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Nonlinear stability of liquid flow down an inclined plane.

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Abstract A modified energy method is proposed to study the nonlinear stability of Poiseuille flow with upper free boundary S_t, down an inclined plane. Sufficient conditions on Reynolds, and Weber numbers, on the slope of the plane, together with the periodicity along the line of maximum slope are given. All constants are computable.
Smallness condition on initial data is required.
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1. Introduction

Given an inclined infinite layer of viscous heavy liquid with upper free boundary, a steady laminar motion develops parallel to the flat bottom of the layer. We name this motion *Poiseuille Free Boundary* PFB flow because of its (half) parabolic velocity profile. In flows over an inclined plane the free surface introduces additional interesting effects of surface tension and gravity. These effects change the character of the instability in a parallel flow, M.K. Smith [24].
Benjamin [1], and Yih [31], have solved the linear stability problem of a uniform film on a inclined plane, such uniform film becomes unstable to long wave disturbances, much larger than the depth of the film, when the Reynolds number defined in (3.17) exceeds a critical value $R_c := (5/4) \cot \beta$ where β is the inclination angle.
Instability takes place in the form of an infinitely long wave, however *surface waves of finite wavelengths are observed*, see e.g. Yih [31]. To give more realistic results, Benjamin [1], and Yih [32], [33] constructed the solution by expansion in a power series of a given parameter. The authors use expansions of different parameters, and different surface tension coefficients. In all these papers it turns out that the periodicity α_1 in the line of maximum slope

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is an important parameter in the stability analysis. After these pioneering investigations several asymptotic expansions and approximations have been proposed, cf. [30], [10], [14], [11], [29]. Introducing a long-wave parameter λ, the basic equations are expanded and a nonlinear evolution equation of the film thickness is obtained that is referred to as a *surface, or long-wave equation*. The above-mentioned papers study how exponentially varying linear waves are modified by the nonlinear effect, namely how small amplitudes, due to nonlinearities, amplify in the course of time.
However up to date tractable asymptotic methods are still lacking, this justifies the attempt for understanding the mechanism causing the surface wave instability, namely the instability of the plane interface between liquid and air also called Kelvin-Helmholtz stability [3], [7], [8], [24]. Another investigation on Kelvin-Helmholtz instability is the study of rotational and irrotational effects of viscosity cf. [15], [16]. *To date, rigorous nonlinear stability theory has not yet been studied*.

In this note we assume that above the liquid there is a uniform pressure due to the air at rest, and the liquid is moving with respect to the air[1], and we investigate nonlinear stability of PFB providing *a rigorous formulation of the problem by the classical direct Lyapunov method assuming periodicity in the plane*. Sufficient conditions on non dimensional Reynolds R, Weber W numbers, on periodicity α_1, on depth of the layer ℓ, and on inclination angle β are computed ensuring Kelvin-Helmholtz *nonlinear stability*. We use *a modified energy method which provides physically meaningful sufficient conditions ensuring nonlinear exponential stability*[2], cf. [17], [20]. The result is achieved in the class of regular solutions occurring in simply connected domains having cone property [3].

We begin by deriving the energy conservation law and deduce, under some assumptions, that the *area of the moving surface Γ_t has finite measure for all time $t \in (0, \infty)$*. Precisely two regularity lemmas are proven in section 3. In Lemma 3.1 smallness is assumed on the slope of the inclined plane, on the ratio of periodicity number a_1 over the depth of the layer ℓ, and on Reynolds R and Weber W numbers, in Lemma 3.2 assumptions of smallness are required only on initial data. Notice that the boundedness of surface area is not true in general because of the destabilizing effect of the component of the gravity along the line of maximum slope.

Next we study *nonlinear stability of PFB flow by perturbing initial data*. As known stability of the steady motion S_b with respect to initial data means control for all time t, in a given norm X, of the difference between S_b and the motion $S(t)$ developing in correspondence to initial data $S(0) = S_b + S_0$. Such difference $S'(t) = S(t) - S_b$ is called perturbation. For free boundary

[1] No wind is blowing over the free surface. If viscosity is neglected the pressure inside the liquid furnishes a instability mechanism.

[2] With the term exponential stability we mean that in a certain spatial norm, function only of time, the perturbation decays to zero with exponential rate

[3] A domain Ω is said to enjoy to cone property if there exists a finite cone C such that each point $x \in \Omega$ is a vertex of a cone C_x contained in Ω and congruent to C.

problems it is difficult to compare two flows because functions defined in the variable volume Ω_t, as velocity, pressure, cannot be compared at different times t_1, t_2 just because they are defined in different domains Ω_{t_1}, Ω_{t_2}. It is customary to reduce the problem into a fixed domain [13], such procedure introduces heavy nonlinearities. Here we propose an alternative definition of perturbation in Eulerian variables, see also [18], [19]. It is worth of notice that the evolution equation for the function η governing the motion of the free boundary is of hyperbolic type, say transport equation. Nevertheless using the free work equation that will be defined later [18], [21], [20], we find a dissipative term for the function η.

As matter of fact, the *technical novelties* in this paper are essentially two.

First novelty concerns the definition of perturbation and we extend the ideas introduced in [18], [19], where stability is studied in Eulerian coordinates.
We remark that we propose a natural definition of perturbation in Eulerian coordinates. Notice that in Eulerian coordinates the external force $\mathbf{f}$ is function of the domain Ω_t where the motion occurs. Specifically, let Ω_t, be the domain occupied by the fluid during its motion. Thus the force $\mathbf{f}$ is expressed through the product of constant gravity acceleration g, times the vertical unit vector downward directed $\mathbf{k}$, times the characteristic function of Ω_t. Therefore the force $\mathbf{f}$ is a unknown function because the domain of definition is unknown.
We remark that the linear equations, obtained by linearization of our scheme around the basic Poiseuille flow, do coincide with the usual linear equations, cf. [32].

Second novelty concerns the computation of the distance between the perturbed surface Γ_t and the basic surface given by the rectangle $\Sigma = (0, \alpha_1) \times (0, \alpha_2)$, with α_2 periodicity in the horizontal direction, defined as

$$d^2(\Gamma_t, \Sigma) := |\Gamma_t| - |\Sigma|.$$

To control the measure d in terms of the perturbation η to the constant line $z = 0$ we introduce a new functional

$$\|\eta\|_X^2 := \int_\Sigma \frac{|\nabla'\eta|^2}{\sqrt{1 + |\nabla'\eta|^2}} dx',$$

where $x' \equiv (x_1, x_2) \in \Sigma$ and ∇' means the derivatives with respect to x'. We prove that $\|\eta\|_X$ is equivalent to $d(\Gamma_t, \Sigma)$, see also (2.3). Notice that $\|\eta\|_X$ is always a real positive function of t. It is amazing to realize how in two dimensional domains the term[4] $\|\eta\|_X$, plus the dissipative shear rate term $\|S(\mathbf{u})\|_{L^2(\Omega_t)}$ may increase all nonlinear terms in the energy inequality for suitably small values of R, W, α_1/ℓ, β.

Methods

To study nonlinear stability of PFB flow, we combine the *classical energy method* with the *free work equation* [17], [20].

[4] $\|\eta\|_X^2$ multiplied by a surface tension coefficient is equivalent to the energy of perturbation.

Hypotheses
a) Reynolds number must satisfy the classical stability condition for shear flows (5.1);
b) inequality (5.22) must be true.
Results
Under hypotheses on the slope of plane, on the ratio between the periodicity along the line of maximum slope α_1 and the depth of the layer, on the Reynolds R, and Weber W numbers assumed in a), b), *the energy of perturbations decays exponentially to zero*, for perturbations satisfying suitably smallness conditions at initial time.

Condition (5.22) is quite intriguing, and the analysis of clear physical restrictions together with comparison with linear results should be developed. This is the object of a paper in preparation. Roughly speaking we may say that W and R must be suitably small, this furnishes a physically reasonable condition. Even though our stability conditions may be considered quite conservative, we remark that for the first time computable critical Reynolds and Weber numbers ensuring nonlinear stability are furnished.

The scheme of the paper is the following. In section 2 mathematical preliminaries are introduced. In section 3 equations of motions are introduced and Poiseuille flow with free boundary PFB is given. Thus the problem of motion in non-dimensional form is formulated, and the energy equation is deduced. Also under suitable conditions either on R, W or on initial data the energy identity provides an "*a priori estimate*" for all time $t \in (0, \infty)$. In section 4 the definition of perturbation is given and the system of equations for perturbation is written. Next in section 5 the stability theorem is stated and proved using simple direct methods. In the appendix, section 6, inequalities used in the stability proofs have been proved.
We end introduction recalling existence of global unsteady solutions, proved by Nishida et al. [13], [12], for existence of a drop of liquid see also [22].

2. Mathematical Preliminaries

2..1 The geometry

Given a infinite layer $\mathcal{L}$ of viscous liquid, bounded below and above by two inclined planes Π_u (up), Π_d (down) we consider rectilinear coordinates with the reference frame $\mathcal{R} =: \{O, x_1, x_2, x_3\} = \{O, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$, with the origin O on Π_u. The x_2 axis is the horizontal line on Π_u crossing O; the x_1 axis is the line on Π_u orthogonal to x_2 and it is called line of greater slope, here $\boldsymbol{e}_1$ downward directed; the x_3 axis is orthogonal to Π_u. We fix the lower plane of the layer at $x_3 = -\ell$.
The rectangle $\Sigma = (0, \alpha_1) \times (0, \alpha_2)$ on Π_u denotes a periodicity cell, by abuse of notations we Σ will denote also the section $\Sigma = (0, \alpha_1) \times (0, \alpha_2) \times \{0\}$. The index $'$ is added to denote quantities calculated on Σ, furthermore ∇' means derivatives along the coordinate lines x_1, x_2, namely

$$\nabla' \equiv \Big(\partial_1, \partial_2\Big), \qquad \zeta_{,1} := \partial_1 \zeta \quad \zeta_{,2} := \partial_2 \zeta.$$

Let $\mathcal{L}_t$ be the liquid layer obtained from $\mathcal{L}$ perturbing Π_u in periodical way, with periodicity Σ. The upper surface Γ_t over Σ admits a cartesian representation. By Ω_t we denote the subset of $\mathcal{L}_t$

$$\Omega_t =: \{(x', x_3) \in R^3 : \quad x' \in \Sigma, \quad x_3 \in (-\ell, \zeta) \quad \zeta = \zeta(x', t)\}, \quad t \in (0, \infty).$$

Ω_t has the upper surface Γ_t

$$\Gamma_t = \{(x', x_3) \in R^3 : \quad x' \in \Sigma, \quad x_3 = \zeta(x', t)\}, \qquad t \in (0, \infty),$$

where ζ is a *unknown* scalar function.
The exterior to Ω_t is occupied by the air at rest and is denoted by $\widehat{\Omega}_t$.

Through the paper we shall use the assumption that Γ_t is **strongly Lipschitz**[5]. The constraint

$$|\zeta| < \frac{|\ell|}{2},$$

implies that Γ_t doesn't touch the bottom, and the domain is simply connected.

The unit normal $\mathbf{n}$ has components $(-\nabla'\eta, 1)/\sqrt{1 + |\nabla'\eta|^2}$, where $\sqrt{1 + |\nabla'\eta|^2}$ is the metric element, and it is given by

$$\mathbf{n} = \frac{1}{\sqrt{1 + |\nabla'\eta|^2}} \Big[-\eta_{,1} \boldsymbol{e}_1 - \eta_{,2}\, \boldsymbol{e}_2 + \boldsymbol{e}_3 \Big]. \tag{2.1}$$

Finally the doubled mean curvature $\mathcal{H}(\zeta)$[6] of Γ_t at x' is given by

$$\mathcal{H}(\zeta) \;=\; \nabla' \cdot \Big(\frac{\nabla'\zeta}{\sqrt{1 + |\nabla'\zeta|^2}} \Big).$$

For **two dimensional domains** we shall use the following hypothesis
H1 - Hypothesis on Γ_t

The curve Γ_t may be decomposed as the union of numerable parts Γ_{ti}, i in a set of indices $\mathcal{I}$, each of them is in normal form with respect to variable x_2 namely it holds

$$\Gamma_t = \int_0^{a_1} \sqrt{1 + |\partial_{x_1}\eta(x_1, t)|^2}\, dx_1 = \bigcup_{i \in \mathcal{I}} \int_{\Sigma_{ti}} \sqrt{1 + |\partial_{x_2}\xi(x_2, t)|^2}\, dx_2.$$

[5] Γ_t is **strongly Lipschitz** if there exist two positive numbers δ and L such that for all $(x_1', \eta(x_1', t)), (x_2', \eta(x_2', t)) \in \Gamma_t$, it happens

$$|x_1' - x_2'| < \delta, \quad \longrightarrow \qquad |\eta(x_1', t) - \eta(x_2', t)| < L|x_1' - x_2'|.$$

[6] Any regular surface S, at any point x admits two main circles γ_1 (the greatest), γ_2 (the smallest) tangent to S in x. Denoting by r_1, r_2 the radii of the two circles, the sum of the two curvatures $1/r_1$ and $1/r_2$ is the doubled mean curvature of S at x.

2..2 Lebesgue and Hilbert spaces

$L^p(\Omega_t)$, $p > 1$ denote the Lebesgue spaces of functions in $L^p(\Omega_t)$, and $H^1(\Omega_t)$ denotes the Hilbert space of functions that, together with their first spatial derivatives, belong to $L^2(\Omega_t)$. For functions f in these spaces the following norms are introduced

$$\begin{aligned}
\|f\|_{L^p(\Omega_t)} &= \left(\int_{\Omega_t} f^p dv\right)^{1/p}, \\
\|f\|_{L^\infty(\Omega_t)} &= \sup_{\Omega_t} |f|, \\
\|f\|_{H^1(\Omega_t)} &= \left(\int_{\Omega_t} (f^2 + |\nabla f|^2) dv\right)^{1/2}.
\end{aligned}$$

Analogously, $L^p(\Sigma)$, $p > 1$ denote the Lebesgue spaces of functions in $L^p(\Sigma)$, and $H^1(\Sigma)$ denotes the Hilbert space of functions that, together with their first spatial derivatives, belong to $L^2(\Sigma)$. For functions f in these spaces the following norms are introduced

$$\begin{aligned}
\|f\|_{L^p(\Sigma)} &= \left(\int_{\Sigma} f^p dx'\right)^{1/p}, \\
\|f\|_{L^\infty(\Sigma)} &= \sup_{\Sigma} |f|, \\
\|f\|_{H^1(\Sigma)} &= \left(\int_{\Sigma} (f^2 + |\nabla' f|^2) dx'\right)^{1/2}.
\end{aligned}$$

2..3 The functional space X

Let X denotes the subspace of functions $\eta \in H^1(\Sigma)$ satisfying

$$\|\eta\|_X := \Big(\int_\Sigma \frac{|\nabla'\eta|^2}{\sqrt{1+|\nabla'\eta|^2}}\, dx'\Big)^{1/2} < \infty.$$

Next Lemma shows that the quantity

$$E_\eta := \int_\Sigma \Big(\sqrt{1+|\nabla'\eta|^2} - 1\Big) dx',$$

is equivalent to $\|\eta\|_X^2$.

