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SOME ADVANCES IN LIFE TESTING AND RELIABILITY

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UNDER THE GUIDANCE

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CERTIFICATE

This is to certify that Ms. Sumeet Arora, a candidate for the award of Doctor of Philosophy in Statistics, under the faculty of Science, Saurashtra University, Rajkot, has worked under my guidance for the period required by the university. The work embodied in thesis is not submitted previously to this or any other university for Ph.D. or any other university for Ph.D. or any other degree or diploma. This is her original research work.

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DECLARATION

I, hereby, declare that the thesis entitled "SOME ADVANCES IN LIFE TESTING AND RELIABILITY" is my own work conducted under the supervision of ***Dr. G.C. Bhimani.***

I further declare that to the best of my knowledge the thesis does not contain any part of any work which has been submitted for the award of any degree or diploma either in this University or any other University or examining body in India or any other country. Wherever the references have been made to previous works of other, it has been clearly indicated as such and included in the Bibliography

Signature of Candidate

Sumeet Arora

October 2009

EXPRESSION OF GRATITUDE

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Sumeet Arora

CONTENTS

CHAPTER 1

INTRODUCTION

1.1 History of Life-testing and Reliability

Reliability theory in nineteenth century was primarily used as a tool to help maritime and life insurance companies figure profitable rates to charge their customers apart from the mainstream of probability and statistics. In today's technological world nearly everyone depends upon the continued functioning of a wide array of complex machinery and equipment for their everyday health, safety, mobility and economic welfare. We expect our cars, computers, electrical appliances, lights, televisions etc. to function whenever we need them day after day, year after year. When they fail the results can be disastrous which lead to injury, loss of life and/or costly lawsuits.

Moreover, repeated failure leads to annoyance, inconvenience and a lasting customer dissatisfaction that can play havoc with the company's market position. It takes a longtime for a company to build up a reputation for reliability and only a short time to be branded as "unreliable" after shipping a flawed product. Continual assessment of new product, reliability and ongoing control of the reliability of everything shipped are vital requirements in today's competitive business arena.

The everyday usage term "quality of a product" is taken for granted to mean its inherent degree of excellence. In industry, this is made more precise by defining quality to be "conformance to requirements at the start of use". Assuming that the product specifications adequately capture customer requirements, the quality level can now be precisely measured by the fraction of units that meet specifications after a week of operation? Or after a month or at the end of a one-year warranty period? That is where "reliability" steps in. Quality is a snapshot at the start of life and reliability is a motion picture of day-by-day operation. The quality level might be described by a single fraction defective. To describe reliability fallout a probability model that describes the fraction fallout over time is needed. This is known as life distribution model. A life distribution does find its frequent application in the engineering and biomedical sciences.

The times to the occurrences of events, which are of interest for some population of individuals, are termed as "life times". Some times the events of interest are deaths of individual or may be a survival time measured from some particular starting point. In some instances "life time" is used in a figurative sense. Mathematically, one can think of "life time" as merely meaning "non-negative valued variable". For e.g. manufactured items such as mechanical or electronic components are often subjected to life tests in order to obtain information on their endurance. This involves subjecting items in operation, often in a laboratory setting and observing them until they fail. In such situation it is common to refer to life times as “failure times”, since when an item ceases operating satisfactorily, it is said to have “failed”.

The theoretical population models that are used to describe unit life times are known as lifetime distribution models. The population is generally considered to be all of the possible unit life times that could be manufactured based on a particular design, choice of materials and manufacturing process. A random sample of size n from this population is the collection of failure times observed for a randomly selected group of n units. A lifetime distribution model can be any probability density function f (t) defined over the range of time from t = 0 to t = ∞. The corresponding cumulative distribution function F (t) is a very useful function as it gives the probability that a randomly selected unit will fail by time t. The data, to which statistical methods are applied in order that parameters of interest can be estimated in their reliability context, usually result from life tests. A typical life test is one in which prototypes of the item or organism of interest is subjected to stresses and environmental conditions demonstrate the intended operating conditions. During the test successive times to failures are noted. Since the failures occur in order, the theory of order statistics plays an important role in the analysis of the life test data.

