MHD THREE-DIMENSIONAL STAGNATION-POINT FLOW OF A MICROPOLAR FLUID

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ABSTRACT. The steady three-dimensional stagnation-point flow of an electrically conducting micropolar fluid in the absence and in the presence of a uniform external electromagnetic field $(\mathbf{E}_0, \mathbf{H}_0)$ is analyzed and some physical situations are examined.
In particular, we proved that if we impress an external magnetic field $\mathbf{H}_0$, and we neglect the induced magnetic field, then the steady MHD three-dimensional stagnation-point flow of such a fluid is possible if, and only if, $\mathbf{H}_0$ has the direction parallel to one of the axes.
In all cases it is shown that the governing nonlinear partial differential equations admit similarity solutions. Moreover in the presence of an external magnetic field $\mathbf{H}_0$, it is found that the flow of a micropolar fluud has to satisfy an ordinary differential problem whose solution depend on $\mathbf{H}_0$ through the Hartmann number M.
Finally, the skin-friction components along the axes are computed.
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1. INTRODUCTION

An important example of flow paste a body, where the three velocity components appear, is the three-dimensional stagnation-point flow.
The simplest example of such a flow occurs when a jet of fluid impinges on a rigid wall.
The steady three-dimensional stagnation-point flow of a Newtonian fluid has been studied by Homman, Howarth, Davey ([6], [7], [1]).
For such a flow, similarity transformations reduce the Navier-Stokes equations to a system of nonlinear ODE, to which suitable boundary conditions have to be appended.
The ODE system obtained depends upon a parameter which is a measure of three-dimensionality.
The aim of this paper is to study how the steady three-dimensional stagnation-point flow of an electrically conducting micropolar fluid is influenced by a uniform external electromagnetic field $(\mathbf{E}_0, \mathbf{H}_0)$.
We recall that the micropolar fluids introduced by Eringen ([3]) physically represent fluids consisting of rigid randomly oriented particles suspended in a viscous medium which have an intrinsic rotational micromotion (for example biological fluids in thin vessels, polymeric suspensions, slurries, colloidal fluids). Extensive reviews of the theory and its applications can be found in [4] and [8].
As it is customary in literature, we assume that at infinity, the velocity $\mathbf{v}$ and the pressure p of a micropolar fluid approach the flow of an inviscid fluid for which the stagnation-point is shifted from the origin ([2], [11], [9] and [10]), and the microrotation $\mathbf{w}$ is given by $\mathbf{w} = \frac{1}{2}\nabla \times \mathbf{v}$, i.e. the micropolar fluid behaves like a classical fluid far from the wall.
For this reason, first of all, we study the steady three-dimensional stagnation-point flow of an inviscid fluid, and in the presence of an external magnetic field $\mathbf{H}_0$ we prove that neglecting the induced magnetic field, $\mathbf{H}_0$ has to be parallel to one of the three axes. We

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recall that neglecting the induced magnetic field is customary in the literature, when the magnetic Reynolds number for the flow is very small, so that the induced magnetic field is negligible in comparison with the imposed field.
Then we consider the same problems for a micropolar fluid.
Taking into account the result obtained for an inviscid fluid, we find that the flow of a micropolar fluid has to satisfy an ordinary differential problem whose solution depend on $\mathbf{H}_0$ through the Hartmann number M, if $\mathbf{H}_0$ is applied.
Moreover we analyze the skin-friction components τ_1, τ_3 along x_1 and x_3 axes, which are of physically interesting.

The paper is organized in this way:
In Section 2, we study the three-dimensional stagnation-point flow in the absence of a uniform external electromagnetic field. The subsection 2.1 is devoted to inviscid fluid, 2.2 to micropolar fluid.
Section 3 is dedicated to treat the case in the presence of a uniform external electromagnetic field. In section 3.1, we prove theorem 2, which shows that if we impress an external magnetic field $\mathbf{H}_0$, and we neglect the induced magnetic field, then the steady MHD three-dimensional stagnation-point flow of such a fluid is possible if, and only if, $\mathbf{H}_0$ has the direction parallel to one of the axes.
In section 3.2 we determine the nonlinear ODE problems for a micropolar fluid due to the theorem 2.

