Robustness of Estimators and Tests with Directional Data

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DECLARATION BY THE CANDIDATE

The thesis embodies the research work carried out by me at the Department of Statistics, Saurashtra University, Rajkot, Gujarat, India and submitted to Faculty of Science, Saurashtra University, Rajkot and the results of this work have not been submitted to any other university for the award of *Doctor of Philosophy*.

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This is to certify that contents of the thesis entitled "**Robustness of Estimators and Tests with Directional Data**" submitted for the Ph.D degree in Statistics of Saurashtra University, Rajkot is the original research work of Mr. K.C. Mahesh carried out under the joint guidance and supervision.

We further certify that the work has not been submitted either partially or fully to any other University/Institute for the award of any degree.

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CONTENTS

PAPERS PUBLISHED/UNDER PUBLICATION

1. Laha, A.K & Mahesh, K.C. (2011) *SB-robustness of Directional Mean for Circular Distributions*, Journal of Statistical Planning and Inference, 141, 1269-1276.

2. Laha, A.K & Mahesh, K.C. (2010) *SB-robust Estimator for the Concentration Parameter of Circular Normal Distribution*, Statistical Papers (to appear) www.springerlink.com/index/Y086661212502670.pdf .

3. Laha, A.K, Mahesh, K.C & Ghosh, D.K (2011), SB-*robust Estimators of the Parameters of the Wrapped Normal Distribution* (submitted).

4. Laha, A.K, Mahesh, K.C (2011), *Robustness of Tests for Directional Mean* (submitted).

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5. *Directional Data – An Introduction* presented at National Level Seminar on Recent Trends in Bio-Medical Statistics, Department of Statistics, Gujarat University, Ahmedabad, 2007.

6. *SB-robustness of Directional mean for circular distributions* presented at National Conference on Statistical Inference and Inferential Aspects of Industrial Statistics, Department of Statistics, Pune University, 2008.

7. *SB-robust Estimation of Parameters of Circular Normal Distribution* presented at 1st IIMA International Conference on Advanced Data *Analysis*, Business Analytics and Intelligence, IIM-A, 2009.

8. *SB-robust Estimation of the Concentration Parameter of Circular Normal Distribution* presented at the Seventh International Triennial Calcutta Symposium on Probability and Statistics Department of Statistics, University of Calcutta, 2009.

9. *On the robustness of tests for mean direction of circular normal distribution: A breakdown approach* presented at 2nd IIMA International Conference on Advanced Data *Analysis*, Business Analytics and Intelligence, IIM-A, 2011.

Chapter 1

Introduction

1.1 Circular Data Analysis

In many fields of study the measurements are directions. These measurements can be angles as in the case of measurement of wind direction or can be observations on a sphere as for example, on the surface of the earth with each point being identified with a latitude-longitude pair. These kinds of data are often termed as directional data. Directional data in two or three dimensions arise quite frequently in many natural, physical and social sciences like Biology, Medicine, Ecology, Geology, Meteorology, Image Analysis, Political Science, Finance, Demography etc. A biologist may be interested in measuring the direction of flight of a bird (see Schmidt-Koenig, 1965 and Batschelet, 1981) or the orientation of an animal. In medical applications circadian rhythms are often analysed as they control characteristics like sleep-wake cycles, hormonal pulsatility, body temperature, mental alertness, reproductive cycles etc. Because of the periodic nature of biological rhythm data it can be put into the frame work of circular data analysis (see Proschan and Follmann, 1997). Medical professionals have shown keen interests in topics such as chronobiology, chronotherapy, and the study of the biological clock (see Morgan, 1990 and Hrushesky, 1994). Jammalamadaka et al. (1986) discuss an interesting medical application where the angle of knee flexion was measured to assess the recovery of orthopaedic patients. Recently, Gavin et al (2003) discuss that circular data can be used to analyse cervical orthoses in flexion and extension. In geology significant interest is shown in the study of paleocurrents to infer the direction of flow of rivers in the past (see Sengupta and Rao, 1967). Ginsberg (1986) and Wallin (1986) discuss the application of angular data in ecological and behavioural studies of animal orientation and habitat selection and also in ecological field studies (see Cain,1989). Apart from wind direction, other types of circular data arising in meteorology include the time of day at which thunderstorms occurs and the times of year at which heavy rains occur (see Mardia and Jupp, 2000). Nikolaidis and Pitas

(1994, 1995) discuss how angular data can be used in vector direction estimation in the area of colour image processing and image sequence processing and the detection of edges on the hue colour component which can be useful in cases like colour object recognition, colour image segmentation etc. Gill and Hangartner (2010) discuss the application of circular data in political science in which they develop a circular regression model for domestic terrorism in which political nature of entities like attacking groups, target groups etc. is an important factor. Also they studied in the context of German Bundestag elections how parties make direction decisions, in a two dimensional ideological space, relative to each other in response to social and political pressure from the electorate. Recently, SenGupta (2011) discuss how circular data can be used to analyse high volatile financial data. In demography, circular data arises in the studies like geographic marital patterns (Coleman and Haskey, 1986), occupational relocation in the same city (Clark and Burt, 1980), and settlement trends (Upton, 1986b). Spherical data arises in the study of paleomagnetism, study of astronomical objects, image analysis, signal processing etc. More examples of applications of circular and spherical data analysis can be found in Fisher (1993), Fisher, et.al (1987), Jammalamadaka & SenGupta (2001) and Mardia & Jupp (2000).

Two dimensional directions can be represented as angles measured with respect to some suitably chosen "zero direction" that is, the starting point and a "sense of rotation" that is, whether clockwise or anti-clockwise. Since a direction has no magnitude, these can be conveniently represented as points on the circumference of a unit circle centered at origin or as unit vectors connecting the origin to these points. Because of this circular representation, observations on such two dimensional directions are called circular data. They are commonly summarized as locations on a unit circle or as angles over a 360° or 2π radians range, with the end-points of each range corresponding to a specified location on the circle. The numerical representation of a two dimensional direction as an angle or a unit vector is not unique since the angular value depends on the choice of zero direction and the sense of rotation. For example, 60 degree by a mathematician who takes East as zero direction and anti-clockwise as positive direction comes out to be 30 degree to a geologist who takes North as zero direction and clockwise as the positive direction.

Therefore it is important to make sure that our conclusions are a function of given observations and do not depend on the arbitrary choice of origin and sense of rotation. Directional data analysis is substantially different from the standard "linear" statistical analysis of univariate or multivariate data.

For circular data the arithmetic mean as well as standard deviation is not useful as they suffer from their strong dependence on the choice of zero direction and the sense of rotation. This emphasizes the fact that in circular data one has to look at measures which are invariant under the choices of origin and sense of rotation. An appropriate and meaningful measure of the mean direction for a set of directions is obtained by treating the data as unit vectors and using the direction of their resultant vector. In circular data analysis the basic statistics of interests for inference purposes are the sums of *sines* and *cosines*, and the resultant length given by $C = \sum_{i=1}^{n} \cos\theta_i$, $S = \sum_{i=1}^{n} \sin\theta_i$, and $R = \sqrt{C^2 + S^2}$, where θ_i's are independently and identically distributed random variables from some model. The mean direction of a set of angular observations, say, $\theta_1, \theta_2, \ldots, \theta_n$ is given by $\overline{\theta}_0 = \arctan^*\left(\frac{S}{C}\right)$, where S and C are defined earlier and $\arctan^*$ is the quadrant - specific inverse of the tangent function which is defined as

$$\arctan^*\left(\frac{S}{C}\right) = \begin{cases} \arctan\left(\frac{S}{C}\right) & \text{if} \quad C > 0 \quad S \geq 0 \\ \pi/2 & \text{if} \quad C = 0 \quad S > 0 \\ \arctan\left(\frac{S}{C}\right) + \pi & \text{if} \quad C < 0 \\ \arctan\left(\frac{S}{C}\right) + 2\pi & \text{if} \quad C \geq 0 \quad S < 0 \end{cases} \qquad \ldots (1.1)$$

(see Jammalamadaka & SenGupta, 2001, p.13). When both C=0 and S=0, a circular mean cannot be defined which indicates that the data is spread evenly or uniformly over the circle, with no concentration towards any direction. It should also be noted that circular mean direction is rotationally invariant. The length of the resultant vector $R = \sqrt{C^2 + S^2}$ is a useful measure for unimodal data of how concentrated the data is towards the circular mean direction. A right analogue of the usual sample variance can be obtained by using appropriate circular distance. A popular measure for sample circular dispersion is $D_v = n - R$ (see Jammalamadaka & SenGupta, 2001, p.14). It measures the dispersion of the sample relative to the center through the sample mean direction. If R is close to 0 then dispersion is large whereas the values of R close to n imply that the observations have small dispersion or more concentration towards the center.

A circular random variable is a map $\Theta : \Omega \to T$ from a suitable probability space Ω to the circle T. The basic quantity characterizing the distribution of X is the probability for X to take values in a certain subset of the circle i.e. in an arc $[\alpha_1, \alpha_2]$ with $\alpha_1, \alpha_2 \in T$. A circular distribution is a probability distribution whose total probability is concentrated on the circumference of a unit circle. The probability density function $f(\theta)$ of a circular random variable θ has the following basic properties:

1) $f(\theta) \geq 0$

2) $\int_0^{2\pi} f(\theta) d\theta = 1$ and

3) $f(\theta) = f(\theta + 2\pi k)$ for any integer k (i.e f is periodic).

The popular circular distributions include Circular Normal (CN), Wrapped Normal (WN), Wrapped Cauchy (WC), Circular Uniform (CU), Cardiod etc. Circular normal (or von-Mises) distribution is most popular circular distribution for applied work. This distribution was introduced as a statistical model by von-Mises (1918) and was discussed earlier by Langevin (1905), in the context of physics. The CN distribution has been extensively studied and inference techniques for this distribution are well developed. This is the model of choice for circular data in most applied problems. A circular random variable Θ is said to have a von-Mises or CN distribution with mean direction μ and concentration κ if it has the probability density function (p.d.f):

$$f(\theta; \mu, \kappa) = \frac{1}{2\pi I_0(\kappa)} e^{\kappa \cos(\theta - \mu)}, 0 \leq \theta < 2\pi \text{ where } 0 \leq \mu < 2\pi \text{ and } \kappa > 0. \quad \text{... (1.2)}$$

where $I_0(\kappa)$ is the modified Bessel function of the first kind and order zero and is given by $I_0(\kappa) = \frac{1}{2\pi}\int_0^{2\pi} e^{\kappa\cos\theta} d\theta = \sum_{r=0}^{\infty}\left(\frac{\kappa}{2}\right)^{2r}\left(\frac{1}{r!}\right)^2$. This distribution is symmetric about μ and is unimodal. We will denote this distribution as $CN(\mu, \kappa)$. If $\kappa = 0$ then $CN(\mu, \kappa)$ can be approximated by the circular uniform distribution which has no preferred direction and when $\kappa \geq 2$, $CN(\mu, \kappa)$ can be approximated by the $WN(\mu, \rho)$, which is a symmetric unimodal distribution obtained by wrapping a $N(\mu, \sigma^2)$ distribution around the circle. Similarly, $CN(\mu, \kappa)$ can closely be approximated by the $WC(\mu, A(\kappa))$ (see Mardia and Jupp, 2000, p.38). Another interesting property of $CN(\mu, \kappa)$ distribution is that, for sufficiently large κ, the CN distribution can be approximated by a linear normal distribution. The trigonometric moments of the circular normal distribution can be obtained by the relation $\varphi_p = A_p(\kappa)e^{ip\mu}$, where p is an integer and $A_p(\kappa) = I_p(\kappa)I_0^{-1}(\kappa)$. The length ρ of the first trigonometric moment is given by $A(\kappa) = I_1(\kappa)I_0^{-1}(\kappa)$. By virtue of symmetry of the CN density, the central trigonometric moments are $\alpha_p^* = A_p(\kappa)\cos p\mu$. The function $A(\kappa)$ has many interesting properties like:

1) $0 \leq A(\kappa) \leq 1$

2) $A(\kappa) \to 0$ as $\kappa \to 0$ and $A(\kappa) \to 1$ as $\kappa \to \infty$ and

3) $A'(\kappa) \equiv \frac{\partial A(\kappa)}{\partial \kappa} = \left(1 - \frac{A(\kappa)}{\kappa} - A^2(\kappa)\right) \geq 0$,

i.e. $A(\kappa)$ is a strictly increasing function of κ so that $\hat{\kappa}$, may be obtained as a unique solution of $A(\kappa) = I_1(\kappa)I_0^{-1}(\kappa)$. The maximum likelihood estimate of the parameters μ and κ are given by $\hat{\mu} = \arctan^*\left(\frac{S}{C}\right) = \bar{\theta}_0$ and $A(\kappa) = I_1(\kappa)I_0^{-1}(\kappa)$. The maximum likelihood estimate of μ remains the same whether or not κ is known. On the other hand, the maximum likelihood estimate of κ is different when μ is known. In this case, maximum likelihood estimate of κ is given by

$$\hat{\kappa} = A^{-1}\left(\frac{V}{n}\right), V > 0 \text{ where } V = \sum_{i=1}^{n} \cos(\theta_i - \mu)$$

(see Jammalamadaka & SenGupta, 2001, p.86-88). Clearly, $\hat{\kappa} = 0$ for $V \leq 0$.

One of the well-known probability distributions on the circle is the wrapped normal distribution obtained by wrapping the normal distribution with parameters μ and σ^2 on to the unit circle having the p.d.f

$$f(\theta; \mu, \rho) = \frac{1}{2\pi}\left\{1 + 2\sum_{p=1}^{\infty} (\rho)^{p^2} \cos p(\theta - \mu)\right\}, 0 \leq \theta < 2\pi, 0 \leq \mu < 2\pi, 0 < \rho < 1 \qquad \text{... (1.3)}$$

This distribution is symmetric about μ and is unimodal with mode at μ. We will denote this distribution as $WN(\mu, \rho)$. The parameter μ is called the mean direction and the parameter ρ is called the concentration parameter. The parameters of the wrapped normal distribution $WN(\mu,\rho)$ arise naturally when wrapping $N(\mu,\sigma^2)$ onto the circle where $\rho = \exp(-\sigma^2/2)$ (Mardia and Jupp, 2000, p.50). This distribution is a member of the wrapped stable family of distributions (see Mardia and Jupp, 2000, p. 52). Excellent surveys on the sampling distributions – samples being drawn from a von-Mises (or circular normal) distribution- of circular statistics is given in Mardia and Jupp (2000) and Jammalamadaka & SenGupta (2001).

1.2 Statistical Functionals

A statistical functional $T(F)$ is a mapping defined on a space of distribution functions with image space $\Re$ (or a set of categories or higher dimensional Euclidean space) and domain includes all empirical distribution functions. Many quantities of interests to statisticians can be expressed as statistical functional $T(F)$ where F is the distribution of the data. The natural estimate of $T(F)$ is often $T(F_n)$ where F_n is the sample distribution function. Many commonly used statistics give rise to statistical functionals in the following way: Suppose $T_n = T_n(X_1, X_2, ..., X_n)$ is a statistic which can be expressed as a functional T of the empirical distribution function F_n i.e. $T_n = T(F_n)$ where T does not depend upon the sample size n. The following examples of some commonly used statistical functionals arise in the above manner. The sample mean functional $T(F_n) = \int_{\Re} x dF_n$, the sample variance functional $V(F_n) = \int_{\Re} (x - T(F_n))^2 dF_n$, the median functional $Q_{0.5}(F_n) = \inf\left\{q : \int_{-\infty}^{q} dF_n \geq 0.5\right\}$. The corresponding population versions can be obtained by replacing F_n with F to get $T(F) = \int_{\Re} x dF$, $V(F) = \int_{\Re} (x - T(F))^2 dF$, and $Q_{0.5}(F) = \inf\left\{q : \int_{-\infty}^{q} dF \geq 0.5\right\}$ respectively. The idea of statistical functional can be extended to a directional set up as follows. Given a random sample of size n from $CN(\mu, \kappa)$ and $WN(\mu, \rho)$ distributions, the parameter μ is estimated as $\hat{\mu} = \arctan^*(S/C)$ where arctan* is the quadrant - specific inverse of the tangent function defined by (1.1). The corresponding functional form is $\arctan^*\left(\frac{E_F(\sin\theta)}{E_F(\cos\theta)}\right)$ where F is the underlying distribution. The parameter ρ is estimated as $\hat{\rho} = \sqrt{C^2 + S^2}$ and the functional form of the estimator is $\sqrt{E_F^2(\cos\theta) + E_F^2(\sin\theta)}$. The parameter κ is estimated as $\hat{\kappa} = A^{-1}(\hat{\rho})$ and the corresponding functional form of the estimator is $A^{-1}\left(\sqrt{E_F^2(\cos\theta) + E_F^2(\sin\theta)}\right)$. Some excellent accounts on statistical functionals can be found in von Mises (1947), Serfling (2002), and Wasserman (2006). In this thesis we have used the functional form of the estimators to study their robustness.

1.3 Robustness with Circular Data

The problem of robustness probably goes back to the prehistory of statistics; it has only been in recent decades that attempts have been made to formalize the problem beyond limited ad hoc measures towards robustness theory. It was Tukey (1960) who demonstrated the drastic non robustness of the mean and also investigated some useful alternatives. His work made robust estimation a general research area. The first attempts towards a comprehensive theory of robustness are by Huber (1964, 1965, and 1968) and Hampel (1968). Huber’s (1964) paper formed the basis for a theory of robust estimation. According to Huber (1981) it is desired that any statistical procedure should be robust in the sense that small deviations from the model assumptions should affect the performance only slightly. Another approach to robust statistics is the infinitesimal approach. In showing how an estimator responds to the introduction of a new observation, Hampel (1974) introduced the concept of influence curve (IC) or influence function (IF). It allows us to assess the relative influence of individual observations towards the value of an estimate or test statistics. It also allows us an immediate and simple, heuristic assessment of the asymptotic properties of an estimate. All statistical methods rely on a number of assumptions either explicit or implicit. In reality, it often happens that one or more of these assumptions fails to hold. One common phenomenon seen while analysing many datasets is the presence of one or a few observations in the dataset which are very different from the rest. These observations are termed as outliers and it is expected that a good statistical procedure would not be adversely affected by these small number of ‘deviant’ observations. Such statistical procedures are termed as robust procedures.

Robust inference includes both robust estimation and robust testing. Robustness of estimates has been extensively studied in the literature. The main approaches towards robust estimation include influence function approach due to Hampel and Huber’s minimax approach. Some well known class of robust estimates for location and scale are M-estimates, R-estimates, L-estimates etc. The second aspect of robust inference is robust testing. The purpose of robust testing is twofold: i) robustness of validity i.e. the level of a test should be stable under small, arbitrary departures from the null hypothesis and ii) robustness of efficiency i.e the test still should have a good power under small arbitrary departures from specified alternatives. Unfortunately many classical tests do not satisfy these criteria. For example, the F-test for comparing two variances is not robust. The classical t-test and F-test for linear models are relatively robust with respect to level, but they lack robustness of efficiency (see Huber and Ronchetti, 2009, pp.297-298). Huber (1965) defined the censored probability ratio test and showed that this test is robust in a well defined minimax sense. But this approach hold for fixed sample size and a given neighbourhood, it is difficult to generalize for more complex models. A feasible alternative is the infinitesimal approach in which the influence of contamination on the level and on the power is examined asymptotically by means of quantities like level influence function (LIF) and power influence function (PIF). Excellent surveys on robust inference can be found in Huber (1981), Hampel et al. (1986), Staudte and Sheather (1990), Marona, Martin and Yohai (2006).

The statistical procedures for analyzing circular data are significantly different from those for linear data (see Mardia and Jupp, 2000 and Fisher, 1993). The outlier problem in the directional data set up is somewhat different from that in the linear case. In linear data, sample mean and median are the estimates of population mean and median. Since sample mean is more sensitive to outliers it is non robust. In directional data analysis one might expect fewer outlier problems to arise, because on the circle there is only restricted room for an observation to out lie. According to Jammalamadaka and SenGupta (2001), how far an observation is from the mean in directional set up, should be judged by using appropriate "circular distance". Due to bounded support of angular data, outliers can be detected only when the observations are sufficiently concentrated around a particular point. The angular deviation given by $\mathrm{arc}(\theta_i, \alpha) = \pi - |\pi - |\theta_i - \alpha||$ between a data point and the population sample mean or median direction can be used to identify whether the observation is outlying or not. The robustness properties of statistical procedures for circular data have not been studied as thoroughly as those for linear data. Practically, in the applications of the circular normal distribution the parameters μ and κ need to be estimated from the data and hence, it is important to study the robustness of these estimators to outliers.

Several tests for mean direction and concentration parameter have been developed in the literature for both κ known and unknown cases. When κ is known one has the choice of using the likelihood ratio test based on the test statistic $w = 2n\kappa(\overline{R} - \overline{C})$, or an unbiased conditional likelihood test based on w, or a score test based on the test statistic $t = n\kappa A^{-1}(\kappa)\bar{S}^2$ where $\bar{C} = n^{-1}\sum_{i=1}^{n}\cos\theta_i$, $\bar{S} = n^{-1}\sum_{i=1}^{n}\sin\theta_i$ and $\bar{R} = \sqrt{\bar{C}^2 + \bar{S}^2}$. In the latter case, both the tests based on w and t can be modified by replacing κ by its estimated value $\hat{\kappa}$ (see Mardia and Jupp, 2000, pp.119-123). The likelihood ratio test for concentration parameter κ of circular normal distribution assuming unknown mean direction has been discussed by Mardia and Jupp, (2000) and a test based on a complete sufficient statistic has been discussed by Jammalamadaka and SenGupta (2001). But the robustness aspects of these tests have not been explored in the literature.

The Ph.D thesis entitled *"Robustness of Estimators and Tests with Directional Data"* contains nine chapters.

In chapter 2 we give the literature review. Here we briefly discuss various methods and techniques that are available in the literature in the areas of circular data and robust estimation and robust testing.

In chapter 3 we discuss the SB-robustness of directional mean for the circular normal distribution. A paper entitled "SB-robustness of Directional Mean for Circular Distributions" based on this chapter has been published in *Journal of Statistical Planning and Inference*, 141, 1269-1276, March 2011 co-authored with Laha, A.K.

In chapter 4 we discuss the SB-robustness of concentration parameter for the circular normal distribution. A paper entitled "SB-robust Estimator for the Concentration Parameter of Circular Normal Distribution" based on this chapter has been published in *Statistical Papers* July 2010 co-authored with Laha, A.K. This paper is available online. The link is given in the reference.

In chapter 5 we have developed robust tests for the directional mean of circular normal distribution using breakdown function approach. A paper entitled "Robustness of Tests for Directional Mean" based on this chapter is communicated for publication co-authored with Laha, A.K.

In chapter 6 we introduced robust tests for the concentration parameter of circular normal distribution.

In chapter 7 we discuss SB-robustness of parameters of the wrapped normal distribution. A paper entitled "SB-robust Estimators of the Parameters of the Wrapped Normal Distribution" based on this chapter is communicated for publication co-authored with Laha, A.K. & Ghosh, D.K.

In chapter 8 we conclude and discuss some scope of further work on this topic.

In chapter 9 we give the references that are used to complete the thesis.

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Chapter 2

Literature Review

2.1 Directional Data

The standard texts on directional data are Mardia (1972) and Fisher (1993). Batschelet (1981) gives a less mathematical account of applications of circular data to the analysis of biological data. Fisher, Lewis and Embleton (1987) give an account of methods for the analysis of spherical data. Mardia & Jupp (2000) discuss both two and three dimensional data. Jammalamadaka and SenGupta (2001) give a comprehensive account of circular data analysis.

2.2 Outliers in Directional Data

In practice, the observed directions may contain one or more data points which appear to be peculiar, not representative or inconsistent relative to the main part of data. This is commonly referred to as the outlier (a.k.a. slippage, discordancy or spuriousity) problem. In directional data analysis one might expect fewer outlier problems to arise, because on the circle there is only restricted room for an observation to be an outlier. How far an observation is from the mean in directional set up should be judged by using appropriate “circular distance”. Thus, unlike in the linear case, outliers here need not be too large or too small, but could be in the “central” part of the data. When the data follows a rather broad distribution on the circle, a small amount of contamination would not be noticed and would have little effect on estimates of location or spread (Lenth 1981). An excellent survey on outliers was provided by Barnett & Lewis (1994) (see Chapter 11).

Collett (1980) discusses a method of identifying surprising observations in a sample of directional data and describe possible tests of discordancy. According to him an outlier is an observation say, θ_k such that $\max\{\xi_i\} = \xi_k$, where $\xi_i = \pi - |\pi - \theta_i^*|$ is the angular deviation of θ_i from the sample mean direction $\overline{\theta}$ and $\theta_i^* = \theta_i - \overline{\theta} \pmod{2\pi}, i = 1,2,\ldots,n$. He proposed four statistics namely L, C, D, and M for assessing the possible discordancy of a single angular outlier of which L and C are defined with particular reference to a von-Mises model and the other two could be used for other models. SenGupta and Laha (2001) treated outliers in circular data due to "slip" in recording and considered slippage problem as a problem of outlier detection.

2.3 Robustness of Estimators with Directional Data

The word "robustness" is used in statistics to convey the notion that the estimator is insensitive to different things like a) small deviations from the distributional assumptions or b) gross errors. A minor error in the mathematical model should cause only a small error in the final conclusions. But unfortunately, this does not always hold. Most of the common statistical methods are excessively sensitive to minor deviations from the theoretical assumptions.

Huber (1964) introduced a flexible class of estimators called M-estimators as a generalization of MLEs. He introduced the "gross error model" $F(x-\theta) = (1-\varepsilon)G(x-\theta) + \varepsilon H(x-\theta)$ assuming that a known fraction $\varepsilon\ (0 \le \varepsilon < 1)$ of the data may consist of gross errors with an arbitrary unknown distribution $H(x-\theta)$ instead of having a strict parametric model $G(x-\theta)$ for known G. His idea is to optimize the worst that can happen over the neighbourhood of the model, as measured by the asymptotic variance of the estimator.

Hampel (1968, 1974) introduced the concept of influence curve or influence function. It allows us to assess the relative influence of individual observations towards the value of an estimate or test statistics. The influence curve (IC) or influence function (IF) of the functional T at the underlying distribution F is defined as

$$IF(x;T,F) = \lim_{\varepsilon \to 0} \frac{[T\{(1-\varepsilon)F + \varepsilon\delta_x\} - T(F)]}{\varepsilon} \quad \ldots (2.1)$$

where δ_x denote the degenerate distribution assigning probability one to the point x.

The gross error sensitivity (g.e.s.) of the estimator T at F is defined as (Hampel, 1974)

$$\gamma(T,F) = \sup_x [IF(x;T,F)] \quad \ldots (2.2)$$

If $\gamma(T,F)$ is finite then the estimator is said to be bias-robust (or B-robust) at F (Rousseeuw, 1981). It gives the maximum asymptotic bias under the gross error model. If the influence function is bounded or the g.e.s has finite lower bound then the estimator is robust. Hampel (1968, 1971) introduced the concept of breakdown point of sequence of estimators generalizing an idea of Hodges (1967). When $\Theta = \Re$ the breakdown point is defined as:

$$\varepsilon^* = \sup\{\varepsilon \le 1 : \text{there exist } r_\varepsilon \text{ such that } \pi(F,G) < \varepsilon \Rightarrow G(\{|T_n| \le r_\varepsilon\}) \to 1 \text{ as } n \to \infty\}$$

where $\pi(F,G)$ is the Prohorov distance (for definition see Hampel et.al, 1986) of two probability distributions F and G belongs to set of all distributions $\Im(\chi)$, and χ is the sample space such that $\chi \subset \Re$. It should be noted that Prohorov distance can be replaced by the Levy distance or the bounded Lipschitz metric, or even by the total variation distance or the Kolmogorov distance or even by the gross error model (for definitions see Huber, 1981).

Wehrly and Shine (1981) showed that for a unimodal symmetric circular distribution F, the directional mean (a.k.a. circular mean) of F defined as $\arctan^*\left(\frac{E_F(\sin\theta)}{E_F(\cos\theta)}\right)$ where $\arctan^*$ is the quadrant - specific inverse of the tangent function which is defined by (1.1) is B-robust, in the sense that it has bounded sensitivity to fixed amounts of contamination. They derived the influence function of the circular mean as $IF(\theta;T,F)=\rho^{-1}\sin(\theta-\mu_0), 0<\rho\le 1$ where ρ is the concentration parameter. For any value of ρ, the influence curve and its first derivative are bounded by $\pm\rho^{-1}$. Thus, the circular mean has a bounded sensitivity to fixed amounts of contamination and to local shifts. Lenth (1981) discussed M-estimators for directional data and studied their performance through simulation. He adapted the established technique for robust estimation of location parameter for use in directional data. He argued that a periodic version of any of the commonly used ψ functions can be used to define a comparable estimator of angular location and also the proposed estimators appear to perform at efficiency levels similar to those of ordinary M-estimators in the linear case. He defines the circular M-estimator $\hat{\mu}$ of directional location as a solution to $\sum_{i=1}^{n}\rho(t(\theta_i-\hat{\mu};\kappa))=\text{minimum}$, where $t(\theta_i-\hat{\mu};\kappa)$ is a periodic function that in some sense "standardizes" the values of $(\theta_i-\hat{\mu})$ according to the concentration parameter κ.

Ko and Guttorp (1988) argued that the notion of robustness based on finiteness of g.e.s. needs to be modified when we deal with bounded parameter spaces because g.e.s. commonly approximates the maximum bias and this is bounded on a bounded parameter space. They introduced the notion of standardized influence function (SIF) and standardized gross error sensitivity (s.g.e.s). The SIF of a functional T with respect to a functional S is defined as

$$SIF(x;T,F,S)=\frac{IF(x;T,F)}{S(F)},\ S(F)\neq 0. \qquad \ldots (2.3)$$

where F is the underlying distribution and the s.g.e.s of T with respect to the functional S at the family of distributions $\Im$ is defined as

$$\gamma^*(T,\Im,S)=\sup_{\Im}\sup_{x}[SIF(x;T,F,S)] \qquad \ldots (2.4)$$

If $\gamma^*(T,\Im,S)$ is finite then the estimator is said to be standardized bias robust (or SB-robust) at the family of distributions $\Im$. Usually the functional S is taken to be a suitable dispersion measure and hence the notion of SB-robustness depends on the choice of the dispersion measure used. They also give a set of desirable conditions that a measure of dispersion S on a (q-1)-dimensional sphere Ω_q on $\Re^q$ should satisfy. Let X and Y be two random unit vectors with unimodal distributions F and G with modal vectors $T(X)$ and $T(Y)$ respectively. A real-valued functional S is called a dispersion on Ω_q in Ko and Guttorp (1988) if

a) $S(F) \le S(G)$ whenever $d(Y,T(Y))$ is stochastically larger than $d(X,T(X))$ where $d(\cdot,\cdot)$ is a metric on Ω_q.
b) $S(F) = S(G)$, if $Y = \Gamma X$ for an orthogonal matrix Γ. ... (2.5)
c) $S(\delta_c) = 0$, if c is a fixed point on Ω_q.

Using a particular choice of dispersion measure they shows that if $\Theta \sim F = CN(\mu,\kappa)$ the directional mean $T(F) = \arctan^*\left(\frac{E_F(\sin\Theta)}{E_F(\cos\Theta)}\right)$ and the concentration parameter $K(F) = A^{-1}(\rho_F)$ are not SB-robust at the family of distributions $\Im = \{CN(\mu,\kappa) : \kappa > 0\}$ where $\rho_F = \sqrt{E_F^2(\cos\Theta) + E_F^2(\sin\Theta)} = A(\kappa)$.

He & Simpson (1992) introduced distance based breakdown function for general parametric family of distributions $\{F_\mu, \mu \in \Theta\}$, as:

$$\varepsilon^*(\delta) = \inf\{\varepsilon_\mu^* : d(\mu_0,\mu) \ge \delta\} \text{ for } \delta \in [0,\delta^*) \text{ with } \delta^* = \sup_{\mu\in\Theta} d(\mu_0,\mu)$$

where $\varepsilon_\mu^* = \inf\{\varepsilon > 0 : T((1-\varepsilon)F + \varepsilon G) = \mu \text{ for some } G\}$ (see He, Simpson and Portnoy(1990)) and $T(F)$ is a Fisher consistent estimating functional for μ i.e. $T(F_\mu) = \mu$ for any $\mu \in \Theta$. The corresponding breakdown point, breakdown slope and the g.e.s with respect to any distance measure defined by them respectively are:

$$\varepsilon^* = \sup\{\varepsilon^*(\delta) : \delta \in [0,\delta^*)\},\ \beta^* = \lim_{\delta\to 0}\left\{\frac{\varepsilon^*(\delta)}{\delta}\right\} \text{ and } \gamma_{GE}^+ = \sup_x \limsup_{\varepsilon\to 0}\left[\frac{d\{T(F), T(F_{\varepsilon,x})\}}{\varepsilon}\right]$$

where $F_{\varepsilon,x} = (1-\varepsilon)F + \varepsilon\Delta_x$ and Δ_x is the distribution of a point mass at x. They defined the notion of SB-robustness by standardising with respect to the Kullback-Leibler distance and conclude that an estimate is SB-robust if its Kullback-Leibler-standardized breakdown slope is bounded away from 0 uniformly in $\kappa \in (0,\infty)$.This standardisation includes both the dispersed case $\kappa \to 0$ and the concentrated case $\kappa \to \infty$. With the above notion of SB-robustness, they showed that for von-Mises distributions both the directional median and symmetrically trimmed mean on the circle are SB-robust for any $\alpha > 0$ where α is the trimming proportion.

Ko (1992) developed a simple robust estimator of the concentration parameter of the von Mises distribution on the q-dimensional unit sphere which for $q = 2$ gives an estimator for the concentration parameter κ on the circle defined as $K_m(F) = \left(\frac{\Phi^{-1}(0.75)}{\text{med}|\Theta - \theta_0|} \right)^2$ where Φ is the c.d.f. of standard normal distribution which is analogous to the estimator based on median absolute deviation in case of linear data. He proved that with a reasonable choice of dispersion functional this estimator is SB-robust at the family of distributions $\Im = \{CN(\mu,\kappa) : \kappa > 0\}$ for the concentration parameter κ. Oteino (2002) proposed that the circular analogue of Hodges-Lehmann estimator (i.e. the circular median of the pair wise circular means) provides an alternative estimate of preferred direction. The new measure of preferred direction is a robust compromise between circular mean and circular median which is asymptotically more efficient than the circular median and its asymptotic efficiency relative to the circular mean is quite comparable. He showed that, for a von-Mises distribution with $\kappa \geq 2$, the influence function of the circular Hodges-Lehmann estimator is bounded and hence is a robust estimator for the preferred direction. Also, he showed that, for a von-Mises distribution with $\kappa \geq 2$, the circular Hodges-Lehmann estimator say, $\hat{\theta}^c_{HL} = \text{circular median}\left(\bar{\theta}^c_{1,1}, \bar{\theta}^c_{1,2}, \ldots, \bar{\theta}^c_{n-1,n}, \bar{\theta}^c_{n,n}\right)$ is approximately distributed as $VM\left(\hat{\theta}^c_{HL}, \frac{3n\kappa}{\pi}\right)$, $\bar{\theta}^c_{i,j}$ is the pair wise circular mean of observations θ_i and θ_j defined as $\bar{\theta}^c_{i,j} = \left[\tan^{-1}\left(\frac{\sin\theta_i + \sin\theta_j}{\cos\theta_i + \cos\theta_j} \right) \right], i \leq j \leq n$.

2.4 Robustness of Tests with Directional Data

Huber's second approach to robust statistics is via robustified likelihood ratio test (LRT). In the classical LRT, a single observation (a gross error) can carry the test statistic to infinity in either direction. Huber (1965) defined the censored probability ratio test and showed that this test is robust in a well defined minimax sense. Huber (1968) describes how robust testing method can be used to derive robust confidence intervals and point estimates of location. Heritier and Ronchetti (1994) developed robust tests for testing hypotheses in a general parametric model and study their properties. They derived robust versions of Wald, score and LRTs based on general M-estimators.

