(t,x,\xi)| \leq C_\alpha \langle x\rangle^{-\frac{j-1}{p-1}}\langle\xi\rangle_h^{j-\alpha},\ 2\leq j\leq p-1 \tag{1.14}$$

$$|\operatorname{Im}\partial_\xi^\alpha D_x^\beta a_j(t,x,\xi)| \leq C_\alpha \langle x\rangle^{-\frac{j-[\beta/2]}{p-1}}\langle\xi\rangle_h^{j-\alpha},\ 1\leq \left[\frac{\beta}{2}\right]\leq j-1,\ 3\leq j\leq p-1 \tag{1.15}$$

for all $(t,x,\xi)\in[0,T]\times\mathbb{R}^2$ *and for some* $C_\alpha > 0$.

Then, the Cauchy problem

$$\begin{cases} P(t,x,D_t,D_x)u(t,x) = f(t,x) & (t,x)\in[0,T]\times\mathbb{R} \\ u(0,x) = g(x) & x\in\mathbb{R} \end{cases} \tag{1.16}$$

is well-posed in H^∞ *(with loss of derivatives). More precisely, there exists a constant* $\sigma > 0$ *such that for all* $f\in C([0,T];H^s)$ *and* $g\in H^s$ *there is a unique solution* $u\in C([0,T];H^{s-\sigma})$ *which satisfies the following energy estimate:*

$$\|u(t,\cdot)\|_{s-\sigma}^2 \leq C_s\left(\|g\|_s^2 + \int_0^t \|f(\tau,\cdot)\|_s^2\,d\tau\right) \qquad \forall t\in[0,T], \tag{1.17}$$

for some $C_s > 0$.

Theorem 1.2 is a generalization of Theorem 1.1 of [ABZ] where $a_p(t,D_x) = a_p(t)D_x^p$ with $a_p\in C([0,T];\mathbb{R}^+)$, and $a_j(t,x,D_x) = a_j(t,x)D_x^j$ were differential operators with uniformly bounded complex valued coefficients. In particular, the assumption $a_p(t)\in\mathbb{R}^+$ of [ABZ] is here replaced by the assumption (1.12) that $\partial_\xi a_p$ is a real elliptic symbol (cf. (3.35) in the proof of Theorem 1.2).

Remark 1.3. Formula (1.17) states that a loss of derivatives appears in the solution of (1.16). In the following, it will be clear that the loss comes from (2.6), more precisely from (2.8). If condition (1.13) for $j = p-1$

$$|\operatorname{Im}\partial_\xi^\alpha a_{p-1}(t,x,\xi)| \leq \frac{C}{\langle x\rangle}\langle\xi\rangle_h^{p-1-\alpha}$$

is substituted by the slightly stronger condition

$$|\operatorname{Im}\partial_\xi^\alpha a_{p-1}(t,x,\xi)| \leq \frac{C}{\langle x\rangle^{1+\eta}}\langle\xi\rangle_h^{p-1-\alpha}$$

for some $\eta > 0$, then, by defining

$$\lambda_{p-1}(x,\xi) = M_{p-1}\int_0^x \langle y\rangle^{-1-\eta}\psi\left(\frac{\langle y\rangle}{\langle\xi\rangle_h^{p-1}}\right)dy\,,$$

(cfr. (2.5)), our method gives well-posedness of (1.16) in Sobolev spaces without any loss of derivatives.

The same considerations hold for formula (1.8), which shows a loss of derivatives in the energy estimate for systems of pseudo-differential p-evolution operators. The loss can be avoided by modifying the assumptions

$$|\operatorname{Im}\partial_\xi^\alpha \mu_k^{(p-1)}(t,x,\xi)| \leq C_\alpha\langle x\rangle^{-1}\langle\xi\rangle_h^{p-1-\alpha},\ 1\leq k\leq m$$

into

$$|\operatorname{Im}\partial_\xi^\alpha \mu_k^{(p-1)}(t,x,\xi)| \leq C_\alpha\langle x\rangle^{-1-\eta}\langle\xi\rangle_h^{p-1-\alpha},\ 1\leq k\leq m$$

for some $\eta > 0$.

2. **Preliminary results**

We need first to prove Theorem 1.2. To this aim, by the energy method we write

$$iP = \partial_t + ia_p(t, D_x) + \sum_{j=0}^{p-1} ia_j(t, x, D_x) =: \partial_t + A(t, x, D_x) \tag{2.1}$$

and compute, for a solution $u(t, x)$ of (1.16),

$$\begin{aligned} \frac{d}{dt}\|u\|_0^2 = & \ 2\,\mathrm{Re}\langle \partial_t u, u\rangle = 2\,\mathrm{Re}\langle iPu, u\rangle - 2\,\mathrm{Re}\langle Au, u\rangle \\ \leq & \ \|f\|_0^2 + \|u\|_0^2 - 2\,\mathrm{Re}\langle Au, u\rangle, \end{aligned} \tag{2.2}$$

where $\|\cdot\|_0$ and $\langle\cdot,\cdot\rangle$ denote, respectively, the norm and the scalar product in $L^2(\mathbb{R})$.

We would like to obtain an estimate from below for $\mathrm{Re}\langle Au, u\rangle$ of the form

$$\mathrm{Re}\langle Au, u\rangle \geq -c\|u\|_0^2$$

for some $c > 0$, but such an estimate does not hold true, in general, since

$$2\,\mathrm{Re}\langle Au, u\rangle = \langle (A + A^*)u, u\rangle$$

and $A + A^*$ is an operator with symbol in S^{p-1} (A^* is the formal adjoint of A). To overcome the obstacle, throughout the paper we work as follows:

(1) we construct an appropriate transformation that changes $\partial_t + A$ into $\partial_t + A_\Lambda$, where A_Λ is an operator of the form $A_\Lambda := (e^\Lambda)^{-1}Ae^\Lambda$ for some pseudo-differential operator Λ;
(2) we use sharp-Gårding Theorem and Fefferman-Phong inequality to obtain the estimate from below

$$\mathrm{Re}\langle A_\Lambda u, u\rangle \geq -c\|u\|_0^2$$

for some $c > 0$;
(3) we produce the energy estimate for the transformed equation $(\partial_t + A_\Lambda)v = f_\Lambda$; by this, we obtain the energy estimate (1.17) for the equation $Pu = f$.

This section is devoted to the construction of the transformation in (1) and to his main features. We look for a transformation of the form $e^{\Lambda(x,D_x)}$, where $\Lambda(x, D_x)$ is a pseudo-differential operator of symbol $\Lambda(x, \xi)$ such that:

- $\Lambda(x, \xi)$ is real valued;
- $e^\Lambda \in S^\delta$, $\delta > 0$, so that $e^\Lambda : \ H^\infty \to H^\infty$;
- e^Λ is invertible;
- $(e^\Lambda)^{-1}$ has principal part $e^{-\Lambda}$.

Then, in Section 3, we consider the Cauchy problem

$$\begin{cases} P_\Lambda v = f_\Lambda \\ v(0, x) = g_\Lambda \end{cases} \tag{2.3}$$

for $P_\Lambda := (e^\Lambda)^{-1}Pe^\Lambda$, $f_\Lambda := (e^\Lambda)^{-1}f$ and $g_\Lambda := (e^\Lambda)^{-1}g$. There we show that (2.3) is well posed in Sobolev spaces; since well-posedness of (2.3) is equivalent to that of (1.16) for

$$u(t, x) = e^{\Lambda(x,D_x)}v(t, x),$$

from the energy estimate for v we gain the desired energy estimate (1.17) for u which proves Theorem 1.2. In the energy estimate for u a loss of $\sigma = 2\delta$ derivatives will appear, due to the fact that the transformations $e^{\pm\Lambda}$ are of positive order δ.
Finally, in Section 4 we prove our main Theorem 1.1 by applying Theorem 1.2.

Let us now construct the operator $\Lambda(x, D_x)$ by defining its symbol

$$\Lambda(x,\xi) := \lambda_{p-1}(x,\xi) + \lambda_{p-2}(x,\xi) + \ldots + \lambda_1(x,\xi) \tag{2.4}$$

with

$$\lambda_{p-k}(x,\xi) := M_{p-k}\int_0^x \langle y\rangle^{-\frac{p-k}{p-1}}\psi\left(\frac{\langle y\rangle}{\langle\xi\rangle_h^{p-1}}\right)dy\langle\xi\rangle_h^{-k+1}, \quad 1\le k\le p-1, \tag{2.5}$$

where the constants $M_{p-k} > 0$ will be chosen later on and $\psi \in C_0^\infty(\mathbb{R})$ satisfy:

$$0 \le \psi(y) \le 1 \qquad \forall y \in \mathbb{R}$$
$$\psi(y) = \begin{cases} 1 & |y| \le \frac{1}{2} \\ 0 & |y| \ge 1. \end{cases}$$

The construction (2.4), (2.5) is similar to the one in [ABZ]. In what follows we list some properties of the just constructed function Λ, that will be used in §3 to prove Theorem 1.2; proofs of these properties heavily use the following immediate features of Λ:

- $\psi(\langle y\rangle/\langle\xi\rangle_h^{p-1})$ is zero outside
$$E_\psi := \{y \in \mathbb{R} : \ \langle y\rangle \le \langle\xi\rangle_h^{p-1}\}.$$
- the derivatives $\psi^{(k)}(\langle y\rangle/\langle\xi\rangle_h^{p-1})$, $k \ge 1$ are zero outside
$$E'_\psi := \{y \in \mathbb{R} : \ \frac{1}{2}\langle\xi\rangle_h^{p-1} \le \langle y\rangle \le \langle\xi\rangle_h^{p-1}\}.$$

This is very useful to give estimates of the derivatives of $\Lambda(x,\xi)$.

Lemma 2.1. *There exist positive constants C, δ and $\delta_{\alpha,\beta}$, independent on h, such that*

$$|\Lambda(x,\xi)| \le C + \delta\log\langle\xi\rangle_h \tag{2.6}$$
$$|\partial_\xi^\alpha D_x^\beta\Lambda(x,\xi)| \le \delta_{\alpha,\beta}\langle\xi\rangle_h^{-\alpha}, \qquad \forall \alpha+\beta \ge 1. \tag{2.7}$$

Remark 2.2. We remark that the positive constant δ in (2.6) is explicitly determined; this is very important since we are going to show that the loss of derivatives is exactly $\sigma = 2\delta$. The precise value of δ is obtained in formula (2.10).

