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FIRST-COUNTABILITY AND ITS GENERALIZATIONS

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Dedicated to my adored parents...

(Statement under O.Ph.D.7 of Saurashtra University, Rajkot)

Declaration

I hereby declare that the contents embodied in this thesis is the bonafide record of investigations carried out by me under the supervision of Dr. D. K. Thakkar, Professor and Head, Department of Mathematics, Saurashtra University, RAJKOT. The investigations reported here have not been submitted in part or full for the award of any degree or diploma to any other Institution or University.

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CERTIFICATE

This is to certify that the thesis entitled **First-Countability and its Generalizations** submitted by **Narendra P Shrimali** to the **Saurashtra University**, RAJKOT for the award of the degree of Doctor of Philosophy in Mathematics is bonafide record of research work carried out by him under my supervision. The contents embodied in the thesis have not been submitted in part or full to any other Institution or University for the award of any degree or diploma.

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CONTENTS

Chapter-0

Introduction

0.1 Introduction

The important and worth knowing results of General Topology (Point-Set Topology) viz. metrization theorems, Tychonoff theorem and Extension theorems were known by 1940. Inspite of this, serious research continues to this date in this innocent branch of Pure mathematics. It is also observed that deeper results and generalizations in point-set topology soon settles on the thin boundary of mathematics, foundational logic and descriptive set-theory. Consequently they become out of reach of mathematicians.

The axiom of first countability appears as an important condition in the following well-known results,

"*Every first countable Hausdorff topological group is metrizable*" [3], [10]

Similar result,

"*A first countable Hausdorff topological vector space is metrizable*" [11]

is a well-known result.

0.2 Review of Literature

The following deep and elegant topological characterization of $\mathbb{Q}$ is not very well-known among mathematicians.

> *"Every countable, first countable, regular space without isolated points is homeomorphic to $\mathbb{Q}$"* [15]

In an attempt of understanding this axiom well and of course a natural mathematical instinct of generalizing it to strengthen known results has provided several notions which generalize this axiom in several different ways. e.g. [5], [6], [17]. Of these generalizations we restrict our attention to only Fréchet-Urysohn spaces and sequential spaces [1], [3], [5], [6], [17]. Our main focus is on construction of such spaces (mainly countable spaces). We also note here that introduction to each chapter contains the review of literature relevant to the topic of the chapter in the disguised form.

0.3 Chapter wise Summary

The work presented in this thesis is divided into seven chapters.

Chapter-1 contains three examples (Here we denote them as X_1, X_2, X_3) of countable spaces with exactly one nonisolated point [16]. We give alternative descriptions of these examples. Also we characterize them topologically. As an application of these results, we give examples of spaces (X, τ_1) and (X, τ_2) which are homeomorphic, however τ_2 is strictly finer than τ_1.

Chapter-2 is about a well-known example (Here we denote it as X_4) which is known as sequential fan[16]. We give alternative descriptions of it and characterize it topologically.

In Chapter-3 we consider three topological spaces (Here we denote them as X_5, X_6, X_7) which can be constructed just like sequential fan [16]. Here we prove that X_5 and X_6 are not homeomorphic to X_4. Also we prove that X_5 and X_6 are not homeomorphic and X_7 is homeomorphic to X_6.

Chapter-4 is concerned with filters on $\mathbb{N}$ [18], [20]. We define filters on $\mathbb{N}$ and using these filters we construct countable spaces with exactly one nonisolated point. Our main task in this chapter is to prove these spaces are nonhomeomorphic.

Chapter-5 is about a well-known example F_ω, which was constructed by S. P. Franklin and M. Rajagopalan [4]. We give our construction of the space F_ω which is a countable, Fréchet-Urysohn, homogeneous, regular but a non first countable space. At the end of this chapter we give an alternative answer to the question - if F_ω supports a compatible group structure - asked by S. P. Franklin and M. Rajagopalan [4].

Chapter-6 is concerned with Ψ-spaces which were first introduced by J. R. Isbell [7]. Here we construct a space using Ψ-space notions which is not homeomorphic to the sequential fan though it is a countable, Fréchet-Urysohn space with unique limit point. In fact this space is not homeomorphic to any one of the examples $X_1, X_2, X_3, X_4, X_5, X_6$.

Chapter-7 contains some concluding remarks.

Chapter-1

Spaces with unique nonisolated point and their topological characterizations

1.1 Introduction

In this chapter we shall discuss three examples of countable topological spaces with exactly one nonisolated point[16]. Also we shall give alternative descriptions of theses examples and characterize them. At the end of this chapter, as an application of these results we shall give examples of spaces (X, τ_1) and (X, τ_2) which are homeomorphic, however τ_2 is strictly finer than τ_1.

1.2 Example 1 (The Space X_1)

$X_1 = \{\frac{1}{n} | n \in \mathbb{N}\} \cup \{0\}$ as a subspace of $\mathbb{R}$. This is an example of a countable, compact, T_1-space, with exactly one nonisolated point.

1.2.1 Alternative intrinsic description of X_1

As before, let $X_1 = \{\frac{1}{n} | n \in \mathbb{N}\} \cup \{0\}$. Without any reference of $\mathbb{R}$, we may define a topology τ_1 as follows:

1. Any subset of X_1 not containing 0 is open.

2. Suppose $0 \in O \subseteq X_1$, then O is open if and only if O contains all but finitely many points of X_1.

   It is easy to verify that τ_1 is a topology on X_1.

Theorem 1.2.1. (X_1, τ_1) *is a countable, compact,* T_1*-space, with exactly one nonisolated point (limit point).*

Proof:

(1) Clearly, X_1 is countable and 0 is its unique nonisolated point.

(2) X_1 is compact:

Let $\mathscr{A} = \{O_\alpha | \alpha \in J\}$ be any open cover of X_1. Then there exists $\beta \in J$ such that $0 \in O_\beta$. Since O_β is open and $0 \in O_\beta$, O_β contains all but finitely many points of X_1, say, $\frac{1}{n_1}, ..., \frac{1}{n_k}$. For each $i, \exists O_{\alpha_i} \in \mathscr{A}$ such that $\frac{1}{n_i} \in O_{\alpha_i}$. Thus $\{O_\beta, O_{\alpha_1}, ..., O_{\alpha_k}\}$ is a finite subcover of X_1. This shows that X_1 is compact.

(3) X_1 is a T_1-space:

Let x, y be two distinct points of X_1. If x, y are both non-zero then $\{x\}$ and $\{y\}$ are required open sets. If one of them is zero, we may assume without loss of generality that $x = 0$, then we may consider $\{y\}$ and $X_1 \backslash \{y\}$ as required open sets. Hence X_1 is a T_1-space. ∎

Remark 1.2.1.

1. Actually one observes that any such space is Hausdorff. That is, if X is any T_1-space with exactly one nonisolated point

then X is also Hausdorff.

2. (X_1, τ_1) as defined above and X_1 as a subspace of $\mathbb{R}$ are the same topological spaces.

Yet another such example:

Let X be any countable discrete space. Note that any two countable discrete spaces are homeomorphic. Consider $X^* = X \cup \{\infty\}$, the one point compactification of X. Then X^* also is an example of a countable, compact, T_1-space with unique nonisolated point ∞.

1.2.2 Characterization of X_1

Theorem 1.2.2. *Let X be any countable, compact, T_1-space with exactly one nonisolated point (note that such a space is metrizable). Then X is homeomorphic to X_1.*

Proof:

Since X is countable, we can write $X = \{x_0, x_1, x_2, ...\}$, where x_0 is the nonisolated point of X. Clearly, any subset of X not containing x_0 is open in X.

Claim: If U is an open set containing x_0, then U must contain all but finitely many points of X, that is, $X \backslash U$ is finite.

Suppose $X\backslash U$ is infinite. Then $\mathscr{A} = \{U\} \cup \big\{\{x_k\} | x_k \notin U\big\}$ is an open cover of X that has no finite subcover which contradicts the compactness of X. This proves our claim.

Now consider the mapping $f : X \to X_1$ given by

$$f(x_n) = \frac{1}{n}, \quad n = 1, 2, ...$$

$$f(x_0) = 0$$

Clearly, f is one-one and onto.

Claim: f is continuous.

Let U be any open set in X_1. If $0 \notin U$, then $x_0 \notin f^{-1}(U)$, so that $f^{-1}(U)$ is open in X. If $0 \in U$, then U contains all but finitely many points of X_1. Since f is one-one, $f^{-1}(U)$ also contains all but finitely many points of X, i.e. $X\backslash f^{-1}(U)$ is a finite set. Therefore $X\backslash f^{-1}(U)$ is closed because X is a T_1 space and hence $f^{-1}(U)$ is open in X. This shows that f is continuous.

Claim: f is open (i.e. here f^{-1} is continuous).

Let U be any open set in X. If $x_0 \notin U$ then $0 \notin f(U)$, hence it is open in X_1. If $x_0 \in U$, then U contains all but finitely many points of X. Therefore $f(U)$ also contains all but finitely many points of X_1 and $0 \in f(U)$. Hence $f(U)$ is open in X_1. This proves our claim.

Thus f is a homeomorphism and therefore X and X_1 are homeomorphic. ∎

This theorem shows that X_1 is the unique example of a countable, compact, metrizable space with exactly one nonisolated point, up to homeomorphic spaces.

1.3 Example 2 (The Space X_2)

$X_2 = \{0, 1, \frac{1}{2}, \frac{1}{3}, ...\} \cup \mathbb{N} = X_1 \cup \mathbb{N}$ as a subspace of $\mathbb{R}$. This is an example of a countable, locally compact, noncompact, T_1-space with exactly one nonisolated point.

1.3.1 Alternative intrinsic description of X_2

As before, let $X_2 = X_1 \cup \mathbb{N} = \{0, 1, \frac{1}{2}, \frac{1}{3}, ...\} \cup \mathbb{N}$. Without any reference of $\mathbb{R}$, we may define a topology τ_2 on X_2 as follows:

1. Any subset of X_2 not containing 0 is open.
2. Suppose $0 \in O \subseteq X_2$, then O is open in X_2 if and only if O contains all but finitely many points of X_1.

Theorem 1.3.1. *(X_2, τ_2) is a countable, locally compact, noncompact, T_1-space with exactly one nonisolated point.*

Proof:

(1) X_2 is countable because X_1 and $\mathbb{N}$ are countable. Clearly, 0 is the unique nonisolated point of X_2 .

(2) X_2 is locally compact:

Let us first characterize the compact subsets of X_2. Let C be any compact subset of X_2.

If $0 \notin C$, then we assert that C is finite. Suppose that C is infinite, then $\mathscr{A} = \big\{\{x\}|x \in C\big\}$ is an open cover of C that has no finite subcover, which contradicts the compactness of C. This proves our assertion.

If $0 \in C$, then we assert that $C \cap \mathbb{N}$ is finite. Suppose that $C \cap \mathbb{N}$ is infinite, then $\mathscr{A} = \{X_1\} \cup \big\{\{x\}|x \in C \cap \mathbb{N}\big\}$ is an open cover of C by sets open in X_2 that has no finite subcover, which contradicts the compactness of C. This proves our assertion.

Now we show that X_2 is locally compact. Let x be any point of X_2. If $x = 0$ then X_1 is a compact open subset of X_2 which contains 0. If $x \neq 0$ then $\{x\}$ itself is compact open subset of X_2 which contains x. Thus X_2 is locally compact.

(3) X_2 is not compact:

$\mathscr{A} = \big\{\{x\}|x \in \mathbb{N}\big\} \cup \{X_1\}$ is an open cover of X_2 that has no finite subcover. Therefore X_2 is not compact.

(4) X_2 is a T_1-space:

Let $x, y \in X_2$ and let $x \neq y$. If x and y both nonzero then $\{x\}$

and $\{y\}$ are required open sets. If one of them is zero, we may assume without loss of generality that $x = 0$, then we may consider $\{y\}$ and $X_2 \backslash \{y\}$ as required open sets. Hence X_2 is T_1-space. ∎

1.3.2 Characterization of X_2

Remark 1.3.1. (X_2, τ_2) defined above and X_2 as a subspace of $\mathbb{R}$ are the same topological spaces.

Theorem 1.3.2. *Let X be any countable, locally compact, noncompact, T_1-space with exactly one nonisolated point (note that such a space is metrizable). Then X is homeomorphic to X_2.*

Proof: Let x_0 denote the nonisolated point of X. Clearly, any subset of X not containing x_0 is open in X. Since X is locally compact, there exists a compact subset C containing x_0, which contains an open set V of x_0. As x_0 is the nonisolated point, V contains infinitely many points of X and therefore C is infinite. Also $X \backslash C$ is infinite otherwise X is compact. Since X is countable, C and $X \backslash C$ are countable, so we can write C and $X \backslash C$ as follows:

$$C = \{x_0, x_1, x_2, x_3, ...\}$$

$$X \backslash C = \{y_1, y_2, y_3, ...\}$$

Observe that C and $X\backslash C$ are clopen subsets of X.

Claim: Suppose $x_0 \in U \subseteq X$, then U is open in X if and only if U contains all but finitely many points of C.

Suppose that U is open in X containing x_0. We want to show that U contains all but finitely many points of C. If $C\backslash U$ is infinite then $\mathscr{A} = \{U\} \cup \big\{\{x\} | x \in C\backslash U\big\}$ is an open cover of C by sets open in X. Clearly, $\mathscr{A}$ has no finite subcollection that covers C, therefore C is not compact. Which is a contradiction to the fact C is compact. Therefore our supposition is false and we conclude that U contains all but finitely many points of C.

Conversely, suppose that U contains all but finitely many points of C. We want to show that U is open in X. If $(X\backslash C) \cap U = \phi$, then $X\backslash U = (C\backslash U) \cup (X\backslash C)$. Since $C\backslash U$ is finite and X is a T_1 space, $C\backslash U$ is closed in X. Also $X\backslash C$ is closed in X. Therefore, in this case $X\backslash U$ is closed and hence U is oepn in X.

If $(X\backslash C) \cap U \neq \phi$, then $X\backslash U = (C\backslash U) \cup ((X\backslash C)\backslash U)$. Since $C\backslash U$ and $(X\backslash C)\backslash U$ both are closed in X, $X\backslash U$ is closed in X. Hence U is open in X.

Now we show that X and X_2 are homeomorphic.

Define a mapping $f : X \longrightarrow X_2$ as follows:

$$\begin{aligned} f(x_0) &= 0 \\ f(x_n) &= \frac{1}{n}, \quad n = 1, 2, ... \\ f(y_n) &= n, \quad n = 1, 2, ... \end{aligned}$$

Claim: f is a homeomorphism.

(1) Clearly, f is one-one and onto.

(2) f is continuous.

Let U be any open set of X_2. If $0 \notin U$, then $x_0 \notin f^{-1}(U)$, so that $f^{-1}(U)$ is open in X. If $0 \in U$, then $x_0 \in f^{-1}(U)$. Since U contains all but finitely many points of X_1, $f^{-1}(U)$ contains all but finitely many points of C. Therefore $f^{-1}(U)$ is open in X by above claim. This shows that f is continuous.

(3) f is open.

Let U be any open set of X. If $x_0 \notin U$, then $0 \notin f(U)$ and hence $f(U)$ is open in X_2. If $x_0 \in U$, then $0 \in f(U)$. Because U contains all but finitely many points of C, $f(U)$ contains all but finitely many points of X_1. Therefore $f(U)$ is open in X_2. This shows that f is open.

From (1), (2) and (3), f is a homeomorphism. ∎

This theorem shows that X_2 is the unique example of a countable, locally compact, noncompact, metrizable space with exactly one nonisolated point, upto homeomorphic spaces.

1.4 Example 3 (The Space X_3)

$X_3 = \{0\} \cup \{\frac{1}{m} + \frac{1}{n} | m, n \in \mathbb{N}$ and $n > m(m-1)\}$ as a subspace of $\mathbb{R}$.

Let $\frac{1}{m} + \frac{1}{n} \in X_3$ then $\frac{1}{m} + \frac{1}{n} = \frac{1}{m} + \frac{1}{m(m-1)+k}$, for some $k \in \mathbb{N}$.

Since $\{\frac{1}{m} + \frac{1}{n}\} = \left(\frac{1}{m} + \frac{1}{m(m-1)+k+1}, \frac{1}{m} + \frac{1}{m(m-1)+k-1}\right) \cap X_3$, $\{\frac{1}{m} + \frac{1}{n}\}$ is open in X_3.

Theorem 1.4.1. *The space X_3 is a countable, non-locally-compact, second countable, metrizable with exactly one nonisolated point.*

Proof:

(1) Clearly, X_3 is countable and 0 is its unique nonisolated point.

(2) X_3 is not locally compact.

Suppose that there exists a compact subset C and an open set O in X_3 such that $0 \in O \subseteq C$. Then there exists a basis element $(a, b) \cap X_3$ of X_3 such that $0 \in (a, b) \cap X_3 \subseteq O$.

Since $\frac{1}{m}$ goes to 0, there exists a positive integer M such that $\frac{1}{m} \in (a, b), \forall\ m \geq M$. Choose $m > M$ then $\frac{1}{m} \in (a, b)$.

$\therefore K = \{\frac{1}{m} + \frac{1}{n} | n \in \mathbb{N}, n > m(m-1)\} \subseteq O$.

$$\begin{aligned} \bar{K}^{X_3} &= \bar{K}^{\mathbb{R}} \cap X_3 \\ &= \left(K \cup \left\{\frac{1}{m}\right\}\right) \cap X_3 \quad \left(\because \bar{K}^{\mathbb{R}} = K \cup \left\{\frac{1}{m}\right\}\right) \\ &= K. \end{aligned}$$

(Here $\bar{K}^{X_3}$ and $\bar{K}^{\mathbb{R}}$ denotes the closure of K in X_3 and closure of K in $\mathbb{R}$ respectively.)

Thus, K is a closed subset of a compact set C. Therefore K is compact. But K as a subspace of X_3 is discrete therefore K is not compact. So we have a contradiction. Hence X_3 is not locally compact.

(3) X_3 is second countable and metrizable.

The space X_3 being a subspace of $\mathbb{R}$ is second countable and metrizable. ∎

1.4.1 Alternative descriptions of X_3

1. Alternative description of X_3:

$Y = \{(\frac{1}{m}, \frac{1}{n}) | m, n \in \mathbb{N}\} \cup \{(0,0)\}$ as a subspace of $\mathbb{R}^2$.

