Heat Convection of Compressible Viscous Fluids. I.

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Abstract. The stationary problem for the heat convection of compressible fluid is considered around the equilibrium solution with the external forces in the horizontal strip domain $z_0 < z < z_0 + 1$ and it is proved that the solution exists uniformly with respect to $z_0 \geq Z_0$. The limit system as $z_0 \to +\infty$ is the Oberbeck-Boussinesq equations.

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1. Introduction

As well known compressible viscous and heat conductive fluid flows, in a region of space $\Omega \subset R^n$, $n = 2, 3$, are described by the following Navier-Stokes-Fourier system of equations, $(x_1, \ldots, x_n) \in \Omega$, $t \in (0, T)$:
The mass conservation :

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u^j)}{\partial x_j} = 0 ,$$

The momentum conservation for $i = 1, \ldots, n$:

$$\frac{\partial(\rho u^i)}{\partial t} + \frac{\partial(\rho u^i u^j + p\delta^{ij})}{\partial x_j} = \frac{\partial}{\partial x_j}\{\mu(\frac{\partial u^i}{\partial x_j} + \frac{\partial u^j}{\partial x_i})\} + \frac{\partial}{\partial x_i}\{\mu'(\frac{\partial u^k}{\partial x_k})\} + \rho f_i ,$$

The energy conservation :

$$\frac{\partial(\rho E)}{\partial t} + \frac{\partial(\rho E u^j + p u^j)}{\partial x_j} = \frac{\partial}{\partial x_j}\{\kappa\frac{\partial T}{\partial x_j} + \mu u^k(\frac{\partial u^k}{\partial x_j} + \frac{\partial u^j}{\partial x_k}) + \mu' u^j(\frac{\partial u^k}{\partial x_k})\} + \rho u_j f_j .$$

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In the above equations $\mathbf{u} = (u_1, ..., u_n)$ is the velocity of fluid, E is the total energy, ρ is the density, p is the pressure, $\mathbf{f} = (f_1, ..., f_n)$ is the external force per unit of mass, and μ, μ', k are the viscosity and heat diffusivity coefficients.

The Benard equilibrium solution describes the rest state of a liquid with suitable positive density, and temperature distributions, filling a horizontal layer having the upper plane $x_n = 0$ and the lower plane $x_n = d$, when a positive temperature gradient is prescribed at the planes. Here we use the orthonormal basis $\{\mathbf{e}_1, ..., \mathbf{e}_n\}$ in $\mathbf{R}^n$, $n = 2$ or 3 , and $\mathbf{e}_n$ is considered in the vertically downward direction. The rest state exists for any potential external force, however it is observed only for "small" finite temperature gradients.

The Benard problem concerns the study of onset of convection for the equilibrium state with large potential forces and nonhomogeneous temperature at the boundary. Precisely, by injecting energy into the system, typically the *equilibrium state becomes unstable* above a certain threshold and, as result of this instability, *well-defined space-time structures emerge.* [1], [2], [3] The study of the *critical threshold* and of the *form of emerging structures* constitutes the *Benard*, also called *Rayleigh-Benard problem*, it started in nineteen century. [4]

The Benard problem is quite complicate, thus one creates thermodynamically consistent, *mathematical models* of non-isothermal flow of isotropically thermally expansible Newtonian fluid in the presence of gravity. Between the most studied models we quote the **Boussinesq**, also called *Oberbeck-Boussinesq model*, that describes the thermodynamical response of a linearly viscous fluid that can sustain motions: *mechanically incompressible, thermally compressible.* The Boussinesq model receives good agreement with experiments for thin layers. *Our aim* is to propose an *approximating model* to Benard problem as alternative to the **Boussinesq** *model*, [5], [6], [7], [9], [10]. Still in this first series of papers we just consider thin layers and construct an alternate model called **Benard-Boussineq model**. The alternative is naturally suggested by rewriting the full system of equations considering as independent variables the velocity, the temperature and the pressure.

We proceed by treating the following steps:

(a) **the existence and uniqueness of a solution** for the heat conducting compressible linear and nonlinear steady system when basic unknown functions are velocity, temperature and pressure;

(b) **the comparison** between the critical Rayleigh numbers of compressible Benard problem $\mathcal{R}_C$ and the classical one $\mathcal{R}_B$ of Boussinesq approximation, and the study of eigenvalues for the linearized system; c.f. [5]

(c) **the comparison** between the critical Rayleigh number of compressible Benard-Boussineq problem $\mathcal{R}_{BB}$ and the classical one $\mathcal{R}_B$ of Boussinesq approximation, and the comparison of the corresponding energies;

(d) **the existence and uniqueness of a solution** for the heat conducting compressible linear and nonlinear unsteady system when basic unknown functions are velocity, temperature and pressure.

In this paper we confine our attention to step(a), and we study the existence and uniqueness of a solution for the heat conducting compressible linear and nonlinear steady system when basic unknown functions are velocity, temperature and pressure. This choice is made in order to compare a new model called compressible Benard-Boussineq model with the incompressible Boussineq model. Concerning existence theorems we quote the papers by Matsumura Nishida [8] where an analogous result has been obtained assuming smallness on the external forces, and those by Mucha, Novotny, Pokorny [14], [15], [13], where potential forces may be large however theorems of existence of regular solutions require a constitutive equation for the pressure that differs from the classical one of ideal fluids. Indeed our result may result interesting because it fills this gap of large potential forces and classical pressure law in the case of uniqueness of the rest state, and for these layers. In a paper in preparation it will be shown an existence theorem of steady compressible flows, considering the pressure for ideal fluids, in the general case of non homogeneous boundary conditions for the temperature, large potential forces and small non potential forces and small heat sources [18].
The plan of the paper is the following: in section 2 we reduce the system into a suitable non-dimensional form where we distinguish between linear and nonlinear terms; in section 3 we prove the main theorem.