Proof. Direct computations give

$$|\lambda_{p-1}(x,\xi)| \le M_{p-1}\log 2 + M_{p-1}(p-1)\log\langle\xi\rangle_h, \tag{2.8}$$
$$|\lambda_{p-k}(x,\xi)| \le M_{p-k}\frac{p-1}{k-1}\langle x\rangle^{\frac{k-1}{p-1}}\langle\xi\rangle_h^{-k+1}\chi_{E_\psi}(x) \le M'_{p-k}, \tag{2.9}$$

for $M'_{p-k} = M_{p-k}\frac{p-1}{k-1}$, and χ_{E_ψ} the characteristic function of E_ψ. Since

$$|\Lambda(x,\xi)| \le |\lambda_{p-1}(x,\xi)| + \sum_{k=2}^{p-1}|\lambda_{p-k}(x,\xi)|,$$

estimates (2.8) and (2.9) give (2.6) for

$$\delta = (p-1)M_{p-1} \tag{2.10}$$

and $C = M_{p-1}\log 2 + \sum_{k=2}^{p-1} M'_{p-k}$.

Now, with the aim to prove (2.7), we derive some useful estimates for the functions λ_{p-k}, $1 \le k \le p-1$. We first give estimates of the derivatives of the function $\psi\left(\langle y\rangle/\langle\xi\rangle_h^{p-1}\right)$. For $\beta \ge 1$ we have:

$$\left|\partial_x^\beta \psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^{p-1}}\right)\right| = \left|\sum_{\substack{r_1+\ldots+r_q=\beta\\ r_i\in\mathbb{N}\setminus\{0\}}} C_{q,r}\psi^{(q)}\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^{p-1}}\right)\partial_x^{r_1}\frac{\langle x\rangle}{\langle\xi\rangle_h^{p-1}}\cdots\partial_x^{r_q}\frac{\langle x\rangle}{\langle\xi\rangle_h^{p-1}}\right| \le c_\beta\langle x\rangle^{-\beta} \tag{2.11}$$

since we are in the region $\langle x\rangle \le \langle\xi\rangle_h^{p-1}$; similarly, for $\alpha \ge 1$:

$$\left|\partial_\xi^\alpha \psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^{p-1}}\right)\right| \le c_\alpha\langle\xi\rangle^{-\alpha}; \tag{2.12}$$

finally, for $\alpha \ge 1$ and $\beta \ge 1$, by (2.11) and (2.12):

$$\begin{aligned}\left|\partial_\xi^\alpha\partial_x^\beta \psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^{p-1}}\right)\right| &\le \sum_{\alpha_0+\ldots\alpha_q=\alpha} c_{\alpha_0,\ldots,\alpha_q}\sum_{\substack{r_1+\ldots+r_q=\beta\\ r_i\in\mathbb{N}\setminus\{0\}}} C_{q,r}\left|\partial_\xi^{\alpha_0}\left(\psi^{(q)}\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^{p-1}}\right)\right)\right|\cdot\\ &\quad\cdot\left|\partial_\xi^{\alpha_1}\partial_x^{r_1}\frac{\langle x\rangle}{\langle\xi\rangle_h^{p-1}}\cdots\partial_\xi^{\alpha_q}\partial_x^{r_q}\frac{\langle x\rangle}{\langle\xi\rangle_h^{p-1}}\right|\\ &\le c_{\alpha,\beta}\langle x\rangle^{-\beta}\langle\xi\rangle_h^{-\alpha}.\end{aligned} \tag{2.13}$$

In order to prove (2.7), let us first consider the case $\alpha = 0$. For $\beta \ge 1$ and $1 \le k \le p-1$

$$\begin{aligned}\partial_x^\beta\lambda_{p-k}(x,\xi) &= M_{p-k}\partial_x^{\beta-1}\left[\langle x\rangle^{-\frac{p-k}{p-1}}\psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^{p-1}}\right)\right]\langle\xi\rangle_h^{-k+1}\\ &= M_{p-k}\left[(\partial_x^{\beta-1}\langle x\rangle^{-\frac{p-k}{p-1}})\psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^{p-1}}\right)\right.\\ &\quad\left.+\sum_{\beta_1=1}^{\beta-1}\binom{\beta-1}{\beta_1}(\partial_x^{\beta-1-\beta_1}\langle x\rangle^{-\frac{p-k}{p-1}})\partial_x^{\beta_1}\psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^{p-1}}\right)\right]\langle\xi\rangle_h^{-k+1}.\end{aligned}$$

By (2.11) there exist positive constants c_β and $C_{k,\beta}$ such that for $\beta \ge 1$ and $1 \le k \le p-1$:

$$\begin{aligned}|\partial_x^\beta\lambda_{p-k}(x,\xi)| &\le M_{p-k}c_\beta\langle x\rangle^{-\frac{p-k}{p-1}-\beta+1}\langle\xi\rangle_h^{-k+1}\chi_{E_\psi}(x)\\ &\le C_{k,\beta}\langle x\rangle^{\frac{k-1}{p-1}-\beta}\langle\xi\rangle_h^{-k+1}\chi_{E_\psi}(x) \le C_{k,\beta}\langle x\rangle^{-\beta} \le C_{k,\beta}\end{aligned} \tag{2.14}$$

For the case $\alpha \ge 1$ and $1 \le k \le p-1$, let us compute (for $\beta = 0$):

$$\begin{aligned}\partial_\xi^\alpha\lambda_{p-k}(x,\xi) &= M_{p-k}\sum_{\alpha_1=1}^{\alpha}\binom{\alpha}{\alpha_1}\int_0^x\langle y\rangle^{-\frac{p-k}{p-1}}\partial_\xi^{\alpha_1}\left[\psi\left(\frac{\langle y\rangle}{\langle\xi\rangle_h^{p-1}}\right)\right]dy\,\partial_\xi^{\alpha-\alpha_1}\langle\xi\rangle_h^{-k+1}\\ &\quad+M_{p-k}\int_0^x\langle y\rangle^{-\frac{p-k}{p-1}}\psi\left(\frac{\langle y\rangle}{\langle\xi\rangle_h^{p-1}}\right)dy\,\partial_\xi^\alpha\langle\xi\rangle_h^{-k+1}.\end{aligned} \tag{2.15}$$

Now, for $k=1$, since $\langle y\rangle^\varepsilon\psi^{(q)}(y)$ is bounded for every $\varepsilon>0$, we obtain that:

$$
\begin{aligned}
|\partial_\xi^\alpha\lambda_{p-1}(x,\xi)| &\leq M_{p-1}\int_0^x\frac{1}{\langle y\rangle}\left|\partial_\xi^\alpha\left[\psi\left(\frac{\langle y\rangle}{\langle\xi\rangle_h^{p-1}}\right)\right]\right|dy\\
&\leq M_{p-1}\sum_{\substack{r_1+\ldots+r_q=\alpha\\ r_i\in\mathbb{N}\setminus\{0\}}}C_{q,r}\int_0^x\frac{1}{\langle y\rangle}\left|\psi^{(q)}\left(\frac{\langle y\rangle}{\langle\xi\rangle_h^{p-1}}\right)\right|\langle\xi\rangle_h^{-\alpha}dy\\
&\leq M_{p-1}\sum_{\substack{r_1+\ldots+r_q=\alpha\\ r_i\in\mathbb{N}\setminus\{0\}}}C_{q,r}\int_0^x\frac{1}{\langle y\rangle^{1+\epsilon}}\sup_{\mathbb{R}}\left|\frac{\langle y\rangle^\epsilon}{\langle\xi\rangle^{\epsilon(p-1)}}\cdot\psi^{(q)}\left(\frac{\langle y\rangle}{\langle\xi\rangle_h^{p-1}}\right)\right|\chi_{E'_\psi}(y)\,dy\cdot\langle\xi\rangle_h^{\epsilon(p-1)-\alpha}\\
&\leq M_{p-1}c'_\alpha\langle x\rangle^{-\epsilon}\chi_{E'_\psi}(x)\langle\xi\rangle_h^{\epsilon(p-1)-\alpha}\\
&\leq M_{p-1}c'_\alpha\langle\xi\rangle_h^{-\alpha}.
\end{aligned}
\tag{2.16}
$$

For $2\leq k\leq p-1$, by (2.15) and (2.12):

$$
\begin{aligned}
|\partial_\xi^\alpha\lambda_{p-k}(x,\xi)| &\leq M_{p-k}c_\alpha\int_0^x\langle y\rangle^{-\frac{p-k}{p-1}}dy\,\chi_{E_\psi}(x)\langle\xi\rangle_h^{-k+1-\alpha}\\
&\leq C_\alpha M_{p-k}\langle x\rangle^{\frac{k-1}{p-1}}\langle\xi\rangle_h^{-k+1-\alpha}\chi_{E_\psi}(x)\\
&\leq C_\alpha M_{p-k}\langle\xi\rangle_h^{-\alpha}.
\end{aligned}
\tag{2.17}
$$

Let us finally assume $\alpha,\beta\geq1$ and compute:

$$
\begin{aligned}
\partial_\xi^\alpha\partial_x^\beta\lambda_{p-k}(x,\xi) = {}& M_{p-k}\sum_{\substack{\alpha_1+\alpha_2=\alpha\\ \beta_1+\beta_2=\beta-1\\ \alpha_1\cdot\beta_2>0}}\binom{\alpha}{\alpha_1}\binom{\beta-1}{\beta_1}\partial_x^{\beta_1}\langle x\rangle^{-\frac{p-k}{p-1}}\partial_\xi^{\alpha_1}\partial_x^{\beta_2}\psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^{p-1}}\right)\partial_\xi^{\alpha_2}\langle\xi\rangle_h^{-k+1}\\
&+M_{p-k}\partial_x^{\beta-1}\langle x\rangle^{-\frac{p-k}{p-1}}\cdot\psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^{p-1}}\right)\cdot\partial_\xi^\alpha\langle\xi\rangle_h^{-k+1}.
\end{aligned}
$$

From (2.13), for $\alpha,\beta\geq1$

$$
|\partial_\xi^\alpha\partial_x^\beta\lambda_{p-k}(x,\xi)|\leq C_{\alpha,\beta}\langle x\rangle^{\frac{k-1}{p-1}-\beta}\langle\xi\rangle_h^{-\alpha-k+1}\chi_{E_\psi}(x)\leq C_{\alpha,\beta}\langle\xi\rangle_h^{-\alpha}.
\tag{2.18}
$$

Summing up, estimates (2.14), (2.16), (2.17) and (2.18) give

$$
\begin{aligned}
|\partial_\xi^\alpha\partial_x^\beta\lambda_{p-k}(x,\xi)| &\leq C_{\alpha,\beta}M_{p-k}\langle x\rangle^{\frac{k-1}{p-1}-\beta}\langle\xi\rangle_h^{-\alpha-k+1}\chi_{E_\psi}(x)\\
&\leq\delta_{\alpha,\beta}\langle\xi\rangle_h^{-\alpha}\qquad\forall 1\leq k\leq p-1,\ \alpha+\beta\geq1,
\end{aligned}
\tag{2.19}
$$

that is (2.7) by construction (2.4). □

In the sequel we shall need also the following Lemmas; for their proofs please refer to [ABZ].