He, Simpson, and Portnoy (1990) introduced in the context of linear data the concept of breakdown robustness of tests. They developed the idea of power and level breakdown functions of a test statistic and gives a unified analysis that combines both local (influence function) and global (breakdown point) stability. These breakdown functions are invariant to one-to-one transformation. Let $\theta_1, \theta_2, \ldots, \theta_n$ be independent observations from a distribution $F_\theta \; (\theta \in \Theta)$, and suppose we desire to test $H_0 : \theta = 0$ against $H_1 : \theta \neq 0$ in a location model $\{F_\theta(x) = F_0(x - \theta), \theta \in R\}$. Given a distribution F, and let $G_\varepsilon(F) = (1-\varepsilon)F + \varepsilon\delta_x$. Then the Level Breakdown Function (LBF) of a test functional T is defined as

$$\varepsilon_\theta^{**}(T) = \inf\{\varepsilon > 0 : T(G_\varepsilon(F_0)) = T(F_\theta) \text{ for some } x\} \qquad \ldots (2.6)$$

and the Power Breakdown Function (PBF) of T is defined as

$$\varepsilon_\theta^{*}(T) = \inf\{\varepsilon > 0 : T(G_\varepsilon(F_\theta)) = T(F_0) \text{ for some } x\}. \qquad \ldots (2.7)$$

The level breakdown point (LBP) is defined as $\varepsilon^{**} = \sup_\theta(\varepsilon_\theta^{**})$ and the power breakdown point (PBP) is defined as $\varepsilon^{*} = \sup_\theta(\varepsilon_\theta^{*})$. The LBF of T at $\theta \, (= \varepsilon_\theta^{**}(T))$ gives the least proportion of contamination of F_0 by some value x which makes the value of the functional evaluated at this contaminated distribution equal to that of the functional evaluated at F_θ. In such situations we will say that level breakdown of T has occurred. Similarly, the PBF of T at $\theta(=\varepsilon^*_\theta(T))$ gives the least proportion of contamination of F_θ by some value x which makes the value of the functional evaluated at this contaminated distribution equal to that of the functional evaluated at F_0. In such a situation we will say that power breakdown of T has occurred. To interpret the LBP intuitively suppose that the level of contamination (η) is less than LBP. Then we can conclude that there exist θ for which level breakdown does not occur. Similar interpretation can be given for the PBP.

Lambert (1981) introduced influence functions for testing by using Hampel's influence function to transformed p-values which describes the effect of an observation and an underlying distribution on the behaviour of a test. She also discusses the relationship between the influence function of the p-value and the influence function of the test statistic. Lambert (1982) introduced the idea of qualitative robustness of tests and showed that in the normal set-up the z-test and Student's t-test are not qualitatively robust whereas the sign, Wilcoxon, Huber censored likelihood and normal scores test are qualitatively robust. Her definition of qualitative robustness can be applied to both conditional and unconditional tests.

Markatou and He (1994) introduced three classes of testing procedures - drop in dispersion, Wald type, and score type tests- based on one step high breakdown point bounded influence estimators for testing sub hypotheses in linear models. They showed that these tests have bounded influence functions. Perez (1993) discusses an interesting notion of robustness for one-tailed tests for the location for a specific class of distributions using the tail-ordering of distributions within this class. Reider (1978) looked at the maximum size and minimum power of a test evaluated asymptotically over certain neighbourhoods in order to obtain quantitative results about the influence of outliers on tests. Rousseeuw and Ronchetti (1979, 1981) introduced the notion of influence function for tests which incorporates estimators that are not Fisher consistent. They modify Hampel's influence function and defined the influence function for testing as follows:

$$IF_{test}(x;T,F_\theta)=IF(x;U,F_\theta)=\lim_{\varepsilon\to 0}\frac{U\{(1-\varepsilon)F_\theta+\varepsilon\delta_x\}-U(F_\theta)}{\varepsilon}$$

where $U(F_\theta)=\xi^{-1}(T(F_\theta))=\theta$ and δ_x is the probability measure which puts mass 1 in the point x. They also examine the asymptotic influence of contamination on the level and power of a test and introduced the concept of level influence function (LIF) and power influence function (PIF) having the following definition.

$$\begin{aligned}&\text{level}(\varepsilon)=\alpha_0+\varepsilon\int LIF(z;T,F_{\theta_0})dG(z)+o(\varepsilon) \text{ and}\\ &\text{power}(\varepsilon)=\beta_0+\varepsilon\int PIF(z;T,F_{\theta_0})dG(z)+o(\varepsilon)\end{aligned}\quad \text{, where}$$

$$LIF(z;T,F_{\theta_0})=\frac{\phi(\Phi^{-1}(1-\alpha_0))IF(z;T,F_{\theta_0})}{(V(T,F_{\theta_0}))^{1/2}},\; PIF(z;T,F_{\theta_0})=\frac{\phi(\Phi^{-1}(1-\alpha_0)-\delta\sqrt{E})IF(z;T,F_{\theta_0})}{(V(T,F_{\theta_0}))^{1/2}},$$

α_0 and β_0 are respectively the asymptotic level and power, $\Phi^{-1}(1-\alpha_0)$ is the $1-\alpha_0$ quantile of the standard normal distribution Φ and ϕ is its density, $E=\dfrac{(\xi'(\theta_0))^2}{V^{-1}(T,F_{\theta_0})}$ is the Pitman's efficacy of the test, $\xi(\theta)=T(F_\theta)$ and $V(T,F_{\theta_0})=\int IF(z;T,F_{\theta_0})^2 dF_{\theta_0}(z)$ is the asymptotic variance of T. The LIF and PIF actually describe the influence of a small amount of contamination at some point z on the asymptotic level and power of the test. Ronchetti (1997) gives an excellent survey on robustness and influence functions. Ylvisaker (1977) introduced the twin concepts of resistance to acceptance and resistance to rejection of a test to quantitatively measure the robustness of a test in the presence of outliers. He defines test resistance as one minus the fraction of observations that determine the test decision regardless the value of the other observations in the sample.

Stephens (1962) developed different exact and approximate tests for direction for circular distributions. He has proposed several approximate tests for polar vectors and test for randomness and discussed the accuracy of these tests in details. Stephens (1969) introduced tests for the modal vector and the concentration parameter of the von-Mises distribution on the circle. He developed exact tests for the following hypotheses: $H_0 : \kappa = \kappa_0$ whether the modal vector is known or not and $H_0 : A_0 = A$ when κ is known where A_0 is the modal vector based on R, the resultant length of a sample of vectors and on X, the component of R on A_0 when this is known or hypothesized. He proposed new χ^2 approximations for the percentage points of R and of X discussed the accuracy of these approximations. Upton (1973) developed some single sample tests for the von-Mises distribution. He introduced three tests for specified direction and two tests for specified distribution based on likelihood ratio. Watson and Williams (1956) developed some significance tests on the circle and sphere based on the fundamental property of sufficient statistics for direction and homogeneity. Robustness of these tests in the context of directional data has not been explored in the literature and is a part of subject matter of this thesis (see chapters 5 and 6).

Chapter 3

Robust Estimator for Mean Direction of Circular Normal Distribution

3.1 Introduction

In this chapter we study the robustness of the directional mean (a.k.a. circular mean) for different families of circular distributions. A circular random variable Θ is said to have a von-Mises or CN distribution with mean direction μ and concentration κ if it has the probability density function (p.d.f):

$$f(\theta;\mu,\kappa)=\frac{1}{2\pi I_0(\kappa)}e^{\kappa\cos(\theta-\mu)}, 0\le\theta<2\pi \text{ where } 0\le\mu<2\pi \text{ and } \kappa>0$$

where $I_0(\kappa)$ is the modified Bessel function of the first kind and order zero. We show that the directional mean is robust in the sense of finite standardized gross error sensitivity (SB-robust) for the following families- (1) mixture of two circular normal distributions, (2) mixture of wrapped normal and circular normal distributions, and (3) mixture of two wrapped normal distributions. We also show that the directional mean is not SB-robust for the family of all circular normal distributions with varying concentration parameter. We define the circular trimmed mean (for definition see section 3.3) and prove that it is SB-robust for this family. In general the property of SB-robustness of an estimator at a family of probability distributions is dependent on the choice of the dispersion measure. An estimator T may be SB-robust at the family of distributions $\Im$ for one choice of dispersion measure while it may not be SB-robust at the family of distributions $\Im$ for another choice of dispersion measure. For example, for the family of distributions $\Im=\{CN(\mu,\kappa): m>\kappa>0\}$ Ko and Guttorp (1988) uses $\sqrt{1-A(\kappa)}$ as the dispersion measure whereas He and Simpson (1992) suggests using $(\kappa A(\kappa))^{-\frac{1}{2}}$ as the dispersion measure. It can be easily seen that the directional mean is SB-robust at the family of distributions $\Im$ when the measure of dispersion is $(\kappa A(\kappa))^{-\frac{1}{2}}$ and not SB-robust at the family of distributions $\Im$ when the measure of dispersion is $\sqrt{1-A(\kappa)}$. We introduce the concept of equivalent dispersion measures and prove that if an estimator is SB-robust for one dispersion measure then it is SB- robust for equivalent dispersion measures. The anomaly shown above occurs due to the fact that $\sqrt{1-A(\kappa)}$ and $(\kappa A(\kappa))^{-\frac{1}{2}}$ are not equivalent measures of dispersion for the family of distributions $\Im$. In this chapter we have developed four lemmas and five theorems on the robust estimator for mean direction of circular normal distribution.

The organization of this chapter is as follows: In Section 3.2 we discuss the SB-robustness of the directional mean and show that the directional mean is not SB-robust at the family of all circular normal distributions with varying $\kappa > 0$. However, we find that the directional mean is SB-robust at some mixture families- namely mixture of two circular normal distributions with differing concentration parameters, mixture of a circular normal and a wrapped normal distribution, and mixture of two wrapped normal distributions with differing mean resulting length (for definition and properties of wrapped normal distribution see Mardia and Jupp, 2000 pp. 50-51). In Section 3.3, we give the definition of circular $\gamma-$trimmed mean and show that it is SB-robust for the family of circular normal distributions with varying $\kappa > 0$. In Section 3.4, we define the notion of equivalent measures of dispersion for a family of circular distributions and study equivalence of different dispersion measures. In Section 3.5, we compare the performance of the three dispersion measures used in Section 3.4.

Definition 1: Let $\Im = \{F_\mu : \mu \in \Theta\}$ be a family of distributions on the unit circle where $\Theta = [0, 2\pi)$ and μ is the central direction. Suppose $T(F)$ is an estimator for μ, i.e. $T(F) = \mu$ for any $\mu \in \Theta$. The circular distance between the directions θ and μ which is denoted by $d(\theta, \mu)$ is defined as

$$d(\theta,\mu) = \min[(\theta-\mu),\ 2\pi-(\theta-\mu)]$$
$$= \pi - |\,\pi - |\,\theta-\mu\,||,\ \ 0 \le d(\theta,\mu) \le \pi\ \ \forall\ \ \theta,\mu \in \Theta. \qquad \ldots (3.1)$$

The circular distance between two angles is defined as the shorter of the two arc lengths on a unit circle between the two points on it which represent these two angles. This measure of dispersion can be considered in some ways a natural measure of dispersion on the circle.

3.2 SB-robustness of the Directional Mean

Let θ be a circular random variable having cumulative distribution function (c.d.f) F and let $G_\varepsilon = (1-\varepsilon)F + \varepsilon\delta_x, 0 < \varepsilon < 1$. The directional mean μ of the circular distribution F is defined implicitly as the solution of $\tan\mu = \dfrac{E_F(\sin\theta)}{E_F(\cos\theta)}$. The estimating functional of μ is $T(F) = \arctan^*\left(\dfrac{E_F(\sin\theta)}{E_F(\cos\theta)}\right)$. Ko and Guttorp (1988) proves that the directional mean is B-robust but not SB-robust at the family of distributions $\Im = \{CN(\mu,\kappa) : \kappa > 0\}$ when the measure of dispersion is $S(F) = \sqrt{(1-\rho)}$ where $\rho = \sqrt{E^2(\cos\theta) + E^2(\sin\theta)}$. In Theorem 3.1 below we show that the directional mean has the same properties when the dispersion measure is $S(F) = E_F(d(\theta,\mu))$.

Theorem 3.1: *The directional mean* $T(F) = \arctan^*\left(\dfrac{E_F(\sin\theta)}{E_F(\cos\theta)}\right)$ *is B-robust but not SB-robust at the family of distributions* $\Im = \{CN(\mu,\kappa) : \kappa > 0\}$ *when the measure of dispersion is* $S(F) = E_F(d(\theta,\mu))$ *where* $F \in \Im$*.*

The following Lemma 1 and Lemma 2 were used to prove the above theorem.

Lemma 1: *Suppose* θ *is a circular random variable such that* $\theta \sim F_{\tilde{\mu}}$ *where F is a* *unimodal circular distribution with mode* $\tilde{\mu}$. *Then* $S(F)=E_F(d(\theta,\tilde{\mu}))$ *is a measure of dispersion on the unit circle.*

Proof:

Let θ and μ be any two angles on the unit circle. Using the definition (3.1) we have

$E_F(d(\theta,\mu))=\int d(\theta,\mu)dF$, where F is the cumulative distribution function on the circle.

Now define $S(F)=E_F(\pi-|\pi-|X-T(X)||)=E_F(d(X,T(X)))$. Here we show that $S(F)$ satisfies the set of conditions (2.5) given in chapter 2.

a) *To prove* $S(F)\le S(G)$.

Let $F^*(u)=P(d(X,T(X))\le u)$ and $G^*(u)=P(d(Y,T(Y))\le u)$ be the distribution functions of $d(X,T(X))$ and $d(Y,T(Y))$ respectively.

Then it is sufficient to prove that $E_F(d(X,T(X)))\le E_G(d(Y,T(Y)))$ whenever $G^*(u)\ge F^*(u)$. Since $d(X,T(X))$ and $d(Y,T(Y))$ is non-negative we have,

$$E_F(d(X,T(X))) =\int_0^\pi P(d(X,T(X))>u)\,du =\int_0^\pi (1-P(d(X,T(X))\le u))\,du=\int_0^\pi (1-F^*(u))\,du$$

and

$$E_G(d(Y,T(Y)))=\int_0^\pi P(d(Y,T(Y))>u)\,du =\int_0^\pi (1-P(d(Y,T(Y))\le u))\,du=\int_0^\pi (1-G^*(u))\,du.$$

Since $G^*(u)>F^*(u)$ implies that

$$\begin{aligned}(1-F^*(u))\le(1-G^*(u)) &\Rightarrow \int_0^\pi (1-F^*(u))\,du\le\int_0^\pi (1-G^*(u))\,du\\ &\Rightarrow E_F(d(X,T(X)))\le E_G(d(Y,T(Y)))\\ &\Rightarrow S(F)\le S(G).\end{aligned}$$

b) *To prove* $S(F)=S(G)$.

In Ω_2**,** we have $\Gamma = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, $X = (\cos\alpha, \sin\alpha)^T$ and $T(X) = (\cos\mu, \sin\mu)^T$.

Therefore $Y=\Gamma X = \begin{bmatrix} \cos(\theta+\alpha) \\ \sin(\theta+\alpha) \end{bmatrix}$ and $T(Y) = \begin{bmatrix} \cos(\theta+\mu) \\ \sin(\theta+\mu) \end{bmatrix}$.

But $S(F) = E_F(d(X,T(X))) = \pi - |\pi - |\alpha-\mu||$. Now,

$$\begin{aligned} S(G) &= E_F(d(Y,T(Y))) = \pi - |\pi - |(\theta+\alpha)-(\theta+\mu)|| \\ &= \pi - |\pi - |\alpha-\mu|| \\ &= S(F) \\ &\Rightarrow S(F)=S(G). \end{aligned}$$

c) To prove $S(\delta_c)=0$, if c is a fixed point on Ω_q.

Since c is a fixed point on Ω_2**,** by taking $X=T(X)=c$, $S(\delta_c)=E_F(\pi-|\pi-|c-c||)=0$.

Hence the lemma.

Lemma 2: *Let* $\theta \sim CN(\mu,\kappa)$. *Then* $E_F(\cos\theta)=\rho\cos\mu$, $E_F(\sin\theta)=\rho\sin\mu$ *and*

$$S(F)=\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{A_{2n+1}(\kappa)}{(2n+1)^2}.$$

Proof:

By definition, we have

$$\begin{aligned} E_F(\cos\theta) &= \frac{1}{2\pi I_0(\kappa)}\int_0^{2\pi}\cos\theta\, e^{\kappa\cos(\theta-\mu)}d\theta = \frac{1}{2\pi I_0(\kappa)}\int_0^{2\pi}\cos(\theta-\mu+\mu)\, e^{\kappa\cos(\theta-\mu)}d\theta \\ &= \frac{\cos\mu}{2\pi I_0(\kappa)}\int_0^{2\pi}\cos(\theta-\mu)\, e^{\kappa\cos(\theta-\mu)}d\theta. \end{aligned}$$

But we know that $\frac{1}{2\pi}\int_0^{2\pi}\cos p(\theta-\mu)\, e^{\kappa\cos(\theta-\mu)}d\theta = I_p(\kappa)\ \forall\, p = 1, 2, 3, \ldots$. Then we get

$E_F(\cos\theta) = A(\kappa)\cos\mu$ where for $CN(\mu, \kappa)$, we have $A(\kappa) = I_0^{-1}(\kappa)I_1(\kappa) = \rho$. Hence, $E_F(\cos\theta) = \rho\cos\mu$.

Similarly, $E_F(\sin\theta) = \frac{1}{2\pi I_0(\kappa)}\int_0^{2\pi}\sin\theta\, e^{\kappa\cos(\theta-\mu)}d\theta = \frac{\sin\mu}{2\pi I_0(\kappa)}\int_0^{2\pi}\sin(\theta-\mu)\, e^{\kappa\cos(\theta-\mu)}d\theta$.

Since, $\frac{1}{2\pi}\int_0^{2\pi}\sin p(\theta-\mu)\, e^{\kappa\cos(\theta-\mu)}d\theta = I_p(\kappa)\ \forall\, p = 1, 2, 3, \ldots$ we have $E_F(\sin\theta) = A(\kappa)\sin\mu$. Hence, $E_F(\sin\theta) = \rho\sin\mu$.

By Lemma 1 $S(F) = E_F(d(\theta,\mu))$ is a dispersion measure on the circle where $d(\theta,\mu)$ is defined by (3.1). Note that the expected circular distance $S(F) = E_F(d(\theta,\mu))$ does not depend on μ. Hence we can without loss of generality assume $\mu = 0$ for computing $S(F)$. Now (3.1) can be written as:

$$d(\theta,\mu) = \begin{cases} \theta & \text{if } \theta \le \pi \\ (2\pi-\theta) & \text{if } \theta > \pi \end{cases}.$$

Hence, $S(F) = E_F(d(\theta)) = \frac{1}{2\pi I_0(\kappa)}\left[\int_0^{\pi}\theta\, e^{\kappa\cos\theta}d\theta + \int_{\pi}^{2\pi}(2\pi\text{-}\theta)\, e^{\kappa\cos\theta}d\theta\right]$.

By making the substitution $(2\pi-\theta) = \lambda$ and using $e^{\kappa\cos\theta} = I_0(\kappa) + 2\sum_{p=1}^{\infty}I_p(\kappa)\cos p\theta$ (Abramowitz & Stegun, 1965, p.376, 9.6.34) $S(F)$ can be written as

$$S(F) = \frac{1}{\pi I_0(\kappa)}\left[I_0(\kappa)\int_0^{\pi}\lambda\, d\lambda + 2\sum_{p=1}^{\infty}I_p(\kappa)\int_0^{\pi}\lambda\cos p\lambda\, d\lambda\right]. \qquad \ldots (3.2)$$

Using integration by parts and simplifying (3.2) we get

$$S(F) = \frac{\pi}{2} + \frac{2}{\pi I_0(\kappa)}\sum_{p=1}^{\infty}I_p(\kappa)\left\{\frac{-\pi\sin p\pi}{p} + \frac{\cos p\pi - 1}{p^2}\right\}.$$

Since $\sin p\pi = 0 \;\; \forall\, p$, $\cos p\pi = -1 \;\; \forall\, p = 1, 3, 5, ...$ and $A_p(\kappa) = I_p(\kappa) I_0^{-1}(\kappa) \;\; \forall\, p > 0$, we have

$$S(F) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{A_{2n+1}(\kappa)}{(2n+1)^2} \; . \qquad \ldots (3.3)$$

Hence the lemma.

Proof of the theorem:

Let $F \in \Im$, $G_\varepsilon = (1-\varepsilon)F + \varepsilon\delta_x$, $0 \le \varepsilon \le 1$ and $\mu_\varepsilon = T((1-\varepsilon)F + \varepsilon\delta_x)$, $0 \le x < 2\pi$. Then we have,

$$\tan\mu_\varepsilon = \frac{E_{G_\varepsilon}(\sin\theta)}{E_{G_\varepsilon}(\cos\theta)} = \frac{E_{T\{(1-\varepsilon)F+\varepsilon\delta_x\}}(\sin\theta)}{E_{T\{(1-\varepsilon)F+\varepsilon\delta_x\}}(\cos\theta)}$$
$$= \frac{(1-\varepsilon)E_F(\sin\theta) + \varepsilon\sin x}{(1-\varepsilon)E_F(\cos\theta) + \varepsilon\cos x}.$$

Dividing both numerator and denominator by $E_F(\cos\theta)$ and using Lemma 2 we get

$$\tan\mu_\varepsilon = \frac{\rho(1-\varepsilon)\sin\mu + \varepsilon\sin x}{\rho(1-\varepsilon)\cos\mu + \varepsilon\cos x} \;\; \Rightarrow \mu_\varepsilon = \arctan^*\left\{\frac{\rho(1-\varepsilon)\sin\mu + \varepsilon\sin x}{\rho(1-\varepsilon)\cos\mu + \varepsilon\cos x}\right\}.$$

Consider $\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\tan(\mu_\varepsilon - \mu) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left[\frac{\tan\mu_\varepsilon - \tan\mu}{1 + \tan\mu_\varepsilon \tan\mu}\right]$.

Substituting the value of $\tan\mu_\varepsilon$ and simplifying, we get

$$(\mu_\varepsilon - \mu) \cong \arctan^*\left[\frac{\varepsilon\sin(x-\mu)}{\rho} + o(\varepsilon)\right] \quad \text{where } \lim_{\varepsilon\to 0}\frac{o(\varepsilon)}{\varepsilon} = 0 \, .$$

Using the Taylor series expansion of $\tan^{-1}(t)$ and applying limit as $\varepsilon \to 0$ we get the influence function of the directional mean as

$$IF(x;T,F) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}(\mu_\varepsilon - \mu) = \frac{\sin(x-\mu)}{\rho} \;\; ; \; \rho > 0 \, . \qquad \ldots (3.4)$$

Now, the gross error sensitivity (g.e.s.) of the estimator T at F is given by

$$\gamma(T,F) = \sup_{0\le x<2\pi} |IF(x;T,F)| = \frac{|\sin(x-\mu)|}{\rho} < \infty .$$

Hence, we see that the directional mean is B-robust.

Now by using (3.3) and (3.4) we get the SIF of the functional T as:

$$SIF(x;T,F,S) = \frac{IF(x;T,F)}{S(F)} = \frac{\sin(x-\mu)}{\rho\left\{\frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{A_{2n+1}(\kappa)}{(2n+1)^2}\right\}} \; ; \rho > 0 . \qquad \dots (3.5)$$

Using (3.5), the s.g.e.s of T at F is given by:

$$\begin{aligned}
\gamma^{*}(T,\Im,S) &= \sup_{\kappa>0}\sup_{0\le x<2\pi}[SIF(x;T,F,S)] \\
&= \sup_{\kappa>0}\left(\left(A(\kappa)\left\{\frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{A_{2n+1}(\kappa)}{(2n+1)^2}\right\}\right)^{-1} \sup_{0\le x<2\pi}(\sin(x-\mu))\right). \\
&= \sup_{\kappa>0}\left(\left(A(\kappa)\left\{\frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{A_{2n+1}(\kappa)}{(2n+1)^2}\right\}\right)^{-1}\right) , \sup_{0\le x<2\pi}\sin(x-\mu) = 1 \\
&= \infty , \text{ since as } \kappa\to\infty, A(\kappa)\to 1 \text{ and } \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{A_{2n+1}(\kappa)}{(2n+1)^2} \to \frac{\pi}{2} .
\end{aligned}$$

Hence the theorem.

Now we explore the properties of the directional mean for some mixture families of distributions.

In Theorem 3.2 we prove that the directional mean is SB-robust for the following families of distributions-

(i) $\Im_1 = \{\alpha CN(\mu,\kappa_1) + (1-\alpha)CN(\mu,\kappa_2) : 0 \le \alpha < 1, 0 < m < \kappa_1, \kappa_2 < M\}$ (mixture of two circular normal distributions)

(ii) $\Im_2 = \{\alpha WN(\mu, A(\kappa_1)) + (1-\alpha)CN(\mu,\kappa_2) : 0 \le \alpha < 1, 0 < m < \kappa_1, \kappa_2 < M\}$ (mixture of wrapped normal and circular normal distributions) and

(iii) $\Im_3 = \{\alpha WN(\mu,\rho_1) + (1-\alpha)WN(\mu,\rho_2) : 0 \le \alpha < 1, 0 < m < \rho_1,\rho_2 < M\}$ (mixture of two wrapped normal distributions) where $A(\kappa) = I_1(\kappa)I_0^{-1}(\kappa)$.

The parameters of the wrapped normal distribution in (ii) above are chosen as μ and $A(\kappa)$ since for large κ, $CN(\mu,\kappa)$ distribution is well approximated by $WN(\mu, A(\kappa))$ (Mardia and Jupp, 2000 p. 38). The parameters of the wrapped normal distribution $WN(\mu,\rho)$ arise naturally when wrapping $N(\mu,\sigma^2)$ onto the circle where $\rho = \exp(-\sigma^2/2)$ (Mardia and Jupp, 2000, p.50).

Theorem 3.2: *The directional mean* $T(F) = \arctan^*\left(\frac{E_F(\sin\theta)}{E_F(\cos\theta)}\right)$ *is SB-robust for the families:*

(a) $\Im_1 = \{\alpha CN(\mu,\kappa_1) + (1-\alpha)CN(\mu,\kappa_2) : 0 \le \alpha < 1, 0 < m < \kappa_1, \kappa_2 < M\}$

(b) $\Im_2 = \{\alpha WN(\mu, A(\kappa_1)) + (1-\alpha)CN(\mu,\kappa_2) : 0 \le \alpha < 1, 0 < m < \kappa_1,\kappa_2 < M\}$ *and*

(c) $\Im_3 = \{\alpha WN(\mu,\rho_1) + (1-\alpha)WN(\mu,\rho_2) : 0 \le \alpha < 1, 0 < m < \rho_1,\rho_2 < M\}$

when the measure of dispersion is $S(F) = E_F(d(\theta,\mu))$ *where* $F \in \Im$*.*

The following Lemma 3 is used prove the above theorem.

Lemma 3:

(a) *Let* $H_\alpha = \alpha CN(\mu,\kappa_1) + (1-\alpha)CN(\mu,\kappa_2)$*. Then* $\beta_1 = (\alpha A(\kappa_1) + (1-\alpha)A(\kappa_2)) \cos\mu$ *and*

$$S(H_\alpha) = \alpha\left\{\frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{A_{2n+1}(\kappa_1)}{(2n+1)^2}\right\} + (1-\alpha)\left\{\frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{A_{2n+1}(\kappa_2)}{(2n+1)^2}\right\}.$$

(b) *Let* $H_\alpha = \alpha WN(\mu, A(\kappa_1)) + (1-\alpha)CN(\mu,\kappa_2)$*. Then* $\beta_2 = (\alpha A(\kappa_1) + (1-\alpha)A(\kappa_2)) \cos\mu$

and $S(H_\alpha) = \alpha\left\{\frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(A(\kappa_1))^{(2n+1)^2}}{(2n+1)^2}\right\} + (1-\alpha)\left\{\frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{A_{2n+1}(\kappa_2)}{(2n+1)^2}\right\}.$

(c) *Let* $H_\alpha = \alpha WN(\mu,\rho_1) + (1-\alpha)WN(\mu,\rho_2)$. *Then* $\beta_3 = (\alpha\,\rho_1 + (1-\alpha)\rho_2)\cos\mu$ *and*

$$S(H_\alpha) = \alpha\left\{\frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(\rho_1)^{(2n+1)^2}}{(2n+1)^2}\right\} + (1-\alpha)\left\{\frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(\rho_2)^{(2n+1)^2}}{(2n+1)^2}\right\}.$$

Proof:

(a) By definition, we have $\beta_1 = E_{H_\alpha}(\cos\theta) = \alpha E_{F_1}(\cos\theta) + (1-\alpha)E_{F_2}(\cos\theta)$. But under the distribution $CN(\mu, \kappa_i)$, we have

$$E_{F_i}(\cos\theta) = \frac{1}{2\pi I_0(\kappa_i)}\int_0^{2\pi}\cos\theta\, e^{\kappa_i\cos(\theta-\mu)}d\theta = A(\kappa_i)\cos\mu, i = 1, 2.$$

Hence, $\beta_1 = E_{H_\alpha}(\cos\theta) = (\alpha\, A(\kappa_1) + (1-\alpha)A(\kappa_2))\cos\mu$. … (3.6)

Note that the expected circular distance $S(H_\alpha) = E_{H_\alpha}(d(\theta,\mu))$ does not depend on μ.

Hence we can without loss of generality assume $\mu = 0$ for computing $S(H_\alpha)$. Now,

$S(H_\alpha) = E_{H_\alpha}(d(\theta)) = \alpha E_{F_1}(d(\theta)) + (1-\alpha)E_{F_2}(d(\theta))$. … (3.7)

Using (3.3) of Lemma 2, we get $E_{F_i}(d(\theta)) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{A_{2n+1}(\kappa_i)}{(2n+1)^2}$ for all $i = 1, 2$. Thus, from (3.7) we get

$$S(H_\alpha) = \alpha\left\{\frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{A_{2n+1}(\kappa_1)}{(2n+1)^2}\right\} + (1-\alpha)\left\{\frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{A_{2n+1}(\kappa_2)}{(2n+1)^2}\right\}. \quad \ldots (3.8)$$

(b) By definition we have $\beta_2 = E_{H_\alpha}(\cos\theta) = \alpha E_{F_1}(\cos\theta) + (1-\alpha)E_{F_2}(\cos\theta)$. But under the distribution $WN(\mu, A(\kappa_1))$ we have

$$E_{F_1}(\cos\theta) = \frac{1}{2\pi}\int_0^{2\pi}\cos\theta\left\{1+2\sum_{p=1}^{\infty}(A(\kappa_1))^{p^2}\cos p(\theta-\mu)\right\}d\theta$$
$$= \frac{1}{\pi}\sum_{p=1}^{\infty}(A(\kappa_1))^{p^2}\int_0^{2\pi}\cos\theta\cos p(\theta-\mu)d\theta$$
$$= \frac{A(\kappa_1)}{\pi}\int_0^{2\pi}\cos\theta\cos(\theta-\mu)d\theta + \frac{1}{\pi}\sum_{p=2}^{\infty}(A(\kappa_1))^{p^2}\int_0^{2\pi}\cos\theta\cos p(\theta-\mu)d\theta$$
$$= \frac{A(\kappa_1)\cos\mu}{2\pi}\int_0^{2\pi}\cos^2\theta, \quad \text{since } \int_0^{2\pi}\cos m\theta\cos n\theta d\theta = \begin{cases}0 & \text{for } m\neq n\\ \pi & \text{for } m=n\end{cases}$$
$$= A(\kappa_1)\cos\mu .$$

For $CN(\mu, \kappa_2)$ distribution, by using part (a) we have $E_{F_2}(\cos\theta) = A(\kappa_2)\cos\mu$.

Therefore, $\beta_2 = E_{H_\alpha}(\cos\theta) = (\alpha\, A(\kappa_1) + (1-\alpha)A(\kappa_2))\cos\mu$. … (3.9)

Under the distribution $WN(\mu, A(\kappa_1))$ and making the substitution $(2\pi-\theta)=\lambda$ we have

$$E_{F_1}(d(\theta)) = \frac{1}{2\pi}\int_0^{2\pi}(\pi - |\pi - \theta|)\left\{1+2\sum_{p=1}^{\infty}(A(\kappa_1))^{p^2}\cos p\theta\right\}d\theta$$
$$= \frac{1}{\pi}\int_0^{\pi}\theta\left\{1+2\sum_{p=1}^{\infty}(A(\kappa_1))^{p^2}\cos p\theta\right\}d\theta$$
$$= \frac{\pi}{2} + \frac{2}{\pi}\sum_{p=1}^{\infty}(A(\kappa_1))^{p^2}\int_0^{\pi}\theta\cos p\theta$$
$$= \frac{\pi}{2} + \frac{2}{\pi}\sum_{p=1}^{\infty}(A(\kappa_1))^{p^2}\left\{\frac{\pi\sin p\pi}{p} + \frac{\cos p\pi - 1}{p^2}\right\}$$
$$= \frac{\pi}{2} + \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(A(\kappa_1))^{(2n+1)^2}}{(2n+1)^2}, \text{ since } \sin p\pi = 0 \ \forall p \text{ and } \cos p\pi = -1 \ \forall p = 1, 3, \ldots.$$

Hence using (3.7) we get

$$S(H_\alpha) = \alpha\left\{\frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(A(\kappa_1))^{(2n+1)^2}}{(2n+1)^2}\right\} + (1-\alpha)\left\{\frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{A_{2n+1}(\kappa_2)}{(2n+1)^2}\right\}. \quad \text{... (3.10)}$$

(c) By definition we have $\beta_3 = E_{H_\alpha}(\cos\theta) = \alpha E_{F_1}(\cos\theta) + (1-\alpha)E_{F_2}(\cos\theta)$. But under the distribution $WN(\mu, \rho_i)$ where $i = 1, 2$ we have

$$E_{F_i}(\cos\theta)=\frac{1}{2\pi}\int_0^{2\pi}\cos\theta\left\{1+2\sum_{p=1}^{\infty}(\rho_i)^{p^2}\cos p(\theta-\mu)\right\}d\theta$$
$$=\frac{1}{\pi}\sum_{p=1}^{\infty}(\rho_i)^{p^2}\int_0^{2\pi}\cos\theta\cos p(\theta-\mu)d\theta$$
$$=\frac{\rho_i}{\pi}\int_0^{2\pi}\cos\theta\cos(\theta-\mu)d\theta+\frac{1}{\pi}\sum_{p=2}^{\infty}(\rho_i)^{p^2}\int_0^{2\pi}\cos\theta\cos p(\theta-\mu)d\theta$$
$$=\frac{\rho_i\cos\mu}{2\pi}\int_0^{2\pi}\cos^2\theta \quad \text{since} \int_0^{2\pi}\cos m\theta\cos n\theta d\theta=\begin{cases}0 & \text{for } m\neq n\\ \pi & \text{for } m=n\end{cases}$$
$$=\rho_i\cos\mu \text{ where } i=1,2.$$

Therefore, $\beta_3=E_{H_\alpha}(\cos\theta)=(\alpha\,\rho_1+(1-\alpha)\rho_2)\cos\mu$. … (3.11)

Under the distribution $WN(\mu,\rho_i)$ where $i=1,2$ and making the substitution $(2\pi-\theta)=\lambda$ we have

$$E_{F_i}(d(\theta))=\frac{1}{2\pi}\int_0^{2\pi}(\pi-|\pi-\theta|)\left\{1+2\sum_{p=1}^{\infty}(\rho_i)^{p^2}\cos p\theta\right\}d\theta$$
$$=\frac{1}{\pi}\int_0^{\pi}\theta\left\{1+2\sum_{p=1}^{\infty}(\rho_i)^{p^2}\cos p\theta\right\}d\theta$$
$$=\frac{\pi}{2}+\frac{2}{\pi}\sum_{p=1}^{\infty}(\rho_i)^{p^2}\int_0^{\pi}\theta\cos p\theta$$
$$=\frac{\pi}{2}+\frac{2}{\pi}\sum_{p=1}^{\infty}(\rho_i)^{p^2}\left\{\frac{\pi\sin p\pi}{p}+\frac{\cos p\pi-1}{p^2}\right\}$$
$$=\frac{\pi}{2}+\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(\rho_i)^{(2n+1)^2}}{(2n+1)^2}, \quad \text{since } \sin p\pi=0\ \forall p \text{ and } \cos p\pi=-1\ \forall p=1,3,\ldots.$$

Hence using (3.7) we get

$$S(H_\alpha)=\alpha\left\{\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(\rho_1)^{(2n+1)^2}}{(2n+1)^2}\right\}+(1-\alpha)\left\{\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(\rho_2)^{(2n+1)^2}}{(2n+1)^2}\right\}. \quad \text{… (3.12)}$$

Hence the lemma.

Proof of the theorem:

(a) Let $H_\alpha\in\Im_1$ where $H_\alpha=\alpha CN(\mu,\kappa_1)+(1-\alpha)CN(\mu,\kappa_2)$.