Lemma 2.1 *Let η be a function in Σ having zero mean value. The following inequalities hold*

$$\frac{1}{2}\int_\Sigma \frac{|\nabla'\eta|^2}{\sqrt{1+|\nabla'\eta|^2}} dx' \leq \int_\Sigma \Big(\sqrt{1+|\nabla'\eta|^2} - 1\Big) dx' \leq \int_\Sigma \frac{|\nabla'\eta|^2}{\sqrt{1+|\nabla'\eta|^2}} dx', \tag{2.2}$$

$$\|\nabla'\eta\|_{L^1(\Sigma)} \leq \|\eta\|_X |\Gamma_t|^{1/2}.$$

Proof The first line of (2.2) states the equivalence between the quantity $|\Gamma_t|-|\Sigma|$ and the squared norm $\|\eta\|^2_X$. It easily follows multiplying numerator and denominator of the integrand in

$$|\Gamma_t| - |\Sigma| = \int_\Sigma \Big(\sqrt{1+|\nabla'\eta|^2} - 1\Big) dx' \tag{2.3}$$

by $\sqrt{1+|\nabla'\eta|^2}\Big(\sqrt{1+|\nabla'\eta|^2}+1\Big)$, and employing the algebraic inequality

$$\frac{1}{2} \le \frac{w}{1+w} \le 1, \qquad\qquad w > 1,$$

with $w = \sqrt{1+|\nabla'\eta|^2}$.

The inequality $(2.2)_2$ follows by the Schwartz inequality

$$\int_\Sigma |\nabla'\eta| dx' \le \Big(\int_\Sigma \frac{|\nabla'\eta|^2}{\sqrt{1+|\nabla'\eta|^2}} dx'\Big)^{1/2} |\Gamma_t|^{1/2}.$$

For two dimensional domains it is trivial to check that it holds

$$\sup_{0,a_1} |\eta| \le c\|\nabla'\eta\|_{L^1(0,a_1)} \le c\|\eta\|_X |\Gamma_t|^{1/2}. \tag{2.4}$$

Indeed, recalling that η has zero mean value in Σ, we deduce

$$\eta(x_1\, t) = \eta(y_1,\, t) + \int_{y_1}^{x_1} \partial_s \eta(s),\, t) d\, s.$$

Integration over $y_1 \in \Sigma$ of this relation yields

$$a_1 \eta(x_1,\, t) = \int_0^{a_1}\int_{y_1}^{x_1} \partial_s \eta(s,\, t) d\, s \le a_1 \int_\Sigma |\partial_s \eta| ds \le a_1 \|\nabla'\eta\|_{L^1(\Sigma)}, \tag{2.5}$$

from which (2.4) follows.

3. The Physical Problem

In this section we formulate the physical problem, and compute the Poiseuille flow, then transform all problem in nondimensional variables. We end by computing the energy equation.

3..1 Equations of motion in Ω_t

Let us consider a fluid filling the unknown domain Ω_t. Given the regular field ϕ, defined in R^3, the equations governing incompressible fluid flows are:

$$\begin{aligned} &\partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} = \nabla\cdot\mathbf{T}(\mathbf{v},q) + \nabla\phi, \\ &\nabla\cdot\mathbf{v} = 0, \qquad\qquad \mathbf{x}\in\Omega_t,\ t\in(0,\infty), \end{aligned} \tag{3.1}$$

where Ω_t, $\mathbf{v}$, q are the unknown domain, velocity, pressure. Furthermore the gravity force $\nabla\phi$ has potential given by

$$\phi = g(\sin\beta\, x_1 - \cos\beta\, x_3),$$

where g is the gravity acceleration. To (3.1) we add the state equations

$$\begin{aligned} &\mathbf{T}(\mathbf{v},q) = -q\mathbf{I} + \nu S(\mathbf{v}), \qquad S(\mathbf{v}) = \nabla\mathbf{v} + \nabla\mathbf{v}^T, \\ &q = q(\mathbf{x},t), \qquad\qquad \mathbf{x}\in\Omega_t,\ t\in(0,\infty), \end{aligned} \tag{3.2}$$

where $\nu > 0$ is the kinematic viscosity.

3..2 Boundary conditions

Boundary conditions on a free surface are of two kinds: *kinematical, and dynamical.* The kinematic condition is expressed at the free surface by

$$\frac{\partial_\tau \zeta}{\sqrt{1+|\nabla'\zeta|^2}} = \mathbf{v}\cdot\mathbf{n}, \qquad (x',t)\in \Sigma\times(0,\infty). \tag{3.3}$$

Equation (3.3) governs the evolution of Γ_t.

For a vector field $\mathbf{v}$ we set

$$v_n := \mathbf{v}\cdot\mathbf{n}, \qquad \mathbf{v}_\tau := \mathbf{v} - v_n\mathbf{n}. \tag{3.4}$$

The nonslip condition at bottom implies that $\mathbf{v}$ must vanish on Σ, i.e.

$$\mathbf{v}(x',-\ell,t) = 0, \qquad (x',t)\in \Sigma\times(0,\infty). \tag{3.5}$$

The air above Γ_t is supposed at rest, and it is acting over the fluid with given uniform pressure p_e. On Γ_t we prescribe the dynamical jump condition on stresses defined on both sides of the free surface

$$\mathbf{T}(\mathbf{v},q)\cdot\mathbf{n} = k\mathcal{H}(\zeta)\mathbf{n} - p_e\mathbf{n}, \qquad \text{on}\, \Gamma_t, \tag{3.6}$$

the exterior pressure p_e is a given constant.

3..3 Initial conditions

To deal with unsteady incompressible motions, to (3.1), (3.5), (3.6) we add the initial conditions:

$$\begin{aligned} &\zeta(x',0) = \zeta_0(x'), && x' \in \Sigma, \\ &\mathbf{v}(\mathbf{x},0) = \mathbf{v}_0(\mathbf{x}), && \mathbf{x} \in \Omega_0. \end{aligned} \tag{3.7}$$

For existence theorems of regular solutions one must also require the **compatibility conditions**

$$\nabla \cdot \mathbf{v}_0(x) = 0, \qquad x \in \Omega_0, \qquad \mathbf{v}_0(x',-\ell,0) = 0, \qquad x' \in \Sigma. \tag{3.8}$$

3..4 A steady exact solution

The basic domain Ω_b occupied by the fluid is a piece of a flat layer, having uniform depth ℓ. Let be given the gravity force $\nabla\phi$, and the constant external pressure p_e, cf. [32], [33], [1].
Take $p_e = 0$.
The PFB flow solves the following problem: to find the depth ℓ of the rectangular parallelepiped $\Sigma \times (-\ell, 0)$, the steady velocity field $\mathbf{v}_b(x_3) = v_b(x_3)\, \boldsymbol{e}_1$, and pressure $q_b = q_b(x_3)$, namely solutions to

$$\mathbf{v}_b \cdot \nabla \mathbf{v}_b = \nabla \cdot \mathbf{T}(\mathbf{v}_b, q_b) + \nabla\phi, \qquad \mathbf{x} \in \Omega_b, \tag{3.9}$$

$$\nabla \cdot \mathbf{v}_b = 0, \qquad \mathbf{x} \in \Omega_b, \tag{3.10}$$

$$\mathbf{v}_b(x', -\ell) = 0, \qquad x' \in \Sigma, \tag{3.11}$$

$$0 = v_{b3}(x', 0), \qquad x' \in \Sigma, \tag{3.12}$$

$$\mathbf{T}(\mathbf{v}_b, q_b) \cdot \boldsymbol{e}_3 = 0, \qquad \text{on } \Gamma_b. \tag{3.13}$$

Equation (3.12) is the equation of motion of Γ_b, it states that it is the plane $x_3 = 0$.
It is easy to verify that an exact solution is given by

$$\begin{aligned} &v_b(x_3) = \frac{g}{2\nu} \sin\beta\, (\ell^2 - x_3^2), && x_3 \in (-\ell, 0) \\ &q_b(x_3) = -g \cos\beta\, x_3\,, && x_3 \in (-\ell, 0) \\ &\zeta_b(x') = \ell, && x' \in \Sigma. \end{aligned} \tag{3.14}$$

We may use (3.14) to compute the mean flow $\overline{v_b}$ and we get

$$\overline{v_b} := \frac{1}{\ell} \frac{g}{2\nu} \sin\beta \int_{-\ell}^{0} (\ell^2 - x_3^2) dx_3 = \frac{g\ell^2}{3\nu} \sin\beta. \tag{3.15}$$

Remark 3.1 *In order to avoid indeterminacy we suppose that the horizontal velocity $\mathbf{V}_b'$ of points of free surface Γ_b coincides with the velocity $\mathbf{v}_b$ of the fluid particles at $x_3 = 0$.*

3..5 IBVP in non dimensional form

$U_*,\, \ell, T_*, P_*$ denote typical dimensional quantities.
Given an external pressure p_e, a gravity acceleration g, and an inclination angle β, we take as typical length ℓ the depth of the layer, and as typical velocity U_* the mean flow $\overline{v_b}$ computed in (3.15) divided by $\sin\beta$,

$$U_* := \frac{g\ell^2}{3\nu}.$$

The non-dimensional quantities will be denoted by

$$\begin{aligned} &x = \frac{x_1}{\ell}, \quad y = \frac{x_2}{\ell}, \quad z = \frac{x_3}{\ell}, \quad t = \frac{t}{T_*} \\ &\eta = \frac{\zeta}{\ell} \quad \mathbf{U} = \frac{\mathbf{v}}{U_*}, \quad P = \frac{q}{P_*}. \end{aligned} \tag{3.16}$$

We shall assume

$$T_* = \frac{\ell}{U_*} \qquad P_* = \frac{1}{U_*^2}.$$

Since no confusion arises in the sequel we use the notation

$$x' \equiv (x,\, y).$$

We continue to name by ∇ the gradient with respect to non-dimensional variables, and by $\mathcal{H}(\eta)$ the non-dimensional curvature. Furthermore we introduce the Weber[7], Reynolds numbers

$$W := \frac{U_*^2\ell}{k}, \qquad R = \frac{U_*\ell}{\nu}, \tag{3.17}$$

where k is the surface tension, and we set

$$\mathbb{S} := \frac{g\ell}{U_*^2}\sin\beta, \qquad \mathbb{C} := \frac{g\ell}{U_*^2}\cos\beta.$$

It results

$$\begin{aligned} &\Phi = \frac{\ell}{U_*^2}\phi = \Big(\mathbb{S}\,x - \mathbb{C}\,z\Big), \qquad x \in (0, a_1), \quad z \in (-1, \eta), \\ &R\,\mathbb{S} = 3\,\sin\beta, \qquad R\,\mathbb{C} = 3\cos\beta, \qquad \frac{\mathbb{C}}{\mathbb{S}} = \cot\beta. \end{aligned} \tag{3.18}$$

[7] We notice that there are different definitions of Weber number, we follow that given in [9]

3..6 Position of the problem

Below, non-dimensional equations of motion, boundary conditions, initial conditions are listed.

Equations of motion
Using the above positions, and (3.18) the motion equations can be rewritten as follows

$$\begin{aligned} &\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla \mathbf{U} = \frac{1}{R} \nabla \cdot S(\mathbf{U}) - \nabla P + \nabla \Phi, && (3.19)\\ &\nabla \cdot \mathbf{U} = 0, && \mathbf{x} \in \Omega_t,\ t \in (0, \infty),\\ &\frac{\partial_t \eta}{\sqrt{1 + |\nabla' \eta|^2}} =: \mathbf{V} \cdot n = \mathbf{U} \cdot n, && x' \in \Sigma,\ t \in (0, \infty), \end{aligned}$$

where $\eta(x', t)$, $\mathbf{U} = \mathbf{U}(\mathbf{x}, t)$, $P = P(\mathbf{x}, t)$, are the unknown surface, velocity, pressure. Furthermore,

$$\mathbf{V} = \mathbf{U}_b + \partial_t \eta$$

represents the velocity of the points of Γ_t, $S(\mathbf{U}) = \nabla \mathbf{U} + \nabla \mathbf{U}^T$. Notice that Φ is defined by (3.18) in Ω_t, and it is zero outside Ω_t.

Boundary conditions

The no-slip condition at bottom implies that $\mathbf{U}$ must vanish on Σ, for $z = -1$, i.e.

$$\begin{aligned} &\mathbf{U}(x', -1, t) = 0, && x' \in \Sigma,\ t \in (0, \infty), \quad (3.20)\\ &\mathbf{U}(\mathbf{x}, t) \text{ periodic on} && \partial\Sigma \times (-1, 0),\ t \in (0, \infty),\\ &\frac{1}{R} S(\mathbf{U}) \cdot \mathbf{n} - P\mathbf{n} = \frac{1}{W} \mathcal{H}(\eta)\mathbf{n}, && x' \in \Sigma,\ t \in (0, \infty). \end{aligned}$$

We remark that condition $(3.20)_2$ assumes $\eta(x', t) = 0$ on $\partial\Sigma$. This assumption doesn't infer a loss of generality.

Initial conditions

To deal with unsteady incompressible motions, (3.20) we add to (3.19) the initial conditions:

$$\begin{aligned} &\eta(x', 0) = \eta_0(x'), && x' \in \Sigma,\\ &\mathbf{U}(\mathbf{x}, 0) = \mathbf{U}_0(\mathbf{x}), && \mathbf{x} \in \Omega_0. \end{aligned} \quad (3.21)$$

The non-dimensional PFB solution
We write the Poiseuille free boundary flow as exact steady solution of non-dimensional system (3.19). The basic domain Ω_b occupied by the fluid is a flat layer, which has now unitary depth, the exterior to Ω_b, $\widehat{\Omega}_b$ is the rectangular channel $z > 0$, with cross section Σ.
The gravity force has non-dimensional form

$$\nabla \Phi = \mathbb{S}\, \boldsymbol{e}_1 - \mathbb{C}\, \boldsymbol{e}_3.$$

We write the non-dimensional velocity field $\mathbf{U}_b(\mathbf{x}) = U_b(z)\boldsymbol{e}_1$, pressure $P_b(\mathbf{x}) = P(z)$, in Ω_b. We set by D the derivative with respect to z. Fields $\mathbf{U}_b$, P_b, satisfy the following boundary value problem

$$\begin{aligned} 0 &= \frac{1}{R} D^2 U_b \boldsymbol{e}_1 - \nabla P_b + \nabla \Phi, \quad (x', z) \in \Sigma \times (-1, 0) \\ S(\mathbf{U}_b) &= D U_b (\boldsymbol{e}_1 \otimes \boldsymbol{e}_3 + \boldsymbol{e}_3 \otimes \boldsymbol{e}_1), \qquad S(\mathbf{U}_b(0)) = 0, \end{aligned} \tag{3.22}$$

The exact solution, namely the PFB flow is given by

$$\begin{aligned} U_b(z) &= \frac{3}{2} \sin\beta\,(1 - z^2), & z \in (-1, 0) \\ P_b(z) &= -\mathbb{C} z\,, & z \in (-1, 0). \end{aligned} \tag{3.23}$$

3..7 Energy equation for PFB

Here we derive the well known total energy equation in Eulerian coordinates. Precisely we prove