2. In the absence of a uniform external electromagnetic field

2.1. Inviscid Fluids. Consider the steady three-dimensional stagnation-point flow of an inviscid, homogeneous, incompressible fluid filling the region $\mathcal{S}$, given by

$$\mathcal{S} = \{\mathbf{x} \in \mathbb{R}^3 : (x_1, x_3) \in \mathbb{R}^2, x_2 > 0\}. \tag{1}$$

The boundary of $\mathcal{S}$ is a rigid, fixed wall.
The equations governing such a flow in the absence of external mechanical body forces are:

$$\begin{aligned} &\rho \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p, \\ &\nabla \cdot \mathbf{v} = 0, \qquad\qquad \text{in } \mathcal{S} \end{aligned} \tag{2}$$

where $\mathbf{v}$ is the velocity field, p is the pressure, ρ is the mass density (constant > 0).

To equations (2) we append the usual boundary condition for $\mathbf{v}$:

$$v_2 = 0 \quad \text{at} \quad x_2 = 0. \tag{3}$$

We are interested in the three-dimensional plane stagnation-point flow, so that

$$v_1 = ax_1, \quad v_2 = -a(1+b)x_2, \quad v_3 = bax_3, \qquad (x_1, x_2, x_3) \in \mathcal{S}, \tag{4}$$

with a, b constants ($a > 0, b > -1, b \neq 0$). We impose that $v_2 < 0$, so that $b > -1$.
As is easy to verify, the pressure has the following form:

$$p = -\rho \frac{a^2}{2}[x_1^2 + (1+b)^2 x_2^2 + b^2 x_3^2] + p_0 = -\rho \frac{v^2}{2} + p_0, \tag{5}$$

where the constant p_0 is the value of the pressure at the stagnation point (i.e. the origin). We note that the pressure assumes its maximum at the stagnation point.

Remark 1. *If $b = 1$, the velocity is axial symmetric with respect to x_2 axes:*

$$v_1 = ax_1, \quad v_2 = -2ax_2, \quad v_3 = ax_3.$$

Remark 2. *In order to study three-dimensional stagnation-point flow for micropolar fluids, it is convenient to consider a more general flow. More precisely, we suppose the fluid impinging on the flat plane* $x_2 = C$ *and*

$$v_1 = ax_1, \quad v_2 = -a(1+b)(x_2 - C), \quad v_3 = bax_3, \qquad (x_1, x_3) \in \mathbb{R}^2, \quad x_2 \geq C, \tag{6}$$

with C *positive constant.*
In this way, the stagnation point on the wall is not the origin but the point $(0, C, 0)$*.*
As it is easy to verify, in this case the pressure field is given by:

$$p = -\rho\frac{a^2}{2}[x_1^2 + (1+b)^2(x_2 - C)^2 + b^2x_3^2] + p_0 = -\rho\frac{v^2}{2} + p_0, \tag{7}$$

and the pressure assumes its maximum value at the stagnation-point.

2.2. **Micropolar Fluids.** Consider the steady three-dimensional stagnation-point flow of a homogeneous incompressible micropolar fluid towards a flat surface coinciding with the plane $x_2 = 0$.
In the absence of external mechanical body forces and body couples, the governing equations for such a fluid are ([8])

$$\begin{aligned} &\mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho}\nabla p + (\nu + \nu_r)\triangle \mathbf{v} + 2\nu_r(\nabla \times \mathbf{w}), \\ &\nabla \cdot \mathbf{v} = 0, \\ &I\mathbf{v} \cdot \nabla \mathbf{w} = \lambda \triangle \mathbf{w} + \lambda_0 \nabla(\nabla \cdot \mathbf{w}) - 4\nu_r \mathbf{w} + 2\nu_r(\nabla \times \mathbf{v}) \quad \text{in } \mathcal{S}, \end{aligned} \tag{8}$$

where $\mathbf{v}$ is the velocity field, p is the pressure, ρ is the mass density (constant > 0), $\mathbf{w}$ is the microrotation field, ν is the kinematic newtonian viscosity coefficient, ν_r is the microrotation viscosity coefficient, λ, λ_0 (positive constants) are material parameters related to the coefficient of angular viscosity and I is the microinertia coefficient.
We notice that in [3], [4], eqs. (8) are slightly different, as they are deduced as a special case of much more general model of microfluids. For the details, we refer to [8], p.23.
As far as the boundary conditions are concerned, we prescribe the appropriate boundary condition for the velocity $\mathbf{v}$ and the microrotation $\mathbf{w}$, i.e.

$$\mathbf{v}|_{x_2=0} = \mathbf{0}, \quad \mathbf{w}|_{x_2=0} = \mathbf{0} \text{ (strict adherence condition)}. \tag{9}$$