Lemma 2.3. *Let $\Lambda(x,\xi)$ satisfy (2.6) and (2.7). Then there exists $h_0\geq1$ such that for $h\geq h_0$ the operator $e^{\Lambda(x,D_x)}$, with symbol $e^{\Lambda(x,\xi)}\in S^\delta$, is invertible and*

$$
(e^\Lambda)^{-1}=e^{-\Lambda}(I+R)
\tag{2.20}
$$

where I is the identity operator and R is of the form $R=\sum_{n=1}^{+\infty}r^n$ with principal symbol

$$
r_{-1}(x,\xi)=\partial_\xi\Lambda(x,\xi)D_x\Lambda(x,\xi)\in S^{-1}.
$$

Lemma 2.4. *Let $\Lambda(x,\xi)$ satisfy (2.7) and $h\geq 1$ be fixed large enough to get (2.20). Then*

$$|\partial_\xi^\alpha e^{\pm\Lambda(x,\xi)}|\leq C_\alpha\langle\xi\rangle_h^{-\alpha}e^{\pm\Lambda(x,\xi)}\qquad\forall\alpha\in\mathbb{N}\tag{2.21}$$

$$|D_x^\beta e^{\pm\Lambda(x,\xi)}|\leq C_\beta\langle x\rangle^{-\beta}e^{\pm\Lambda(x,\xi)}\qquad\forall\beta\in\mathbb{N}.\tag{2.22}$$

Lemma 2.5. *Let $A(t,x,D_x)$ be the operator in (2.1), Λ satisfying (2.7), $h\geq h_0$ and R as in (2.20).*

Then the operator

$$A_\Lambda(t,x,D_x):=(e^{\Lambda(x,D_x)})^{-1}A(t,x,D_x)e^{\Lambda(x,D_x)}\tag{2.23}$$

can be written as

$$\begin{aligned}A_\Lambda(t,x,D_x)=\;&e^{-\Lambda(x,D_x)}A(t,x,D_x)e^{\Lambda(x,D_x)}\\&+\sum_{m=0}^{p-2}\frac{1}{m!}\sum_{n=1}^{p-1-m}e^{-\Lambda(x,D_x)}A^{n,m}(t,x,D_x)e^{\Lambda(x,D_x)}+A_0(t,x,D_x),\end{aligned}\tag{2.24}$$

where $A_0(t,x,D_x)$ has symbol $A_0(t,x,\xi)\in S^0$ and

$$\sigma(A^{n,m}(t,x,D_x))=\partial_\xi^m r^n(x,\xi)D_x^m A(t,x,\xi)\in S^{p-m-n}.\tag{2.25}$$

Lemma 2.6. *Let Λ be defined by (2.4), with λ_{p-k} satisfying (2.19). Then, for $m\geq 1$,*

$$e^{-\Lambda}D_x^m e^{\Lambda}=\sum_{s=0}^{p-2}f_{-s}(\lambda_{p-1},\dots,\lambda_{p-s-1})+f_{-p+1}(\lambda_{p-1},\dots,\lambda_1)\tag{2.26}$$

for some $f_{-p+1}\in S^{-p+1}$ depending on $\lambda_{p-1},\dots,\lambda_1$ and $f_{-s}\in S^{-s}$ depending only on λ_{p-1}, ..., λ_{p-s-1}, and not on $\lambda_{p-s},\dots,\lambda_1$, such that

$$|\partial_\xi^\alpha\partial_x^\beta f_{-s}|\leq C_{\alpha,\beta,s}\frac{\langle\xi\rangle_h^{-s-\alpha}}{\langle x\rangle^{\frac{p-1-s}{p-1}+\beta}}\qquad\forall\alpha,\beta\geq 0,\tag{2.27}$$

for some $C_{\alpha,\beta,s}>0$.

We conclude this Section by recalling the sharp-Gårding Theorem and the Fefferman-Phong inequality, the two main tools we are going to use in proving Theorem 1.2, referring respectively to [KG] and [FP] for proofs.

Theorem 2.7 (Sharp-Gårding). *Let $A(x,\xi)\in S^m$ with $\operatorname{Re}A(x,\xi)\geq 0$. There exist pseudo-differential operators $Q(x,D_x)$ and $R(x,D_x)$ with symbols, respectively, $Q(x,\xi)\in S^m$ and $R(x,\xi)\in S^{m-1}$, such that*

$$\begin{aligned}&A(x,D_x)=Q(x,D_x)+R(x,D_x)\\&\operatorname{Re}\langle Q(x,D_x)u,u\rangle\geq 0\qquad\forall u\in H^m\\&R(x,\xi)\sim\psi_1(\xi)D_xA(x,\xi)+\sum_{\alpha+\beta\geq 2}\psi_{\alpha,\beta}(\xi)\partial_\xi^\alpha D_x^\beta A(x,\xi),\end{aligned}\tag{2.28}$$

with $\psi_1,\psi_{\alpha,\beta}$ real valued functions, $\psi_1\in S^{-1}$ and $\psi_{\alpha,\beta}\in S^{(\alpha-\beta)/2}$. As a consequence, there exists $c>0$ such that it holds the well-known sharp-Gårding inequality

$$\operatorname{Re}\langle A(x,D_x)u,u\rangle\geq -c\|u\|^2_{(m-1)/2}.\tag{2.29}$$

Theorem 2.8 (Fefferman-Phong inequality). *Let $A(x,\xi)\in S^m$ with $A(x,\xi)\geq 0$. There exists $c>0$ such that*

$$\operatorname{Re}\langle A(x,D_x)u,u\rangle\geq -c\|u\|^2_{(m-2)/2}.\tag{2.30}$$

3. The scalar energy estimate

Let $\Lambda(x, D_x)$ be the operator constructed in (2.4), (2.5). Fix $h \geq 1$ large enough so that the operator e^{Λ} is invertible, and (2.20) holds. As described in Section 2, we set $A_\Lambda = (e^\Lambda)^{-1}Ae^\Lambda$ with

$$A(t,x,D_x) = \sum_{j=0}^{p} ia_j(t,x,D_x)$$

and $a_p = a_p(t, D_x)$. To prove Theorem 1.2 we need an estimate of the form

$$\mathrm{Re}\langle A_\Lambda v, v\rangle \geq -c\|v\|_0^2 \qquad \forall v(t,\cdot) \in H^\infty$$

for some $c > 0$. Such an estimate will be obtained by choosing the constants $M_{p-1}, \ldots, M_1$ in a suitable way and by several applications of sharp-Garding and Fefferman-Phong inequalities. In what follows, we state and prove some useful lemmas. Then, we give the proof of Theorem 1.2. Throughout this section, we work with the more simple operator $e^{-\Lambda}Ae^{\Lambda}$; then, at the end of the proof, we recover by Lemma 2.5 the full operator $A_\Lambda = (e^\Lambda)^{-1}Ae^\Lambda$.

Lemma 3.1. *Let us consider the operator $e^{-\Lambda}Ae^{\Lambda}$. Its terms of order $p-k$, denoted by $(e^{-\Lambda}Ae^{\Lambda})\big|_{\mathrm{ord}(p-k)}$, satisfy for $1 \leq k \leq p-1$:*

$$\left|\mathrm{Re}(e^{-\Lambda}Ae^{\Lambda})\big|_{\mathrm{ord}(p-k)}(t,x,\xi)\right| \leq C_{(M_{p-1},\ldots,M_{p-k})}\langle x\rangle^{-\frac{p-k}{p-1}}\langle\xi\rangle_h^{p-k} \tag{3.1}$$

for a constant $C_{(M_{p-1},\ldots,M_{p-k})} > 0$ depending only on $M_{p-1},\ldots,M_{p-k}$ and not on $M_{p-k-1},\ldots,M_1$.

Proof. We compute first

$$\begin{aligned}\sigma\left(A(t,x,D_x)e^{\Lambda(x,D_x)}\right) &= \sum_{m\geq 0}\frac{1}{m!}\partial_\xi^m A(t,x,\xi)D_x^m e^{\Lambda(x,\xi)}\\ &= \sum_{m=0}^{p-1}\sum_{j=m+1}^{p}\frac{1}{m!}\partial_\xi^m\left(ia_j(t,x,\xi)\right)D_x^m e^{\Lambda(x,\xi)} + \bar{A}_0\end{aligned}$$

$\bar{A}_0 \in S^0$. Then, for some $A_0 \in S^0$ (which may change from one equality to the other) we have:

$$\begin{aligned}\sigma(e^{-\Lambda}Ae^{\Lambda}) &= \sum_{\alpha\geq 0}\frac{1}{\alpha!}\partial_\xi^\alpha e^{-\Lambda}D_x^\alpha\left(\sum_{m=0}^{p-1}\sum_{j=m+1}^{p}\frac{1}{m!}\partial_\xi^m\left(ia_j(t,x,\xi)\right)D_x^m e^{\Lambda(x,\xi)} + \bar{A}_0\right)\\ &= \sum_{m=0}^{p-1}\sum_{j=m+1}^{p}\sum_{\alpha=0}^{j-1-m}\frac{1}{\alpha!}\frac{1}{m!}(\partial_\xi^\alpha e^{-\Lambda})\sum_{\beta=0}^{\alpha}\binom{\alpha}{\beta}(\partial_\xi^m D_x^\beta(ia_j(t,x,\xi)))(D_x^{m+\alpha-\beta}e^\Lambda) + A_0\\ &= \sum_{m=0}^{p-1}\sum_{j=m+1}^{p}\frac{1}{m!}(e^{-\Lambda}D_x^m e^\Lambda)(\partial_\xi^m(ia_j(t,x,\xi)))\\ &\quad+\sum_{m=0}^{p-2}\sum_{j=m+1}^{p}\sum_{\alpha=1}^{j-1-m}\sum_{\beta=0}^{\alpha}\frac{1}{\alpha!}\frac{1}{m!}\binom{\alpha}{\beta}(\partial_\xi^\alpha e^{-\Lambda})(\partial_\xi^m D_x^\beta(ia_j(t,x,\xi)))(D_x^{m+\alpha-\beta}e^\Lambda) + A_0.\end{aligned} \tag{3.2}$$

Put now

$$A_I := \sum_{m=0}^{p-1} \sum_{j=m+1}^{p} \frac{1}{m!}(e^{-\Lambda} D_x^m e^{\Lambda})(\partial_\xi^m (ia_j(t,x,\xi))), \tag{3.3}$$

$$A_{II} := \sum_{m=0}^{p-2} \sum_{j=m+1}^{p} \sum_{\alpha=1}^{j-1-m} \sum_{\beta=0}^{\alpha} \frac{1}{\alpha!}\frac{1}{m!}\binom{\alpha}{\beta}(\partial_\xi^\alpha e^{-\Lambda})(\partial_\xi^m D_x^\beta (ia_j(t,x,\xi)))(D_x^{m+\alpha-\beta} e^{\Lambda}). \tag{3.4}$$