Theorem 3.1 *Solutions to* (3.19) *satisfy the following energy identity*

$$\frac{d}{dt} E(t) = -\frac{1}{R} \|S(\mathbf{U})\|^2_{L^2(\Omega_t)}, \tag{3.24}$$

where

$$\begin{aligned} E(t) &:= \frac{1}{2} \left(\|\mathbf{U}(t)\|^2_{L^2(\Omega_t)} + \frac{1}{W} |\Gamma_t| + E_\eta(t) \right), \\ E_\eta(t) &:= \frac{1}{W} |\Gamma_t| + \mathbb{C} \|\eta\|^2_{L^2(\Sigma)} - 2\mathbb{S} \int_\Sigma x\,\eta\,dx'. \end{aligned} \tag{3.25}$$

Proof Multiply $(3.19)_1$ by $\mathbf{U}$, integrate over Ω_t, and use $(3.19)_3$, and the transport theorem given in subsection 6.4 of the appendix. Integration by parts in the resulting equation yields yields

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \|\mathbf{U}\|^2_{L^2(\Omega_t)} \\ &= -\frac{1}{2R} \|S(\mathbf{U})\|^2_{L^2(\Omega_t)} + \int_{\Gamma_t} \Big(\mathbf{U} \cdot \frac{1}{R} S(\mathbf{U})\mathbf{n} - P \mathbf{U} \cdot \mathbf{n} \Big) dS + \int_{\Omega_t} \nabla \Phi \cdot \mathbf{U}\, dv. \end{aligned} \tag{3.26}$$

Employing dynamical boundary condition $(3.20)_2$, we deduce the following identity

$$\int_{\Gamma_t} \Big(\mathbf{U} \cdot \frac{1}{R} S(\mathbf{U})\mathbf{n} - p \mathbf{U} \cdot \mathbf{n} \Big)\, dS = \frac{1}{W} \int_{\Gamma_t} \mathcal{H}(\eta) \mathbf{U} \cdot \mathbf{n}\, dS. \tag{3.27}$$

Use of $(3.19)_3$ in the integral at r.h.s. of (3.27) yields

$$\int_{\Gamma_t} \mathcal{H}(\eta) \mathbf{u} \cdot \mathbf{n}\, dS = \int_{\Gamma_t} \nabla' \cdot \Big(\frac{\nabla' \eta}{\sqrt{1 + |\nabla' \eta|^2}} \Big) \frac{\partial_t \eta}{\sqrt{1 + |\nabla' \eta|^2}}\, dS. \tag{3.28}$$

By periodicity it vanishes the term over $\partial\Gamma_t$ of the first integral at r.h.s. of (3.28), hence integrations by parts in (3.28) implies

$$\int_{\Gamma_t} \mathcal{H}(\eta)\mathbf{u}\cdot\mathbf{n}\,dS = -\frac{d}{dt}\int_{\Sigma}\sqrt{1+|\nabla'\eta|^2}\;\;dx'\,, \tag{3.29}$$

and we conclude

$$\int_{\Gamma_t}\Big(\mathbf{U}\cdot\frac{1}{R}S(\mathbf{U})\mathbf{n}-p\mathbf{U}\cdot\mathbf{n}\Big)\;dS\;=-\frac{1}{W}\frac{d}{dt}\int_{\Sigma}\sqrt{1+|\nabla'\eta|^2}dx' =: -\frac{1}{W}\frac{d}{dt}|\Gamma_t|. \tag{3.30}$$

To control the power term $\nabla\Phi\cdot\mathbf{U}$ we employ the condition that the force is conservative and write it as time derivative of potential energy[8]. Precisely by $(3.19)_3$ we get

$$\begin{aligned}&\int_{\Omega_t}\nabla\Phi\cdot\mathbf{U}\,dv=\int_{\Gamma_t}(\mathbb{S}\,x-\mathbb{C}\,\eta)\mathbf{n}\cdot\mathbf{U}\,dS=\\&\int_{\Sigma}(\mathbb{S}\,x-\mathbb{C}\,\eta)\,\partial_t\eta\,dx'=\frac{d}{dt}\Big(\,\mathbb{S}\int_{\Sigma}x\,\eta\,dx'-\frac{\mathbb{C}}{2}\,\|\eta\|^2_{L^2(\Sigma)}\Big)\,.\end{aligned} \tag{3.31}$$

Substituting (3.30) and (3.31) into (3.26) it yields (3.24).

Integrating in time (3.24) we deduce in particular

$$E(t)+\frac{1}{R}\int_0^t\|S(\mathbf{U})(s)\|^2_{L^2(\Omega_s)}ds\le\;E_0:=E(0), \tag{3.32}$$

where $E(t)$ is not positive definite because of the last term in E_η which contains the destabilizing energy due to gravity along $\boldsymbol{e}_1$.

3..8 Boundedness of the moving surface area

Below we study the term $E_\eta(t)$, we wish to find explicit sufficient conditions either on W, R, a_1[9], β, or on initial data, ensuring $E_\eta\ge 0$. We begin by defining new variables

$$(\mathbb{X},\mathbb{Y})\equiv\Big(\|\eta\|_{L^2(\Sigma)},|\Gamma_t|^{1/2}\Big). \tag{3.33}$$

Thus E_η can be written as

$$E_\eta(t):=\frac{1}{W}\mathbb{Y}^2+\mathbb{C}\mathbb{X}^2-2\,\mathbb{S}\int_{\Sigma}x\,\eta\,dx',\qquad\forall t. \tag{3.34}$$

The first two Lemmas furnish explicit sufficient conditions on W, R, a_1, β, ensuring $E_\eta>0$.

[8] Notice that we adopt a different procedure for the proof of stability of the rest in section 5., see (5.4).

[9] Here a_1 denotes the ratio between the wave length α_1 along the line of greatest slope x_1, and the depth of the layer

Lemma 3.1 *Let*

$$\mathbb{S}^2 W\, {a_1}^2 \;<\; 3\,\mathbb{C}, \tag{3.35}$$

then

$$E_\eta(t) := \frac{1}{W}\mathbb{Y}^2 + \mathbb{C}\mathbb{X}^2 - 2\,\mathbb{S}\int_\Sigma x\,\eta\,dx' \geq 0, \qquad \forall t, \tag{3.36}$$

and

$$|\Gamma_t| \leq \; W\,E_0. \tag{3.37}$$

Proof (3.36) easily follows from the Schwartz, isoperimetric inequalities and

$$|\Sigma| \leq |\Gamma_t| = \mathbb{Y}^2.$$

Actually it holds

$$\int_\Sigma x\,\eta\,dx' \leq \frac{a_1\sqrt{(a_1 a_2)}}{\sqrt{3}}\|\eta\|_{L^2(\Sigma)} \leq \frac{a_1}{\sqrt{3}}\|\eta\|_{L^2(\Sigma)}|\Sigma|^{1/2} \leq \frac{a_1}{\sqrt{3}}\mathbb{X}\mathbb{Y}.$$

By (3.35) it is

$$E_\eta := \frac{1}{W}\mathbb{Y}^2 + \mathbb{C}\mathbb{X}^2 - 2\,\mathbb{S}\int_\Sigma x\,\eta\,dx' \geq \frac{1}{W}\mathbb{Y}^2 + \;\mathbb{C}\,\mathbb{X}^2 - 2\,\mathbb{S}\frac{a_1}{\sqrt{3}}\mathbb{X}\mathbb{Y}.$$

Hence E_η is positive definite iff

$$\frac{\mathbb{C}}{W} > \frac{\mathbb{S}^2 a_1^2}{3},$$

that is equivalent to (3.35). Thus recalling the definition of $E(t)$ we deduce

$$\frac{1}{W}|\Gamma_t| \leq \; E(t) \leq E_0, \tag{3.38}$$

and Lemma is proved.

Next we achieve the same result (3.36) when at initial time the L^1 norm of η, and the total energy E_0 are sufficiently small, precisely we prove

Lemma 3.2 *Let*

$$\begin{aligned} &\mathbb{S}W\|\eta_0\|_{L^1(\Sigma)} < \frac{a_2}{2}, \\ &E_0 \;<\; \frac{a_2}{a_1}\frac{\mathbb{C}}{4\mathbb{S}^2\,W^2}, \end{aligned} \tag{3.39}$$

where a_i denote periodicity in x_i, wave-lengths, then it holds the estimate (3.37).

Proof Let us begin again with the isoperimetric inequality

$$-\int_{\Sigma} x\,\eta\,dx' \leq a_1\|\eta\|_{L^1(\Sigma)} \leq \|\eta\|_{L^1(\Sigma)}\frac{|\Sigma|}{a_2} \leq \|\eta\|_{L^1(\Sigma)}\frac{|\Gamma_t|}{a_2} \leq \frac{1}{a_2}\|\eta\|_{L^1(\Sigma)}Y^2, \tag{3.40}$$

which substituted in (3.25) implies

$$E_\eta \geq \mathbb{C}\mathbb{X}^2 + \left(\frac{1}{W} - \frac{2\mathbb{S}}{a_2}\|\eta\|_{L^1(\Sigma)}\right)Y^2. \tag{3.41}$$

In order to let $E_\eta(t)$ positive definite we assume at initial time relation $(3.39)_1$ to hold, thus $E_\eta(0)$ is positive definite. By continuity in time relation $(3.39)_2$ will continue to hold in a time interval $(0,\bar{t})$. This yields

$$\|\mathbf{U}(t)\|^2_{L^2(\Omega_t)} + \mathbb{C}\|\eta(t)\|^2_{L^2(\Sigma)} + \frac{1}{W}|\Gamma_t| \leq E_0, \qquad t < \bar{t}. \tag{3.42}$$

By use of Schwartz inequality we get

$$\|\eta(t)\|_{L^1(\Sigma)} \leq \sqrt{\Sigma}\|\eta(t)\|_{L^2(\Sigma)},$$

and deduce

$$\frac{\mathbb{C}}{|\Sigma|}\|\eta(t)\|^2_{L^1(\Sigma)} \leq E_0. \tag{3.43}$$

Hence employing $(3.39)_2$ we get

$$\|\eta(t)\|_{L^1(\Sigma)} < \frac{a_2}{2\mathbb{S}W}, \qquad t(\in (0,\bar{t}], \tag{3.44}$$

which ensures that $(3.39)_1$ is true at $\bar{t}$, therefore for all time. Moreover from (3.41) we deduce that E_η is positive for all time t which substituted in $(3.25)_1$ implies the estimate (3.37).

Concluding remarks
Lemma 3.1 requires
smallness on periodicity length a_1, inclination angle β, Weber number W.
Lemma 3.2 requires
that (3.39) must be satisfied for suitably small initial data.

Remark 3.2 *(a) Both conditions* (3.35), (3.39) *imply smallness conditions on* a_1 *that represents the ratio between the wave length along the line of greatest slope, and the depth of the layer.*
(b) Conditions (3.39) *may be satisfied requiring smallness on initial data only.*

(c) Conditions (3.39) *do not bound* a_2[10] *because it appears also at left hand side, specifically one may write*

$$\begin{aligned} &\int_0^{a_1} \frac{\int_0^{a_2} |\eta| dy}{a_2} dx < \frac{1}{2\mathbb{S}W}, \\ &\frac{E_0}{a_2} < \frac{\mathbb{C}}{4a_1 \mathbb{S}^2 W^2}. \end{aligned} \tag{3.45}$$

Next step is to deduce an "a priori" estimate for the surface area through constants independent of time.

Corollary 3.2 *Under assumptions either of Lemma 3.1, or 3.2 the surface area is bounded for all time by a constant depending on initial data only.*

If the domain is two dimensional, (2.4) *yields that also the depth of the layer is bounded for all time by a constant depending on initial data only.*

Integrating over time (3.37) we get

$$\begin{aligned} &\frac{1}{2}\left(\|\mathbf{U}\|^2_{L^2(\Omega_t)} + \frac{1}{W}|\Gamma_t| \right) \leq E_0, \\ &E_0 := \frac{1}{2}\left(\|\mathbf{U}_0\|^2_{L^2(\Omega_0)} + \mathbb{C}\|\eta_0\|^2_{\Sigma} + \frac{2}{W}|\Gamma_0| \right) - \mathbb{S}\int_{\Sigma} x\eta_0 \, dx'. \end{aligned} \tag{3.46}$$

In particular it holds for all t

$$|\Gamma_t| \leq c_0, \qquad c_0 := 2W\, E_0. \tag{3.47}$$

We remark that if $\Sigma = (0, a_1)$, under hypothesis of Lemma 3.1, or 3.2 from(2.4) the following inequalities hold

$$\begin{aligned} &\sup_{\Sigma} |\eta| \leq \|\nabla'\eta\|_{L^1(\Sigma)} \leq \|\eta\|_X |\Gamma_t|^{1/2} \leq \sqrt{c_0}\|\eta\|_X \\ &\sup_{\Sigma} |\eta| \leq c_0. \end{aligned} \tag{3.48}$$

Therefore, if either (3.35), or (3.39) hold the free surface doesn't touch the bottom, as required.

[10] This is in agreement with the result of Yih [34] who has extended to flows with free surfaces, interfaces, or density stratification, the Squire result concerning stability of threedimensional disturbances in unidirectional flows between rigid boundaries [28].

4. Equations of perturbations

The subject of this section is the definition of perturbation to kinetic field $\mathbf{U}$, to pressure field P, and to the height 1 of the layer.

We recall that stability of the steady motion $(U, P, 1)$ with respect to initial data means control, for all time t in a given norm, of the difference between S_b and the motion $S(t)$ corresponding to initial data $S(0) = S_b + S'_0$. Such difference $S(t) = S(t) - S_b$ is called perturbation. For free boundary problems it is difficult to compare two flows because the volume variables, as velocity, and pressure, are defined in domains which change by changing the motion. It is customary to reduce the problem into a fixed domain [13]. Here we propose an alternative definition of perturbation in Eulerian variables, see also [18], [19].

We explicitly remark that in the linear case stability results are independent of definition of perturbations.

We also write the problem that perturbations must satisfy.

4..1 Definition of perturbation $(\widetilde{\mathbf{u}}, \widetilde{p})$ to $(\widetilde{\mathbf{U}}_b, \widetilde{P}_b)$.