By doing similar computations as in the proof of Theorem 3.1 it can be shown that

$$\left(\mu^* - \mu\right) \cong \arctan^* \left\{ \frac{\varepsilon \sin(x-\mu)}{\beta_1 \ \sec\mu} + o(\varepsilon) \right\} \text{ where } \beta_1 = E_{H_\alpha}(\cos\theta).$$

Hence $IF(x;T,H_\alpha) = \dfrac{\sin(x-\mu)}{\beta_1 \sec\mu}$; $\mu \neq \dfrac{\pi}{2}, \dfrac{3\pi}{2}$. ... (3.13)

Using lemma 3(a) and (3.13) we get

$$SIF(x;T,H_\alpha,S) = \frac{\sin(x-\mu)\left(\left(\alpha\, A(\kappa_1) + (1-\alpha)A(\kappa_2)\cos\mu\right)\right)^{-1} \sec^{-1}\mu}{\left(\alpha\left\{\frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{A_{2n+1}(\kappa_1)}{(2n+1)^2}\right\} + (1-\alpha)\left\{\frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{A_{2n+1}(\kappa_2)}{(2n+1)^2}\right\}\right)};\ \mu \neq \frac{\pi}{2}, \frac{3\pi}{2}$$

$$= \frac{\sin(x-\mu)}{\left(\lambda_1\alpha^2 + \lambda_2\alpha + \lambda_3\right)\sec\mu};\ \mu \neq \frac{\pi}{2}, \frac{3\pi}{2}. \qquad \text{...(3.14)}$$

where λ_i's $(i = 1,2,3)$ are constants involving κ_1 and κ_2.

Note that the numerator of (3.14) is a bounded function of x and the denominator of (3.14) is a product of two linear functions of α which are both non-zero in the closed interval [0, 1]. Since the denominator is positive and a continuous function of α in the closed interval [0, 1], it is bounded away from zero in [0, 1]. Hence,

$$\gamma^*(T,\Im_1,S) = \sup_{\substack{0\le\alpha<1,\\ 0<m<\kappa<M}} \sup_{0\le x<2\pi} \left\{ \frac{\sin(x-\mu)}{\left(\lambda_1\alpha^2 + \lambda_2\alpha + \lambda_3\right)\sec\mu} \right\} < \infty. \qquad \text{... (3.15)}$$

This proves that the directional mean is SB-robust at the family of distributions $\Im_1$.

(b) Let $H_\alpha \in \Im_2$ where $H_\alpha = \alpha WN(\mu, A(\kappa_1)) + (1-\alpha)CN(\mu,\kappa_2)$. By doing similar computations as in the proof of Theorem 3.1 it can be shown that

$$\left(\mu^* - \mu\right) \cong \arctan^* \left\{ \frac{\varepsilon \sin(x-\mu)}{\beta_2 \ \sec\mu} + o(\varepsilon) \right\} \text{ where } \beta_2 = E_{H_\alpha}(\cos\theta).$$

Hence, $IF(x;T,H_\alpha)=\dfrac{\sin(x-\mu)}{\beta_2\sec\mu}$; $\mu\neq\dfrac{\pi}{2},\dfrac{3\pi}{2}$. ... (3.16)

Thus using lemma 3(b) and (3.16) we get,

$$SIF(x;T,H_\alpha,S)=\frac{\sin(x-\mu)\left(\left(\alpha\,A(\kappa_1)+(1-\alpha)A(\kappa_2)\cos\mu\right)\right)^{-1}\sec^{-1}\mu}{\left(\alpha\left\{\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(A(\kappa_1))^{(2n+1)^2}}{(2n+1)^2}\right\}+(1-\alpha)\left\{\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{A_{2n+1}(\kappa_2)}{(2n+1)^2}\right\}\right)};\ \mu\neq\frac{\pi}{2},\frac{3\pi}{2}$$

$$=\frac{\sin(x-\mu)}{\left(\nu_1\alpha^2+\nu_2\alpha+\nu_3\right)\sec\mu};\ \mu\neq\frac{\pi}{2},\frac{3\pi}{2}. \quad \text{... (3.17)}$$

where ν_i's $(i=1,2,3)$ are constants involving κ_1 and κ_2.

Note that the numerator of (3.17) is a bounded function of x and the denominator of (3.17) is a product of two linear functions of α which are both non-zero in the closed interval [0, 1]. Since the denominator is positive and a continuous function of α in the closed interval [0, 1], it is bounded away from zero in [0, 1]. Hence,

$$\gamma^*(T,\Im_2,S)\ =\ \sup_{\substack{0\le\alpha<1,\\ 0<m<\kappa<M}}\ \sup_{0\le x<2\pi}\left\{\frac{\sin(x-\mu)}{\left(\nu_1\alpha^2+\nu_2\alpha+\nu_3\right)\sec\mu}\right\}\ <\ \infty. \quad \text{… (3.18)}$$

This proves that the directional mean is SB-robust at the family of distributions $\Im_2$.

(c) Let $H_\alpha\in\Im_3$ where $H_\alpha=\alpha WN(\mu,\rho_1)+(1-\alpha)WN(\mu,\rho_2)$. By doing similar computations as in the proof of Theorem 3.1 it can be shown that

$$\left(\mu^*-\mu\right)\cong\arctan^*\left\{\frac{\varepsilon\sin(x-\mu)}{\beta_3\ \sec\mu}+o(\varepsilon)\right\}\ \text{ where } \beta_3=E_{H_\alpha}(\cos\theta).$$

Hence, $IF(x;T,H_\alpha)=\dfrac{\sin(x-\mu)}{\beta_3\sec\mu}$; $\mu\neq\dfrac{\pi}{2},\dfrac{3\pi}{2}$. ... (3.19)

Thus by using lemma 3(c) and (3.19) we get,

$$SIF(x;T,H_\alpha,S)=\frac{\sin(x-\mu)((\alpha\rho_1+(1-\alpha)\rho_2)\cos\mu)^{-1}\sec^{-1}\mu}{\left(\alpha\left\{\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(\rho_1)^{(2n+1)^2}}{(2n+1)^2}\right\}+(1-\alpha)\left\{\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(\rho_2)^{(2n+1)^2}}{(2n+1)^2}\right\}\right)};\ \mu\neq\frac{\pi}{2},\frac{3\pi}{2}$$

$$=\frac{\sin(x-\mu)}{(\delta_1\alpha^2+\delta_2\alpha+\delta_3)\sec\mu};\ \mu\neq\frac{\pi}{2},\frac{3\pi}{2}. \qquad \text{... (3.20)}$$

where δ_i's $(i=1,2,3)$ are constants involving ρ_1 and ρ_2.

Note that the numerator of (3.20) is a bounded function of x and the denominator of (3.20) is a product of two linear functions of α which are both non-zero in the closed interval [0, 1]. Since the denominator is positive and a continuous function of α in the closed interval [0, 1], it is bounded away from zero in [0, 1]. Hence,

$$\gamma^*(T,\Im_3,S) = \sup_{\substack{0\le\alpha<1,\\ 0<m<\kappa<M}} \sup_{0\le x<2\pi}\left\{\frac{\sin(x-\mu)}{(\nu_1\alpha^2+\nu_2\alpha+\nu_3)\sec\mu}\right\} < \infty. \qquad \text{... (3.21)}$$

This proves that the directional mean is SB-robust at the family $\Im_3$.

Hence the theorem.

3.3 Robustness of the Circular Trimmed Mean

We have seen in Theorem 3.1 that directional mean is B-robust but not SB-robust. In this section we give a definition of γ-circular trimmed mean and prove that it is SB-robust for the family of distributions $\Im=\{CN(\mu,\kappa):\kappa>0\}$ (Theorem 3.3).

Definition 2: Suppose θ is a circular random variable with p.d.f $f(\theta)$ and $0\le\gamma<0.5$ is fixed. Let α, β be two points on the unit circle satisfying

(i) $\int_\beta^\alpha f(\theta)\,d\theta =$ 1-2γ, and

(ii) $d_1(\alpha,\beta) \le d_1(\mu,\nu)$ for all μ,ν satisfying $\int_{\nu}^{\mu} f(\theta)\, d\theta =$ 1-2γ, where $d_1(\phi,\xi)$ is the length of the arc starting from ξ and ending at ϕ traversed in the anticlockwise direction.

Then the circular γ - trimmed mean (γ -CTM) is defined as

$$\mu_\gamma \; = \; \arg\left[\frac{1}{(1-2\gamma)}\int_{\beta}^{\alpha} e^{i\theta} f(\theta)\, d\theta\right] \qquad \ldots (3.22)$$

where γ is the trimming proportion.

Theorem 3.3: Let $0 \le \gamma < 0.5$. *The* γ-CTM (μ_γ) *is SB-robust at the family of distributions* $\Im = \{CN(\mu,\kappa) : \kappa > 0\}$ *when the measure of dispersion is* $S(F) = E_{\gamma,F}(d(\theta,\mu))$ *where* $F \in \Im$.

The following Lemma 4 is used to prove above theorem.

Lemma 4: *Suppose* $\theta \sim CN(\mu,\kappa)$. *Then* $\rho_\gamma = c\left[S_{1,\mu} + \sum_{p=2}^{\infty} A_p(\kappa)\left(\frac{S_{p+1,\mu}}{(p+1)} + \frac{S_{p-1,\mu}}{(p-1)}\right)\right]$ *and*

$S(F) = \frac{1}{2\pi(1-2\gamma)}\left[\alpha^2(\kappa) + 4\sum_{p=1}^{\infty} A_p(\kappa)\left\{\frac{\alpha(\kappa)S_{p,0}}{p} + \frac{C_{p,0}-1}{p^2}\right\}\right]$ *where F is the c.d.f of* $CN(\mu,\kappa)$, $S_{p,\nu} = \sin[p(\alpha(\rho)-\nu)]$, $C_{p,\nu} \; = \; \cos[p(\alpha(\kappa)-\nu)]$ *and* $c = \cos\mu(\pi(1-2\gamma)^{-1})$.

Proof:

Note that α,β depend on κ and in what follows we will make this dependence explicit by writing them as $\alpha(\kappa)$ and $\beta(\kappa)$ respectively. Then by definition, we have,

$$\rho_\gamma = E_{\gamma,F}(\cos\theta) = \frac{1}{2\pi I_0(\kappa)(1-2\gamma)}\int_{\beta(\kappa)}^{\alpha(\kappa)} \cos\theta\, e^{\kappa\cos(\theta-\mu)} d\theta .$$

Let $(\theta-\mu) = \upsilon$. Then $\rho_\gamma = 2c_1 \cos\mu \int_{0}^{\alpha(\kappa)-\mu} \cos\upsilon\, e^{\kappa\cos\upsilon} d\upsilon$ where $c_1 = [2\pi I_0(\kappa)(1-2\gamma)]^{-1}$.

Now using $e^{\kappa\cos\upsilon} = I_0(\kappa) + 2\sum_{p=1}^{\infty} I_p(\kappa)\cos p\upsilon$ we get

$$\rho_\gamma = 2c_1\cos\mu\left[I_0(\kappa)\int_0^{\alpha(\kappa)-\mu}\cos\upsilon\, d\upsilon + 2\sum_{p=1}^{\infty} I_p(\kappa)\int_0^{\alpha(\kappa)-\mu}\cos\upsilon\cos p\upsilon d\upsilon\right]. \qquad \ldots (3.23)$$

But using $\int\cos\upsilon\cos p\upsilon d\upsilon = (p^2-1)^{-1}\{p\cos\upsilon\sin p\upsilon - \cos p\upsilon\sin\upsilon\}$ and simplifying we get $\int_0^{\alpha(\kappa)-\mu}\cos\upsilon\cos p\upsilon d\upsilon = \frac{1}{2}\left\{\frac{S_{p+1,\mu}}{(p+1)} + \frac{S_{p-1,\mu}}{(p-1)}\right\}$. Hence from (3.23) we get

$$\rho_\gamma = \frac{\cos\mu}{\pi(1-2\gamma)}\left[S_{1,\mu} + \sum_{p=2}^{\infty} A_p(\kappa)\left\{\frac{S_{p+1,\mu}}{(p+1)} + \frac{S_{p-1,\mu}}{(p-1)}\right\}\right]. \qquad \ldots (3.24)$$

Since $S(F) = E_{\gamma,F}(d(\theta,\mu))$ does not depend on μ, we can without loss of generality assume μ=0 for computing S(F). Using the substitution $(2\pi-\theta)=\lambda$ we have,

$$S(F) = E_{\gamma,F}(d(\theta)) = \frac{1}{2\pi(1-2\gamma)I_0(\kappa)}\left[\int_0^{\alpha(\kappa)}\theta\, e^{\kappa\cos\theta}d\theta + \int_{\beta(\kappa)}^{\alpha(\kappa)}(2\pi-\theta)\, e^{\kappa\cos\theta}d\theta\right]$$
$$= \frac{1}{\pi(1-2\gamma)I_0(\kappa)}\int_0^{\alpha(\kappa)}\lambda\, e^{\kappa\cos\lambda}d\lambda.$$

Now using the identity $e^{\kappa\cos\lambda} = I_0(\kappa) + 2\sum_{p=1}^{\infty} I_p(\kappa)\cos p\lambda$ and simplifying we get

$$S(F) = \frac{1}{\pi(1-2\gamma)I_0(\kappa)}\left[I_0(\kappa)\int_0^{\alpha(\kappa)}\lambda\, d\lambda + 2\sum_{p=1}^{\infty} I_p(\kappa)\int_0^{\alpha(\kappa)}\lambda\cos p\lambda d\lambda\right]$$
$$= \frac{1}{2\pi(1-2\gamma)}\left[\alpha^2(\kappa) + 4\sum_{p=1}^{\infty} A_p(\kappa)\left\{\frac{\alpha(\kappa)\sin[p\alpha(\kappa)]}{p} + \frac{\cos[p\alpha(\kappa)]-1}{p^2}\right\}\right].$$

Thus, $S(F) = \frac{1}{2\pi(1-2\gamma)}\left[\alpha^2(\kappa) + 4\sum_{p=1}^{\infty} A_p(\kappa)\left\{\frac{\alpha(\kappa)S_{p,0}}{p} + \frac{C_{p,0}-1}{p^2}\right\}\right]$. … (3.25)

Hence the lemma.

Proof of the theorem:

Let $F \in \Im$ and $T_\gamma(F) = \arctan^*\left(\frac{E_{\gamma,F}(\sin\theta)}{E_{\gamma,F}(\cos\theta)}\right)$ be the estimating functional for μ_γ.

Define $\mu_{\gamma,\varepsilon} = T_\gamma\{(1-\varepsilon)F + \varepsilon\delta_x\}$. Then,

$$\mu_{\gamma,\varepsilon} = \begin{cases} \arg\left[(1-\varepsilon)\int_\beta^\alpha e^{i\theta} f(\theta)d\theta + \varepsilon\, e^{ix}\right]; & \text{if} \quad x \in (\beta,\alpha) \\ \arg\left[(1-\varepsilon)\int_\beta^\alpha e^{i\theta} f(\theta)d\theta\right]; & \text{if} \quad x \notin (\beta,\alpha) \end{cases} \qquad \text{… (3.26)}$$

where (β,α) is the arc starting at β and ending at α traversed in the anticlockwise direction. The above relation (3.26) can be written as:

$$\mu_{\gamma,\varepsilon} = \begin{cases} \arg\{(1-\varepsilon)E_{\gamma,F}(\cos\theta) + \varepsilon\cos x + i\,((1-\varepsilon)E_{\gamma,F}(\sin\theta) + \varepsilon\sin x)\} & \text{if} \quad x \in (\beta,\alpha) \\ \arg\{(1-\varepsilon)E_{\gamma,F}(\cos\theta) + i\,((1-\varepsilon)E_{\gamma,F}(\sin\theta))\} & \text{if} \quad x \notin (\beta,\alpha) \end{cases}.$$

Therefore,

$$\tan\mu_{\gamma,\varepsilon} = \begin{cases} \dfrac{(1-\varepsilon)E_{\gamma,F}(\sin\theta) + \varepsilon\sin x}{(1-\varepsilon)E_{\gamma,F}(\cos\theta) + \varepsilon\cos x} & \text{if } x \in (\beta,\alpha) \\ \dfrac{E_{\gamma,F}(\sin\theta)}{E_{\gamma,F}(\cos\theta)} & \text{if } x \notin (\beta,\alpha) \end{cases}$$

$$\Rightarrow \mu_{\gamma,\varepsilon} = \begin{cases} \arctan^*\left[\dfrac{\rho_\gamma(1-\varepsilon)\tan\mu_\gamma + \varepsilon\sin\, x}{\rho_\gamma(1-\varepsilon) + \varepsilon\cos\, x}\right] & \text{if} \quad x \in (\beta,\alpha) \\ \mu_\gamma & \text{if} \quad x \notin (\beta,\alpha) \end{cases}$$

where $\rho_\gamma = E_{\gamma,F}(\cos\theta)$. Thus, as in Theorem 3.1 we can show that

$$IF(x;\, T_\gamma, F) = \begin{cases} 0 & \text{if } x \notin (\beta,\alpha) \\ \dfrac{\sin(x-\mu_\gamma)}{\rho_\gamma\ \sec\mu_\gamma} & \text{if } x \in (\beta,\alpha) \end{cases} \qquad \text{… (3.27)}$$

where $\mu_\gamma \neq \frac{\pi}{2}, \frac{3\pi}{2}$ and $\rho_\gamma > 0$. Hence by using Lemma 4 and (3.27) we get

$$\text{SIF}(x;\, T_\gamma,\, F,\, S\,) = \frac{\sin(x - \mu_\gamma)\, I(x)}{\rho_\gamma S(F)} \quad \text{where } I(x) = \begin{cases} 1 & \text{if } x \in (\beta(\kappa), \alpha(\kappa)) \\ 0 & \text{otherwise} \end{cases} . \quad \ldots (3.28)$$

We can show that s.g.e.s is bounded with respect to the dispersion functional S at F by directly looking at the integrals of ρ_γ and $S(F)$ for both $\kappa \to 0$ and $\kappa \to \infty$ as follows. We know that, as $\kappa \to 0, I_0(\kappa) \to 1$ and the circular normal distribution tends to circular uniform distribution with density function $f(\theta) = (2\pi)^{-1}, 0 \le \theta < 2\pi$. Therefore,

$\frac{1}{2\pi} \lim_{\kappa \to 0} \frac{1}{I_0(\kappa)} \int_0^{\alpha(\kappa)} e^{\kappa\cos(\theta-\mu)} d\theta = \frac{(1-2\gamma)}{2} \Rightarrow \alpha(0) = \pi(1-2\gamma)$. Since $\alpha(\kappa) + \beta(\kappa) = 2\pi$, we have $\beta(0) = \pi(1+2\gamma)$. Hence,

$$\lim_{\kappa \to 0} \rho_\gamma = [2\pi(1-2\gamma)]^{-1} \int_0^{\alpha(0)} \cos\theta \, d\theta = \frac{\sin[\pi(1-2\gamma)]}{\pi(1-2\gamma)}, \; 0 \le \gamma < 0.5 .$$

Now as $\kappa \to 0$, and letting $(2\pi - \theta) = \lambda$, we get,

$$\lim_{\kappa \to 0} S(F) = \frac{1}{\pi(1-2\gamma)} \int_0^{\alpha(0)} \lambda d\lambda = \frac{\pi(1-2\gamma)}{2} .$$

Also we note that the numerator of the expression on the right hand side of (3.28) is bounded. Hence,

$$\lim_{\kappa \to 0} \sup_{0 \le x < 2\pi} \text{SIF}(x;\, T_\gamma,\, \Im,\, S\,) < \infty . \quad \ldots (3.29)$$

Hill(1976) proved that when $\mu = 0$ and κ is large, $\alpha(\kappa)$ can be expanded asymptotically as

$$\alpha(\kappa) = \frac{\chi}{\sqrt{\kappa}} + \frac{\chi^3 + 3\chi}{24\,\kappa\sqrt{\kappa}} + \frac{3\chi^5 + 20\chi^3 + 45\chi}{640\,\kappa^2\sqrt{\kappa}} + \ldots$$

where $\chi = \Phi^{-1}(1-\gamma)$. Using the fact $\theta \sim CN(\mu,\kappa) \Rightarrow \theta - \mu (\text{mod}\, 2\pi) \sim CN(0,\kappa)$ we get $\alpha(\kappa) \to \mu$ as $\kappa \to \infty$. By symmetry of the circular normal distribution about μ we can also conclude that $\beta(\kappa) \to \mu$ as $\kappa \to \infty$. Thus for any $x \ne \mu,\; 0 \le x < 2\pi,$ there exists an M >0, such that if $\kappa > M,\; x \notin (\beta(\kappa), \alpha(\kappa))$. Therefore,

$$\lim_{\kappa \to \infty} \sup_{0 \le x < 2\pi} SIF(x; T_\gamma, F, S) = 0. \quad \ldots (3.30)$$

Thus, using (3.29) and (3.30), we can conclude that

$$\gamma^*(T_\gamma, \Im, S) = \sup_{\kappa > 0} \sup_{0 \le x < 2\pi} [SIF(x; T_\gamma, F, S)] < \infty.$$

Hence the theorem.

3.4 Equivalent Measures of Dispersion

The notion of SB-robustness heavily depends on the choice of the dispersion measure used. Thus an estimator T may be SB-robust at the family of distributions $\Im$ for one choice of dispersion measure while it may not be SB-robust at the family of distributions $\Im$ for another choice of dispersion measure. For example, the directional mean is SB-robust for μ at the family of distributions $\Im = \{CN(\mu, \kappa); \kappa > 0\}$ for the dispersion measure $S(F) = (\kappa A(\kappa))^{-\frac{1}{2}}$ (He and Simpson, 1992) but is not SB-Robust at the family of distributions $\Im$ for the dispersion measure $S(F) = (1 - A(\kappa))^{\frac{1}{2}}$ (Ko and Guttorp, 1988) and also for the dispersion measure $S(F) = E_F(d(\theta, \mu))$ (see Theorem 3.1 above). In this section we study this aspect of SB-robustness in some detail.

We begin with the definition of equivalent measures of dispersion:

Definition 3: Suppose S_1 and S_2 are two dispersion measures defined on the family of distributions $\Im$. Then S_1 and S_2 are said to be equivalent measures of dispersion for the family of distributions $\Im$ if $\sup_{F \in \Im} R(F)$ and $\sup_{F \in \Im} R^{-1}(F)$ are both finite, where $R(F) = S_1(F) S_2^{-1}(F)$.

We shall use the notation $S_1 \overset{\Im}{\sim} S_2$ to denote that S_1 and S_2 are equivalent measures of dispersion for the family of distributions $\Im$. Theorem 3.4 below is a consequence of the above definition and we prove that $\overset{\Im}{\sim}$ is an equivalence relation on the class of all dispersion measures for the family of distributions $\Im$ and the property of SB-robustness of the estimator is preserved when equivalent measures of dispersion for the family of distributions $\Im$ are considered.

Theorem 3.4: (a) $\overset{\Im}{\sim}$ *is an equivalence relation on the class of all dispersion measures for the family of distributions* $\Im$.

b) *Suppose* S_1 *and* S_2 *are two equivalent measures of dispersion for the family of distributions* $\Im$. *Suppose that the estimating functional T is SB-robust at the family of distributions* $\Im$ *when the measure of dispersion is* S_2. *Then, T is also SB-robust at the family of distributions* $\Im$ *when the measure of dispersion is* S_1.

Proof:

a) Let Ψ denote the set of all dispersion measures for the family of distributions $\Im$. It immediately follows from the above definition that $\overset{\Im}{\sim}$ is reflexive and symmetric. To prove $\overset{\Im}{\sim}$ is transitive suppose that $S_1, S_2, S_3 \in \Psi$, $S_1 \overset{\Im}{\sim} S_2$ and $S_2 \overset{\Im}{\sim} S_3$. Let $R_{ij}(F) = S_i(F)S_j^{-1}(F) \;\; ; \forall\, i < j \;, i, j = 1,\ 2,\ 3$. Note that $R_{13}(F) = R_{12}(F)R_{23}(F)$.

Thus, $\sup_{\Im} R_{13}(F) = \sup_{\Im}[R_{12}(F)\ R_{23}(F)] \le \sup_{\Im} R_{12}(F) \sup_{\Im} R_{23}(F) < \infty$, since both $R_{12}(F)$ and $R_{23}(F)$ are non negative. Therefore, $S_1 \overset{\Im}{\sim} S_3$ proving $\overset{\Im}{\sim}$ is transitive.

b) From the definition of SIF it follows that $SIF(x;T,F,S_1) = R^{-1}(F)\, SIF(x;T,F,S_2)$. Then,

$$\gamma^*(T,\Im,S_1) = \sup_{\Im} \sup_{x} [SIF(x;T,F,S_1)] = \sup_{\Im} R^{-1}(F) \sup_{x} [SIF(x;T,F,S_2)].$$

Since $\lambda(T,F,S_2) = \sup_x [SIF(x;T,F,S_2)]$, we have $\gamma^*(T,\Im,S_1) = \sup_{\Im} R^{-1}(F)\,\lambda(T,F,S_2)$. Again since S_1 and S_2 are two equivalent measures of dispersion for the family of distributions $\Im$ we have $\sup_{\Im} R^{-1}(F) = k < \infty$. Then,

$$\sup_{\Im} R^{-1}(F)\lambda(T,F,S_2) \le k \sup_{\Im} \lambda(T,F,S_2) \;\; \forall\, F \in \Im \;.$$

Since T is SB-robust at the family of distributions $\Im$ when the measure of dispersion is S_2 implies $\gamma^*(T,\Im,S_1) \le k\,\gamma^*(T,\Im,S_2) < \infty$.

Hence the theorem.

In Theorem 3.5 we prove the equivalence of dispersion measures discussed in Ko and Guttorp (1988), He and Simpson (1992) and Lemma 1 above for the families of distributions $\Im^* = \{CN(0,\kappa);\kappa > m\}$ and $\tilde{\Im} = \{CN(0,\kappa);\,\kappa > 0\}$.

Theorem 3.5: *(a) Consider the family of distributions* $\Im^* = \{CN(0,\kappa);\kappa > m\}$ *and define for* $F \in \Im^*$, $S_1(F) = \sqrt{1 - A(\kappa)}$, $S_2(F) = E_F(d(\theta,0))$ *and* $S_3(F) = (\kappa A(\kappa))^{-\frac{1}{2}}$*. Then* S_1, S_2 *and* S_3 *are equivalent measures of dispersion for the family of distributions* $\Im^*$*.*

(b) Now consider the family of ditributions $\tilde{\Im} = \{CN(0,\kappa);\,\kappa > 0\}$*. Then the following are true: (1)* $S_1 \overset{\tilde{\Im}}{\sim} S_2$ *(2)* $S_2 \overset{\tilde{\Im}}{\nsim} S_3$ *(3)* $S_1 \overset{\tilde{\Im}}{\nsim} S_3$.

Proof: **Part (a)**

(a) Let $S_1(F) = h(\kappa) = \sqrt{1 - A(\kappa)}$, $S_2(F) = g(\kappa) = \dfrac{1}{2\pi I_0(\kappa)}\displaystyle\int_0^{2\pi}(\pi - |\pi - \theta|)\,e^{\kappa\cos\theta}d\theta$ and $R(F) = r(\kappa) = \dfrac{h(\kappa)}{g(\kappa)}$. We know that for large κ, $A(\kappa)$ can be expanded asymptotically as

$$A(\kappa)=1-\frac{1}{2\kappa}-\frac{1}{8\kappa^2}-\frac{1}{8\kappa^3}-o(\kappa^{-3}) \qquad \text{... (3.31)}$$

(See Mardia & Jupp, 2000, p.40). Using (3.31) we get,

$$h(\kappa)=\frac{1}{\sqrt{\kappa}}\sqrt{\frac{1}{2}+\frac{1}{8\kappa}+\frac{1}{8\kappa^2}+o(\kappa^{-3})} \quad . \qquad \text{... (3.32)}$$

The $CN(0,\kappa)$ density $f(\theta)$ can be expanded around the standard normal density (ϕ) as $f(\alpha)=\phi(\alpha)\left[1+\frac{(\alpha^4-3)}{24\kappa}+\frac{(5\alpha^8-8\alpha^6-30\alpha^4-315)}{5760\kappa^2}+.......\right]$ where $\alpha=\theta\sqrt{\kappa}$ (Hill, 1976). Using this $g(\kappa)$ reduces to:

$$g(\kappa)=\frac{1}{\sqrt{\kappa}}\int_0^{\pi\sqrt{\kappa}}\alpha\,\phi(\alpha)\left[1+\frac{(\alpha^4-3)}{24\kappa}+o(\kappa^{-2})\right]d\alpha . \qquad \text{... (3.33)}$$

Therefore, using (3.32) and (3.33) $r(\kappa)$ can be written as

$$r(\kappa)=\frac{h(\kappa)}{g(\kappa)}=\frac{\sqrt{\frac{1}{2}+\frac{1}{8\kappa}+\frac{1}{8\kappa^2}+o(\kappa^{-3})}}{\int_0^{\pi\sqrt{\kappa}}\alpha\,\phi(\alpha)\left[1+\frac{(\alpha^4-3)}{24\kappa}+o(\kappa^{-2})\right]d\alpha} .$$

Then, a simple calculation shows that $\lim_{\kappa\to\infty} r(\kappa)=\sqrt{\pi}$. Hence, $\sup_{\Im^*} R(F)=\sup_{\kappa>m} r(\kappa)<\infty$

and $\sup_{\Im^*} R^{-1}(F)<\infty$. Thus, $S_1 \overset{\Im^*}{\sim} S_2$.

Now, let $S_2(F)=h(\kappa)=E_F(d(\theta,0))$ and $S_3(F)=g(\kappa)=(\kappa A(\kappa))^{-\frac{1}{2}}$. Using the asymptotic expansions of $A(\kappa)$ for large κ and the density $f(\theta)$, we get:

$$r(\kappa)=\frac{\int_0^{\pi\sqrt{\kappa}}\alpha\,\phi(\alpha)\left[1+\frac{(\alpha^4-3)}{24\kappa}+o(\kappa^{-2})\right]d\alpha}{\sqrt{1-\frac{1}{2\kappa}-\frac{1}{8\kappa^2}-\frac{1}{8\kappa^3}--o(\kappa^{-4})}} .$$

A simple calculation shows $\lim_{\kappa\to\infty} r(\kappa) = \frac{1}{\sqrt{2\pi}}$. Hence, $\sup_{\Im^*} R(F) = \sup_{\kappa>m} r(\kappa) < \infty$ and $\sup_{\Im^*} R^{-1}(F) < \infty$. Hence $S_2 \overset{\Im^*}{\sim} S_3$. Since $\overset{\Im^*}{\sim}$ is an equivalence relation (Theorem 5.1above) we have $S_1 \overset{\Im^*}{\sim} S_3$.

Part (b)

1) Let $h(\kappa), g(\kappa)$ and $r(\kappa)$ be as defined in the proof of part (a) above. By simple calculations we have $\lim_{\kappa\to\infty} r(\kappa) = \sqrt{\pi}$ and $\lim_{\kappa\to 0} r(\kappa) = \frac{2}{\pi}$. Thus $\sup_{\kappa>0} r(\kappa)$ and $\sup_{\kappa>0} r^{-1}(\kappa)$are both finite implying that $S_1 \overset{\tilde{\Im}}{\sim} S_2$.

2) Now as $\kappa \to 0$, $h(\kappa) \to \frac{\pi}{2}$ and $g(\kappa) \to \infty$. Therefore, $\sup_{\kappa>0} r(\kappa) = 0$ but $\sup_{\kappa>0} r^{-1}(\kappa) = \infty$.

Hence, $S_2 \overset{\tilde{\Im}}{\nsim} S_3$.

3) Suppose $S_1 \overset{\tilde{\Im}}{\sim} S_3$. Since $S_1 \overset{\tilde{\Im}}{\sim} S_2$ and $\overset{\tilde{\Im}}{\sim}$ is an equivalence relation we have $S_2 \overset{\tilde{\Im}}{\sim} S_3$ which is a contradiction to the fact that $S_2 \overset{\tilde{\Im}}{\nsim} S_3$ shown above. Hence, $S_1 \overset{\tilde{\Im}}{\nsim} S_3$.

Hence the theorem.

3.5 Comparison of Different Dispersion Measures

We have numerically evaluated the three dispersion measures S_1, S_2 and S_3 for different values of κ which are tabulated in Table 1. A graphical comparison of the above three measures of dispersion is also provided in Figure 1. Figure 1 given below is based on Table 1.

Table1: Comparison of dispersion measures.

κ	$A(\kappa)$	$(1-A(\kappa))^{1/2}$	$(\kappa A(\kappa))^{-1/2}$	$E_F(d(\theta))$
0.25	0.12	0.94	5.68	1.41
0.50	0.24	0.87	2.87	1.26
0.75	0.35	0.81	1.95	1.12
1.00	0.45	0.74	1.50	1.00
2.00	0.70	0.55	0.85	0.67
3.00	0.80	0.45	0.65	0.51
4.00	0.86	0.37	0.54	0.43
5.00	0.89	0.33	0.47	0.38
6.00	0.91	0.30	0.43	0.34
7.00	0.93	0.27	0.39	0.31
8.00	0.94	0.25	0.37	0.29
9.00	0.94	0.24	0.34	0.27
10.00	0.95	0.23	0.32	0.26
12.00	0.96	0.21	0.30	0.23
14.00	0.96	0.19	0.27	0.22
16.00	0.97	0.18	0.25	0.20
18.00	0.97	0.17	0.24	0.19
20.00	0.97	0.16	0.23	0.18
25.00	0.98	0.14	0.20	0.16
30.00	0.98	0.13	0.18	0.15
40.00	0.99	0.11	0.16	0.13
50.00	0.99	0.10	0.14	0.11
60.00	0.99	0.09	0.13	0.10
70.00	0.99	0.08	0.12	0.10
80.00	0.99	0.08	0.11	0.09
90.00	0.99	0.07	0.11	0.08
100.00	1.00	0.07	0.10	0.08

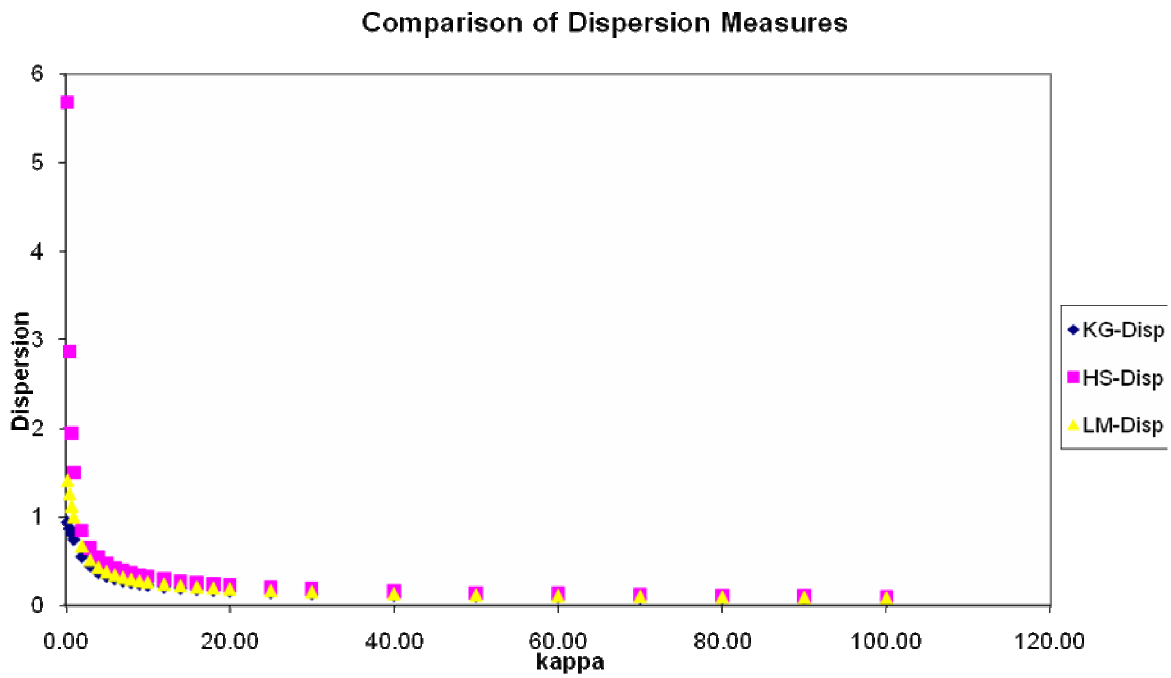

Figure 1: Comparison of the three measures of dispersion: $S_1 = \sqrt{1 - A(\kappa)}$ (KG-Disp), $S_2 = E_F(d(\theta))$ (LM-Disp) and $S_3 = (\kappa A(\kappa))^{-\frac{1}{2}}$ (HS-Disp).

From Figure 1 it can be seen that the three measures behave similarly for large κ, but for values of κ close to 0 the behaviour of the dispersion measure S_3 is very different from that of S_1 and S_2 which can also be observed from the above Table 1. This intuitively explains why the directional mean is SB-robust at the family of distributions $\Im = \{CN(\mu, \kappa) : \kappa > 0\}$ when the measure of dispersion is $(\kappa A(\kappa))^{-\frac{1}{2}}$ and not SB-robust at the family of distributions $\Im$ when the measure of dispersion is $\sqrt{1 - A(\kappa)}$ or $E_F(d(\theta, \mu))$.