Other boundary conditions are possible. We refer to Eringen ([3], p.17-18) for a complete discussion. In our studies we will always assume the strict adherence condition.
We search $\mathbf{v}$, $\mathbf{w}$ in the following form

$$\begin{aligned} &v_1 = ax_1 f'(x_2), \quad v_2 = -a[f(x_2) + bg(x_2)], \quad v_3 = bax_3 g'(x_2), \\ &w_1 = bx_3 F(x_2), \quad w_2 = 0, \quad w_3 = x_1 G(x_2), \qquad (x_1, x_2, x_3) \in \mathcal{S}, \end{aligned} \tag{10}$$

where f, g, F, G are unknown sufficiently regular functions.
The conditions (9) supply

$$\begin{aligned} &f(0) = 0, \quad f'(0) = 0, \quad g(0) = 0, \quad g'(0) = 0, \\ &F(0) = 0, \quad G(0) = 0. \end{aligned} \tag{11}$$

Moreover, as is customary when studying the stagnation-point flow for viscous fluids, we assume that at infinity, the flow approaches the flow of an inviscid fluid, whose velocity is

given by (6).
Therefore, to (10) we must append also the following conditions

$$\lim_{x_2\to+\infty} f'(x_2) = 1, \quad \lim_{x_2\to+\infty} g'(x_2) = 1,$$
$$\lim_{x_2\to+\infty} F(x_2) = 0, \quad \lim_{x_2\to+\infty} G(x_2) = 0. \tag{12}$$

Conditions $(12)_{3,4}$ mean that at infinity, $\mathbf{w} = \dfrac{1}{2}\nabla\times\mathbf{v} = \mathbf{0}$, i.e. the micropolar fluid behaves like an inviscid fluid.
The asymptotic behaviour of f and g at infinity are:

$$f \sim x_2 - A, \quad g \sim x_2 - B \quad \text{as} \quad x_2 \to \infty, \tag{13}$$

where $A,\ B$ are related to the constant C in the following way:

$$f + bg \sim x_2 - A + (x_2 - B)b = (1+b)(x_2 - C) \quad \Rightarrow \quad A + bB = C(1+b). \tag{14}$$

In order to determine p, f, g, F, G we substitute (10) in $(8)_{1,3}$. We note that the incompressibly condition $(8)_2$ is automatically satisfied. After some calculations, we arrive at

$$ax_1\left[(\nu+\nu_r)f''' + af''(f+bg) - af'^2 + \frac{2\nu_r}{a}G'\right] = \frac{1}{\rho}\frac{\partial p}{\partial x_1},$$
$$-(\nu+\nu_r)a(f''+bg'') - a^2(f'+bg')(f+bg) + 2\nu_r(bF-G) = \frac{1}{\rho}\frac{\partial p}{\partial x_2},$$
$$abx_3\left[(\nu+\nu_r)g''' + ag''(f+bg) - abg'^2 - \frac{2\nu_r}{a}F'\right] = \frac{1}{\rho}\frac{\partial p}{\partial x_3},$$
$$b\{\lambda F'' + Ia[F'(f+bg) - bFg'] - 2\nu_r(2F - ag'')\} = 0,$$
$$\lambda G'' + Ia[G'(f+bg) - Gf'] - 2\nu_r(2G + af'') = 0. \tag{15}$$

Since we are interested in three-dimensional flow, we assume $b \neq 0$ and so equation $(15)_4$ can be replaced by

$$\lambda F'' + Ia[F'(f+bg) - bFg'] - 2\nu_r(2F - ag'') = 0.$$

Then, by integrating $(15)_2$, we find

$$p(x_1,x_2,x_3) = -\frac{1}{2}\rho a^2[f(x_2)+bg(x_2)]^2 - \rho a(\nu+\nu_r)[f'(x_2)+bg'(x_2)]$$
$$+ 2\nu_r\rho\int_0^{x_2}[bF(s) - G(s)]ds + P(x_1,x_3),$$

where the function $P(x_1,x_3)$ is determined supposing that, far from the wall, the pressure p has the same behaviour as for an inviscid fluid, whose velocity and pressure are given by (6) and (7) respectively.
Therefore, under the assumption $F, G \in L^1([0,+\infty))$, by virtue of (12), (13), we get

$$P(x_1,x_3) = -\rho\frac{a^2}{2}({x_1}^2 + b^2x_3^2) + p_0,$$

where p_0 is a suitable constant. Finally, the pressure field assumes the form

$$p(x_1,x_2,x_3) = -\rho\frac{a^2}{2}\{x_1^2 + [f(x_2)+bg(x_2)]^2 + b^2x_3^2\} - \rho a(\nu+\nu_r)[f'(x_2)+bg'(x_2)]$$
$$+ 2\nu_r\rho\int_0^{x_2}[bF(s) - G(s)]ds + p_0. \tag{16}$$