2. Reduction

We follow Spiegel's dimensionalization with the vertical axis ($\boldsymbol{e}_3$) pointing downward and set

$$\mathcal{R}^2 = \frac{P^2 \beta R_* c_p (m+1)^3 d^{2m+3}}{g^2 \mu \kappa}, \qquad z_0 = \frac{T_u}{\beta_0 d}, \quad \beta_0 = \frac{T_l - T_u}{d},$$

where d is the width of the layer. ([5], [7], [10]). The non-dimensional system is the following in the horizontal domain $z_0 < z < z_0 + 1$

$$\frac{\partial \rho}{\partial t} + \mathcal{R} \nabla \cdot (\rho \boldsymbol{u}) = 0,$$

$$\frac{1}{\mathcal{P}_r} \rho \frac{\partial \boldsymbol{u}}{\partial t} - \Delta \boldsymbol{u} - \frac{1}{3} \nabla(\nabla \cdot \boldsymbol{u}) + \frac{\mathcal{R}}{b\gamma(m+1)} \nabla p + \frac{\mathcal{R}}{\mathcal{P}_r} \rho \boldsymbol{u} \cdot \nabla \boldsymbol{u} = \frac{\mathcal{R}}{b\gamma} \rho \boldsymbol{e}_3,$$

$$\rho \frac{\partial T}{\partial t} - \Delta T + \mathcal{R}(\gamma - 1) p \nabla \cdot \boldsymbol{u} + \mathcal{R} \rho \boldsymbol{u} \cdot \nabla T = \frac{2 g b \gamma}{\beta_0 c_v} \{ D : D - \frac{1}{3} (\nabla \cdot \boldsymbol{u})^2 \}.$$

The equilibrium solution heated from below in the horizontal strip domain $z_0 < z < z_0 + 1$ is the stratified heat conduction state :

$$\overline{\boldsymbol{u}} = \boldsymbol{0}, \quad \overline{\rho} = z^m, \quad \overline{T} = z, \quad \overline{p} = z^{m+1},$$

where and hereafter we use the state equation of fluid $p = \rho T$.

We consider the stationary problem around the equilibrium solution with the external forces in the horizontal strip domain, where the Dirichlet zero boundary condition is supposed to the velocity and the temperature on the horizontal boundaries $z = z_0 , z_0 + 1$.
The mass conservation :

$$\mathcal{R}\nabla \cdot (\rho_* \boldsymbol{u}_*) = 0 ,$$

The momentum conservation :

$$-\Delta \boldsymbol{u}_* - \frac{1}{3}\nabla(\nabla \cdot \boldsymbol{u}_*) + \frac{\mathcal{R}}{b\gamma(m+1)}\nabla p_* + \frac{\mathcal{R}}{\mathcal{P}_r}\rho_* \boldsymbol{u}_* \cdot \nabla \boldsymbol{u}_* = \frac{\mathcal{R}}{b\gamma}\rho_* \boldsymbol{e}_3 + \mathcal{R}\rho_* \boldsymbol{f_e} ,$$

The energy conservation :

$$-\Delta T_* + \mathcal{R}(\gamma - 1) p_* \nabla \cdot \boldsymbol{u}_* + \mathcal{R}\rho_* \boldsymbol{u}_* \cdot \nabla T_* = \frac{2\, g\, b\, \gamma}{\beta_0\, c_v}\{ D : D - \frac{1}{3}(\nabla \cdot \boldsymbol{u}_*)^2\} + \mathcal{R}\rho_* h_e .$$

The boundary conditions :

$$\boldsymbol{u}_*(z_0) = \boldsymbol{u}_*(z_0+1) = \mathbf{0} , \quad T_*(z_0) = z_0 , \quad T_*(z_0+1) = z_0 + 1 .$$

and the side condition : $\int \rho_* = \mathit{constant}$.

We use the unknown variables $p_*, \boldsymbol{u}_* = (u, v, w), T_*$ and rewrite the mass conservation for the perturbation from the equilibrium state by

$$\boldsymbol{u}_* \to \boldsymbol{u} , \quad p_* \to z^{m+1} + p , \quad T_* \to z + \theta , \quad \rho_* \to z^m + \rho ,$$

where $p = (z+\theta)\rho + z^m \theta$, i.e., $\rho = (p - z^m \theta)/(z + \theta)$:

$$\mathcal{R}\nabla \cdot ((z^m + \rho)\boldsymbol{u}) = \mathcal{R}\nabla \cdot \left(z^m \boldsymbol{u} + \frac{p - z^m \theta}{z + \theta}\boldsymbol{u} \right) = 0 .$$

Then we have for the unknowns $p , \boldsymbol{u} = (u, v, w) , \theta$

$$\mathcal{R}\left(\nabla \cdot \boldsymbol{u} + \frac{m\, w}{z} + \nabla \cdot (\frac{p\,\boldsymbol{u}}{z^m (z+\theta)}) \right) = \mathcal{R}\left((\nabla \cdot (\frac{\theta\,\boldsymbol{u}}{z+\theta}) - \frac{m}{z^{m+1}}\frac{p\, w}{(z+\theta)} + \frac{m}{z}\frac{\theta\, w}{(z+\theta)} \right) .$$

Namely we have the following mass conservation equation after a decomposition of the third term in the left-hand side

$$\mathcal{R}\left(\nabla \cdot \boldsymbol{u} + \frac{m\, w}{z} + \nabla \cdot (\frac{p\,\boldsymbol{u}}{z^{m+1}}) \right) = \mathcal{R}\, g ,$$

where

$$
\begin{aligned}
g &= \frac{z^{m+1} + p}{z^{m+1}(z+\theta)} \, \boldsymbol{u} \cdot \nabla \theta - \frac{m\,p\,w}{z^{m+2}} - \frac{z^{m+1} + p}{z^{m+2}(z+\theta)} \, \theta \, w \\
&= \frac{1}{z} \boldsymbol{u} \cdot \nabla \theta + g_0 \;, \\
g_0 &= \frac{\theta}{z(z+\theta)} \boldsymbol{u} \cdot \nabla \theta + \frac{p}{z^{m+1}(z+\theta)} \, \boldsymbol{u} \cdot \nabla \theta - \frac{m\,p\,w}{z^{m+2}} \\
&\qquad - \frac{z^{m+1} + p}{z^{m+2}(z+\theta)} \, \theta \, w \;.
\end{aligned}
$$

The momentum equation is the following :

$$
\begin{aligned}
&-\Delta \boldsymbol{u} - \frac{1}{3} \nabla (\nabla \cdot \boldsymbol{u}) + \frac{\mathcal{R}}{b\gamma(m+1)} \nabla p - \frac{\mathcal{R}}{b\,\gamma} \frac{p - z^m \theta}{z} \boldsymbol{e}_3 \\
&\qquad\qquad = - \frac{\mathcal{R}}{\mathcal{P}_r} z^m \, \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \boldsymbol{f} \;,
\end{aligned}
$$

where

$$
\begin{aligned}
\boldsymbol{f} &= - \frac{\mathcal{R}}{\mathcal{P}_r} \frac{p - z^m \theta}{z+\theta} \, \boldsymbol{u} \cdot \nabla \boldsymbol{u} - \frac{\mathcal{R}}{b\,\gamma} \frac{p - z^m \theta}{z+\theta} \frac{\theta}{z} \boldsymbol{e}_3 \\
&\qquad + \mathcal{R} \left(z^m + \frac{p - z^m \theta}{z+\theta} \right) \boldsymbol{f_e} \;.
\end{aligned}
$$