We have

$$\sigma(e^{-\Lambda} A e^{\Lambda}) = A_I + A_{II} + A_0. \tag{3.5}$$

We consider first A_{II}, where $\alpha \geq 1$. In the case $m+\alpha-\beta \geq 1$, from (2.19) we get:

$$\begin{aligned}
&|(\partial_\xi^\alpha e^{-\Lambda})(\partial_\xi^m D_x^\beta ia_j)(D_x^{m+\alpha-\beta} e^{\Lambda})| \\
&\leq c\, \langle \xi \rangle_h^{j-m} \cdot \left| \partial_\xi^\alpha \prod_{k=1}^{p-1} e^{-\lambda_{p-k}} \right| \cdot \left| \partial_x^{m+\alpha-\beta} \prod_{k'=1}^{p-1} e^{\lambda_{p-k'}} \right| \\
&= c\, \langle \xi \rangle_h^{j-m} \sum_{\alpha_1+\ldots+\alpha_{p-1}=\alpha} \frac{\alpha!}{\alpha_1!\cdots\alpha_{p-1}!} \prod_{k=1}^{p-1} |\partial_\xi^{\alpha_k} e^{-\lambda_{p-k}}| \cdot \sum_{\substack{\gamma_1+\ldots+\gamma_{p-1}=\\ m+\alpha-\beta}} \frac{(m+\alpha-\beta)!}{\gamma_1!\cdots\gamma_{p-1}!} \prod_{k'=1}^{p-1} |\partial_x^{\gamma_{k'}} e^{\lambda_{p-k'}}| \\
&\leq c' \sum_{\substack{\alpha_1+\ldots+\alpha_{p-1}=\alpha \\ \gamma_1+\ldots+\gamma_{p-1}=m+\alpha-\beta \\ r_1+\ldots+r_{q_k}=\alpha_k;\, r_i,\alpha_k\geq 1 \\ s_1+\ldots+s_{p_{k'}}=\gamma_{k'};\, s_i,\gamma_{k'}\geq 1}} \prod_{k,k'=1}^{p-1} M_{p-k}^{q_k} \frac{\langle x \rangle^{\frac{k-1}{p-1} q_k}}{\langle \xi \rangle_h^{\alpha_k + q_k(k-1)}} \cdot M_{p-k'}^{p_{k'}} \frac{\langle x \rangle^{\frac{k'-1}{p-1} p_{k'} - \gamma_{k'}}}{\langle \xi \rangle_h^{p_{k'}(k'-1)}} \langle \xi \rangle_h^{j-m} \qquad (3.6)
\end{aligned}$$

for some $c, c' > 0$.

Each term of (3.6) has order $j-m-\alpha-\sum_{k=1}^{p-1} q_k(k-1) - \sum_{k'=1}^{p-1} p_{k'}(k'-1)$ and decay in x of the form

$$\langle x \rangle^{\frac{\sum_{k=1}^{p-1} q_k(k-1) + \sum_{k'=1}^{p-1} p_{k'}(k'-1)}{p-1} - m - \alpha + \beta} \leq \langle x \rangle^{-\frac{j-m-\alpha-\sum_{k=1}^{p-1} q_k(k-1) - \sum_{k'=1}^{p-1} p_{k'}(k'-1)}{p-1}}$$

since $-(p-1)(m+\alpha-\beta) \leq -j+m+\alpha$ for $m+\alpha-\beta \geq 1$.

Note also that $j-m-\alpha-\sum_{k=1}^{p-1} q_k(k-1) - \sum_{k'=1}^{p-1} p_{k'}(k'-1) \leq p-k-1$ and $j-m-\alpha-\sum_{k=1}^{p-1} q_k(k-1) - \sum_{k'=1}^{p-1} p_{k'}(k'-1) \leq p-k'-1$, so that whenever M_{p-k} or $M_{p-k'}$ appear in (3.6), then the order is at most $p-k-1$ and $p-k'-1$ respectively.

In the case $m+\alpha-\beta = 0$, by (2.19) we have, for all $0 \leq \beta \leq j-1$ with $1 \leq j \leq p-1$:

$$\begin{aligned}
&|\operatorname{Re}[(\partial_\xi^\alpha e^{-\Lambda})(\partial_\xi^m D_x^\beta ia_j) e^{\Lambda}]| \leq |\partial_\xi^\alpha e^{-\Lambda}| \cdot |\operatorname{Im} \partial_\xi^m D_x^\beta a_j| e^{\Lambda} \\
&= \sum_{\alpha_1+\ldots+\alpha_{p-1}=\alpha} \frac{\alpha!}{\alpha_1!\cdots\alpha_{p-1}!} \cdot \prod_{k=1}^{p-1} \left(\sum_{\substack{r_1+\ldots+r_{q_k}=\alpha_k \\ r_i,\alpha_k\geq 1}} C_{q,k} |\partial_\xi^{r_1} \lambda_{p-k}| \cdots |\partial_\xi^{r_{q_k}} \lambda_{p-k}| \right) \cdot |\operatorname{Im} \partial_\xi^m D_x^\beta a_j| \\
&\leq C \sum_{\substack{\alpha_1+\ldots+\alpha_{p-1} \\ =\alpha}} \prod_{k=1}^{p-1} \sum_{\substack{r_1+\ldots+r_{q_k}=\alpha_k \\ r_i,\alpha_k\geq 1}} M_{p-k}^{q_k} \langle x \rangle^{\frac{k-1}{p-1} q_k} \langle \xi \rangle_h^{-\alpha_k - q_k(k-1)} \cdot |\operatorname{Im} \partial_\xi^m D_x^\beta a_j| \qquad (3.7)
\end{aligned}$$

for some $C > 0$. Now, for

$$\gamma(\beta) = \begin{cases} 0 & \beta = 0 \\ 1 & \beta = 1 \\ \left[\dfrac{\beta}{2}\right] & \beta \geq 2 \end{cases} \tag{3.8}$$

and $\min\{\beta+1, 3\} \leq j \leq p-1$ we have that (3.7) becomes, because of (1.13)-(1.15):

$$|\operatorname{Re}[(\partial_\xi^\alpha e^{-\Lambda})(\partial_\xi^m ia_j)e^\Lambda]| \leq C' \sum_{\substack{\alpha_1+\ldots+\alpha_{p-1}\\ =\alpha}} \prod_{k=1}^{p-1} \sum_{\substack{r_1+\ldots+r_{q_k}=\alpha_k\\ r_i,\alpha_k \geq 1}} M_{p-k}^{q_k} \langle x \rangle^{\frac{k-1}{p-1}q_k} \langle \xi \rangle_h^{-\alpha_k - q_k(k-1)} \cdot \tag{3.9}$$
$$\cdot \langle x \rangle^{-\frac{j-\gamma(\beta)}{p-1}} \langle \xi \rangle_h^{j-m}$$

Each term of (3.9) is a symbol of order $j - m - \alpha - \sum_{k=1}^{p-1} q_k(k-1)$ and has decay in x of the form:

$$\langle x \rangle^{\frac{\sum_{k=1}^{p-1} q_k(k-1) - j + \gamma(\beta)}{p-1}} \leq \langle x \rangle^{-\frac{j-m-\alpha-\sum_{k=1}^{p-1} q_k(k-1)}{p-1}} \quad \text{if } \min\{\beta+1, 3\} \leq j \leq p-1,$$

since $\gamma(\beta) \leq \beta = \alpha + m$.

Here again $j - m - \alpha - \sum_{k=1}^{p-1} q_k(k-1) \leq p-k-1$ and hence M_{p-k} appears in (3.9) only when the order is at most $p-k-1$.

Summing up, formulas (3.6) and (3.9) give that the terms of order $p-k$ of A_{II}, denoted by $A_{II}|_{\mathrm{ord}(p-k)}$, satisfy:

$$\left|\operatorname{Re} A_{II}|_{\mathrm{ord}(p-k)}\right| \leq C \langle x \rangle^{-\frac{p-k}{p-1}} \langle \xi \rangle_h^{p-k} \tag{3.10}$$

for some $C > 0$.

Moreover, $\operatorname{Re} A_{II}|_{\mathrm{ord}(p-k)}$ depends only on $M_{p-1}, \ldots, M_{p-k+1}$ and not on $M_{p-k}, \ldots, M_1$.

We consider then

$$\begin{aligned} A_I &= \sum_{m=0}^{p-1} \sum_{j=m+1}^{p} \frac{1}{m!} (\partial_\xi^m (ia_j))(e^{-\Lambda} D_x^m e^\Lambda) \\ &= \sum_{k=0}^{p-1} \sum_{m=0}^{k} \frac{1}{m!} (\partial_\xi^m (ia_{p-k+m}))(e^{-\Lambda} D_x^m e^\Lambda) \\ &= ia_p + \sum_{k=1}^{p-1} \left(ia_{p-k} + \sum_{m=1}^{k} \frac{1}{m!} (\partial_\xi^m (ia_{p-k+m}))(e^{-\Lambda} D_x^m e^\Lambda) \right). \end{aligned} \tag{3.11}$$

Note that $D_x \Lambda = D_x \lambda_{p-1} + D_x \lambda_{p-2} + \ldots + D_x \lambda_1$ with $D_x \lambda_{p-k} \xi^{p-1} \in S^{p-k}$ because of (2.14). Moreover, from Lemma 2.6 it follows that there exist $f_{-s} \in S^{-s}$, for $0 \leq s \leq p-2$, depending only on $\lambda_{p-1}, \ldots, \lambda_{p-s-1}$, and $f_{-p+1} \in S^{-p+1}$ such that, for $\tilde{f}_0 = (\partial_\xi^m a_{p-k+m}) f_{-p+1} \in S^0$,

$$(\partial_\xi^m a_{p-k+m})(e^{-\Lambda} D_x^m e^\Lambda) = \sum_{s=0}^{p-2} f_{-s}(\lambda_{p-1}, \ldots, \lambda_{p-s-1}) \partial_\xi^m a_{p-k+m} + \tilde{f}_0, \tag{3.12}$$

and, from (2.27) for $0 \leq s \leq p-2$,

$$|f_{-s} \partial_\xi^m a_{p-k+m}| \leq \frac{C_s}{\langle x \rangle^{\frac{p-1-s}{p-1}}} \langle \xi \rangle_h^{p-k-s} \leq \frac{C_s}{\langle x \rangle^{\frac{p-k-s}{p-1}}} \langle \xi \rangle_h^{p-k-s} \qquad \forall k \geq 1 \tag{3.13}$$

for some $C_s > 0$. Rearranging the terms of the second addend of A_I in (3.11) and putting together all terms of order $p-k$, we can thus write, because of (3.12), (3.13):

$$A_I = ia_p + \sum_{k=1}^{p-1} (ia_{p-k} + iD_x\lambda_{p-k}\partial_\xi a_p + B_{p-k}) + \tilde{B}_0,$$

for some $\tilde{B}_0 \in S^0$ and $B_{p-k} \in S^{p-k}$ coming from (3.12) and of the form

$$B_{p-k} = \sum_{s=2}^{k} if_{-(k-s)}(\lambda_{p-1}, \dots, \lambda_{p-k+s-1}) \sum_{m=1}^{s} \partial_\xi^m a_{p-s+m}, \quad k = 1, \dots, p-1. \tag{3.14}$$

Notice that $B_{p-k} \in S^{p-k}$ depends only on $\lambda_{p-1}, \dots, \lambda_{p-k+1}$ and not on $\lambda_{p-k}, \dots, \lambda_1$, and moreover

$$|B_{p-k}| \leq \frac{C_k}{\langle x \rangle^{\frac{p-k}{p-1}}} \langle \xi \rangle_h^{p-k} \tag{3.15}$$

for some $C_k > 0$.