We note that p_0 is the pressure at the origin.
In consideration of (16), we obtain the ordinary differential system

$$
\begin{aligned}
&\frac{\nu+\nu_r}{a}f''' + (f+bg)f'' - f'^2 + 1 + \frac{2\nu_r}{a^2}G' = 0,\\
&\frac{\nu+\nu_r}{ba}g''' + \frac{1}{b}(f+bg)g'' - g'^2 + 1 - \frac{2\nu_r}{ba^2}F' = 0,\\
&\lambda F'' + aI[F'(f+bg) - bFg'] - 2\nu_r(2F - ag'') = 0,\\
&\lambda G'' + aI[G'(f+bg) - Gf'] - 2\nu_r(2G + af'') = 0.
\end{aligned}
\tag{17}
$$

To these equations we append the boundary conditions (11), (12).

Remark 3. *If $\nu_r = 0$, then $(17)_1$ and $(17)_2$ are the equations governing the three-dimensional stagnation-point flow of a Newtonian fluid.*
If $b = g = 0$, we observe that $(17)_1$ and $(17)_4$ have the same form as the equations found by Guram and Smith ([5]) for the orthogonal stagnation-point flow of a micropolar fluid.

Remark 4. *If $b = 1$, $f = g$, $F = -G$, axial symmetric case is obtained and f, F have to satisfied the following differential system:*

$$
\begin{aligned}
&\frac{\nu+\nu_r}{a}f''' + 2ff'' - f'^2 + 1 - \frac{2\nu_r}{a^2}F' = 0,\\
&\lambda F'' + aI[2fF' - Ff'] - 2\nu_r(2F - af'') = 0,
\end{aligned}
\tag{18}
$$

together with the boundary conditions $(11)_{1,2,5}$ $(12)_{1,3}$.

We summarize the previous results in the following

Theorem 1. *Let a homogeneous, incompressible, micropolar fluid occupy the region $\mathcal{S}$. The steady three-dimensional stagnation-point flow of such a fluid has the following form :*

$$
\begin{aligned}
\mathbf{v} =& [ax_1 f'(x_2)]\mathbf{e}_1 - a[f(x_2) + bg(x_2)]\mathbf{e}_2 + [bax_3 g'(x_2)]\mathbf{e}_3,\\
\mathbf{w} =& bx_3 F(x_2)\mathbf{e}_1 + x_1 G(x_2)\mathbf{e}_3,\\
p =& -\rho\frac{a^2}{2}\left\{x_1^2 + [f(x_2) + bg(x_2)]^2 + b^2 x_3^2\right\} - \rho a(\nu+\nu_r)[f'(x_2) + bg'(x_2)]\\
&+ 2\nu_r\rho\int_0^{x_2}[bF(s) - G(s)]ds + p_0, \qquad (x_1, x_2, x_3) \in \mathcal{S},
\end{aligned}
$$

where (f, g, F, G) satisfies the problem (17), (11), and (12), provided $F, G \in L^1([0, +\infty))$.

Now we write the system (17), together with the conditions (11), (12), in dimensionless form. To this end we put

$$
\begin{aligned}
&\eta = \sqrt{\frac{a}{\nu+\nu_r}}x_2, \quad \phi(\eta) = \sqrt{\frac{a}{\nu+\nu_r}} f\left(\sqrt{\frac{\nu+\nu_r}{a}}\eta\right),\\
&\gamma(\eta) = \sqrt{\frac{a}{\nu+\nu_r}} g\left(\sqrt{\frac{\nu+\nu_r}{a}}\eta\right), \quad \Phi(\eta) = \frac{2\nu_r}{a^2}\sqrt{\frac{a}{\nu+\nu_r}} F\left(\sqrt{\frac{\nu+\nu_r}{a}}\eta\right),\\
&\Gamma(\eta) = \frac{2\nu_r}{a^2}\sqrt{\frac{a}{\nu+\nu_r}} G\left(\sqrt{\frac{\nu+\nu_r}{a}}\eta\right).
\end{aligned} \tag{19}
$$