Since $\left\{(\frac{1}{m}, \frac{1}{n})\right\} = \left((\frac{1}{m+1}, \frac{1}{m-1}) \times (\frac{1}{n+1}, \frac{1}{n-1})\right) \cap Y$, $\left\{(\frac{1}{m}, \frac{1}{n})\right\}$ is open in Y.

Theorem 1.4.2. *The space Y is a countable, non-locally-compact, second countable, metrizable space with exactly one nonisolated point.*

Proof:

(1) Clearly, Y is countable and $(0,0)$ is its unique nonisolated point.

(2) <u>Y is not locally compact</u>.

Suppose that there exists a compact subset C of Y and an open set U in Y such that $(0,0) \in U \subseteq C$. Then there exists a basis element $B = ((a,b) \times (c,d)) \cap Y$ of Y such that $(0,0) \in B \subseteq U$.

Choose a point $(\frac{1}{M}, 0) \in (a,b) \times (c,d)$.

Since the sequence $\{(\frac{1}{M}, \frac{1}{n})\}$ converges to $(\frac{1}{M}, 0)$, there exists a positive integer N_0 such that $(\frac{1}{M}, \frac{1}{n}) \in (a,b) \times (c,d)$ for every $n \geq N_0$.

$\therefore K = \{(\frac{1}{M}, \frac{1}{n}) | n \in \mathbb{N}, n \geq N_0\} \subseteq U$.

Now $\bar{K}^Y = K \quad \big(\because \bar{K}^Y = \bar{K}^{\mathbb{R}^2} \cap Y$ and $\bar{K}^{\mathbb{R}^2} = K \cup \{(\frac{1}{M}, 0)\}\big)$

Thus K is a closed subset of a compact set C. Therefore K is compact. But K as a subspace of Y is discrete and hence K is not compact. Which is a contradiction. Thus Y is not locally compact.

(3) <u>Y is second countable and metrizable</u>.

We know that every subspace of a second countable space is second countable. Therefore Y as a subspace of $\mathbb{R}^2$ is second countable. Also as a subspace of $\mathbb{R}^2$, Y is metrizable. ∎

2. Alternative description of X_3:

Let us put $E = \mathbb{Q}$; the set of rational numbers. The topology on $E = \mathbb{Q}$ is defined as follows:

1. Any subset of $E = \mathbb{Q}$ not containing 0 is open.
2. Suppose $0 \in O \subseteq E = \mathbb{Q}$, then O is open if and only if there exists $\epsilon > 0$ such that $0 \in (-\epsilon, \epsilon) \cap \mathbb{Q} \subseteq O$.

Thus we are considering on the rationals the enlarged topology from the usual topology by declaring all rationals other than zero as open.

Theorem 1.4.3. *The space $E = \mathbb{Q}$ is a countable, non-locally-compact, regular, second countable, metrizable space with exactly one nonisolated point.*

Proof:

(1) Obviously, $E = \mathbb{Q}$ is countable and 0 is its unique nonisolated point.

(2) E is not locally compact:

Suppose there exists a compact subset C of $E = \mathbb{Q}$ and an open set O of $E = \mathbb{Q}$ such that $0 \in O \subseteq C$. Then there exists $\epsilon > 0$ such that $0 \in (-\epsilon, \epsilon) \cap \mathbb{Q} \subseteq O$. Choose an irrational number i in $(-\epsilon, \epsilon)$. Then there exist a sequence (r_n) of nonzero rationals such that (r_n) converges to i.

Put $A = \{r_n | n = 1, 2, ...\}$. Clearly, A is an infinite set. Since there is a $\delta > 0$ such that $((-\delta, \delta) \cap \mathbb{Q}) \cap A = \phi, 0 \notin \bar{A}^E$. Therefore $\bar{A}^E = A$. Thus A is a closed subset of a compact set C and hence A is compact. But since A as a subspace of $E = \mathbb{Q}$ is discrete, A cannot be compact. Which is a contradiction. Hence $E = \mathbb{Q}$ is not locally compact.

(3) <u>E is second countable</u>:

Let $\mathscr{B} = \{\{x\} | x \in \mathbb{Q}, x \neq 0\} \cup \{(a, b) \cap \mathbb{Q} | a, b \in \mathbb{Q}, 0 \in (a, b)\}$.

Then $\mathscr{B}$ is a countable collection of open sets of E.

<u>Claim</u>: $\mathscr{B}$ is a basis.

Let O be any open set of E.

If $0 \neq x \in O$, then $\exists \{x\} \in \mathscr{B}$ such that $x \in \{x\} \subseteq O$.

If $0 \in O$, then $\exists \epsilon > 0$ such that $0 \in (-\epsilon, \epsilon) \cap \mathbb{Q} \subseteq O$. Now $-\epsilon < 0 < \epsilon, \exists\, a, b \in \mathbb{Q}$ such that $-\epsilon < a < 0 < b < \epsilon$ and hence $(a, b) \cap \mathbb{Q} \subseteq (-\epsilon, \epsilon) \cap \mathbb{Q} \subseteq O$. Thus we have $B = (a, b) \cap \mathbb{Q}$ in $\mathscr{B}$ such that $0 \in B \subseteq O$. Hence $\mathscr{B}$ is a basis for $E = \mathbb{Q}$. Thus E is second countable.

(4) E is regular:

Let A be any closed subset of E and x be a point of E such that $x \notin A$. If $x \neq 0$, then $\{x\}$ and $E \backslash \{x\}$ are required disjoint open sets containing x and A respectively. If $x = 0$ then $E \backslash A$ and A are required disjoint open sets containing x and A respectively.

(5) E is metrizable:

Because E is regular and second countable, E is metrizable. ∎

3. Alternative description of X_3:

Let $W = \mathbb{N} \cup \{0\} = \left(\bigcup_{m=1}^{\infty} A_m \right) \cup \{0\}$, where $|A_m| = \aleph_0$, $\mathbb{N} = \bigcup_{m=1}^{\infty} A_m$, and $A_m \cap A_n = \phi$ if $m \neq n$.

The topology on W is described in the following way:

(i) Any subset of W not containing 0 is open.

(ii) If $0 \in O \subseteq W$, then O is open if and only if O contains all but finitely many full A_ms.

Theorem 1.4.4. *The space W is a countable, non-locally-compact, regular, second countable metrizable space with exactly one nonisolated point.*

Proof:

(1) Clearly, W is countable and 0 is its unique nonisolated point.

(2) W is not locally compact:

Claim: If C is a compact subset of W, then C contains only finitely many points from each A_m.

Suppose there exists at least one A_m such that $C \cap A_m$ is infinite. Then $\mathscr{A} = \{W \backslash A_m\} \cup \Big\{\{x\}/x \in C \cap A_m\Big\}$ is an open cover of C by sets open in W that has no finite subcover which is a contradiction to the fact that C is compact. This proves our claim.

Now we prove W is not locally compact. Suppose that W is a locally compact space then there exists a compact subset C of W and an open set O of W such that $0 \in O \subseteq C$. Since O is an open set containing 0, O contains all but finitely many full A_ms. On the other hand, C being a compact subset of W, by above discussion it will contain only finitely many points from each A_m. Therefore O cannot be contained in C. Thus W is not a locally compact space.

(3) W is regular:

Let $x \in W$ and let A be any closed subset of W with $x \notin A$. If $x \neq 0$, then $\{x\}$ and $W \backslash \{x\}$ are required disjoint open sets containing x and A respectively. If $x = 0$, then $W \backslash A$ and A

are required disjoint open sets containing x and A respectively. Thus we conclude that W is regular.

(4) W is second countable:

Let $\mathscr{B} = \big\{\{x\}|x \in \mathbb{N}\big\} \cup \big\{\{0\} \cup B_k \ / \ B_k = \bigcup_{m=k}^{\infty} A_m, k \in \mathbb{N}\big\}$.

Clearly, $\mathscr{B}$ is a countable collection of open sets of W.

Claim: $\mathscr{B}$ is a basis for the topology on W.

Let V be any open set in W and let $x \in V$. If $x \neq 0$, then there exists $B = \{x\}$ in $\mathscr{B}$ such that $x \in B \subseteq V$. If $x = 0$, then V contains all but finitely many full A_ms, say, $A_1, A_2, ..., A_k$. Then there exists $B = \{0\} \cup B_{k+1}$ in $\mathscr{B}$ such that $x \in B \subseteq V$. Thus $\mathscr{B}$ is a basis for the topology on W. Hence W is second countable.

(5) W is metrizable:

Since W is regular and second countable, W is metrizable. ∎

1.4.2 Characterization of X_3

Theorem 1.4.5. *Let X be any countable, non-locally-compact, metrizable space having exactly one nonisolated point, then X is homeomorphic to X_3.*

Proof: Here we prove X is homeomorphic to W (W is the alternative description of X_3).

Since X is countable, we can write $X = \{x_0, x_1, ..x_n, ...\}$, where x_0 is the nonisolated point. Clearly, any subset of X not containing x_0 is open in X. Since X is not locally compact, X is not compact. Then there exists an open cover $\mathscr{A}_1 = \{O_\alpha | \alpha \in J\}$ of X such that $\mathscr{A}_1$ has no finite subcover. Since $\mathscr{A}_1$ is an open cover of X, there exists $\beta \in J$ such that $x_0 \in O_\beta$. Since X is not locally compact, O_β cannot be compact. Also O_β contains infinitely many points of X as x_0 is the nonisolated point of X. Furthermore, $X \backslash O_\beta$ is infinite otherwise $\mathscr{A}_1$ has a finite subcover. Moreover, $X \backslash O_\beta$ cannot be covered by finite number of O_αs otherwise $\mathscr{A}_1$ has a finite subcover. Now X is a metrizable space and O_β is open in X containing x_0, then there exists $\epsilon_1 > 0, \epsilon_1 < 1$ such that $B(x_0, \epsilon_1) \subseteq O_\beta$ and $d(x_0, x_1) > \epsilon_1$. As $X \backslash O_\beta$ is infinite, $X \backslash B(x_0, \epsilon_1)$ is infinite. Observe that $B(x_0, \epsilon_1)$ is clopen. Also observe that $B(x_0, \epsilon_1)$ cannot be compact as X is not locally compact. Put $B_1 = X \backslash B(x_0, \epsilon_1)$. Since $B(x_0, \epsilon_1)$ is not compact, there exists an open cover $\mathscr{A}_2 = \{G_\alpha | \alpha \in J\}$ of $B(x_0, \epsilon_1)$ by sets open in X that has no finite subcover. There exists $\delta \in J$ such that $x_0 \in G_\delta$. As G_δ is open, $\exists\ \epsilon_2 > 0, \epsilon_2 < \frac{1}{2}$, such that $\epsilon_2 < \epsilon_1, \epsilon_2 < d(x_0, x_2), B(x_0, \epsilon_2) \subseteq G_\delta$, $B(x_0, \epsilon_2) \subseteq B(x_0, \epsilon_1)$. $B(x_0, \epsilon_1)$ is not compact, therefore

$B(x_0, \epsilon_1) \backslash B(x_0, \epsilon_2)$ is infinite. Observe that $B(x_0, \epsilon_2)$ is clopen but not compact.

Put $B_2 = B(x_0, \epsilon_1) \backslash B(x_0, \epsilon_2)$.

<u>Claim</u>: $B_1 \cap B_2 = \phi$.

Suppose $x_j \in B_1 \cap B_2$ for some j.

$\Rightarrow x_j \in B_1$ and $x_j \in B_2$.

$x_j \in B_1 \Rightarrow x_j \notin B(x_0, \epsilon_1)$. ($\because B_1 = X \backslash B(x_0, \epsilon_1)$)

and $x_j \in B_2 \Rightarrow x_j \in B(x_0, \epsilon_1)$.

- a contradiction.

This proves our claim.

Continuing in this way, we have $B(x_0, \epsilon_{n-1}), 0 < \epsilon_{n-1} < \frac{1}{n-1}$, which is clopen but not compact. Therefore, there exists an open cover $\mathscr{A}_{n-1} = \{U_\alpha | \alpha \in J\}$ of $B(x_0, \epsilon_{n-1})$ by sets open in X that has no finite subcover. Since $\mathscr{A}_{n-1}$ is an open cover of $B(x_0, \epsilon_{n-1})$, $\exists\ k \in J$ such that $x_0 \in U_k$. Then $\exists\ \epsilon_n > 0, \epsilon_n < \frac{1}{n}$ such that $\epsilon_n < \epsilon_{n-1}$ and $B(x_0, \epsilon_n) \subseteq U_k$. Clearly, $B(x_0, \epsilon_n) \subseteq B(x_0, \epsilon_{n-1})$. Observe that $B(x_0, \epsilon_n)$ is clopen but not compact and $B(x_0, \epsilon_{n-1}) \backslash B(x_0, \epsilon_n)$ is infinite. Put $B_n = B(x_0, \epsilon_{n-1}) \backslash B(x_0, \epsilon_n)$

<u>Claim</u>: $X = \left(\bigcup\limits_{n=1}^{\infty} B_n \right) \cup \{x_0\}$.

Let $x \in X (x \neq x_0)$, then $x = x_j$ for some j.

Since $x \neq x_0, 0 < d(x_0, x_j) < \infty$.

If $d(x_0, x_j) \geq \epsilon_1$, then $x_j \notin B(x_0, \epsilon_1)$.

Therefore, $x_j \in X \backslash B(x_0, \epsilon_1) = B_1$.

If $d(x_0, x_j) < \epsilon_1$, then there exists unique m such that

$\epsilon_m \leq d(x_0, x_j) < \epsilon_{m-1}$

$(\because \epsilon_1 > \epsilon_2 > \epsilon_3 > ... \; and \quad \lim_{n \longrightarrow \infty} \epsilon_n = 0)$.

$\therefore x_j \in B(x_0, \epsilon_{m-1})$ but $x_j \notin B(x_0, \epsilon_m)$.

$\therefore x_j \in B(x_0, \epsilon_{m-1}) \backslash B(x_0, \epsilon_m) = B_m$.

$\therefore X \subseteq \left(\bigcup_{n=1}^{\infty} B_n \right) \cup \{x_0\}$.

Claim: $B_m \cap B_n = \phi$, if $m \neq n$.

Assume $m < n$.

We have, $B_m = B(x_0, \epsilon_{m-1}) \backslash B(x_0, \epsilon_m)$

and $B_n = B(x_0, \epsilon_{n-1}) \backslash B(x_0, \epsilon_n)$

. Suppose that $x_j \in B_m \cap B_n$.

Then $x_j \in B_m$ and $x_j \in B_n$.

$$\begin{aligned} x_j \in B_m \;&\Rightarrow\; x_j \in B(x_0, \epsilon_{m-1}) \text{ but } x_j \notin B(x_0, \epsilon_m) \\ &\Rightarrow\; x_j \notin B(x_0, \epsilon_k) \;\; \forall k \geq m \;\; (\because \epsilon_k < \epsilon_m, \forall k \geq m) \end{aligned}$$

In particular $x_j \notin B(x_0, \epsilon_{n-1})$ $(\because m < n, m \leq n-1)$

$$x_j \in B_n \Rightarrow x_j \in B(x_0, \epsilon_{n-1})$$

- a contradiction.

Thus we get,

$X = \left(\bigcup_{n=1}^{\infty} B_n\right) \cup \{x_0\}$, where each B_n is countable and B_ns are pairwise disjoint.

Claim: Suppose $x_0 \in U \subseteq X$, then U is open in X if and only if there exists a positive integer N such that $\forall n \geq N, B(x_0, \epsilon_n) \subseteq U$.

Suppose that U is open in X containing x_0. Then $\exists \epsilon > 0$, such that $B(x_0, \epsilon) \subseteq U$. Since $\epsilon > 0$, $\exists$ a positive integer N such that $\forall n \geq N, \frac{1}{n} < \epsilon$. Here $\epsilon_n < \frac{1}{n}$, therefore $\epsilon_n \longrightarrow 0$ as $n \longrightarrow \infty$. So $B(x_0, \epsilon_n) \subseteq B(x_0, \frac{1}{n}) \subseteq B(x_0, \epsilon) \subseteq U \;\; \forall n \geq N$.

Conversely, suppose that for a fixed positive integer N, $B(x_o, \epsilon_n) \subseteq U$ for every $n \geq N$.

Then $U = B(x_0, \epsilon_N) \cup \left(\bigcup_{\substack{x \in U \\ x \notin B(x_0, \epsilon_N)}} \{x\} \right)$.

Therefore, U as a union of open sets in X, U is open in X.

This proves our claim.

Equivalently, we can see that if $x_0 \in U$, then U is open in X if and only if $\exists$ +ve integer N such that $\forall n > N, B_n \subseteq U$.

Finally, we prove that X is homeomorphic to W.

Define $f : X \longrightarrow W$ as follows:

$$f(x_0) = 0$$

$$f(B_n) = A_n$$

(any one-to-one correspondence between B_n and A_n)

(1) clearly f is one-one and onto.

(2) f is continuous.

Let U be any open set in W. If $0 \notin U$, then $x_0 \notin f^{-1}(U)$, so that $f^{-1}(U)$ is open in X. If $0 \in U$, then $x_0 \in f^{-1}(U)$. Since U contains all but finitely many full A_is, $f^{-1}(U)$ contains all but finitely many full B_is. Therefore $f^{-1}(U)$ is open in X. Thus f is continuous.

(3) f is open.

Let U be any open set in X. If $x_0 \notin U$, then $0 \notin f(U)$, so that $f(U)$ is open in W. If $x_0 \in U$, then $0 \in f(U)$. Since $x_0 \in U$ and U is open, $\exists$ +ve integer N such that $\forall n > N, B_n \subseteq U$. Therefore $f(U)$ contains all A_n for every $n > N$. So $f(U)$ is open in W.

From (1), (2) and (3), f is a homeomorphism. ∎

This theorem shows that X_3 is the unique example of a countable, non-locally-compact, metrizable space having exactly one nonisolated point.

1.5 Application

Finally, we end this chapter with the following application of the results that we have proved.