Literature related to statistical methods used in the analysis of life test data lies scattered in a number of professional journals and books. Reliability studies frequently involve testing of items (say n in number) that are designed to last for long periods of time. In such studies, constraints in the form of truncation and / or censoring would be deemed essential as means of obtaining information within reasonable time limitations; while there are several means of censorship (see Gajjar and Khatri (1969)) two types are commonly used. These are referred as Type-I and Type-II censorships. Type-I censorship or censoring occurs when the researcher sets a time limit on terminating the life test, even though some of the test items remain operational. Type-II censoring occurs when the life test is terminated at the particular (the r^{th}, say $r < n$) failure. In Type-I censoring the number of failures and all the failure times are random variables, the number of failures being considered fixed. Type-II censoring has the advantage of providing more or less uniform amount of information in repeated sampling with the disadvantage that the length of testing time varies from test to test. Type-I censoring provides a constant length of testing time in repeated sampling with amount of information varying from test to test. One advantage of Type-I censoring is that it simplifies the problem of test scheduling in a production process where information from periodic production of lots has to be obtained at regular intervals.

1.2 PRE-WORK

There is an extensive body of literature concerning properties of several estimators that are proposed for estimating parameters of probability models commonly used in reliability studies under Type-II censoring. Though some work in the area of reliability and life testing has been done under Type-I censoring but it is not as extensive as that under Type-II censoring. The early work concerning estimation of parameters from continuous life time distributions such as Normal, Exponential, Weibull, Extreme Value distributions and discrete life time distribution particularly Geometric distribution based on single stage Type-I and Type-II censoring was initiated by Gupta(1952), Epstein and Sobel (1953, 1954), Lieblein and Zelen (1956), Bartholomew (1957, 1963), Cohen (1965), Tiku (1967) and others. Recently, rather extensively the work has been studied by Yaqub and Khan (1981), Patel and Gajjar (1990), Cohen (1991), Balakrishnan and Cohen (1991). These authors have all considered lifetime studies in industrial as well as actuarial (human life time) contexts, in parametric and non-parametric cases.

In several situations, the initial censoring results only in withdrawal of a portion of the surviving items. Those which remain on test continue under further observation until an ultimate failure or until a subsequent stage of censoring is performed. For sufficiently large samples censoring is done through several stages. This leads to progressive censoring of Type-I or Type-II. Progressive censoring can be adopted for several reasons. Progressively censored sample arise, for instance, when certain items must be withdrawn from a life test prior to failure for use as test objects in related experimentation. They may also result from a compromise between the need for more rapid testing and the desire to include at least some extreme life spans in the sample data. When the test facilities are limited and when prolonged life tests are cost-prohibitive, the early censoring of a substantial number of items from the test frees facilities for other tests while items which are allowed to continue on test until subsequent failures provide information on extreme sample values.

Cohen (1963) considered Type-I progressively censored samples in case of Normal and Exponential distributions and obtained maximum likelihood estimates of the parameters of these distributions with the assumption that the parameters remain the same at each stage of censoring. But there are situations where it might be reasonable to assume that the parameters of a distribution under consideration might change at each stage of censoring. The justification of this reasoning likes in the fact that the surviving items entering the subsequent stage are checked and overhauled eliminating or repairing minor defects wherever possible. It may be noted that due to different parameters at different stages of censoring it leads to estimating parameters from truncated censored distributions. Srivastava (1967), Gajjar and Khatri

(1969), Patel and Gajjar (1979) and Patel (1991) have considered Type-I progressively censored and group-censored samples from Exponential, Weibull, Inverse Gaussian, Log-normal, Power series and Logistic distributions with different parameters at different stages of censoring and obtained maximum likelihood estimates of the parameters. The maximum likelihood estimates or estimating equations obtained by Gupta (1952) and Cohen (1963) can be deduced as special cases from these results.

1.3 PRESENT WORK

In order to obtain information about the reliability or warranty period of manufactured items such as electrical or electronics, components are often put on life tests and life times are observed periodically. A model is specified to represent the distribution of life times and statistical inferences are made on the basis of this model. The lifetime models may be discrete or continuous. The widely-used continuous lifetime models are Exponential, Weibull, Rayleigh, Lognormal distributions etc, whereas Geometric distribution, a discrete analogue of Exponential distribution, is used as discrete lifetime failure model.

In life testing experiments, usually the items are checked by destroying them and/or are very costly. This limits the number of items we can test. In these situations the life test may be terminated at the pre-determined number of failures. For instance, we may put N items on the test and terminate the experiment when a pre-assigned number of items, say r (< N) have failed. The samples obtained from such an experiment are called right Type-II censored samples. Another way to get censored data is to observe largest lifetimes. The lifetimes of first (N-s) components are missing; such a censoring is called left Type-II censoring. Moreover, if left and right Type-II censoring situations arise together, this is known as doubly Type-II censoring scheme.