So system (20) can be written as

$$
\begin{aligned}
&\phi''' + (\phi + b\gamma)\phi'' - \phi'^2 + 1 + \Gamma' = 0,\\
&\gamma''' + (\phi + b\gamma)\gamma'' - b\gamma'^2 + b - \Phi' = 0,\\
&\Phi'' + c_3\Phi'(\phi + b\gamma) - \Phi(c_3 b\gamma' + c_2) + c_1\gamma'' = 0,\\
&\Gamma'' + c_3\Gamma'(\phi + b\gamma) - \Gamma(c_3\phi' + c_2) - c_1\phi'' = 0.
\end{aligned} \tag{20}
$$

where

$$
c_1 = \frac{4\nu_r^2}{\lambda a}, \quad c_2 = \frac{4\nu_r(\nu+\nu_r)}{\lambda a}, \quad c_3 = \frac{I}{\lambda}(\nu+\nu_r).
$$

The boundary conditions in dimensionless form become:

$$
\begin{aligned}
&\phi(0) = 0, \quad \phi'(0) = 0,\\
&\gamma(0) = 0, \quad \gamma'(0) = 0,\\
&\Phi(0) = 0, \quad \Gamma(0) = 0,\\
&\lim_{\eta\to+\infty} \phi'(\eta) = 1, \quad \lim_{\eta\to+\infty} \gamma'(\eta) = 1,\\
&\lim_{\eta\to+\infty} \Phi(\eta) = 0, \quad \lim_{\eta\to+\infty} \Gamma(\eta) = 0.
\end{aligned} \tag{21}
$$

Remark 5. *It is physically interesting to determine the skin-friction components τ_1, τ_3 along x_1 and x_3 axes :*

$$
\begin{aligned}
&\tau_1 = (\mu+\mu_r)\left(\frac{\partial v_1}{\partial x_2}\right)_{x_2=0} = \rho(\nu+\nu_r)^{1/2}a^{3/2}x_1\phi''(0),\\
&\tau_3 = (\mu+\mu_r)\left(\frac{\partial v_3}{\partial x_2}\right)_{x_2=0} = \rho(\nu+\nu_r)^{1/2}ba^{3/2}x_3\gamma''(0).
\end{aligned} \tag{22}
$$

3. In the presence of a uniform external electromagnetic field

3.1. **Inviscid Fluids.**

Consider the steady plane MHD flow of an inviscid, homogeneous, incompressible, electrically conducting fluid near a stagnation point occupying the region $\mathcal{S}$ given by (1). The boundary of $\mathcal{S}$, having the equation $x_2 = 0$, is a rigid, fixed, non-electrically conducting wall.

The equations governing such a flow in the absence of external mechanical body forces are:

$$\begin{aligned}
&\rho \mathbf{v}\cdot\nabla\mathbf{v} = -\nabla p + \mu_e(\nabla\times\mathbf{H})\times\mathbf{H},\\
&\nabla\cdot\mathbf{v} = 0,\\
&\nabla\times\mathbf{H} = \sigma_e(\mathbf{E}+\mu_e\mathbf{v}\times\mathbf{H}),\\
&\nabla\times\mathbf{E} = \mathbf{0},\quad \nabla\cdot\mathbf{E} = 0,\quad \nabla\cdot\mathbf{H} = 0, \qquad \text{in } \mathcal{S}
\end{aligned} \tag{23}$$

where $\mathbf{v}$ is the velocity field, p is the pressure, $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic fields, respectively, ρ is the mass density (constant > 0), μ_e is the magnetic permeability, σ_e is the electrical conductivity (μ_e, σ_e = constants > 0).
We suppose that an external uniform magnetic field $\mathbf{H}_0$ is impressed and that the electric field is absent. As it is customary in the literature, we further assume that the magnetic Reynolds number for the flow is very small, so that the induced magnetic field is negligible in comparison with the imposed field. Then

$$(\nabla\times\mathbf{H})\times\mathbf{H} \simeq \sigma_e\mu_e(\mathbf{v}\times\mathbf{H}_0)\times\mathbf{H}_0. \tag{24}$$

Now our aim is to prove the following:

Theorem 2. *Let a homogeneous, incompressible, electrically conducting inviscid fluid occupy the region $\mathcal{S}$. If we impress an external magnetic field $\mathbf{H}_0$, and we neglect the induced magnetic field, then the steady MHD three-dimensional stagnation-point flow of such a fluid is possible for all $b(> -1)$ if, and only if, $\mathbf{H}_0$ has the direction parallel to one of the axes.*