Considerations of the following application arose when we were confronted with the problem of showing that the lower limit topology on $\mathbb{R}$ is not homeomorphic to the standard topology on $\mathbb{R}$. The fact that the lower-limit topology on $\mathbb{R}$ is strictly finer than the standard topology on $\mathbb{R}$, is not the valid argument. The valid argument is by showing some topological property that is enjoyed by the lower limit topology (viz. in this case the second countable [9]). To logically convince the student one has to give an example on a set X topologies τ_1 and τ_2, where τ_2 is strictly finer than τ_1 but (X, τ_1) and (X, τ_2) are homeomorphic.

Let $X = A \cup B \cup C \cup \{x_0\}$

where A, B, C are countably infinite disjoint sets and $x_0 \notin A \cup B \cup C$.

Define a topology τ_1 on X as follows:

(1) Each subset of X not containing x_0 is in τ_1.

(2) Suppose $x_0 \in O \subseteq X$, then O is in τ_1 iff O contains all but finitely many points of $A \cup B$.

Define a topology τ_2 on X as follows:

(1) Each subset of X not containing x_0 is in τ_2.

(2) Suppose $x_0 \in O \subseteq X$, then O is in τ_2 iff O contains all but finitely many points of A.

First we check that τ_1 is a topology on X.

(1) Clearly, $\phi, X \in \tau_1$.

(2) Let $\{O_\alpha | \alpha \in J\}$ be subcollection of τ_1.

Put $O = \bigcup\limits_{\alpha \in J} O_\alpha$. To prove $O \in \tau_1$

Suppose that for at least one β, O_β contains x_0, then O contains x_0. Therefore O contains all but finitely many points of $A \cup B$ as O_β is open containing x_0. Thus $O \in \tau_1$. Suppose that no O_α contains x_0 then $x_0 \notin O$ and hence it is in τ_1.

(3) Let $O_1, O_2, ..., O_k$ are nonempty members of τ_1. We have to show that $O = \bigcap\limits_{i=1}^{k} O_i$ is in τ_1. If $x_0 \in O$, then $x_0 \in O_i$, $i = 1, 2, ..., k$. Since each O_i contains all but finitely many points of $A \cup B$, O contains all but finitely many points of $A \cup B$. Therefore $O \in \tau_1$. If $x_0 \notin \bigcap\limits_{i=1}^{k} O_i = O$, then by definition of τ_1, $O \in \tau_1$.

From (1), (2) and (3), we conclude that τ_1 is a topology on X.

Similarly one can prove that τ_2 is also topology on X.

Claim: τ_2 is strictly finer than τ_1.

Let $O \in \tau_1$. If $x_0 \notin O$ then by definition of $\tau_2, O \in \tau_2$. If $x_0 \in O$, then O contains all but finitely many points of $A \cup B$ and hence it contains all but finitely many points of A. This shows that $O \in \tau_2$. Thus τ_2 is finer than τ_1. On the other hand, let $O = A \cup \{x_0\}$ then $O \in \tau_2$ but $O \notin \tau_1$. Thus τ_2 is strictly finer than τ_1.

Claim: (X, τ_1) and (X, τ_2) are homeomorphic.

One can easily prove that (X, τ_1) and (X, τ_2) are countable, locally compact but not compact spaces with exactly one nonisolated point.

Therefore (X, τ_1) and (X, τ_2) are homeomorphic.

$\big(\because (X, \tau_1)$ and (X, τ_2) both are homeomorphic to $X_2\big)$.

Chapter-2

Topological Characterization
of
Sequential Fan

2.1 Introduction

Here we are going to discuss some alternative descriptions of sequential fan [16]. Also we shall characterize the sequential fan.

2.2 Definitions

Definition 2.2.1. A space X is said to have a *countable basis* at x if there is a countable collection $\mathscr{B}$ of neighborhoods of x (open sets containing x) such that each neighborhood of x contains at least one of the elements of $\mathscr{B}$. A space that has a countable basis at each of its points is said to satisfy the *first countability axiom*, or to be *first countable*.

Note that every metrizable space satisfies this axiom [9].

The most useful fact concerning spaces that satisfy this axiom is the fact that in such a space, convergent sequences are adequate to detect limit points of sets and to check continuity of functions[9].

Definition 2.2.2. A topological space X is called a *Fréchet-Urysohn space (or Fréchet space)* if for every $A \subseteq X$ and every $x \in \bar{A}$ there exists a sequence (x_n) of points of A converging to x.

Note that every metric space is Fréchet-Urysohn. More generally, any first countable space is Fréchet-Urysohn.

Definition 2.2.3. A topological space X is said to be *hemicompact* if X has a countable cover $\mathscr{C}$ of compact subspaces such that if K is a compact subset of X then there exists a $C \in \mathscr{C}$ for which $K \subseteq C$.

2.3 Sequential fan

Consider countably many disjoint copies of a convergent sequence (i.e. copies of $\{1/n : n \in \mathbb{N}\} \cup \{0\}$ as subsets of real line) and identify the limit points, denote this new identified point by 0 and the resulting space by the set X_4. New space X_4 is called *sequential fan*.

Some authors have used different notations for sequential fan. For example, F was used by S.P.Franklin and M. Rajagopalan [4], $S(w)$ by Shou Lin [8], and S_ω by Yoshio Tanaka [19].

We also want to warn the reader, that notation S_ω was also used for a space that was constructed by A. V. Arhangel'skii and S. P. Franklin [1], which has the following properties:

(i) Countable

(ii) Sequential

(iii) Zero-dimensional

(iv) Homogeneous

(v) Hausdorff space

(vi) Sequential order is ω_1 (the first uncountable ordinal)

And so clearly it is different from the sequential fan.

The quotient topology on X_4 is described in the following way :

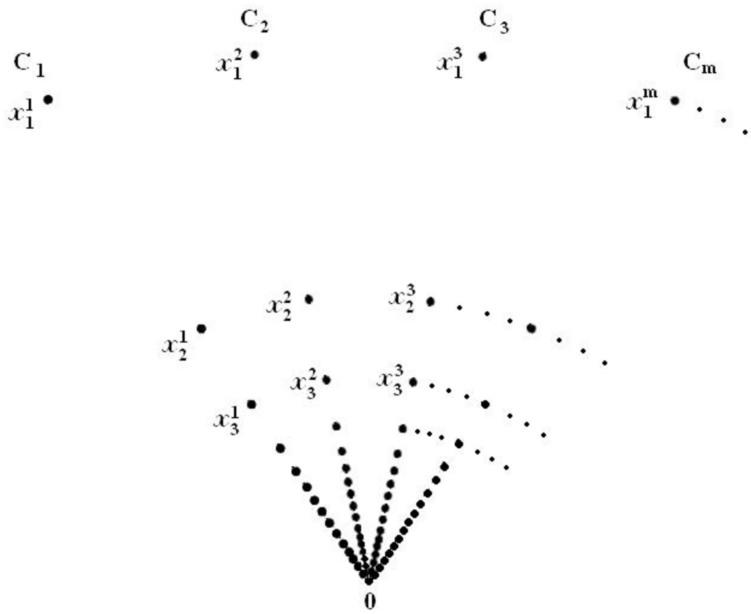
1. Any subset of X_4 not containing 0 is open.

2. Suppose $0 \in U \subseteq X_4$, then U is open if and only if U contains all but finitely many points of each copy of the convergent sequence.

We can write sequential fan X_4 as a set in the following way :

$$X_4 = \left(\bigcup_{j=1}^{\infty} C_j \right) \cup \{0\},$$

where C_j is the set of points of convergent sequence. (That is, $C_j = \{x_i^j | i = 1, 2, ...\}$). So we can visualize the sequential fan as in the following picture.



X_4

[Note that sequential fan X_4 is not a subspace of $\mathbb{R}^2$]

Lemma 2.3.1. *Sequential fan X_4 is not first countable.*

Proof: As above

$$X_4 = \left(\bigcup_{j=1}^{\infty} C_j\right) \cup \{0\}$$

Let $\{O_i\}$ be any countable neighborhood base of X_4 at a point 0, where

$O_i = \left(\bigcup_{j=1}^{\infty} O_{ij}\right) \cup \{0\},\ O_{ij} \subseteq C_j,\ C_j \backslash O_{ij}$ is finite for each j.

For each i, choose a point x_i of O_{ii} and set $V_i = O_{ii} \backslash \{x_i\}$.

Then $V = \left(\bigcup V_i\right) \cup \{0\}$ is an open set containing 0.

Since $O_{ii} \nsubseteq V_i,\ O_i \nsubseteq V$ for each i.

This proves that sequential fan is not first countable.

Lemma 2.3.2. *Sequential fan X_4 is a Fréchet-Urysohn space.*

Proof: Let A be any subset of X_4 . Let x be a limit point of A, then x must be 0 (Otherwise $\{x\}$ would be open in X_4 and therefore $\{x\} \cap A$ does not contain any point of A other than x, so x cannot be a limit point of A).

Claim: If 0 is a limit point of A, then there exists a copy C_k such that A contains infinitely many points of C_k.

Suppose that A contains only finitely many points of each C_j. Then $U = (X_4 \backslash A) \cup \{0\}$ is an open set containing 0 and it does not contain any point of A. Therefore 0 cannot be a limit point of A. Which contradicts our supposition. This proves our claim.

Put the points of $C_k \cap A$ in a form of a sequence (x_n). Let U be any open set containing 0. Then U contains all but finitely many points of each C_j. Therefore $\exists$ +ve integer N such that $\forall n \geq N, x_n \in U$. Thus (x_n) converges to 0. This shows that sequential fan is Fréchet-Urysohn.

Lemma 2.3.3. *Sequential fan X_4 is hemicompact.*

Proof: Let us first characterize the compact subsets of X_4.

Claim : If K is a compact subset of X_4 not containing 0, then K must be finite.

Suppose K is infinite and $0 \notin K$. Then $\mathscr{A} = \{\{x\} | x \in K\}$ is an open cover of K that has no finite subcover. Thus K must be finite.

Claim: If $0 \in K$, then K is compact iff K intersects finitely many C_js.

Suppose that K is compact. We have to show that K intersects finitely many C_js. Suppose there is an infinite subset D of $\mathbb{N}$ such that $K \cap C_i \neq \phi, \forall i \in D$. Set $B = \{x_i | x_i \in K \cap C_i, \forall i \in D\}$. Then $U = X_4 \backslash B$ is open in X_4 and $\mathscr{A} = \{U\} \cup \{\{x\} | x \in B\}$ is an open cover of K by sets open in X_4 that has no finite subcover. Which contradicts the compactness of K. Therefore our supposition is false and we conclude that K intersects finitely many C_js.

Conversely, suppose that K intersects finitely many C_js. We want to show that K is compact. Suppose $\mathscr{A} = \{O_\alpha\}_{\alpha \in J}$ is any open cover of K by sets open in X_4. Since $0 \in K, \exists \beta \in J$ such that $0 \in O_\beta$. O_β contains all but finitely many points of each C_j as O_β is open in X_4. Since K intersects finitely many C_js, O_β contains all but finitely many points of K, say, $x_1, x_2, ..., x_n$.
For each $x_i,\ \exists O_{\alpha_i} \in \mathscr{A}$ such that $x_i \in O_{\alpha_i}$. Thus $\{O_{\alpha_1}, ..., O_{\alpha_n}, O_\beta\}$ is a finite subcover of K. Hence K is compact.

Now we show that X_4 is hemicompact.
Set $K_i = (C_1 \cup C_2 \cup ... \cup C_i) \cup \{0\}$. By above discussion K_i is compact. Then $\mathscr{B} = \{K_i | i = 1, 2, ...\}$ is a countable collection of compact subsets of X_4. Let $0 \neq x \in X_4$, then $x \in C_i$ for some i. So $x \in K_i$. Thus $X_4 = \bigcup\limits_{i=1}^{\infty} K_i$ (Note that $0 \in K_i$ for each i). Therefore $\mathscr{B}$ is a countable cover of X_4. Now suppose that K is any compact subset of X_4. Then K is either finite or infinite by above discussion. In both the cases K intersects finitely many C_js, therefore $K \subseteq K_i$ for some i. This shows that X_4 is a hemicompact space.

Lemma 2.3.4. *Sequential fan X_4 is a T_1 - space.*

Proof: Let x and y be two distinct points of X_4. If x and y both are nonzero then $\{x\}$ and $\{y\}$ are required open sets. If one of them is zero, then we may assume without loss of generality that $x = 0$ and $y \neq 0$. In this case we may consider $\{y\}$ and $X_4 \backslash \{y\}$ as open sets containing y and x respectively. Thus X_4 is a T_1 - space.

Lemma 2.3.5. *Sequential fan X_4 is a non-locally-compact space.*

Proof: In the proof of Lemma 2.3.4, we have characterized compact subsets of the sequential fan. By this lemma any compact subset C of X_4 intersects finitely many C_js. Therefore we cannot find an open set O containing 0 such that $0 \in O \subseteq C$.

2.4 Alternative descriptions of sequential fan

1. Alternative description of sequential fan:

$$\begin{aligned} \text{Let } Y &= \mathbb{N} \times \mathbb{N} \cup \{0\} \\ &= \left(\bigcup_{j=1}^{\infty} A_j \right) \cup \{0\}, \end{aligned}$$

where $A_j = \{j\} \times \mathbb{N}$ and here we call it column of Y.

The topology on Y is described as follows:

1. Any subset of Y not containing 0 is open.

2. Suppose $0 \in U \subseteq Y$, then U is open in Y if and only if U contains all but finitely many points of each A_j.

Remark 2.4.1. The proofs of the properties of the alternative descriptions of sequential fan X_4, are almost similar to the proofs of these properties for X_4. But we are giving them, with the understanding that not every one will be at home with all the descriptions.

Theorem 2.4.1. *The space Y is a countable, T_1, Fréchet-Urysohn, hemicompact, non first countable , non-locally-compact having exactly one limit point.*

Proof:

(1) Clearly, Y is countable and 0 is its unique limit point.

(2) The space Y is not first countable:

Suppose that $\{O_i\}_{i=1}^{\infty}$ is any countable neighborhood base at a point 0, where $O_i = \left(\bigcup\limits_{j=1}^{\infty} O_{ij}\right) \cup \{0\}, O_{ij} \subseteq A_j, A_j \backslash O_{ij}$ is finite for each j. For each i, choose a point x_i of O_{ii} and set $V_i = O_{ii} \backslash \{x_i\}$. Then $V = \left(\bigcup\limits_{i=1}^{\infty} V_i\right) \cup \{0\}$ is an open set containing 0. Since $O_{ii} \nsubseteq V_i, O_i \nsubseteq V$ for each i. This shows that Y is not first countable.

(3) Y is a Fréchet-Urysohn space:

Let A be any subset of Y and let x be a limit point of A. Then x must be 0 (Otherwise $\{x\}$ would be open in Y and therefore $\{x\} \cap A$ does not contain any point of A other than x, so x cannot be a limit point of A).

Claim: If 0 is a limit point of A then there exists a column A_k such that A contains infinitely many points of A_k.

Suppose A contains finitely many points of each column. Set $U = (\mathbb{N} \times \mathbb{N} \backslash A) \cup \{0\}$. Then U is an open set such that $0 \in U$ and $U \cap A = \phi$, which is a contradiction to the fact that 0 is a limit point of A. This Proves our claim.

Put the points of $A_k \cap A$ in a form of a sequence (x_n). Let U be any open set containing 0. Then U contains all but finitely many points of each A_j. Therefore $\exists$ a positive integer N_0 such that $\forall n \geq N_0, x_n \in U$. Thus, (x_n) converges to 0. This shows that Y is Fréchet-Urysohn.

(4) The space Y is hemicompact:

Let us first characterize the compact subsets of Y.

Claim: If K is a compact subset of Y with $0 \notin K$, then it must be finite.

Suppose K is infinite with $0 \notin K$. Then $A = \{\{x\}|x \in K\}$ is an open cover that has no finite subcover. Which contradicts the compactness of K. This proves our claim.

Claim: Suppose $0 \in K$, then K is compact if and only if K intersects finitely many columns.

Suppose there is an infinite subset D of $\mathbb{N}$ such that $K \cap A_i \neq \phi, \forall i \in D$. Set $B = \{x_i|x_i \in K \cap A_i, \forall i \in D\}$. Then $U = Y \backslash B$ is open in Y and $\mathscr{A} = \{U\} \cup \{\{x_i\}|x_i \in B\}$ is an open cover of K by sets open in Y that has no finite subcover. Which contradicts the compactness of K.

Conversely, suppose that K intersects finitely many columns of Y. We want to show that K is compact. Let $\mathscr{A} = \{O_\alpha|\alpha \in J\}$ be any open cover of K by sets open in Y. Since $0 \in K, \exists\ \beta \in J$, $\ni\ 0 \in O_\beta$. Since K intersects finitely many columns and O_β being open, O_β contains all but finitely many points of K, say, $x_1, x_2, ..., x_n$. For each x_i, we have $O_{\alpha_i} \in \mathscr{A}$ such that $x_i \in O_{\alpha_i}$. Then $\{O_{\alpha_1}, ..., O_{\alpha_n}, O_\beta\}$ is a finite subcover of K. Therefore K is compact. This proves our claim.

Now we show that Y is hemicompact.

Set $K_i = A_1 \cup A_2 \cup .... \cup A_i \cup \{0\}$. By above discussion K_i is compact in Y. Then $\mathscr{B} = \{K_i \mid i = 1, 2, ...\}$ is a countable collection of compact subsets of Y. Let $0 \neq x \in Y$, then $x \in A_i$ for some i. So, $x \in K_i$. Thus $Y = \bigcup\limits_{i=1}^{\infty} K_i$ (Note that $0 \in K_i$ for each i). Therefore $\mathscr{B}$ is a countable cover of Y. Suppose that K is any compact subset of Y. Then K is either finite or infinite by above discussion. In both the cases K intersects finitely many columns, therefore $K \subseteq K_i$ for some i. This proves that Y is hemicompact.

(5) The space Y is not locally compact:

As any compact subset C containing 0 intersects finitely many columns of Y, we cannot find an open set O containing 0 such that $0 \in O \subseteq C$.

(6) Y is a T_1 - space:

Let x and y be any two distinct points of Y. If x and y both are nonzero then $\{x\}$ and $\{y\}$ are required open sets. If one of them is zero, we may assume without loss of generality that $x = 0$ and $y \neq 0$. In this case we may consider $\{y\}$ and $Y \backslash \{y\}$ as required open sets. Thus Y is a T_1 - space.