Chapter 4

Robust Estimator for Concentration Parameter of Circular Normal Distribution

4.1 Introduction

In this chapter we discuss robust estimation of the concentration parameter (κ) of the circular normal (CN) distribution. A circular random variable Θ is said to have a von-Mises or CN distribution with mean direction μ and concentration κ if it has the probability density function (p.d.f):

$$f(\theta;\mu,\kappa)=\frac{1}{2\pi I_0(\kappa)}e^{\kappa\cos(\theta-\mu)}, 0\le\theta<2\pi \text{ where } 0\le\mu<2\pi \text{ and } \kappa>0$$

where $I_0(\kappa)$ is the modified Bessel function of the first kind and order zero. It is known that the m.l.e of the concentration parameter is not B-robust at the family of all circular normal distributions with fixed mean direction (μ) and varying $\kappa > 0$. We show that the usual estimator $K(F)=A^{-1}(\rho_F)$ of the concentration parameter is not SB-robust with respect to the dispersion measure $S(F)=E_F(d(\theta,\mu))$ where $\rho_F=\sqrt{E_F^2(\cos\theta)+E_F^2(\sin\theta)}=A(\kappa)$ and $A(\kappa)=I_0^{-1}(\kappa)I_1(\kappa)$. We next show that $T(F)=g^{-1}[E_F(d(\theta,\mu)]$ is B-robust but is not SB-robust estimator of κ. We propose a new estimator (see Section 4.3 for definition) for κ and show that it is B-robust and SB-robust at the family of distributions $\{CN(\mu,\kappa): m\le\kappa\le M\}$ where m and M are two arbitrary constants. We also obtained the limiting cases of the dispersion measures $S(F)=E_{\gamma,F}(d(\theta,\mu))$ as both $\gamma\to 0$ and $\gamma\to 0.5$ which are given in the form of Lemma 4.

Ko (1992) suggested that a reasonable choice of $S(F)$ is the inverse of Fisher information or equivalently the Cramer-Rao lower bound for the standard error of the estimating functional T. For $F = CN(\mu, \kappa)$ the inverse of the Fisher information for the concentration parameter κ is $(A'(\kappa))^{-1/2}$. In Lemma 5 we show that this choice of $S(F)$ is not a good one as it is not a dispersion measure. Thus, the claim that the estimator $K_m(F)$ (see chapter 2, p. 17) is SB-robust at the family of distributions $\Im$ is not appropriate though the result is technically not wrong since the definition of SB-robustness does not require the functional S to be a dispersion measure. In this chapter we have developed five lemmas and three theorems on the robust estimator for concentration parameter of circular normal distribution.

This chapter is organised as follows. In Section 4.2 we discuss the robustness of two estimators of κ and in Section 4.3 we propose a new SB-robust estimator for κ.

4.2 Robustness of the Estimator for the Concentration Parameter

As mentioned in chapter 1 Ko and Guttorp (1988) has shown that the traditional estimator for κ, $A^{-1}(\rho_F)$, is not SB-robust at the family of distributions $\Im = \{CN(\mu, \kappa) : \kappa > 0\}$ with respect to the dispersion measure $S(F) = (1 - \rho_F)^{1/2}$. In Theorem 4.1 below we show that $A^{-1}(\rho_F)$, is not SB-robust at the family of distributions $\Im$ with respect to the dispersion measure $S(F) = E_F(d(\theta, \mu))$ where $d(\theta, \mu)$ is defined by (3.1) in chapter 3.

***Theorem 4.1**: Suppose $\Theta \sim CN(\mu, \kappa)$ where $\kappa > 0$ and μ is the mean direction. Then $T(F) = A^{-1}(\rho_F)$ is an estimating functional for κ which is not SB-robust at the family of distributions $\Im = \{CN(\mu, \kappa) : \kappa > 0\}$ with respect to $S(F) = E_F(d(\theta, \mu))$ where $F \in \Im$.*

Proof:

Since $\Theta \sim CN(\mu,\kappa)$ and μ is the mean direction, we can write

$$\mu = \int e^{i\theta} dF(\theta) = E_F(\cos\theta) + iE_F(\sin\theta) \text{ and}$$

$$\rho_F = \sqrt{E_F^2(\cos\theta) + E_F^2(\sin\theta)} = A(\kappa) = \frac{I_1(\kappa)}{I_0(\kappa)}.$$

Let $T(F)$ be an estimating function for κ. Then we can write $\kappa = T(F) = A^{-1}(\rho_F)$.

Let $\kappa_\varepsilon = T\{(1-\varepsilon)F + \varepsilon\delta_x\}$. Then we can write:

$$\kappa_\varepsilon = A^{-1}\left(\sqrt{E^2_{(1-\varepsilon)F+\delta_x}(\cos\theta) + E^2_{(1-\varepsilon)F+\delta_x}(\sin\theta)}\right) \qquad \text{... (4.1)}$$

Using Lemma 2 in chapter 3, (4.1) reduces to

$$\kappa_\varepsilon = A^{-1}\left(\sqrt{\rho_F^2 - 2\rho_F\varepsilon(\rho_F - \cos(x-\mu)) + \varepsilon^2(\rho_F^2 - 2\rho_F\cos(x-\mu) + 1)}\right).$$

Now, by L'Hôpital's rule we have $IF(x;T,F) = \lim\limits_{\varepsilon\to 0}\left(\dfrac{\kappa_\varepsilon - \kappa}{\varepsilon}\right) = \lim\limits_{\varepsilon\to 0}\dfrac{d}{d\varepsilon}(\kappa_\varepsilon)$. ... (4.2)

Using $A^{-1'}(y) = \left[A'(A^{-1}(y))\right]^{-1}$ and $A'(\kappa) = 1 - A^2(\kappa) - \kappa^{-1}A(\kappa)$, we get from (4.2)

$$IF(x;T,F) = \lim_{\varepsilon\to 0}\frac{d}{d\varepsilon}(\kappa_\varepsilon) = \left[\frac{A^{-1}(\rho_F)(\cos(x-\mu) - \rho_F)}{A^{-1}(\rho_F)(1-\rho_F^2) - \rho_F}\right]. \qquad \text{... (4.3)}$$

By Lemma 1 in chapter 3, $S(F) = E_F(d(\theta,\mu))$ is a dispersion measure on the circle where $\Theta \sim CN(\mu,\kappa)$ and again by using Lemma 2 in chapter 3 we get,

$$S(F) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{A_{2n+1}(\kappa)}{(2n+1)^2} \text{ where } A_p(\kappa) = \frac{I_p(\kappa)}{I_0(\kappa)} \ \forall\, p > 0. \qquad \text{... (4.4)}$$

Hence by using (4.3) and (4.4) the standardized influence function (SIF) of T with respect to the dispersion functional S is given by

$$\mathrm{SIF}(x;T,F,S)=\left[\frac{A^{-1}(\rho_F)\,(\cos(x-\mu)-\rho_F)\left\{\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{A_{2n+1}(\kappa)}{(2n+1)^2}\right\}^{-1}}{\left\{(1-\rho_F^2)A^{-1}(\rho_F)-\rho_F\right\}}\right].$$

and the s.g.e.s of T is given by $\gamma^*(T,\Im,S)=\sup_{\kappa>0}\sup_{x}[\mathrm{SIF}(x;T,F,S)]$.

Now as $\kappa\to\infty$, both $A(\kappa)\to 1$, $A_{2n+1}(\kappa)\to 1$ for every n so that $\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{A_{2n+1}(\kappa)}{(2n+1)^2}\to\frac{\pi}{2}$ and hence $S(F)\to 0$ and hence $\gamma^*(T,\Im,S)$ is not finite which implies that standardized influence function is not bounded.

Hence the theorem.

Without loss of generality, let us assume $\Theta\sim CN(\mu,\kappa)$. In Theorem 4.2 below we propose a new estimator of κ, $T(F)=g^{-1}[E_F(d(\theta))]$ where $g(\kappa)=E_F(d(\theta))$ and show that this new estimator also is not SB-robust with respect to the dispersion measure $S(F)=E_F(d(\theta))$.

Theorem 4.2: *Let* $\Theta\sim CN(\mu,\kappa)$. *Define* $d(\theta)=\pi-|\pi-\theta|$ *and* $g(\kappa)=E_F(d(\theta))$. *Then* $T(F)=g^{-1}[E_F(d(\theta))]$ *is B-robust but is not SB-robust for* κ *at the family of distributions* $\Im=\{CN(0,\kappa):\kappa>0\}$ *with respect to* $S(F)=E_F(d(\theta))$ *where* $F\in\Im$.

The following Lemma 1 is used to prove the above theorem.

Lemma 1: *Let* $\Theta\sim CN(\mu,\kappa)$ *and define* $g(\kappa)=E_F(d(\theta))$ *where* $d(\theta)=\pi-|\pi-\theta|$. *Then*

$$g'(\kappa)=-\left[\frac{1}{\pi}+\frac{2}{\pi}\left(\sum_{n=0}^{\infty}\frac{A_{2n+2}(\kappa)}{(2n+1)^2}+\sum_{n=1}^{\infty}\frac{A_{2n}(\kappa)}{(2n+1)^2}\right)\right]-A(\kappa)g(\kappa).$$

Proof:

Since $S(F)=g(\kappa)=E_F(d(\theta,\mu))$ does not depend on μ we can without loss of generality assume $\mu=0$ for computing $g(\kappa)$. Hence $g(\kappa)=E_F(d(\theta))$. Now for $\Theta \sim CN(0,\kappa)$, by Lemma 2 in chapter 3 we have,

$$g(\kappa)=\frac{1}{2\pi I_0(\kappa)}\int_0^{2\pi}(\pi-|\pi-\theta|)e^{\kappa\cos\theta}d\theta=\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{A_{2n+1}(\kappa)}{(2n+1)^2}.$$

Differentiating with respect to κ we get

$$g'(\kappa)=\frac{1}{2\pi}\left[\frac{I_0(\kappa)\int_0^{2\pi}(\pi-|\pi-\theta|)\cos\theta\, e^{\kappa\cos\theta}\,d\theta-I_1(\kappa)\int_0^{2\pi}(\pi-|\pi-\theta|)\,e^{\kappa\cos\theta}d\theta}{(I_0(\kappa))^2}\right]$$

$$=\frac{1}{2\pi I_0(\kappa)}\int_0^{2\pi}(\pi-|\pi-\theta|)\cos\theta\, e^{\kappa\cos\theta}\,d\theta - A(\kappa)g(\kappa).$$

By making the substitution $(2\pi-\theta)=\lambda$ and using $e^{\kappa\cos\theta}=I_0(\kappa)+2\sum_{p=1}^{\infty}I_p(\kappa)\cos p\theta$ (Abramowitz & Stegun, 1965, p.376, 9.6.34) $g'(\kappa)$ can be written as

$$g'(\kappa)=\frac{1}{\pi I_0(\kappa)}\left[I_0(\kappa)\int_0^{\pi}\lambda\cos\lambda d\lambda+2\sum_{p=1}^{\infty}I_p(\kappa)\int_0^{\pi}\lambda\cos\lambda\cos p\lambda\, d\lambda\right]-A(\kappa)g(\kappa)$$

$$=\frac{1}{\pi I_0(\kappa)}\left[-I_0(\kappa)+\sum_{p=1}^{\infty}I_p(\kappa)\int_0^{\pi}\lambda\{\cos(p+1)\lambda+\cos(p-1)\lambda\}\,d\lambda\right]-A(\kappa)g(\kappa)$$

$$=\frac{1}{\pi I_0(\kappa)}\left[-I_0(\kappa)+\sum_{p=1}^{\infty}I_p(\kappa)\int_0^{\pi}\lambda\cos(p+1)\lambda d\lambda+\sum_{p=2}^{\infty}I_p(\kappa)\int_0^{\pi}\lambda\cos(p-1)\lambda d\lambda\right]-A(\kappa)g(\kappa)$$

$$=\frac{1}{\pi I_0(\kappa)}\left[-I_0(\kappa)+\sum_{p=2}^{\infty}I_p(\kappa)\left\{\frac{\cos(p+1)\pi-1}{(p+1)^2}+\frac{\cos(p-1)\pi-1}{(p-1)^2}\right\}\right]-A(\kappa)g(\kappa),\text{ since }\sin p\pi=0\ \forall p$$

$$=\frac{1}{\pi I_0(\kappa)}\left[-I_0(\kappa)-2\left\{\sum_{n=0}^{\infty}\frac{I_{2n+2}(\kappa)}{(2n+1)^2}+\sum_{n=1}^{\infty}\frac{I_{2n}(\kappa)}{(2n+1)^2}\right\}\right]-A(\kappa)g(\kappa),\text{ since }\cos p\pi=-1\ \forall p$$

$$=-\left[\frac{1}{\pi}+\frac{2}{\pi}\left(\sum_{n=0}^{\infty}\frac{A_{2n+2}(\kappa)}{(2n+1)^2}+\sum_{n=1}^{\infty}\frac{A_{2n}(\kappa)}{(2n+1)^2}\right)\right]-A(\kappa)g(\kappa).$$

Hence the lemma.

Proof of the theorem:

Let $\kappa_\varepsilon = T\{(1-\varepsilon)F + \varepsilon\delta_x\}$. Then, we can write

$$\begin{aligned}\kappa_\varepsilon = T\{(1-\varepsilon)F + \varepsilon\delta_x\} &= g^{-1}[(1-\varepsilon)E_F\{d(\theta)\} + \varepsilon d(x)] \\ &= g^{-1}[g(\kappa) + \varepsilon(d(x) - g(\kappa))].\end{aligned}$$

Using the Taylor series expansion of g^{-1} around the point $g(\kappa)$, we have

$$\begin{aligned}\kappa_\varepsilon &= g^{-1}(g(\kappa)) + \frac{\varepsilon(d(x)-g(\kappa))}{g'(g^{-1}(g(\kappa)))} + O(\varepsilon^2) \\ &= \kappa + \frac{\varepsilon(d(x)-g(\kappa))}{g'(\kappa)} + O(\varepsilon^2).\end{aligned} \qquad \ldots (4.5)$$

From (4.2) we get,

$$IF(x;T,F) = \lim_{\varepsilon\to 0}\left(\frac{\kappa_\varepsilon - \kappa}{\varepsilon}\right) = \left(\frac{d(x)-g(\kappa)}{g'(\kappa)}\right); g'(\kappa) \neq 0.$$

Note that as $\kappa \to \infty, A(\kappa) \to 1$, $A_{2n+2}(\kappa) \to 1$, $A_{2n}(\kappa) \to 1$, for every n. Thus from Lemma 1 we get $g(\kappa) \to 0$ and hence for large values of $\kappa, g'(\kappa) < 0$. Since $\gamma(T,F) = \sup_x | IF(x;T,F) | < \infty$, we have $g^{-1}[E_F(d(\theta))]$ is B-robust for the concentration parameter κ at the family of distributions $\Im$. Now let $S(F) = E_F(d(\theta))$. Since as $\kappa \to \infty, g(\kappa) \to 0$ and $g'(\kappa) < 0$, we get the s.g.e.s $\gamma^*(T,\Im,S)$ is not finite and hence $g^{-1}[E_F(d(\theta))]$ is not SB-robust for the concentration parameter κ at the family of distributions $\Im$.

Hence the theorem.

4.3 A New SB-robust Estimator for the Concentration Parameter

In Theorem 4.1 we see that the usual estimator for the concentration parameter for κ is not SB-robust at the family of distributions $\Im = \{CN(\mu,\kappa) : \kappa > 0\}$. In this section we propose a new estimator for the concentration parameter κ.

Definition 1: Let $f(\theta;\mu,\kappa)$ be the p.d.f. of $CN(\mu,\kappa)$ distribution and $\alpha(\kappa)$ and $\beta(\kappa)$ be symmetrically placed around μ such that $\int_{\beta(\kappa)}^{\alpha(\kappa)} f(\theta;\mu,\kappa)d\theta = 1-2\gamma$ where γ is the trimming proportion such that $\gamma\in[0,\ 0.5)$. Define,

$$g^*(\kappa) = E_{\gamma,F}(d(\theta,\mu)) = \int_{\beta(\kappa)}^{\alpha(\kappa)} d(\theta,\mu) f(\theta;\mu,\kappa)d\theta .$$

Then the new estimator for the concentration parameter κ is defined as

$$T_\gamma(F) = g^{*-1}\left[E_{\gamma,F}(d(\theta,\mu))\right]. \qquad \text{... (4.6)}$$

Without loss of generality we assume $\mu = 0$. Since the circular normal distribution is symmetric about $\mu = 0$ we have $\beta(\kappa) = 2\pi - \alpha(\kappa)$. Then $T_\gamma(F) = g^{*-1}\left[E_{\gamma,F}(d(\theta))\right]$ where $g^*(\kappa) = (1-2\gamma)^{-1}\int_{\beta(\kappa)}^{\alpha(\kappa)} d(\theta)\,dF$, γ is the trimming proportion such that $\gamma\in[0,\ 0.5)$ and $\alpha(\kappa)$ and $\beta(\kappa)$ are such that $(1-2\gamma)^{-1}\int_{\beta(\kappa)}^{\alpha(\kappa)} dF = 1$. In Theorem 4.3 below, we prove that $T_\gamma(F)$ is SB-robust at the family of distributions $\Im = \{CN(0,\kappa) : 0 < m \le \kappa \le M\}$ with respect to the dispersion measure $S(F) = E_{\gamma,F}(d(\theta))$. In Lemma 4 we discuss the limiting cases of S(F) as both $\gamma \to 0$ and $\gamma \to 0.5$ and in Lemma 5 we show that the choice of $S(F) = (A'(\kappa))^{-1/2}$ is not a good one as it is not a dispersion measure.

Theorem 4.3: *Let* $\Theta \sim CN(0,\kappa)$*. Define* $d(\theta) = \pi - |\pi - \theta|$ *and* $g^*(\kappa) = E_{\gamma,F}(d(\theta))$*. Then* $T_\gamma(F) = g^{*-1}\left[E_{\gamma,F}(d(\theta))\right]$ *is SB-robust at the family of distributions* $\Im = \{CN(0,\kappa) : 0 < m \le \kappa \le M\}$ *with respect to the dispersion measure* $S(F) = E_{\gamma,F}(d(\theta)) = (1-2\gamma)^{-1}\int_{\beta(\kappa)}^{\alpha(\kappa)} d(\theta)\,dF$.

The following Lemma 2 and Lemma 3 were used to prove the above theorem.

Lemma 2: *Let* $\Theta \sim CN(\mu, \kappa)$ *and define* $g^*(\kappa) = E_{\gamma,F}(d(\theta))$ *where* $d(\theta) = \pi - |\pi - \theta|$.

Then $g^{*'}(\kappa) = \dfrac{\alpha(\kappa)\alpha'(\kappa)e^{\kappa\cos\alpha(\kappa)}}{\pi(1-2\gamma)I_0(\kappa)} - A(\kappa)g^*(\kappa)$.

Proof:

Since $S(F) = g^*(\kappa) = E_{\gamma,F}(d(\theta,\mu))$ does not depend on μ we can without loss of generality assume $\mu = 0$ for computing $g^*(\kappa)$. Hence $g^*(\kappa) = E_{\gamma,F}(d(\theta))$. Using the substitution $(2\pi - \theta) = \lambda$ we have

$$g^*(\kappa) = \frac{1}{(1-2\gamma)}\int_{\beta(\kappa)}^{\alpha(\kappa)} d^*(\theta)dF(\theta) = \frac{1}{2\pi(1-2\gamma)I_0(\kappa)}\left[\int_0^{\alpha(\kappa)} \theta\, e^{\kappa\cos\theta}\, d\theta + \int_{\beta(\kappa)}^{2\pi}(2\pi-\theta)\; e^{\kappa\cos\theta}d\theta\right]$$
$$= \frac{1}{\pi(1-2\gamma)I_0(\kappa)}\int_0^{\alpha(\kappa)}\lambda\, e^{\kappa\cos\lambda}d\lambda.$$

Using Leibnitz rule for differentiation of a definite integral whose limits are functions of the variable with respect to which differentiation is being performed we get:

$$g^{*'}(\kappa) = \frac{1}{2\pi(1-2\gamma)(I_0(\kappa))^2}\left[I_0(\kappa)\{\alpha(\kappa)e^{\kappa\cos\alpha(\kappa)}\alpha'(\kappa)\} - \left\{\int_0^{\alpha(\kappa)}\theta\, e^{\kappa\cos\theta}\, d\theta\right\}I_0'(\kappa)\right].$$

But since $I_0'(\kappa) = I_1(\kappa)$ and $A(\kappa) = \dfrac{I_1(\kappa)}{I_0(\kappa)}$, we have

$$g^{*'}(\kappa) = \frac{\alpha(\kappa)\alpha'(\kappa)e^{\kappa\cos\alpha(\kappa)}}{\pi(1-2\gamma)I_0(\kappa)} - A(\kappa)g^*(\kappa).$$

Hence the lemma.

Lemma 3: Suppose $\kappa_1 > \kappa_2$. If x and y are such that $F_1(x) = F_2(y) = 0.5 - \gamma$ then $x<y$ where F_1 and F_2 are the distribution functions under $CN(\mu, \kappa_1)$ and $CN(\mu, \kappa_2)$.

Proof:

Case1: $0 < x, y < \pi/2$.

If $0 < \theta < \pi/2$, then we have $e^{\kappa_1 \cos\theta} > e^{\kappa_2 \cos\theta}$. Now suppose y < x. Then

$$\int_0^y e^{\kappa_1 \cos\theta} d\theta > \int_0^y e^{\kappa_2 \cos\theta} d\theta \;\Rightarrow \int_0^x e^{\kappa_1 \cos\theta} d\theta > \int_0^y e^{\kappa_2 \cos\theta} d\theta \Rightarrow (0.5-\gamma) > (0.5-\gamma)$$

which is a contradiction and hence $y \geq x$. If y = x, $(0.5-\gamma) > (0.5-\gamma)$ again a contradiction and hence x < y.

Case2: $x, y > \pi/2$ or $0 < y < \pi/2 < x < \pi$.

If $0 < \theta < \pi/2$, then we have $e^{\kappa_1 \cos\theta} > e^{\kappa_2 \cos\theta}$. This implies that:

$$\int_0^y e^{\kappa_1 \cos\theta} d\theta > \int_0^y e^{\kappa_2 \cos\theta} d\theta \;\Rightarrow \int_0^x e^{\kappa_1 \cos\theta} d\theta > \int_0^y e^{\kappa_2 \cos\theta} d\theta \Rightarrow (0.5-\gamma) > (0.5-\gamma).$$

Again a contradiction and hence $y < \pi/2 < x$ cannot happen.

Case3: $x > y$.

In this case we can write:

$$\int_0^x e^{\kappa_1 \cos\theta} d\theta = \int_0^{\pi/2} e^{\kappa_1 \cos\theta} d\theta + \int_{\pi/2}^x e^{\kappa_1 \cos\theta} d\theta \quad \text{and}$$

$$\int_0^y e^{\kappa_2 \cos\theta} d\theta = \int_0^{\pi/2} e^{\kappa_2 \cos\theta} d\theta + \int_{\pi/2}^y e^{\kappa_2 \cos\theta} d\theta$$

$$\Rightarrow C_1 + \int_{\pi/2}^x e^{\kappa_1 \cos\theta} d\theta = L\left(C_2 + \int_{\pi/2}^y e^{\kappa_2 \cos\theta} d\theta \right)$$

$$\Rightarrow C_1 - LC_2 = L\int_{\pi/2}^y e^{\kappa_2 \cos\theta} d\theta - \int_{\pi/2}^x e^{\kappa_1 \cos\theta} d\theta$$

$$\Rightarrow t = L\int_{\pi/2}^{y} e^{\kappa_2\cos\theta}d\theta - \left[\int_{\pi/2}^{y} e^{\kappa_1\cos\theta}d\theta + \int_{y}^{x} e^{\kappa_1\cos\theta}d\theta\right]$$

$$\Rightarrow \frac{-1}{I_0(\kappa_1)}\int_{y}^{x} e^{\kappa_1\cos\theta}d\theta = \frac{t}{I_0(\kappa_1)} + \frac{1}{I_0(\kappa_1)}\int_{\pi/2}^{y} e^{\kappa_1\cos\theta}d\theta - \frac{1}{I_0(\kappa_2)}\int_{\pi/2}^{y} e^{\kappa_2\cos\theta}d\theta$$

$$\Rightarrow \frac{-1}{I_0(\kappa_1)}\int_{y}^{x} e^{\kappa_1\cos\theta}d\theta = \frac{t}{I_0(\kappa_1)} + h(y) \qquad \text{... (4.7)}$$

where $t = C_1 - LC_2$, $L = \frac{I_0(\kappa_1)}{I_0(\kappa_2)}$, $C_1 = \int_0^{\pi/2} e^{\kappa_1\cos\theta}d\theta$ and $C_2 = \int_0^{\pi/2} e^{\kappa_2\cos\theta}d\theta$.

Now we have to show that the RHS of (4.7) is positive so that the contradiction will establish x < y. Also $t(I_0(\kappa_1))^{-1}$ is positive and hence it is sufficient to prove that h(y) is a strictly decreasing function. It is obvious from (4.7) that h(0) = 0. Since $\kappa_1 > \kappa_2 \Rightarrow I_0(\kappa_1) > I_0(\kappa_2)$ we have

$$\frac{-1}{I_0(\kappa_1)}\int_{\pi/2}^{y} e^{\kappa_2 \cos y}dy > \frac{-1}{I_0(\kappa_2)}\int_{\pi/2}^{y} e^{\kappa_2 \cos y}dy \Rightarrow h(y) > \frac{1}{I_0(\kappa_1)}\int_{\pi/2}^{y} e^{\kappa_1 \cos y}dy - \frac{1}{I_0(\kappa_2)}\int_{\pi/2}^{y} e^{\kappa_2 \cos y}dy .$$

Since $\cos y$ is negative in $(\pi/2, \pi)$ and for $\kappa_1 > \kappa_2$, $\left(e^{\kappa_1\cos y} - e^{\kappa_2\cos y}\right) < 0$, we have

$$h(y) < \frac{1}{I_0(\kappa_1)}\int_{\pi/2}^{y}\left(e^{\kappa_1\cos y} - e^{\kappa_2\cos y}\right)dy < 0 .$$

Hence the lemma.

Remark 1: Let $\alpha(\kappa)$ be defined as $F(\alpha(\kappa))$=0.5 – γ where F is the c.d.f of $CN(0,\kappa)$ distribution. Then from Lemma 3 we conclude that $\alpha(\kappa_1) < \alpha(\kappa_2)$ for $\kappa_1 > \kappa_2$ i.e. $\alpha(\kappa)$ is a decreasing function of κ and $\alpha'(\kappa) < 0$.

Proof of the theorem:

Let $g^*(\kappa) = E_{\gamma,F}\left(d^*(\theta)\right) = (1-2\gamma)^{-1}\int_{\beta(\kappa)}^{\alpha(\kappa)} d^*(\theta)dF$. Then we can write

$\kappa^* = T_\gamma(F) = g^{*-1}\left[E_{\gamma,F}\left(d^*(\theta)\right)\right]$. Therefore,

$$\kappa^*_\varepsilon = T_\gamma\{(1-\varepsilon)F + \varepsilon\delta_x\} = \begin{cases} g^{*-1}\left[(1-\varepsilon)E_{\gamma,F}\left(d^*(\theta)\right) + c\varepsilon\, d^*(x)\right] & ; x \in (\beta(\kappa), \alpha(\kappa)) \\ g^{*-1}\left[(1-\varepsilon)E_{\gamma,F}\left(d^*(\theta)\right)\right] & ; x \notin (\beta(\kappa), \alpha(\kappa)) \end{cases}$$

$$= \begin{cases} g^{*-1}\left[(1-\varepsilon)g^*(\kappa) + c\varepsilon\, d^*(x)\right] & ; x \in (\beta(\kappa), \alpha(\kappa)) \\ g^{*-1}\left[(1-\varepsilon)g^*(\kappa)\right] & ; x \notin (\beta(\kappa), \alpha(\kappa)) \end{cases}$$

where $c = [\pi(1-2\gamma)]^{-1}$.

When $x \notin (\beta(\kappa), \alpha(\kappa))$, using Taylor series expansion of g^{*-1} around $g^*(\kappa)$ we get

$$\kappa^*_\varepsilon = \kappa - \frac{\varepsilon\, g^*(\kappa)}{g^{*\prime}(\kappa)} + O(\varepsilon^2). \qquad \dots (4.8)$$

From (4.8) the influence function is given by:

$$IF(x;T_\gamma,F) = \frac{-g^*(\kappa)}{g^{*\prime}(\kappa)} \text{ when } x \notin (\beta(\kappa), \alpha(\kappa)). \qquad \dots (4.9)$$

When $x \in (\beta(\kappa), \alpha(\kappa))$, we have

$$\kappa^*_\varepsilon = \kappa + \frac{\varepsilon\left(c\, d^*(x) - g^*(\kappa)\right)}{g^{*\prime}(\kappa)} + O(\varepsilon^2). \qquad \dots (4.10)$$

From (4.10), the influence function is given by:

$$IF(x;T,F) = \frac{c\, d^*(x) - g^*(\kappa)}{g^{*\prime}(\kappa)} \text{ when } x \in (\beta(\kappa), \alpha(\kappa)). \qquad \dots (4.11)$$

Combining (4.9) and (4.11), we get

$$IF(x;T_\gamma,F) = \begin{cases} \left(c\, d^*(x) - g^*(\kappa)\right)\left[g^{*\prime}(\kappa)\right]^{-1} & ; x \in (\beta(\kappa), \alpha(\kappa)) \\ -g^*(\kappa)\left[g^{*\prime}(\kappa)\right]^{-1} & ; x \notin (\beta(\kappa), \alpha(\kappa)) \end{cases}. \qquad \dots (4.12)$$

Hence, $\gamma(T_\gamma, F) = \sup_x |IF(x; T_\gamma, F)| = \sup_{\kappa>0} \left\{ \frac{-g^*(\kappa)}{g^{*'}(\kappa)}, \frac{c\,\alpha(\kappa) - g^*(\kappa)}{g^{*'}(\kappa)} \right\}$.

Since $\gamma(T_\gamma, F)$ is independent of x, and $0 < m \le \kappa \le M$ we can conclude that $g^{*-1}\left[E_{\gamma,F}(d^*(\theta))\right]$ is B-robust at the family of distributions $\Im$.

Now let $S(F) = E_{\gamma,F}(d(\theta))$. Then using (4.12) the standardized influence function (SIF) can be written as:

$$SIF(x; T_\gamma, F, S) = \begin{cases} \dfrac{(c\,d^*(x) - g^*(\kappa))}{g^{*'}(\kappa) g^*(\kappa)} & ; x \in (\beta(\kappa), \alpha(\kappa)) \\ -\left(g^{*'}(\kappa)\right)^{-1} & ; x \notin (\beta(\kappa), \alpha(\kappa)) \end{cases}.$$

Since $g^*(\kappa)$ is strictly positive and $g^{*'}(\kappa)$ is strictly negative and bounded away from zero for $0 < m \le \kappa \le M$ by Lemma 2 and remark 1of Lemma 3, we can conclude that the s.g.e.s $\gamma^*(T_\gamma, \Im, S)$ is finite and hence $g^{*-1}\left[E_{\gamma,F}(d^*(\theta))\right]$ is a SB- robust estimator for the concentration parameter of the circular normal distribution.

Hence the theorem.

The following Lemma 4 gives the limiting cases of the dispersion measures $S(F) = E_{\gamma,F}(d(\theta,\mu))$ as both $\gamma \to 0$ and $\gamma \to 0.5$.

Lemma 4: *(a) As* $\gamma \to 0.5$, $E_{\gamma,F}(d(\theta)) \to 0$ *and (b) As* $\gamma \to 0, E_{\gamma,F}(d(\theta)) \to E_F(d(\theta))$.

Proof:

(a) We have $E_{\gamma,F}(d(\theta)) = \dfrac{1}{2\pi I_0(\kappa)(1-2\gamma)} \displaystyle\int_{\beta(\kappa)}^{\alpha(\kappa)} (\pi - |\pi - \theta|) e^{\kappa\cos\theta} d\theta$ ... (4.13) where $\beta(\kappa) = 2\pi - \alpha(\kappa)$. We write $\alpha(\kappa) = \alpha_\kappa(\gamma)$ since for fixed κ, $\alpha(\kappa)$ depends on γ and we want to make the dependence on γ explicit. Then we can write:

$$E_{\gamma,F}(d(\theta)) = \frac{2c}{(1-2\gamma)} \int_0^{\alpha_\kappa(\gamma)} \theta\, e^{\kappa\cos\theta} d\theta \quad \text{where } c = [\pi I_0(k)]^{-1}$$

$$\cong \frac{2c}{(1-2\gamma)} \alpha_\kappa^2(\gamma)\; e^{\kappa\cos\alpha_\kappa(\gamma)}, \text{since for small x } \int_0^x f(x)\,dx \;\cong\; x^2 f(x).$$

Following Hill (1976), when $\mu = 0$ and κ is large, $\alpha_\kappa(\gamma)$ can be expanded asymptotically as

$$\begin{aligned} \alpha_\kappa(\kappa) &= \frac{\chi}{\sqrt{\kappa}} + \frac{\chi^3 + 3\chi}{24\,\kappa\sqrt{\kappa}} + \frac{3\chi^5 + 20\chi^3 + 45\chi}{640\,\kappa^2\sqrt{\kappa}} + \ldots \\ &= \frac{\chi}{\sqrt{\kappa}}\left[1 + \frac{\chi^2 + 3}{24\,\kappa} + \frac{3\chi^4 + 20\chi^3 + 45}{640\,\kappa^2} + \ldots\right] \qquad \text{... (4.14)} \\ &\cong \frac{\chi}{\sqrt{\kappa}}. \end{aligned}$$

where $\chi = \Phi^{-1}(1-\gamma)$ and $\Phi(.)$ is the c.d.f of standard normal distribution. Using the relation $\Phi^{-1}(p) = \sqrt{2}\,\mathrm{erf}^{-1}(2p-1)$ where $\mathrm{erf}(.)$ is the error function and the asymptotic expansion of inverse of the error function (see Carlitz, 1962) given by

$$\mathrm{erf}^{-1}(x) = \frac{\sqrt{\pi}}{2} x\left[1 + \frac{\pi}{12}x^2 + \frac{7\pi^2}{480}x^4 + \frac{127\pi^3}{40320}x^6 + \ldots\right],$$

we get $$\Phi^{-1}(1-\gamma) = \sqrt{\frac{\pi}{2}}\,(1-2\gamma)\left[1 + \frac{\pi}{12}(1-2\gamma)^2 + \frac{7\pi^2}{480}(1-2\gamma)^4 + \frac{127\pi^3}{40320}(1-2\gamma)^6 + \ldots\right].$$

Now from (4.13) we get,

$$\alpha_\kappa(\gamma) = \sqrt{\frac{\pi}{2\kappa}}\,(1-2\gamma)\left[1 + \frac{\pi}{12}(1-2\gamma)^2 + O((1-2\gamma)^4)\right].$$

Therefore,

$$E_{\gamma,F}(d(\theta)) = c_1(1-2\gamma)\left[1 + \frac{\pi(1-2\gamma)^2}{12} + O((1-2\gamma)^4)\right]\; e^{\kappa\cos\left\{\sqrt{\frac{\pi}{2\kappa}}(1-2\gamma)\left[1+\frac{\pi(1-2\gamma)^2}{12}+O((1-2\gamma)^4)\right]\right\}}$$

where $c_1 = [\sqrt{2\pi\kappa}\, I_0(\kappa)]^{-1}$. Thus, as $\gamma \to 0.5$, $(1\text{-}2\gamma) \to 0$ and hence $E_{\gamma,F}(d(\theta)) \to 0$.

(b) Note that when $\gamma \to 0, \alpha_\kappa(\gamma) \to \pi$ and $\beta_\kappa(\gamma) \to -\pi$. Hence, from (4.12), we get

$$\lim_{\gamma\to 0} E_{\gamma,F}(d(\theta)) = \frac{1}{2\pi I_0(\kappa)} \int_0^{2\pi} (\pi - |\pi - \theta|)\, e^{\kappa\cos\theta} d\theta = E_F(d(\theta)).$$

Hence the lemma.

The following Lemma 5 shows that Ko's choice of S is not a dispersion measure.

Lemma 5: *Let* $X \sim F$*, T(X) be the modal vector and* $d(\theta,\mu) = \cos^2(\theta - \mu)$*. Then* $S(F) = E_F^{-1/2}(d(X, T(X)))$ *is not a dispersion measure on the unit circle.*

Proof:

Consider a circular random variable θ on the unit circle (T) and let F be the distribution function on T. Consider $d(\theta,\mu) = \cos^2(\theta - \mu)$ where θ and μ are any two angles on T. Let X and Y be two random unit vectors with unimodal distributions F and G with modal vectors $T(X)$ and $T(Y)$ respectively. Further, let $S(F) = E_F^{-1/2}(d(X, T(X)))$. Then $S(F)$ is a dispersion on T, if it satisfies the set of conditions (2.5) given in chapter 2.