Proof. For brevity sake, we will denote by $\mathbf{H}$ the external magnetic field:

$$\mathbf{H} = H_1\mathbf{e}_1 + H_2\mathbf{e}_2 + H_3\mathbf{e}_3, \tag{25}$$

where $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is the canonical base of $\mathbb{R}^3$.
On substituting the approximation (24) in $(23)_1$, taking into account that the velocity field $\mathbf{v}$ is given by (4), we get:

$$\begin{aligned}
\frac{\partial p}{\partial x_1} &= -\rho a^2 x_1 + \sigma_e a[bB_1B_3x_3 - (B_2^2+B_3^2)x_1 - (1+b)B_1B_2x_2],\\
\frac{\partial p}{\partial x_2} &= -\rho a^2(1+b)^2 x_2 + \sigma_e a[B_1B_2x_1 + (1+b)(B_1^2+B_3^2)x_2 + bB_2B_3x_3],\\
\frac{\partial p}{\partial x_3} &= -\rho a^2 b^2 x_3 + \sigma_e a[-(1+b)B_2B_3x_2 - b(B_1^2+B_2^2)x_3 + B_1B_3x_1],
\end{aligned} \tag{26}$$

where $\mathbf{B} = \mu_e\mathbf{H}$.

It is possible to find a function $p = p(x_1, x_2, x_3)$ satisfying (26) if and only if

$$\frac{\partial^2 p}{\partial x_i\partial x_j} = \frac{\partial^2 p}{\partial x_j\partial x_i}, \quad i,j = 1,2,3,\ i\neq j. \tag{27}$$

On the other hand we have

$$\frac{\partial^2 p}{\partial x_1\partial x_2} = -\sigma_e a(1+b)B_1B_2, \quad \frac{\partial^2 p}{\partial x_2\partial x_1} = \sigma_e aB_1B_2, \tag{28}$$

$$\frac{\partial^2 p}{\partial x_1\partial x_3} = \sigma_e abB_1B_3, \quad \frac{\partial^2 p}{\partial x_3\partial x_1} = \sigma_e aB_1B_3, \tag{29}$$

$$\frac{\partial^2 p}{\partial x_2\partial x_3} = \sigma_e abB_2B_3, \quad \frac{\partial^2 p}{\partial x_3\partial x_2} = -\sigma_e a(1+b)B_2B_3. \tag{30}$$

Therefore, since b is arbitrary (> -1), conditions (28), (29), (30) are satisfied if and only if $\mathbf{B} = B\mathbf{e}_1$ or $\mathbf{B} = B\mathbf{e}_2$ or $\mathbf{B} = B\mathbf{e}_3$.
Finally if $\mathbf{B} = B\mathbf{e}_1$ we deduce

$$p(x_1, x_2, x_3) = -\rho\frac{v^2}{2} + \frac{a}{2}\sigma_e B^2[(1+b)x_2^2 - bx_3^2] + p_0; \tag{31}$$

if $\mathbf{B} = B\mathbf{e}_2$ we deduce

$$p(x_1, x_2, x_3) = -\rho\frac{v^2}{2} - \frac{a}{2}\sigma_e B^2[x_1^2 + bx_3^2] + p_0; \tag{32}$$

if $\mathbf{B} = B\mathbf{e}_3$ we deduce

$$p(x_1, x_2, x_3) = -\rho\frac{v^2}{2} - \frac{a}{2}\sigma_e B^2[x_1^2 - (1+b)x_2^2] + p_0. \tag{33}$$

□

Remark 6. *The results obtained in theorem 2 hold for any b. We remark that if $b = 1$, it is possible to consider also the magnetic field parallel to the plane Ox_1x_3, while if $b = -\frac{1}{2}$ it is possible to consider the magnetic field parallel to the plane Ox_2x_3.*
In the sequel we will consider only the results which hold for any b.

3.2. **Micropolar Fluids.** Consider the steady three-dimensional stagnation-point flow of an electrically conducting homogeneous incompressible micropolar fluid towards a flat surface coinciding with the plane $x_2 = 0$, the flow being confined to the region $\mathcal{S}$. In the absence of external mechanical body forces and body couples, the MHD equations for such a fluid are ([8])

$$\begin{aligned}
&\mathbf{v}\cdot\nabla\mathbf{v} = -\frac{1}{\rho}\nabla p + (\nu+\nu_r)\triangle\mathbf{v} + 2\nu_r(\nabla\times\mathbf{w}) + \frac{\mu_e}{\rho}(\nabla\times\mathbf{H})\times\mathbf{H},\\
&\nabla\cdot\mathbf{v} = 0,\\
&I\mathbf{v}\cdot\nabla\mathbf{w} = \lambda\triangle\mathbf{w} + \lambda_0\nabla(\nabla\cdot\mathbf{w}) - 4\nu_r\mathbf{w} + 2\nu_r(\nabla\times\mathbf{v}) = \mathbf{0}
\end{aligned} \tag{34}$$

together with $(23)_3 - (23)_6$, and boundary conditions (11), (12).
In order to study the influence of a uniform external electromagnetic field, we continue to use the approximation (24), where $\mathbf{v}$ is given by $(10)_{1,2,3}$. As follows from theorem 2, we consider the three cases demonstrated in this theorem.