Let $\widehat{\Omega}_t$ denote the domain exterior to Ω_t that is supposed to be filled by air at rest. Our aim is a definition of perturbed fields, say $\widetilde{\mathbf{u}}$, $\widetilde{p}$, to $\widetilde{\mathbf{U}}$, $\widetilde{P}_b$ satisfying (3.19), (3.20). We achieve partially this goal by introducing the following four sub-domains functions of time:

$$A_1(t) = \{\mathbf{x} \in R^3_+ : \qquad \mathbf{x} \in \Omega_t \cap \Omega_b\};$$

$$A_2(t) = \{\mathbf{x} \in R^3_+ : \qquad \mathbf{x} \in \widehat{\Omega}_t \cap \widehat{\Omega}_b\};$$

$$A_3(t) = \{\mathbf{x} \in \widehat{\Omega}_t : \qquad z < 0\};$$

$$A_4(t) = \{\mathbf{x} \in \Omega_t : \qquad z > 0\}.$$

To define the boundaries of the sets A_i we introduce the positions

$$S_{-1} := S\{(x', -1) : x' \in \Sigma\},$$

which represents the bottom of Ω_t,

$$S_l := S\{(x', z) : x' \in \partial\Sigma, \quad z \in (-1, 0)\},$$

that represents the lateral surface. Both S_b, and S_l are fixed in time.
The set Σ_0 in the plane $z = 0$ is divided in two parts:

$$\Sigma_{-t} = \{(x', 0) : \qquad x' \in \Sigma, \quad \text{if} \quad \eta(x', t) < 0\},$$

$$\Sigma_{+t} \;=\; \{(x',0): \qquad x' \in \Sigma: \quad \text{if} \quad \eta(x',t) \;>\; 0\}.$$

At infinity we consider

$$S_\infty \;=\; \lim_{z\to\infty} \{(x',z): \qquad x' \in \Sigma,\},$$

The free boundary Γ_t is given by the union of the following two subsets:

$$\Gamma_t \;=\; \{(x',\eta) \in \Gamma_t: \quad x' \in \Sigma_{-t}\} \cup \{x',\eta \in \Gamma_t: \quad x' \in \Sigma_{+t}\} = \Gamma_{-t} \cup \Gamma_{+t}.$$

The boundaries of the subsets A_i, $i = 1, ..., 4$ are constituted by the union of the following portions of surfaces

$$\partial A_1(t) = S_0 \cup\, S_l \cup \Sigma_{+t} \cup \Gamma_{-t};$$

$$\partial A_2(t) = S_\infty \cup\, S_l \cup \Gamma_{+t} \cup \Sigma_{-t}\,;$$

$$\partial A_3(t) = \Sigma_{-t} \cup \Gamma_{-t}\,;$$

$$\partial A_4(t) = \;\Sigma_{+t} \cup\, \Gamma_{+t}\,.$$

Boundaries of the sets A_i are oriented with normal $\mathbf{N}$ directed toward the exterior of A_i. Concerning the normals, denoting by $\boldsymbol{e}_3$ the normal to the layer oriented toward the vacuum region, and by $\mathbf{n}$ the normal to Γ_t oriented toward the vacuum region, we have

$$\begin{array}{ll} \mathbf{N} = \boldsymbol{e}_3 \text{ normal to } \partial A_1(t) \cap \Sigma_0, & \mathbf{N} = \mathbf{n} \text{ normal to } \Gamma_{-t}; \\ \mathbf{N} = -\mathbf{n} \text{ normal to } \partial A_2(t) \cap \Gamma_{+t}, & \mathbf{N} = -\boldsymbol{e}_3 \text{ normal to } \partial A_2(t) \cap \Sigma_{-t}; \\ \mathbf{N} = -\mathbf{n} \text{ normal to } \partial A_3(t) \cap \Gamma_{-t}, & \mathbf{N} = \boldsymbol{e}_3 \text{ normal to } \partial A_3(t) \cap \Sigma_{-t}; \\ \mathbf{N} = -\boldsymbol{e}_3 \text{ normal to } \partial A_4(t) \cap \Sigma_{+t}, & \mathbf{N} = \mathbf{n} \text{ normal to } \partial A_4(t) \cap \Gamma_{+t}. \end{array} \tag{4.1}$$

Remark 4.1 *To have an naive idea of the sets A_i we have drown a picture in Figure 1, which shows the simplified case of just four connected sets. If one recalls the coordinates introduced by Hanzawa [27]*

$$\mathbf{x} = \mathbf{Y} + \rho(\mathbf{Y},t)\mathbf{N}(\mathbf{y}), \qquad \mathbf{Y} \in \Gamma_b, \quad \mathbf{N} = \boldsymbol{e}_3,$$

one may recover also regularity properties of the surface.

Remark 4.2 *If the number of sets where $\eta > 0$ is denumerable, denoting by $\mathcal{I}$ a set of natural numbers, and let A_i^α, $\alpha \in \mathcal{I}$ the connected domains of kind A_i of figure 1, it will happen*

$$A_i := \cup_{\alpha\in\mathcal{I}} A_i^\alpha.$$

This happens if the boundary is analytic.

For less regular boundaries, the definitions should be understood in a weaker sense cf. Plotnikov and Starovoitov, [23].

We are now in the position to define the perturbation field $\widetilde{\mathbf{u}}$ in each domain Ω_t, $\widehat{\Omega}_t$. Here the basic velocity and pressure fields $\mathbf{U}$, $\widehat{\mathbf{U}}$, P_b, $\widehat{P}_b$, are just defined in Ω_b, $\widehat{\Omega}_b$ respectively.

We define the perturbation to velocity and pressure as follows

$$\widetilde{\mathbf{u}}(\mathbf{x},t) = \begin{cases} \mathbf{u}(\mathbf{x},t) & \mathbf{x} \in \Omega_t, \\ 0 & \mathbf{x} \in \widehat{\Omega}_t. \end{cases}$$

$$\widetilde{p}(\mathbf{x},t) = \begin{cases} p(\mathbf{x},t) & \mathbf{x} \in \Omega_t, \\ \widehat{p} & \mathbf{x} \in \widehat{\Omega}_t. \end{cases}$$

Using this natural definition for $\widetilde{\mathbf{u}}$, and $\widetilde{p}$, we define $\widetilde{\mathbf{U}}$, and $\widetilde{P}$ as follows

$$\widetilde{\mathbf{U}}(\mathbf{x},t) = \begin{cases} \mathbf{U}_b + \mathbf{u}(\mathbf{x},t) & \mathbf{x} \in A_1(t), \\ 0 & \mathbf{x} \in A_2(t), \\ \mathbf{U}_b(\mathbf{x},t) & \mathbf{x} \in A_3(t), \\ \mathbf{u}(\mathbf{x},t) & \mathbf{x} \in A_4(t). \end{cases} \tag{4.2}$$

$$\widetilde{P}(\mathbf{x},t) = \begin{cases} P_b(z) + p(\mathbf{x},t) & \mathbf{x} \in A_1(t), \\ 0 & \mathbf{x} \in A_2(t), \\ P_b & \mathbf{x} \in A_3(t), \\ p(\mathbf{x},t) & \mathbf{x} \in A_4(t). \end{cases} \tag{4.3}$$

Since we study stability with respect to initial data we do not perturb the external force which will be given by

$$\widetilde{f}_b(\mathbf{x}) = \begin{cases} \nabla\Phi & \mathbf{x} \in A_1(t) \cup A_3(t) = \Omega_b, \\ 0 & \mathbf{x} \in A_2 \cup A_3(t) = \widehat{\Omega}_b. \end{cases} \tag{4.4}$$

Notice that for the moving fluid we should define the gravity force $\mathbf{f}_t$ in the form

$$\widetilde{\mathbf{f}}_t(\mathbf{x},t) = \begin{cases} \nabla\Phi & \mathbf{x} \in A_1(t) \cup A_4(t) = \Omega_t, \\ 0 & \mathbf{x} \in A_2(t) \cup A_3(t) = \widehat{\Omega}_t. \end{cases} \tag{4.5}$$

4..2 The equations of perturbation

We wish to study stability of the PFB flow. To this end we must study the evolution of a motion $(\mathbf{U}, P)$ with initial data $(\mathbf{U}_0, \eta_0)$ different from $(\mathbf{U}_b, 0)$. We are in the three-dimensional parallelepiped $\mathcal{R} = (0, a_1) \times (0, a_2) \times (-1, 0) = \Sigma \times (-1, 0)$, and set $x' \in \Sigma$, $z \in (-1, \eta)$.

Let us recall that $\mathbf{V} = \mathbf{U}_b + \eta_t \boldsymbol{e}_3$, where $\eta_t \boldsymbol{e}_3$ denotes the perturbation to the velocity $\mathbf{V}_b = \mathbf{U}_b$ of the flat boundary Σ_b. Taking the difference between equation (3.19) and (3.22), with (3.23) we obtain, for $t \in (0,\infty)$, the solutions $\mathbf{u}$, p, η to the following perturbation problem

$$
\begin{aligned}
&\partial_t \mathbf{u} + \mathbf{U}\cdot\nabla\mathbf{u} = -\,u_2\,DU_b\boldsymbol{e}_1 + \frac{1}{R}\nabla\cdot\, S(\mathbf{u}) - \nabla\, p + \frac{1}{R}D^2U_b\boldsymbol{e}_1 - \nabla\, P_b + \nabla\Phi, && A_1(t),\\
&\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = \frac{1}{R}\nabla\cdot\, S(\mathbf{u}) - \nabla\, p + \nabla\Phi, && A_4,\\
&\partial_t\eta = \mathbf{u}\cdot\mathbf{n}\sqrt{1+|\nabla'\eta|^2}, && \Gamma_t,\\
&\nabla\cdot\mathbf{u} = 0, && A_1\cup\, A_4,\\
&\frac{1}{R}S(\mathbf{u})\mathbf{n} - p\mathbf{n} = \frac{1}{W}\mathcal{H}(\eta)\mathbf{n} - \frac{1}{R}S(\mathbf{U}_b(\eta))\mathbf{n} + P_b(\eta)\mathbf{n}, && \Gamma_t^-,\\
&\frac{1}{R}S(\mathbf{u})\mathbf{n} - p\mathbf{n} = \frac{1}{W}\mathcal{H}(\eta)\mathbf{n}, && \Gamma_t^+,\\
&\mathbf{u}(x',-1,t) = 0, && x'\in\Sigma.
\end{aligned}
\tag{4.6}
$$

Notice that

$$
\frac{1}{R}D^2U_b\boldsymbol{e}_1 - \nabla\, P_b + \nabla\Phi = 0, \qquad A_1\cup\, A_3. \tag{4.7}
$$

Elementary calculations give on $\Gamma_t\cap\overline{\Omega_b}$

$$
DU_b = -3\,\sin\beta\, z,
$$

$$
S(\mathbf{U}_b)\mathbf{n} = -\frac{3}{2}\,\sin\beta\;\eta\;\Big(\boldsymbol{e}_3\otimes\boldsymbol{e}_1 + \boldsymbol{e}_1\otimes\,\boldsymbol{e}_3\Big)\cdot\frac{(-\nabla'\eta + \boldsymbol{e}_3)}{\sqrt{1+|\nabla'\eta|^2}} = -\frac{3}{2}\sin\beta\;\eta\,\frac{\Big(\boldsymbol{e}_1 - \eta_x\,\boldsymbol{e}_3\Big)}{\sqrt{1+|\nabla'\eta|^2}}.
$$

Employing these identities in (4.6) we deduce

$$
\begin{aligned}
&\partial_t \mathbf{u} + \mathbf{U}\cdot\nabla\mathbf{u} = \mathbb{S}\,R\,z\,u_3\,\boldsymbol{e}_1 + \frac{1}{R}\nabla\cdot\, S(\mathbf{u}) - \nabla p\,, && A_1,\\
&\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = \frac{1}{R}\nabla\cdot\, S(\mathbf{u}) - \nabla p + \nabla\Phi, && A_4,\\
&\mathbf{u}\cdot\mathbf{n} = \frac{\partial_t\eta}{\sqrt{1+|\nabla'\eta|^2}}, && \Sigma,\\
&\nabla\cdot\mathbf{u} = 0, \qquad\qquad \text{in} && A_1\cup\, A_4,\\
&\frac{1}{R}S(\mathbf{u})\mathbf{n} - p\mathbf{n} = \frac{1}{W}\mathcal{H}(\eta)\mathbf{n} - \frac{\mathbb{S}}{2}\,\eta\,\frac{\Big(\boldsymbol{e}_1 - \eta_x\,\boldsymbol{e}_3\Big)}{\sqrt{1+|\nabla'\eta|^2}} - \mathbb{C}\,\eta\,\mathbf{n}, && \Sigma_{-t},\\
&\frac{1}{R}S(\mathbf{u})\mathbf{n} - p\mathbf{n} = \frac{1}{W}\mathcal{H}(\eta)\mathbf{n}, && \Sigma_{+t},\\
&\mathbf{u}(x',-1,t) = 0, && x'\in\Sigma,
\end{aligned}
\tag{4.8}
$$

with periodicity in x'.

Remark 4.3 *We stress the analogy between our formula* $(4.8)_6$ *and formulas (iii) (iv) on p.323 of [31]. Actually, the domain* Σ_{-t} *coincides with* Σ *in the linearized version, moreover multiplying* $(4.8)_{5,6}$ *times tangent and normal unit vectors we obtain two scalar equations whose linear parts coincide just with formulas (iii) (iv) on p.323 of [31]. In particular there exists a linear tangent shear stress cf. [24] equation (2.2f).*

5. Nonlinear Stability

Aim of this section is the proof of a nonlinear stability result. Precisely we shall prove

Theorem 5.1 *Assume that there exist regular global in time solutions to* (4.8)*. Also we assume that in three dimensional domains* η *is uniformly bounded.*
Let the non-dimensional numbers a_1, a_2, W, $\mathbb{S}$, β *be such that either* (3.35) *is satisfied, or let initial data satisfy* (3.39)*. If*

$$1 - C_1\,\mathbb{S}\,R^2 \,>\, 0, \tag{5.1}$$

where C_1 *is function of embedding constants, and inequality* (5.22) *is satisfied, then the PFB* (3.23) *is nonlinearly exponentially stable, provided the initial height* η_0 *satisfies condition* (5.28)*, and the initial energy* E_0 *satisfies* (5.30)*.*

The results are achieved by usual Lyapunov method, we give the proof.

5..1 Energy of perturbation

We multiply $(4.8)_{1,2}$ by $\mathbf{u}$, integrate over A_1, A_4 respectively, and add the resulting equations, integrating by parts, employing Lemma 6.8 it yields

$$\frac{d}{dt}\int_{\Omega_t}\frac{\mathbf{u}^2}{2}\,dv = -\frac{1}{2R}\|S(\mathbf{u})\|^2_{L^2(\Omega_t)} + \int_{\Gamma_t}\Big(\mathbf{u}\cdot\frac{1}{R}S(\mathbf{u})\mathbf{n} - p\mathbf{u}\cdot\mathbf{n}\Big)dS$$
$$+\,\mathbb{S}\,R\int_{A_1} z\,u_1u_3\,dv + \int_{A_4}\nabla\Phi\,\cdot\mathbf{u}\,dv. \tag{5.2}$$

Boundary conditions $(4.8)_{6,7}$ infer the following identity

$$\int_{\Gamma_t}\Big(\mathbf{u}\cdot\frac{1}{R}S(\mathbf{u})\mathbf{n} - p\mathbf{u}\cdot\mathbf{n}\Big)\,dS\,=$$
$$\frac{1}{W}\int_{\Gamma_t}\mathcal{H}(\eta)\mathbf{u}\cdot\mathbf{n}\,dS - \frac{\mathbb{S}}{2}\int_{\Gamma_t^-}\eta\,\mathbf{u}\cdot\frac{(\boldsymbol{e}_1 - \eta_x\,\boldsymbol{e}_3)}{\sqrt{1+|\nabla'\eta|^2}}\,dS - \mathbb{C}\int_{\Gamma_t^-}\eta\,\mathbf{u}\cdot\mathbf{n}\,dS. \tag{5.3}$$

We also have by (3.29)

$$\frac{1}{W}\int_{\Gamma_t}\mathcal{H}(\eta)\mathbf{u}\cdot\mathbf{n}\,dS = -\frac{1}{W}\frac{d}{dt}\int_{\Sigma}(\sqrt{1+|\nabla'\eta|^2}-1)dx'.$$