2. Alternative description of sequential fan

Let $W = \mathbb{N} \cup \{0\} = \left(\bigcup_{m=1}^{\infty} A_m\right) \cup \{0\}$, where $\{A_m | m = 1, 2, ...\}$ is a partition of $\mathbb{N}$ and $|A_m| = \aleph_0$.

The topology on W is described in the following way:

1. Any subset of W not containing 0 is open.
2. Suppose $0 \in O \subseteq W$, then O is open if and only if O contains all but finitely many points of each A_i.

Theorem 2.4.2. *The space W is a countable, T_1, Fréchet-Urysohn, hemi-compact space having exactly one limit point, which is non-first-countable and non-locally compact.*

Proof:

(1) Clearly, W is countable and 0 is its unique limit point.

(2) W is a T_1 - space:

Let x and y be two distinct points of W. If x and y are both nonzero then $\{x\}$ and $\{y\}$ are required open sets. If one of them is zero, then we may assume without loss of generality that $x = 0$ and $y \neq 0$. In this case consider $\{y\}$ and $W \backslash \{y\}$ as required open sets. Thus W is a T_1 - space.

(3) <u>W is not a first countable space</u>:

Suppose $\{O_i\}_{i=1}^{\infty}$ is any neighborhood base at a point 0, where $O_i = \left(\bigcup_{j=1}^{\infty} O_{ij}\right) \cup \{0\}, O_{ij} \subseteq A_j, A_j \backslash O_{ij}$ is finite for each j.

For each i, choose a point x_i of O_{ii} and set $V_i = O_{ii} \backslash \{x_i\}$. Then $V = \left(\bigcup_{i=1}^{\infty} V_i\right) \cup \{0\}$ is an open set containing 0. Since $O_{ii} \nsubseteq V_i, O_i \nsubseteq V$ for each i. This proves that W is not first countable.

(4) <u>W is a Fréchet-Urysohn space</u>:

Let A be any subset of W and let x be a limit point of A, then x must be 0.

<u>Claim</u>: If 0 is a limit point of A, then there exists some A_k such that A contains infinitely many points of A_k.

Suppose A contains finitely many points of each A_j.

Set $U = (\mathbb{N} \backslash A) \cup \{0\}$. Then U is an open set such that $0 \in U$ and $U \cap A = \phi$, which is a contradiction to the fact that 0 is a limit point of A. Thus our supposition is wrong. This proves our claim.

Put the points of $A_k \cap A$ in a form of a sequence (x_n). Let U be any open set containing 0. Then U contains all but finitely many points of each A_j. Therefore there exists a positive integer N_0 such that $\forall n \geq N_0, x_n \in U$. Thus (x_n) converges to 0. This shows that W is a Fréchet-Urysohn space.

(5) W is a hemicompact space:

Let us first characterize the compact subsets of W.

Claim: If K is a compact subset of W with $0 \notin K$, then it must be finite.

Suppose K is infinite with $0 \notin K$. Then $\mathscr{A} = \{\{x\} | x \in K\}$ is an open cover that has no finite subcover. Which contradicts the compactness of K.

Claim: Suppose that $0 \in K \subseteq W$, then K is compact if and only if K intersects finitely many A_js.

Suppose there is an infinite subset D of $\mathbb{N}$ such that $K \cap A_i \neq \phi, \ \forall i \in D$. Set $B = \{x_i | x_i \in K \cap A_i, \forall i \in D\}$. Then $U = W \backslash B$ is open in W and $\mathscr{A} = \{U\} \cup \{\{x\} | x \in B\}$ is an open cover of K by sets open in W that has no finite subcover. Which is a contradiction to the fact that K is compact. Thus K intersects finitely many A_js.

Conversely, suppose that K intersects finitely many A_js. We want to show that K is compact. Suppose that $\mathscr{A} = \{O_\alpha\}_{\alpha \in J}$ is any open cover of K by sets open in W. Since $0 \in K$, $\exists \ \beta \in J$

such that $0 \in O_\beta$. Since K intersects finitely many A_js and O_β being open, O_β contains all but finitely many points of K, say, $x_1, x_2, ..., x_n$. For each x_i, we have $O_{\alpha_i} \in \mathscr{A}$ such that $x_i \in O_{\alpha_i}$. Then $\{O_{\alpha_1}, ..., O_{\alpha_n}, O_\beta\}$ is a finite subcover of K. Therefore K is compact.

Now we show that W is hemicompact.

Set $K_i = A_1 \cup A_2 \cup ... \cup A_i \cup \{0\}$. By above discussion K_i is compact in W. Then $\mathscr{B} = \{K_i | i = 1, 2, ...\}$ is a countable collection of compact subsets of W. Let $0 \neq x \in W$, then $x \in A_i$ for some i. Therefore $x \in K_i$. Thus $W = \bigcup\limits_{i=1}^{\infty} K_i$. (Note that $0 \in K_i$ for each i.) Therefore $\mathscr{B}$ is a countable cover of W. Suppose that K is any compact subset of W. Then K is either finite or infinite by above discussion. In both the cases K intersects finitely many A_js, therefore $K \subseteq K_i$ for some i. This proves that W is hemicompact.

(6) The space W is non-locally-compact:

As any compact subset C containing 0 intersects finitely many A_js, we cannot find an open set O containing 0 such that $0 \in O \subseteq C$.

2.5 Topological characterization of sequential fan

Theorem 2.5.1. *If X is a countable, hemicompact, non-locally-compact, T_1, Fréchet-Urysohn space having exactly one limit point which is not first countable, then X is homeomorphic to the sequential fan X_4.*

Proof:

Let y_0 denote the limit point of X. Clearly, any subset of X not containing y_0 is open in X. Since X is a Fréchet-Urysohn space, there exists a sequence (y_n) such that (y_n) converges to y_0. Therefore $\{y_n|n \in \mathbb{N}\} \cup \{y_0\}$ is a compact subset of X. This shows that X has at least one compact subset.

Now suppose that $\mathscr{B} = \{K_i \subseteq X|K_i \text{ is compact}, i = 1, 2, ...\}$ is a countable cover of X. X is hemicompact, therefore for any compact subset K of X there exist $K_i \in \mathscr{B}$ such that $K \subseteq K_i$. Let $C_1 = K_1, C_2 = K_1 \cup K_2, ..., C_i = K_1 \cup ... \cup K_i, ...$ Then C_i is compact, for each i, because finite union of compact sets is compact. Thus $\mathscr{C} = \{C_i|i = 1, 2, ...\}$ is a countable cover of X such that $C_1 \subseteq C_2 \subseteq ....$ Again X is hemicompact, therefore for any compact subset K of X, there is C_i in $\mathscr{C}$ such that $K \subseteq C_i$. Also each C_i is not finite because X is hemicompact and X has at least one infinite compact subset containing y_0. Therefore there is at least one C_i

which is infinite. Suppose C_{i_1} is the first infinite compact subset in $\mathscr{C}$. Clearly, C_{i_1} must contain y_0. If $C_j \backslash C_{j-1}$ is finite for each $j \geq i_1$, then X would be locally compact, a contradiction to the fact that X is not locally compact. Therefore there exists $C_n,\ n > i_1$, such that $C_n \backslash C_{n-1}$ is infinite. Call $C_n = C_{i_2}$. Again by same argument we get $C_m,\ m > i_2$ such that $C_m \backslash C_{m-1}$ is infinite. Call $C_m = C_{i_3}$. Continuing in this way inductively we get $C_{i_4}, C_{i_5}, ...$

$$\begin{aligned} \text{Put } D_1 &= C_1 \cup ... \cup C_{i_1} \\ D_2 &= (C_{i_2} \backslash C_{i_1}) \cup \{y_0\} \\ D_3 &= (C_{i_3} \backslash C_{i_2}) \cup \{y_0\} \\ &\vdots \end{aligned}$$

Thus we get $X = \bigcup\limits_{i=1}^{\infty} D_i$, where $D_i \cap D_j = \{y_0\}$, for $i \neq j$ and each D_i is infinite and contains y_0. Also each D_i is compact.

Claim: Suppose $y_0 \in U \subseteq X$,N then U is open in X if and only if U contains all but finitely many points from each D_i.

Suppose that U is open in X and that $y_0 \in U$. We have to show that U contains all but finitely many points of each D_i. Suppose $\exists\ D_j$ such that $D_j \backslash U$ is infinite . Then $\mathscr{A} = \{U\} \cup \{\{x\} | x \in D_j \backslash U\}$ is an open cover of D_j by sets open in X that has no finite subcover.

This contradicts the compactness of D_j. Thus our supposition is false and we conclude that U contains all but finitely many points of each D_i.

Conversely, suppose that U contains all but finitely many points of each D_i. We want to show that U is open. If $X \backslash U$ is finite, then it is closed because X is T_1. If $X \backslash U$ is infinite, then $X \backslash U$ intersects infinitely many D_i. Now suppose that y_0 is a limit point of $X \backslash U$. Then $\exists$ a seq (y_n) of points of $X \backslash U$ such that (y_n) converges to y_0. Put $B = \{y_n | n \in N\} \cup \{y_0\}$. Clearly, B is compact. Therefore $B \subseteq C_{i_k}$ for some i_k. Also $B \cap D_{i_j} = \phi$ for $j \geq k+1$, which is not possible, because $X \backslash U$ intersects infinitely many D_i. This shows that y_0 cannot be a limit point of $X \backslash U$. Thus $X \backslash U$ is closed. From both the cases we conclude that U is open. This proves our claim.

Now define a mapping $f : X \to X_4$ by $f(y_0) = 0$, and letting f to be any bijection between $D_i \backslash \{y_0\}$ and C_i so that $f(D_i \backslash \{y_0\}) = C_i, \ \forall i$.

<u>Claim</u>: f is a homeomorphism.

(1) Clearly, f is one -one and onto.

(2) <u>f is continuous</u>:

Let U be any open set of X_4. If $0 \notin U$, then $y_0 \notin f^{-1}(U)$ and hence $f^{-1}(U)$ is open in X. If $0 \in U$, then U contains all but finitely many points of each C_i. Then $f^{-1}(U)$ contains all but finitely many points of each D_i and $y_0 \in f^{-1}(U)$. Thus by above discussion $f^{-1}(U)$ is open in X.

(3) f is open:

Let U be any open set in X. If $y_0 \notin U$, then $0 \notin f(U)$ and hence it is open in X_4. If $y_0 \in U$, then $0 \in f(U)$ and U contains all but infinitely many points of each D_i. Therefore $f(U)$ contains all but finitely many points of each C_i. Hence $f(U)$ is open in X_4.

From (1), (2) and (3), f is a homeomorphism.

This theorem shows that sequential fan is the unique example of a countable, hemicompact, T_1, Fréchet-Urysohn space having exactly one limit point which is non-first countable and non-locally-compact.

Chapter-3

Sequential fan like spaces

3.1 Introduction

In this chapter we shall discuss about some topological spaces which can be constructed like sequential fan. We observe that some turn out to be new topological spaces (viz. X_5 and X_6 to be discussed in this chapter). While one (viz. X_7 to be discussed in this chapter) turns out to be homeomorphic to X_6.

The important message that we wish to pass on is that each such new construction will not necessarily give the new topological space (i.e. not homeomorphic to the space already discussed).

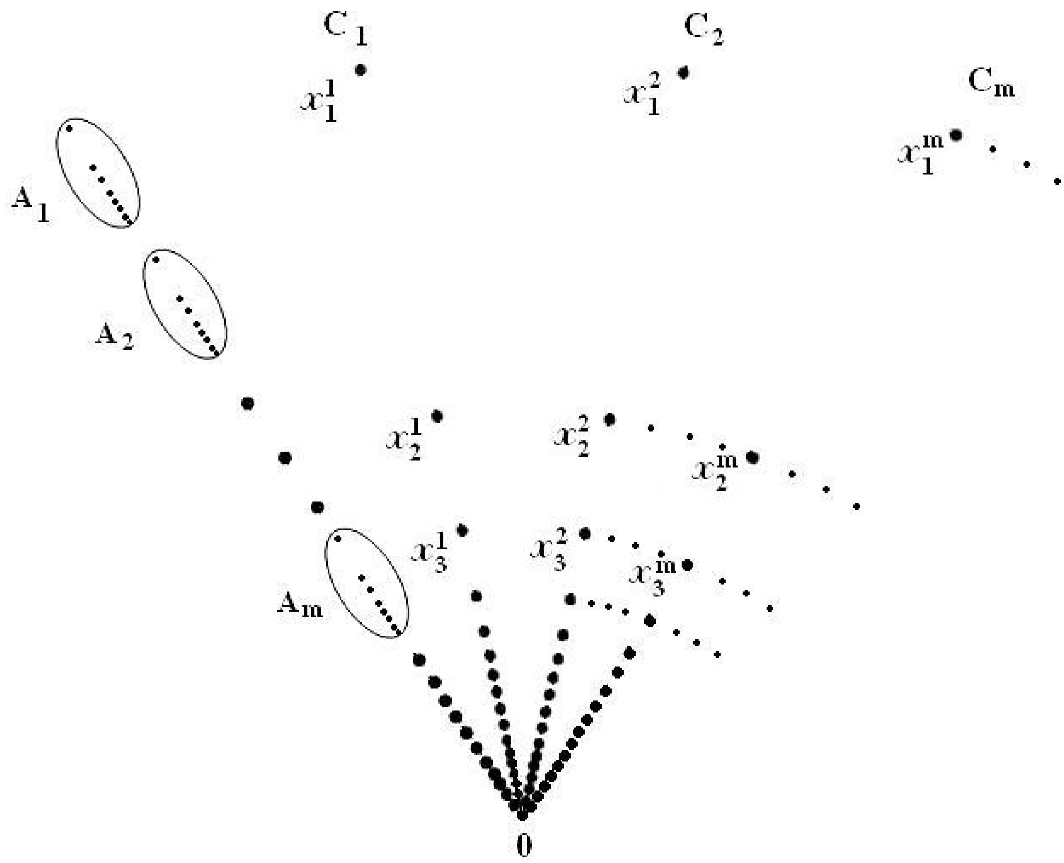
3.2 Examples X_5, X_6 and X_7

(1) Let $X_5 = \{0\} \cup \mathbb{N} = \{0\} \cup \left(\bigcup_{j=1}^{\infty} C_j\right) \cup \left(\bigcup_{i=1}^{\infty} A_i\right)$ where $\{A_i, C_j | i = 1, 2, ..., \quad j = 1, 2, ...\}$ is a family of pairwise disjoint countably infinite sets partitioning the set $\mathbb{N}$. The topology τ_5 on X_5 is defined as follows:

(a) Any subset of X_5 not containing 0 is open.

(b) suppose $0 \in U \subseteq X_5$, then U is open in X_5 if and only if U contains all but finitely many points from each C_i and all but finitely many A_js.

So we can visualize this space in the following picture:



X_5

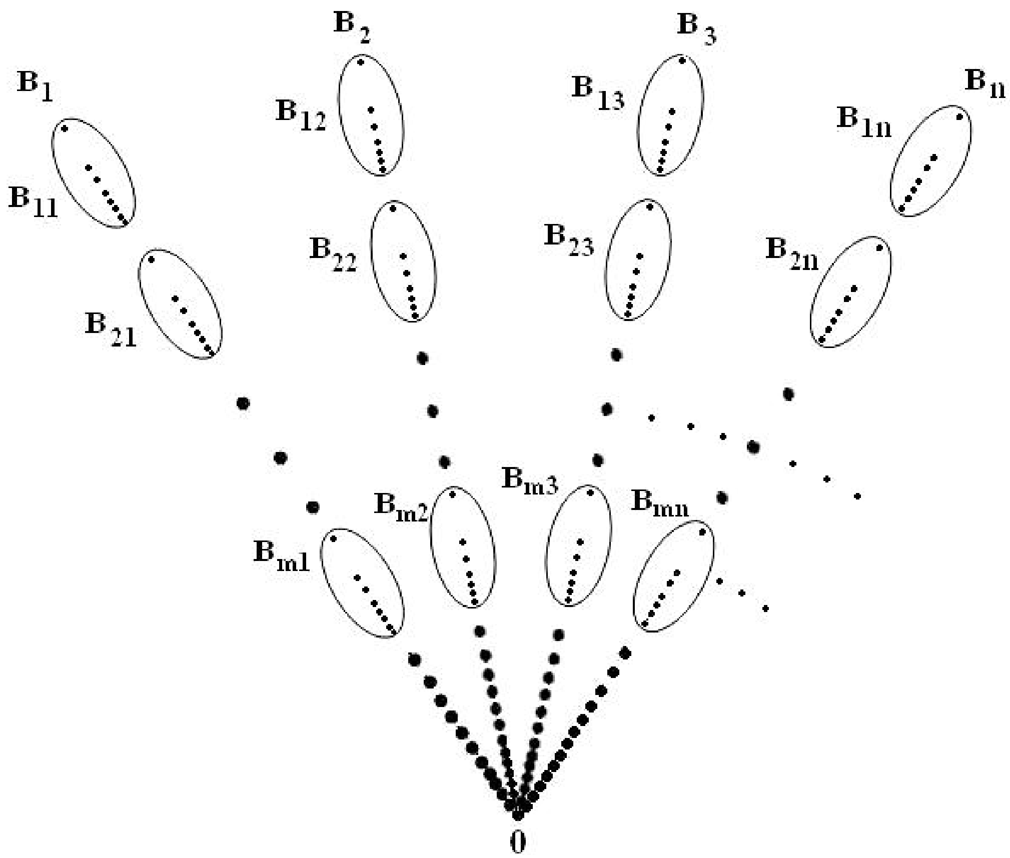
[Note that the space X_5 is not a subspace of $\mathbb{R}^2$]

(2) Let $X_6 = \{0\} \cup \mathbb{N} = \{0\} \cup \left(\bigcup\limits_{j=1}^{\infty} B_j \right)$, and $B_j = \bigcup\limits_{i=1}^{\infty} B_{ij}$ where $\{B_{ij} | i, j = 1, 2, ...\}$ is a family of pairwise disjoint countably infinite sets partitioning the set $\mathbb{N}$. The topology τ_6 on X_6 is defined as follows:

(a) Any subset of X_6 not containing 0 is open.