The energy equation is the following :

$$
-\Delta \theta + \mathcal{R}(\gamma-1) \, z^{m+1} \nabla \cdot \boldsymbol{u} + \mathcal{R} \, z^m \, w = - \mathcal{R} \, z^m \, \boldsymbol{u} \cdot \nabla \theta + h' \;,
$$

$$
\begin{aligned}
h' &= - \mathcal{R} \, \rho \, \boldsymbol{u} \cdot \nabla \theta - \mathcal{R}(\gamma - 1) \, p \nabla \cdot \boldsymbol{u} - \mathcal{R} \rho w \\
&\quad + \frac{2gb\gamma}{\beta_0 c_v} \left(\boldsymbol{D} : \boldsymbol{D} - \frac{1}{3} (\nabla \cdot \boldsymbol{u})^2 \right) + \mathcal{R} \left(z^m + \rho \right) h_e \;.
\end{aligned}
$$

The energy equation can be rewritten by the mass conservation in the following form :

$$
-\Delta \theta + \mathcal{R} \left(1 - m(\gamma-1) \right) z^m \, w = - \mathcal{R} \, \gamma \, z^m \, \boldsymbol{u} \cdot \nabla \theta + h \;,
$$

where

$$
\begin{aligned}
h &= \mathcal{R} \gamma \frac{z^m \, \theta}{z+\theta} \, \boldsymbol{u} \cdot \nabla \theta - \mathcal{R} \gamma \frac{p}{z+\theta} \, \boldsymbol{u} \cdot \nabla \theta \\
&\quad - \mathcal{R}(\gamma-1) \left(\nabla \cdot (p \boldsymbol{u}) + p \nabla \cdot \boldsymbol{u} + (m+1) \frac{p\,w}{z} + z^{m+1} g_0 \right) \\
&\qquad - \mathcal{R} \frac{(p - z^m \theta) \, w}{z+\theta} + \frac{2gb\gamma}{\beta_0 c_v} \left(\boldsymbol{D} : \boldsymbol{D} - \frac{1}{3} (\nabla \cdot \boldsymbol{u})^2 \right) \\
&\qquad\qquad + \mathcal{R} \left(z^m + \frac{p - z^m \theta}{z+\theta} \right) h_e \;.
\end{aligned}
$$

We are considering the system in the horizontal domain $z_0 < z < z_0 + 1$, and we use the following scale for the velocity and the pressure, where

$$L = z_0 + \frac{1}{2} .$$

$$\boldsymbol{u} = \frac{\tilde{\boldsymbol{u}}}{\sqrt{L}} , \quad p = L^{m-1} \tilde{p} , \qquad \boldsymbol{f_e} = \frac{\tilde{\boldsymbol{f}}_{\boldsymbol{e}}}{L} , \quad h_e = \frac{\tilde{h}_e}{\sqrt{L}}.$$

Then we have the following system in the horizontal strip $L - \frac{1}{2} < z < L + \frac{1}{2}$:

$$\mathcal{R} \left(\nabla \cdot \tilde{\boldsymbol{u}} + \nabla \cdot (\frac{L^{m-1}}{z^{m+1}} \tilde{p}\, \tilde{\boldsymbol{u}}) + \frac{m\, \tilde{w}}{z} \right) = \mathcal{R}\, \tilde{g} ,$$

$$- \Delta \tilde{\boldsymbol{u}} - \frac{1}{3} \nabla(\nabla \cdot \tilde{\boldsymbol{u}}) + \frac{\mathcal{R}\, L^{m-\frac{1}{2}}}{b\gamma(m+1)} \nabla \tilde{p} - \frac{\mathcal{R}\, L^{m-\frac{1}{2}}}{b\,\gamma} \frac{\tilde{p}}{z} \boldsymbol{e}_3$$
$$+ \frac{\mathcal{R}}{b\,\gamma} \sqrt{L}\, z^{m-1}\, \theta\, \boldsymbol{e}_3 = - \frac{\mathcal{R}}{\mathcal{P}} \frac{z^m}{\sqrt{L}} \tilde{\boldsymbol{u}} \cdot \nabla \tilde{\boldsymbol{u}} + \tilde{\boldsymbol{f}} ,$$

$$- \Delta \theta + \mathcal{R} \left(1 - m(\gamma - 1) \right) \frac{z^m}{\sqrt{L}} \tilde{w} = - \mathcal{R}\, \gamma \frac{z^m}{\sqrt{L}} \tilde{\boldsymbol{u}} \cdot \nabla \theta + \tilde{h} ,$$

where

$$\tilde{g} = \frac{1}{z} \tilde{\boldsymbol{u}} \cdot \nabla \theta + \tilde{g}_0 ,$$

$$\tilde{g}_0 = - \frac{\theta\, \tilde{\boldsymbol{u}} \cdot \nabla \theta}{z\,(z+\theta)} + \frac{L^{m-1}}{z^{m+1}} \frac{\tilde{p}\, \tilde{\boldsymbol{u}} \cdot \nabla \theta}{(z+\theta)} - \frac{\theta\, \tilde{w}}{z(z+\theta)}$$
$$- \frac{m\, L^{m-1}\, \tilde{p}\, \tilde{w}}{z^{m+2}} - \frac{L^{m-1}\, \tilde{p}\, \tilde{w}\, \theta}{z^{m+2}\,(z+\theta)} ,$$

$$\tilde{\boldsymbol{f}} = \frac{\mathcal{R}}{\mathcal{P}} \frac{z^m\, \theta}{\sqrt{L}\,(z+\theta)} \tilde{\boldsymbol{u}} \cdot \nabla \tilde{\boldsymbol{u}} - \frac{\mathcal{R}}{\mathcal{P}} \frac{L^{m-\frac{3}{2}}\, \tilde{p}}{(z + \theta)} \tilde{\boldsymbol{u}} \cdot \nabla \tilde{\boldsymbol{u}}$$
$$- \frac{\mathcal{R}}{b\,\gamma} \frac{L^{m-\frac{1}{2}}\, \theta}{(z+\theta)} \left(\frac{\tilde{p}}{z} - \frac{z^{m-1}}{L^{m-1}} \theta \right) \boldsymbol{e}_3$$
$$+ \mathcal{R}\, L^{m-\frac{1}{2}} \left((\frac{z}{L})^m \frac{z}{z+\theta} + \frac{\tilde{p}}{L\,(z+\theta)} \right) \tilde{\boldsymbol{f}}_{\boldsymbol{e}} ,$$