Next it holds[11]

$$\begin{aligned}&\int_{A_4}\nabla\Phi\cdot\mathbf{u}\,dv = \mathbb{S}\int_{A_4}\boldsymbol{e}_1\cdot\mathbf{u}\,dv - \mathbb{C}\int_{A_4}\nabla z\cdot\mathbf{u}\,dv\\ &= \mathbb{S}\int_{A_4}u_1\,dv - \mathbb{C}\left(-\int_{\Sigma_+}z\,\boldsymbol{e}_3\cdot\mathbf{u}\,dx'\Big|_{z=0} + \int_{\Gamma_+}\eta\,\mathbf{n}\cdot\mathbf{u}\,dS\right) =: \mathbb{S}\,A_u + \mathbb{C}B_u.\end{aligned} \tag{5.4}$$

We analyze the last term B_u in (5.4) having stabilizing effect to deduce

$$B_u = -\int_{\Gamma_t^+}\eta\,\mathbf{n}\cdot\mathbf{u}\,dS. \tag{5.5}$$

Adding (5.3) to (5.4) we obtain

$$\begin{aligned}&\int_{\Gamma_t}\left(\mathbf{u}\cdot\frac{1}{R}S(\mathbf{u})\mathbf{n} - p\mathbf{u}\cdot\mathbf{n}\right)dS + \int_{\Omega_t}\nabla\Phi\cdot\mathbf{u}\,dv\\ &= -\frac{1}{W}\frac{d}{dt}\int_{\Sigma}(\sqrt{1+|\nabla'\eta|^2}-1)dx' - \frac{\mathbb{S}}{2}\int_{\Gamma_t^-}\eta\,\mathbf{u}\cdot\frac{(\boldsymbol{e}_1-\eta_x\,\boldsymbol{e}_3)}{\sqrt{1+|\nabla'\eta|^2}}\,dS\\ &-\mathbb{C}\int_{\Gamma_t}\eta\,\mathbf{u}\cdot\mathbf{n}\,dS + \mathbb{S}A_u.\end{aligned} \tag{5.6}$$

We observe that, employing (4.8) it follows

$$-\mathbb{C}\int_{\Gamma_t}\eta\,\mathbf{u}\cdot\mathbf{n}\,dS = -\mathbb{C}\int_{\Sigma}\eta\partial_t\eta\,dx' = -\frac{\mathbb{C}}{2}\frac{d}{dt}\int_{\Sigma}\eta^2\,dx'.$$

Hence we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|^2_{L^2(\Omega_t)} + \frac{1}{W}\frac{d}{dt}\int_{\Sigma}(\sqrt{1+|\nabla'\eta|^2}-1)dx' + \frac{\mathbb{C}}{2}\frac{d}{dt}\int_{\Sigma}\eta^2\,dx'\\ &= -\frac{1}{R}\|S(\mathbf{u})\|^2_{L^2(\Omega_t)} + F_1(\mathbf{u},\eta) + N_u(\mathbf{u},\eta),\end{aligned} \tag{5.7}$$

where we have set

$$F_1(\mathbf{u},\eta) = \mathbb{S}\,R\int_{A_1}z\,u_1u_3\,dv - \frac{\mathbb{S}}{2}\int_{\Sigma_{-t}}u_1\eta\,dx' + \mathbb{S}A_u,$$

$$N_u(\mathbf{u},\eta) := \frac{\mathbb{S}}{2}\int_{\Sigma_{-t}}u_3\eta\,\eta_x dx'.$$

[11] Notice the difference of procedure adopted for the power term in formula (5.4), and in formula (3.31).

To deduce a stability result by energy method we need to prove that the time derivative of the total perturbation energy $E(t)$ is negative. Therefore we look for negative right hand side in (5.7). Notice that F_1 has no definite sign and then it should be absorbed in the dissipative term $S(\mathbf{u})$. Unfortunately F_1 cannot be absorbed in $S(\mathbf{u})$ because it contains also the variable η. To recover a dissipative term for η in next subsection we use the free work method, cf. [20].

5..2 Free work equation

In this subsection we introduce the free work equation, cf. [20]. By (4.8) the kinematic equation of the boundary becomes

$$\partial_t \eta = u_3(x', \eta, t) + \mathbf{u}'(x', \eta, t) \cdot \nabla' \eta(x', t), \qquad x' \in \Sigma, \tag{5.8}$$

where the first two terms are linear. Furthermore we notice that the following identity holds

$$\int_{\Gamma_t} \eta(x', t) n_3 dS = \int_{\Sigma} \eta \, dx' = \int_{\Sigma} \left(\int_{-1}^{\eta} dz - \int_{-1}^{0} dz \right) dx' = |\Omega_t| - |\Omega_b| = 0.$$

Therefore the compatibility condition (6.3) is satisfied and we may apply Lemma 6.2. Let us multiply $(4.8)_{1,2}$ by $\mathbf{W}$ given in Lemma 6.2, and integrate over A_1, A_4 respectively, summing the resulting integrals, and employing Lemma 6.9 it yields

$$\frac{d}{dt} \int_{\Omega_t} \mathbf{u} \cdot \mathbf{W} \, dv = \int_{\Gamma_t} \mathbf{n} \cdot \Big(\frac{1}{R} S(\mathbf{u}) - p \mathbf{I} \Big) \mathbf{W} \, dS + \int_{A_4} \mathbf{W} \cdot \nabla \Phi \, dv + L + N, \tag{5.9}$$

where

$$L(\mathbf{u}, \eta) = \int_{\Omega_t} \Big[\mathbf{u} \cdot \partial_t \mathbf{W} - \frac{1}{R} S(\mathbf{u}) \cdot \nabla \mathbf{W} \Big] dv + \mathbb{S} \, R \int_{A_1} z \, W_1 u_3 \, dv, \tag{5.10}$$

$$N_w = \int_{\Omega_t} \mathbf{u} \cdot \nabla \mathbf{W} \cdot \mathbf{u} \, dv. \tag{5.11}$$

Next it holds

$$\begin{aligned} \int_{A_4} \nabla \Phi \cdot \mathbf{W} \, dv &= \mathbb{S} \int_{A_4} x \, \boldsymbol{e}_1 \cdot \mathbf{W} \, dv - \mathbb{C} \int_{A_4} \nabla z \cdot \mathbf{W} \, dv \\ &= \mathbb{S} \int_{A_4} x \, W_1 \, dv - \mathbb{C} \left(- \int_{\Sigma_{+t}} z \, \boldsymbol{e}_3 \cdot \mathbf{W} \, dx' \Big|_{z=0} + \int_{\Gamma_+} \eta \, \mathbf{n} \cdot \mathbf{W} \, dS \right) \\ &=: \mathbb{S} \, A_w - \mathbb{C} \int_{\Sigma_{+t}} \eta^2 \, dx'. \end{aligned} \tag{5.12}$$

Concerning the first term see in the appendix Lemma 6.2. The last term has a dissipative effect.

We analyze now boundary terms contained in (5.9). Boundary conditions $(4.6)_6$, and $(6.4)_3$ for $\mathbf{W}$ furnish

$$\int_{\Gamma_t} \mathbf{W}\cdot\Big(\frac{1}{R}S(\mathbf{u})-p\mathbf{I}\Big)\mathbf{n}\,dS \;=\; \frac{1}{W}\int_{\Gamma_t}\nabla'\cdot\Big(\frac{\nabla'\eta}{\sqrt{1+|\nabla'\eta|^2}}\Big)\mathbf{W}\cdot\mathbf{n}dS$$
$$-\frac{\mathbb{S}}{2}\int_{\Gamma_t^-}\eta\mathbf{W}\cdot\frac{(\boldsymbol{e}_1-\eta_x\,\boldsymbol{e}_3)}{\sqrt{1+|\nabla'\eta|^2}}\,dS-\mathbb{C}\int_{\Gamma_t^-}\eta\,\mathbf{W}\cdot\mathbf{n}\,dS$$
$$=-\frac{1}{W}\int_{\Sigma}\frac{|\nabla'\eta|^2}{\sqrt{1+|\nabla'\eta|^2}}\;dx'\;-\frac{\mathbb{S}}{2}\int_{\Sigma_{-t}}\eta\,(\,W_1-W_3\eta_x\,)\,dx'-\mathbb{C}\int_{\Sigma_{-t}}\eta^2\;dx'. \tag{5.13}$$

Therefore (5.9), together with (5.13), (5.12), yields

$$\frac{d}{dt}\int_{\Omega_t}\mathbf{u}\cdot\mathbf{W}\,dv$$
$$=-\frac{1}{W}\int_{\Sigma}\frac{|\nabla'\eta|^2}{\sqrt{1+|\nabla'\eta|^2}}\;dx'-\mathbb{C}\int_{\Sigma}\eta^2\;dx'\;+L_1(\mathbf{W},\eta)+N_w(\mathbf{W},\eta), \tag{5.14}$$

with

$$L_1(\mathbf{W},\eta):=L(\mathbf{W},\eta)+\mathbb{S}A_w-\frac{\mathbb{S}}{2}\int_{\Sigma_{-t}}W_1\eta\;\;dx',$$

$$N_w:=N+\frac{\mathbb{S}}{2}\int_{\Sigma_{-t}}W_3\eta\,\eta_x\;dx'.$$

Equation (5.14) is known as free work equation, and furnishes the wanted dissipative terms for η.

5..3 A stability result

In the sequel all numbers c_i, d_i, C_1 are embedding constants functions of η only.

Adding (5.7) to (5.14) multiplied by an arbitrary positive parameter γ we get

$$\frac{d}{dt}\mathcal{E}(t)\;\le\;-\,D(t)+\;F_1+\gamma\,L_1+N_u+\gamma\,N_w, \tag{5.15}$$

where

$$\mathcal{E}(t):=\Big[\frac{1}{2}\|\mathbf{u}\|^2_{L^2(\Omega_t)}+\frac{1}{W}\,E_\eta+\;\frac{\mathbb{C}}{2}\,\|\eta\|^2_{\Sigma_t}+\gamma\int_{\Omega_t}\mathbf{u}\cdot\mathbf{W}\,dv\Big],$$

$$D(t):=\;\frac{1}{R}\|S(\mathbf{u})\|^2_{L^2(\Omega_t)}\;+\;\gamma\,\frac{1}{W}\|\eta\|^2_X\;+\;\gamma\,\mathbb{C}\|\eta\|^2_{\Sigma},$$

Now we must estimate the quadratic functional

$$F_1 + \gamma\, L_1,$$

and the nonlinear functional

$$N_u + \gamma\, N_w.$$

Assume that η has an upper bound in space and time, $\|\eta\|_{L^\infty(\Sigma\times(0,\infty))} = h < \infty$.

Remark 5.1 *Notice that assuming the strongest hypothesis $\nabla'\eta$ has an upper bound in space and time, that is $\|\nabla'\eta\|_{L^\infty(\Sigma\times(0,\infty))} = h_1 < \infty$, the quantity $\|\eta\|_X$ becomes equivalent to $\|\nabla\eta\|_{L^2(\Sigma)} = h_1$.*

To estimate $F_1 + \gamma\, L_1$ in terms of $\mathcal{D}$ we shall use Poincare' and Korn's inequalities. In these inequalities the constants depend on h, h_1.

Employing (3.48), Lemmas 6.2, 6.5 in the Appendix, and classical inequalities plus uniform boundedness of η, it is not difficult to prove

$$\begin{aligned}
&F_1 + \gamma\, L_1 \qquad\qquad (5.16)\\
&\quad \le C_1 \mathbb{S} R\, \|S(\mathbf{u})\|^2_{L^2(\Omega_t)} + \frac{C_1}{2}\,\mathbb{S}\|S(\mathbf{u})\|_{L^2(\Omega_t)}\|\eta\|_{\Sigma_{-t}} + \frac{C_1}{2}\,\mathbb{S}\|S(\mathbf{u})\|_{L^2(\Omega_t)}\|\eta\|_X\\
&\quad + \gamma C_1 d_3\|S(\mathbf{u})\|^2_{L^2(\Omega_t)} + \gamma\left(C_1 d_3 + \frac{d_2}{R}\right)\|S(\mathbf{u})\|_{L^2(\Omega_t)}\|\eta\|_X\\
&\quad + \gamma\, C_1 d_1\, \mathbb{S}\, R\,\|S(\mathbf{u})\|_{L^2(\Omega_t)}\|\eta\|_{L^2(\Sigma)} + c_5\mathbb{S}\|\eta\|_{L^2(\Sigma)}\|S(\mathbf{u})\|_{L^2(\Omega_t)} + \gamma\, c_5 \mathbb{S}\|\eta\|_{L^2(\Sigma)}\|\eta\|_X,\\
&N_u + \gamma N_w \le\ C_3\|S(\mathbf{u})\|^2_{L^2(\Omega_t)}\|\eta\|_X,\\
&\Big|\int_{\Omega_t} \mathbf{u}\cdot\mathbf{W}\, dv\Big| \le\ C_2\|\mathbf{u}\|_{L^2(\Omega_t)}\|\eta\|_X,
\end{aligned}$$

where the constants are functions of the data.
Set

$$X = \frac{1}{\sqrt{R}}\|S(\mathbf{u})\|_{L^2(\Omega_t)}, \qquad Y = \frac{1}{\sqrt{W}}\|\eta\|_X, \qquad Z = \sqrt{\mathbb{C}}\|\eta\|_\Sigma. \tag{5.17}$$

Take

$$\gamma^2 < \frac{2}{W c_2^2},$$

where c_2 is function of d_i in the Appendix and Poincare' constant, from $(6.24)_1$ the perturbation energy. With these positions the energy results equivalent to the norm

$$\frac{1}{4}\|\mathbf{u}\|^2_{L^2(\Omega_t)} + \frac{1}{4}Y^2 + \frac{1}{2}Z^2 \le \mathcal{E}(t) \le \|\mathbf{u}\|^2_{L^2(\Omega_t)} + Y^2 + Z^2,$$

and the dissipation becomes equivalent to the following norm

$$\mathcal{D}(t) :=\ X^2 + \gamma\, Y^2 + \gamma\, Z^2.$$

It results

$$
\begin{aligned}
&\mathcal{D}(t) - F_1 - \gamma L_1 \qquad (5.18)\\
&\quad \geq X^2 + \gamma Y^2 + \gamma Z^2 - C_1 \mathbb{S} R^2 X^2 - \frac{C_1}{2}\mathbb{S}\sqrt{\frac{R}{\mathbb{C}}} X Z \\
&\quad - \frac{C_1}{2}\mathbb{S}\sqrt{RW}\, XY - \gamma C_1 d_3 R X^2 - \gamma\Big(C_1 d_3 + \frac{d_2}{R}\Big)\sqrt{RW}\, X Y \\
&\quad - \gamma C_1 d_1 \mathbb{S} R\sqrt{\frac{R}{\mathbb{C}}} XZ - c_5 \mathbb{S}\sqrt{\frac{R}{\mathbb{C}}} XZ - \gamma c_5 \mathbb{S}\sqrt{\frac{W}{\mathbb{C}}} YZ =: \mathcal{Q}_1, \\
&N_u + \gamma N_w \leq c_4 R\sqrt{W}\, X^2 Y.
\end{aligned}
$$

Using hypothesis (5.1)

$$
1 - C_1 \mathbb{S} R^2 > 0,
$$

which is analogous to the classical stability condition of Poiseuille flow, and assuming a further smallness condition on γ

$$
\gamma < \gamma_1 := min\left\{\frac{1 - C_1\mathbb{S} R^2}{C_1 d_3 R}, \frac{C_1 \mathbb{S} R}{2(C_1 d_3 R + d_2)}, \frac{C_1}{2 d_1 R}\right\}, \qquad (5.19)
$$

one may check that the quadratic form $\mathcal{Q}_1$ in (5.18) can be decreased by the quadratic form $\mathcal{Q}$

$$
\mathcal{Q} := r_1 X^2 + \gamma Y^2 + \gamma Z^2 - C_1 \mathbb{S}\sqrt{RW}\, XY - r_2 \mathbb{S}\sqrt{\frac{R}{\mathbb{C}}}\, XZ - \gamma c_5 \mathbb{S}\sqrt{\frac{W}{\mathbb{C}}}\, YZ, \qquad (5.20)
$$

where

$$
r_1 := 1 - C_1 \mathbb{S} R^2 - \gamma C_1 d_3 R, \qquad r_2 := (C_1 + c_5).
$$

By Sylvester criterion, and condition (5.19), a sufficient condition in order the form $\mathcal{Q}$ to be positive definite is given by

$$
\frac{C_1 r_2 c_5}{4\mathbb{C}}\mathbb{S}^3 RW + \frac{C_1^2 W + r_2^2}{4\mathbb{C}}\mathbb{S}^2 R \leq \Big(1 - \frac{c_5^2 \mathbb{S}^2 W}{4\mathbb{C}}\Big) r_1 \gamma_1. \qquad (5.21)
$$

Inequality (5.21) is certainly satisfied if

$$
(C_1 r_2 c_5 \mathbb{S} W + C_1^2 W + r_2^2)\mathbb{S} R \leq \Big(1 - c_5^2 \mathbb{S} W \tan\beta\Big) 4 r_1 \gamma_1 \cot\beta \qquad (5.22)
$$

which implies smallness conditions on Rayleigh R, and Weber W numbers, and on β. Conditions (5.22) are called *stability conditions* .