Note that if $X = c$ with probability 1 then $S(\delta_c) = E_F(\cos^2(c - c)) = 1 \neq 0$ which implies that condition c) of (2.5) is violated. Further, condition a) is also violated as shown below:

Let $F^*(u) = P[d(X, T(X)) \le u]$ and $G^*(u) = P[d(Y, T(Y)) \le u]$ denotes the cumulative distribution functions of $d(X, T(X))$ and $d(Y, T(Y))$ respectively. Since $d(X, T(X))$ and $d(Y, T(Y))$ are both non-negative we have,

$$S(F) = E_F^{-1/2}(d(X,T(X))) = \left[\int_0^{\pi} P[d(X,T(X)) > u]du\right]^{-1/2}$$

$$= \left[\int_0^{\pi} \{1 - P[d(X,T(X)) \le u]du\}\right]^{-1/2} = \left[\int_0^{\pi} (1 - F^*(u))du\right]^{-1/2}$$

Similarly, $E_G^{-1/2}(d(Y,T(Y))) = \left[\int_0^{\pi} (1 - G^*(u))du\right]^{-1/2}$. Now, whenever

$$\begin{aligned} G^*(u) \le F^*(u) \;\Rightarrow\; & (1 - F^*(u)) \le (1 - G^*(u)) \\ & \Rightarrow \left[\int_0^{\pi} (1 - F^*(u))du\right]^{1/2} \le \left[\int_0^{\pi} (1 - G^*(u))du\right]^{1/2} \\ & \Rightarrow \left[\int_0^{\pi} (1 - F^*(u))du\right]^{-1/2} \ge \left[\int_0^{\pi} (1 - G^*(u))du\right]^{-1/2} \\ & \Rightarrow S(F) \ge S(G) \end{aligned}$$

which is a contradiction to condition a) in the definition of dispersion measure.

Hence the lemma.

Remark 2: If $F \sim CN(\mu, \kappa)$ then $S(F) = (A'(\kappa))^{-1/2}$. Hence $(A'(\kappa))^{-1/2}$ is not a dispersion measure on the unit circle.

Chapter 5

Robustness of Tests for Mean Direction of Circular Normal Distribution

5.1 Introduction

In this chapter we adopt the approach of He, Simpson and Portnoy (1990) to study the robustness of three single sample tests (based on different test statistics) for the mean direction of circular normal distribution with p.d.f.

$$f(\theta;\mu,\kappa)=\frac{1}{2\pi I_0(\kappa)}e^{\kappa\cos(\theta-\mu)}, 0\leq\theta<2\pi \text{ where } 0\leq\mu<2\pi \text{ and } \kappa>0$$

where $I_0(\kappa)$ is the modified Bessel function of the first kind and order zero. We study the robustness of the following test functionals: $W(F)=\rho_F-E_F(\cos\theta)$ (the likelihood ratio test statistic (see Mardia and Jupp, 2000, pp. 119-120)), $W_1(F)=\arctan^*\left[\frac{E_F(\sin\theta)}{E_F(\cos\theta)}\right]$ (the directional mean as a test statistic) and $W_\gamma(F)=\arctan^*\left[\frac{E_{\gamma,F}(\sin\theta)}{E_{\gamma,F}(\cos\theta)}\right]$ (the γ-circular trimmed mean as a test statistic, see chapter 3, section 3.3) where γ is the trimming proportion. We consider the testing problem $H_0:\mu=0$ against $H_1:\mu\neq0$ and we assume that the parameter κ is known. Note that tests based on the statistics W_1 and W_γ can be easily constructed as the cut-off points can be easily determined through simulation. We compare the robustness of the W, W_1 and W_γ by studying their LBF and PBF and also their LBP and PBP. In Section 5 we show that W_γ has the best robustness property in the sense that it has the highest LBP and PBP. Also, the PBF of W_γ clearly dominates that of W_1 and W. In this chapter we have developed four lemmas and six theorems on the robustness of tests for mean direction of circular normal distribution.

The organisation of this chapter is as follows. In Section 5.2 we discuss the robustness of likelihood ratio test statistic (W). In Section 5.3 we consider directional mean as a test statistics (W_1) and discuss its robustness. In Section 5.4 we consider circular trimmed mean as a test statistic (W_γ) and its robustness. In Section 5.5 we compare the robustness of the three test statistics.

5.2 Robustness of the Likelihood Ratio Test Statistic

Let $\theta_1, \theta_2, \cdots, \theta_n$ be a random sample from $CN(\mu, \kappa)$ with $\kappa > 0$ and known. Consider the problem of testing $H_0 : \mu = 0$ against $H_1 : \mu \neq 0$. A test statistic for testing H_0 against H_1 is $W = \overline{R} - \overline{C}$ (see Mardia and Jupp, 2000, pp. 119-120) where $\overline{C} = \frac{1}{n}\sum_{i=1}^{n} \cos\theta_i, \overline{S} = \frac{1}{n}\sum_{i=1}^{n} \sin\theta_i$ and $\overline{R} = \sqrt{\overline{C}^2 + \overline{S}^2}$. This corresponds to the functional $W(F) = \rho_F - E_F(\cos\theta)$. Theorem 5.1 below gives the LBF of W. The LBP is obtained numerically.

Theorem 5.1: ***a)*** *The LBF of W is*

$$\varepsilon_\mu^{**}(W) = \inf\{\varepsilon > 0 : \sqrt{y^2 + \varepsilon^2 \sin^2 x} = y + c \text{ for some } 0 \leq x < 2\pi\} \textit{ where}$$

$y = \rho(1-\varepsilon) + \varepsilon \cos x$ *and* $c = 2\rho \sin^2 \frac{\mu}{2}$.

b) *The LBP is* $\varepsilon^{**} = \sup_\mu(\varepsilon_\mu^{**}(W))$.

The following Lemma 1 is used to prove the theorem.

Lemma1: *Let* F_μ *denote the* $CN(\mu, \kappa)$ *distribution then under the assumption that* H_0 *is true we have* $E_{(1-\varepsilon)F_0 + \varepsilon\delta_x}(\cos\theta) = \rho(1-\varepsilon) + \varepsilon \cos x$ *and* $E_{(1-\varepsilon)F_0 + \varepsilon\delta_x}(\sin\theta) = \varepsilon \sin x$.

Proof:

Using Lemma 2 in chapter 3 for $CN(\mu,\kappa)$ distribution and under the assumption that H_0 is true we get $E_{F_0}(\cos\theta)=\rho$ and $E_{F_0}(\sin\theta)=0$. Therefore, we get

$$E_{(1-\varepsilon)F_0+\varepsilon\delta_x}(\cos\theta)=(1-\varepsilon)E_{F_0}(\cos\theta)+\varepsilon\cos x=\rho(1-\varepsilon)+\varepsilon\cos x \text{ and}$$
$$E_{(1-\varepsilon)F_0+\varepsilon\delta_x}(\sin\theta)=(1-\varepsilon)E_{F_0}(\sin\theta)+\varepsilon\cos x=\varepsilon\sin x.$$

Hence the lemma.

Proof of the theorem:

a) Let F_μ denote the $CN(\mu,\kappa)$ distribution and δ_x denote the point mass at x. Let $W_\varepsilon(F)=W((1-\varepsilon)F+\varepsilon\delta_x)$. Consider the functional representations of W and W_ε given by:

$$W(F)=\rho_F-E_F(\cos\theta)=\sqrt{E_F^2(\cos\theta)+E_F^2(\sin\theta)}-E_F(\cos\theta) \text{ and}$$

$$W_\varepsilon(F)=\sqrt{E^2_{(1-\varepsilon)F_0+\varepsilon\delta_x}(\cos\theta)+E^2_{(1-\varepsilon)F_0+\varepsilon\delta_x}(\sin\theta)}-E_{(1-\varepsilon)F_0+\varepsilon\delta_x}(\cos\theta).$$

Let $E_{(1-\varepsilon)F_0+\varepsilon\delta_x}(\cos\theta)=y$. Then using Lemma1 we get:

$$W((1-\varepsilon)F_0+\varepsilon\delta_x)=\sqrt{y^2+\varepsilon^2\sin^2 x}-y. \qquad \text{... (5.1)}$$

Again, easy computation yields:

$$W(F_\mu)=\sqrt{E^2_{F_\mu}(\cos\theta)+E^2_{F_\mu}(\sin\theta)}-E_{F_\mu}(\cos\theta)=c. \qquad \text{... (5.2)}$$

Now using (2.6) in chapter 2, the LBF of W is given by

$$\varepsilon^{**}_\mu(W)=\inf\{\varepsilon>0:\sqrt{y^2+\varepsilon^2\sin^2 x}=y+c \text{ for some } 0\le x<2\pi\}. \qquad \text{... (5.3)}$$

b) In order to obtain LBP we evaluate (5.3) numerically for different values of κ with varying μ. The following Table 2 shows the values of LBP for different values of κ and μ. Figure 2 given below is based on Table 2.

Table 2: LBF for different values of κ and μ .

Mu	$\kappa=1$	$\kappa=2$	$\kappa=4$	$\kappa=10$	Mu	$\kappa=1$	$\kappa=2$	$\kappa=4$	$\kappa=10$
0.00	0.00	0.00	0.00	0.00	3.25	0.62	0.82	0.93	0.97
0.25	0.11	0.16	0.19	0.21	3.50	0.61	0.81	0.91	0.96
0.50	0.20	0.29	0.35	0.37	3.75	0.60	0.79	0.89	0.93
0.75	0.28	0.40	0.46	0.50	4.00	0.56	0.75	0.85	0.89
1.00	0.35	0.48	0.55	0.59	4.25	0.53	0.71	0.80	0.84
1.25	0.40	0.55	0.62	0.65	4.50	0.50	0.66	0.74	0.78
1.50	0.45	0.60	0.68	0.71	4.75	0.46	0.61	0.69	0.72
1.75	0.49	0.65	0.74	0.77	5.00	0.41	0.56	0.63	0.66
2.00	0.53	0.70	0.79	0.83	5.25	0.35	0.49	0.56	0.60
2.25	0.56	0.75	0.84	0.88	5.50	0.29	0.41	0.48	0.51
2.50	0.59	0.78	0.88	0.93	5.75	0.21	0.31	0.36	0.39
2.75	0.61	0.81	0.91	0.96	6.00	0.12	0.18	0.22	0.24
3.00	0.62	0.82	0.93	0.97	6.25	0.01	0.02	0.03	0.03

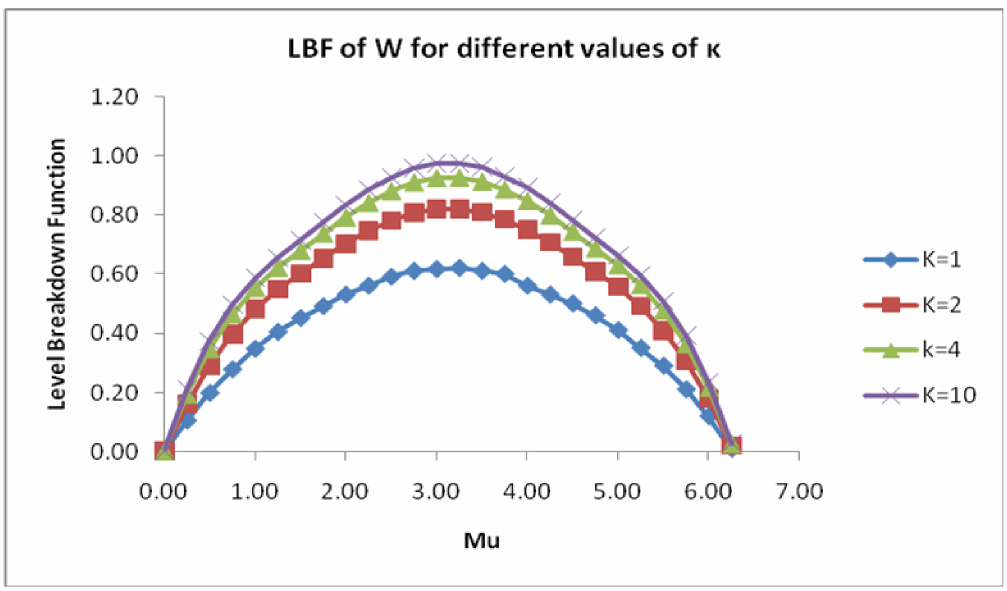

Figure 2: Variation of $\varepsilon_{\mu}^{**}(W)$ with μ and for different values of κ.

From Figure 2 we see that the LBF of W for higher value of κ dominates that of a lower value of κ Thus, we can say the robustness of W with respect to level breakdown increases with the value of κ.This is also supported by the LBP values for different κ given in Table 3 given below.

Table 3: LBP values of W for different κ

Test Functional W	$\kappa = 1$	$\kappa = 2$	$\kappa = 4$	$\kappa = 10$
LBP	0.62	0.82	0.93	0.97

Hence the theorem.

In Theorem 5.2 below we give the PBF and the corresponding PBP of W.

Theorem 5.2: *The PBF of W is* $\varepsilon_\mu^*(W) = \dfrac{\rho|\sin\mu|}{1+\rho|\sin\mu|}$ *and the PBP of* W *is* $\varepsilon^* = \dfrac{\rho}{1+\rho}$.

Proof:

Let F_μ and δ_x be as in the proof of Theorem 5.1. Noting that for $CN(\mu, \kappa)$, $E_{F_\mu}(\cos\theta) = \rho\cos\mu$ and $E_{F_\mu}(\sin\theta) = \rho\sin\mu$ we have

$$\begin{aligned} W((1-\varepsilon)F_\mu + \varepsilon\delta_x) &= \sqrt{E^2_{(1-\varepsilon)F_\mu+\varepsilon\delta_x}(\cos\theta) + E^2_{(1-\varepsilon)F_\mu+\varepsilon\delta_x}(\sin\theta)} - E_{(1-\varepsilon)F_\mu+\varepsilon\delta_x}(\cos\theta) \\ &= \sqrt{\rho^2(1-\varepsilon)^2 + 2\rho\varepsilon(1-\varepsilon)\cos(x-\mu) + \varepsilon^2} - (\rho(1-\varepsilon)\cos\mu + \varepsilon\cos x) \end{aligned}$$

and under H_0 is true $W(F_0) = 0$. Now using (2.7) in chapter 2, the PBF of W is given by

$$\varepsilon_\mu^*(W) = \inf\{\varepsilon > 0 : \varepsilon\sin x + \rho(1-\varepsilon)\sin\mu = 0 \text{ for some } x,\ 0 \le x < 2\pi\}. \qquad \text{... (5.4)}$$

Note that $\varepsilon\sin x + \rho(1-\varepsilon)\sin\mu = 0$ has a solution in $x \in [0,2\pi)$ if and only if $|\Delta| < 1$

where $\Delta = \frac{-\rho(1-\varepsilon)\sin\mu}{\varepsilon}$. Now, $|\Delta| < 1 \Rightarrow \frac{\rho|\sin\mu|}{1+\rho|\sin\mu|} < \varepsilon$, we get the PBF of as

$\varepsilon^*_\mu(W) = \frac{\rho|\sin\mu|}{1+\rho|\sin\mu|}$. Further, the PBP is $\varepsilon^* = \sup_\mu(\varepsilon^*_\mu(W)) = \frac{\rho}{1+\rho}$.

Hence the theorem.

5.3 Robustness of the Directional Mean as a Test Statistic

We now consider directional mean $W_1(F) = \arctan^*\left[\frac{E_F(\sin\theta)}{E_F(\cos\theta)}\right]$ as a test statistic and study its robustness based on its breakdown properties. Theorem 5.3 gives the LBF and the corresponding LBP.

Theorem 5.3: *The LBF of* W_1 *is* $\varepsilon^{**}_\mu(W_1) = \frac{\rho|\tan\mu|}{1+\rho|\tan\mu|}$ *and the LBP of* W_1 *is* $\varepsilon^{**} = 1$.

Proof:

Let F_μ and δ_x be as in the proof of Theorem 5.1. Under H_0 is true and noting that $E_{F_0}(\cos\theta) = \rho$ we have

$$W_1((1-\varepsilon)F_0 + \varepsilon\delta_x) = \arctan^*\left(\frac{E_{(1-\varepsilon)F_0+\varepsilon\delta_x}(\sin\theta)}{E_{(1-\varepsilon)F_0+\varepsilon\delta_x}(\cos\theta)}\right) = \arctan^*\left(\frac{(1-\varepsilon)E_{F_0}(\sin\theta)+\varepsilon\sin x}{(1-\varepsilon)E_{F_0}(\cos\theta)+\varepsilon\cos x}\right)$$

$$= \arctan^*\left(\frac{\rho(1-\varepsilon)\tan\mu_0 + \varepsilon\sin x}{(1-\varepsilon)\rho+\varepsilon\cos x}\right) = \arctan^*\left(\frac{\varepsilon\sin x}{\rho(1-\varepsilon)+\varepsilon\cos x}\right)$$

and $W_1(F_\mu) = \arctan^*\left(\frac{E_{F_\mu}(\sin\theta)}{E_{F_\mu}(\cos\theta)}\right) = \mu$. Therefore, by definition (1.1) the LBF of W_1 is given by:

$$\varepsilon^{**}_\mu(W_1) = \inf\left\{\varepsilon > 0 : \arctan^*\left(\frac{\varepsilon\sin x}{\rho(1-\varepsilon)+\varepsilon\cos x}\right) = \mu \text{ for some } x \in [0, 2\pi)\right\}. \qquad \text{... (5.5)}$$

Since $\arctan^{*}\left(\frac{\varepsilon\sin x}{\rho(1-\varepsilon)+\varepsilon\cos x}\right)=\mu$ has a solution in $x\in[0,2\pi)$ if and only if $|\Delta|<1$ where $\Delta=\frac{\rho(1-\varepsilon)\tan\mu}{\varepsilon}$. Now $|\Delta|<1\Rightarrow\frac{\rho|\tan\mu|}{1+\rho|\tan\mu|}<\varepsilon$, we get the LBF of W_1 as

$\varepsilon_{\mu}^{**}(W_1)=\frac{\rho|\tan\mu|}{1+\rho|\tan\mu|}$.

The LBP of W_1 can be easily computed to be $\varepsilon^{**}=\sup_{\mu}(\varepsilon_{\mu}^{**}(W_1))=1$.

Hence the theorem.

Theorem 5.4 below gives the PBF and the corresponding PBP of W_1.

Theorem 5.4: *The PBF of* W_1 *is* $\varepsilon_{\mu}^{*}(W_1)=\frac{\rho|\sin\mu|}{1+\rho|\sin\mu|}$ *and the PBP of* W_1 *is* $\varepsilon^{*}=\frac{\rho}{1+\rho}$.

Proof:

Let F_{μ} and δ_x be as in the proof of Theorem 5.1. Again noting that for $CN(\mu,\kappa)$, $E_{F_{\mu}}(\cos\theta)=\rho\cos\mu$ and $E_{F_{\mu}}(\sin\theta)=\rho\sin\mu$ we have

$$W_1((1-\varepsilon)F_{\mu}+\varepsilon\delta_x)=\arctan^{*}\left(\frac{E_{(1-\varepsilon)F_{\mu}+\varepsilon\delta_x}(\sin\theta)}{E_{(1-\varepsilon)F_{\mu}+\varepsilon\delta_x}(\cos\theta)}\right)=\arctan^{*}\left(\frac{\rho(1-\varepsilon)\sin\mu+\varepsilon\sin x}{\rho(1-\varepsilon)\cos\mu+\varepsilon\cos x}\right)$$

and under H_0 is true $W_1(F_0)=\arctan^{*}\left(\frac{E_{F_0}(\sin\theta)}{E_{F_0}(\cos\theta)}\right)=0$. Now from definition (1.2) the PBF of W_1 is given by:

$$\varepsilon_{\mu}^{*}(W_1)=\inf\left\{\varepsilon>0:\arctan^{*}\left(\frac{\rho(1-\varepsilon)\sin\mu+\varepsilon\sin x}{\rho(1-\varepsilon)\cos\mu+\varepsilon\cos x}\right)=\mu \text{ for some } x\in[0,2\pi)\right\}. \quad \text{... (5.6)}$$

Since $\arctan^*\left(\frac{\rho(1-\varepsilon)\sin\mu+\varepsilon\sin x}{\rho(1-\varepsilon)\cos\mu+\varepsilon\cos x}\right)=0$ has a solution in $x\in[0,2\pi)$ if and only if $|\Delta|<1$ where $\Delta=\frac{-\rho(1-\varepsilon)\sin\mu}{\varepsilon}$. Now $|\Delta|<1\Rightarrow\frac{\rho|\sin\mu|}{1+\rho|\sin\mu|}<\varepsilon$, we get the PBF of W_1 as $\varepsilon_\mu^*(W_1)=\frac{\rho|\sin\mu|}{1+\rho|\sin\mu|}$.

Further, the PBP is $\varepsilon^*=\sup_\mu(\varepsilon_\mu^*(W_1))=\frac{\rho}{1+\rho}$.

Hence the theorem.

5.4 Robustness of the Circular Trimmed Mean as a Test Statistic

We now consider γ- circular trimmed mean (for definition see chapter 3) as a test statistic and study its robustness based on its breakdown properties. Theorem 5.5 gives the LBF and the corresponding LBP.

Theorem 5.5: *The LBF of W_γ is* $\varepsilon_{\mu,\gamma}^{**}(W_\gamma)=\frac{\rho_{\gamma,0}(1-2\gamma)\tan\mu}{k_1(\mu)+\rho_{\gamma,0}(1-2\gamma)\tan\mu}$ *where* $\rho_{\gamma,0}=E_{\gamma,F_0}(\cos\theta)=(1-2\gamma)^{-1}\int_\tau^\eta\cos\theta\,dF_0$, $k_1(\mu)=\sup\{\sin(x-\mu):x\in(\theta_1,\theta_2)\}$, F_0 *is the cdf of* $CN(0,\kappa)$, $\tau=F_0^{-1}(\gamma)$, $\eta=F_0^{-1}(1-\gamma)$, $\theta_1=F_0^{-1}\left(\frac{\gamma}{1-\varepsilon}\right)$ *and* $\theta_2=F_0^{-1}\left(1-\frac{\gamma}{1-\varepsilon}\right)$ *and the LBP of W_γ is* $\varepsilon^{**}=1$.

The following Lemma 2 and Lemma 3 were used to prove the above theorem.

Lemma 2: *Let* $\mu_{\gamma,\mu}=\arctan^*\left[\frac{E_{\gamma,F\mu}(\sin\theta)}{E_{\gamma,F\mu}(\cos\theta)}\right]$ and $\mu=\arctan^*\left[\frac{E_{F\mu}(\sin\theta)}{E_{F\mu}(\cos\theta)}\right]$. Then

$\mu_{\gamma,\mu}=\mu\ \forall\ \mu\in[-\pi,\pi)$ *where* $0\le\gamma<0.5$.

Proof:

Using the above definition we have,

$$\mu_{\gamma,\mu} = \mu \Rightarrow \frac{E_{\gamma,F\mu}(\sin\theta)}{E_{\gamma,F\mu}(\cos\theta)} = \frac{E_{F\mu}(\sin\theta)}{E_{\mu}(\cos\theta)}$$
$$\Rightarrow E_{\gamma,F\mu}(\sin\theta)E_{F\mu}(\cos\theta) = E_{\gamma,F\mu}(\cos\theta)E_{F\mu}(\sin\theta).$$

Define T_1 to be a circular arc having μ as the centre point which satisfies $\int_{T_1} f_\mu(\theta)d\theta = 1-2\gamma$ and let $T_2 = T_1'$. Then,

$$E_{F\mu}(\cos\theta) = \int_{-\pi}^{\pi} \cos\theta\, f_\mu(\theta)d\theta = \left[\int_{T_1} \cos\theta\, f_\mu(\theta)d\theta + \int_{T_2} \cos\theta\, f_\mu(\theta)d\theta\right]$$
$$= \frac{1}{2\pi I_0(\kappa)}\left[(1-2\gamma)\left\{\frac{1}{(1-2\gamma)}\int_{T_1} \cos\theta\, f_\mu(\theta)d\theta\right\} + \int_{T_2} \cos\theta\, f_\mu(\theta)d\theta\right]$$
$$= (1-2\gamma)E_{\gamma,F\mu}(\cos\theta) + \int_{T_2} \cos\theta\, f_\mu(\theta)d\theta$$
$$\Rightarrow E_{F\mu}(\cos\theta) = (1-2\gamma)E_{\gamma,F\mu}(\cos\theta) + C_1.$$

Similar calculations show that $E_{F\mu}(\sin\theta) = (1-2\gamma)E_{\gamma,F\mu}(\sin\theta) + S_1$ where

$C_1 = \int_{T_2} \cos\theta\, f_\mu(\theta)d\theta$ and $S_1 = \int_{T_2} \sin\theta\, f_\mu(\theta)d\theta$. Writing $T_2 = (\mu+\pi-\alpha_1, \mu+\pi+\alpha_1)$ we get

$$\frac{S_1}{C_1} = \frac{\int_{\mu+\pi-\alpha_1}^{\mu+\pi+\alpha_1} \left(\sin(\theta-\mu)\cos\mu + \cos(\theta-\mu)\sin\mu\right)e^{\kappa\cos(\theta-\mu)}d\theta}{\int_{\mu+\pi-\alpha_1}^{\mu+\pi+\alpha_1} \left(\cos(\theta-\mu)\cos\mu - \sin(\theta-\mu)\sin\mu\right)e^{\kappa\cos(\theta-\mu)}d\theta}$$
$$= \frac{-\cos\mu \int_{-\alpha_1}^{\alpha_1} \sin\nu\, e^{\kappa\cos\nu}d\nu - \sin\mu \int_{-\alpha_1}^{\alpha_1} \cos\nu\, e^{-\kappa\cos\nu}d\nu}{-\cos\mu \int_{-\alpha_1}^{\alpha_1} \cos\nu\, e^{-\kappa\cos\nu}d\nu + \sin\mu \int_{-\alpha_1}^{\alpha_1} \sin\nu\, e^{\kappa\cos\nu}d\nu}$$

Now using the fact that $\int_{-\alpha_1}^{\alpha_1} \sin\nu\, e^{\kappa\cos\nu}d\nu = 0$ we get $\frac{S_1}{C_1} = \tan\mu$.

Hence the lemma.

Lemma 3: *Let* $W_\gamma(G_\varepsilon)=\arctan^*\left[\frac{E_{\gamma,(1-\varepsilon)F_0+\varepsilon\delta_x}(\sin\theta)}{E_{\gamma,(1-\varepsilon)F_0+\varepsilon\delta_x}(\cos\theta)}\right]$. Then $W_\gamma(G_\varepsilon)\neq\mu$ *for* $x\leq\theta_1$ *or* $x\geq\theta_2$ *where* $G_\varepsilon=(1-\varepsilon)F_0+\varepsilon\delta_x$, $\varepsilon<\min(\gamma,1-\gamma)$ *and* $0\leq\gamma<0.5$.

Proof:

When $x\leq\theta_1$, we have, $E_{\gamma,(1-\varepsilon)F_0+\varepsilon\delta_x}(\sin\theta)=(1-\varepsilon)\tilde{E}_{\gamma,F_0}(\sin\theta)=\frac{1-\varepsilon}{1-2\gamma}\int_\lambda^{\theta_2}\sin\theta dF_0$ and

$E_{\gamma,(1-\varepsilon)F_0+\varepsilon\delta_x}(\cos\theta)=(1-\varepsilon)\tilde{E}_{\gamma,F_0}(\cos\theta)=\frac{1-\varepsilon}{1-2\gamma}\int_\lambda^{\theta_2}\cos\theta dF_0$. Thus,

$$W_\gamma(G_\varepsilon)=\arctan^*\left[\frac{\tilde{E}_{\gamma,F_0}(\sin\theta)}{\tilde{E}_{\gamma,F_0}(\cos\theta)}\right].$$

Now by Lemma 1 we have, $W_\gamma(F_\mu)=\arctan^*\left[\frac{E_{\gamma,F_\mu}(\sin\theta)}{E_{\gamma,F_\mu}(\cos\theta)}\right]=\mu$.

But $\tilde{E}_{\gamma,F_0}(\sin\theta)$ and $\tilde{E}_{\gamma,F_0}(\cos\theta)$ can be simplified as:

$$\begin{aligned}\tilde{E}_{\gamma,F_0}(\sin\theta)&=(1-2\gamma)^{-1}\int_\lambda^{\theta_2}\sin\theta dF_0=(1-2\gamma)^{-1}\left(\int_\lambda^{-\theta_2}\sin\theta dF_0+\int_{-\theta_2}^{\theta_2}\sin\theta dF_0\right)\\&=(1-2\gamma)^{-1}\int_\lambda^{-\theta_2}\sin\theta dF_0\end{aligned}$$

and

$$\begin{aligned}\tilde{E}_{\gamma,F_0}(\cos\theta)&=(1-2\gamma)^{-1}\int_\lambda^{\theta_2}\cos\theta dF_0=(1-2\gamma)^{-1}\left[\int_{-\theta_2}^{\theta_2}\cos\theta dF_0+\int_\lambda^{-\theta_2}\cos\theta dF_0\right]\\&=(1-2\gamma)^{-1}\left[2\int_0^{\theta_2}\cos\theta dF_0+\int_\lambda^{-\theta_2}\cos\theta dF_0\right]=(1-2\gamma)^{-1}\left[A(\kappa)-2\int_{\theta_2}^{\pi}\cos\theta dF_0+\int_\lambda^{-\theta_2}\cos\theta dF_0\right]\\&=(1-2\gamma)^{-1}\left[A(\kappa)-\int_{\theta_2}^{\pi}\cos\theta dF_0+\int_{-\pi}^{\lambda}\cos\theta dF_0\right].\end{aligned}$$

Therefore, $\dfrac{\tilde{E}_{\gamma,F_0}(\sin\theta)}{\tilde{E}_{\gamma,F_0}(\cos\theta)} = \dfrac{\int_{\lambda}^{-\theta_2}\sin\theta dF_0}{A(\kappa) - \int_{\theta_2}^{\pi}\cos\theta dF_0 + \int_{-\pi}^{\lambda}\cos\theta dF_0} \neq 0$, since $\int_{\lambda}^{-\theta_2}\sin\theta dF_0 \neq 0$.

Similar computations shows that $W_\gamma(G_\varepsilon) \neq \mu$ when $x \geq \theta_2$.

Hence the lemma.

Proof of the theorem:

Let F_μ and δ_x be as in the proof of Theorem 5.1. Let $G_\varepsilon = (1-\varepsilon)F_0 + \varepsilon\delta_x$, $x \in [-\pi, \pi)$ and $0 \leq \gamma < 0.5$. Then

$$G_\varepsilon(\theta) = \begin{cases} (1-\varepsilon)F_0(\theta) & \text{if } -\pi \leq \theta < x \\ (1-\varepsilon)F_0(\theta) + \varepsilon & \text{if } \theta \leq x < \pi \end{cases}$$

where $F_0(\theta) = \dfrac{1}{2\pi I_0(\kappa)}\int_{-\pi}^{\theta} e^{\kappa\cos\varphi} d\varphi$. Suppose $\theta_1 = F_0^{-1}\left(\dfrac{\gamma}{1-\varepsilon}\right)$ and $\theta_2 = F_0^{-1}\left(1 - \dfrac{\gamma}{1-\varepsilon}\right)$.

Note that since is symmetric about 0 we have $\theta_1 = -\theta_2$. Also we have,

$$W_\gamma(G_\varepsilon) = \arctan^*\left[\frac{E_{\gamma,(1-\varepsilon)F_0+\varepsilon\delta_x}(\sin\theta)}{E_{\gamma,(1-\varepsilon)F_0+\varepsilon\delta_x}(\cos\theta)}\right].$$

Case 1: When $\theta_1 < x < \theta_2$. In this case we have,

$$E_{\gamma,(1-\varepsilon)F_0+\varepsilon\delta_x}(\sin\theta) = \frac{(1-\varepsilon)}{(1-2\gamma)}\int_{\theta_1}^{\theta_2}\sin\theta\, dF_0 + \frac{\varepsilon}{1-2\gamma}\sin x \text{ and} \quad \text{... (5.7)}$$

$$E_{\gamma,(1-\varepsilon)F_0+\varepsilon\delta_x}(\cos\theta) = \frac{(1-\varepsilon)}{(1-2\gamma)}\int_{\theta_1}^{\theta_2}\cos\theta\, dF_0 + \frac{\varepsilon}{1-2\gamma}\cos x. \quad \text{... (5.8)}$$

Now consider $E_{\gamma,F_0}(\sin\theta) = (1-2\gamma)^{-1}\int_{-\tau}^{\eta}\sin\theta\, f(\theta)d\theta$. Using (5.7) and the fact that F_0 is

symmetric, this integral can be written as

$$E_{\gamma,F_0}(\sin\theta) = (1-2\gamma)^{-1}\int_{-\tau}^{\eta}\sin\theta\, f(\theta)d\theta$$

$$= (1-2\gamma)^{-1}\left[\int_{-\tau}^{\theta_1}\sin\theta\, f(\theta)d\theta + \int_{\theta_1}^{\theta_2}\sin\theta\, f(\theta)d\theta + \int_{\theta_2}^{\eta}\sin\theta\, f(\theta)d\theta\right]$$

$$\Rightarrow E_{\gamma,(1-\varepsilon)F_0+\varepsilon\delta_x}(\sin\theta) - \frac{\varepsilon}{(1-2\gamma)}\sin x = (1-\varepsilon)E_{\gamma,F_0}(\sin\theta) - \frac{(1-\varepsilon)}{(1-2\gamma)}\left[\int_{-\tau}^{\theta_1}\sin\theta\, f(\theta)d\theta + \int_{\theta_2}^{\eta}\sin\theta\, f(\theta)d\theta\right]$$

$$\Rightarrow E_{\gamma,(1-\varepsilon)F_0+\varepsilon\delta_x}(\sin\theta) = (1-\varepsilon)E_{\gamma,F_0}(\sin\theta) - \frac{\varepsilon}{(1-2\gamma)}\sin x. \quad \text{...(5.9)}$$

Similarly using (5.8) we get

$$E_{\gamma,(1-\varepsilon)F_0+\varepsilon\delta_x}(\cos\theta) = (1-\varepsilon)E_{\gamma,F_0}(\cos\theta) - \frac{\varepsilon}{(1-2\gamma)}\cos x. \quad \text{... (5.10)}$$

Using (5.9) and (5.10) we get

$$W_\gamma(G_\varepsilon) = \arctan^*\left[\frac{\rho_{\gamma,0}(1-\varepsilon)(1-2\gamma)\tan\mu_{\gamma,0} + \varepsilon\sin x}{\rho_{\gamma,0}(1-\varepsilon)(1-2\gamma) + \varepsilon\cos x}\right]. \quad \text{... (5.11)}$$

Let $\varepsilon < \min(\gamma, 1-\gamma)$, $\lambda = F_0^{-1}\left(\frac{\gamma-\varepsilon}{1-\varepsilon}\right)$, and $\psi = F_0^{-1}\left(\frac{1-\gamma}{1-\varepsilon}\right)$.

Define $\tilde{E}_{\gamma,F_0}(\sin\theta) = \frac{1}{1-2\gamma}\int_{\lambda}^{\theta_2}\sin\theta dF_0$ and $\breve{E}_{\gamma,F_0}(\sin\theta) = \frac{1}{1-2\gamma}\int_{\theta_1}^{\psi}\sin\theta dF_0$.