Setting

$$A^0_{p-k} := ia_{p-k} + iD_x\lambda_{p-k}\partial_\xi a_p \tag{3.16}$$

we write

$$A_I = ia_p + \sum_{k=1}^{p-1}(A^0_{p-k} + B_{p-k}) + \tilde{B}_0. \tag{3.17}$$

Note that $A^0_{p-k}, B_{p-k} \in S^{p-k}$ and, since $\mathrm{Re}(A^0_{p-k}) = -\operatorname{Im} a_{p-k} + \partial_x\lambda_{p-k}\partial_\xi a_p$, from (1.13) with $j = p-k$ and $\alpha = 0$, the first inequality in (2.14) with $\beta = 1$, and (3.15) we have

$$|\operatorname{Re} A^0_{p-k}| + |B_{p-k}| \leq C_k \langle x \rangle^{-\frac{p-k}{p-1}} \langle \xi \rangle_h^{p-k} \tag{3.18}$$

for some $C_k > 0$.

Moreover, A^0_{p-k} depends only on M_{p-k} and B_{p-k} depends only on $M_{p-1}, \dots, M_{p-k+1}$ (and not on $M_{p-k}, \dots, M_1$) as a consequence of (3.14).

Formulas (3.10) and (3.17)-(3.18) together give (3.1) because of (3.5). The proof is completed. □

Lemma 3.2. *Let us consider, for $1 \leq k \leq p-3$ the operator $(e^{-\Lambda} A e^{\Lambda})\big|_{\mathrm{ord}(p-k)}$ and define*

$$R_{p-k} = \psi_1(\xi) D_x \, (e^{-\Lambda} A e^{\Lambda})\big|_{\mathrm{ord}(p-k)} + \sum_{\alpha+\beta\geq 2} \psi_{\alpha,\beta}(\xi)\partial_\xi^\alpha D_x^\beta \, (e^{-\Lambda} A e^{\Lambda})\big|_{\mathrm{ord}(p-k)}, \tag{3.19}$$

with $\psi_1, \psi_{\alpha,\beta}$ as in Theorem 2.7. Denote by $R_{p-k}|_{\mathrm{ord}(p-k-s)}$ the terms of order $p-k-s$ of R_{p-k}, $1 \leq s \leq p-k-1$. Then:

$$\left| \mathrm{Re}(R_{p-k})|_{\mathrm{ord}(p-k-s)} (t, x, \xi) \right| \leq C_{(M_{p-1},\dots,M_{p-k-s})} \langle x \rangle^{-\frac{p-k-s}{p-1}} \langle \xi \rangle_h^{p-k-s} \tag{3.20}$$

for every $1 \leq s \leq p-k-1$ and for a positive constant $C_{(M_{p-1},\dots,M_{p-k-s})}$ depending only on $M_{p-1}, \dots, M_{p-k-s}$ and not on $M_{p-k-s-1}, \dots, M_1$.

Proof. From (3.5), to estimate R_{p-k} we need to give estimates of

$$R(\,A_I|_{\mathrm{ord}(p-k)}) = \psi_1(\xi) D_x \, A_I|_{\mathrm{ord}(p-k)} + \sum_{\alpha+\beta\geq 2} \psi_{\alpha,\beta}(\xi)\partial_\xi^\alpha D_x^\beta \, A_I|_{\mathrm{ord}(p-k)}$$

and

$$R(A_{II}|_{\mathrm{ord}(p-k)}) = \psi_1(\xi)D_x\, A_{II}|_{\mathrm{ord}(p-k)} + \sum_{\alpha+\beta\geq 2}\psi_{\alpha,\beta}(\xi)\partial_\xi^\alpha D_x^\beta\, A_{II}|_{\mathrm{ord}(p-k)}$$

We start by considering $R(A_I|_{\mathrm{ord}(p-k)}) = R(A^0_{p-k}) + R(B_{p-k})$, because of (3.17) for A^0_{p-k} and B_{p-k} defined respectively in (3.16) and (3.14). In computing

$$R(A^0_{p-k}) = \psi_1 D_x A^0_{p-k} + \sum_{\alpha+\beta\geq 2}\psi_{\alpha,\beta}\partial_\xi^\alpha D_x^\beta A^0_{p-k} \tag{3.21}$$

we find

$$\psi_1 D_x A^0_{p-k} = i\psi_1 D_x a_{p-k} + iD_x^2\lambda_{p-k}\psi_1\partial_\xi a_p\,;$$

by (1.14):

$$\begin{aligned}|\operatorname{Re}(\psi_1 D_x A^0_{p-k})| &\leq\; |\operatorname{Im} D_x a_{p-k}|\cdot|\psi_1| \leq \frac{C'}{\langle x\rangle^{\frac{p-k-1}{p-1}}}\langle\xi\rangle_h^{p-k-1}\\ &\leq\; \psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^{p-1}}\right)\frac{C'\langle\xi\rangle_h^{p-k-1}}{\langle x\rangle^{\frac{p-k-1}{p-1}}} + C''\end{aligned} \tag{3.22}$$

since $\psi_1\in S^{-1}$ and $\langle\xi\rangle_h^{p-k-1}/\langle x\rangle^{\frac{p-k-1}{p-1}}$ is bounded on $\mathrm{supp}(1-\psi)$.

We now look at

$$\begin{aligned}\sum_{\alpha+\beta\geq 2}\psi_{\alpha,\beta}\partial_\xi^\alpha D_x^\beta A^0_{p-k} &= \sum_{\alpha+\beta\geq 2}\psi_{\alpha,\beta}\partial_\xi^\alpha D_x^\beta(ia_{p-k} + iD_x\lambda_{p-k}\partial_\xi a_p)\\ &= \sum_{\alpha+\beta\geq 2}\psi_{\alpha,\beta}i\partial_\xi^\alpha D_x^\beta a_{p-k} + \sum_{\alpha+\beta\geq 2}\psi_{\alpha,\beta}\sum_{\alpha_1+\alpha_2=\alpha}\binom{\alpha}{\alpha_1}i\partial_\xi^{\alpha_1}D_x^{\beta+1}\lambda_{p-k}\cdot\partial_\xi^{\alpha_2+1}a_p.\end{aligned} \tag{3.23}$$

Note that the first addend in (3.23) is $\psi_{\alpha,\beta}i\partial_\xi^\alpha D_x^\beta a_{p-k}\in S^{p-k-\frac{\alpha+\beta}{2}}$, so it has to be considered at level $p-k-\frac{\alpha+\beta}{2}$ if $\alpha+\beta$ is even, at level $p-k-\frac{\alpha+\beta}{2}+\frac12$ if $\alpha+\beta$ is odd, thus at level $p-k+\left[-\frac{\alpha+\beta}{2}+\frac12\right]$. Looking also at its decay as $x\to\infty$, we have by (1.14), (1.15), for $p-k\geq 3$ and $\gamma(\beta)$ defined by (3.8):

$$\begin{aligned}|\operatorname{Re}(\psi_{\alpha,\beta}i\partial_\xi^\alpha D_x^\beta a_{p-k})| &\leq\; \langle\xi\rangle_h^{p-k-\frac{\alpha+\beta}{2}}\frac{C}{\langle x\rangle^{\frac{p-k-\gamma(\beta)}{p-1}}}\\ &\leq\; C\psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^{p-1}}\right)\frac{\langle\xi\rangle_h^{p-k+\left[-\frac{\alpha+\beta}{2}+\frac12\right]}}{\langle x\rangle^{\frac{p-k+\left[-\frac{\alpha+\beta}{2}+\frac12\right]}{p-1}}} + C'\end{aligned} \tag{3.24}$$

for some $C'>0$, since

$$-\gamma(b)\geq\left[-\frac{a+b}{2}+\frac12\right]\qquad\forall a,b\geq 0. \tag{3.25}$$

We remark that decay estimates of the form (3.24) are needed until level $p-k-\frac{\alpha+\beta}{2}\geq\frac12$, i.e.

$$0\leq\left[\frac{\beta}{2}\right]\leq p-k-1,\qquad\text{for } p-k\geq 3. \tag{3.26}$$

For the second addend of (3.23) by (2.19) we immediately get:

$$\left|\sum_{\alpha+\beta\geq 2}\psi_{\alpha,\beta}\sum_{\alpha_1+\alpha_2=\alpha}\binom{\alpha}{\alpha_1}i\partial_\xi^{\alpha_1}D_x^{\beta+1}\lambda_{p-k}\cdot\partial_\xi^{\alpha_2+1}a_p\right| \leq \sum_{\alpha+\beta\geq 2}\frac{C_{\alpha,\beta}}{\langle x\rangle^{\frac{p-k}{p-1}+\beta}}\langle\xi\rangle_h^{p-k-\frac{\alpha+\beta}{2}}$$

$$\leq C\langle\xi\rangle_h^{p-k+\left[-\frac{\alpha+\beta}{2}+\frac{1}{2}\right]}\langle x\rangle^{-\frac{p-k+\left[-\frac{\alpha+\beta}{2}+\frac{1}{2}\right]}{p-1}} \tag{3.27}$$

since $\beta(p-1)\geq\left[-\frac{\alpha+\beta}{2}+\frac{1}{2}\right]$.

Summing up, we have obtained, for the second addend of (3.21), that

$$\left|\mathrm{Re}\sum_{\alpha+\beta\geq 2}\psi_{\alpha,\beta}\partial_\xi^\alpha D_x^\beta A^0_{p-k}\right|\leq C\langle\xi\rangle_h^{p-k+\left[-\frac{\alpha+\beta}{2}+\frac{1}{2}\right]}\langle x\rangle^{-\frac{p-k+\left[-\frac{\alpha+\beta}{2}+\frac{1}{2}\right]}{p-1}}\psi+C'$$

for some $C, C'>0$, because of (3.24) and (3.27). Note that only in (3.24) the assumptions (1.14), (1.15) are used. We have thus proved, looking also at (3.22), that $R(A^0_{p-k})$ fulfills the decay estimate in (3.20) and, moreover, it depends only on M_{p-k} and not on M_j for $j\neq p-k$.