A sufficient condition

From the definition of γ_1, it is easy to check that for R suitably small the minimum is achieved at

$$
\gamma_1 = \frac{1}{2 C_1 d_3 R}.
$$

Moreover we may take $\mathbb{S}W$ suitably small such that

$$c_5^2 \mathbb{S}W < \frac{\cot\beta}{2},$$

and condition (5.22) infers

$$\mathbb{S}\, R^2 \leq \frac{2r_1}{2C_1 d_3 (C_1 r_2\, c_5 \mathbb{S}\, W \,+\, C_1^2 W \,+\, r_2^2)} \cot\beta, \tag{5.23}$$

which should be compared with the condition $R < (5/4)\cot\beta$ of linear stability.

If (5.22) is true then it holds

$$-\mathcal{D} + F_1 + \gamma\, L_1 \leq -\alpha\mathcal{D}, \qquad \alpha > 0, \tag{5.24}$$

which furnishes

$$-\mathcal{D} + F_1 + \gamma\, L_1 + N_u + \gamma\, N_w \leq -\alpha\mathcal{D} + \gamma\, c_4 \|\eta\|_X \mathcal{D}, \tag{5.25}$$

where α is a suitable constant. It is trivial to verify that

$$\mathcal{E}(t) \leq\, c_1^2 R\, X^2 + Y^2 + Z^2 \leq \frac{1}{\delta}\mathcal{D}, \tag{5.26}$$

where $1/\delta$ is the maximum between R and γ. Thus estimate (5.24), together with (5.16) yields

$$\frac{d}{dt}\mathcal{E}(t) \,\leq\, -\Big(\alpha - \gamma\, c_4 \|\eta_0\|_X\Big)\mathcal{D}(t). \tag{5.27}$$

Suppose at initial time

$$\|\eta_0\|_X < \frac{\alpha}{2\gamma\, c_4}, \tag{5.28}$$

thus (5.29), (5.26) imply

$$\frac{d}{dt}\mathcal{E}(t) \,\leq\, -\frac{\alpha}{2}\delta\mathcal{E}, \tag{5.29}$$

and $\mathcal{E}(t)$ is initially decreasing. therefore by Gronwall's Lemma, in the time interval $t \in (0, \bar{t})$, it follows

$$\mathcal{E}(t) \leq \mathcal{E}_0 \exp^{-\alpha\delta\, t/2}.$$

Recalling that $\|\eta(t)\|_X < 4W\mathcal{E}(t)$, the stronger hypothesis

$$4W\mathcal{E}(0) < \frac{\alpha}{2\gamma\, c_4} \tag{5.30}$$

ensures $\|\eta(t)\|_X < \frac{\alpha}{2\gamma\, c_4}$ in $\bar{t}$. Hence the estimate

$$\mathcal{E}(t) \leq \mathcal{E}_0 \exp^{-\alpha\delta\, t/2},$$

is verified for all time and furnishes the exponential decay to the steady basic flow PFB.

Remark 5.2 *Notice that the decay is of exponential rate. However, the decay constant $\alpha\delta$ is very small.*

Remark 5.3 ***Two dimensional domain*** *For two dimensional domains the boundary is a line and it is possible to give a bound of $\|\eta\|_{L^\infty(\Sigma)}$ in terms of Γ_t cf.* (2.4)*.*
Furthermore, we don't assume the strongest hypothesis that $\nabla'\eta$ has a supremum, because we may use Lemma 6.5.

In the wake of previous estimates, it is not difficult to prove

$$\begin{aligned}
&F_1 + \gamma\, L_1 \qquad (5.31)\\
&\quad \le c_1 \frac{\mathbb{S}}{2} \|S(\mathbf{u})\|_{L^2(\Omega_t)} \|\eta\|_{\Sigma_{-t}} + \mathbb{S} R\, c_0 c_1^2\, \|S(\mathbf{u})\|^2_{L^2(\Omega_t)} + c_1 c_0 \frac{\mathbb{S}}{2} \|S(\mathbf{u})\|_{L^2(\Omega_t)} \|\eta\|_X\\
&\quad + \gamma d_3 c_1 \|S(\mathbf{u})\|^2_{L^2(\Omega_t)} + \gamma \left(d_3\, c_1 + \frac{d_2}{R}\right) \|S(\mathbf{u})\|_{L^2(\Omega_t)} \|\eta\|_X\\
&\quad + \gamma\, c_0 c_1 d_1\, \mathbb{S}\, R\, \|S(\mathbf{u})\|_{L^2(\Omega_t)} \|\eta\|_{L^2(\Sigma_{-t})}\\
&\quad + \mathbb{S} c_5 \|\eta\|_{L^2(\Sigma)} \|S(\mathbf{u})\|_{L^2(\Omega_t)} + \mathbb{S} c_5 \|\eta\|_{L^2(\Sigma)} \|\eta\|_X,\\
&N_w \le\; c_3 \|S(\mathbf{u})\|^2_{L^2(\Omega_t)} \|\eta\|_X,
\end{aligned}$$

where c_1 is the Poincare'-Korn constant, cf. Lemma 6.1. The constants d_i are all given in Lemma 6.2 in the appendix, and c_2, c_3 depend on a_1. Notice that ***the constants do not depend on the solution, but only on external data!***

Remark 5.4 *The conditions* (5.22), (5.19)*, look very conservative, indeed it was not our intention to furnish optimal stability limits. We just want to show for a free boundary problem a direct stability method that allows the study of nonlinear stability avoiding the method of asymptotic limit.*

6. Appendix

This section is devoted to clarify some mathematical tools that have been employed in the stability proof.

6..1 Some integral inequalities.

The following inequalities hold true.

Lemma 6.1 *Let $\mathbf{u}$ be a solenoidal vector field in $W^{1,2}(\Omega_t)$, with boundary having the cone property. If $\mathbf{u}$ vanishes on a part Γ_t of the boundary of Ω_t, then the following embedding inequalities hold true*

$$\begin{aligned}
&\|\nabla\mathbf{u}\|^2_{L^2(\Omega_t)} \le c_* \|S(\mathbf{u})\|^2_{L^2(\Omega_t)}, && \textit{Korn inequality,}\\
&\|\mathbf{u}\|_{L^2(\Omega_t)} \le c^* \|\nabla\mathbf{u}\|_{L^2(\Omega_t)}\,, && \textit{Poincare' inequality,} \qquad (6.1)\\
&\|\mathbf{u}\|_{L^2(\Gamma_t)} \le \overline{c}\, \|S(\mathbf{u})\|_{L^2(\Omega_t)}, && \textit{Trace inequality}
\end{aligned}$$

where $\overline{c}$ is function of the aperture of the cone and of the diameter of Ω. We set

$$c_1 := \max\{c_*,\, c^*,\, \overline{c}\}. \tag{6.2}$$

This Lemma is proved for a unitary rectangle in [30], in general domains, under different boundary conditions there are several proofs, and we shall omit the proof, see e.g. [26], [6].

Remark 6.1 *We remark that if the domain is strongly Lipschitz $\overline{c}$ hence c_1 is function of the Lipschitz constant.*

6..2 Auxiliary Lemma

Let Ω_t, Ω_b be given three-dimensional layers the former function of time, the latter constant. Let $\eta_b = 0$, $\eta = \eta(x',t)$, $\quad x' \in \Sigma$ describe the fixed surface Γ_b of Ω_b, and the moving surface Γ_t of Ω_t respectively. Given the function $\mathbf{u} \in \Omega_t$, we assume that η satisfies the equation (5.8). It holds the following fundamental Lemma, cf. [21]

Lemma 6.2 *Auxiliary Function Let the field $\eta(x',t)) \in L^2(0,\infty;H^1(\Sigma))$ be a solution to* (5.8)*, and satisfy the condition*

$$\int_\Sigma \eta\, dx' = 0, \tag{6.3}$$

then there exists a vector field $\mathbf{W} \in L^\infty(0,\infty;H^1_0(\Omega_t))$ solution to the following problem

$$\begin{aligned} \nabla\cdot\mathbf{W} &= 0, && x \in \Omega_t,\\ \mathbf{W}(x',0,t) &= 0, && x' \in \Sigma,\\ \mathbf{W}(x',\eta(x',t)) &= \eta\mathbf{n}, && x' \in \Sigma. \end{aligned} \tag{6.4}$$

Moreover, there exist constants d_i, $i = 1,2,3$ depending on η, such that the following estimates hold true:

$$\begin{aligned} &\|\mathbf{W}\|_{L^2(\Omega)} \le d_1\|\eta\|_X\\ &\|\nabla\mathbf{W}\|_{L^2(\Omega)} \le d_2\|\eta\|_X,\\ &\|\partial_t\mathbf{W}\|_{L^2(\Omega)} \le d_3\left(\|S(\mathbf{u})\|_{L^2(\Omega_t)} + \|\eta\|_X\right), \end{aligned} \tag{6.5}$$

where d_i are functions of Ω_t. Also the following inequality holds true

$$\left\|\frac{|\nabla\mathbf{W}}{\sqrt[4]{1+|\nabla'\eta|^2}}\right\|_{L^2(\Omega)} \le d_4\,\|\eta\|_X. \tag{6.6}$$

Proof

Let $\mathbf{b}(x',t) = b_1\boldsymbol{e}_1 + b_2\boldsymbol{e}_2$ be a two dimensional vector such that

$$\nabla' \cdot \mathbf{b} = \eta(x',t).$$

Since the compatibility condition is satisfied we can set

$$\mathbf{b}(x',t) = \nabla'\varphi(x',t), \tag{6.7}$$

where φ is the unique, up to a constant, periodic in Σ solution to the elliptic problem

$$\nabla'^2\varphi(x',t) = \eta, \qquad x' \in \Sigma. \tag{6.8}$$

For equation (6.8) we have explicit construction of the solution. As well known the following estimates hold true

$$\begin{aligned}
&\|\mathbf{b}\|_{L^2(\Sigma)} = \|\varphi\|_{H^1(\Sigma)} \le c\|\eta\|_{H_{-1}(\Sigma)},\\
&\|\nabla'\mathbf{b}\|_{L^2(\Sigma)} = \|\varphi\|_{H^2(\Sigma)} \le c\|\eta\|_{L^2(\Sigma)},\\
&\|\nabla'\mathbf{b}\|_{H^1(\Sigma)} = \|\varphi\|_{H^3(\Sigma)} \le c\|\eta\|_{H^1(\Sigma)},\\
&\|\partial_t\mathbf{b}\|_{H^1(\Sigma)} = \|\partial_t\varphi\|_{H^2(\Sigma)} \le c\|\partial_t\eta\|_{L^2(\Sigma)},
\end{aligned} \tag{6.9}$$

where c increases several embedding constants functions of the domain.

Remark 6.2 *Notice that in* (6.9) *c increases constants functions of boundedness of Ω and of its cone property. We are assuming that the perturbed height is regular enough to ensure c uniformly bounded in time.*

We recall that Lemmas 3.1, 3.2 ensure under suitable assumptions the boundedness of surface area Γ_t, moreover in two dimension boundedness of height follows by (2.4)*.*

*Notice that, even in two dimensions, boundedness of surface area is not enough to ensure **cone property**, that **must be assumed** to hold uniformly in time.*

Since η has zero mean value we may apply Poincare' inequality in inequality $(6.9)_1$ to deduce

$$\|\mathbf{b}\|_{L^2(\Sigma)} \le c\|\eta\|_{L^2(\Sigma)}. \tag{6.10}$$

We define the three-dimensional vector $\mathbf{A}(x',t)$ as

$$\mathbf{A}(x',t) \equiv (-b_2(x',t), b_1(x',t), 0),$$

and we look for a solution in form

$$\mathbf{W} = \nabla \times (\mathbf{A}(x')\chi(x_3)), \tag{6.11}$$

where χ is a cut-off function vanishing for $z < -1/2$, equals one for $z > -1/4$, having first and second time derivatives less than one. Developing the derivative we deduce

$$\mathbf{W} = \chi(x_3)\nabla' \cdot \mathbf{b}(x',t)\boldsymbol{e}_3 - \dot{\chi}\mathbf{b}(x',t) = \chi(x_3)\eta(x',t)\boldsymbol{e}_3 - \dot{\chi}(x_3)\mathbf{b}(x',t), \tag{6.12}$$

where $\dot{\chi}$ denotes the derivative of χ with respect to x_3. It holds

$$\begin{aligned}\nabla \mathbf{W} &= \dot{\chi}(x_3)\eta(x',t)\boldsymbol{e}_3 \otimes \boldsymbol{e}_3 + \chi(x_3)\nabla'\eta(x',t) \otimes \boldsymbol{e}_3 - \\ &\quad \ddot{\chi}(x_3)\boldsymbol{e}_3 \otimes \mathbf{b}(x',t) - \chi(x_3)\nabla' \otimes \nabla'\mathbf{b}(x',t), \\ \partial_t \mathbf{W} &= \chi(x_3)\partial_t\eta(x',t)\boldsymbol{e}_3 - \dot{\chi}(x_3)\partial_t\mathbf{b}(x',t).\end{aligned} \tag{6.13}$$

From the explicit expression of $\mathbf{b}$ in terms of η we may compute time and spatial derivatives of $\mathbf{b}$. Inequalities $(6.5)_{1,2}$ follow by Calderon-Zygmund inequality, using Poincare', Korn's inequalities. Differentiating (6.7) with respect to time, and using (**??**) we infer that both the integrals at left hand side of (6.13) can be bounded by the L^2 norm of $\nabla\mathbf{u}$. Employing Poincare', Korn's inequalities we obtain $(6.5)_3$.
In two dimensions we have $\nabla' = \partial_x$ and $\mathbf{b} = \int^x \eta \boldsymbol{e}_1$.