(b) Suppose $0 \in U \subseteq X_5$, then U is open in X_6 if and only if U contains all but finitely many B_{ij}s from each B_j.

So we can visualize this space in the following picture:



X_6

[Note that the space X_6 is not a subspace of $\mathbb{R}^2$]

(3) Let $X_7 = \{0\} \cup \mathbb{N}$

$$= \{0\} \cup \left(\bigcup_{j=1}^{\infty} B_j \right) \cup \left(\bigcup_{j=1}^{\infty} C_j \right)$$

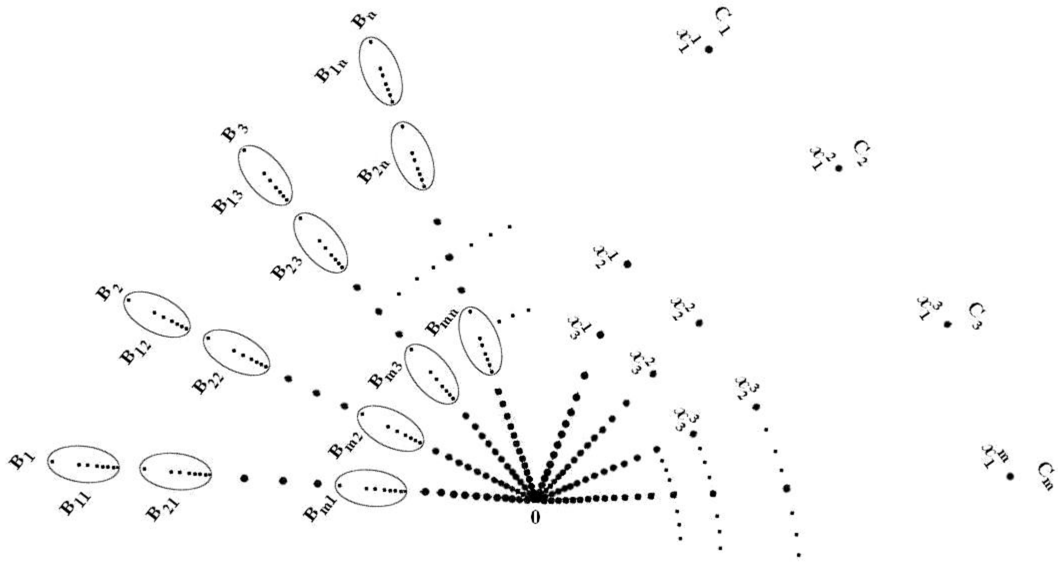
Where $B_j = \bigcup_{i=1}^{\infty} B_{ij}$ and $\{B_{ij}, C_j | i = 1, 2, ..., \ j = 1, 2, ...\}$ is a family of pairwise disjoint countably infinite sets partitioning the set $\mathbb{N}$.

The topology τ_7 on X_7 is defined as follows :

(i) any subset not containing 0 is open.

(ii) Suppose $0 \in U \subseteq X_7$, then U is open in X_7 if and only if U contains all but finitely many points from each C_j and all but finitely many B_{ij}s from each B_j.

So we can visualize this space in the following picture:



X_7

[Note that the space X_7 is not a subspace of $\mathbb{R}^2$]

3.3 Are all these four spaces different ?

Theorem 3.3.1. *The spaces X_5 and X_6 are not homeomorphic to sequential fan X_4.*

Proof:

(1) The space X_5 is not homeomorphic to sequential fan X_4:

Suppose f is a homeomorphism from X_5 to X_4.

Claim: $f(A_m) \cap C_i$ is finite for all i and for all m.

Suppose $f(A_k) \cap C_j$ is infinite for some j and for some k. Then the points of this set form a sequence (y_n) in X_4 which converges to 0 and since f is a homeomorphism we have a sequence (x_n), where $f(x_n) = y_n$ and $x_n \in A_k$, in X_5 must converge to 0. But no sequence in A_k converges to 0, which is a contradiction. Thus our claim is proved.

Let j_1 be the first index such that $f(A_1) \cap C_{j_1}$ is nonempty. Choose a point y_1 in $f(A_1) \cap C_{j_1}$. Let j_2 be the first index after $j_1 (i.e.\ j_2 > j_1)$ such that $f(A_2) \cap C_{j_2}$ is nonempty. Choose a point y_2 in $f(A_2) \cap C_{j_2}$. Continuing in this way we get a sequence (y_n) in X_4. Since f is onto, corresponding to each y_n there exists x_n in A_n such that $f(x_n) = y_n$. Thus we get a sequence (x_n) in X_5.

Claim: The sequence (x_n) converges to 0 in X_5.

Let U be any open set containing 0 in X_5. Then $\exists$ a positive integer N such that $\bigcup_{n \geq N} A_n \subseteq U$. Thus $\forall n \geq N, x_n \in U$. This proves our claim.

Since f is a homeomorphism, the sequence (y_n) must converge to $f(0) = 0$. But $U = X_4 \backslash \{y_1, y_2, ...\}$ is an open set containing 0 in X_4 which does not contain any point of the sequence (y_n). So the sequence (y_n) cannot converge to 0. Which is a contradiction to the fact that f is a homeomorphism. Thus our supposition is false and we conclude that X_5 is not homeomorphic to X_4.

(2) The space X_6 is not homeomorphic to sequential fan X_4:

Suppose f is a homeomorphism from X_6 to X_4.

Claim: $f(B_{ij}) \cap C_k$ is finite for all i, j, k.

Suppose that $f(B_{mn}) \cap C_t$ is infinite for some m, for some n and for some t. Then the points of this set form a sequence (y_n) in X_4 which converges to 0 and since f is a homeomorphism we have a sequence (x_n), where $f(x_n) = y_n$ and $x_n \in B_{mn}$, in X_6 must converge to 0. But no sequence in B_{mn} converges to 0 in X_6, which is a contradiction. This proves our claim.

Let j_1 be the first index such that $f(B_{11}) \cap C_{j_1}$ is nonempty. Choose a point y_1 in $f(B_{11}) \cap C_{j_1}$. Let j_2 be the first index after j_1 (i.e. $j_2 > j_1$) such that $f(B_{21}) \cap C_{j_2}$ is nonempty. Choose a point y_2 in $f(B_{21}) \cap C_{j_2}$. Continuing in this way, we get a sequence (y_n) in X_4. Since f is onto, corresponding to each y_n there exists x_n in B_{n1} such that $f(x_n) = y_n$. Thus we get a sequence (x_n) in X_6.

Claim The sequence (x_n) converges to 0 in X_6.

Let U be any open set containing 0. Then $\exists$ a positive integer N such that $\bigcup\limits_{n\geq N} B_{n1} \subseteq U$. Thus $\forall n \geq N, x_n \in U$ and hence (x_n) converges to 0.

Since f is a homeomorphism, the sequence (y_n) must converge to $f(0) = 0$. But $U = X_4 \backslash \{y_1, y_2, ...\}$ is an open set containing 0 in X_4 which does not contain any point of the sequence (y_n). So the sequence (y_n) cannot converge to 0. Which is a contradiction to the fact that f is a homeomorphism. This shows that X_6 is not homeomorphic to X_4.

Theorem 3.3.2. *The spaces X_5 and X_6 are not homeomorphic.*

Proof: Suppose that f is a homeomorphism from X_5 to X_6.

Claim: $f(A) \not\subseteq \bigcup\limits_{i=1}^{n} B_i$ for any n, where $A = \bigcup\limits_{i=1}^{\infty} A_i$.

Suppose $f(A) \subseteq \bigcup\limits_{i=1}^{k} B_i$ for some k. Consider B_{k+1}. We assert that $f^{-1}(B_{i(k+1)}) \cap C_j$ is finite for all i and for all j. Suppose $f^{-1}(B_{m(k+1)}) \cap C_t$ is infinite for some t and for some m. Then the points of $f^{-1}(B_{m(k+1)}) \cap C_t$ form a sequence (x_n) in C_t which converges to 0 in X_5 and hence whose image sequence $(f(x_n))$ which is a sequence in $B_{m(k+1)}$ must converge to 0 in X_6. Which is a contradiction to the fact that no sequence in $B_{m(k+1)}$ converges to 0 in X_6. Thus our assertion is proved.

Let j_1 be the first index such that at least one point say x_1 in C_{j_1} whose image $f(x_1)$ lies in $B_{1(k+1)}$. This is possible because $f^{-1}(B_{1(k+1)}) \cap C_j$ is finite for all j. Similarly, let j_2 be the first index after j_1 $(i.e.\ j_2 > j_1)$ such that at least one point say x_2 in C_{j_2} whose image $f(x_2)$ lies in $B_{2(k+1)}$. Inductively, we get a sequence (x_n) in X_5 where $x_n \in f^{-1}(B_{n(k+1)}) \cap C_{j_n}$ $(j_1 < j_2 < ...)$ and a sequence $f(x_n)$ in X_6. The sequence (x_n) does not converge to 0 in X_5 because $X_5 \backslash \{x_1, x_2, ...\}$ is an open set containing 0 in X_5 which does not contain any point of sequence (x_n). But the sequence $f(x_n)$ converges to 0 in X_6 because for any open set U containing 0, $\exists$ a positive integer N such that $\forall n \geq N$, $\bigcup\limits_{n \geq N} B_{n(k+1)} \subseteq U$ and hence $\forall n \geq N, f(x_n) \in U$. Thus, we get a contradiction to the fact that f is a homeomorphism. Hence our claim is proved.

Let j_1 be the first index such that $f(A_1) \cap B_{j_1} \neq \phi$. Therefore $f(A_1) \cap B_{i_1 j_1} \neq \phi$ for some i_1 ($\because$ $B_{j_1} = \bigcup\limits_{i=1}^{\infty} B_{ij_1}$). Choose a point $y_1 \in f(A_1) \cap B_{i_1 j_1}$. Then $y_1 = f(x_1)$ for some $x_1 \in A_1$. Put $A^1 = A \backslash A_1$. By the same argument given above we can prove that $f(A^1) \not\subseteq \bigcup\limits_{i=1}^{n} B_i$ for any n. Then $f(A^1) \cap B_j \neq \phi$ for some $j > j_1$. Let j_2 be the first index after j_1 $(i.e.\ j_2 > j_1)$ such that $f(A^1) \cap B_{j_2} \neq \phi$. Therefore $f(A^1) \cap B_{i_2 j_2} \neq \phi$ for some i_2 ($\because$ $B_{j_2} = \bigcup\limits_{i=1}^{\infty} B_{ij_2}$). Choose

$y_2 \in f(A^1) \cap B_{i_2 j_2}$ for some i_2. Then for some k, there exists a point $x_2 \in A_k$ $(k \neq 1)$ such that $f(x_2) = y_2$. Put $A^2 = A \backslash (\bigcup\limits_{i=1}^{k} A_i)$. By the same argument given above we can prove that $f(A^2) \not\subseteq \bigcup\limits_{i=1}^{n} B_i$ for any n. Then $f(A^2) \cap B_j \neq \phi$ for some $j > j_2$. Let j_3 be the first index after j_2 (i.e. $j_3 > j_2$) such that $f(A^2) \cap B_{j_3} \neq \phi$. Therefore $f(A^2) \cap B_{i_3 j_3} \neq \phi$ for some i_3 ($\because$ $B_{j_3} = \bigcup\limits_{i=1}^{\infty} B_{i j_3}$). Choose a point $y_3 \in f(A^2) \cap B_{i_3 j_3}$. Then for some t, where $t \notin \{1, 2, ...k\}$, there exists a point $x_3 \in A_t$ such that $f(x_3) = y_3$. Inductively we get a sequence (x_n) in X_5 and a sequence (y_n) in X_6.

<u>(x_n) converges to 0:</u>

Let U be any open set containing 0. Then $\exists$ a positive integer N such that $\bigcup\limits_{n \geq N} A_n \subseteq U$. Thus $\forall n \geq N, x_n \in U$.

<u>(y_n) does not converge to 0:</u>

$U = X_6 \backslash (\bigcup\limits_{n=1}^{\infty} B_{i_n j_n})$ is an open set containing 0 in X_6 which does not contain any point of sequence (y_n).

Thus the sequence (x_n) converges to 0 in X_5 whereas (y_n) does not converge to 0 in X_6. Which is a contradiction to the fact that f is a homeomorphism. Thus our supposition that X_5 and X_6 are homeomorphic is wrong. Hence the theorem.

Theorem 3.3.3. *The spaces X_7 and X_6 are homeomorphic.*

Proof: We have

$$X_6 = \{0\} \cup \mathbb{N} = \{0\} \cup \left(\bigcup_{j=1}^{\infty} B_j\right),$$

where $B_j = \bigcup_{j=1}^{\infty} B_{ij}$ and $\{B_{ij}|i,j = 1,2,...\}$ is a family of pairwise disjoint countably infinite sets partitioning the set $\mathbb{N}$.

And

$$X_7 = \{0\} \cup \mathbb{N} = \{0\} \cup \left(\bigcup_{j=1}^{\infty} B_j\right) \cup \left(\bigcup_{j=1}^{\infty} C_j\right)$$

where $B_j = \bigcup_{i=1}^{\infty} B_{ij}$ and $\{B_{ij}, C_j|i = 1,2,...,j = 1,2,...\}$ is a family of pairwise disjoint countably infinite sets partitioning the set $\mathbb{N}$.

Since B_{ij} and C_j are countable we can write B_{ij} and C_j as follows:

$$B_{ij} = \{x^i_{kj} \mid k = 1,2,...\} \;\; i,j = 1,2,...$$

$$\text{and } C_j = \{x^j_k \mid k = 1,2,...\} \;\; j = 1,2,...$$

We define a map f from X_7 to X_6 as follows:

$$f(x^j_k) = x^k_{1j}$$

$$f(x^i_{kj}) = x^i_{(k+1)j}$$

$$f(0) = 0$$

(1) Clearly, f is one-one and onto.

(2) <u>f is continuous</u>:

Let U be any open set in X_6. If $0 \notin U$, then $0 \notin f^{-1}(U)$. Therefore $f^{-1}(U)$ is open in X_7. If $0 \in U$, then U contains all but finitely many B_{ij} from each B_j, say,

$B_{11}, B_{21}, ..., B_{\alpha_1 1}, B_{12}, B_{22}, ..., B_{\alpha_2 2}, ..., B_{1j}, B_{2j}, ..., B_{\alpha_j j}, ...$

Then $f^{-1}(U)$ contains 0 and all B_{ij} except

$B_{11}, ..., B_{\alpha_1 1}, B_{12}, B_{22}, ..., B_{\alpha_2 2}, ..., B_{1j}, ..., B_{\alpha_j j}, ...$ and contain all points except finitely many from each C_j which are

$x_1^1, ..., x_{\alpha_1}^1, x_1^2, ..., x_{\alpha_2}^2, ..., x_1^j, ..., x_{\alpha_j}^j ....$

Therefore $f^{-1}(U)$ is open in X_7. Hence f is continuous.

(3) <u>f is an open map</u>:

Let U be any open set in X_7. If $0 \notin U$, then $0 \notin f(U)$. Therefore $f(U)$ is open in X_6. If $0 \in U$, then U contains all but finitely many B_{ij} from each B_j, say, $B_{11}, ..., B_{\alpha_1 1}, B_{12}, ..., B_{\alpha_2 2}, ..., B_{1j}, ..., B_{\alpha_j j}, ...$ and contains all but finitely many points from each C_j, say, $x_1^1, ..., x_{\beta_1}^1, x_1^2, ..., x_{\beta_2}^2, ..., x_1^j, ..., x_{\beta_j}^j, ...$ Without loss of generality we assume $\beta_i \leq \alpha_i$. Then $f(U)$ contains all B_{ij} from each B_j except $B_{11}, ..., B_{\alpha_1 1}, B_{21}, ..., B_{\alpha_2 2}, ..., B_{1j}, ..., B_{\alpha_j j}, ...$ Thus $f(U)$ is open in X_6. Hence f is open.

From (1), (2) and (3), f is a homeomorphism. That is, X_6 and X_7 are homeomorphic.

Chapter-4

Construction of spaces
with
unique nonisolated point
using filters

4.1 Introduction

In this chapter, we shall define filters on $\mathbb{N}$ and use them to construct countable spaces with exactly one nonisolated point. Also we shall show that these spaces are nonhomeomorphic. We establish this using the set-theoretic arguments only.

The original ideas of the techniques employed here have its roots in example [18] which appears here as Y_7.

4.2 Definitions

Definition 4.2.1. A *filter* on a set X is a nonempty collection $\mathscr{F}$ of nonempty subsets of X having the following properties:

(i) If $F, G \in \mathscr{F}$, then $F \cap G \in \mathscr{F}$.

(ii) If $F \in \mathscr{F}$ and $F \subseteq G$, then $G \in \mathscr{F}$.

Definition 4.2.2. A *filter base* on a set X is a nonempty collection $\mathscr{B}$ of nonempty subsets of X with the following property:

(i) If $B, C \in \mathscr{B}$, then $\exists D \in \mathscr{B} \ni D \subseteq B \cap C$.

Definition 4.2.3. If $\mathscr{B}$ is a filter base, then the *filter generated by $\mathscr{B}$* is given by

$$\mathscr{F} = \{F \subseteq X \mid \exists\, B \in \mathscr{B} \ni B \subseteq F\}.$$

One can easily check that $\mathscr{F}$ indeed is a filter.

Definition 4.2.4. A filter $\mathscr{F}$ on X is called a *fixed filter* if $\bigcap \mathscr{F} \neq \phi$ and *free* if $\bigcap \mathscr{F} = \phi$.

Definition 4.2.5. Let Y be a subset of X and $\mathscr{F}$ be a filter on X. Then the *trace of* $\mathscr{F}$ on Y is the filter $\{F \cap Y | F \in \mathscr{F}\}$ on Y.

Definition 4.2.6. A filter $\mathscr{F}$ is an *ultrafilter* if there is no strictly finer filter $\mathscr{G}$ than $\mathscr{F}$. Thus the ultrafilters are the maximal filters.

A very well known and useful result about ultrafilter is the following characterization:

> A filter $\mathscr{F}$ on X is an ultrafilter if and only if for each $E \subseteq X$, either $E \in \mathscr{F}$ or $X \backslash E \in \mathscr{F}$.