We now estimate the other term

$$R(B_{p-k}) = \sum_{s=2}^{k}\sum_{m=1}^{s}R\left(if_{-(k-s)}\partial_\xi^m a_{p-s+m}\right)$$

$$= \sum_{s=2}^{k}\sum_{m=1}^{s}\left[\psi_1 D_x(if_{-(k-s)}\partial_\xi^m a_{p-s+m})+\sum_{\alpha+\beta\geq 2}\psi_{\alpha,\beta}\partial_\xi^\alpha D_x^\beta(if_{-(k-s)}\partial_\xi^m a_{p-s+m})\right] \tag{3.28}$$

for $\psi_1\in S^{-1}$, $\psi_{\alpha,\beta}\in S^{\frac{\alpha-\beta}{2}}$ and B_{p-k} defined by (3.14).

We have from (2.27):

$$|\psi_1 D_x(if_{-(k-s)}\partial_\xi^m a_{p-s+m})| \leq |\psi_1(\partial_x f_{-(k-s)})\partial_\xi^m a_{p-s+m}|+|\psi_1 f_{-(k-s)}\partial_\xi^m\partial_x a_{p-s+m})|\leq$$

$$\leq \langle\xi\rangle_h^{-1}C_{k-s}\left(\frac{1}{\langle x\rangle^{\frac{p-1-k+s}{p-1}+1}}+\frac{1}{\langle x\rangle^{\frac{p-1-k+s}{p-1}}}\right)\langle\xi\rangle_h^{-k+s}\langle\xi\rangle_h^{p-s}$$

$$\leq \frac{C_{k-s}}{\langle x\rangle^{\frac{p-k-1}{p-1}}}\langle\xi\rangle_h^{p-k-1},$$

therefore, for each $2\leq s\leq k$,

$$|\psi_1 D_x(if_{-(k-s)}\partial_\xi^m a_{p-s+m})|\leq c\psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^{p-1}}\right)\frac{\langle\xi\rangle_h^{p-k-1}}{\langle x\rangle^{\frac{p-k-1}{p-1}}}+c' \tag{3.29}$$

for some $c, c'>0$. For the second addend of (3.28) we write

$$\sum_{\alpha+\beta\geq 2}\psi_{\alpha,\beta}\partial_\xi^\alpha D_x^\beta(if_{-(k-s)}\partial_\xi^m a_{p-s+m})$$

$$= \sum_{\alpha+\beta\geq 2}\psi_{\alpha,\beta}\sum_{\alpha'=0}^{\alpha}\sum_{\beta'=0}^{\beta}\binom{\alpha}{\alpha'}\binom{\beta}{\beta'}i(\partial_\xi^{\alpha'}D_x^{\beta'}f_{-(k-s)})(\partial_\xi^{\alpha-\alpha'+m}D_x^{\beta-\beta'}a_{p-s+m})$$

By (2.27) we have that $\psi_{\alpha,\beta}(\partial_\xi^{\alpha'}D_x^{\beta'}f_{-(k-s)})(\partial_\xi^{\alpha-\alpha'+m}D_x^{\beta-\beta'}a_{p-s+m})\in S^{p-k-\frac{\alpha+\beta}{2}}$ and

$$
\begin{aligned}
|\psi_{\alpha,\beta}(\partial_\xi^{\alpha'}D_x^{\beta'}f_{-(k-s)})(\partial_\xi^{\alpha-\alpha'+m}D_x^{\beta-\beta'}a_{p-s+m})| &\leq \frac{C_{k-s}}{\langle x\rangle^{\frac{p-1-k+s}{p-1}+\beta'}}\langle\xi\rangle_h^{p-k-\frac{\alpha+\beta}{2}}\\
&\leq \frac{C_{k-s}}{\langle x\rangle^{\frac{p-k+\left[-\frac{\alpha+\beta}{2}+\frac12\right]}{p-1}}}\langle\xi\rangle_h^{p-k+\left[-\frac{\alpha+\beta}{2}+\frac12\right]}
\end{aligned}
$$

for some $C_{k-s}>0$, since $p-1-k+s\geq p-k$ (being $s\geq 2$) and $\beta'\geq\left[-\frac{\alpha+\beta}{2}+\frac12\right]$.

This, together with (3.29), means that $R(B_{p-k})$ satisfies the decay estimate in (3.20), independently of the conditions on the x-decay of the coefficients.

Now we are going to estimate $R(A_{II}|_{\mathrm{ord}(p-k)})$, where A_{II} is defined in (3.4). We have:

$$
\begin{aligned}
(3.30)\quad & R\left((\partial_\xi^\alpha e^{-\Lambda})(\partial_\xi^m D_x^\beta(ia_j(t,x,\xi)))(D_x^{m+\alpha-\beta}e^{\Lambda})\right)=\\
&\quad\psi_1 D_x\left[(\partial_\xi^\alpha e^{-\Lambda})(\partial_\xi^m D_x^\beta(ia_j(t,x,\xi)))(D_x^{m+\alpha-\beta}e^{\Lambda})\right]\\
&\qquad+\sum_{\alpha'+\beta'\geq 2}\psi_{\alpha',\beta'}\partial_\xi^{\alpha'}D_x^{\beta'}\left[(\partial_\xi^\alpha e^{-\Lambda})(\partial_\xi^m D_x^\beta(ia_j(t,x,\xi)))(D_x^{m+\alpha-\beta}e^{\Lambda}).\right]
\end{aligned}
$$

for $\psi_1\in S^{-1}$ and $\psi_{\alpha',\beta'}\in S^{\frac{\alpha'-\beta'}{2}}$.

In order to avoid further computations analogous to those already made for the estimate of A_I, we make some remarks. When the x-derivatives fall on $(\partial_\xi^\alpha e^{-\Lambda})(D_x^{m+\alpha-\beta}e^{\Lambda})$, the decay in x gets better because of Lemma 2.4, while the level in ξ decreases. When the x-derivatives fall on $\partial_\xi^m D_x^\beta(ia_j)$ the assumptions (1.14) and (1.15) on the coefficients give a decay in $\langle x\rangle$ of order $(j-\gamma(\beta+1))/(p-1)$ in the first addend of (3.30), and of order $(j-\gamma(\beta+\beta'))/(p-1)$ in the second addend of (3.30), with γ the function defined in (3.8); at the same time we have that the level in ξ decreases of 1 in the first addend of (3.30) and of $\alpha'-\frac{\alpha'-\beta'}{2}=\frac{\alpha'+\beta'}{2}$ in the second addend of (3.30). Therefore the assumptions (1.14), (1.15) on the coefficients give that $R(A_{II}|_{\mathrm{ord}(p-k)})$ satisfies the decay estimate in (3.20), since

$$(3.31)\qquad -\gamma(\beta+1)\geq\left[-\frac{\beta}{2}-1+\frac12\right]$$

$$(3.32)\qquad -\gamma(\beta+\beta')\geq\left[-\frac{\beta}{2}-\frac{\alpha'+\beta'}{2}+\frac12\right]$$

because of (3.25) with $b=\beta+1$, $a=1$ and $b=\beta+\beta'$, $a=\alpha'$ respectively. □

Proof of Theorem 1.2

The proof of Theorem 1.2 consists in choosing recursively positive constants $M_{p-1},\dots,M_1$ in such a way that

$$(3.33)\qquad \mathrm{Re}\,(e^{-\Lambda}Ae^{\Lambda})\big|_{\mathrm{ord}(p-k)}+\tilde C\geq 0$$

for some $\tilde C>0$, and applying the sharp-Gårding Theorem 2.7 to terms of order $p-2$, $p-3$, and so on, up to order 3, the Fefferman-Phong inequality to terms of order $p-k=2$ and the sharp-Gårding inequality (2.29) to terms of order $p-k=1$, finally obtaining that

$$e^{-\Lambda}Ae^{\Lambda}=ia_p(t,D_x)+\sum_{s=1}^{p}Q_{p-s}$$

with

$$\begin{aligned}
&\mathrm{Re}\langle Q_{p-s}v, v\rangle \geq 0 \qquad \forall v(t,\cdot) \in H^{p-s}, \quad s = 1, \dots, p-3 \\
&\mathrm{Re}\langle Q_{p-s}v, v\rangle \geq -c\|v\|_0^2 \qquad \forall v(t,\cdot) \in H^{p-s}, \quad s = p-2, p-1 \\
&Q_0 \in S^0.
\end{aligned}$$

At the end of the proof we will show that the result holds not only for $e^{-\Lambda}Ae^{\Lambda}$, but also for the full operator $(e^{\Lambda})^{-1}Ae^{\Lambda}$, finding a constant $c > 0$ such that

$$\mathrm{Re}\langle A_\Lambda v, v\rangle \geq -c\|v\|_0^2 \qquad \forall v(t,\cdot) \in H^\infty.$$

From this, the thesis follows by standard energy arguments.

Lemma 3.1 is fundamental to make these choices possible: it states that all terms of order $p-k$ $(1 \leq k \leq p-1)$ of the operator $e^{-\Lambda}Ae^{\Lambda}$ have the *"right decay at the right level"*, in the sense that they satisfy (3.1); the fact that the constants $C_{(M_{p-1},\dots,M_{p-k})}$ depend only on $M_{p-1}, \dots, M_{p-k}$ and not on $M_{p-k-1}, \dots, M_1$ is very important in the following in the application of the sharp-Gårding Theorem, since we shall choose $M_{p-1}, \dots, M_1$ step by step, and at each step (say "step $p-k$") we need something which depends only on the already chosen $M_{p-1}, \dots, M_{p-k+1}$ and on the new M_{p-k} that we need to choose, and not on the constants $M_{p-k-1}, \dots, M_1$ which will be chosen in the next steps.

Lemma 3.2 states that not only the terms of order $p-k$ of the operator $e^{-\Lambda}Ae^{\Lambda}$, but also remainder terms coming from an application of Theorem 2.7 have the *"right decay at the right level"* (formula (3.20)), with constants $C_{(M_{p-1},\dots,M_{p-k-s})}$ depending only on $M_{p-1}, \dots, M_{p-k-s}$ and not on $M_{p-k-s-1}, \dots, M_1$; this lets the recursive choice of the constants possible.

So, let us start with the proof.