6..3 Proof of estimates in (5.16)

We use the expressions

$$\begin{aligned}\nabla \mathbf{W} &= \dot{\chi}(z)\eta(x',t)\boldsymbol{e}_3 \otimes \boldsymbol{e}_3 + \chi(z)\nabla'\eta(x',t) \otimes \boldsymbol{e}_3 \\ &\quad \ddot{\chi}(z)\boldsymbol{e}_3 \otimes \mathbf{b}(x',t) - \chi(z)\nabla' \otimes \nabla'\mathbf{b}(x',t), \\ \partial_t \mathbf{W} &= \chi(z)\partial_t\eta(x',t)\boldsymbol{e}_3 - \dot{\chi}(z)\partial_t\mathbf{b}(x',t),\end{aligned} \tag{6.14}$$

with $\chi(z)$ a function with $C^2(R)$ norm bounded by 1. Recalling the expression of $\mathbf{b}$ it is clear that it is enough to furnish estimates for the terms

$$\chi(z)\nabla' \otimes \nabla'\mathbf{b}(x',t), \qquad \chi(z)\partial_t\eta(x',t)\boldsymbol{e}_3.$$

We also notice that it holds

$$\begin{aligned}\Big\| \frac{|\nabla \mathbf{W}}{\sqrt[4]{1+|\nabla'\eta|^2}} \Big\|^2_{L^2(\Omega)} &\leq 3\int_{\Sigma} \frac{|\dot{\chi}\eta|^2 + |\chi\nabla'\eta|^2 + |\ddot{\chi}\int^x \eta|^2 + |\ddot{\chi}\int^y \eta|^2 + |\chi\nabla'\eta|^2}{\sqrt{1+|\partial_x\eta|^2}} \\ &\leq 3\Big(\|\eta\|^2_{\Sigma} + \|\eta\|^2_X\Big).\end{aligned} \tag{6.15}$$

First estimate

To estimate the term involving the spatial derivative of $\mathbf{W}$ we limit ourselves to analyze the term with the second order spatial derivatives of $\mathbf{b}$.

$$\begin{aligned}
&\int_{\Omega_t} \chi(z)\Big(\mathbf{u}\otimes\mathbf{u} - \frac{1}{R}S(\mathbf{u})\Big)\cdot\nabla'\otimes\nabla'\mathbf{b}(x',t)\,dv \\
&= \int_\Sigma \Big(\int_{-1}^{\eta} \chi(z)\mathbf{u}\otimes\mathbf{u}\,dz\Big)\cdot\nabla'\otimes\nabla'\mathbf{b}\,dx' - \frac{1}{R}\int_\Sigma \Big(\int_{-1}^{\eta}\chi(z)S(\mathbf{u})\,dz\Big)\cdot\nabla'\otimes\nabla'\mathbf{b}\,dx' \\
&\le \Big(\int_\Sigma \Big|\int_{-1}^{\eta}\chi(z)\mathbf{u}\otimes\mathbf{u}\,dz\Big|^2\sqrt{1+|\nabla'\eta|^2}dx'\Big)^{1/2}\Big(\int_\Sigma \frac{|\nabla'^2\mathbf{b}|^2}{\sqrt{1+|\nabla'\eta|^2}}\,dx'\Big)^{1/2} \\
&+ \frac{1}{R}\Big(\int_\Sigma\Big(\int_{-1}^{\eta} S(\mathbf{u})\,dz\Big)^2\sqrt{1+|\nabla'\eta|^2}dx'\Big)^{1/2}\Big(\int_\Sigma \frac{|\nabla'^2\mathbf{b}|^2}{\sqrt{1+|\nabla'\eta|^2}}\,dx'\Big)^{1/2} \\
&= \Big\{\Big(\int_{\Gamma_t}\Big|\int_{-1}^{\eta}|\mathbf{u}|^2\,dz\Big|^2\,dS\Big)^{1/2} + \frac{1}{R}\Big(\int_\Sigma\Big(\int_{-1}^{\eta}|S(\mathbf{u})|\,dz\Big)^2\,dS\Big)^{1/2}\Big\} \\
&\quad \Big(\int_\Sigma \frac{|\nabla'^2\mathbf{b}|^2}{\sqrt{1+|\nabla'\eta|^2}}\,dx'\Big)^{1/2}
\end{aligned}$$

By trace and Korn's inequalities one deduces

$$\int_{\Omega_t} \chi(z)\Big(\mathbf{u}\otimes\mathbf{u} - \frac{1}{R}S(\mathbf{u})\Big)\cdot\nabla'\otimes\nabla'\mathbf{b}(x',t)\,dv \le \Big(1+\frac{1}{R}\Big)c\|S(\mathbf{u})\|_{L^2(\Omega_t)}\|\eta\|_X, \tag{6.16}$$

where c is function of Ω_t.

In the last step we have used trace, Korn and Poincare' inequalities. In two dimensional domains the constants are functions of data since hypotheses either (3.45) ensure boundedness of surface area Γ_t. This in turn, by (2.4), (**??**), (2.2), and (3.37) implies boundedness of the depth η. In three dimensional domains the constants are functions of unknown depth, that we assume to be bounded.

Second estimate

We begin by writing the term with time derivative of $\mathbf{W}$

$$\int_{\Omega_t} \partial_t\mathbf{W}\cdot\mathbf{u}\,dv = \int_{\Omega_t}\Big(\chi(z)\partial_t\eta(x',t)\,u_3 - \dot\chi(z)\partial_t\mathbf{b}(x',t)\cdot\mathbf{u}\Big)dv. \tag{6.17}$$

To estimate the term involving the time derivative of $\mathbf{W}$ we confine ourselves to study the term involving the time derivative of η.

We analyze only the term with the partial derivative in time of η, employing the property $|\chi(z)| < 1$, $|\dot\chi| < 1$, $\|\eta\|_{L^\infty(\Sigma)} < 1/2$. From (5.8) this term may be increased as follows

$$
\begin{aligned}
&\int_{\Omega_t} \chi(z)\partial_t\eta(x',t)\,u_3 dv \\
&\quad = \int_\Sigma \int_{-1}^{\eta} \chi(z)\,u_3(x',z,t)\Big(u_3(x',\eta,t) + \mathbf{u}'(x',\eta,t)\cdot\nabla'\eta(x',t) + U_b(\eta)\eta_{,x}(x',t)\Big)dv \\
&\quad \le \int_\Sigma \Big(|u_3(x',\eta,t)| + |\mathbf{u}'(x',\eta,t)||\nabla'\eta(x',t)| + |U_b(0)||\eta_{,x}(x',t)|\Big)\int_{-1}^{\eta}|u_3(x',z,t)|dz\,dx' \\
&\quad \le \sqrt{3}\Big[\int_\Sigma \Big(|u_3(x',\eta,t)|^2 + |\mathbf{u}'(x',\eta,t)|^2|\nabla'\eta(x',t)|^2 + |U_b(0)|^2|\eta_{,x}(x',t)|^2\Big)(1+|\nabla'\eta|^2)^{-1/2}dx'\Big]^{1/2} \\
&\quad \times\Big[\int_\Sigma\Big(\int_{-1}^{\eta}|u_3(x',z,t)|dz\Big)^2\sqrt{1+|\nabla'\eta|^2}\,dx'\Big]^{1/2} \le \sqrt{3}\Big(\int_{\Omega_t}|\nabla\,u_3|^2(x',z,t)dv\Big)^{1/2}\times \\
&\Big[\Big(\int_{\Gamma_t}\Big(|u_3(x',\eta,t)|^2 dS\Big)^{1/2} + \Big(\int_{\Gamma_t}|\mathbf{u}'(x',\eta,t)|^2 dS\Big)^{1/2} + |U_b(0)|\left(\int_\Sigma \frac{|\eta_{,x}(x',t)|^2}{\sqrt{1+|\nabla'\eta|^2}}dx'\right)^{1/2}\Big].
\end{aligned}
\tag{6.18}
$$

In (6.18) we have used the trivial inequality

$$|\nabla'\eta| \;<\; \sqrt{1+|\nabla'\eta|^2}$$

We also remark that

$$\left(\int_{\Gamma_t}(|\mathbf{u}'(x',\eta,t)|^2)^2 dS\right)^{1/2} \le \left(\int_{\Omega_t}|S(\mathbf{u}'(x',z,t))|^2 dv\right),$$

$$\int_\Sigma \frac{|\nabla'\eta(x',t)|^4}{(1+|\nabla'\eta|^2)^{3/2}}dx' \le \int_\Sigma \frac{|\nabla'\eta(x',t)|^2}{(1+|\nabla'\eta|^2)^{1/2}}dx'.$$

Thus by (6.18) we deduce

$$\int_{\Omega_t} \chi(z)\partial_t\eta(x',t)\,u_3 dv \le\; c\|S(\mathbf{u})\|_{L^2(\Omega_t}\Big(\|S(\mathbf{u})\|^2_{L^2(\Omega_t)} + \|\eta\|^2_X\Big), \tag{6.19}$$

where we employed the trace, Korn, Poincare' inequalities.

Third estimate

The last term is increased using Schwartz, Poincare' and Korn inequalities, thus employing (3.35), (6.5) we get

$$
\begin{aligned}
|\int_{A_1} z\,V_1\,u_2\,dv| &\le (1+\|\eta\|_{L^\infty(\Sigma_{-t})})c\|S(\mathbf{u})\|_{L^2(\Omega_t)}\|\eta\|_{\Sigma_{-t}} \\
&\le\; c_K(1+\|\eta\|_X)\|S(\mathbf{u})\|_{L^2(\Omega_t)}\|\eta\|_{\Sigma_{-t}}.
\end{aligned}
\tag{6.20}
$$

We furnish some further lemmas useful to obtain an inequality for the linear functional $F_1 + L_1$ in the terms of the dissipation $D(t)$ in (5.15).

Lemma 6.3 *The following inequalities are true*

$$\begin{aligned} A_u &\leq\ a_1\, c\|\eta\|_{L^2(\Sigma)}\|S(\mathbf{u})\|_{L^2(\Omega_t)}, \\ A_w &\leq a_1\, c\|\eta\|_{L^2(\Sigma)}\|\eta\|_X. \end{aligned} \tag{6.21}$$

Both inequalities can be derived by the following inequality

$$\begin{aligned} A_v &= \int_{A_4} x\, v_1(x', z, t)\, dx'dz = \int_{\Sigma_+} x\Big(\int_0^\eta \frac{dz}{dz}\, v_1(x', z, t)dz\Big)dx' \\ &= \int_{\Sigma_+} x\,\Big(z\, v_1(x', z, t)\Big)_0^\eta, dx' - \int_{\Sigma_+} x\Big(\int_0^\eta\ z\partial_z v_1(x', z, t)dz\Big)dx' \\ &\leq a_1\Big(\int_{\Sigma_+}\ \eta^2 dx' \int_{\Sigma_+}\ v_1^2(x', \eta, t)dx'\Big)^{1/2} \\ &\qquad + a_1 c\Big(\int_{\Sigma_+}\ \frac{\eta^3}{3}dx'\Big)^{1/2}\Big(\int_{\Omega_t}\ |\nabla\mathbf{v}|^2(x',\, z, t)dx\Big)^{1/2} \\ &\leq a_1\, c\Big(1 + \sqrt{\frac{\sup\,\eta}{3}}\Big)\|\eta\|_{L^2(\Sigma)}\|\nabla\mathbf{v}\|_{L^2(\Omega_t)}. \end{aligned} \tag{6.22}$$

Actually to obtain the first inequality it is enough to take $\mathbf{v} = \mathbf{u}$, and employ the Korns inequality, to get the second we take $\mathbf{v} = \mathbf{W}$, thus we must use (6.5) of Lemma 6.2.

Lemma 6.4 *The following inequality holds true*

$$\begin{aligned} &\Big|\int_{\Sigma_+}\ \mathbf{u}'\cdot\nabla'\eta\, dx'\Big|\ \leq \Big(\int_{\Sigma_+}\ |\mathbf{u}'|^2\sqrt{1 + |\nabla'\eta|^2}dx' \int_{\Sigma_+}\ \frac{|\nabla'\eta|^2}{\sqrt{1 + |\nabla'\eta|^2}}dx'\Big)^{1/2} \\ &\leq \Big(\int_{\Gamma_+}\ |\mathbf{u}'|^2 dS\Big)^{1/2}\ \|\eta\|_X\ \leq\ c\|S(\mathbf{u})\|_{L^2(\Omega_t)}\|\eta\|_X. \end{aligned} \tag{6.23}$$

We end the subsection with a Lemma true in two dimensional domains Ω_t, which appears to be very interesting.

Lemma 6.5 *Let the curve Γ_t verifies Hypothesis H1, thus the following main inequality is true*

$$\begin{aligned} &\Big|\int_{\Omega_t}\ \partial_{x_2}\, u(x_1, x_2, t)\, \eta_{x_1}(x_1, t)dx_1dx_2\Big|\ \leq\ c\|S(\mathbf{u})\|_{L^2(\Omega_t)}\|\eta\|_X, \\ &\Big|\int_{\Omega_t}\ \partial_{x_1}\, u(x_1, x_2, t)\, \xi_{x_2}(x_2, t)dx_1dx_2\Big|\ \leq\ c\|S(\mathbf{u})\|_{L^2(\Omega_t)}\|\eta\|_X. \end{aligned} \tag{6.24}$$

Proof The proof is elementary, and we shall use the trace embedding inequality

$$\Big(\int_0^{a_1}\ u^2(x_1, \eta, t)\sqrt{1 + \eta_{x_1}^2}dx_1\Big)^{1/2} = \Big(\int_{\Gamma_t}\ u^2(x_1, \eta, t)\, dS\Big)^{1/2}\ \leq\ c\,\|S(\mathbf{u})\|_{L^2(\Omega_t)},$$

where also Korn's inequality has been used. Let us begin with the derivative of u with respect to x_2. Recalling that u is zero for $x_2 = -1$, it holds

$$\begin{aligned}
&\Big|\int_0^{a_1}\int_{-1}^{\eta} \partial_{x_2} u(x_1,x_2,t)\,\eta_{x_1}(x_1,t)dx_1dx_2\Big| \\
&\quad\le \int_0^{a_1} |\eta_{x_1}|(x_1,t)\Big|\int_{-1}^{\eta}\Big|\partial_{x_2} u(x_1,x_2,t)\Big|dx_2\Big|dx_1 \\
&\quad= \int_0^{a_1} \frac{|\eta_{x_1}|}{\sqrt[4]{1+\eta_{x_1}^2}}(x_1,t)\sqrt[4]{1+\eta_{x_1}^2}\,|u|(x_1,\eta,t)dx_1 \\
&\quad\le \Big[\int_0^{a_1} \frac{\eta_{x_1}^2}{\sqrt{1+|\nabla'\eta|^2}}(x_1,t)dx_1 \int_0^{a_1} u^2(x_1,\eta,t)\sqrt{1+|\nabla'\eta|^2}dx_1\Big]^{1/2} \\
&\quad\le c\|S(\mathbf{u})\|_{L^2(\Omega_t)}\|\eta\|_X.
\end{aligned} \tag{6.25}$$

To deal the inequality with the derivative of u with respect to x_1 we employ the hypothesis that Γ_t is in normal form with respect to x_1 and deduce

$$\begin{aligned}
&\Big|\int_{\Omega_t} \partial_{x_1} u(x_1,x_2,t)\,\xi_{x_2}(x_2,t)dx_1dx_2\Big| \\
&\quad= \int_{-1}^{0} \xi_{x_2}(x_2,t)\Big(\sum_{i\in\mathcal{I}}\int_{\Sigma_i} \partial_{x_1} u(x_1,x_2,t)\,dx_1\Big)dx_2 \\
&\quad\le \Big[\int_{-1}^{0} \frac{\xi_{x_2}^2}{\sqrt{1+|\nabla'\xi|^2}}(x_2,t)dx_2 \int_{-1}^{0}\Big(\sum_{i\in\mathcal{I}}\int_{\Sigma_i} u^2(x_2,\xi,t)\sqrt{1+|\nabla'\eta|^2}dx_2\Big)\Big]^{1/2} \\
&\quad\le c\|S(\mathbf{u})\|_{L^2(\Omega_t)}\|\eta\|_X.
\end{aligned} \tag{6.26}$$

Since all terms of the form $\partial_{x_j} u \partial_{x_k}\eta$ can be reduced to one of the forms proposed by Lemma 6.5, as corollary we may state that

$$\Big|\int_{\Omega_t} |S(\mathbf{u})||\eta_x|dv\Big| \le c\|S(\mathbf{u})\|_{L^2(\Omega_t)}\|\eta\|_X.$$

Lemma 6.5 is used to increase the term in L having as integrand $S(\mathbf{u})\cdot\nabla\mathbf{W}$.