$$\tilde{h} = \frac{\mathcal{R}\,\gamma}{\sqrt{L}} \frac{z^m\, \theta}{z+\theta} \tilde{\boldsymbol{u}} \cdot \nabla \theta - \mathcal{R}\,\gamma \frac{L^{m-\frac{3}{2}}\, \tilde{p}}{z + \theta} \tilde{\boldsymbol{u}} \cdot \nabla \theta$$
$$- \mathcal{R}(\gamma - 1)\, L^{m-\frac{3}{2}} \left(\nabla \cdot (\tilde{p}\, \tilde{\boldsymbol{u}}) + \frac{m+1}{z} \tilde{p}\, \tilde{w} + \tilde{p} \nabla \cdot \tilde{\boldsymbol{u}} + L\, (\frac{z}{L})^{m+1} (L\, \tilde{g}_0) \right)$$
$$- \frac{\mathcal{R}}{\sqrt{L}} \frac{z^m\, \theta\, \tilde{w}}{(z+\theta)} - \mathcal{R}\, L^{m-\frac{1}{2}} \frac{\tilde{p}\, \tilde{w}}{L\,(z+\theta)} + \frac{2gb\gamma}{\beta_0 c_v\, L} \left(\boldsymbol{D} : \boldsymbol{D} - \frac{1}{3} (\nabla \cdot \tilde{\boldsymbol{u}})^2 \right)$$
$$- \mathcal{R}\, L^{m-\frac{1}{2}} \left(\frac{\tilde{p}}{L\,(z + \theta)} - \frac{z}{z+\theta} (\frac{z}{L})^m \right) \tilde{h}_e .$$

We use the following Rayleigh number which was pointed out by Spiegel [5]

$$\mathcal{R}_m = \mathcal{R}\, L^{m-\frac{1}{2}} ,$$

then we have the following system that we want to solve

$$\mathcal{R}_m \left(\nabla \cdot \tilde{\boldsymbol{u}} + \frac{m\,\tilde{w}}{z} + \nabla \cdot \left((\frac{L}{z})^{m-1} \frac{\tilde{p}\,\tilde{\boldsymbol{u}}}{z^2} \right) \right) = \mathcal{R}_m\, \tilde{\tilde{g}} , \qquad (2.1)$$

$$\begin{aligned} - \Delta \tilde{\boldsymbol{u}} - \frac{1}{3} \nabla(\nabla \cdot \tilde{\boldsymbol{u}}) + \frac{\mathcal{R}_m}{b\gamma(m+1)} \nabla \tilde{p} - \frac{\mathcal{R}_m}{b\gamma} \frac{\tilde{p}}{z} \boldsymbol{e}_3 \\ + \frac{\mathcal{R}_m}{b\,\gamma} (\frac{z}{L})^{m-1} \theta\, \boldsymbol{e}_3 = - \frac{\mathcal{R}_m}{\mathcal{P}} (\frac{z}{L})^m\, \tilde{\boldsymbol{u}} \cdot \nabla \tilde{\boldsymbol{u}} + \tilde{\tilde{\boldsymbol{f}}} , \qquad (2.2) \end{aligned}$$

$$- \Delta \theta + \mathcal{R}_m (1 - m(\gamma - 1)) (\frac{z}{L})^m\, \tilde{w} = - \mathcal{R}_m\, \gamma\, (\frac{z}{L})^m\, \tilde{\boldsymbol{u}} \cdot \nabla \theta + \tilde{\tilde{h}} , (2.3)$$

where

$$\begin{aligned} \tilde{\tilde{g}} &= \frac{\tilde{\boldsymbol{u}} \cdot \nabla \theta}{z} + \tilde{\tilde{g}}_0 \\ \tilde{\tilde{g}}_0 &= \frac{\theta\, \tilde{\boldsymbol{u}} \cdot \nabla \theta}{z\,(z+\theta)} + (\frac{L}{z})^{m-1} \frac{\tilde{p}\,\tilde{\boldsymbol{u}} \cdot \nabla \theta}{z^2\,(z+\theta)} - \frac{\theta\,\tilde{w}}{z(z+\theta)} \\ &\quad - (\frac{L}{z})^{m-1} \frac{m\,\tilde{p}\,\tilde{w}}{z^3} - (\frac{L}{z})^{m-1} \frac{\tilde{p}\,\theta\,\tilde{w}}{z^3\,(z+\theta)} , \end{aligned}$$

$$\begin{aligned} \tilde{\tilde{\boldsymbol{f}}} &= \frac{\mathcal{R}_m}{\mathcal{P}} ((\frac{z}{L})^m \frac{\theta}{(z+\theta)} - \frac{\tilde{p}}{L\,(z+\theta)})\, \tilde{\boldsymbol{u}} \cdot \nabla \tilde{\boldsymbol{u}} \\ &\quad - \frac{\mathcal{R}_m}{b\,\gamma} \frac{\tilde{p}\,\theta}{z\,(z+\theta)} \boldsymbol{e}_3 + \frac{\mathcal{R}_m}{b\,\gamma} (\frac{z}{L})^{m-1} \frac{\theta^2}{(z+\theta)} \boldsymbol{e}_3 \\ &\quad + \mathcal{R}_m ((\frac{z}{L})^m \frac{z}{z+\theta} + \frac{\tilde{p}}{L\,(z+\theta)})\, \tilde{\boldsymbol{f}}_{\boldsymbol{e}} , \end{aligned}$$

$$\begin{aligned} \tilde{\tilde{h}} = &\ \mathcal{R}_m\, \gamma\, ((\frac{z}{L})^m \frac{\theta}{z+\theta} - \frac{\tilde{p}}{L\,(z+\theta)})\, \tilde{\boldsymbol{u}} \cdot \nabla \theta \\ &- \mathcal{R}_m \frac{\gamma - 1}{L} \left(\nabla \cdot (\tilde{p}\,\tilde{\boldsymbol{u}}) + \frac{m+1}{z} \tilde{p}\,\tilde{w} + \tilde{p}\, \nabla \cdot \tilde{\boldsymbol{u}} \right) \\ &+ \mathcal{R}_m \left((\frac{z}{L})^m \frac{\theta\,\tilde{w}}{(z+\theta)} - \frac{\tilde{p}\,\tilde{w}}{L\,(z+\theta)} \right) + \mathcal{R}_m\, (\gamma - 1)\, (\frac{z}{L})^{m+1} (L\, \tilde{g}_0) \\ &+ \mathcal{R}_m \left(\frac{\tilde{p}}{L\,(z+\theta)} - (\frac{z}{L})^m \frac{z}{z+\theta} \right) \tilde{h}_e \\ &+ \frac{2gb\gamma}{\beta_0 c_v\, L} \left(\boldsymbol{D} : \boldsymbol{D} - \frac{1}{3} (\nabla \cdot \tilde{\boldsymbol{u}})^2 \right) . \end{aligned}$$

We impose the boundary conditions as follows.

$$\tilde{\boldsymbol{u}} = \boldsymbol{0} , \quad \tilde{\theta} = 0 \quad \text{on} \quad z = L - \frac{1}{2} ,\ L + \frac{1}{2} , \qquad (2.4)$$

and the periodic boundary condition with respect to x and y with the period $2\pi/a$ and $2\pi/b$ respectively.