Choice of M_{p-1}. Let us define, with the notations of Lemma 3.1,

$$\begin{aligned}
A_{p-k} :=& \ (e^{-\Lambda}Ae^{\Lambda})\big|_{\mathrm{ord}(p-k)} = A_I\big|_{\mathrm{ord}(p-k)} + A_{II}\big|_{\mathrm{ord}(p-k)} \\
=& \ A^0_{p-k} + B_{p-k} + A_{II}\big|_{\mathrm{ord}(p-k)}, \qquad k = 1, \dots, p-1.
\end{aligned} \tag{3.34}$$

We focus on the real part of A_{p-k}. From (1.12), (1.13)-(1.15), (2.5) we have

$$\begin{aligned}
\mathrm{Re}\, A^0_{p-k} =& \ -\mathrm{Im}\, a_{p-k} + \partial_x\lambda_{p-k}\partial_\xi a_p \\
=& \ M_{p-k}\langle x\rangle^{-\frac{p-k}{p-1}}\psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^{p-1}}\right)\langle\xi\rangle_h^{-k+1}\partial_\xi a_p - \mathrm{Im}\, a_{p-k} \\
\geq& \ C_pM_{p-k}\langle x\rangle^{-\frac{p-k}{p-1}}\langle\xi\rangle_h^{p-k}\psi - C\langle x\rangle^{-\frac{p-k}{p-1}}\langle\xi\rangle_h^{p-k}\psi - C\langle x\rangle^{-\frac{p-k}{p-1}}\langle\xi\rangle_h^{p-k}(1-\psi) \\
\geq& \ \psi\cdot(C_pM_{p-k} - C)\langle x\rangle^{-\frac{p-k}{p-1}}\langle\xi\rangle_h^{p-k} - C''
\end{aligned} \tag{3.35}$$

for some $C'' > 0$ since $\langle\xi\rangle_h^{p-1}/\langle x\rangle$ is bounded on the support of $(1-\psi)$. Then, from (3.35), (3.18) and (3.10):

$$\begin{aligned}
\mathrm{Re}\, A_{p-k} =& \ \mathrm{Re}(A^0_{p-k}) + \mathrm{Re}(B_{p-k}) + \mathrm{Re}(A_{II}\big|_{\mathrm{ord}(p-k)}) \\
\geq& \ \psi(C_pM_{p-k} - C)\langle x\rangle^{-\frac{p-k}{p-1}}\langle\xi\rangle_h^{p-k} - C'' - (C_k + C')\langle x\rangle^{-\frac{p-k}{p-1}}\langle\xi\rangle_h^{p-k},
\end{aligned} \tag{3.36}$$

where the constants C, C', C'', C_k depend only on $M_{p-1}, \dots, M_{p-k+1}$ and not on $M_{p-k}, \dots, M_1$.

In particular, for $k = 1$,

$$\mathrm{Re}\, A_{p-1} \geq \psi(C_pM_{p-1} - C - C_1 - C')\langle x\rangle^{-1}\langle\xi\rangle_h^{p-1} - C''$$

and we can thus choose $M_{p-1} > 0$ sufficiently large, so that

$$\mathrm{Re}\, A_{p-1}(t,x,\xi) \geq -\tilde{C} \qquad \forall (t,x,\xi) \in [0,T] \times \mathbb{R}^2$$

for some $\tilde{C} > 0$. Applying the sharp-Gårding Theorem 2.7 to $A_{p-1}+\tilde{C}$ we can thus find pseudo-differential operators $Q_{p-1}(t,x,D_x)$ and $\tilde{R}_{p-1}(t,x,D_x)$ with symbols $Q_{p-1}(t,x,\xi) \in S^{p-1}$ and $\tilde{R}_{p-1}(t,x,\xi) \in S^{p-2}$ such that

$$\begin{aligned} & A_{p-1} = Q_{p-1} + \tilde{R}_{p-1} - \tilde{C} \\ & \mathrm{Re}\langle Q_{p-1}v, v\rangle \geq 0 \qquad \forall (t,x) \in [0,T]\times\mathbb{R},\ \forall v(t,\cdot) \in H^{p-1}(\mathbb{R}) \\ & \tilde{R}_{p-1}(t,x,\xi) \sim \psi_1(\xi) D_x A_{p-1}(t,x,\xi) + \sum_{\alpha+\beta\geq 2} \psi_{\alpha,\beta}(\xi)\partial_\xi^\alpha D_x^\beta A_{p-1}(t,x,\xi) \end{aligned} \tag{3.37}$$

with $\psi_1 \in S^{-1}$, $\psi_{\alpha,\beta} \in S^{(\alpha-\beta)/2}$, $\psi_1, \psi_{\alpha,\beta} \in \mathbb{R}$.

Therefore, the first application of the sharp-Gårding Theorem 2.7 gives, because of (3.5), (3.34) and (3.37):

$$\begin{aligned} \sigma(e^{-\Lambda}Ae^{\Lambda}) = &\ ia_p + \sum_{k=1}^{p-1} A_{p-k} + A_0' = ia_p + A_{p-1} + \sum_{k=2}^{p-1} A_{p-k} + A_0' \\ = &\ ia_p + Q_{p-1} + \sum_{k=2}^{p-1}(A_I|_{\mathrm{ord}(p-k)} + A_{II}|_{\mathrm{ord}(p-k)} + \tilde{R}_{p-1}\Big|_{\mathrm{ord}(p-k)}) + A_0'' \end{aligned} \tag{3.38}$$

for some $A_0', A_0'' \in S^0$, where $\tilde{R}_{p-1}|_{\mathrm{ord}(p-k)}$ denotes the terms of order $p-k$ of $\tilde{R}_{p-1} := R(A_{p-1})$. We have thus proved that it is possible to choose $M_{p-1} > 0$ such that

$$\sigma(e^{-\Lambda}Ae^{\Lambda}) = ia_p(t,\xi) + Q_{p-1} + \sum_{k=2}^{p-1} (e^{-\Lambda}Ae^{\Lambda})\big|_{\mathrm{ord}(p-k)} + \tilde{R}_{p-1} + A_0, \tag{3.39}$$

where $Q_{p-1}(t,x,D)$ is a positive operator of order $p-1$, $\tilde{R}_{p-1}$ is a remainder of order $p-2$, and $A_0(t,x,D)$ is an operator of order zero.

Choice of $M_{p-2},\dots,M_3$**.** To iterate this process, applying the sharp-Gårding Theorem 2.7 to terms of order $p-2$, $p-3$, and so on, up to order 3, we need to investigate the action of the sharp-Gårding Theorem to each term of the form

$$(e^{-\Lambda}Ae^{\Lambda})\big|_{\mathrm{ord}(p-k)} + S_{p-k},$$

where S_{p-k} denotes terms of order $p-k$ coming from remainders of previous applications of the sharp-Gårding Theorem 2.7, for $p-k \geq 3$. Lemma 3.2 says that remainders of terms of the form $(e^{-\Lambda}Ae^{\Lambda})\big|_{\mathrm{ord}(p-k)}$ have "the right decay at the right level", in the sense of (3.20); in what follows we show that also S_{p-k} (and hence their remainders $R(S_{p-k})$) are sums of terms with "the right decay at the right level". Then we apply the sharp-Gårding Theorem 2.7 to terms of order $p-k$, up to order $p-k=3$.

To estimate S_{p-k} and then $R(S_{p-k})$ we previously need to make some remarks.

From (3.38) with $\tilde{R}_{p-1} = R(A_{p-1})$ we have

$$\begin{aligned} \sigma(e^{-\Lambda}Ae^{\Lambda}) = &\ ia_p + Q_{p-1} + R(A_{p-1}) + \sum_{k=2}^{p-1} A_{p-k} + A_0'' \\ = &\ ia_p + Q_{p-1} + A_{p-2} + R(A_{p-1})\big|_{\mathrm{ord}(p-2)} + \sum_{k=3}^{p-1}(A_{p-k} + R(A_{p-1})\big|_{\mathrm{ord}(p-k)}) + A_0''. \end{aligned}$$

From (3.36) with $k = 2$ and Lemma 3.2 with $k = 1$, we can now choose $M_{p-2} > 0$ sufficiently large so that

$$\mathrm{Re}\left(A_{p-2} + R(A_{p-1})|_{\mathrm{ord}(p-2)}\right)(t,x,\xi) \geq -\tilde{C} \qquad \forall (t,x,\xi) \in [0,T]\times\mathbb{R}^2$$

for some $\tilde{C} > 0$.

Note that A_{p-2} depends on M_{p-1} and M_{p-2}, in the sense of (3.36), while $R(A_{p-1})|_{\mathrm{ord}(p-2)}$ depends only on the already chosen M_{p-1}. Thus, by the sharp-Gårding Theorem 2.7 there exist pseudo-differential operators Q_{p-2} and $\tilde{R}_{p-2}$, with symbols in S^{p-2} and S^{p-3} respectively, such that

$$\begin{aligned} &\mathrm{Re}\langle Q_{p-2}v, v\rangle \geq 0 \qquad \forall v(t,\cdot) \in H^{p-2} \\ &A_{p-2} + R(A_{p-1})\big|_{\mathrm{ord}(p-2)} = Q_{p-2} + \tilde{R}_{p-2}, \end{aligned}$$

with

$$\tilde{R}_{p-2} = R(A_{p-2} + R(A_{p-1})\big|_{\mathrm{ord}(p-2)}) = R(A_{p-2}) + R(R(A_{p-1})\big|_{\mathrm{ord}(p-2)}),$$

so that

$$\begin{aligned} \sigma(e^{-\Lambda}Ae^{\Lambda}) =\ & ia_p + Q_{p-1} + Q_{p-2} + R(A_{p-2}) + R(R(A_{p-1})\big|_{\mathrm{ord}(p-2)}) \\ & + \sum_{k=3}^{p-1}(A_{p-k} + R(A_{p-1})\big|_{\mathrm{ord}(p-k)}) + A_0'' \\ =\ & ia_p + Q_{p-1} + Q_{p-2} \\ & + \left(A_{p-3} + R(A_{p-1})\big|_{\mathrm{ord}(p-3)} + R(A_{p-2})\big|_{\mathrm{ord}(p-3)} + R^2(A_{p-1})\big|_{\mathrm{ord}(p-3)}\right) \\ & + \sum_{k=4}^{p-1}\left(A_{p-k} + R(A_{p-1})\big|_{\mathrm{ord}(p-k)} + R(A_{p-2})_{\mathrm{ord}(p-k)} + R^2(A_{p-1})\big|_{\mathrm{ord}(p-k)}\right) + A_0''. \end{aligned}$$

To proceed analogously for the terms of order $p-3$, then $p-4$ and so on up to order 3, we thus need to estimate, for $p-k \geq 3$ and $s \geq 2$:

$$R^s(A_{p-k}) = R^s(A^0_{p-k}) + R^s(B_{p-k}) + R^s(A_{II}|_{\mathrm{ord}(p-k)}).$$

The arguments are analogous to those already made for the discussion of $R(A^0_{p-k})$, $R(B_{p-k})$ and $R(A_{II}|_{\mathrm{ord}(p-k)})$ in Lemma 3.2. Indeed, in the remainders of the sharp-Gårding Theorem 2.7 we have a first addend with some $\tilde{\psi}_1 \in S^{-1}$ and where some derivatives D_x appears and a second addend with some $\psi_{\alpha',\beta'} \in S^{\frac{\alpha'-\beta'}{2}}$ and where some derivatives $\partial_\xi^{\alpha'}D_x^{\beta'}$ appear.

When the x-derivatives fall on λ_{p-j} the decay in x gets better by (2.14), while the level in ξ decreases, so that we still have the "right decay".

When the x-derivatives fall on the coefficients then the assumptions (1.13)-(1.15) still give the "right decay" since the level in ξ decreases of $\frac{\alpha'+\beta'}{2}$ (for $\alpha' = \beta' = 1$ in the first addend) and because of (3.31) and (3.32).