3.2.1. *CASE I.*

$$\mathbf{H}_0 = H_0\mathbf{e}_1.$$

$$(\nabla\times\mathbf{H})\times\mathbf{H} \simeq \sigma_e\mu_e a\{[H_0^2(f+bg)]\mathbf{e}_2 - bH_0^2 g' x_3\mathbf{e}_3\} \tag{35}$$

As we can see from $(34)_3$, equations $(15)_{4,5}$ are not modified by the presence of the external magnetic field.

We substitute (35) in $(34)_1$ to obtain

$$ax_1\left[(\nu+\nu_r)f'''+af''(f+bg)-af'^2+\frac{2\nu_r}{a}G'\right]=\frac{1}{\rho}\frac{\partial p}{\partial x_1},$$

$$-(\nu+\nu_r)a(f''+bg'')-a^2(f'+bg')(f+bg)+2\nu_r(bF-G)+\frac{\sigma_e a}{\rho}B_0^2(f+bg)=\frac{1}{\rho}\frac{\partial p}{\partial x_2},$$

$$abx_3\left[(\nu+\nu_r)g'''+ag''(f+bg)-abg'^2-\frac{2\nu_r}{a}F'-\frac{\sigma_e}{\rho}B_0^2g'\right]=\frac{1}{\rho}\frac{\partial p}{\partial x_3}. \tag{36}$$

Then, by integrating $(36)_2$, we find

$$p(x_1,x_2,x_3)=-\frac{1}{2}\rho a^2[f(x_2)+bg(x_2)]^2-\rho a(\nu+\nu_r)[f'(x_2)+bg'(x_2)]$$
$$+2\nu_r\rho\int_0^{x_2}[bF(s)-G(s)]ds+\sigma_e aB_0^2\int_0^{x_2}[f(s)+bg(s)]ds+P(x_1,x_3),$$

where the function $P(x_1,x_3)$ is determined supposing that, far from the wall, the pressure p has the same behaviour as for an inviscid fluid, whose velocity is given by (6) and the pressure is given by (31) replacing x_2 by x_2-C.
Therefore, under the assumption $F,G\in L^1([0,+\infty))$, by virtue of (12), (13), we get

$$P(x_1,x_3)=-\rho\frac{a^2}{2}({x_1}^2+b^2x_3^2)-\frac{a}{2}\sigma_eB_0^2bx_3^2+p_0,$$

where p_0 is a suitable constant. Finally, the pressure field assumes the form

$$p(x_1,x_2,x_3)=-\rho\frac{a^2}{2}\{x_1^2+[f(x_2)+bg(x_2)]^2+b^2x_3^2\}-\rho a(\nu+\nu_r)[f'(x_2)+bg'(x_2)]$$
$$+2\nu_r\rho\int_0^{x_2}[bF(s)-G(s)]ds+\sigma_e aB_0^2\{\int_0^{x_2}[f(s)+bg(s)]ds-\frac{b}{2}x_3^2\}+p_0. \tag{37}$$

We note that p_0 is the pressure at the origin.
In consideration of (37), we obtain the ordinary differential system

$$\frac{\nu+\nu_r}{a}f'''+(f+bg)f''-f'^2+1+\frac{2\nu_r}{a^2}G'=0,$$
$$\frac{\nu+\nu_r}{a}g'''+(f+bg)g''-bg'^2+b-\frac{2\nu_r}{a^2}F'+M^2(1-g')=0, \tag{38}$$

where $M^2=\dfrac{\sigma_eB_0^2}{\rho a}$ is the Hartmann number. To these equations we append equations $(17)_{3,4}$ and the boundary conditions (11), (12).
As far as the other two cases are concerned, if we proceed as previously we get