We furnish now the proof of a simple inequality which needs use of notations $x' \equiv (x_1, x_2)$.

Lemma 6.6 *Let η has zero mean value over Σ, then it holds*

$$\Big|\int_{\Sigma} x_1\,\eta(x_1,x_2)\,dx_1dx_2\Big| \le\ {a_1}^2\,|\Gamma_t|. \tag{6.27}$$

We observe that, owing the fact that η has zero mean value over Σ, it holds

$$\int_0^{a_2} \eta(a_1,x_2)\,dx_2 \ \le\ \int_{\Sigma}|\eta_{x_1}|\,dx' \tag{6.28}$$

Hence we deduce the following inequality

$$\Big|\int_0^{a_1}\int_0^{a_2} x_1\,\eta(x_1,x_2)\,dx_1dx_2\Big| \le a_1^2\int_\Sigma |\eta_{x_1}(x')|dx' \le {a_1}^2\int_\Sigma \sqrt{1+|\nabla'\eta|^2}\,dx' \le\ {a_1}^2\,|\Gamma_t|. \tag{6.29}$$

Lemma 6.7 *Let η has zero mean value over Σ, than it holds*

$$|A| \le \int_{\Sigma_+} x\Big(\int_0^\eta \partial_z u_3(x',z,t)dz\Big)dx' - \int_{\Sigma_+} x\,\mathbf{u}'\cdot\nabla'\eta\,dx' \tag{6.30}$$

$$|A| = \int_{\Sigma_+} x\Big(\int_0^\eta \partial_z u_3(x',z,t)dz\Big)dx' - \int_{\Sigma_+} x\,\mathbf{u}'\cdot\nabla'\eta\,dx' \tag{6.31}$$

6..4 Transport theorems

Below we prove a version of the classical Reynolds transport theorem for the transport of a time derivative out of a three-dimensional domain, and a new version of such theorem for a two-dimensional domain.

Lemma 6.8 *The following Reynolds transport theorem holds*

$$\int_{A_1}\Big\{\partial_t\mathbf{u}+\mathbf{U}\cdot\nabla\mathbf{u}\Big\}\cdot\mathbf{u}\,dv + \int_{A_4}\Big\{\partial_t\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}\Big\}\cdot\mathbf{u}\,dv = \frac{1}{2}\frac{d}{dt}\int_{\Omega_t}\mathbf{u}^2\,dv. \tag{6.32}$$

Proof We first notice that by Reynolds transport theorem it holds

$$\begin{aligned}
\int_{A_1}\partial_t\mathbf{u}^2\,dv &= \frac{d}{dt}\int_{A_1}\mathbf{u}^2\,dv - \int_{\Sigma_{+t}}\mathbf{u}^2\,\mathbf{V}\cdot\boldsymbol{e}_3\,dS - \int_{\Gamma_{-t}}\mathbf{u}^2\,\mathbf{V}\cdot\mathbf{n}\,dS,\\
\int_{A_1}\nabla\cdot\Big(\mathbf{u}^2\mathbf{U}\Big)\,dv &= \int_{\Sigma_{+t}}\boldsymbol{e}_3\cdot\Big(\mathbf{u}^2\mathbf{U}\Big)\,dS + \int_{\Gamma_{-t}}\mathbf{n}\cdot\Big(\mathbf{u}^2\mathbf{U}\Big)\,dS,\\
\int_{A_4}\partial_t\mathbf{u}^2\,dv &= \frac{d}{dt}\int_{A_4}\mathbf{u}^2\,dv + \int_{\Sigma_{+t}}\mathbf{u}^2\,\mathbf{V}\cdot\boldsymbol{e}_3\,dS - \int_{\Gamma_{+t}}\mathbf{u}^2\,\mathbf{V}\cdot\mathbf{n}\,dS,\\
\int_{A_4}\nabla\cdot\Big(\mathbf{u}^2\mathbf{u}\Big)\,dv &= -\int_{\Sigma_{+t}}\boldsymbol{e}_3\cdot\Big(\mathbf{u}^2\mathbf{u}\Big)\,dS + \int_{\Gamma_{+t}}\mathbf{n}\cdot\Big(\mathbf{u}^2\mathbf{u}\Big)\,dS,
\end{aligned} \tag{6.33}$$

where $\mathbf{V}$ is the velocity of the boundary. We add the four equations to get

$$\begin{aligned}
&2\int_{A_1}\Big\{\partial_t\mathbf{u}+\mathbf{U}\cdot\nabla\mathbf{u}\Big\}\cdot\mathbf{u}\,dv + 2\int_{A_4}\Big\{\partial_t\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}\Big\}\cdot\mathbf{u}\,dv = \\
&\quad\frac{d}{dt}\int_{A_1\cup A_4}\mathbf{u}^2\,dv - \int_{\Gamma_t}\mathbf{u}^2\,\mathbf{V}\cdot\mathbf{n}\,dS\\
&+\int_{\Sigma_{+t}}\boldsymbol{e}_3\cdot\Big(\mathbf{u}^2\mathbf{U}_b\Big)\,dS + \int_{\Gamma_t}\mathbf{n}\cdot\Big(\mathbf{u}^2\mathbf{u}\Big)\,dS + \int_{\Gamma_{-t}}\mathbf{n}\cdot\Big(\mathbf{u}^2\mathbf{U}_b\Big)\,dS.
\end{aligned} \tag{6.34}$$

It holds $\mathbf{U}_b \cdot \boldsymbol{e}_3 = 0$ on Σ_{+t}. Furthermore the definition of $\mathbf{U}$ yields

$$\int_{\Gamma_t} \mathbf{n} \cdot \Big(\mathbf{u}^2 \mathbf{u}\Big)\, dS + \int_{\Gamma_{-t}} \mathbf{n} \cdot \Big(\mathbf{u}^2 \mathbf{U}_b\Big)\, dS = \int_{\Gamma_t} \mathbf{n} \cdot \Big(\mathbf{u}^2 \mathbf{U}\Big)\, dS. \tag{6.35}$$

Since on Γ_t the impermeability condition imposes $\mathbf{U} \cdot \mathbf{n} = \mathbf{V} \cdot \mathbf{n}$ from (6.34), (6.35) we deduce (6.32).

Lemma 6.9 *The following Reynolds transport theorem holds*

$$\int_{A_1} \Big\{\partial_t \mathbf{u} + \mathbf{U} \cdot \nabla \mathbf{u}\Big\} \cdot \mathbf{W}\, dv + \int_{A_4} \Big\{\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}\Big\} \cdot \mathbf{W}\, dv =$$
$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_t} \mathbf{u} \cdot \mathbf{W}\, dv - \int_{A_1} \Big\{\partial_t \mathbf{W} + \mathbf{U} \cdot \nabla \mathbf{W}\Big\} \cdot \mathbf{u}\, dv - \int_{A_4} \Big\{\partial_t \mathbf{W} + \mathbf{u} \cdot \nabla \mathbf{W}\Big\} \cdot \mathbf{u}\, dv. \tag{6.36}$$

Proof We first notice that by Reynolds transport theorem, integration by parts yields

$$\int_{A_1} \partial_t \mathbf{u} \cdot \mathbf{W}\, dv = \frac{d}{dt}\int_{A_1} \mathbf{u} \cdot \mathbf{W}\, dv - \int_{A_1} \partial_t \mathbf{W} \cdot \mathbf{u}\, dv$$
$$- \int_{\Sigma_{+t}} \mathbf{u} \cdot \mathbf{W}\, \mathbf{V} \cdot \boldsymbol{e}_3\, dS - \int_{\Gamma_{-t}} \mathbf{u} \cdot \mathbf{W}\, \mathbf{V} \cdot \mathbf{n}\, dS,$$
$$\int_{A_1} \nabla \cdot \Big(\mathbf{u} \cdot \mathbf{W}\, \mathbf{U}\Big)\, dv = \int_{\Sigma_{+t}} \boldsymbol{e}_3 \cdot \Big(\mathbf{u} \cdot \mathbf{W}\, \mathbf{U}\Big)\, dS + \int_{\Gamma_{-t}} \mathbf{n} \cdot \Big(\mathbf{u} \cdot \mathbf{W}\, \mathbf{U}\Big)\, dS,$$
$$\int_{A_4} \partial_t \mathbf{u} \cdot \mathbf{W}\, dv = \frac{d}{dt}\int_{A_4} \mathbf{u} \cdot \mathbf{W}\, dv - \int_{A_4} \partial_t \mathbf{W} \cdot \mathbf{u}\, dv$$
$$+ \int_{\Sigma_{+t}} \mathbf{u} \cdot \mathbf{W}\, \mathbf{V} \cdot \boldsymbol{e}_3\, dS - \int_{\Gamma_{+t}} \mathbf{u} \cdot \mathbf{W}\, \mathbf{V} \cdot \mathbf{n}\, dS,$$
$$\int_{A_4} \nabla \cdot \Big(\mathbf{u} \cdot \mathbf{W}\, \mathbf{u}\Big)\, dv = - \int_{\Sigma_{+t}} \boldsymbol{e}_3 \cdot \Big(\mathbf{u} \cdot \mathbf{W}\, \mathbf{u}\Big)\, dS + \int_{\Gamma_{+t}} \mathbf{n} \cdot \Big(\mathbf{u} \cdot \mathbf{W}\, \mathbf{u}\Big)\, dS. \tag{6.37}$$

where $\mathbf{V}$ is the velocity of the boundary. We add the four equations to get

$$\int_{A_1} \Big\{\partial_t \mathbf{u} + \mathbf{U} \cdot \nabla \mathbf{u}\Big\} \cdot \mathbf{W}\, dv + \int_{A_4} \Big\{\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}\Big\} \cdot \mathbf{W}\, dv =$$
$$\frac{d}{dt}\int_{A_1 \cup A_4} \mathbf{u} \cdot \mathbf{W}\, dv - \int_{A_1 \cup A_4} \Big\{\partial_t \mathbf{W} + \mathbf{U} \cdot \nabla \mathbf{W}\Big\} \cdot \mathbf{u}\, dv - \int_{\Gamma_t} \mathbf{u} \cdot \mathbf{W}\, \mathbf{V} \cdot \mathbf{n}\, dS$$
$$+ \int_{\Sigma_{+t}} \boldsymbol{e}_3 \cdot \Big(\mathbf{u} \cdot \mathbf{W}\mathbf{U}_b\Big)\, dS + \int_{\Gamma_t} \mathbf{n} \cdot \Big(\mathbf{u} \cdot \mathbf{W}\mathbf{u}\Big)\, dS + \int_{\Gamma_{-t}} \mathbf{n} \cdot \Big(\mathbf{u} \cdot \mathbf{W}\mathbf{U}_b\Big)\, dS. \tag{6.38}$$

It holds $\mathbf{U}_b \cdot \boldsymbol{e}_3 = 0$ on Σ_{+t}. Furthermore the definition of $\mathbf{U}$ yields

$$\int_{\Gamma_t} \mathbf{n} \cdot \Big(\mathbf{u} \cdot \mathbf{W}\mathbf{u}\Big)\, dS + \int_{\Gamma_{-t}} \mathbf{n} \cdot \Big(\mathbf{u} \cdot \mathbf{W}\mathbf{U}_b\Big)\, dS = \int_{\Gamma_t} \mathbf{n} \cdot \Big(\mathbf{u}^2 \mathbf{U}\Big)\, dS. \tag{6.39}$$

Since on Γ_t the impermeability condition imposes $\mathbf{U} \cdot \mathbf{n} = \mathbf{V} \cdot \mathbf{n}$ from (6.38), (6.39) we deduce (6.36).

Lemma below is given for reasons of completeness, it will not be used in the proof of stability.

Lemma 6.10 *Let $f(\eta(y,t))$ be a function vanishing for $\eta = 0$, then we may carry the time derivative out of the integral sign over the variable domain Σ_{-t}, defined in subsection 4.1, to obtain*

$$\int_{\Sigma_{-t}} \partial_t f(\eta)\, dx' \;= \frac{d}{dt}\int_{\Sigma_{-t}} f(\eta)\, dx'. \tag{6.40}$$

Proof Applying the transport theorem we find

$$\frac{d}{dt}\int_{\Sigma_{-t}} f(\eta(x',t))\, dx' = \frac{d}{dt}\Big\{\sum_{i=1}^{N}\int_{\Sigma_{it}} f(\eta(x',t))\, dx'\Big\}$$
$$= \sum_{i=1}^{N}\int_{\Sigma_{it}} \partial_t f(\eta(x',t))\, dx' + \sum_{i=1}^{N}\int_{\partial\Sigma_{it}} f(\eta(x',t))V_i\, dx' = \sum_{i=1}^{N}\int_{\Sigma_{it}} \partial_t f(\eta(x',t))\, dx', \tag{6.41}$$

where V_i are the normal components of velocities of points of $\partial\Sigma_{it}$, and we have employed the property that η, vanishes at boundaries $\partial\Sigma_{it}$. Furthermore since $f(\eta)$ vanishes for $\eta = 0$, also $f(\eta)$ is zero at boundaries $\partial\Sigma_{it}$. Therefore in

$$\int_{\Sigma_{-t}} \partial_t f(\eta)\, dx', \tag{6.42}$$

we may carry out the time derivative and we get (6.40) as requested.

As corollary of this Lemma for $f(\eta) = \eta^2/2$ it yields

$$-\int_{\Sigma_{-t}} \eta\, \mathbf{u}\cdot\mathbf{n}\sqrt{1+|\nabla'\eta|^2}\, dx' \;= -\frac{d}{dt}\int_{\Sigma_{-t}} \frac{\eta^2}{2}\, dx'. \tag{6.43}$$

Notice that

$$\int_{\Sigma_{-t}} f(\eta)\eta_x\, dx' = 0,$$

where f is a regular function of η whose antiderivative F, $F'(x) = f(x)$, vanishes for $\eta = 0$.

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