Therefore, remainders coming from the sharp-Gårding Theorem 2.7 always have the "right decay".

This shows that we can apply again and again the sharp-Gårding Theorem 2.7 until we find pseudo-differential operators $Q_{p-1}, Q_{p-2}, \ldots, Q_3$ of order $p-1, p-2, \ldots, 3$ respectively and all positive definite, such that

$$\sigma(e^{-\Lambda}Ae^{\Lambda}) = ia_p + Q_{p-1} + Q_{p-2} + \ldots Q_3 + \sum_{k=p-2}^{p-1}(A_{p-k} + S_{p-k}) + \tilde{A}_0$$

for some $\tilde{A}_0 \in S^0$ and S_{p-k} coming from remainders of the sharp-Gårding theorem.

Choice of M_2 and M_1. Let us write

$$A_2 + S_2 = T_2 + iT_2'$$

with $T_2 = \mathrm{Re}(A_2 + S_2)$ and $T_2' = \mathrm{Im}(A_2 + S_2)$. As in the previous steps we choose $M_2 > 0$ such that

$$T_2 = \mathrm{Re}(A_2 + S_2) \geq 0$$

(up to a constant that we can put in $\tilde{A}_0$). Then, by the Fefferman-Phong inequality (2.30), we get that

$$\mathrm{Re}\langle T_2 v, v\rangle \geq -c\|v\|_0^2 \tag{3.40}$$

for some $c > 0$, without any remainder.

On the other hand, we write

$$iT_2' = \frac{iT_2' + (iT_2')^*}{2} + \frac{iT_2' - (iT_2')^*}{2},$$

where

$$\mathrm{Re}\langle \frac{iT_2' - (iT_2')^*}{2} u, u\rangle = 0, \tag{3.41}$$

while $iT_2' + (iT_2')^*$ has a real principal part of order 1, has the "right decay" and does not depend on M_1. Therefore we can choose $M_1 > 0$ sufficiently large so that

$$\mathrm{Re}\left(\frac{iT_2' + (iT_2')^*}{2} + A_1 + S_1\right) \geq 0$$

and hence, by the sharp-Gårding inequality (2.29) for $m = 1$,

$$\mathrm{Re}\langle\left(\frac{iT_2' + (iT_2')^*}{2} + A_1 + S_1\right) v, v\rangle \geq -c\|v\|_0^2. \tag{3.42}$$

By (3.40), (3.41) and (3.42) we finally get

$$\sigma(e^{-\Lambda} A e^{\Lambda}) = ia_p + \sum_{s=1}^{p-3} Q_{p-s} + (A_2 + S_2) + (A_1 + S_1) + \tilde{A}_0$$

with

$$\mathrm{Re}\langle Q_{p-s} v, v\rangle \geq 0 \qquad \forall v(t,\cdot) \in H^{p-s},\ s = 1, 2, \ldots, p-3$$
$$\mathrm{Re}\langle (A_2 + S_2 + A_1 + S_1) v, v\rangle \geq -c\|v\|_0^2 \qquad \forall v(t,\cdot) \in H^2.$$

Estimates for the operator A_Λ. We finally look at the full operator A_Λ in (2.23); by (2.24), (2.25) we notice that $A^{n,m}$ is of the same kind of A with $\partial_\xi^m r^n D_x^m a_j$ instead of a_j. This implies that we have m more x-derivatives on a_j, but the level in ξ decreases of $-n-m < -m$, so that we argue as for $\sigma(e^{-\Lambda} A e^{\Lambda})$ and find that also

$$\sigma(e^{-\Lambda} A^{n,m} e^{\Lambda}) = \sum_{s=0}^{p} Q_{p-s}^{n,m}$$

with $Q_0^{n,m} \in S^0$ and

$$\mathrm{Re}\langle Q_{p-s}^{n,m} v, v\rangle \geq -C_{n,m}\|v\|_0^2 \qquad \forall v(t,\cdot) \in H^{p-s}\ 1 \leq s \leq p-1$$

for some $C_{n,m} > 0$.

Since every $Q \in S^0$ also satisfies

$$\mathrm{Re}\langle Qv, v\rangle \geq -c\|v\|_0^2 \qquad \forall v \in H^0$$

for some $c > 0$, by Lemma 2.5 we finally have that

$$\operatorname{Re}\langle A_\Lambda v, v\rangle \geq -c\|v\|_0^2 \qquad \forall v(t,\cdot) \in H^\infty \tag{3.43}$$

for some $c > 0$, and hence if $v \in C([0,T]; L^2)$ is a solution of (2.3), by (2.2) with A_Λ instead of A we get that

$$\begin{aligned}\frac{d}{dt}\|v\|_0^2 \leq\ & \|f_\Lambda\|_0^2 + \|v\|_0^2 - 2\operatorname{Re}\langle A_\Lambda v, v\rangle \\ \leq\ & (2c+1)(\|f_\Lambda\|_0^2 + \|v\|_0^2).\end{aligned}$$

By standard arguments we deduce that, for all $s \in \mathbb{R}$, if $v \in C([0,T]; H^s)$,

$$\|v(t,\cdot)\|_s^2 \leq c'\left(\|g_\Lambda\|_s^2 + \int_0^t \|f_\Lambda(\tau,\cdot)\|_s^2 d\tau\right) \qquad \forall t \in [0,T], \tag{3.44}$$

for some $c' > 0$.

Since $e^{\pm\Lambda} \in S^\delta$, for $u = e^\Lambda v$ we finally have, from (3.44) with $s - \delta$ instead of s:

$$\begin{aligned}\|u\|_{s-2\delta}^2 \leq\ & c_1\|v\|_{s-\delta}^2 \leq c_2\left(\|g_\Lambda\|_{s-\delta}^2 + \int_0^t \|f_\Lambda\|_{s-\delta}^2 d\tau\right) \\ \leq\ & c_3\left(\|g\|_s^2 + \int_0^t \|f\|_s^2 d\tau\right)\end{aligned}$$

for some $c_1, c_1, c_3 > 0$.

This proves the existence of a solution $u \in C([0,T]; H^\infty(\mathbb{R}))$ of (1.16) which satisfies (1.17) for $\sigma = 2\delta = 2(p-1)M_{p-1}$. □

Remark 3.3. For the choice of $M_{p-1}, \ldots, M_3$ we made use of the sharp-Gårding Theorem 2.7 obtaining, at each step, a new remainder given by (2.28). On the contrary, for the choice of M_2 and M_1 we made use of, respectively, the Fefferman-Phong inequality (2.30) and the sharp-Gårding inequality (2.29), where no new remainders appear. This lets us save some conditions on the coefficients a_1 and a_2, for which we required, indeed, only conditions (1.13) and (1.13)-(1.14) respectively, in the statement of Theorem 1.2.

4. Energy estimate for systems: proof of Theorem 1.1

Let us now consider the operator L in (1.1) and the transformed operator $L_\Lambda := (e^\Lambda)^{-1} L e^\Lambda$, for Λ defined by (2.4), (2.5):

$$\begin{aligned}L_\Lambda =\ & (e^\Lambda)^{-1} D_t e^\Lambda + (e^\Lambda)^{-1}\begin{pmatrix}\mu_1 & & \\ & \ddots & \\ & & \mu_m\end{pmatrix} e^\Lambda + (e^\Lambda)^{-1} R e^\Lambda \\ =\ & D_t + \begin{pmatrix}(e^\Lambda)^{-1}\mu_1 e^\Lambda & & \\ & \ddots & \\ & & (e^\Lambda)^{-1}\mu_m e^\Lambda\end{pmatrix} + R_\Lambda\end{aligned}$$

with $R_\Lambda(t, x\xi) \in S^0$. Setting

$$A_\Lambda = \begin{pmatrix}A_1 & & \\ & \ddots & \\ & & A_m\end{pmatrix}, \qquad A_j = i(e^\Lambda)^{-1}\mu_j e^\Lambda, \quad 1 \leq j \leq m$$

we can thus write

$$L_\Lambda = D_t - iA_\Lambda + R_\Lambda.$$

As is §2 we substitute the Cauchy problem (1.9) by

$$\begin{cases} L_\Lambda V(t,x) = F_\Lambda(t,x) & (t,x)\in[0,T]\times\mathbb{R} \\ V(0,x) = G_\Lambda(x) & x\in\mathbb{R} \end{cases} \tag{4.1}$$

for $F_\Lambda = (e^\Lambda)^{-1}F$ and $G_\Lambda = (e^\Lambda)^{-1}G$.

Proving the energy estimate for V we can then deduce the energy estimate for $U = e^\Lambda V$ solution of (1.9). For a solution V of (4.1) we have:

$$\begin{aligned} \frac{d}{dt}|||V|||_0^2 =&\ 2\,\mathrm{Re}\langle\langle V', V\rangle\rangle = 2\,\mathrm{Re}\langle\langle iF_\Lambda, V\rangle\rangle - 2\,\mathrm{Re}\langle\langle A_\Lambda V, V\rangle\rangle - 2\,\mathrm{Re}\langle\langle iR_\Lambda V, V\rangle\rangle \\ \le&\ C(|||F_\lambda|||_0^2 + |||V|||_0^2) - 2\,\mathrm{Re}\langle\langle A_\Lambda V, V\rangle\rangle \end{aligned} \tag{4.2}$$

for some $C > 0$, where for given vectors $U = (U_1, \dots, U_m)$ and $V = (V_1, \dots, V_m)$ we denote $\langle\langle U, V\rangle\rangle := \sum_{j=1}^m \langle U_j, V_j\rangle$. Note that every A_j is of the same form as (2.23), so that by (3.43):

$$\mathrm{Re}\langle\langle A_\Lambda V, V\rangle\rangle = \sum_{j=1}^m \mathrm{Re}\langle A_j V_j, V_j\rangle \ge -c\sum_{j=1}^m \|V_j\|_0^2 = -c|||V|||_0^2.$$

Substituting in (4.2) we obtain, by standard arguments, the energy estimate for V

$$|||V(t,\cdot)|||_s^2 \le C\left(|||V(0)|||_s^2 + \int_0^t |||F_\Lambda(\tau,\cdot)|||_s^2 d\tau\right)$$

for some $C > 0$, and hence the desired energy estimate for $U = e^\Lambda V$:

$$\begin{aligned} |||U(t,\cdot)|||_{s-2\delta}^2 =&\ |||e^\Lambda V|||_{s-2\delta}^2 \le C_1|||V|||_{s-\delta}^2 \\ \le&\ C_2\left(|||V(0)|||_{s-\delta}^2 + \int_0^t |||F_\Lambda(\tau,\cdot)|||_{s-\delta}^2 d\tau\right) \\ \le&\ C_3\left(|||U(0)|||_s^2 + \int_0^t |||F(\tau,\cdot)|||_s^2 d\tau\right) \end{aligned}$$

for some $C_1, C_2, C_3 > 0$, since $e^\Lambda \in S^\delta$.

This concludes the proof of Theorem 1.1. □

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