Remark 2.1 It is worth of notice an important property of system (2.1), (2.2) and (2.3). We are considering the domain

$$\left| \frac{z}{L} - 1 \right| \le \frac{1}{2L}$$

and it is clear which terms of the system remain as the limit of $L \to \infty$.

3. Existence

We aim to prove existence of a regular steady solution to heat-conducting compressible fluids (2.1)-(2.4) in correspondence of small external forces $\tilde{\boldsymbol{f}}_{\boldsymbol{e}}$, $\tilde{h}_e$ for $L \ge L_0$ and for $0 \le \mathcal{R}_m < \mathcal{R}_c$. A difficulty to construct the solution is the treatment of a derivative-loss of the third nonlinear term on the left hand side of the mass conservation law (2.1). In order to overcome the difficulty we follow the arguments of Heywood and Padula [11] and Bause, Heywood, Novotny and Padula [12].

We use the Hilbert-Sobolev spaces $H^l = W^{l,2}$, $l = 0, 1, 2, 3$ with the Dirichelet boundary conditions and the periodic boundary conditions. Hereafter we omit the tilde of the above system of equations (2.1)–(2.3) and (2.4).

Theorem

There exist constants L_0 and $\mathcal{R}_c$ such that for any $L \ge L_0$, $0 \le \mathcal{R}_m < \mathcal{R}_c$ and for small external forces there exist the solutions for (2.1)-(2.3), (2.4)*, and that the solutions converge to the solutions of Boussinesq equations as $L \to \infty$.*

We consider the resolvent system for the steady linearized equations of (2.1), (2.2), (2.3) in the following form.

$$\lambda \frac{\pi}{z^m} + \mathcal{R}_m \left(\nabla \cdot \boldsymbol{u} + \frac{m\,w}{z} \right) = G_l \, , \tag{3.1}$$

$$\lambda \frac{\boldsymbol{u}}{\mathcal{P}} - \Delta \boldsymbol{u} - \frac{1}{3} \nabla (\nabla \cdot \boldsymbol{u}) + \frac{\mathcal{R}_m}{b\gamma(m+1)} \nabla \pi - \frac{\mathcal{R}_m}{b\gamma} \frac{\pi}{z} \boldsymbol{e}_3 + \frac{\mathcal{R}_m}{b\gamma} (\frac{z}{L})^{m-1} \theta \, \boldsymbol{e}_3 = \boldsymbol{F}_l \, , \tag{3.2}$$

$$\lambda \gamma \theta - \Delta \theta + \mathcal{R}_m (1 - m(\gamma - 1)) (\frac{z}{L})^m w = H_l \, . \tag{3.3}$$

We consider the problem in the domain

$$\Omega = \{ (x, y, z) \,|\, 0 \le x \le 2\pi/a \, , \; 0 \le y \le 2\pi/b, \; L - \frac{1}{2} \le z \le L + \frac{1}{2} \}$$

with the Dirichlet zero boundary conditions for $\boldsymbol{u}$ and θ on the boundaries $z = L - \frac{1}{2}$, $L + \frac{1}{2}$ and the periodic boundary condition with respect to x, y . Here we notice that

$$z^m G_l = \nabla \cdot \mathcal{G}_l$$

for some vector function $\mathcal{G}_l$, which vanishes on the horizontal boundaries.

Lemma 1

There exist constants L_0 , $\mathcal{R}_0$ *such that if*

$$R_e \lambda \geq 0 , \quad 0 \leq \mathcal{R}_m < \mathcal{R}_0 \quad \text{and} \quad L \geq L_0$$

then the resolvent exists and the solution $(\pi , \boldsymbol{u} , \theta)$ *satisfies the compatibility condition* $\int \pi = 0$ *and for* $\lambda = 0$ *the following estimates.*
If $G_l \in H^1$, $\boldsymbol{F_l} \in \boldsymbol{L^2}$, $H_l \in L^2$, *then we have*

$$|| \pi ||_1 , \; || \boldsymbol{u} ||_2 , \; || \theta ||_2 \leq C_1 (|| G_l ||_1 + || \boldsymbol{F_l} || + || H_l ||) .$$

If $G_l \in H^2$, $\boldsymbol{F_l} \in \boldsymbol{H^1}$, $H_l \in H^1$, *then we have*

$$|| \pi ||_2 , \; || \boldsymbol{u} ||_3 , \; || \theta ||_3 \leq C_2 (|| G_l ||_2 + || \boldsymbol{F_l} ||_1 + || H_l ||_1) .$$

If the first eigenvalue with the largest real part crosses the imaginary axis at $\mathcal{R}_m = \mathcal{R}_c (L)$, *then the above estimate extends to* $0 \leq \mathcal{R}_m < \mathcal{R}_c (L)$. *The square* $\mathcal{R}_c(L)^2$ *tends to the critical Rayleigh number of the Oberbeck-Boussinesq equation* $\mathcal{R}_B$ *as* $L \to +\infty$ [16], [10], [17].

Lemma 2 Estimates for the nonlinear terms of (2.1)-(2.3).

We assume that

$$z \geq L - \frac{1}{2} \geq 2 , \quad \boldsymbol{u} \Big|_{z = L - \frac{1}{2}, L + \frac{1}{2}} = 0 ,$$

and that

$$|| p ||_2 , \; || \boldsymbol{u} ||_3 , \; || \theta ||_3 \leq 1 ,$$

then g *vanishes on the horizontal boundaries and we have the estimates*

$$|| g || \leq \frac{c_0}{L} (|| p || + || \boldsymbol{u} ||_2) (|| \boldsymbol{u} ||_2 + || \theta ||_2)$$

$$|| g ||_1 \leq \frac{c_1}{L} (|| p ||_1 + || \boldsymbol{u} ||_2) (|| \boldsymbol{u} ||_3 + || \theta ||_3)$$

$$|| g ||_2 \leq \frac{c_2}{L} (|| p ||_2 + || \boldsymbol{u} ||_2) (|| \boldsymbol{u} ||_3 + || \theta ||_3)$$

$$|| \boldsymbol{f} || \leq \frac{c_0}{L} (|| p || + || \boldsymbol{u} ||_2 + || \theta ||_2) (|| \boldsymbol{u} ||_2 + || \theta ||_2) + c_0 || \boldsymbol{f_e} ||$$

$$|| \boldsymbol{f} ||_1 \leq \frac{c_1}{L} (|| p ||_1 + || \boldsymbol{u} ||_2 + || \theta ||_2) (|| \boldsymbol{u} ||_2 + || \theta ||_2) + c_1 || \boldsymbol{f_e} ||_1$$

$$|| h || \leq \frac{c_0}{L} (|| p ||_1 + || \boldsymbol{u} ||_2) (|| \boldsymbol{u} ||_2 + || \theta ||_2) + c_0 || h_e ||$$

$$|| h ||_1 \leq \frac{c_1}{L} (|| p ||_2 + || \boldsymbol{u} ||_2) (|| \boldsymbol{u} ||_3 + || \theta ||_3) + c_1 || h_e ||_1$$

We begin the iteration to prove the existence of the solution of the nonlinear system (2.1)-(2.3) uniformly bounded for large L .