Case 2: When $x \le \theta_1$. In this case we have,

$$E_{\gamma,(1-\varepsilon)F_0+\varepsilon\delta_x}(\sin\theta) = (1-\varepsilon)\tilde{E}_{\gamma,F_0}(\sin\theta) = \frac{1-\varepsilon}{1-2\gamma}\int_{\lambda}^{\theta_2}\sin\theta dF_0 \text{ and}$$

$$E_{\gamma,(1-\varepsilon)F_0+\varepsilon\delta_x}(\cos\theta) = (1-\varepsilon)\tilde{E}_{\gamma,F_0}(\cos\theta) = \frac{1-\varepsilon}{1-2\gamma}\int_{\lambda}^{\theta_2}\cos\theta dF_0.$$

Therefore, we get $W_\gamma(G_\varepsilon) = \arctan^*\left[\frac{\tilde{E}_{\gamma,F_0}(\sin\theta)}{\tilde{E}_{\gamma,F_0}(\cos\theta)}\right]$. ... (5.12)

Case 3: When $x \ge \theta_2$. In this case we have,

$$E_{\gamma,(1-\varepsilon)F_0+\varepsilon\delta_x}(\sin\theta) = (1-\varepsilon)\breve{E}_{\gamma,F_0}(\sin\theta) = \frac{1-\varepsilon}{1-2\gamma}\int_{\theta_1}^{\psi}\sin\theta dF_0 \text{ and}$$

$$E_{\gamma,(1-\varepsilon)F_0+\varepsilon\delta_x}(\cos\theta) = (1-\varepsilon)\breve{E}_{\gamma,F_0}(\cos\theta) = \frac{1-\varepsilon}{1-2\gamma}\int_{\theta_1}^{\psi}\cos\theta dF_0 .$$

Therefore, we get $W_\gamma(G_\varepsilon) = \arctan^*\left[\frac{\breve{E}_{\gamma,F_0}(\sin\theta)}{\breve{E}_{\gamma,F_0}(\cos\theta)}\right].$ ... (5.13)

Using (5.11), Lemma 2 and Lemma 3, the LBF of W_γ is given by

$\varepsilon^{**}_{\mu,\gamma}(W_\gamma) = \inf\{\varepsilon > 0 : W_\gamma(G_\varepsilon) = \mu \text{ for some } x \in [-\pi,\pi)\}$. … (5.14)

Now using (5.12) and (5.13), (5.14) reduces to

$$\begin{aligned}\varepsilon^{**}_{\mu,\gamma}(W_\gamma) &= \inf\left\{\varepsilon > 0 : \frac{\varepsilon\sin x}{\rho_{\gamma,0}(1-\varepsilon)(1-2\gamma)+\varepsilon\cos x} = \tan\mu \text{ for some } x \in (\theta_1,\theta_2)\right\}\\ &= \inf\{\varepsilon > 0 : \varepsilon\sin(x-\mu) = \rho_{\gamma,0}(1-\varepsilon)(1-2\gamma)\tan\mu \text{ for some } x \in (\theta_1,\theta_2)\}.\end{aligned}$$

Let, $k_1(\mu) = \sup\{\sin(x-\mu) : x \in (\theta_1,\theta_2)\}$ and $k_2(\mu) = \inf\{\sin(x-\mu) : x \in (\theta_1,\theta_2)\}$ such that $k_2(\mu) < k_1(\mu)$. Also let $\tau_1 = k_1(\mu) + \rho_{\gamma,0}(1-2\gamma)\tan\mu$ and $\tau_2 = k_2(\mu) + \rho_{\gamma,0}(1-2\gamma)\tan\mu$. Then, $\varepsilon\sin(x-\mu) = \rho_{\gamma,0}(1-\varepsilon)(1-2\gamma)\tan\mu$ has a solution in $x \in (\theta_1,\theta_2)$ if and only if

$$\frac{\rho_{\gamma,0}(1-2\gamma)\tan\mu}{k_1(\mu)+\rho_{\gamma,0}(1-2\gamma)\tan\mu} < \varepsilon < \frac{\rho_{\gamma,0}(1-2\gamma)\tan\mu}{k_2(\mu)+\rho_{\gamma,0}(1-2\gamma)\tan\mu} \text{ when } \tau_1,\tau_2 > 0,$$

$$\frac{\rho_{\gamma,0}(1-2\gamma)\tan\mu}{k_1(\mu)+\rho_{\gamma,0}(1-2\gamma)\tan\mu} < \varepsilon \text{ and } \varepsilon < \frac{-\rho_{\gamma,0}(1-2\gamma)\tan\mu}{k_2(\mu)+\rho_{\gamma,0}(1-2\gamma)\tan\mu} \text{ when } \tau_1 > 0, \tau_2 < 0 \text{ and}$$

$$\frac{\rho_{\gamma,0}(1-2\gamma)\tan\mu}{k_2(\mu)+\rho_{\gamma,0}(1-2\gamma)\tan\mu} < \varepsilon < \frac{\rho_{\gamma,0}(1-2\gamma)\tan\mu}{k_1(\mu)+\rho_{\gamma,0}(1-2\gamma)\tan\mu} \text{ when } \tau_1,\tau_2 < 0.$$

Therefore, $\varepsilon^{**}_{\mu,\gamma}(W_\gamma) = \dfrac{\rho_{\gamma,0}(1-2\gamma)\tan\mu}{k_1(\mu)+\rho_{\gamma,0}(1-2\gamma)\tan\mu}$.

Further, the LBP of W_γ is $\varepsilon^{**} = \sup_\mu\left(\dfrac{\rho_{\gamma,0}(1-2\gamma)\tan\mu}{k_1(\mu)+\rho_{\gamma,0}(1-2\gamma)\tan\mu}\right) = 1.$

Hence the theorem.

Theorem 5.6 gives the PBF of W_γ and the corresponding PBP is obtained numerically.

Let $c_1(\mu)=\mu+F_0^{-1}(\gamma)$ and $c_2(\mu)=\mu+F_0^{-1}(1-\gamma)$. Define

$\phi_\mu = \sup_x\{\sin x : \text{x lies on the arc } c_1 \text{ to } c_2 \text{ of the unit circle traversed anticlockwise}\}$ and $\psi_\mu = \inf_x\{\sin x : \text{x lies on the arc } c_1 \text{ to } c_2 \text{ of the unit circle traversed anticlockwise}\}$. Let F_μ and δ_x be as in the proof of Theorem 5.1.

Theorem 5.6: ***(a)*** *The PBF of* W_γ *is*

$$\varepsilon^*_{\mu,\gamma}(W_\gamma)=\begin{cases} \max\left(0,\ \min\left\{\dfrac{\lambda_\mu}{\lambda_\mu-\psi_\mu},1\right\}\right) & \text{if} \quad \lambda_\mu \ge \phi_\mu \\ \min\left(1,\ \max\left\{\dfrac{\lambda_\mu}{\lambda_\mu-\psi_\mu},\dfrac{\lambda_\mu}{\lambda_\mu-\phi_\mu},0\right\}\right) & \text{if} \quad \psi_\mu \le \lambda_\mu \le \phi_\mu \\ \max\left(0,\ \min\left\{\dfrac{\lambda_\mu}{\lambda_\mu-\phi_\mu},1\right\}\right) & \text{if} \quad \lambda_\mu \le \psi_\mu \end{cases}$$

where $\lambda_\mu = 2C_\gamma \sin\mu$, $C_\gamma = \int_0^{\nu_2}\cos\nu\, f_0(\nu)\,d\nu$, $\nu_2 = F_0^{-1}(1-\gamma)$, *and* F_0 *is the cdf of* $CN(0,\kappa)$.

(b) *The PBP of* W_γ *is given by* $\varepsilon^* = \sup_\mu(\varepsilon^*_{\mu,\gamma}(W_\gamma))$.

The following Lemma 4 is used to prove the theorem.

Lemma 4: *Suppose that* $\theta \sim CN(0,\kappa)$. *Then* $C_\gamma = \int_0^{\theta^*}\cos\theta\, f_0(\theta)d\theta > 0$ *where* $\theta\in(-\pi,\pi)$, $\theta^* = F_0^{-1}(1-\gamma)$ *and* $0 \le \gamma < 0.5$.

Proof:

Suppose $\theta^* < \frac{\pi}{2}$. Then $\cos\theta > 0 \; \forall \; 0 \le \theta \le \theta^*$ which implies $C_\gamma > 0$. When $\theta^* > \frac{\pi}{2}$,

let $\theta^* = \frac{\pi}{2} + \delta$ and $\beta^* = \frac{\pi}{2} - \delta$. Then, $\int_{\beta^*}^{\pi/2} \cos\alpha \, e^{\kappa\cos\alpha} d\alpha > \left| \int_{\pi/2}^{\theta^*} \cos\beta \, e^{\kappa\cos\beta} d\beta \right|$. Therefore,

$$\int_0^{\theta^*} \cos\alpha \, e^{\kappa\cos\alpha} d\alpha = \int_0^{\beta^*} \cos\alpha \, e^{\kappa\cos\alpha} d\alpha + \int_{\beta^*}^{\frac{\pi}{2}} \cos\alpha \, e^{\kappa\cos\alpha} d\alpha + \int_{\frac{\pi}{2}}^{\theta^*} \cos\alpha \, e^{\kappa\cos\alpha} d\alpha > 0.$$

Hence the lemma.

Proof of the theorem:

Let $G_\varepsilon = (1-\varepsilon)F_\mu + \varepsilon\delta_x$, $x \in [\mu-\pi, \mu+\pi)$ and $0 \le \gamma < 0.5$. Then

$$G_\varepsilon(\theta) = \begin{cases} (1-\varepsilon)F_\mu(\theta) & \text{if } \mu - \pi \le \theta < x \\ (1-\varepsilon)F_\mu(\theta) + \varepsilon & \text{if } \theta \le x < \mu + \pi \end{cases}$$

where $F_\mu(\theta) = \frac{1}{2\pi I_0(\kappa)} \int_{\mu-\pi}^{\theta} e^{\kappa\cos\varphi} d\varphi = F_0(\theta - \mu)$.

Case 1: When $c_1(\mu) < x < c_2(\mu)$.

In this case we have, $W_\gamma(G_\varepsilon) = \arctan^*\left[\frac{E_{\gamma,G_\varepsilon}(\sin\theta)}{E_{\gamma,G_\varepsilon}(\cos\theta)}\right]$ and $W_\gamma(F_\mu) = \mu$. Note that $\mu = 0$ under H_0 giving $W_\gamma(F_0) = 0$. Therefore,

$$\begin{aligned} W_\gamma(G_\varepsilon) = 0 &\Rightarrow \int_{c_1(\mu)}^{c_2(\mu)} \sin\theta dG_\varepsilon(\theta) = 0 \\ &\Rightarrow (1-\varepsilon)S_\gamma + \varepsilon \sin x = 0 \end{aligned}$$

where $S_\gamma = \int_{c_1(\mu)}^{c_2(\mu)} \sin\theta \, f_\mu(\theta) d\theta = \cos\mu \int_{\nu_1}^{\nu_2} \sin\nu \, f_0(\nu) d\nu + \sin\mu \int_{\nu_1}^{\nu_2} \cos\nu \, f_0(\nu) d\nu$, $\theta - \mu = \nu$ and

$\nu_1 = c_1(0) = F_0^{-1}(\gamma), \nu_2 = c_2(0) = F_0^{-1}(1-\gamma)$. Since f_0 is symmetric about zero, $\nu_1 = -\nu_2$, and $\sin\theta$ is odd function we have

$$S_\gamma = 2\sin\mu \int_0^{\nu_2} \cos\nu\, f_0(\nu) d\nu = 2\, C_\gamma \sin\mu = \lambda_\mu .$$

By Lemma 3, we have $C_\gamma > 0$. Again,

$$\begin{aligned}\varepsilon^*_{\mu,\gamma}(W_\gamma) &= \inf\{\varepsilon > 0 : W_\gamma(G_\varepsilon) = 0 \text{ for some } x \in (c_1(\mu), c_2(\mu))\} \\ &= \inf\{\varepsilon > 0 : (1-\varepsilon)\lambda_\mu + \varepsilon \sin x = 0 \text{ for some } x \in (c_1(\mu), c_2(\mu))\}.\end{aligned}$$

Now, $(1-\varepsilon)\lambda_\mu + \varepsilon \sin x = 0 \Rightarrow x = \sin^{-1}\left(\frac{-\lambda_\mu(1-\varepsilon)}{\varepsilon}\right) = \sin^{-1}(\Delta)$ where $\Delta = \frac{-\lambda_\mu(1-\varepsilon)}{\varepsilon}$.

Then the equation has a solution in x if and only if $\psi_\mu \le \Delta \le \phi_\mu$. This yields:

$$\max\left(0,\ \min\left\{\frac{\lambda_\mu}{\lambda_\mu - \psi_\mu}, 1\right\}\right) \le \varepsilon \le \max\left(0,\ \min\left\{\frac{\lambda_\mu}{\lambda_\mu - \phi_\mu}, 1\right\}\right); \text{ if } \lambda_\mu - \phi_\mu > 0$$

$$\min\left(1, \max\left\{\frac{\lambda_\mu}{\lambda_\mu - \psi_\mu}, \frac{\lambda_\mu}{\lambda_\mu - \phi_\mu}, 0\right\}\right) \le \varepsilon; \text{ if } \lambda_\mu - \phi_\mu < 0 \text{ and } \lambda_\mu - \psi_\mu > 0 \text{ and}$$

$$\max\left(0,\ \min\left\{\frac{\lambda_\mu}{\lambda_\mu - \phi_\mu}, 1\right\}\right) \le \varepsilon \le \max\left(0,\ \min\left\{\frac{\lambda_\mu}{\lambda_\mu - \psi_\mu}, 1\right\}\right); \text{ if } \lambda_\mu - \psi_\mu < 0.$$

Case 2: When $x < c_1(\mu)$ and $x > c_2(\mu)$.

In this case we have, $W_\gamma(G_\varepsilon) = (1-\varepsilon)\lambda_\mu \ne 0$. Since W_γ does not involve x, for any ε there exist no solution for which $x < c_1(\mu)$ and $x > c_2(\mu)$. Noting $-2C_\gamma \le \lambda_\mu \le 2C_\gamma$,

we have

$$\varepsilon^*_{\mu,\gamma}(W_\gamma) = \begin{cases} \max\left(0,\ \min\left\{\frac{\lambda_\mu}{\lambda_\mu - \psi_\mu}, 1\right\}\right) & \text{if} \quad 2C_\gamma \ge \lambda_\mu \ge \phi_\mu \\ \min\left(1,\ \max\left\{\frac{\lambda_\mu}{\lambda_\mu - \psi_\mu}, \frac{\lambda_\mu}{\lambda_\mu - \phi_\mu}, 0\right\}\right) & \text{if} \quad \psi_\mu \le \lambda_\mu \le \phi_\mu \\ \max\left(0,\ \min\left\{\frac{\lambda_\mu}{\lambda_\mu - \phi_\mu}, 1\right\}\right) & \text{if} \quad -2C_\gamma \le \lambda_\mu \le \psi_\mu. \end{cases} \quad \text{... (5.15)}$$

b) In order to obtain the PBP of W_γ we numerically evaluate (5.15) for different values of κ. The following Table 4 shows PBP for different values of μ and κ. Figure 3 given below is based on Table 4.

Table 4: PBP of W_γ for different values of μ and κ.

Mu	$\kappa = 1$	$\kappa = 2$	$\kappa = 4$	$\kappa = 10$	Mu	$\kappa = 1$	$\kappa = 2$	$\kappa = 4$	$\kappa = 10$
0.00	0.00	0.00	0.00	0.00	3.25	0.48	0.07	0.11	0.19
0.25	0.68	0.16	0.26	0.44	3.50	0.75	0.23	0.37	0.64
0.50	0.80	0.31	0.52	0.93	3.75	0.83	0.37	0.64	1.00
0.75	0.85	0.45	0.81	1.00	4.00	0.87	0.51	0.97	1.00
1.00	0.88	0.61	1.00	1.00	4.25	0.90	0.69	1.00	1.00
1.25	0.91	0.81	1.00	1.00	4.50	0.92	0.93	1.00	1.00
1.50	0.93	1.00	1.00	1.00	4.75	0.93	1.00	1.00	1.00
1.75	0.92	0.96	1.00	1.00	5.00	0.91	0.83	1.00	1.00
2.00	0.90	0.71	1.00	1.00	5.25	0.89	0.62	1.00	1.00
2.25	0.87	0.53	1.00	1.00	5.50	0.86	0.46	0.84	1.00
2.50	0.83	0.38	0.67	1.00	5.75	0.81	0.32	0.54	0.98
2.75	0.76	0.25	0.40	0.69	6.00	0.70	0.18	0.29	0.49
3.00	0.54	0.10	0.15	0.24	6.25	0.22	0.02	0.03	0.06
3.14	0.01	0.00	0.00	0.00	6.28	0.03	0.00	0.00	0.01

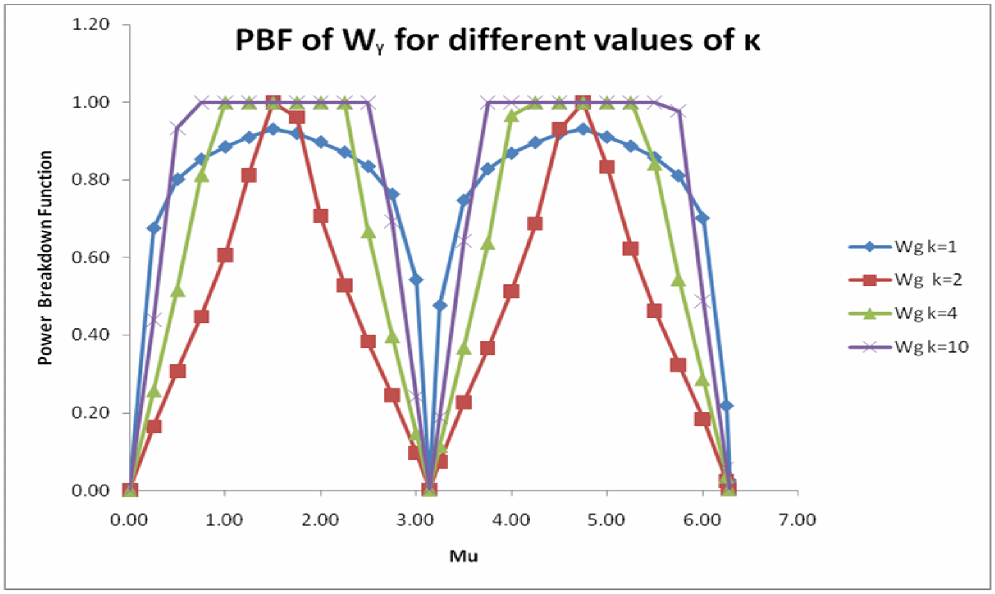

Figure 3: Variation of $\varepsilon_{\mu}^{*}(W_{\gamma})$ with μ and for different values of κ.

From Figure 3 we see that the PBF is periodic about π. It can be seen that as κ increases the PBF for values of μ outside a neighbourhood of 0 and a neighbourhood of π is very close to one which is also reflected in Table 5 given below.

Table 5: The PBP values of W_{γ} for different κ

Test Functional W_{γ}	$\kappa = 1$	$\kappa = 2$	$\kappa = 4$	$\kappa = 10$
PBF	0.93	1	1	1

Hence the theorem.

5.5 Comparison of Robustness of Different Test Statistics

Here we gives the comparisons between the LBP and PBP of different tests for different values of κ. From Table 6 below we can see that the circular trimmed mean has higher LBP and PBP for all $\kappa = 1, 2, 4, 10$. Hence it appears that the circular trimmed mean is a more robust test statistic compared to likelihood ratio test statistic and directional mean as test statistic.

Table 6: Comparison between the LBP and PBP of different tests for different κ.

Test Functional	LBP for different values of κ				PBP for different values of κ			
	$\kappa=1$	$\kappa=2$	$\kappa=4$	$\kappa=10$	$\kappa=1$	$\kappa=2$	$\kappa=4$	$\kappa=10$
W	0.62	0.82	0.93	0.97	0.31	0.41	0.46	0.49
W_1	1	1	1	1	0.31	0.41	0.46	0.49
W_{γ}	1	1	1	1	0.93	1	1	1

As pointed out in He, Simpson and Portnoy (1990) a comparison of LBP and PBP may not be enough for deciding on the robustness of the test statistics. A more detailed comparison can be done by comparing the LBF's and PBF's. From Figure 4 it can be seen that the LBF of W_1 and W_{γ} do not dominate the LBF of W for all values of μ. The LBF of the three test statistics take similar values in the neighbourhood of zero and then the LBF values of W_1 and W_{γ} becomes larger than that of W. However, this trend is not continued. We see that the LBF values of W are larger than that of W_1 and W_{γ} in a zone around π. Thus, we can say that W_1 and W_{γ} has similar or better robustness with respect to level breakdown than W locally around 0. The following Table 7 shows LBFs of the test functionals W, W_1 and W_{γ} for $\kappa=1$. Figure 4 given below is based on Table 7.

Table 7: LBFs of the test functionals W, W_1 and W_{γ} for $\kappa=1$.

Mu	W	W_1	W_{γ}	Mu	W	W_1	W_{γ}
0.00	0.00	0.00	0.00	3.50	0.61	0.14	0.16
0.25	0.11	0.10	0.11	3.75	0.60	0.24	0.26

0.50	0.20	0.20	0.22	4.00	0.56	0.34	0.37
0.75	0.28	0.29	0.32	4.25	0.53	0.47	0.50
1.00	0.35	0.41	0.44	4.50	0.50	0.67	0.70
1.25	0.40	0.57	0.60	4.75	0.46	0.92	0.93
1.50	0.45	0.86	0.88	5.00	0.41	0.60	0.63
1.75	0.49	0.71	0.74	5.25	0.35	0.43	0.46
2.00	0.53	0.49	0.52	5.50	0.29	0.31	0.33
2.25	0.56	0.36	0.38	5.75	0.21	0.21	0.23
2.50	0.59	0.25	0.27	6.00	0.12	0.11	0.13
2.75	0.61	0.16	0.17	6.25	0.01	0.01	0.02
3.00	0.62	0.06	0.07				
3.25	0.62	0.05	0.05				

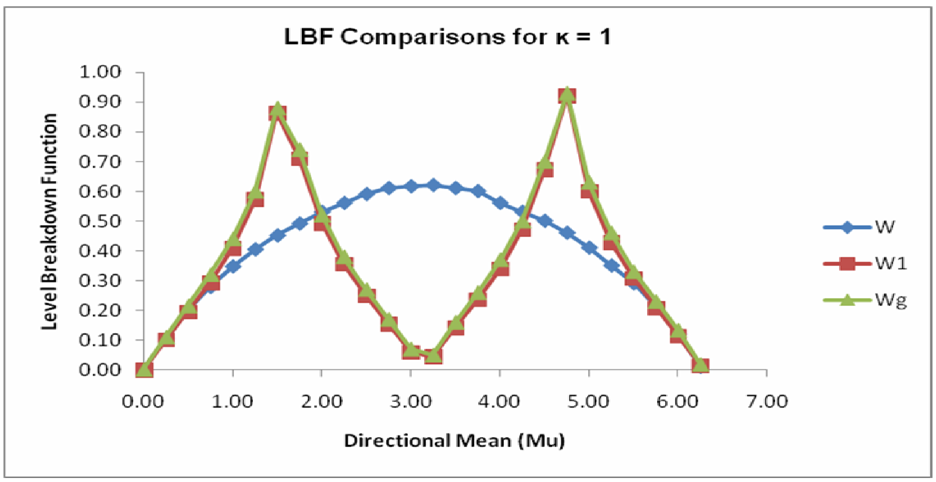

Figure 4: Graphical comparison of LBFs of the three test statistics for $\kappa = 1$

From Figure 5 (a), (b), (c) and (d) we see that the PBF of W_γ clearly dominates that of W_1 and W for $\kappa = 1, 2, 4$ and 10. Therefore we can say with reasonable confidence that W_γ has superior power breakdown property in comparisons to W_1 and W. Based on the study of LBFs, PBFs, LBPs and PBPs, we can conclude that the circular trimmed mean as a test statistic has superior robustness properties compared to the classical LRT statistic and directional mean as a test statistic. We have tabulated the PBFs of the aforesaid test functional for different values of μ and for fixed κ. Figures 5, 6, 7 and 8 given below are respectively based on Tables 8, 9, 10, and 11.

Table 8: PBFs of the test functionals W, W_1 and W_γ for $\kappa = 1$

$\kappa = 1$							
Mu	W	W1	W_γ	Mu	W	W1	W_γ
0.00	0.00	0.00	0.00	3.25	0.05	0.05	0.48
0.25	0.10	0.10	0.68	3.50	0.14	0.14	0.75
0.50	0.18	0.18	0.80	3.75	0.20	0.20	0.83
0.75	0.23	0.23	0.85	4.00	0.25	0.25	0.87
1.00	0.27	0.27	0.88	4.25	0.29	0.29	0.90
1.25	0.30	0.30	0.91	4.50	0.30	0.30	0.92
1.50	0.31	0.31	0.93	4.75	0.31	0.31	0.93
1.75	0.31	0.31	0.92	5.00	0.30	0.30	0.91
2.00	0.29	0.29	0.90	5.25	0.28	0.28	0.89
2.25	0.26	0.26	0.87	5.50	0.24	0.24	0.86
2.50	0.21	0.21	0.83	5.75	0.18	0.18	0.81
2.75	0.15	0.15	0.76	6.00	0.11	0.11	0.70
3.00	0.06	0.06	0.54	6.25	0.01	0.01	0.22
3.14	0.00	0.00	0.01	6.28	0.00	0.00	0.03

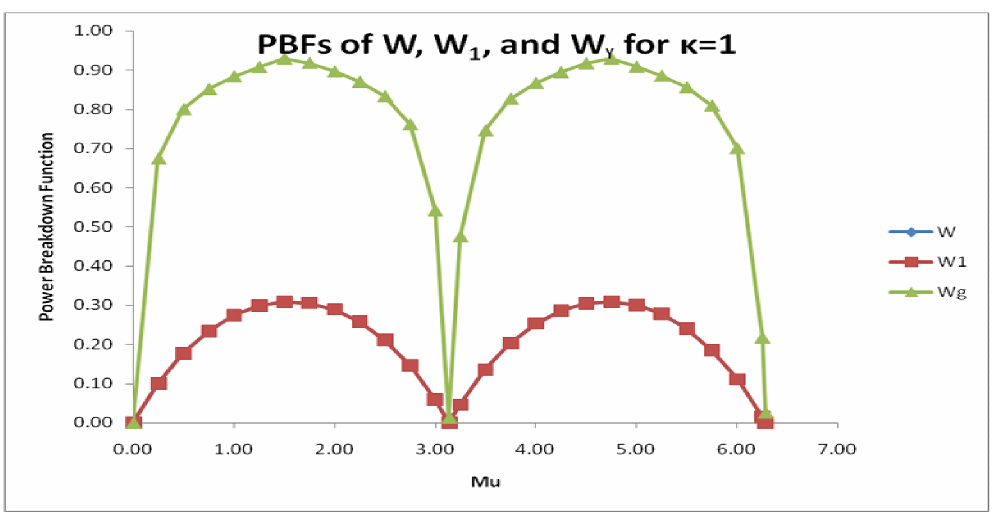

Figure 5: Graphical comparison of PBFs of the three test statistics for $\kappa = 1$

Table 9: PBFs of the test functionals W, W_1 and W_γ for $\kappa = 2$

$\kappa = 2$							
Mu	W	W1	W_γ	Mu	W	W1	W_γ
0.00	0.00	0.00	0.00	3.25	0.07	0.05	0.07
0.25	0.15	0.09	0.16	3.50	0.20	0.12	0.23
0.50	0.25	0.15	0.31	3.75	0.29	0.16	0.37
0.75	0.32	0.18	0.45	4.00	0.35	0.19	0.51
1.00	0.37	0.20	0.61	4.25	0.38	0.21	0.69
1.25	0.40	0.21	0.81	4.50	0.41	0.22	0.93
1.50	0.41	0.22	1.00	4.75	0.41	0.22	1.00
1.75	0.41	0.22	0.96	5.00	0.40	0.21	0.83
2.00	0.39	0.21	0.71	5.25	0.37	0.20	0.62
2.25	0.35	0.19	0.53	5.50	0.33	0.18	0.46
2.50	0.29	0.17	0.38	5.75	0.26	0.15	0.32
2.75	0.21	0.13	0.25	6.00	0.16	0.10	0.18
3.00	0.09	0.06	0.10	6.25	0.02	0.02	0.02
3.14	0.00	0.00	0.00	6.28	0.00	0.00	0.00

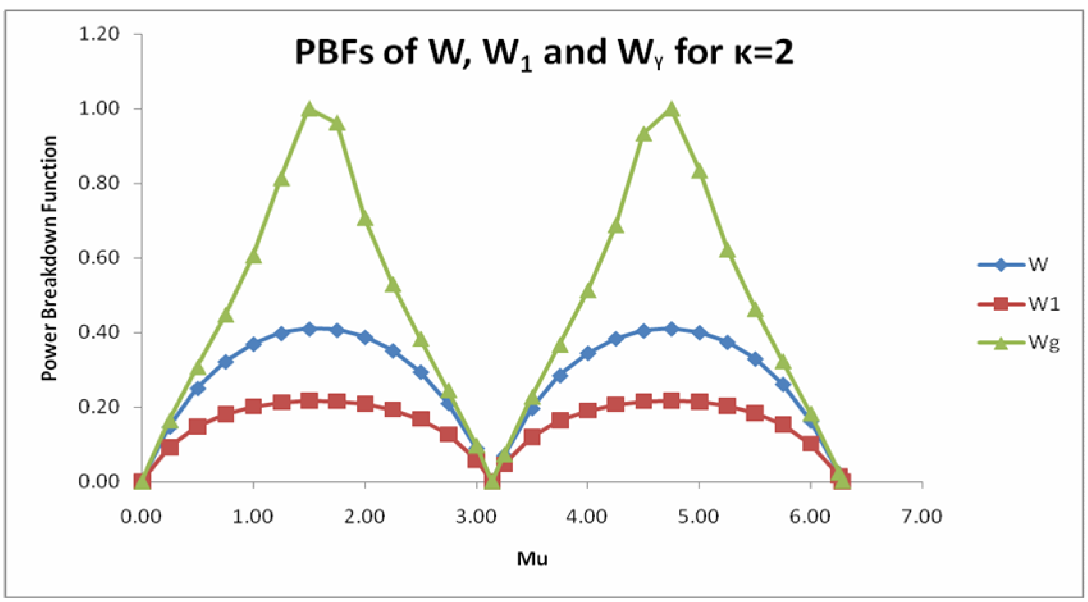

Figure 6: Graphical comparison of PBFs of the three test statistics for $\kappa = 2$

Table 10: PBFs of the test functionals W, W_1 and W_γ for $\kappa = 4$

$\kappa = 4$							
Mu	W	W1	W_γ	Mu	W	W1	W_γ
0.00	0.00	0.00	0.00	3.25	0.09	0.09	0.11
0.25	0.18	0.18	0.26	3.50	0.23	0.23	0.37
0.50	0.29	0.29	0.52	3.75	0.33	0.33	0.64
0.75	0.37	0.37	0.81	4.00	0.40	0.40	0.97
1.00	0.42	0.42	1.00	4.25	0.44	0.44	1.00
1.25	0.45	0.45	1.00	4.50	0.46	0.46	1.00
1.50	0.46	0.46	1.00	4.75	0.46	0.46	1.00
1.75	0.46	0.46	1.00	5.00	0.45	0.45	1.00
2.00	0.44	0.44	1.00	5.25	0.43	0.43	1.00
2.25	0.40	0.40	1.00	5.50	0.38	0.38	0.84
2.50	0.34	0.34	0.67	5.75	0.31	0.31	0.54
2.75	0.25	0.25	0.40	6.00	0.19	0.19	0.29
3.00	0.11	0.11	0.15	6.25	0.03	0.03	0.03
3.14	0.00	0.00	0.00	6.28	0.00	0.00	0.00

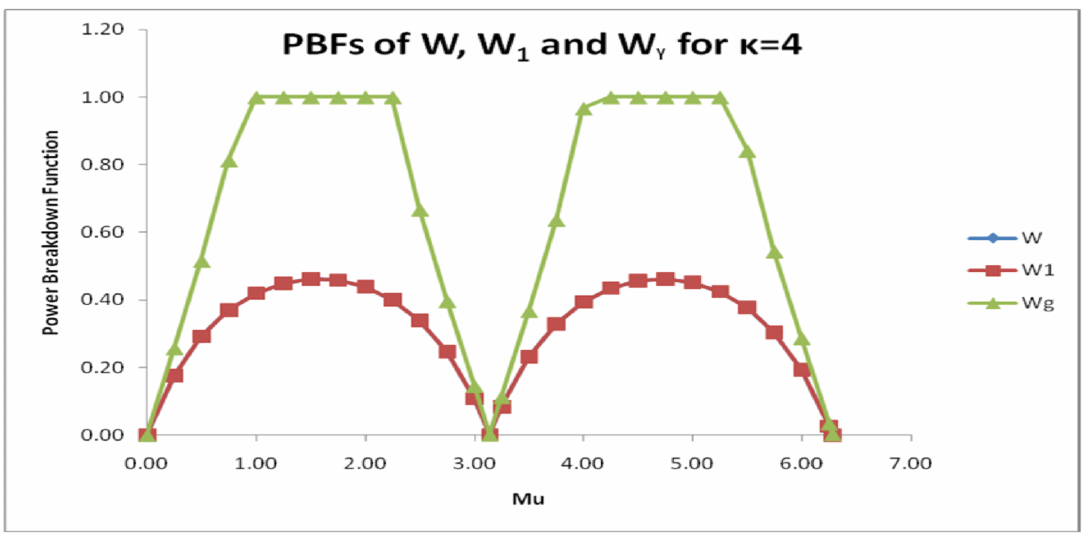

Figure 7: Graphical comparison of PBFs of the three test statistics for $\kappa = 4$

Table 11: PBFs of the test functionals W, W_1 and W_γ for $\kappa = 10$

$\kappa = 10$							
Mu	W	W1	W_γ	Mu	W	W1	W_γ
0.00	0.00	0.00	0.00	3.25	0.09	0.09	0.19
0.25	0.19	0.19	0.44	3.50	0.25	0.25	0.64
0.50	0.31	0.31	0.93	3.75	0.35	0.35	1.00
0.75	0.39	0.39	1.00	4.00	0.42	0.42	1.00
1.00	0.44	0.44	1.00	4.25	0.46	0.46	1.00
1.25	0.47	0.47	1.00	4.50	0.48	0.48	1.00
1.50	0.49	0.49	1.00	4.75	0.49	0.49	1.00
1.75	0.48	0.48	1.00	5.00	0.48	0.48	1.00
2.00	0.46	0.46	1.00	5.25	0.45	0.45	1.00
2.25	0.42	0.42	1.00	5.50	0.40	0.40	1.00
2.50	0.36	0.36	1.00	5.75	0.33	0.33	0.98
2.75	0.27	0.27	0.69	6.00	0.21	0.21	0.49
3.00	0.12	0.12	0.24	6.25	0.03	0.03	0.06
3.14	0.00	0.00	0.00	6.28	0.00	0.00	0.01

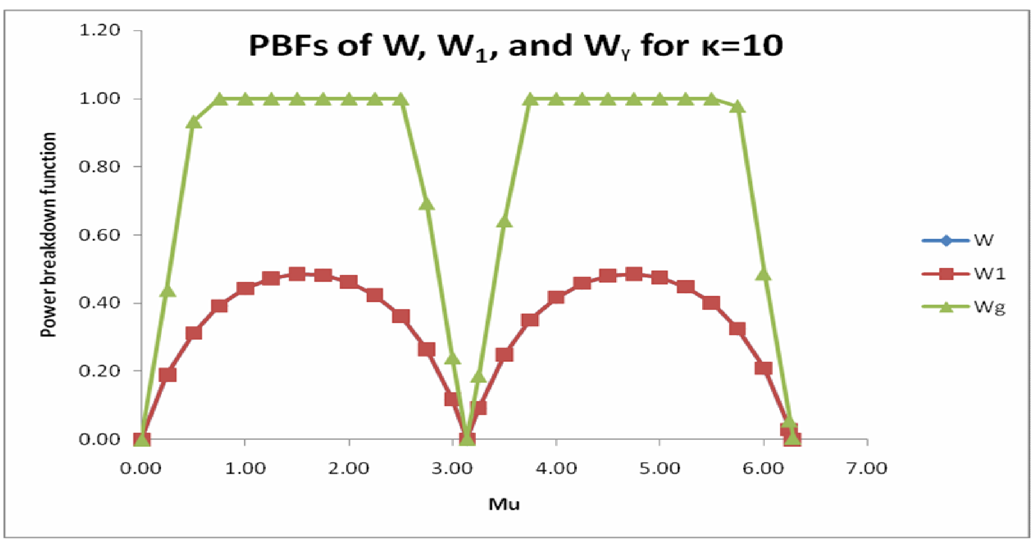

Figure 8: Graphical comparison of PBFs of the three test statistics for $\kappa = 10$

Note: In all the above figures W_g **stands for** W_γ.

Chapter 6

Robustness of Tests for Concentration Parameter of Circular Normal Distribution

6.1 Introduction

Several tests for concentration parameter κ of circular normal distribution have been developed in the literature (see Mardia and Jupp, 2000) but the robustness aspect of these tests has not been explored in the literature. In this chapter we study the robustness of the following test functionals: $V(F) = E_F(\cos\theta)$ (Jammalamadaka and SenGupta, 2001, p.123) and $V_\gamma(F) = g^{*-1}\left[E_{\gamma,F}(d(\theta))\right]$ (for definition see chapter 4, section 4.3) where $\gamma \in [0, 0.5)$ is the trimming proportion and $d(\theta) = \pi - |\pi - |\theta||$. We adopt the approach of He, Simpson and Portnoy (1990) to study the robustness of two single sample tests for the concentration parameter κ of the circular normal distribution with p.d.f

$$f(\theta;\mu,\kappa) = \frac{1}{2\pi I_0(\kappa)} e^{\kappa\cos(\theta-\mu)}, 0 \le \theta < 2\pi \text{ where } 0 \le \mu < 2\pi \text{ and } \kappa > 0$$

where $I_0(\kappa)$ is the modified Bessel function of the first kind and order zero. Assuming that the parameter μ is known we consider the testing problem $H_0 : \kappa = \kappa_0$ against $H_1 : \kappa \ne \kappa_0$ where $\kappa_0 > 0$. We compare the robustness of the V and V_γ by studying their LBF and PBF and also their LBP and PBP. In this chapter we have developed one theorem on the robustness of tests for concentration parameter of circular normal distribution.