3.2.2. *CASE II.*

$$\mathbf{H}_0=H_0\mathbf{e}_2.$$

$$p(x_1,x_2,x_3)=-\rho\frac{a^2}{2}\{x_1^2+[f(x_2)+bg(x_2)]^2+b^2x_3^2\}-\rho a(\nu+\nu_r)[f'(x_2)+bg'(x_2)]$$
$$+2\nu_r\rho\int_0^{x_2}[bF(s)-G(s)]ds-\sigma_e aB_0^2(x_1^2+bx_3^2)+p_0. \tag{39}$$

$$\frac{\nu+\nu_r}{a}f''' + (f+bg)f'' - f'^2 + 1 + \frac{2\nu_r}{a^2}G' + M^2(1-f') = 0,$$
$$\frac{\nu+\nu_r}{a}g''' + (f+bg)g'' - bg'^2 + b - \frac{2\nu_r}{a^2}F' + M^2(1-g') = 0. \quad (40)$$

3.2.3. *CASE III.*

$$\mathbf{H}_0 = H_0\mathbf{e}_3.$$

$$p(x_1,x_2,x_3) = -\rho\frac{a^2}{2}\{x_1^2 + [f(x_2)+bg(x_2)]^2 + b^2x_3^2\} - \rho a(\nu+\nu_r)[f'(x_2)+bg'(x_2)] + 2\nu_r\rho\int_0^{x_2}[bF(s)-G(s)]ds + \sigma_e a B_0^2\{\int_0^{x_2}[f(s)+bg(s)]ds - \frac{x_1^2}{2}\} + p_0. \quad (41)$$

$$\frac{\nu+\nu_r}{a}f''' + (f+bg)f'' - f'^2 + 1 + \frac{2\nu_r}{a^2}G' + M^2(1-f') = 0,$$
$$\frac{\nu+\nu_r}{a}g''' + (f+bg)g'' - bg'^2 + b - \frac{2\nu_r}{a^2}F' = 0. \quad (42)$$

Thus, we obtain the following:

Theorem 3. *Let a homogeneous, incompressible, electrically conducting micropolar fluid occupy the region $\mathcal{S}$. If we impress the external magnetic field $\mathbf{H}_0$ parallel to one of the axes and if we neglect the induced magnetic field, then the steady MHD three-dimensional stagnation-point flow of such a fluid has the form*

$$v_1 = ax_1f'(x_2), \quad v_2 = -a[f(x_2)+bg(x_2)], \quad v_3 = bax_3g'(x_2),$$
$$w_1 = bx_3F(x_2), \quad w_2 = 0, \quad w_3 = x_1G(x_2), \quad \mathbf{E} = \mathbf{0},$$

and

(1) *if $\mathbf{H}_0 = H_0\mathbf{e}_1$, the pressure field is given by (37) and (f,g,F,G) satisfies problem (38), $(17)_{3,4}$, (11), and (12), provided $F,G \in L^1([0,+\infty))$;*

(2) *if $\mathbf{H}_0 = H_0\mathbf{e}_2$, the pressure field is given by (39) and (f,g,F,G) satisfies problem (40), $(17)_{3,4}$, (11), and (12), provided $F,G \in L^1([0,+\infty))$;*

(3) *if $\mathbf{H}_0 = H_0\mathbf{e}_3$, the pressure field is given by (41) and (f,g,F,G) satisfies problem (42), $(17)_{3,4}$, (11), and (12), provided $F,G \in L^1([0,+\infty))$.*

Remark 7. *If $b=1$, $f=g$, $F=-G$, $\mathbf{H}_0 = H_0\mathbf{e}_2$, the axisymmetric case is obtained.*

The relations (19) allow us to rewrite equation (38) in dimensionless form:

$$\phi''' + (\phi+b\gamma)\phi'' - \phi'^2 + 1 + \Gamma' = 0,$$
$$\gamma''' + (\phi+b\gamma)\gamma'' - b\gamma'^2 + b - \Phi' + M^2(1-\gamma') = 0;$$

equation (40) in

$$\phi''' + (\phi + b\gamma)\phi'' - {\phi'}^2 + 1 + \Gamma' + M^2(1 - \phi') = 0,$$

$$\gamma''' + (\phi + b\gamma)\gamma'' - b{\gamma'}^2 + b - \Phi' + M^2(1 - \gamma') = 0;$$

equation (42) in

$$\phi''' + (\phi + b\gamma)\phi'' - {\phi'}^2 + 1 + \Gamma' + M^2(1 - \phi') = 0,$$

$$\gamma''' + (\phi + b\gamma)\gamma'' - b{\gamma'}^2 + b - \Phi' = 0.$$

Of course we obtain three different ordinary differential problems by adjoin equations $(20)_{3,4}$ and the boundary conditions (21).

Remark 8. *The skin-friction components τ_1, τ_3 along x_1 and x_3 axes are given by (22). Actually $\phi''(0), \gamma''(0)$ depend on M.*

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