Given the functions for step k , $k = 0, 1, 2, \cdots$ with

$$p^{(0)} = 0 \,,\; \boldsymbol{u}^{(\mathbf{0})} = \mathbf{0} \,,\; \theta^{(0)} = 0 \,,$$

we define

$$\begin{aligned} G^{(k)} &= -\mathcal{R}_m \left(\nabla \cdot ((\frac{L}{z})^{m-1} \frac{p^{(k)}\, \boldsymbol{u}^{(\boldsymbol{k})}}{z^2}) + g^{(k)} \right), \quad g^{(k)} = g(\, p^{(k)} ,\, \boldsymbol{u}^{(\boldsymbol{k})} ,\, \theta^{(k)} \,) \\ \boldsymbol{F}^{(\boldsymbol{k})} &= -\frac{\mathcal{R}_m}{\mathcal{P}_r} (\frac{z}{L})^m \boldsymbol{u}^{(\boldsymbol{k})} \cdot \nabla \boldsymbol{u}^{(\boldsymbol{k})} + \boldsymbol{f}^{(\boldsymbol{k})} \,, \quad \boldsymbol{f}^{(\boldsymbol{k})} = \boldsymbol{f}(\, p^{(k)} ,\, \boldsymbol{u}^{(\boldsymbol{k})} ,\, \theta^{(k)} \,) \,, \\ H^{(k)} &= -\mathcal{R}_m \,\gamma\, (\frac{z}{L})^m \boldsymbol{u}^{(\boldsymbol{k})} \cdot \nabla \theta^{(k)} + h^{(k)} \,, \quad h^{(k)} = h(\, p^{(k)} ,\, \boldsymbol{u}^{(\boldsymbol{k})} ,\, \theta^{(k)} \,) \,, \end{aligned}$$

where $G^{(k)}$ vanishes at horizontal boundaries. We solve the linear boundary value problem for the unknowns $\pi^{(k+1)}$, $\boldsymbol{u}^{(\boldsymbol{k+1})}$, $\theta^{(k+1)}$

$$\mathcal{R}_m \left(\nabla \cdot \boldsymbol{u}^{(\boldsymbol{k+1})} + \frac{m\, w^{(k+1)}}{z} \right) = G^{(k)} \,, \tag{3.4}$$

$$\begin{aligned} &- \Delta \boldsymbol{u}^{(\boldsymbol{k+1})} - \frac{1}{3} \nabla \nabla \cdot \boldsymbol{u}^{(\boldsymbol{k+1})} + \frac{\mathcal{R}_m}{b\,\gamma\,(m+1)} \nabla \pi^{(k+1)} \\ &\qquad - \frac{\mathcal{R}_m}{b\,\gamma\, z} \pi^{(k+1)} \boldsymbol{e}_3 + \frac{\mathcal{R}_m}{b\,\gamma} (\frac{z}{L})^{m-1} \theta^{(k+1)} \boldsymbol{e}_3 = \boldsymbol{F}^{(\boldsymbol{k})} \,, \end{aligned} \tag{3.5}$$

$$- \Delta \theta^{(k+1)} + \mathcal{R}_m \,(\, 1 - m(\gamma - 1)\,)\, (\frac{z}{L})^m w^{(k+1)} = H^{(k)} \,, \tag{3.6}$$

$$\boldsymbol{u}^{(\boldsymbol{k+1})} \Big|_{z = L - \frac{1}{2},\, L + \frac{1}{2}} = \mathbf{0} \,, \quad \theta^{(k+1)} \Big|_{z = L - \frac{1}{2},\, L + \frac{1}{2}} = 0 \,. \tag{3.7}$$

The transport equation is introduced as follows. We define

$$E^{(k+1)} = \frac{\mathcal{R}_m}{b\,\gamma\,(m+1)} \pi^{(k+1)} - \frac{4\,\mathcal{R}_m}{3} \left(\nabla \cdot \boldsymbol{u}^{(\boldsymbol{k+1})} + \frac{m\, w^{(k+1)}}{z} \right) . \tag{3.8}$$

Then $E^{(k+1)}$ satisfies the following

$$\begin{aligned} \Delta E^{(k+1)} &= -\frac{4\,\mathcal{R}_m}{3} \Delta \left(\frac{m\, w^{(k+1)}}{z} \right) \\ &\quad + \frac{\mathcal{R}_m}{b\,\gamma} \frac{\partial}{\partial z} \left(\frac{\pi^{(k+1)}}{z}\right) - \frac{\mathcal{R}_m}{b\,\gamma} \frac{\partial}{\partial z} \left((\frac{z}{L})^{m-1} \theta^{(k+1)} \right) + \nabla \cdot \boldsymbol{F}^{(\boldsymbol{k})} \,. \end{aligned}$$

The pressure of $k+1$ step is obtained by the following transport equation

$$\begin{aligned} \frac{\mathcal{R}_m}{b\,\gamma\,(m+1)} p^{(k+1)} &+ \frac{4\,\mathcal{R}_m}{3} \nabla \cdot \left((\frac{L}{z})^{m-1} \frac{p^{(k+1)}\, \boldsymbol{u}^{(\boldsymbol{k+1})}}{z^2} \right) \\ &= E^{(k+1)} + \frac{4\,\mathcal{R}_m}{3} g^{(k)} \,. \end{aligned} \tag{3.9}$$

If we use the estimates of Lemma 1 for the solutions of linear system (3.4) - (3.6), we have the following estimate for $E^{(k+1)}$.