4.3 Filters on $\mathbb{N}$

Now we describe the following filters on $\mathbb{N}$, which we are going to use to construct countable spaces with unique nonisolated point.

(1) $\mathscr{F}_1 = \{F \subseteq N / N \backslash F$ is finite$\}$.

Note that the topology of any countable space with unique nonisolated point can be transfered on the set $X = \mathbb{N} \cup \{x_0\}$, where $x_0 \notin \mathbb{N}$, in such a way that x_0 is the unique nonisolated point. Thus, when we are considering countable spaces with unique nonisolated point, we may assume without loss of generality that the underlying set is $X = \mathbb{N} \cup \{x_0\}$. There is a nice one-to-one correspondence between the filters on $\mathbb{N}$ containing $\mathscr{F}_1$ and the traces on $\mathbb{N}$ of the neighborhood filters of x_0 arising out of T_1-topologies on X. This we capture in the following theorem.

Theorem 4.3.1. *Suppose $X = \mathbb{N} \cup \{x_0\}$ is a countable T_1-space with unique nonisolated point, then there exists a filter $\mathscr{F}$ on $\mathbb{N}$ which contains the filter $\mathscr{F}_1$.*

Conversely, if there is a filter $\mathscr{F}$ on $\mathbb{N}$ containing $\mathscr{F}_1$, then there exists a topology on $X = \mathbb{N} \cup \{x_0\}$ such that X with this topology is a T_1 -space with unique nonisolated point.

Proof:

Suppose $X = \mathbb{N} \cup \{x_0\}$ is a countable T_1-space with unique nonisolated point x_0.

Define $\mathscr{F} = \{F \subseteq \mathbb{N} / F \cup \{x_0\} \text{is open in } X\}$.

Let us check that $\mathscr{F}$ is a filter on $\mathbb{N}$.

(i) Let $F_1, F_2 \in \mathscr{F}$. Then $F_1 \cup \{x_0\}$ and $F_2 \cup \{x_0\}$ are open in X. Therefore $(F_1 \cup \{x_0\}) \cap (F_2 \cup \{x_0\}) = (F_1 \cap F_2) \cup \{x_0\}$

is open in X. Hence $F_1 \cap F_2 \in \mathscr{F}$.

(ii) Let $F \in \mathscr{F}$ and $G \subseteq \mathbb{N}$ such that $F \subseteq G$.

Since $F \in \mathscr{F}, F \cup \{x_0\}$ is open in X.

Now $G = F \cup (G \backslash F)$

$$\therefore G \cup \{x_0\} = (F \cup \{x_0\}) \cup (G \backslash F) \quad (1)$$

Since $F \cup \{x_0\}$ and $G \backslash F$ which is a subset of $\mathbb{N}$, are open in $X, G \cup \{x_0\}$ is open in X(by(1)). Therefore $G \in \mathscr{F}$.

Thus from (i) and (ii) $\mathscr{F}$ is a filter on $\mathbb{N}$.

Now it remains to show that $\mathscr{F}_1 \subseteq \mathscr{F}$.

Let $F \in \mathscr{F}_1$.

$\Rightarrow N \backslash F$ is finite.

$\Rightarrow N \backslash (F \cup \{x_0\})$ is finite.

$\Rightarrow N \backslash (F \cup \{x_0\})$ is closed in X. ($\because X$ is T_1)

$\Rightarrow F \cup \{x_0\}$ is open in X.

$\Rightarrow F \in \mathscr{F}$.

Thus, $\mathscr{F}_1 \subseteq \mathscr{F}$.

Conversely, suppose $\mathscr{F}$ is a filter on $\mathbb{N}$ such that $\mathscr{F}_1 \subseteq \mathscr{F}$.

Define $\tau = \mathscr{P}(\mathbb{N}) \cup \{U / U = F \cup \{x_0\},$ where $F \in \mathscr{F}\}$, where $\mathscr{P}(\mathbb{N})$ is a power set of $\mathbb{N}$.

First, let us check that τ is a topology on $X = \mathbb{N} \cup \{x_0\}$.

(i) Clearly, $\phi, X \in \tau$.

(ii) Let $\{U_\alpha\}_{\alpha \in J}$ be any subcollection of τ.

To prove $U = \bigcup_{\alpha \in J} U_\alpha$ is in τ.

If all $U_\alpha \subseteq N$ then $U = \bigcup_{\alpha \in J} U_\alpha \subseteq N$ and hence $U \in \tau$.

If at least one $\beta, U_\beta = F \cup \{x_0\}$ where $F \in \mathscr{F}$.

$$\text{Then } U = \left(\bigcup_{\substack{\alpha \in J \\ \alpha \neq \beta}} U_\alpha \right) \cup (F \cup \{x_0\})$$

$$= \left(F \cup (\bigcup_{\alpha \neq \beta} U_\alpha) \right) \cup \{x_0\} \in \tau$$

$$\left(\because F \subseteq F \cup \left(\bigcup_{\substack{\alpha \in J \\ \alpha \neq \beta}} U_\alpha \right) \text{ and } F \in \mathscr{F} \right)$$

Thus $U = \bigcup_{\alpha \in J} U_\alpha \in \tau$.

(iii) Let $U_1, U_2, ..., U_n \in \tau$.

If for each $i = 1, 2, ..., n, U_i = F_i \cup \{x_0\}$.

Then $\bigcap_{i=1}^{n} U_i = \left(\bigcap_{i=1}^{n} F_i \right) \cup \{x_0\} \in \tau \ (\because \bigcap F_i \in \mathscr{F})$.

If for at least one $\beta, U_\beta \subseteq \mathbb{N}$ then $\bigcap_{i=1}^{n} U_i \subseteq \mathbb{N}$.

Therefore $\bigcap_{i=1}^{n} U_i \in \tau$.

Thus from (i), (ii) and (iii), τ is a topology on $X = \mathbb{N} \cup \{x_0\}$.

Now we show that (X, τ) is T_1.

Let x, y be any two distinct points of X. If both x and y are in $\mathbb{N}$, then $\{x\}$ and $\{y\}$ are required open sets. If one of the points is

x_0, then we may assume without any loss of generality $x = x_0$ and $y \in \mathbb{N}$. Now $F = \mathbb{N}\backslash\{y\} \in \mathscr{F}_1$, because $\mathbb{N}\backslash F$ is finite ($\mathbb{N}\backslash F = \{y\}$). Since $\mathscr{F}_1 \subseteq \mathscr{F}, F \in \mathscr{F}$. Therefore $F \cup \{x_0\}$ is open in X. Thus $F \cup \{x_0\}$ and $\{y\}$ are required open sets. Hence X is a T_1- space.

Now we construct filters finer than $\mathscr{F}_1$.

(2) $\mathscr{F}_2 = \{F \subseteq \mathbb{N}/O\backslash F$ is finite$\}$, where $O = \{x_1, x_2, ...\}$ is the set of all odd numbers.

i.e. $\mathscr{F}_2 = \{F \subseteq \mathbb{N}/\exists N \ni x_j \in F, \forall j \geq N\}$.

(3) First we write $\mathbb{N}$ as:

$$\mathbb{N} = \bigcup_{j=1}^{\infty} A_j, \quad |A_j| = \aleph_0, A_i \cap A_j = \phi, \text{for } \ i \neq j.$$

In other words we are considering a partition $\{A_j | j = 1, 2, ...\}$ of $\mathbb{N}$ consisting of countably infinite subsets.

Define

$$\mathscr{F}_3 = \{F \subseteq \mathbb{N}/\exists N \ni A_j \subseteq F \ \ \forall j \geq N\}$$

(4) As in (3), write $\mathbb{N} = \bigcup\limits_{j=1}^{\infty} C_j$, where

$$C_j = \{x_{ij}/i = 1, 2, ...\}, C_i \cap C_j = \phi \ \text{ for } \ i \neq j.$$

Define

$$\mathscr{F}_4 = \{F \subseteq \mathbb{N}/ \ \text{for each } \ j, \ \exists N_j \ni x_{ij} \in F, \forall i \geq N_j\}$$

That is

$$\mathscr{F}_4 = \Big\{F \subseteq \mathbb{N} \ / \ F \text{ contains all but finitely many points from each } C_j\Big\}.$$

(5) Write $\mathbb{N} = \left(\bigcup\limits_{i=1}^{\infty} A_i\right) \cup \left(\bigcup\limits_{j=1}^{\infty} C_j\right)$, where $\{A_i, C_j/i, j = 1, 2, ...\}$ is a family of pairwise disjoint countably infinite sets partitioning the set $\mathbb{N}$ and as before $C_j = \{x_{ij}|i = 1, 2, ...\}$.

Define

$$\mathscr{F}_5 = \Big\{F \subseteq \mathbb{N} \ / \ \exists N \ni A_i \subseteq F, \ \ \forall i \geq N \text{ and for each } j, \ \ \exists N_j \ni x_{ij} \in F, \forall i \geq N_j\Big\}.$$

That is

$$\mathscr{F}_5 = \Big\{F \subseteq N \ / \ F \text{ contains all but finitely many points from each } C_j \text{ and all but finitely many full } A_i\Big\}.$$

(6) Write $\mathbb{N} = \bigcup\limits_{j=1}^{\infty} B_j$, where $B_j = \bigcup\limits_{i=1}^{\infty} B_{ij}$.

$\{B_{ij}/i, j = 1, 2, ...\}$ is a family of pairwise disjoint countably infinite sets partitioning the set $\mathbb{N}$.

Define

$$\mathscr{F}_6 = \Big\{F \subseteq \mathbb{N}/\text{for each } j \ \exists \ N_j \ni B_{ij} \subset F \ \ \forall \ i \geq N_j\Big\}$$

i.e. $$\mathscr{F}_6 = \Big\{F \subseteq \mathbb{N} \ / \text{ for each } j \text{ all but finitely many } B_{ij} \text{ are contained in } F\Big\}.$$

One can easily check that $\mathscr{F}_i(i = 1, 2, ..., 6)$ are filters on $\mathbb{N}$.

(7) Let $\mathscr{F}_7$ be any free ultrafilter on $\mathbb{N}$.

Next, let us construct topological spaces $Y_i(i = 1, 2, ..., 7)$ using above filters on $\mathbb{N}$.

4.4 Construction of spaces $Y_i(i = 1, 2, ..., 7)$ using filters on $\mathbb{N}$

Put $Y_i = \mathbb{N} \cup \{\mathscr{F}_i\}$.

The topology τ_i on $Y_i(i = 1, 2, .., 7)$ is defined in the following way:

$\tau_i = \mathscr{P}(\mathbb{N}) \cup \{F \cup \{\mathscr{F}_i\}/F \in \mathscr{F}_i\}$, where $\mathscr{P}(\mathbb{N})$ is a power set of $\mathbb{N}$.

That is, each subset of $\mathbb{N}$ is open in Y_i and $F \cup \{\mathscr{F}_i\}$ is open in Y_i if and only if $F \in \mathscr{F}_i$.

It is easy to verify that τ_i is a topology on $Y_i(i = 1, 2, ..., 7)$.

4.5 Are theses spaces different ?

Theorem 4.5.1. *The space Y_i is not homeomorphic to Y_j if $i \neq j$ and $i, j \in \{1, 2, ..., 7\}$*

Proof:

(1) <u>Y_1 is not homeomorphic to Y_2</u>:

Suppose $f : Y_1 = \mathbb{N} \cup \{\mathscr{F}_1\} \to Y_2 = \mathbb{N} \cup \{\mathscr{F}_2\}$ is a homeomorphism. Now $O \cup \{\mathscr{F}_2\}$ is open in $Y_2(\because\ O \in \mathscr{F}_2)$. Therefore $f^{-1}(O) \cup \{\mathscr{F}_1\}$ is open in Y_1 as f is a homeomorphism and so $f^{-1}(O) \in \mathscr{F}_1$. Hence $\mathbb{N}\backslash f^{-1}(O)$ is finite. Now $\mathbb{N} = O \cup E$, where O is the set of odd number and E is the set of even numbers.

$\Rightarrow f^{-1}(\mathbb{N}) = f^{-1}(O) \cup f^{-1}(E)$

$\Rightarrow \mathbb{N} = f^{-1}(O) \cup f^{-1}(E)$ (1)

Also, $O \cap E = \phi$

$\Rightarrow f^{-1}(O) \cap f^{-1}(E) = \phi$ (2)

From (1) and (2)

$\mathbb{N}\backslash f^{-1}(O) = f^{-1}(E)$

$\therefore \mathbb{N}\backslash f^{-1}(O)$ is infinite. ($\because f^{-1}(E)$ is infinite.)

Which is a contradiction to the fact that $\mathbb{N}\backslash f^{-1}(O)$ is finite. Therefore our supposition that f is a homeomorphism is false and hence we conclude that Y_1 is not homeomorphic to Y_2.

(2) Y_1 is not homeomorphic to Y_3:

Suppose $f : Y_1 = \mathbb{N} \cup \{\mathscr{F}_1\} \to Y_3 = \mathbb{N} \cup \{\mathscr{F}_3\}$ is a homeomorphism. Set $F = \mathbb{N}\backslash A_1$. Then $F \cup \{\mathscr{F}_3\}$ is open in $Y_3(\because\ F \in \mathscr{F}_3)$. Therefore $f^{-1}(F) \cup \{\mathscr{F}_1\}$ is open in Y_1. Thus $f^{-1}(F) \in \mathscr{F}_1$ and so $\mathbb{N}\backslash f^{-1}(F)$ is finite.

$$\begin{aligned}\mathbb{N}\backslash f^{-1}(F) &= \mathbb{N}\backslash f^{-1}(\mathbb{N}\backslash A_1)\\ &= \mathbb{N}\backslash(\mathbb{N}\backslash f^{-1}(A_1))\\ &= f^{-1}(A_1)\end{aligned}$$

Since A_1 is infinite and f is onto, $f^{-1}(A_1)$ is infinite, that is, $\mathbb{N}\backslash f^{-1}(F)$ is infinite. which is a contradiction to the fact that $\mathbb{N}\backslash f^{-1}(F)$ is finite. Thus our supposition that f is a homeomorphism is false and hence we conclude that Y_1 and Y_3 are not homeomorphic.

(3) Y_1 is not homeomorphic to Y_4:

Suppose $f : Y_1 = \mathbb{N} \cup \{\mathscr{F}_1\} \to Y_4 = \mathbb{N} \cup \{\mathscr{F}_4\}$ is a homeomorphism. Choose $y_j \in C_j$ for each j and put $A = \{y_j | j = 1, 2, ...\}$. Then $\mathbb{N}\backslash A$ is in $\mathscr{F}_4$ and so $(\mathbb{N}\backslash A) \cup \{\mathscr{F}_4\}$ is open in Y_4. As f is a homeomorphism $f^{-1}(\mathbb{N}\backslash A) \cup \{\mathscr{F}_1\}$ is open in Y_1. Therefore $f^{-1}(\mathbb{N}\backslash A)$ is in $\mathscr{F}_1$ and hence $\mathbb{N}\backslash f^{-1}(\mathbb{N}\backslash A)$ is finite.

$$\begin{aligned}\mathbb{N}\backslash f^{-1}(\mathbb{N}\backslash A) &= \mathbb{N}\backslash(f^{-1}(\mathbb{N})\backslash f^{-1}(A))\\ &= \mathbb{N}\backslash(\mathbb{N}\backslash f^{-1}(A))\\ &= f^{-1}(A)\end{aligned}$$

As A is infinite and f is onto, $f^{-1}(A)$ is infinite, that is, $\mathbb{N}\backslash f^{-1}(\mathbb{N}\backslash A)$ is infinite. Which is a contradiction to the fact that

$\mathbb{N}\backslash f^{-1}(\mathbb{N}\backslash A)$ is finite. Thus our supposition that f is a homeomorphism is false and hence we conclude that Y_1 and Y_4 are not homeomorphic.

(4) Y_1 is not homeomorphic to Y_5:

Suppose $f : Y_1 = \mathbb{N} \cup \{\mathscr{F}_1\} \rightarrow Y_5 = \mathbb{N} \cup \{\mathscr{F}_5\}$ is a homeomorphism. Choose $y_j \in C_j$ for each j and put $A = \{y_j | j = 1, 2, ...\}$. Then $\mathbb{N}\backslash A$ is in $\mathscr{F}_5$ and so $(\mathbb{N}\backslash A) \cup \{\mathscr{F}_5\}$ is open in Y_5. As f is a homeomorphism $f^{-1}(\mathbb{N}\backslash A) \cup \{\mathscr{F}_1\}$ is open in Y_1.

Therefore $f^{-1}(\mathbb{N}\backslash A)$ is in $\mathscr{F}_1$ and hence $\mathbb{N}\backslash f^{-1}(\mathbb{N}\backslash A)$ is finite.

$$\begin{aligned}
\mathbb{N}\backslash f^{-1}(\mathbb{N}\backslash A) &= \mathbb{N}\backslash(f^{-1}(\mathbb{N})\backslash f^{-1}(A)) \\
&= \mathbb{N}\backslash(\mathbb{N}\backslash f^{-1}(A)) \\
&= f^{-1}(A)
\end{aligned}$$

Since A is infinite and f is onto, $f^{-1}(A)$ is infinite, that is, $\mathbb{N}\backslash f^{-1}(\mathbb{N}\backslash A)$ is infinite. Which is a contradiction to the fact that $\mathbb{N}\backslash f^{-1}(\mathbb{N}\backslash A)$ is finite. Thus our supposition that f is a homeomorphism is false and hence we conclude that Y_1 and Y_5 are not homeomorphic.