The organisation of this chapter is as follows. In Section 6.2 we discuss the robustness of the complete sufficient statistic V and the trimmed estimator V_γ. In Section 6.3 we compare the LBFs and PBFs of both V and V_γ.

6.2 Robustness of the Test for the Concentration Parameter

Let $\theta_1, \theta_2, \cdots, \theta_n$ be a random sample from $CN(\mu, \kappa)$ with μ known. Without loss of generality we assume $\mu = 0$. Consider the hypothesis testing problem: $H_0 : \kappa = \kappa_0$ against $H_1 : \kappa \neq \kappa_0$ where $\kappa_0 > 0$ and fixed. A complete sufficient test statistic for testing H_0 against H_1 is $V = \sum_{i=1}^{n} \cos\theta_i$ (see Jammalamadaka and SenGupta, 2001, p.123). A UMPU test for the two sided alternative is based on V having critical region $V \le \nu_1$ or $V \ge \nu_2$ with $P_0(V \le \nu_1) + P_0(V \ge \nu_2) = \alpha$. The corresponding functional form of V is $V(F) = E_F(\cos\theta)$.

Let F_0 denote the $CN(0, \kappa)$ distribution and δ_x denote the point mass at x, $t_1 = t_1(\kappa) = \dfrac{A(\kappa_0) - A(\kappa)}{1 + A(\kappa_0)}$ and $t_2 = t_2(\kappa) = \dfrac{A(\kappa) - A(\kappa_0)}{1 - A(\kappa_0)}$ where $A(\kappa) = \dfrac{I_1(\kappa)}{I_0(\kappa)}$. Also we have $E_{F_0}(\cos\theta) = A(\kappa)$ under the null. The following Theorem 6.1 below gives the LBF and LBP of V.

Theorem 6.1: *The LBF of V is* $\varepsilon_\kappa^{**}(V) = \min(1,\ \max(t_1, t_2, 0))$ *and the LBP of V is* $\varepsilon^{**} = 1$.

Proof:

Consider the test functional $V(F) = E_F(\cos\theta)$ and let $G_\varepsilon = (1-\varepsilon)F_0 + \varepsilon\delta_x$ is the contaminated model under the assumption that H_0 is true. Then we have,

$V(G_\varepsilon) = V((1-\varepsilon)F_0 + \varepsilon\delta_x) = (1-\varepsilon)A(\kappa_0) + \varepsilon\cos x$ and $V(F_\kappa) = A(\kappa)$. Using (2.6) in chapter 2, the LBF of V is given by

$$\begin{aligned} \varepsilon_\kappa^{**}(V) &= \inf\{\varepsilon > 0 : V(G_\varepsilon) = V(F_\kappa) \text{ for some } x \in [0, 2\pi)\} \\ &= \inf\{\varepsilon > 0 : (1-\varepsilon)A(\kappa_0) - A(\kappa) + \varepsilon\cos x = 0 \text{ for some } x \in [0, 2\pi)\}. \end{aligned} \quad \text{... (6.1)}$$

Now $(1-\varepsilon)A(\kappa_0)-A(\kappa)+\varepsilon\cos x=0$ has a solution if and only if $|\Delta|\le 1$ where $\Delta=\varepsilon^{-1}(A(\kappa)-(1-\varepsilon)A(\kappa_0))$. Solving for ε we get

$$\varepsilon\ge\frac{A(\kappa_0)-A(\kappa)}{1+A(\kappa_0)}\text{ and }\varepsilon\ge\frac{A(\kappa)-A(\kappa_0)}{1-A(\kappa_0)}\Rightarrow\varepsilon\ge\max\left\{\frac{A(\kappa_0)-A(\kappa)}{1+A(\kappa_0)},\ \frac{A(\kappa)-A(\kappa_0)}{1-A(\kappa_0)}\right\}.$$

Thus the LBF of V is given by $\varepsilon_\kappa^{**}(V)=\min(1,\ \max(t_1,t_2,0))$. Further, the LBP of V is $\varepsilon^{**}=\sup_\mu(\varepsilon_\mu^{**})=1$.

Hence the theorem.

Let F_0 denote the $CN(0,\kappa)$ distribution and δ_x denote the point mass at x, $t_3=t_3(\kappa)=\frac{A(\kappa)-A(\kappa_0)}{1+A(\kappa)}$ and $t_4=t_4(\kappa)=\frac{A(\kappa_0)-A(\kappa)}{1-A(\kappa)}$ where $A(\kappa)=\frac{I_1(\kappa)}{I_0(\kappa)}$. Also we have $E_{F_\kappa}(\cos\theta)=A(\kappa)$. The following Theorem 6.2 below gives the PBF and PBP of V.

Theorem 6.2: *The PBF of V is* $\varepsilon_\kappa^{*}(V)=\min(1,\max(t_3,t_4,0))$ *and the PBP of V is* $\varepsilon^{*}=1$.

Proof:

Let $G_\varepsilon=(1-\varepsilon)F_\kappa+\varepsilon\delta_x$. Then we have, $V(G_\varepsilon)=V((1-\varepsilon)F_\kappa+\varepsilon\delta_x)=(1-\varepsilon)A(\kappa)+\varepsilon\cos x$ and $V(F_0)=A(\kappa_0)$. Using (2.7) in chapter 2, the PBF of V is given by

$$\begin{aligned}\varepsilon_\kappa^{*}(V)&=\inf\{\varepsilon>0:V(G_\varepsilon)=V(F_0)\text{ for some }x\in[0,2\pi)\}\\&=\inf\{\varepsilon>0:(1-\varepsilon)A(\kappa)-A(\kappa_0)+\varepsilon\cos x=0\text{ for some }x\in[0,2\pi)\}.\end{aligned}\quad\text{... (6.2)}$$

Now $(1-\varepsilon)A(\kappa)-A(\kappa_0)+\varepsilon\cos x=0$ has a solution if and only if $|\Delta_1|\le 1$ where $\Delta_1=\varepsilon^{-1}(A(\kappa_0)-(1-\varepsilon)A(\kappa))$. Solving for ε we get

$$\varepsilon \geq \frac{A(\kappa)-A(\kappa_0)}{1+A(\kappa)} \text{ and } \varepsilon \geq \frac{A(\kappa_0)-A(\kappa)}{1-A(\kappa)} \Rightarrow \varepsilon \geq \max\left\{\frac{A(\kappa)-A(\kappa_0)}{1+A(\kappa)}, \frac{A(\kappa_0)-A(\kappa)}{1-A(\kappa)}\right\}.$$

Thus the PBF of V is given by $\varepsilon_\kappa^*(V) = \min(1, \max(t_3, t_4, 0))$. Further, the PBP of V is given by $\varepsilon^* = \sup_\mu(\varepsilon_\mu^*) = 1$.

Hence the theorem.

We now consider γ- trimmed estimator as a test statistic and study its robustness based on its breakdown properties. We define the new estimator as: $V_\gamma(F) = g^{*^{-1}}[E_{\gamma,F}(d(\theta))]$ where $E_{\gamma,F}(d\theta) = (1-2\gamma)^{-1}\int_{\beta(\kappa)}^{\alpha(\kappa)} d(\theta)dF$ such that $\alpha(\kappa)+\beta(\kappa) = 2\pi$ and $d(\theta)$ defined earlier. Let F_0 denote the $CN(0,\kappa)$ distribution and δ_x denote the point mass at x. Then by using the definitions of LBF and PBF we get

$$\varepsilon_\kappa^{**}(V_\gamma) = \begin{cases} \inf\{\varepsilon > 0 : (1-\varepsilon)g^*(\kappa_0) + \varepsilon d(x) = g^*(\kappa) \text{ for some } x \in (\beta_\gamma(\kappa_0), \alpha_\gamma(\kappa_0))\} \\ \inf\{\varepsilon > 0 : (1-\varepsilon)g^*(\kappa_0) = g^*(\kappa) \text{ for some } x \notin (\beta_\gamma(\kappa_0), \alpha_\gamma(\kappa_0))\} \end{cases} \quad \text{... (6.3)}$$

and

$$\varepsilon_\kappa^{*}(V_\gamma) = \begin{cases} \inf\{\varepsilon > 0 : (1-\varepsilon)g^*(\kappa) + \varepsilon d(x) = g^*(\kappa_0) \text{ for some } x \in (\beta_\gamma(\kappa_0), \alpha_\gamma(\kappa_0))\} \\ \inf\{\varepsilon > 0 : (1-\varepsilon)g^*(\kappa) = g^*(\kappa_0) \text{ for some } x \notin (\beta_\gamma(\kappa_0), \alpha_\gamma(\kappa_0))\}. \end{cases} \quad \text{... (6.4)}$$

Since both the expressions (6.3) and (6.4) are not in a closed form, we evaluate both LBF and PBF numerically for different values of kappa. The findings are summarised in the graphs along with the graphs of LBF and PBF of V.

6.3. Comparisons

A graphical comparison between the two test statistics for different values of κ is given below. From the figures 9 and 10 it is seen that the functional V is better than V_γ in terms of both its power breakdown property and also level breakdown property. Thus we may conclude that V has better robustness property than V_γ. Figures 9 and

10 are based on the Tables 12 and 13 respectively.

Table 12: Combined LBFs of V and V_γ for different κ

κ	V	V_γ	κ	V	V_γ	κ	V	V_γ	κ	V	V_γ
0.01	0.31	0.43	0.50	0.14	0.19	1.35	0.20	0.16	3.00	0.66	0.50
0.02	0.30	0.43	0.55	0.13	0.17	1.40	0.22	0.18	3.25	0.69	0.52
0.03	0.30	0.42	0.60	0.11	0.15	1.45	0.25	0.20	3.50	0.71	0.54
0.04	0.29	0.41	0.65	0.10	0.13	1.50	0.27	0.22	3.75	0.73	0.56
0.05	0.29	0.41	0.70	0.08	0.11	1.55	0.29	0.24	4.00	0.75	0.57
0.06	0.29	0.4	0.75	0.07	0.09	1.60	0.31	0.26	4.50	0.78	0.61
0.07	0.28	0.4	0.80	0.05	0.07	1.65	0.33	0.28	5.00	0.81	0.62
0.08	0.28	0.4	0.85	0.04	0.05	1.70	0.35	0.30	5.50	0.83	0.64
0.09	0.28	0.39	0.90	0.03	0.03	1.75	0.37	0.30	6.00	0.84	0.66
0.10	0.27	0.38	0.95	0.01	0.02	1.80	0.39	0.30	6.50	0.85	0.68
0.15	0.26	0.36	1.00	0.00	0.01	1.85	0.41	0.32	7.00	0.87	0.69
0.20	0.24	0.33	1.05	0.03	0.02	1.90	0.42	0.33	7.50	0.87	0.70
0.25	0.22	0.31	1.10	0.06	0.02	1.95	0.44	0.34	8.00	0.88	0.71
0.30	0.21	0.29	1.15	0.09	0.07	2.00	0.45	0.35	8.50	0.89	0.72
0.35	0.19	0.26	1.20	0.12	0.10	2.25	0.52	0.40	9.00	0.90	0.72
0.40	0.17	0.24	1.25	0.15	0.12	2.50	0.58	0.44	9.50	0.90	0.73
0.45	0.16	0.22	1.30	0.17	0.14	2.75	0.62	0.47	10.00	0.91	0.74

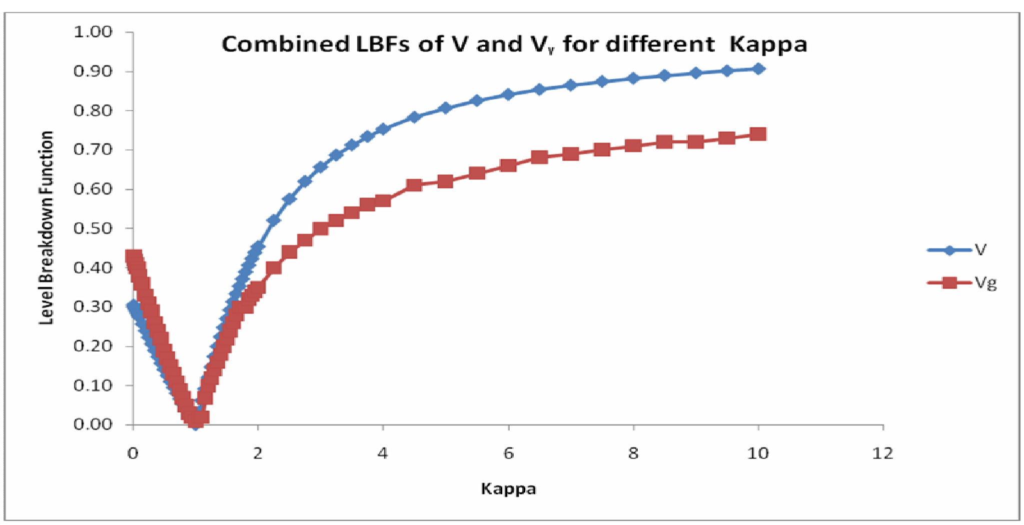

Figure 9: Graphical comparisons of LBFs for different values of κ

Table 13: Combined PBFs of V and V_γ for different κ

κ	V	V_γ	κ	V	V_γ	κ	V	V_γ	κ	V	V_γ
0.01	0.44	0.42	0.50	0.27	0.25	1.35	0.25	0.12	3.00	1.00	0.63
0.02	0.44	0.42	0.55	0.25	0.23	1.40	0.29	0.13	3.25	1.00	0.69
0.03	0.44	0.41	0.60	0.22	0.20	1.45	0.33	0.14	3.50	1.00	0.74
0.04	0.44	0.41	0.65	0.20	0.18	1.50	0.37	0.15	3.75	1.00	0.81
0.05	0.43	0.41	0.70	0.17	0.16	1.55	0.41	0.17	4.00	1.00	0.86
0.06	0.43	0.40	0.75	0.15	0.14	1.60	0.46	0.19	4.50	1.00	0.99
0.07	0.43	0.40	0.80	0.12	0.11	1.65	0.50	0.21	5.00	1.00	
0.08	0.42	0.40	0.85	0.09	0.08	1.70	0.55	0.22	5.50	1.00	
0.09	0.42	0.40	0.90	0.06	0.05	1.75	0.59	0.24	6.00	1.00	
0.10	0.42	0.40	0.95	0.03	0.03	1.80	0.64	0.26	6.50	1.00	
0.15	0.40	0.39	1.00	0.00	0.01	1.85	0.69	0.27	7.00	1.00	
0.20	0.39	0.37	1.05	0.03	0.02	1.90	0.73	0.29	7.50	1.00	
0.25	0.37	0.35	1.10	0.07	0.03	1.95	0.78	0.31	8.00	1.00	
0.30	0.35	0.33	1.15	0.10	0.05	2.00	0.83	0.32	8.50	1.00	
0.35	0.33	0.31	1.20	0.14	0.06	2.25	1.00	0.40	9.00	1.00	
0.40	0.31	0.29	1.25	0.17	0.08	2.50	1.00	0.48	9.50	1.00	
0.45	0.29	0.27	1.30	0.21	0.10	2.75	1.00	0.55	10.00	1.00	

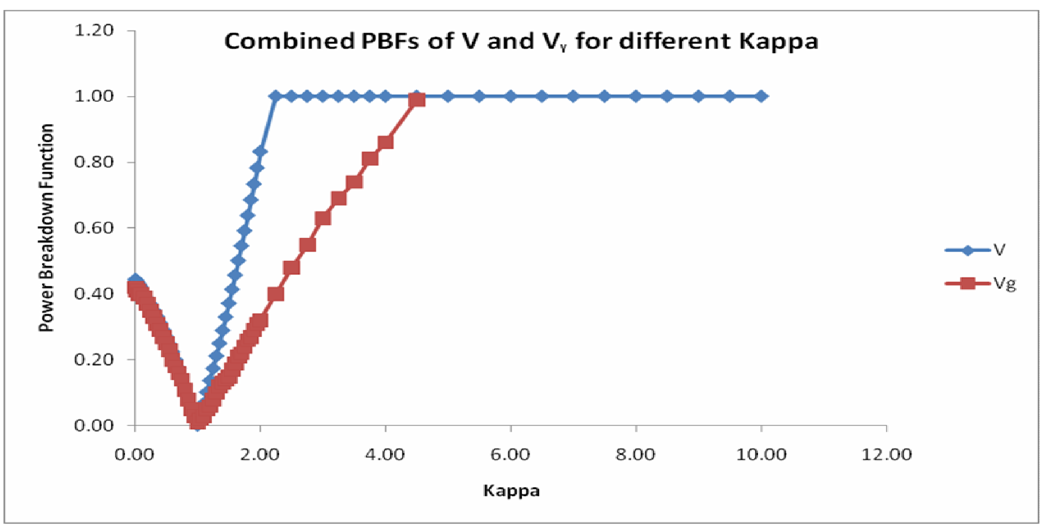

Figure 10: Graphical comparisons of PBFs for different values of κ

Note: In all the above figures V_g **stands for** V_γ.

Chapter 7

Robust Estimators of Parameters of Wrapped Normal Distribution

7.1 Introduction

In this chapter, we focus on the wrapped normal distribution and discuss SB-robustness of the directional mean, the $\gamma-$circular trimmed mean and the concentration parameter at various families of distributions using several different dispersion measures. We study the equivalence of different dispersion measures with respect to (w.r.t.) wrapped normal family of distributions with p.d.f.

$$f(\theta;\mu,\rho)=\frac{1}{2\pi}\left\{1+2\sum_{p=1}^{\infty}(\rho)^{p^2}\cos p(\theta-\mu)\right\}, 0\leq\theta<2\pi, 0\leq\mu<2\pi, 0<\rho<1$$

where $\rho=\exp\left(-\sigma^2/2\right)$. In this chapter we have developed three lemmas, nine theorems and five corollaries on the robust estimators of the parameters of wrapped the normal distribution.

The organization of the chapter is as follows: In Section 7.2 we study the equivalence of different dispersion measures w.r.t. wrapped normal family of distributions. In Section 7.3 we discuss the SB-robustness of the directional mean for the family of wrapped normal distributions w.r.t. different dispersion measures. In Section 7.4 we discuss the SB-robustness of $\gamma-$ circular trimmed mean and show that it is SB-robust for the family of wrapped normal distributions w.r.t. different dispersion measures. In Section 7.5 we discuss the SB-robustness of the usual estimator $\rho_F=\sqrt{E_F^2(\cos\theta)+E_F^2(\sin\theta)}$ of the concentration parameter ρ. We introduce a new estimator for ρ namely $T_\gamma(F)=h^{-1}(E_{\gamma,F}(d(\theta,\mu)))$ and discuss its SB-robustness.

7.2 Equivalent Measures of Dispersion for the Family of Wrapped Normal Distributions

The property of SB-robustness of an estimator T at a family of distributions $\Im$ is in general dependent of the choice of the measure of dispersion. In Section 3.4 of chapter 3, the notion of equivalent measures of dispersion for a family of distributions $\Im$ is introduced and it is shown that the property of SB-robustness of an estimator T at a family of distributions $\Im$ is preserved when we are working with measures of dispersion which are equivalent to one another. This greatly facilitates the study of SB-robustness of an estimator for different measures of dispersion. In Theorem 7.1 below we prove the equivalence of some dispersion measures on the circle for the families of distributions $\Im^* = \{WN(0,\rho); 0 < m < \rho < 1\}$ and $\tilde{\Im} = \{WN(0,\rho); 0 < \rho < 1\}$.

Theorem 7.1:.a) *Consider the family of distributions* $\Im^* = \{WN(0,\rho); 0 < m < \rho < 1\}$ *and define for* $F \in \Im^*$, $S_1(F) = \sqrt{1-\rho}$, $S_2(F) = E_F[d(\theta,0)]$, $S_3(F) = \left(\rho A^{-1}(\rho)\right)^{-\frac{1}{2}}$ *and* $S_4(F) = E_{\gamma,F}[d(\theta,0)]$. *Then* S_1, S_2, S_3 *and* S_4 *are equivalent measures of dispersion for the family of distributions* $\Im^*$.

b) *Now consider the family of distributions* $\tilde{\Im} = \{WN(0,\rho); 0 < \rho < 1\}$. *Then the following are true:*

(1) $S_1 \overset{\tilde{\Im}}{\sim} S_2$ *(2)* $S_2 \overset{\tilde{\Im}}{\sim} S_4$ *and (3)* $S_2 \overset{\tilde{\Im}}{\nsim} S_3$

Proof: **Part a)**

Let $S_1(F) = h(\rho) = \sqrt{1-\rho}$, $S_2(F) = g(\rho) = \frac{1}{2\pi}\int_0^{2\pi}\left(\pi - |\pi-\theta|\right)\left\{1 + 2\sum_{p=1}^{\infty}(\rho)^{p^2}\cos p\theta\right\}d\theta$ and $R(F) = r(\rho) = \frac{h(\rho)}{g(\rho)}$. Using the fact that, as $\rho \to 1$, $WN(0,\rho)$ can be well approximated by $CN(0, A^{-1}(\rho))$ (see Jones and Pewsey (2005), Collett and Lewis (1981) and

Stephens (1963)) and Hill's (1976) approximation for the quantile of the circular normal distribution we get

$$h(\rho)=\frac{1}{\sqrt{A^{-1}(\rho)}}\sqrt{\frac{1}{2}+\frac{1}{8(A^{-1}(\rho))}+\frac{1}{8(A^{-1}(\rho))^2}+o\left((A^{-1}(\rho))^{-3}\right)} \text{ and}$$

$$g(\rho)=\frac{1}{\sqrt{A^{-1}(\rho)}}\int_0^{2\pi\sqrt{A^{-1}(\rho)}}\alpha\ \phi(\alpha)\left[1+\frac{(\alpha^4-3)}{24(A^{-1}(\rho))}+o\left((A^{-1}(\rho))^{-2}\right)\right]d\alpha.$$

A simple calculation using the fact as $\rho\to 1, A^{-1}(\rho)\to\infty$ shows that $\lim_{\rho\to 1} r(\rho)=\sqrt{\pi}$.

Hence, $\sup_{\Im^*} R(F)=\sup_{m<\rho<1} r(\rho)<\infty$ and $\sup_{\Im^*} R^{-1}(F)<\infty$. Thus, $S_1\overset{\Im^*}{\sim}S_2$.

Now consider the dispersion measures S_2 and S_3. Similar computations as above show that

$$\frac{S_2(F)}{S_3(F)}=r(\rho)=\frac{\int_0^{2\pi\sqrt{A^{-1}(\rho)}}\alpha\ \phi(\alpha)\left[1+\frac{(\alpha^4-3)}{24\kappa}+o(\kappa^{-2})\right]d\alpha}{(\rho)^{-\frac{1}{2}}}.$$

It is now straightforward to see that $\lim_{\rho\to 1} r(\rho)=\frac{1}{\sqrt{2\pi}}$. Thus, $\sup_{\Im^*} R(F)=\sup_{m<\rho<1} r(\rho)<\infty$ and $\sup_{\Im^*} R^{-1}(F)<\infty$. Hence $S_2\overset{\Im^*}{\sim}S_3$.

Again, consider

$$\frac{S_2(F)}{S_4(F)}=r(\rho)=\frac{\int_0^{\pi\sqrt{A^{-1}(\rho)}}\alpha\,\phi(\alpha)\left[1+\frac{(\alpha^4-3)}{24(A^{-1}(\rho))}+o\left((A^{-1}(\rho))^{-2}\right)\right]d\alpha}{\frac{1}{(1-2\gamma)}\int_0^{\alpha(\rho)\sqrt{A^{-1}(\rho)}}\alpha\ \phi(\alpha)\left[1+\frac{(\alpha^4-3)}{24(A^{-1}(\rho))}+o\left((A^{-1}(\rho))^{-2}\right)\right]d\alpha}.$$

Now using the facts that as $\rho\to 1, A^{-1}(\rho)\to\infty$ and $\alpha(\rho)\sqrt{A^{-1}(\rho)}\to\Phi^{-1}(1-\gamma)$, and writing $\Phi^{-1}(1-\gamma)=\chi$ we get $\lim_{\rho\to 1} r(\rho)=2(1-2\gamma)\left(2-\exp\left(\frac{-\chi^2}{2}\right)\right)^{-1}$. Thus $S_2\overset{\Im^*}{\sim}S_4$.

Since $\overset{\mathfrak{I}^*}{\sim}$ is an equivalence relation (Laha and Mahesh (2011)) we can conclude that S_1, S_2, S_3 and S_4 are equivalent measures of dispersion for the family of distributions $\mathfrak{I}^*$.

Part b)

1) Let $R(F) = \frac{S_1(F)}{S_2(F)}$. Using Hill's (1976) expansion and the fact that as $\rho \to 0$, $A^{-1}(\rho) \to 0$, it is straightforward to check that $\lim_{\rho \to 0} R(F) = \frac{2}{\pi}$. Also, $\lim_{\rho \to 1} R(F) = \sqrt{\pi}$ (proved in part a) above). Hence we can conclude that, $0 < \sup_{\tilde{\mathfrak{I}}} R(F) < \infty$ and $\sup_{\tilde{\mathfrak{I}}} R^{-1}(F) < \infty$. Thus, $S_1 \overset{\tilde{\mathfrak{I}}}{\sim} S_2$.

2) Let $r(\rho) = R(F) = \frac{S_2(F)}{S_4(F)} = \frac{h(\rho)}{g(\rho)}$. As $\rho \to 0$, $\alpha(\rho) \to \pi(1-2\gamma)$ and hence $g(\rho) \to \frac{(1-2\gamma)}{4}$. Also, as $\rho \to 0$, $h(\rho) \to \frac{\pi}{2}$. Therefore, $\lim_{\rho \to 0} r(\rho) = \frac{2\pi}{1-2\gamma}$. Using the fact that $\lim_{\rho \to 1} r(\rho)$ is finite (proved in part a) above) we conclude that $0 < \sup_{0<\rho<1} r(\rho) < \infty$ which implies, $\sup_{0<\rho<1} r^{-1}(\rho) < \infty$. Hence, $S_2 \overset{\tilde{\mathfrak{I}}}{\sim} S_4$.

3) Let $r(\rho) = R(F) = \frac{S_2(F)}{S_3(F)} = \frac{h(\rho)}{g(\rho)}$**.** Now as $\rho \to 0$, $h(\rho) \to \frac{\pi}{2}$ and $g(\rho) \to \infty$. Therefore $\sup_{0<\rho<1} r(\rho) = 0$ but $\sup_{0<\rho<1} r^{-1}(\rho) = \infty$. Hence, $S_2 \overset{\tilde{\mathfrak{I}}}{\nsim} S_3$.

Hence the theorem.

Theorem 7.2: *If an estimator T is SB-robust with respect to the dispersion measure* $S_1(F) = \sqrt{1-\rho}$ *at the family of distributions* $\tilde{\mathfrak{I}} = \{WN(0,\rho); 0 < \rho < 1\}$ *then it is SB-robust with respect to the dispersion measure* $S_3(F) = (\rho A^{-1}(\rho))^{-\frac{1}{2}}$ *where* $F \in \tilde{\mathfrak{I}}$.

Proof:

Let $S_1(F) = h(\rho) = \sqrt{1-\rho}$, $S_3(F) = g(\rho) = \left(\rho A^{-1}(\rho)\right)^{-\frac{1}{2}}$. Using the asymptotic expansion of $A^{-1}(\rho)$ near unity (see Watson 1983, appendix A.2 with p=2) we get:

$$g(\rho) = \sqrt{(1-\rho)\left(3-\rho+o\left((1-\rho)^3\right)\right)}.$$

Therefore, $\lim_{\rho\to 1}\frac{h(\rho)}{g(\rho)} = \frac{1}{\sqrt{2}}$. Now as $\rho \to 0$, $g(\rho) \to \infty$ and $h(\rho) \to 1$ $\Rightarrow \lim_{\rho\to 0}\frac{h(\rho)}{g(\rho)} = 0$.

Since $\frac{h(\rho)}{g(\rho)}$ is a continuous function of ρ, $\sup_{0<\rho<1}\frac{h(\rho)}{g(\rho)}$ is finite. Again since T is SB-robust at $\tilde{\Im}$ with respect to S_1, we have

$$\gamma^*\left(T,\tilde{\Im},S_1\right) = \sup_{\Im}\sup_{x} SIF(x;T,F,S_1) < \infty.$$

Therefore, we can write,

$$\begin{aligned}\gamma^*\left(T,\tilde{\Im},S_3\right) &= \sup_{\Im}\sup_{x} SIF(x;T,F,S_3)\\ &= \sup_{\Im}\sup_{x}\left[\frac{S_1(F)}{S_3(F)}SIF(x;T,F,S_1)\right]\\ &= \sup_{\Im}\left[\frac{S_1(F)}{S_3(F)}g^*(T,F,S_1)\right]\\ &< \infty\end{aligned}$$

where $g^*(T,F,S_1) = \sup_x SIF(T,F,S_1)$.

Hence the theorem.

7.3 SB-robustness of the Directional Mean

Let θ be a circular random variable having c.d.f. F and let $G_\varepsilon = (1-\varepsilon)F + \varepsilon\delta_x, 0 < \varepsilon < 1$. The directional mean μ of the circular distribution F is defined implicitly as the solution of $\tan\mu = \dfrac{E_F(\sin\theta)}{E_F(\cos\theta)}$. The corresponding estimating functional of μ is $T(F) = \arctan^*\left(\dfrac{E_F(\sin\theta)}{E_F(\cos\theta)}\right)$. In Theorems 7.3 to 7.5 given below establishes the SB-robustness of directional mean for family of wrapped normal distribution for different choices of dispersion measures.

Theorem 7.3: *The directional mean* $T(F) = \arctan^*\left(\dfrac{E_F(\sin\theta)}{E_F(\cos\theta)}\right)$ is *SB-robust at the family of distributions* $\Im^{**} = \{WN(0,\rho); 0 < m < \rho < M < 1\}$ *when the measure of dispersion is* $S_1(F) = \sqrt{1-\rho}$ *where* $F \in \Im^{**}$*.*

Proof:

Using the expression of $IF(x;T,F)$ given in Wehrly and Shine (1981) we get

$$SIF(x;T,F,S_1) = \frac{IF(x;T,F)}{S_1(F)} = \frac{\sin(x-\mu)}{\rho\sqrt{1-\rho}}; 0 < \rho < 1.$$

Then, $\gamma^*(T,\Im^{**},S_1) = \sup\limits_{0<m<\rho<M<1}\left(\left(\rho\{\sqrt{1-\rho}\}\right)^{-1}\right) < \infty$.

Hence the theorem.

Corollary 1: Since $T(F)$ is SB-robust at the family of distributions $\Im^{**}$ with respect to the dispersion measure $S_1(F)$, it is also SB-robust at the family of distributions $\Im^{**}$ with respect to the dispersion measures $S_2(F)$, $S_3(F)$ and $S_4(F)$ using Theorem 7.1(a) and noting that $\Im^{**} \subset \Im^*$.

Theorem 7.4: *The directional mean* $T(F) = \arctan^*\left(\frac{E_F(\sin\theta)}{E_F(\cos\theta)}\right)$ *is not SB-robust at the family of distributions* $\tilde{\Im}_1 = \{WN(\mu,\rho); 0 < \rho < 1\}$ *when the measure of dispersion is* $S_2(F) = E_F(d(\theta,\mu))$ *where* $F \in \tilde{\Im}_1$.

The following Lemma 1 is used to prove the theorem.

Lemma 1: *Suppose* $\theta \sim WN(\mu,\rho)$. *Then* $E_F(\cos\theta) = \rho\cos\mu$, $E_F(\sin\theta) = \rho\sin\mu$ *and* $S_2(F) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(\rho)^{(2n+1)^2}}{(2n+1)^2}$.

Proof:

Since $\theta \sim WN(\mu,\rho)$, by definition we have,

$$E_F(\cos\theta) = (2\pi)^{-1}\int_0^{2\pi}\cos\theta\left\{1 + 2\sum_{p=1}^{\infty}(\rho)^{p^2}\cos p(\theta-\mu)\right\}d\theta$$
$$= (\pi)^{-1}\sum_{p=1}^{\infty}(\rho)^{p^2}\int_0^{2\pi}\cos\theta\cos p(\theta-\mu)d\theta$$
$$= \frac{\rho}{\pi}\int_0^{2\pi}\cos\theta\cos p(\theta-\mu)d\theta + (\pi)^{-1}\sum_{p=2}^{\infty}(\rho)^{p^2}\int_0^{2\pi}\cos\theta\cos p(\theta-\mu)d\theta$$
$$= \frac{\rho}{\pi}\int_0^{2\pi}\cos\theta\cos p(\theta-\mu)d\theta, \text{ since } \int_0^{2\pi}\cos m\theta\cos n\theta d\theta = \begin{cases} 0 & \text{for } m \neq n \\ \pi & \text{for } m = n \end{cases}$$
$$= \frac{\rho\cos\mu}{\pi}\int_0^{2\pi}\cos^2\theta d\theta$$
$$= \rho\cos\mu .$$

Similarly we can prove that $E_F(\sin\theta) = \rho\sin\mu$.

Also note that $S_2(F) = E_F(d(\theta,\mu))$ does not depend on μ. Hence we can without loss of generality assume μ=0 for computing $S_2(F)$. Using the substitution $(2\pi - \theta) = \lambda$, and the fact that $\cos p\pi = -1$ $\forall$ odd values of p we have,

$$S_2(F) = E_F(d(\theta)) = (2\pi)^{-1}\int_0^{2\pi}(\pi - |\pi-\theta|)\left\{1 + 2\sum_{p=1}^{\infty}(\rho)^{p^2}\cos p\theta\right\}d\theta$$

$$= (\pi)^{-1}\int_0^{\pi}\theta\left\{1 + 2\sum_{p=1}^{\infty}(\rho)^{p^2}\cos p\theta\right\}d\theta = \frac{\pi}{2} + 2\sum_{p=1}^{\infty}(\rho)^{p^2}\left\{\frac{\cos p\pi - 1}{p^2}\right\}$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(\rho)^{(2n+1)^2}}{(2n+1)^2}.$$

Hence the lemma.

Proof of the theorem:

Let $F \in \tilde{\Im}_1, G_\varepsilon = (1-\varepsilon)F + \varepsilon\delta_x, 0 \le \varepsilon \le 1$ and $\mu_\varepsilon = T((1-\varepsilon)F + \varepsilon\delta_x),\ 0 \le x < 2\pi$. Then we have,

$$\tan\mu_\varepsilon = \frac{E_{G_\varepsilon}(\sin\theta)}{E_{G_\varepsilon}(\cos\theta)} = \frac{E_{T\{(1-\varepsilon)F+\varepsilon\delta_x\}}(\sin\theta)}{E_{T\{(1-\varepsilon)F+\varepsilon\delta_x\}}(\cos\theta)} = \frac{(1-\varepsilon)E_F(\sin\theta) + \varepsilon\sin x}{(1-\varepsilon)E_F(\cos\theta) + \varepsilon\cos x}.$$

Dividing both numerator and denominator by $E_F(\cos\theta)$ and using lemma 1 we get

$$\tan\mu_\varepsilon = \frac{\rho(1-\varepsilon)\sin\mu + \varepsilon\sin x}{\rho(1-\varepsilon)\cos\mu + \varepsilon\cos x} \Rightarrow \mu_\varepsilon = \arctan^*\left\{\frac{\rho(1-\varepsilon)\sin\mu + \varepsilon\sin x}{\rho(1-\varepsilon)\cos\mu + \varepsilon\cos x}\right\}.$$

Consider $\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\tan(\mu_\varepsilon - \mu) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\frac{\tan\mu_\varepsilon - \tan\mu}{1 + \tan\mu_\varepsilon\tan\mu}\right].$

Substituting the value of $\tan\mu_\varepsilon$ and simplifying, we get

$$(\mu_\varepsilon - \mu) \cong \arctan^*\left[\frac{\varepsilon\sin(x-\mu)}{\rho} + o(\varepsilon)\right] \quad \text{where } \lim_{\varepsilon\to 0}\frac{o(\varepsilon)}{\varepsilon} = 0.$$

Using the Taylor's series expansion of $\tan^{-1}(t)$ and applying limit as $\varepsilon \to 0$ we get the influence function of the directional mean as

$$IF(x;T,F) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}(\mu_\varepsilon - \mu) = \frac{\sin(x-\mu)}{\rho}\ ;\ \rho > 0. \qquad \ldots (7.1)$$

Now, the gross error sensitivity (g.e.s) of the estimator T at F is given by

$$\gamma(T,F) = \sup_{0\le x<2\pi} |IF(x;T,F)| = \frac{|\sin(x-\mu)|}{\rho} < \infty .$$

Hence, we see that the directional mean is B-robust at the family of distributions $\tilde{\Im}_1$.