Lemma 3

The linear system of equations (3.4), (3.5), (3.6) has a solution for $\mathcal{R}_m < \mathcal{R}_c(L)$, *where* $\mathcal{R}_c(L)^2$ *is the critical Rayleigh-Bénard number for the system. The solution satisfies the estimates as follows.*

If $G^{(k)} \in H^1$, $\boldsymbol{F^{(k)}} \in \boldsymbol{L^2}$, $H^{(k)} \in L^2$, *then we have*

$$|| \pi^{(k+1)} ||_1 \ , \quad || \boldsymbol{u^{(k+1)}} ||_2 \ , \quad || \theta^{(k+1)} ||_2 \quad \leq \quad C_1 \, (\, || G^{(k)} ||_1 + || \boldsymbol{F^{(k)}} || + || H^{(k)} || \,) \ .$$

If $G^{(k)} \in H^2$, $\boldsymbol{F^{(k)}} \in \boldsymbol{H^1}$, $H^{(k)} \in H^1$, *then we have*

$$|| \pi^{(k+1)} ||_2 \ , \quad || \boldsymbol{u^{(k+1)}} ||_3 \ , \ || \theta^{(k+1)} ||_3 \quad \leq \quad C_2 \, (\, || G^{(k)} ||_2 + || \boldsymbol{F^{(k)}} ||_1 + || H^{(k)} ||_1 \,) \ .$$

Therefore we have

$$\begin{aligned} || E^{(k+1)} ||_1 \quad & \leq \quad C \, C_1 \, (\, || G^{(k)} ||_1 + || \boldsymbol{F^{(k)}} || + || H^{(k)} || \,) \ , \\ || \Delta \, E^{(k+1)} || \quad & \leq \quad C \, C_1 \, (\, || G^{(k)} ||_1 + || \boldsymbol{F^{(k)}} ||_1 + || H^{(k)} || \,) \ , \\ || E^{(k+1)} ||_2 \quad & \leq \quad C \, C_2 \, (\, || G^{(k)} ||_2 + || \boldsymbol{F^{(k)}} ||_1 + || H^{(k)} ||_1 \,) \ . \end{aligned}$$

The norm $|| p^{(k+1)} ||_2$ and $|| G^{(k+1)} ||_2$ is recovered by the transport equation (3.9) as follows.

Lemma 4

There exists a constant C_T *such that if* $C_T \, \dfrac{|| \boldsymbol{u^{(k+1)}} ||_3}{L^2} \ \leq \ \dfrac{1}{2}$ *is fulfilled, then the transport equation has the solution* $p^{(k+1)}$ *and it satisfies the following estimates :*

$$\begin{aligned} & || p^{(k+1)} ||_1 \quad \leq \quad C_3 \, (\, || G^{(k)} ||_1 + || \boldsymbol{F^{(k)}} || + || H^{(k)} || \,) \ , \\ & || p^{(k+1)} ||_2 \quad \leq \quad C_4 \, (\, || G^{(k)} ||_2 + || \boldsymbol{F^{(k)}} ||_1 + || H^{(k)} ||_1 \,) \\ & \leq \ C_4 \left(|| \boldsymbol{F^{(k)}} ||_1 + || H^{(k)} ||_1 + || g^{(k)} ||_2 + || \nabla \cdot \left((\frac{L}{z})^{m-1} \, \frac{p^{(k)} \, \boldsymbol{u^{(k)}}}{z^2} \right) ||_2 \right) \ , \end{aligned}$$

$$|| \Delta \, p^{(k+1)} || \quad , \quad || \Delta \, \nabla \cdot \left((\frac{L}{z})^{m-1} \, \frac{p^{(k+1)} \, \boldsymbol{u^{(k+1)}}}{z^2} \right) ||$$

$$\leq \ C_5 \left(\ || G^{(k)} ||_1 + || \boldsymbol{F^{(k)}} ||_1 + || H^{(k)} || \ + \ || g^{(k)} ||_2 \ + \ C_G \ \frac{|| \boldsymbol{u^{(k+1)}} ||_3 \, || p^{(k+1)} ||_2}{L^2} \right)$$

Therefore if we notice the Dirichlet zero boundary condition for the velocity and so that $G^{(k)}\,,\ k=0,1,\cdots$ vanish on the boundaries, we have

$$\begin{aligned}
||\,\nabla\cdot\left((\frac{L}{z})^{m-1}\,\frac{p^{(k+1)}\,\boldsymbol{u^{(k+1)}}}{z^2}\right)||_2 \;&\le\; C\,||\,\Delta\,\nabla\cdot\left((\frac{L}{z})^{m-1}\,\frac{p^{(k+1)}\,\boldsymbol{u^{(k+1)}}}{z^2}\right)||\\
&\le\; C\,C_5\,\Big(\,||\,G^{(k)}\,||_1+||\,\boldsymbol{F^{(k)}}\,||_1+||\,H^{(k)}\,||\;+\;||\,g^{(k)}\,||_2\\
&\qquad\qquad + C_G\,\frac{||\,\boldsymbol{u^{(k+1)}}\,||_3\,||\,p^{(k+1)}\,||_2}{L^2}\Big)\\
&\le\; C_6\,\Big(\,||\,G^{(k)}\,||_1+||\,\boldsymbol{F^{(k)}}\,||_1+||\,H^{(k)}\,||+||\,g^{(k)}\,||_2\,\Big)
\end{aligned}$$

$$+\,\frac{C_6\,C_G}{L^2}\,\left(\,||\,\boldsymbol{F^{(k)}}\,||_1+||\,H^{(k)}\,||_1+||\,g^{(k)}\,||_2+||\,\nabla\cdot\left((\frac{L}{z})^{m-1}\,\frac{p^{(k)}\,\boldsymbol{u^{(k)}}}{z^2}\right)||_2\,\right)^2 .$$

Now we proceed to obtain the uniform bounds for

$$||\,p^{(k+1)}\,||_2\,,\quad ||\,\boldsymbol{u^{(k+1)}}\,||_3\,,\quad ||\,\theta^{(k+1)}\,||_3\ .$$

Let us assume that for small R and S

$$||\,p^{(k)}\,||_2\,,\ \ ||\,\boldsymbol{u^{(k)}}\,||_3\,,\ \ ||\,\theta^{(k)}\,||_3\ \le\ R\,,\qquad ||\,\boldsymbol{f_e}\,||_1\,,\ \ ||\,h_e\,||_1\ \le\ S\ =\ R^2\,,$$

and also that

$$||\,\nabla\cdot\left((\frac{L}{z})^{m-1}\,\frac{p^{(k)}\,\boldsymbol{u^{(k)}}}{z^2}\right)||_2\quad\le\quad C_n\,R^2\ .$$

Then we know by lemma 2 that for $L\,\ge\,2.5$

$$||\,L\,g^{(0)}\,||_2\,,\quad ||\,g^{(k)}\,||_2\,,\quad ||\,G^{(k)}\,||_1\ \le\ C\,R^2\quad,\quad ||\,G^{(k)}\,||_2\ \le\ C_n\,R^2\;+\;C\,R^2\,,$$
$$||\,\boldsymbol{F^{(k)}}\,||_1\quad,\qquad ||\,H^{(k)}\,||_1\ \le\ C\,R^2\;+\;C_0\,S\ .$$