(5) Y_1 is not homeomorphic to Y_6:

Suppose $f : Y_1 = \mathbb{N} \cup \{\mathscr{F}_1\} \rightarrow Y_6 = \mathbb{N} \cup \{\mathscr{F}_6\}$ is a homeomorphism. For a fixed i, choose $y_j \in B_{ij}$ for each j and put $A = \{y_j | j = 1, 2, ...\}$. Then $\mathbb{N} \backslash A$ is in $\mathscr{F}_6$ and so $(\mathbb{N} \backslash A) \cup \{\mathscr{F}_6\}$ is open in Y_6. As f is a homeomorphism $f^{-1}(\mathbb{N} \backslash A) \cup \{\mathscr{F}_1\}$ is open in Y_1. Therefore $f^{-1}(\mathbb{N} \backslash A) \in \mathscr{F}_1$ and hence $\mathbb{N} \backslash f^{-1}(\mathbb{N} \backslash A)$ is finite.

$$\mathbb{N} \backslash f^{-1}(\mathbb{N} \backslash A) = f^{-1}(A)$$

Since A is infinite and f is onto, $f^{-1}(A)$ is infinite, that is, $\mathbb{N} \backslash (f^{-1}(\mathbb{N} \backslash A))$ is infinite. Which is a contradiction to the fact that $\mathbb{N} \backslash f^{-1}(\mathbb{N} \backslash A)$ is finite. Thus our supposition that f is a homeomorphism is false and hence we conclude that Y_1 and Y_6 are not homeomorphic.

(For Y_1 is not homeomorphic to Y_7 , see (16) which follows).

(6) Y_2 is not homeomorphic to Y_3:

Suppose $f : Y_2 = \mathbb{N} \cup \{\mathscr{F}_2\} \to Y_3 = \mathbb{N} \cup \{\mathscr{F}_3\}$ is a homeomorphism. Now $O \cup \{\mathscr{F}_2\}$ is open in Y_2 as $O \in \mathscr{F}_2$. Therefore $f(O) \cup \{\mathscr{F}_3\}$ is open in Y_3. Then $\exists N$ such that $A_j \subseteq f(O)$, $\forall j \geq N$, that is, $\bigcup\limits_{j=N}^{\infty} A_j \subseteq f(O)$. In particular, $A_N \subseteq f(O)$ and so, $f^{-1}(A_N) \subseteq O$.

As A_N is infinite and f is onto $f^{-1}(A_N)$ is infinite. We also have $\bigcup\limits_{j=N+1}^{\infty} A_j \subseteq \bigcup\limits_{j=N}^{\infty} A_j \subseteq f(O)$ and therefore $f^{-1}\left(\bigcup\limits_{j=N+1}^{\infty} A_j\right) \subseteq O$.

Since $\bigcup\limits_{j=N+1}^{\infty} A_j \in \mathscr{F}_3$, $\left(\bigcup\limits_{j=N+1}^{\infty} A_j\right) \cup \{\mathscr{F}_3\}$ is open in Y_3. Then $f^{-1}\left(\bigcup\limits_{j=N+1}^{\infty} A_j\right) \cup \{\mathscr{F}_2\}$ is open in Y_2. But then $f^{-1}\left(\bigcup\limits_{j=N+1}^{\infty} A_j\right) \in \{\mathscr{F}_2\}$ and hence $O \backslash f^{-1}\left(\bigcup\limits_{j=N+1}^{\infty} A_j\right)$ is finite.

Now $A_N \cap \left(\bigcup\limits_{j=N+1}^{\infty} A_j\right) = \phi$

$\Rightarrow f^{-1}(A_N) \cap f^{-1}\left(\bigcup\limits_{j=N+1}^{\infty} A_j\right) = \phi$

$\Rightarrow f^{-1}(A_N) \subseteq O \backslash f^{-1}\left(\bigcup\limits_{j=N+1}^{\infty} A_j\right)$

Which is a contradiction to the fact that no subset of a finite set can be infinite. Thus our supposition that f is a homeomorphism is false and hence we conclude that Y_2 and Y_3 are not homeomorphic.

(7) <u>Y_2 is not homeomorphic to Y_4</u>:

Suppose $f : Y_2 = \mathbb{N} \cup \{\mathscr{F}_2\} \rightarrow Y_4 = \mathbb{N} \cup \{\mathscr{F}_4\}$ is a homeomorphism. Now $O \cup \{\mathscr{F}_2\}$ is open in Y_2 as $O \in \mathscr{F}_2$. Therefore $f(O) \cup \{\mathscr{F}_4\}$ is open in Y_4 and so $f(O) \in \mathscr{F}_4$. Then $\exists N_j \ni x_{ij} \in f(O),\ \forall i \geq N_j$, that is, $f(O)$ contains all but finitely many points from each C_j. Choose $y_j \in f(O) \cap C_j$ for each j and put $A = \{y_j | j = 1, 2, ...\}$. Then $U = \left(\bigcup\limits_{j=1}^{\infty} (f(O) \cap C_j)\right) \backslash A$ is in $\mathscr{F}_4$.

So that, $U \cup \{\mathscr{F}_4\}$ is open in Y_4. As f is a homeomorphism, $f^{-1}(U) \cup \{\mathscr{F}_2\}$ is open in Y_2 and hence $O \backslash f^{-1}(U)$ is finite.

We know that $\quad U \cap A = \phi$

$$\Rightarrow f^{-1}(U) \cap f^{-1}(A) = \phi$$

$$\Rightarrow f^{-1}(A) \subseteq O \backslash f^{-1}(U)$$

Since A is infinite and f is onto, $f^{-1}(A)$ is infinite. Thus we have a contradiction to the fact that no subset of a finite set can be infinite. Thus our supposition that f is a homeomorphism is false and therefore we conclude that Y_2 and Y_4 are not homeomorphic.

(8) Y_2 is not homeomorphic to Y_5:

Suppose $f : Y_2 = \mathbb{N} \cup \{\mathscr{F}_2\} \to Y_5 = \mathbb{N} \cup \{\mathscr{F}_5\}$ is a homeomorphism.

Now $O \cup \{\mathscr{F}_2\}$ is open in Y_2 as $O \in \mathscr{F}_2$. Therefore $f(O) \cup \{\mathscr{F}_5\}$ is open in Y_5 and so $f(O) \in \mathscr{F}_5$. Then $\exists N \ni A_i \subseteq f(O), \forall i \geq N$ and for each $j, \exists N_j \ni x_{ij} \in f(O), \forall i \geq N_j$.

Choose $y_j \in f(O) \cap C_j$ and put $A = \{y_j | j = 1, 2, ...\}$. By definition of $\mathscr{F}_5, U = Y_5 \backslash A$ is in $\mathscr{F}_5$, so that $U \cup \{\mathscr{F}_5\}$ is open in Y_5. As f is a homeomorphism, $f^{-1}(U) \cup \{\mathscr{F}_2\}$ is open in Y_2 and hence $O \backslash f^{-1}(U)$ is finite.

We know that $\quad U \cap A = \phi$

$$\Rightarrow f^{-1}(U) \cap f^{-1}(A) = \phi$$

$$\Rightarrow f^{-1}(A) \subseteq O \backslash f^{-1}(U)$$

Since A is infinite and f is onto, $f^{-1}(A)$ is infinite. Thus we have a contradiction to the fact that no subset of a finite set can be infinite. Thus our supposition that f is a homeomorphism is false and therefore we conclude that Y_2 and Y_5 are not homeomorphic.

(9) Y_2 is not homeomorphic to Y_6:

Suppose $f : Y_2 = \mathbb{N} \cup \{\mathscr{F}_2\} \rightarrow Y_6 = \mathbb{N} \cup \{\mathscr{F}_6\}$ is a homeomorphism. Now $O \cup \{\mathscr{F}_2\}$ is open in Y_2 as $O \in \mathscr{F}_2$. Therefore $f(O) \cup \{\mathscr{F}_6\}$ is open in Y_6 and so $f(O) \in \mathscr{F}_6$. Then for each $j, \exists N_j$ such that $B_{ij} \subseteq f(O) \quad \forall i \geq N_j$. Choose B_{tj} for each j such that $B_{tj} \subseteq f(O)$, where $t > N_j$ for each j. Put $A = \bigcup_{j=1}^{\infty} B_{tj}$. Clearly, $A \subseteq f(O)$. Then $U = \mathbb{N} \backslash A$ is in $\mathscr{F}_6$ and hence $U \cup \{\mathscr{F}_6\}$ is open in Y_6. As f is a homeomorphism,$f^{-1}(U) \cup \{\mathscr{F}_2\}$ is open in Y_2 and hence $O \backslash f^{-1}(U)$ is finite.

Now $\quad U \cap A = \phi$

$$\Rightarrow f^{-1}(U) \cap f^{-1}(A) = \phi$$

$$\Rightarrow f^{-1}(A) \subseteq O \backslash f^{-1}(U)$$

Since A is infinite and f is onto, $f^{-1}(A)$ is infinite. Thus we have a contradiction to the fact that no subset of a finite set can be infinite. Thus our supposition that f is a homeomorphism is false and therefore we conclude that Y_2 and Y_6 are not homeomorphic.

(10) Y_3 is not homeomorphic to Y_4:

Suppose $f : Y_3 = \mathbb{N} \cup \{\mathscr{F}_3\} \to Y_4 = \mathbb{N} \cup \{\mathscr{F}_4\}$ is a homeomorphism.

Claim: $f(A_m) \cap C_j$ is finite for each j and for each m.

Suppose $f(A_t) \cap C_k$ is infinite for some k and for some t.

Put $F = \mathbb{N} \backslash f^{-1}(f(A_t) \cap C_k)$

But $F = \mathbb{N} \backslash f^{-1}(f(A_t) \cap C_k)$

$$= \left(\bigcup_{j=1}^{\infty} A_j \right) \backslash \{x \in A_t / f(x) \in f(A_t) \cap C_k\}$$

By definition of $\mathscr{F}_3$, $F \in \mathscr{F}_3$ and so $F \cup \{\mathscr{F}_3\}$ is open in Y_3. As f is a homeomorphism $f(F) \cup \{\mathscr{F}_4\}$ is open in Y_4.

$$\begin{aligned} \text{But } f(F) &= f\left(\mathbb{N} \backslash f^{-1}(f(A_t) \cap C_k)\right) \\ &= f(\mathbb{N}) \backslash (f(A_t) \cap C_k) \\ &= \mathbb{N} \backslash (f(A_t) \cap C_k) \end{aligned}$$

$\therefore f(F) \notin \mathscr{F}_4$ (by definition of $\mathscr{F}_4$).

$\therefore f(F) \cup \{\mathscr{F}_4\}$ is not open in Y_4.

Which is a contradiction to the fact that f is a homeomorphism. This proves our claim.

Now let j_1 be the first index such that $f(A_1) \cap C_{j_1}$ is nonempty. Choose a point y_1 in $f(A_1) \cap C_{j_1}$. Let j_2 be the first index after j_1 (i.e. $j_2 > j_1$) such that $f(A_2) \cap C_{j_2}$ is nonempty. Choose a point y_2 in $f(A_2) \cap C_{j_2}$. Continuing in this way we get a set $A = \{y_1, y_2, ...\}$. Since f is onto, corresponding to each y_i there exists x_i in A_i such that $f(x_i) = y_i$. By definition of $\mathscr{F}_4, \mathbb{N}\backslash A \in \mathscr{F}_4$. Therefore $(\mathbb{N}\backslash A) \cup \{\mathscr{F}_4\}$ is open in Y_4. As f is a homeomorphism, $f^{-1}(\mathbb{N}\backslash A) \cup \{\mathscr{F}_3\}$ is open in Y_3. But for each $i, x_i \notin f^{-1}(\mathbb{N}\backslash A)$, so that $A_i \nsubseteq f^{-1}(\mathbb{N}\backslash A)$ for each i. Hence $f^{-1}(\mathbb{N}\backslash A) \notin \mathscr{F}_3$. Therefore $f^{-1}(\mathbb{N}\backslash A) \cup \{\mathscr{F}_3\}$ is not open in Y_3. which is a contradiction to the fact that f is a homeomorphism. Thus we conclude that Y_3 and Y_4 are not homeomorphic.

(11) Y_3 is not homeomorphic to Y_5:

Suppose $f : Y_3 = \mathbb{N} \cup \{\mathscr{F}_3\} \to Y_5 = \mathbb{N} \cup \{\mathscr{F}_5\}$ is a homeomorphism.

Claim: $f(A_m) \cap C_j$ is finite for each j and for each m.

Suppose $f(A_t) \cap C_k$ is infinite for some k and for some t.

Put $F = \mathbb{N}\backslash f^{-1}(f(A_t) \cap C_k)$.

$\therefore F = \left(\bigcup_{j=1}^{\infty} A_j \right) \backslash \{x \in A_t / f(x) \in f(A_t) \cap C_k\}$

By definition of $\mathscr{F}_3, F \in \mathscr{F}_3$ and so $F \cup \{\mathscr{F}_3\}$ is open in Y_3.

As f is a homeomorphism, $f(F) \cup \{\mathscr{F}_5\}$ is open in Y_5.

$$\begin{aligned} \text{But } f(F) &= f\left(\mathbb{N} \backslash f^{-1}(f(A_t) \cap C_k)\right) \\ &= f(\mathbb{N}) \backslash (f(A_t) \cap C_k) \\ &= \mathbb{N} \backslash (f(A_t) \cap C_k) \end{aligned}$$

$\therefore f(F) \notin \mathscr{F}_5$ (by definition of $\mathscr{F}_5$).

$\therefore f(F) \cup \{\mathscr{F}_5\}$ is not open in Y_5.

Which is a contradiction to the fact that f is a homeomorphism. Thus our claim is proved.

Now let j_1 be the first index such that $f(A_1) \cap C_{j_1}$ is nonempty. Choose a point y_1 in $f(A_1) \cap C_{j_1}$. Let j_2 be the first index after j_1 (i.e. $j_2 > j_1$) such that $f(A_2) \cap C_{j_2}$ is nonempty. Choose a point y_2 in $f(A_2) \cap C_{j_2}$. Inductively we get a set $A = \{y_1, y_2, ...\}$. Since f is onto, corresponding to each y_i there exists x_i in A_i such that $y_i = f(x_i)$. By definition of $\mathscr{F}_4, \mathbb{N} \backslash A \in \mathscr{F}_5$. Therefore $(\mathbb{N} \backslash A) \cup \{\mathscr{F}_5\}$ is open in Y_5. As f is a homeomorphism, $f^{-1}(\mathbb{N} \backslash A) \cup \{\mathscr{F}_3\}$ is open in Y_3. But for each i, $x_i \notin f^{-1}(\mathbb{N} \backslash A)$, so that $A_i \not\subseteq f^{-1}(\mathbb{N} \backslash A)$ for each i. Hence $f^{-1}(\mathbb{N} \backslash A) \notin \mathscr{F}_3$.

Therefore $f^{-1}(\mathbb{N}\backslash A) \cup \{\mathscr{F}_3\}$ cannot be open in Y_3. Which is a contradiction to the fact that f is a homeomorphism. Thus we conclude that Y_3 and Y_5 are not homeomorphic.

(12) Y_3 is not homeomorphic to Y_6:

Suppose $f : Y_3 = \mathbb{N} \cup \{\mathscr{F}_3\} \to Y_6 = \mathbb{N} \cup \{\mathscr{F}_6\}$ is a homeomorphism.

Claim: For each m and for each j, $f(A_m)$ intersects only finitely many B_{ij}.

Suppose for some t and for some k, there exists an infinite subset D of $\mathbb{N}$ such that $f(A_t) \cap B_{nk} \neq \phi$ for $\forall\ n \in D$. Choose $y_n \in f(A_t) \cap B_{nk}, \forall\ n \in D$. Then we get a sequence (y_n) in Y_6 which converges to $\mathscr{F}_6$. Corresponding to each y_n there exist $x_n \in A_t$ such that $f(x_n) = y_n$ and hence we have a sequence (x_n) must converge to $\mathscr{F}_3$. But there is no sequence in A_t which converges to $\mathscr{F}_3$. Which is a contradiction. Thus our claim is proved.

Now let j_1 be the first index such that $f(A_1) \cap B_{ij_1} \neq \phi$ for finitely many i. Choose a point $y_1 \in f(A_1) \cap B_{ij_1}$ for some i. Let j_2 be the first index after j_1 (i.e. $j_2 > j_1$) such that $f(A_2) \cap B_{ij_2} \neq \phi$ for finitely many i. Choose $y_2 \in f(A_2) \cap B_{ij_2}$ for some i. Continuing in this way, we get a set $A = \{y_1, y_2, y_3, ...\}$. Corresponding to each y_i there exists x_i in A_i such that $f(x_i) = y_i$. Clearly, $\mathbb{N}\backslash A \in \mathscr{F}_6$. Therefore $(\mathbb{N}\backslash A) \cup \{\mathscr{F}_6\}$ is open in Y_6 and hence, $f^{-1}(\mathbb{N}\backslash A) \cup \{\mathscr{F}_3\}$ is open in Y_3 as f is a homeomorphism. But $x_i \notin f^{-1}(\mathbb{N}\backslash A)$ for each i, therefore $A_i \nsubseteq f^{-1}(\mathbb{N}\backslash A)$ for each i. So that, $f^{-1}(\mathbb{N}\backslash A) \notin \mathscr{F}_3$ and hence $f^{-1}(\mathbb{N}\backslash A) \cup \mathscr{F}_3$ is not open in Y_3. which is a contradiction to the fact that f is a homeomorphism. Thus we conclude that Y_3 and Y_6 are not homeomorphic.

(13) Y_4 is not homeomorphic to Y_5:

By similar argument given in (11) (i.e. Y_3 is not homeomorphic to Y_5), we can prove Y_4 is not homeomorphic to Y_5.

(14) Y_4 is not homeomorphic to Y_6:

By similar argument given in (11) (i.e. Y_3 is not homeomorphic to Y_5), we can prove Y_4 is not homeomorphic to Y_6.

(15) Y_5 is not homeomorphic to Y_6:

By similar argument given in (11) (i.e. Y_3 is not homeomorphic to Y_5), we can prove Y_5 is not homeomorphic to Y_6.

(16) Y_7 is not homeomorphic to $Y_i, i = 1, 2, ...6$:

This follows from the fact that each of the filter $\mathscr{F}_i$ $(i = 1, 2, ..., 6)$ contains a set F such that $F = A \cup B, |A| = |B| = \aleph_0, A \cap B = \phi$. But neither A nor B is in $\mathscr{F}_i$, while $\mathscr{F}_7$ being an ultrafilter when $H \in \mathscr{F}_7$ and $H = C \cup D$, then either C or D must be in $\mathscr{F}_7$.

Remark 4.5.1. Some of the spaces considered here can be shown to be non-homeomorphic to some other spaces by considering certain topological properties. But we have not gone into it here. Instead we have totally relied on elementary set-theoretic arguments only.