Again using Lemma 1 and (7.1) we get:

$$SIF(x;T,F,S_2) \ = \ \sin(x-\mu)\left\{\rho\left(\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(\rho)^{(2n+1)^2}}{(2n+1)^2}\right)\right\}^{-1} ; 0<\rho<1 .$$

Now the standardized gross error sensitivity (s.g.e.s) of T is given by:

$$\gamma^*(T,\Im,S_2) = \sup_{0<\rho<1}\ \sup_{0\le x<2\pi}[SIF(x;T,F,S_2)] = \sup_{0<\rho<1}\left(\left(\rho\left\{\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(\rho)^{(2n+1)^2}}{(2n+1)^2}\right\}\right)^{-1}\right)$$

$$= \infty \quad \text{since as } \rho \to 1,\ \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(\rho)^{(2n+1)^2}}{(2n+1)^2} \to \frac{\pi}{2}.$$

Hence the theorem.

Corollary 2: Since $T(F)$ is not SB-robust at the family of distributions $\tilde{\Im}$ with respect to the dispersion measure $S_2(F)$, it is also not SB-robust at the family of distributions $\tilde{\Im}$ with respect to the dispersion measures $S_1(F)$ and $S_4(F)$ by Theorem 7.1(b).

Theorem 7.5: *The directional mean* $T(F) = \arctan^*\left(\frac{E_F(\sin\theta)}{E_F(\cos\theta)}\right)$ is *not SB-robust at the family of distributions* $\tilde{\Im}_1 = \{WN(\mu,\rho); 0<\rho<1\}$ *when the measure of dispersion is* $S_3(F) = (\rho A^{-1}(\rho))^{-\frac{1}{2}}$ *where* $F \in \tilde{\Im}_1$.

Proof:

Using the expression of the influence function given in Wehrly and Shine (1981) we get $\mathrm{SIF}(x;T,F,S_3) = \left(\frac{\sin(x-\mu)}{\rho}\right)\left(\frac{\rho}{A^{-1}(\rho)}\right)^{-\frac{1}{2}}; 0<\rho<1$. Now using the asymptotic expansion of $A^{-1}(\rho)$ near unity (Watson 1983, appendix A.2 with p=2) we get $\left(\frac{\rho}{A^{-1}(\rho)}\right)^{-\frac{1}{2}} \to 0$ as $\rho \to 1$. Since $\sup \sin(x-\mu) = 1$ where $x \in [0, 2\pi)$, we have

$$\gamma^*(T,\Im,S_3) = \sup_{0<\rho<1} \sup_{0\le x<2\pi} [\mathrm{SIF}(x;T,F,S_3)] = \infty .$$

Hence the theorem.

7.4 Robustness of the Circular Trimmed Mean

We have seen in Theorems 7.4 and 7.5 that the directional mean is not SB-robust at the family of distributions $\tilde{\Im}_1 = \{WN(\mu,\rho); 0<\rho<1\}$ for all choices of the dispersion measures $S_i(F), i=1,...,4$. In Theorem 7.6 below we prove that γ-circular trimmed mean (for definition see chapter 3, section 3.3) is SB-robust for the family of distributions $\tilde{\Im}_1$ when the measure of dispersion is $S_4(F)$.

Theorem 7.6: Let $0 \le \gamma < 0.5$. *The* γ-CTM (μ_γ) *is SB-robust at the family of distributions* $\tilde{\Im}_1 = \{WN(\mu,\rho); 0<\rho<1\}$ *when the measure of dispersion is* $S_4(F) = E_{\gamma,F}(d(\theta,\mu))$ *where* $F \in \tilde{\Im}_1$.

The following Lemma 2 is used to prove the theorem.

Lemma 2: *Suppose* $\theta \sim WN(\mu,\rho)$. *Then* $\nu_\gamma = c\left[S_{1,0} + \cos\mu \sum_{p=2}^{\infty} (\rho)^{p^2} \left(\frac{S_{p+1,\mu}}{(p+1)} + \frac{S_{p-1,\mu}}{(p-1)}\right)\right]$

and $S_4(F)=\dfrac{1}{2\pi(1-2\gamma)}\left[\alpha^2(\rho)+4\sum_{p=1}^{\infty}(\rho)^{p^2}\left\{\dfrac{\alpha(\rho)S_{p,0}}{p}+\dfrac{C_{p,0}-1}{p^2}\right\}\right]$ *where* $c=\left(\pi(1-2\gamma)^{-1}\right)$, $S_{p,\nu}=\sin[p\,(\alpha(\rho)-\nu)]$ *and* $C_{p,\nu}=\cos[p\,(\alpha(\rho)-\nu)]$.

Proof:

Note that α,β depend on ρ and in what follows we will make this dependence explicit by writing them as $\alpha(\rho)$ and $\beta(\rho)$ respectively. By definition we have

$$\nu_\gamma=E_{\gamma,F}(\cos\theta)=\left(2\pi(1-2\gamma)^{=1}\right)\left[\int_{\beta(\rho)}^{\alpha(\rho)}\cos\theta d\theta+2\sum_{p=1}^{\infty}(\rho)^{p^2}\int_{\beta(\rho)}^{\alpha(\rho)}\cos\theta\cos p(\theta-\mu)d\theta\right]$$
$$=\left(2\pi(1-2\gamma)^{=1}\right)\left[I_1+2\sum_{p=1}^{\infty}(\rho)^{p^2}I_2\right]$$

where $I_1=\int_{\beta(\rho)}^{\alpha(\rho)}\cos\theta d\theta$ and $I_2=\int_{\beta(\rho)}^{\alpha(\rho)}\cos\theta\cos p(\theta-\mu)d\theta$. Then clearly $I_1=2\sin\alpha(\rho)$.

Put $(\theta-\mu)=\eta$ so that $d\theta=d\eta$ and the limits of integration changes to $\alpha(\rho)-\mu$ and $\beta(\rho)-\mu$. Noting that $\sin\theta$ is an odd function and $\cos\theta$ is an even function we get

$$I_2=\int_{\beta(\rho)-\mu}^{\alpha(\rho)-\mu}\cos(\eta+\mu)\cos p\eta d\eta=2\cos\mu\int_{0}^{\alpha(\rho)-\mu}\cos\eta\cos p\eta d\eta=\cos\mu\left\{\frac{S_{p+1,\mu}}{(p+1)}+\frac{S_{p-1,\mu}}{(p-1)}\right\}.$$

Substituting the values of I_1 and I_2 in the above expression for ν_γ we get,

$$\nu_\gamma=c\left[S_{1,0}+\cos\mu\sum_{p=2}^{\infty}(\rho)^{p^2}\left(\frac{S_{p+1,\mu}}{(p+1)}+\frac{S_{p-1,\mu}}{(p-1)}\right)\right].$$

We note that $S_2(F)=E_F(d(\theta,\mu))$ does not depend on μ. Hence we can without loss of generality assume μ=0 for computing $S_2(F)$. Using the substitution $(2\pi-\theta)=\lambda$, we get

$$S_4(F)=E_{\gamma,F}(d(\theta)) = \left[\pi(1-2\gamma)^{-1}\right]\int_0^{\alpha(\rho)}\theta\left\{1+2\sum_{p=1}^{\infty}(\rho)^{p^2}\cos p\theta\right\}d\theta$$

$$=\left[\pi(1-2\gamma)^{-1}\right]\left\{\int_0^{\alpha(\rho)}\theta d\theta+2\sum_{p=1}^{\infty}(\rho)^{p^2}\int_0^{\alpha(\rho)}\cos p\theta d\theta\right\}$$

$$=\left[\pi(1-2\gamma)^{-1}\right]\left\{\frac{\alpha^2(\rho)}{2}+2\sum_{p=1}^{\infty}(\rho)^{p^2}\left(\frac{\alpha(\rho)S_{p,0}}{p}+\frac{C_{p,0}-1}{p^2}\right)\right\}$$

$$=\frac{1}{2\pi(1-2\gamma)}\left[\alpha^2(\rho)+4\sum_{p=1}^{\infty}(\rho)^{p^2}\left\{\frac{\alpha(\rho)S_{p,0}}{p}+\frac{C_{p,0}-1}{p^2}\right\}\right].$$

Hence the lemma.

Proof of the theorem:

Let $F\in\tilde{\Im}_1$ and $T_\gamma(F) = \arctan^*\left[\frac{E_{\gamma,F}(\sin\theta)}{E_{\gamma,F}(\cos\theta)}\right]$ be the estimating functional for μ_γ.

Define $\mu_{\gamma,\varepsilon} = T_\gamma\{(1-\varepsilon)F+\varepsilon\delta_x\}$. Then,

$$\mu_{\gamma,\varepsilon}=\begin{cases}\arg\left[(1-\varepsilon)\int_\beta^\alpha e^{i\theta}f(\theta)d\theta+\varepsilon\, e^{ix}\right]; & \text{if} \quad x\in(\beta,\alpha)\\ \arg\left[(1-\varepsilon)\int_\beta^\alpha e^{i\theta}f(\theta)d\theta\right]; & \text{if} \quad x\notin(\beta,\alpha)\end{cases} \qquad \dots (7.2)$$

where (β,α) is the arc starting at β and ending at α traversed in the anticlockwise direction. The above relation (7.2) can be written as:

$$\mu_{\gamma,\varepsilon}=\begin{cases}\arg\{(1-\varepsilon)E_{\gamma,F}(\cos\theta)+\varepsilon\cos x+i\,((1-\varepsilon)E_{\gamma,F}(\sin\theta)+\varepsilon\sin x)\} & \text{if} \quad x\in(\beta,\alpha)\\ \arg\{(1-\varepsilon)E_{\gamma,F}(\cos\theta)+i\,((1-\varepsilon)E_{\gamma,F}(\sin\theta))\} & \text{if} \quad x\notin(\beta,\alpha)\end{cases}.$$

Therefore,

$$\tan\mu_{\gamma,\varepsilon}=\begin{cases}\dfrac{(1-\varepsilon)E_{\gamma,F}(\sin\theta)+\varepsilon\sin x}{(1-\varepsilon)E_{\gamma,F}(\cos\theta)+\varepsilon\cos x} & \text{if} \;\; x\in(\beta,\alpha)\\ \dfrac{E_{\gamma,F}(\sin\theta)}{E_{\gamma,F}(\cos\theta)} & \text{if} \;\; x\notin(\beta,\alpha)\end{cases}$$

$$\Rightarrow \mu_{\gamma,\varepsilon} = \begin{cases} \arctan^{*}\left[\dfrac{\nu_\gamma(1-\varepsilon)\tan\mu_\gamma + \varepsilon \sin\ x}{\nu_\gamma(1-\varepsilon)+\varepsilon\cos\ x}\right] & \text{if} \quad x \in (\beta,\alpha) \\ \mu_\gamma & \text{if} \quad x \notin (\beta,\alpha) \end{cases}$$

where $\nu_\gamma = E_{\gamma,F}(\cos\theta)$. Thus, following the similar steps as in Theorem 3.1 of chapter 3, we get

$$IF(x;\ T_\gamma, F) = \begin{cases} \dfrac{\sin(x-\mu_\gamma)}{\nu_\gamma\ \sec\mu_\gamma} & \text{if } x \in (\beta,\alpha) \\ 0 & \text{if } x \notin (\beta,\alpha) \end{cases} \qquad \ldots (7.3)$$

where $\mu_\gamma \neq \frac{\pi}{2}, \frac{3\pi}{2}$. Now, define $I(x) = \begin{cases} 1 & \text{if } x \in (\beta(\rho),\alpha(\rho)) \\ 0 & \text{otherwise} \end{cases}$.

Again using Lemma 2, the standardized influence function (SIF) can be written as

$$SIF(x;\ T_\gamma,\ F,\ S_4\ ) = \frac{\sin(x - \mu_\gamma)\, I(x)}{\nu_\gamma S_4(F)} \qquad \ldots (7.4)$$

We can show that the s.g.e.s is bounded with respect to the dispersion functional S at F by directly looking at the integrals of ρ_γ and $S(F)$ for both $\rho \to 0$ and $\rho \to 1$ as follows: We know that, as $\rho \to 0$, the wrapped normal distribution tends to circular uniform distribution with density function $f(\theta) = (2\pi)^{-1}, 0 \leq \theta < 2\pi$. Therefore,

$$\frac{1}{2\pi}\lim_{\rho\to 0}\int_0^{\alpha(\kappa)}\left\{1 + 2\sum_{p=1}^{\infty}(\rho)^{p^2}\cos p(\theta-\mu)\right\}d\theta = \frac{(1-2\gamma)}{2} \Rightarrow \alpha(0) = \pi(1-2\gamma).$$

Since $\alpha(\rho) + \beta(\rho) = 2\pi$, we have $\beta(0) = \pi(1+2\gamma)$. Hence,

$$\lim_{\rho\to 0}\rho_\gamma = [\pi(1-2\gamma)]^{-1}\int_0^{\alpha(0)}\cos\theta\, d\theta = \frac{\sin[\pi(1-2\gamma)]}{\pi(1-2\gamma)},\ 0 \leq \gamma < 0.5\,. \qquad \ldots (7.5)$$

Now as $\rho \to 0$, and letting $(2\pi - \theta) = \lambda$, we get,

$$\lim_{\rho\to 0} S(F) = \frac{1}{\pi(1-2\gamma)}\int_0^{\alpha(0)}\lambda d\lambda = \frac{\pi(1-2\gamma)}{2}\,. \qquad \ldots (7.6)$$

It is obvious that the numerator of the expression on the right hand side of (7.4) is bounded, and hence by using (7.5) and (7.6) we can conclude that:

$$\lim_{\rho\to 0}\sup_{0\le x<2\pi} SIF(x;\, T_\gamma,\, F,\, S_4) < \infty. \qquad \text{... (7.7)}$$

Since $WN(\mu,\rho)$ distribution can be well approximated by $CN(\mu, A^{-1}(\rho))$ as $\rho \to 1$, applying Hill's (1976) expansion we have

$$\alpha(A^{-1}(\rho)) = \frac{\chi}{\sqrt{A^{-1}(\rho)}} + \frac{\chi^3+3\chi}{24\, A^{-1}(\rho)\sqrt{A^{-1}(\rho)}} + \frac{3\chi^5+20\chi^3+45\chi}{640\,(A^{-1}(\rho))^2\sqrt{A^{-1}(\rho)}} + \ldots$$

where $\chi = \Phi^{-1}(1-\gamma)$. Using the fact $\theta \sim WN(\mu,\rho) \Rightarrow \theta-\mu (\mathrm{mod}\, 2\pi) \sim WN(0,\rho)$ we get $\alpha(\rho)\to\mu$ as $\rho\to 1$. By symmetry of the wrapped normal distribution about μ we can also conclude that $\beta(\rho)\to\mu$ as $\rho\to 1$. Thus for any $x\ne\mu,\ 0\le x<2\pi$, there exists an M >0, such that if $1\ge\rho>M,\ x\notin(\beta(\rho),\alpha(\rho))$.

Hence, $\lim_{\rho\to 1}\sup_{0\le x<2\pi} SIF(x;\, T_\gamma,\, F,\, S_4) = 0.$ … (7.8)

Thus, using (7.7) and (7.8), we can conclude that

$$\gamma^*(T_\gamma,\, \tilde{\Im}_1,\, S_4) = \sup_{0<\rho<1}\ \sup_{0\le x<2\pi}\left[SIF(x;\, T_\gamma,\, F,\, S_4)\right] < \infty.$$

Hence the theorem.

Corollary 3: Since $T_\gamma(F)$ is SB-robust at the family of distributions $\tilde{\Im}_1$ with respect to the dispersion measure $S_4(F)$, it is also SB-robust at the family of distributions $\tilde{\Im}_1$ with respect to the dispersion measures $S_1(F)$ and $S_2(F)$ by Theorem 7.1(b). Also it is SB-robust at the family of distributions $\tilde{\Im}_1$ with respect to the dispersion measure $S_3(F)$ by Theorem 7.2.

7.5 Robustness of the Estimate of the Concentration Parameter

Laha & Mahesh (2010) introduced a new estimator for the concentration parameter with reference to circular normal distribution. Here we will investigate the SB-robustness of the concentration parameter ρ of the wrapped normal distribution. In Theorem 7.7 below we prove that the natural estimator of ρ is not SB-robust at the family of distributions $\tilde{\Im}_1 = \{WN(\mu,\rho); 0 < \rho < 1\}$ with respect to the dispersion measure $S_2(F) = E_F(d(\theta,\mu))$.

Theorem 7.7: *Suppose* $\Theta \sim WN(\mu,\rho)$ *where* $\rho > 0$ *and* μ *is the mean direction. Then* $\tilde{T}(F) = \sqrt{E_F^2(\cos\theta) + E_F^2(\sin\theta)}$ *is an estimating functional for* ρ *which is not SB-robust at the family of distributions* $\tilde{\Im}_1 = \{WN(\mu,\rho); 0 < \rho < 1\}$ *with respect to* $S_2(F) = E_F(d(\theta,\mu))$ $F \in \tilde{\Im}_1$.

Proof:

Since $\Theta \sim WN(\mu,\kappa)$ and μ is the mean direction, we can write

$\mu = \int e^{i\theta} dF(\theta) = E_F(\cos\theta) + iE_F(\sin\theta)$ and $\rho_F = \sqrt{E_F^2(\cos\theta) + E_F^2(\sin\theta)}$. Let $\tilde{T}(F)$ be an estimating function for ρ such that $\tilde{T}(F) = \rho_F = \sqrt{E_F^2(\cos\theta) + E_F^2(\sin\theta)}$.

Let $\rho_\varepsilon = \tilde{T}((1-\varepsilon)F + \varepsilon\delta_x)$. Then we can write using Lemma1,

$$\rho_\varepsilon = \sqrt{\rho_F^2 - 2\rho_F\varepsilon(\rho_F - \cos(x-\mu)) + \varepsilon^2(\rho_F^2 - 2\rho_F\cos(x-\mu) + 1)}.$$

But $IF(x;\tilde{T},F) = \lim_{\varepsilon\to 0}\left(\dfrac{\rho_\varepsilon - \rho_F}{\varepsilon}\right) = (\cos(x-\mu) - \rho_F)$. Using, (3.4) we get

$$\mathrm{SIF}\left(x;\tilde{T},F,S_2\right)=\frac{\left(\cos(x-\mu)-\rho_F\right)}{\left[\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(\rho_F)^{(2n+1)^2}}{(2n+1)^2}\right]}.$$

As $\rho\to 1$, $\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(\rho_F)^{(2n+1)^2}}{(2n+1)^2}\to\frac{\pi}{2}$ and hence $S_2(F)\to 0$. Therefore, $\gamma^*\left(\tilde{T},\tilde{\mathfrak{I}}_1,S_2\right)$ is not finite which implies that standardized influence function is not bounded. Thus $\tilde{T}(F)$ is not SB-robust at the family of distributions $\tilde{\mathfrak{I}}_1$.

Hence the theorem.

Corollary 4: Since $\tilde{T}(F)$ is not SB-robust at the family of distributions $\tilde{\mathfrak{I}}_1$ with respect to the dispersion measure $S_2(F)$, it is also not SB-robust at the family of distributions $\tilde{\mathfrak{I}}_1$ with respect to the dispersion measures $S_1(F)$ and $S_4(F)$ by Theorem 7.1(b).

Theorem 7.8: *The functional* $\tilde{T}(F)=\sqrt{E_F^2(\cos\theta)+E_F^2(\sin\theta)}$ for ρ is *not SB-robust at the family of distributions* $\tilde{\mathfrak{I}}_1=\{WN(\mu,\rho);0<\rho<1\}$ *when the measure of dispersion is* $S_3(F)=\left(\rho A^{-1}(\rho)\right)^{-\frac{1}{2}}$ *where* $F\in\tilde{\mathfrak{I}}_1$.

Proof:

We have $\mathrm{SIF}\left(x;\tilde{T},F,S_3\right)=\left(\cos(x-\mu)-\rho_F\right)\left(\frac{\rho}{A^{-1}(\rho)}\right)^{-1/2};0<\rho<1$. Using the asymptotic expansion of $A^{-1}(\rho)$ near unity (Watson 1983, appendix A.2 with p=2) we get

$\left(\frac{\rho}{A^{-1}(\rho)}\right)^{-1/2}\to 0$ as $\rho\to 1$. Thus, $\gamma^*\left(\tilde{T},\tilde{\mathfrak{I}}_1,S_3\right)=\infty$.

Hence the theorem.

Next we introduce a new estimator for ρ.

Definition 1: Let γ be the trimming proportion such that $\gamma \in [0,\ 0.5)$ and $\alpha(\rho)$, $\beta(\rho)$ are such that

(i) $\int_{\beta(\rho)}^{\alpha(\rho)} f(\theta)\, d\theta = 1\text{-}2\gamma$, and

(ii) $d_1(\alpha(\rho), \beta(\rho)) \le d_1(\mu, \nu)$ for all μ, ν satisfying $\int_{\nu}^{\mu} f(\theta)\, d\theta = 1\text{-}2\gamma$, where $d_1(\phi, \xi)$ is the length of the arc starting from ξ and ending at ϕ traversed in the anticlockwise direction.

Then the new estimator for ρ is defined as $T_\gamma(F) = g^{*-1}\left[E_{\gamma,F}(d(\theta, \mu))\right]$ where $g^*(\rho) = (1-2\gamma)^{-1} \int_{\beta(\rho)}^{\alpha(\rho)} d(\theta, \mu)\, dF$.

Without loss of generality we assume $\mu = 0$. Since the wrapped normal distribution is symmetric about $\mu = 0$ we have $\beta(\rho) = 2\pi - \alpha(\rho)$. Then $T_\gamma(F) = g^{*-1}\left[E_{\gamma,F}(d(\theta))\right]$ where $g^*(\rho) = (1-2\gamma)^{-1} \int_{\beta(\rho)}^{\alpha(\rho)} d(\theta)\, dF$. In Theorem 7.9 below, we prove that $T_\gamma(F)$ is SB-robust at the family of distributions $\Im^{**} = \{WN(0,\rho); 0 < m < \rho < M < 1\}$ with respect to the dispersion measure $S_4(F) = (1-2\gamma)^{-1} \int_{\beta(\rho)}^{\alpha(\rho)} d(\theta)\, dF$.

Theorem 7.9: *Let* $\Theta \sim WN(0, \rho)$, $d(\theta, 0) = d^*(\theta) = \pi - |\pi\text{-}\theta|$, *and* $g^*(\rho) = E_{\gamma,F}(d(\theta))$. *Then* $T_\gamma(F) = g^{*-1}\left[E_{\gamma,F}(d^*(\theta))\right]$ *is SB-robust at the family of distributions* $\Im^{**} = \{WN(0,\rho); 0 < m < \rho < M < 1\}$ *with respect to the dispersion measure* $S_4(F) = E_{\gamma,F}(d^*(\theta))$ *where* $F \in \Im^{**}$.

The following Lemma 3 is used to prove the theorem.

Lemma 3: *Let* $\Theta \sim WN(0, \rho)$ *and define* $g^*(\rho) = E_{\gamma,F}(d(\theta))$ *where* $d^*(\theta) = \pi - |\pi\text{-}\theta|$.

Then, $g^{*(1)}(\rho)=\dfrac{\alpha(\rho)\alpha^{(1)}(\rho)\left\{1+2\sum_{p=1}^{\infty}(\rho)^{p^2}\cos p\alpha(\rho)\right\}}{\pi(1-2\gamma)}$ *where* $g^{*(1)}(.)$ and $\alpha^{(1)}(.)$ *are the derivatives of* $g^{*}(.)$ and $\alpha(.)$ *respectively.*

Proof:

Since $S_4(F)=g^{*}(\rho)=E_{\gamma,F}\left(d^{*}(\theta,\mu)\right)$ does not depend on μ we can without loss of generality assume $\mu=0$ for computing $g(\kappa)$. Hence $g^{*}(\rho)=E_{\gamma,F}\left(d^{*}(\theta)\right)$. Using the substitution $(2\pi-\theta)=\lambda$ we have

$$
\begin{aligned}
g^{*}(\rho) &= E_{\gamma,F}\left(d^{*}(\theta)\right)=(2\pi(1-2\gamma))^{-1}\int_{0}^{2\pi}(\pi-|\pi-\theta|)\left\{1+2\sum_{p=1}^{\infty}(\rho)^{p^2}\cos p\theta\right\}d\theta \\
&= (2\pi(1-2\gamma))^{-1}\left[\int_{0}^{\alpha(\rho)}\theta\left\{1+2\sum_{p=1}^{\infty}(\rho)^{p^2}\cos p\theta\right\}d\theta+\int_{\beta(\rho)}^{2\pi}(2\pi-\theta)\left\{1+2\sum_{p=1}^{\infty}(\rho)^{p^2}\cos p\theta\right\}d\theta\right] \\
&= (\pi(1-2\gamma))^{-1}\int_{0}^{\alpha(\rho)}\lambda\left\{1+2\sum_{p=1}^{\infty}(\rho)^{p^2}\cos p\lambda\right\}d\lambda
\end{aligned}
$$

Using Leibnitz's rule for differentiation under the integral sign we get

$$g^{*(1)}(\rho)=\frac{\alpha(\rho)\alpha^{(1)}(\rho)\left\{1+2\sum_{p=1}^{\infty}(\rho)^{p^2}\cos p\alpha(\rho)\right\}}{\pi(1-2\gamma)}.$$

Hence the lemma.

Proof of the theorem:

Let $g^{*}(\rho)=E_{\gamma,F}\left(d^{*}(\theta)\right)=\dfrac{1}{1-2\gamma}\displaystyle\int_{\beta(\rho)}^{\alpha(\rho)}d(\theta)\,dF$. We can write $\rho_F^{*}=T_{\gamma}(F)=g^{*-1}\left[E_{\gamma,F}\left(d^{*}(\theta)\right)\right]$.

Also let $\rho_{\varepsilon}^{*}=T_{\gamma}\{(1-\varepsilon)F+\varepsilon\delta_x\}$. Then,

$$\rho_\varepsilon^* = \begin{cases} g^{*-1}\left[(1-\varepsilon)g^*(\rho)+c\varepsilon\, d^*(x)\right] & ;\ x \in (\beta(\rho),\ \alpha(\rho)) \\ g^{*-1}\left[(1-\varepsilon)g^*(\rho)\right] & ;\ x \notin (\beta(\rho),\ \alpha(\rho)) \end{cases}$$

where $c = [\pi(1-2\gamma)]^{-1}$.

Case 1: When $x \notin (\beta(\rho),\ \alpha(\rho))$.

Using Taylor series expansion of g^{*-1} around $g^*(\rho)$ we get

$$\rho_\varepsilon^* = \rho_F - \frac{\varepsilon\, g^*(\rho)}{g^{*(1)}(\rho)} + O(\varepsilon^2). \qquad \text{… (7.9)}$$

Thus, from (7.9) we get that the influence function is given by:

$$IF(x;T_\gamma,F) = \lim_{\varepsilon\to 0}\left(\frac{\rho_\varepsilon^* - \rho_F^*}{\varepsilon}\right) = \frac{-g^*(\rho)}{g^{*(1)}(\rho)}. \qquad \text{… (7.10)}$$

Case 2: When $x \in (\beta(\rho),\ \alpha(\rho))$.

$$\rho_\varepsilon^* = \rho_F + \frac{\varepsilon\left(c\, d^*(x) - g^*(\rho)\right)}{g^{*(1)}(\rho)} + O(\varepsilon^2). \qquad \text{… (7.11)}$$

From (7.11), the influence function is given by:

$$IF(x;T_\gamma,F) = \lim_{\varepsilon\to 0}\left(\frac{\rho_\varepsilon^* - \rho_F^*}{\varepsilon}\right) = \frac{c\, d^*(x) - g^*(\rho)}{g^{*(1)}(\rho)}. \qquad \text{… (7.12)}$$

Combining (7.10) and (7.12), we get

$$IF(x;T_\gamma,F) = \begin{cases} \dfrac{c\, d^*(x) - g^*(\rho)}{g^{*(1)}(\rho)}, & x \in (\beta(\rho),\ \alpha(\rho)) \\ \dfrac{-g^*(\rho)}{g^{*(1)}(\rho)}, & x \notin (\beta(\rho),\ \alpha(\rho)) \end{cases}. \qquad \text{... (7.13)}$$

Now in this case, $S_4(F) = E_{\gamma,F}(d^*(\theta)) = g^*(\rho)$. Then using (7.13) the standardized influence function (SIF) can be written as:

$$SIF(x;T_\gamma,F,S)=\begin{cases}\dfrac{\left(c\,d^*(x)-g^*(\rho)\right)}{g^{*'}(\rho)*g^*(\rho)}, & x\in(\beta(\rho),\alpha(\rho))\\ -\left[g^{*'}(\rho)\right]^{-1}, & x\notin(\beta(\rho),\alpha(\rho))\end{cases}. \quad \ldots (7.14)$$

By Lemma 3 we have,

$$g^{*(1)}(\rho)=\frac{\alpha(\rho)\alpha^{(1)}(\rho)\left\{1+2\sum_{p=1}^{\infty}(\rho)^{p^2}\cos p\alpha(\rho)\right\}}{\pi(1-2\gamma)}. \quad \ldots (7.15)$$

Since $g^*(\rho)$ is strictly positive and $g^{*(1)}(\rho)$ is strictly negative and bounded away from 0 for $0<m<\rho<M<1$, we can conclude that the s.g.e.s $\gamma^*\left(T_\gamma,\Im^{**},S_4\right)$ is finite and hence $g^{*-1}\left[E_{\gamma,F}\left(d^*(\theta)\right)\right]$ is a SB- robust estimator for the concentration parameter of the wrapped normal distribution.

Hence the theorem.

Corollary 5: Since $T_\gamma(F)$ is SB-robust at the family of distributions $\Im^{**}$ with respect to the dispersion measure $S_4(F)$, it is also SB-robust at the family of distributions $\Im^{**}$ with respect to the dispersion measures $S_1(F), S_2(F)$ and $S_3(F)$ by Theorem 7.1(a).

Chapter 8

Conclusions

8.1 Conclusion

In this thesis entitled “Robustness of estimators and tests with circular data” we first studied the SB-robustness of the directional mean at different family of circular normal distributions and its mixtures.

We proved that a trimmed version of directional mean is SB-robust at the family of circular normal distributions. Also we introduced the concept of equivalent dispersion measures and prove that if an estimator is SB-robust for one measure of dispersion then it is SB-robust for equivalent dispersion measures.

We proved that the usual estimator of the concentration parameter for circular normal distribution is not SB-robust. We introduced a new trimmed estimator for the concentration parameter and proved that this new estimator is SB-robust at the family of circular normal distributions.

We proved that the trimmed version of the directional mean is a robust test statistic in the sense of breakdown properties in comparison with likelihood ratio test statistic and the directional mean as a test statistic.

We study the robustness of the tests of the concentration parameter of circular normal distribution and proved that the complete sufficient statistic V has better robustness property than V_γ in the sense of level and power breakdown properties.

We studied the SB-robustness of the directional mean with respect to different dispersion measures and at different family of wrapped normal distributions. We proved that the trimmed version of directional mean is SB-robust at the family of wrapped normal distributions. Also we proved that the usual estimator of the concentration parameter for wrapped normal distribution is not SB-robust. But a trimmed estimator is SB-robust at the family of wrapped normal distributions.

8.2 Scope of Further Work

The results obtained in this thesis can be extended to the case of other circular distributions like wrapped Cauchy distribution, Kato and Jones (2010) distribution etc. One of the well known distributions on the circle is the wrapped Cauchy distribution obtained by wrapping the Cauchy distribution $C(a, \mu)$ onto the unit circle. The p.d.f of wrapped Cauchy distribution with parameters μ and ρ is given by

$$f(\theta;\ \mu, \rho) = \frac{1}{2\pi} \frac{(1-\rho^2)}{(1+\rho^2 - 2\rho\cos(\theta-\mu))};\ 0 \le \theta, \mu < 2\pi,\ 0 < \rho < 1$$

where μ is the mean direction and $\rho = e^{-a}$ is the concentration parameter. The corresponding functional form of μ and ρ are respectively given by $T(F) = \arctan^*\left(\frac{E_F(\sin\theta)}{E_F(\cos\theta)}\right)$ and $\rho_F = \sqrt{E_F^2(\cos\theta) + E_F^2(\sin\theta)}$ where F is the underlying distribution. The robustness of the above functionals can be established using the techniques discussed in this thesis.

Kato and Jones (2010) recently proposed a four parameter symmetric family of distribution on the circle. As a special case, they derived a three parameter symmetric family of distributions with probability density function

$$f(\theta) = \frac{1-r^2}{2\pi I_0(\kappa)} \exp\left\{\frac{\kappa(1+r^2)\cos(\theta-\mu) - 2r}{1+r^2 - 2r\cos(\theta-\mu)}\right\} \times \frac{1}{1+r^2 - 2r\cos(\theta-\mu)}; 0 \le \theta < 2\pi$$

where $0 \le \mu < 2\pi$, $\kappa > 0$, and $-1 < r < 1$. This distribution is symmetric about $\theta = \mu$ and $\mu + \pi$ and is unimodal when $0 \le r < 1$. The parameter μ is the mean direction and κ, r - the concentration parameters of $CN(\mu, \kappa)$ and $WC(\mu, r)$ distributions respectively. The corresponding functional form of the estimator of μ is given by $T(F) = \arctan^{*}\left(\frac{E_F(\sin\theta)}{E_F(\cos\theta)}\right)$. The above model includes the von-Mises $(r = 0)$, wrapped Cauchy $(\kappa = 0)$ and uniform distributions $(\kappa = r = 0)$ as special cases. As $\kappa \to \infty$, the Kato-Jones distribution tends to $N(\mu, \omega_r)$ where $\omega_r = \frac{1-r}{1+r}$. The robustness of the above functional can be established using the techniques discussed in this thesis.

The results of this thesis can also be extended to spherical distributions like the Fisher-Bingham distribution. Kent (1982) proposed a five parameter distribution on the unit sphere $\Omega_3 = \left\{X \in \Re^3 : x_1^2 + x_2^2 + x_3^2 = 1\right\}$ which he calls the Fisher-Bingham distribution with p.d.f

$$f(X) = \frac{\exp\left\{\kappa\gamma'_{(1)}X + \beta\left((\gamma'_{(2)}X)^2 - (\gamma'_{(3)}X)^2\right)\right\}}{2\pi\sum_{j=0}^{\infty} I_{2j+\frac{1}{2}}(\kappa)\frac{\Gamma\left(j+\frac{1}{2}\right)}{\Gamma(j+1)}\beta^{2j}\left(\frac{\kappa}{2}\right)^{-2j-\frac{1}{2}}}; X \in \Re^3, \kappa \ge 0, \beta \ge 0$$

where $\Gamma = (\gamma_{(1)}\ \gamma_{(2)}\ \gamma_{(3)})$ is a 3×3 orthogonal matrix. The parameters can be interpreted as follows: κ represents the concentration, β describes the ovalness, $\gamma_{(1)}$ is the mean direction or pole, $\gamma_{(2)}$ is the major axis, and $\gamma_{(3)}$ is the minor axis. The functional form of the estimate of $\gamma_{(1)}$ is given by $T(F) = \frac{(E_F(X_1)\ \ E_F(X_2)\ \ E_F(X_3))'}{\|(E_F(X_1)\ \ E_F(X_2)\ \ E_F(X_3))\|}$ where F is the underlying distribution. The robustness of the above functional can be established using the techniques discussed in this thesis.

In the real line context one can derive similar results in the case of distributions with bounded support like Kumaraswamy distribution. Jones (2009) explored a two parameter family of distributions in the open interval $(0,1)$ (which he called Kumaraswamy's distribution) which has many similarities to the beta distribution and a number of advantages in terms of tractability. The corresponding probability density function is with p.d.f

$$g(x) = \alpha\beta x^{\alpha-1}(1-x^{\alpha})^{\beta-1}; 0 < x < 1,$$

where α and β are two positive shape parameters. Unlike the Beta distribution, Kumaraswamy's distribution admits a closed form c.d.f. which is given by $G(x) = 1-(1-x^{\alpha})^{\beta}; 0 < x < 1$. This distribution is unimodal, uniantimodal, increasing, decreasing or constant depending on the values of α and β. Some interesting limiting distributions namely, Weibull, Generalized exponential, and extreme value distributions can be derived from Kumaraswamy distribution with parameters α and β by letting $\beta \to \infty$, $\alpha \to \infty$ and both $\alpha, \beta \to \infty$ respectively by using suitable normalized transformations. The functional of the mean of Kumaraswamy's distribution is given by $T(F) = E_F(X)$ where F is the underlying distribution. The robustness of the above functional can be established using the techniques discussed in this thesis.

Chapter 9

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