Then it follows from lemma 3 and lemma 4 that

$$||\,\pi^{(k+1)}\,||_2\,,\quad ||\,\boldsymbol{u^{(k+1)}}\,||_3\,,\quad ||\,\theta^{(k+1)}\,||_3$$
$$\le\ C\,C_2\,\big(\,C_n\,R^2\;+C\,R^2\;+\;C_0\,S\,\big)\ =\ C_7\,\big(\,C_n\,R^2\;+C\,R^2\;+\;C_0\,S\,\big)\ ,$$

$$\begin{aligned}
||\,p^{(k+1)}\,||_2\ &\le\ C\,C_4\,\big(\,C_n\,R^2+C\,R^2\;+\;C_0\,S\,\big)\\
&=\ C_8\,\big(\,C_n\,R^2+C\,R^2\;+\;C_0\,S\,\big)\ .
\end{aligned}$$

$$\begin{aligned}
||\,\nabla\cdot\left((\frac{L}{z})^{m-1}\,\frac{p^{(k+1)}\,\boldsymbol{u^{(k+1)}}}{z^2}\right)||_2\ &\le\ C\,C_6\,(\,C\,R^2\;+\;C_0\,S\,)\\
&\quad+\ \frac{C\,C_6\,C_G}{L^2}\,C_7\,C_8\,(\,C\,R^2\;+\;C_0\,S\;+\;C_n\,R^2\,)^2\\
&\le\ C\,C_6\,R^2\left(\,C\;+\;C_0\;+\;C_7\,C_8\,(\,C\;+\;C_0\;+\;C_n\,)^2\,\frac{C_G\,R^2}{L^2}\right)\\
&\le\ 2\,C\,C_6\,(\,C+C_0\,)\,R^2\ =\ C_n\,R^2\ .
\end{aligned}$$

Therefore we have the uniform estimate :

Proposition 1

Let $C_9 = \max\{ C_7 , C_8 \}$, $S = R^2$ *and* $C_n = 2\, C\, C_6 \,(C + C_0)$. *If* S *and* $\dfrac{1}{L}$ *are small such that* $C_9 \,(C + C_0 + C_n)\, R \leq 1$,

$$C_7 \, C_8 \, (C + C_0 + C_n)^2 \, \frac{C_G \, R^2}{L^2} \leq C + C_0 \ ,$$

$$|| \, p^{(k)} \, ||_2 \ , \quad || \, \boldsymbol{u^{(k)}} \, ||_3 \ , \quad || \, \theta^{(k)} \, ||_3 \leq R \ ,$$

$$|| \, \nabla \cdot \left((\frac{L}{z})^{m-1} \, \frac{p^{(k)} \, \boldsymbol{u^{(k)}}}{z^2} \right) ||_2 \leq C_n \, R^2 \ .$$

then

$$|| \, p^{(k+1)} \, ||_2 \ , \quad || \, \boldsymbol{u^{(k+1)}} \, ||_3 \ , \quad || \, \theta^{(k+1)} \, ||_3 \leq R \ .$$

$$|| \, \nabla \cdot \left((\frac{L}{z})^{m-1} \, \frac{p^{(k+1)} \, \boldsymbol{u^{(k+1)}}}{z^2} \right) ||_2 \leq C_n \, R^2 \ .$$

Remark. Notice that for $L > L_0$ given by Lemma 1, Lemma 4 and Proposition 1 we may prove the existence and uniqueness only for $\mathcal{R}$ suitably small.

To prove the convergence of the iteration we consider the difference

$$p^{(k+1)} - p^{(k)} \ , \quad \boldsymbol{u}^{(k+1)} - \boldsymbol{u}^{(k)} \ , \quad \theta^{(k+1)} - \theta^{(k)} \ .$$

They satisfy the similar system of equations to (3.4), (3.5), (3.6) and (3.9) and so they have the similar estimates.

$$\begin{aligned} & || \, \pi^{(k+1)} - \pi^{(k)} \, ||_1 \ , \quad || \, \boldsymbol{u^{(k+1)}} - \boldsymbol{u^{(k)}} ||_2 \ , \quad || \, \theta^{(k+1)} - \theta^{(k)} ||_2 \\ & \qquad \leq C_{10} \, (C \, R + C_0 \, S) \, (|| \, p^{(k)} - p^{(k-1)} \, ||_1 \\ & \qquad\qquad + || \, \boldsymbol{u^{(k)}} - \boldsymbol{u^{(k-1)}} ||_2 + || \, \theta^{(k)} - \theta^{(k-1)} ||_2 \,) \ , \\ & || \, p^{(k+1)} - p^{(k)} ||_1 \\ & \qquad \leq C_{11} \, (C \, R + C_0 \, S) \, (|| \, p^{(k)} - p^{(k-1)} ||_1 \\ & \qquad\qquad + || \, \boldsymbol{u^{(k)}} - \boldsymbol{u^{(k-1)}} ||_2 + || \, \theta^{(k)} - \theta^{(k-1)} ||_2 \,) \ . \end{aligned}$$

Therefore we have the convergence estimate :

Proposition 2

Let $C_{12} = \max\{ C_{10} , C_{11} \}$ *and* $S = R^2$. *If* S *is small such that*

$$C_{12} \, (C \, R + C_0 \, S) \leq \frac{1}{2}$$

then

$$\begin{aligned} & || \, p^{(k+1)} - p^{(k)} \, ||_1 + || \, \boldsymbol{u^{(k+1)}} - \boldsymbol{u^{(k)}} ||_2 + || \, \theta^{(k+1)} - \theta^{(k)} ||_2 \\ & \leq \frac{1}{2} \, (|| \, p^{(k)} - p^{(k-1)} \, ||_1 + || \, \boldsymbol{u^{(k)}} - \boldsymbol{u^{(k-1)}} ||_2 + || \, \theta^{(k)} - \theta^{(k-1)} ||_2 \,) \ . \end{aligned}$$

Thus as $k \to \infty$ by (3.4),(3.5),(3.8),(3.9) we have the limit functions for small S and large L uniformly with respect to $L \geq L_0$

$$p^{(\infty)} = \pi^{(\infty)} \in H^2 , \quad \boldsymbol{u}^{(\infty)} \in H^3 , \quad \theta^{(\infty)} \in H^3 ,$$

which are solutions to the stationary equations (2.1)-(2.3) and (2.4) for each L . Therefore by lemma 2 with the uniform estimates the limit functions as $L = z_0 + \frac{1}{2} \to +\infty$ satisfies the stationary Oberbeck-Boussinesq system with the small external forces for $\mathcal{R}_m < \mathcal{R}_c$. Theorem is completely proved.

Thus we have a justification of approximation by Oberbeck-Boussinesq equations for the system of the compressible fluids in the case of stationary solutions with small external forces as $L = z_0 + \dfrac{1}{2} = \dfrac{T_l + T_u}{2\,\beta_0\,d} \to +\infty$.

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