Chapter-5

Does F_{ω} support a compatible group structure?

5.1 Introduction

S. P. Franklin and M. Rajagopalan [4] defined a topological space F_ω, which is an example of a countable, homogeneous, regular, Fréchet-Urysohn space which is not first countable. They also ask a question if it supports a compatible group structure making it into a topological group. Desai [2] answered this question in negation using the π -weight of the space. We here, are giving another proof of the same result using the idea of a diagonal sequence [10]. We also give another description of the space F_ω. Such questions and many related problems are discussed recently also by many e.g. [12],[13],[14].

5.2 Definitions

Definition 5.2.1. A topological space X is called *homogeneous* if the status of each point in the space is same. Technically, a space X is called *homogeneous* if for each pair of points x and y in X there is a homeomorphism $h : X \to X$ such that $h(x) = y$.

Definition 5.2.2. A topological space X which is also a group is called a *topological group* if the multiplication and the inversion are continuous functions (i.e. the mapping $g_1 : X \times X \to X$ given by

$g_1(x, y) = xy$ and the mapping $g_2 : X \to X$ given by $g_2(x) = x^{-1}$ are continuous).

If G is a group and if there is a topology with respect to which G becomes a topological group, we say that topology is compatible with the given group structure of G. Similarly, if X is a topological space and if there is a group structure on X with respect to which X becomes a topological group, we say that the group structure is compatible with the given topology on X.

Note: Every topological group G is a homogeneous space as $h : G \to G$ defined by $h(t) = t(x^{-1}y)$, for each pair of points x, y in G, is a homeomorphism which maps x to y.

5.3 Diagonal Sequence Condition

Peter J Nyikos stated the following condition [10]. Here we shall refer to it as the diagonal sequence condition.Thus a topological space X is said to satisfy the diagonal sequence condition if $x \in X$ and $(x_n^m)_{n=1}^{\infty}$ for each positive integer m is a sequence converging to x, it is possible to choose a sequence $\big(m(k)\big)_{k=1}^{\infty}$ of distinct positive integers and $\big(j(k)\big)_{k=1}^{\infty}$ of positive integers in such a way that the diagonal sequence $\left(x_{j(k)}^{m(k)}\right)_{k=1}^{\infty}$ converges to x.

Note: One can check easily that every metric space and more generally every first countable space satisfies the diagonal sequence condition.

Also note that the sequential fan X_4 does not satisfy the diagonal sequence condition. Actually the diagonal sequence condition fails at the unique nonisolated point of the sequential fan because of the following reason.

One can consider each of the disjoint copies of convergent sequence as the sequence $(x_n^m)_{n=1}^{\infty}$ for each fixed m. Thus $(x_n^m)_{n=1}^{\infty}$ converges to 0 of sequential fan X_4. However no diagonal sequence $\left(x_{j(k)}^{m(k)}\right)_{k=1}^{\infty}$ of $(x_n^m)_{n=1}^{\infty}$ can converge to 0 as the complement U of the range of the set of the diagonal sequence is an open set containing 0 in X_4 which does contain any point of the diagonal sequence.

Next theorem shows that diagonal sequence condition is hereditary.

Theorem 5.3.1. *If X satisfies diagonal sequence condition then each subspace Y of X satisfies this condition.*

Proof:

First we observe that if Y is a subspace of X and if (y_n) is a sequence in Y and $a \in Y$, then (y_n) converges to a in Y if and only if (y_n) converges to a in X.

Now, let $x \in Y$. Let $(x_n^m)_{n=1}^{\infty}$ be a sequence in Y converging to x for each positive integer m. Since Y is a subspace of X and the diagonal sequence condition is satisfied for X, there exists a sequence $\big(m(k)\big)_{k=1}^{\infty}$ of distinct positive integers and a sequence $\big(j(k)\big)_{k=1}^{\infty}$ of positive integers in such a way that the diagonal sequence $\left(x_{j(k)}^{m(k)}\right)_{k=1}^{\infty}$ converges to x in X. Now because of our earlier observation sequence $\left(x_{j(k)}^{m(k)}\right)_{k=1}^{\infty}$ converges to x in Y. ∎

5.4 The Construction of F_ω

The original construction of the space F_ω in [4] by S. P. Franklin and M. Rajagopalan is very concise and is aimed for the expert topologist. In our attempt to overcome certain difficulties in understanding their construction, we have now come up with totally different way of looking at F_ω. Here is our construction of F_ω.

We start with the set :

$$\begin{aligned} F_\omega &= \bigcup_{n=0}^{\infty} \mathbb{N}^n \\ &= \mathbb{N}^0 \cup \mathbb{N} \cup (\mathbb{N}\times\mathbb{N}) \cup (\mathbb{N}\times\mathbb{N}\times\mathbb{N}) \cup ..., \end{aligned}$$

where $\mathbb{N}^0 = \{0\}$

Let $a, b \in F_\omega$, then $a = (a_1, ..., a_m)$ and $b = (b_1, ..., b_n)$. We say that $b \leq a$, if $m \leq n$ and $a_i = b_i$ for each $i = 1, 2, ..., m$. Under this order relation F_ω is a partially ordered set. Now consider the sequential fan topology on the set $\mathbb{N} \cup \{0\}$. Let $U \subseteq \mathbb{N}$ such that $U \cup \{0\}$ is open in $\mathbb{N} \cup \{0\}$ and let

$$U_p^* = \Big\{y \in F_\omega / y \leq (p, i) \text{ for some } i \in U\Big\} \cup \Big\{(p, i)/i \in U\Big\} \cup \{p\}.$$

Then $\mathscr{B} = \Big\{U_p^*/p \in F_\omega, U \cup \{0\} \text{ is open in } \mathbb{N} \cup \{0\}\Big\}$ is a basis of F_ω.

Theorem 5.4.1. *F_ω has the following properties:*

(i) Countable

(ii) Sequential fan X_4 is a subspace of F_ω (i.e. $\mathbb{N} \cup \{0\}$ with sequential fan topology is a subspace of F_ω)

(iii) Homogeneous

(iv) Not first countable

(v) Fréchet-Urysohn

(vi) Regular

Proof:

(i), (ii) and (iii) follow from the construction of F_ω.

(iv) If F_ω is first countable, then its subspace $\mathbb{N} \cup \{0\}$ with sequential fan topology is also first countable. A contradiction to the fact that $\mathbb{N} \cup \{0\}$ with sequential fan topology is not first countable.

(v) Let A be any subset of F_ω. Without any loss of generality, let us assume that $0 \in \bar{A}$. Let $I(m) = \{y \in F_\omega / y < m\}$ and let $B = \{m \in \mathbb{N} / I(m) \cap A \neq \phi\} \subseteq \mathbb{N}$.

Claim: B is infinite.

Suppose B is finite. Then $U = (\mathbb{N} \backslash B) \cup \{0\}$ is open in $\mathbb{N} \cup \{0\}$. And

$$U_0^* = \{y \in F_\omega / y < i \text{ for some } i \in U\} \cup U \cup \{0\}$$

is open in F_ω containing 0. Clearly $U_0^* \cap A = \phi$. Which is a contradiction to the fact that $0 \in \bar{A}$. This proves our claim.

Claim: $0 \in \bar{B}$ ($\bar{B}$ is a closure of B in $\mathbb{N} \cup \{0\}$).

Suppose $0 \notin \bar{B}$. Then there exists an open set V containing 0 in $\mathbb{N} \cup \{0\}$ such that $V \cap B = \phi$.

Therefore for each $m \in V, I(m) \cap A = \phi$.

Now,

$$V_0^* = \{y \in F_\omega / y < i \text{ for some } i \in V\} \cup V \cup \{0\}$$

is an open set containing 0 in F_ω which cannot intersects A.

Thus we have a contradiction to the fact that $0 \in \bar{A}$. This proves our claim.

Since $\mathbb{N} \cup \{0\}$ has sequential fan topology, there exists a sequence (z_n) in B such that (z_n) converges to 0. As $z_n \in B$, $I(z_n) \cap A \neq \phi$. Form a sequence (x_n) in F_ω, where $x_n \in I(z_n) \cap A$. Then (x_n) converges to 0 in F_ω because (z_n) converges to 0 in $\mathbb{N} \cup \{0\}$. This shows that F_ω is a Fréchet-Urysohn space.

(vi) Since U_p^* is clopen for each $p \in F_\omega$, F_ω is regular. ∎

Remark 5.4.1. In the construction of F_ω we have used a sequential fan topology on the set $\mathbb{N} \cup \{0\}$ and then at each point of F_ω we define the space using the topology of $\mathbb{N} \cup \{0\}$. One could have started with any countable space with unique limit point and transferring the topology to $\mathbb{N} \cup \{0\}$ and then using this space $\mathbb{N} \cup \{0\}$ one can construct a space like F_ω. It is clear that some of the properties of F_ω constructed with this change will depend on the space $\mathbb{N} \cup \{0\}$.

5.5 Answer to the question

Peter J Nyikos proved the following result in his paper [10]. He has considered a slightly more general situation, admitting also non-Hausdorff spaces. Since we are interested in Hausdorff spaces only, the proof is little more transparent, so we present it here.

Theorem 5.5.1. *Every Fréchet-Urysohn topological group satisfies diagonal sequence condition.*

Proof:

Let G be a Fréchet-Urysohn topological group. Assume G is Hausdorff. Let e denote the identity of a topological group G, it is enough to concentrate on e. So let for each positive integer m, $(x_n^m)_{n=1}^{\infty}$ be a sequence converging to e. Now for each positive integer k, the sequence $\left(x_k^1\, x_n^k\right)_{n=1}^{\infty}$ converges to x_k^1, and so the union A of the ranges of all these sequences has e in its closure. By hypothesis there is a sequence S in A converging to e. Since group G is Hausdorff, S can only meet each sequence in finitely many terms, hence must have a subsequence of the form $\left(x_{k(i)}^1\, x_{n(i)}^{k(i)}\right)_{i=1}^{\infty}$ converging to e, with $k(i) \neq k(j)$ when $i \neq j$. But $\left((x_{k(i)}^1)^{-1}\right)_{i=1}^{\infty}$ converges to e, and thus $\left((x_{k(i)}^1)^{-1}\, x_{k(i)}^1\, x_{n(i)}^{k(i)}\right)_{i=1}^{\infty} = \left(x_{n(i)}^{k(i)}\right)_{i=1}^{\infty}$ converges to e as desired. ∎

Now we are able to provide an alternative answer to the question asked by S. P. Franklin and M. Rajagopalan [4] .

Theorem 5.5.2. *F_ω does not support a compatible group structure.*

Proof:

Suppose F_ω supports a group structure. Since F_ω is Fréchet-Urysohn F_ω must satisfy the diagonal sequence condition. Since Sequential fan X_4 is a subspace of F_ω, it must satisfy the diagonal sequence condition. Which is a contradiction to the fact that sequential fan does not satisfy the diagonal sequence condition, as has already been observed. Hence the theorem. ∎

Chapter-6

An application of Ψ-spaces

6.1 Introduction

Ψ-spaces were first introduced by J. R. Isbell and were also discussed by L. Gillman and M. Jerison in their famous book Rings of Continuous Functions [7]. S. P. Franklin used Ψ-space to give an example of a compact, Hausdorff, sequential, non Fréchet-Urysohn space [6]. Here we make use of the Ψ-space to construct a space which is not homeomorphic to the sequential fan (in fact not homeomorphic to any one of the examples $X_1, X_2, X_3, X_4, X_5, X_6$), though it is a countable, Fréchet-Urysohn space with unique limit point.

6.2 Definitions

Definition 6.2.1. Two sets A and B are said to be *almost disjoint* if their intersection is finite.

Definition 6.2.2. A *pairwise almost disjoint family* (abbreviated as p.a.d. family) on a set X is a collection $\mathscr{F}$ of infinite subsets of X such that $A \cap B$ is finite for any two distinct members A, B in $\mathscr{F}$. For example, partition of $\mathbb{N}$ is a p.a.d. family on $\mathbb{N}$.

A maximal p.a.d. family (abbreviated as MAD family) on a set X is a p.a.d. family on X properly contained in no p.a.d. family on

X. For example, $\mathscr{F} = \{O, E\}$, where O is the set of odd numbers and E is the set of even numbers, is a maximal p.a.d. family on $\mathbb{N}$.

6.3 Construction of an example

Ψ-space:

Let $\mathscr{F}$ be a p.a.d. family of infinite subsets of $\mathbb{N}$. Let $\{\omega_F / F \in \mathscr{F}\}$ be a new set of distinct points and define $\Psi = \mathbb{N} \cup \{\omega_F / F \in \mathscr{F}\}$ with the following topology: each subset of $\mathbb{N}$ is open; while $U \subseteq \Psi$ containing ω_F is open if and only if U contains all but finitely many points of F.

Easy application of Zorn's lemma shows that every such $\mathscr{F}$ is contained in a maximal p.a.d. family of infinite subsets of $\mathbb{N}$. The spaces Ψ for such maximal p.a.d. families were first introduced by J. R. Isbell and considered in [7]. They appeared in [6] to provide an example of a compact, Hausdorff, sequential, non Fréchet-Urysohn space.

Frank Siwiec gave six examples which are countable spaces with exactly one nonisolated point [16]. Three of them are metrizable which appear here in Chapter-1 as X_1, X_2, X_3 and in Chapter-4 as Y_1, Y_2, Y_3. Fourth one is the sequential fan and the other two contains a copy of sequential fan. These are nonhomeomorphic. Here we construct an example of a space having exactly one nonisolated point which contains no copy of sequential fan and hence it cannot be homeomorphic to sequential fan or two spaces like sequential fan. This space is obtained as a $\widetilde{\Psi}$-space out of a definite Ψ-space.

For our convenience, we shall take $\mathbb{Q}$ the set of rational numbers and a p.a.d. family on $\mathbb{Q}$.

Example:

Let $\Psi = \mathbb{Q} \cup \{\omega_F / F \in \mathscr{F}\}$, and let $\widetilde{\Psi}$ be the quotient space of Ψ by identifying all ω_F to one point, say ω. Our aim is to find a particular p.a.d. family $\mathscr{F}$ on $\mathbb{Q}$ such that $\widetilde{\Psi}$ and sequential fan are not homeomorphic. We construct $\mathscr{F}$ in the following way.

Consider the unit interval $I = [0, 1]$ of $\mathbb{R}$ with usual topology. For every $r \in [0, 1]$, we can choose rational sequences (x_n^r) and (y_n^r) such that $A_r = \{x_n^r / n = 1, 2, 3, ...\}$ and $B_r = \{y_n^r / n = 1, 2, 3, ...\}$ are disjoint and both sequences converge to r.

Set $\mathscr{G} = \{A_r / r \in [0, 1]\} \cup \{B_r / r \in [0, 1]\}$, which is an uncountable p.a.d. family on $\mathbb{Q}$. Then there exists a maximal p.a.d. family $\mathscr{M}$ on $\mathbb{Q}$ such that $\mathscr{G} \subseteq \mathscr{M}$. Let $\mathscr{F} = \mathscr{M} \backslash \{B_r / r \in [0, 1]\}$. Now we show that under this family $\mathscr{F}$ the quotient space $\widetilde{\Psi}$ is not homeomorphic to sequential fan. Suppose $\widetilde{\Psi}$ contains a copy of the sequential fan. Then there exists F_1 (say) in $\mathscr{F}$ such that we have a sequence (x_n^1) of points of F_1 which converges to some r_1 in $[0,1]$. Similarly, there exists F_2 (say) in $\mathscr{F}$ such that we have a sequence (x_n^2) of points of F_2 which converges to some r_2 in $[0,1]$. Continuing in this way, we get, because of compactness of $[0,1]$, a sequence (r_n) in $[0,1]$ which will converge to some r_0 in $[0,1]$. Now we can choose a point z_k in (x_n^k) for each k such that we have a diagonal sequence (z_k) in $\widetilde{\Psi}$ with $G_{r_0} \cap A_{r_0} = \phi$, $G_{r_0} \cap B_{r_0} = \phi$, where $G_{r_0} = \{z_k/k = 1,2,3,...\}$ and sequence (z_k) converges to r_0. Also $G_{r_0} \cap A_r, G_{r_0} \cap B_r$ are finite for all $r \in [0,1]$ and therefore $G_{r_0} \in \mathscr{F}$. Thus we get a diagonal sequence converging to ω, a contradiction to the fact that no diagonal sequence converges in a sequential fan. Thus $\widetilde{\Psi}$ does not contain a copy of sequential fan, hence $\widetilde{\Psi}$ and sequential fan are not homeomorphic.

Chapter-7

Concluding Remarks

Can there be a countable, Fréchet-Urysohn topological group other than $\mathbb{Q}$? As we have noted earlier the answer is yes under certain set-theoretic assumptions. In the light of this there are number of directions one can try to contribute something. These directions are as follows:

(1) Can we get a countable Fréchet-Urysohn topological group other than $\mathbb{Q}$ without any set-theoretic assumptions ?

(2) Try to evolve processes by which one embeds a countable space with unique nonisolated point into a Fréchet-Urysohn homogeneous space. We have already one such process due to S. P. Franklin and M. Rajagopalan [4] about which we have dealt with at length in chapter-5.

(3) Is it possible to think of a condition general than First countability but stronger than Fréchet-Urysohn property which together with countability and homogeneity forces the space to be $\mathbb{Q}$? (Of course we assume all spaces to be regular.) Affirmative to answer this will really be a very good result that will generalize

Sierpiński's result.

(4) Can we construct countable homogeneous space with each sequential order $< \omega_1$? If yes, can they be topological groups ?

(5) Does the plus topology on $\mathbb{Q} \times \mathbb{Q}$ support a compatible group structure ?

(6) If we use the space $\tilde{\Psi}$ (Chapter-6) as a countable space with unique nonisolated point instead of a sequential fan in the construction of F_ω, do we get a countable homogeneous space which can be made into a topological group ? In other words, does this countable homogeneous space support a compatible group structure ?

It is our humble submission that the results which we have obtained in this thesis are the byproduct of our constant focus on above mentioned problems throughout the journey of the Ph.D. programm. Apart from this it is our hope that some of the examples that we have constructed can be used in the discussion related to product of Fréchet-Urysohn spaces, the topic which we have not touched at all in this thesis.

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