Estimation based on classical inferences has been found to be extremely useful for a variety of problems. This thesis is concerned with the problem of estimation under progressive Type-I and Type-II, and progressive Type-I interval censoring schemes.

Suppose an item with failure rate X follows the distribution F(X|θ) with density function f(X|θ) for θ is a vector valued parameter in a real parameter space Ω. Suppose X has the distribution function $F(X|\theta_i)$ in the time interval $(N_{i-1},N_i]$ for i=1,2,…,k (k>1) with $N_0 = 0$ and $N_k = \infty$.

Let n items are placed on a life test without replacement and let n_i be the number of items that withdrawn from the test immediately after the censoring time N_{i-1}, i=2,3,…,k so that $r_{(k)} = n^{(k)} - n_k$; where $n^{(k)}$ denotes the number of item entering the k^{th} stage of an experiment. Also, let $X_1^{(i)} \le X_2^{(i)} \le \ldots \le X_{n_i}^{(i)}$ be the times of failure for i=1,2,…,k (k>1) then the likelihood function for k-stage Type-I progressive censoring without replacement is given by

$$L \alpha \prod_{i=1}^{k} \left\{ \prod_{j=1}^{n_i} f_i\left(x_j^{(i)}\right) \right\} \prod_{1=1}^{k} \left[1-F_i\left(N_i\right)\right]^{r_i} .$$

Taking X as non-negative integer valued random variable and N_i's can be chosen to be non-negative integers, a problem of estimating parameters at different stages of censoring can be considered. The method of maximum likelihood can be employed to estimate the properties of different types of estimators like MLE, shrinkage estimator, minimum mean square error estimator, and almost unbiased estimator can be investigated. Patel and Patel (2003, 2005a, 2005b, 2005c, 2006) have consider estimation of parameters of geometric life time distribution under progressive Type-I and Type-II censoring with mixture as well as competing risk models.

A generalization of Type-II censoring is progressive Type-II censoring. According to Balakrishnan and Aggrawala (2000) under progressive Type-II censoring scheme a total of n units are placed on a life test, only m are completely observed until failure. At the time of first failure, R_1 of the n-1 surviving units are randomly withdrawn from the test. At the time of next failure R_2 of the n-2- R_1 surviving units are censored, and so on. Finally, at the time of n^{th} failure all the remaining R_m = n-m-ΣR_i surviving units are censored.

Again a more generalization of such progressive Type-II censoring scheme is discussed by Lawless (1982). In this scheme, the first n_1 failures in a sample of n items are observed. Then r_1 of the remaining n-n_1 working items are withdrawn from the experiment, leaving n- n_1- r_1 on the test. When further n_2 items have failed r_2 of the still working items are withdrawn and so on. Finally, the experiment is terminated at the end of n_k^{th} failure.

Let $X_1^{(i)}, X_2^{(i)}, \ldots, X_{n_i}^{(i)}$ are the failure times during the i^{th} stage of censoring i= 1, 2,…,k and $X_{n_1}^{(1)}, X_{n_2}^{(2)}, \ldots, X_{n_k}^{(k)}$ are the censoring times for k-stage respectively.

Then the likelihood function for k-stage Type-II progressive censoring without replacement is given by

$$L=\prod_{i=1}^{k}\frac{n^{(i)}!}{\left(n^{(i)}-r_i\right)!}\left\{\prod_{j=1}^{n_i} f_i\left(x_j^{(i)}\right)\right\}\prod_{1=1}^{k}\left[1-F_i\left(x_{n_i}^{(i)}\right)\right]^{r_i}.$$

where f (.) and F(.) are composite probability density function and cumulative distribution function of life time random variable respectively.

Using the method of maximum likelihood estimation of the parameters expected waiting time of the test, expected total time of the test, sample size to minimize the total cost of the test can be considered for discrete or continuous lifetimes models. Patel and Patel (2007) have used progressive Type-II censored sample for geometric life time model. Gajjar and Patel (2008) have considered estimation for a mixture of exponential distribution based on progressive Type-II censored sample.

In this thesis the length biased exponential distribution, reciprocal exponential distribution, generalized half logistic distribution are used as life time models. The thesis may be divided into three categories viz:

(1) Estimation of the parameters under Type-I and Type-II progressive censoring scheme when samples are drawn from

(a) Length biased exponential distribution

(b) Reciprocal exponential distribution

(c) Generalized half logistic distribution

(2) Estimation of the parameters under progressive interval Type-I censoring scheme when samples are drawn from

(a) Reciprocal exponential distribution

(3) Bayesian estimation for parameters for

(a) Length biased exponential distribution

(b) Type-II generalized half logistic distribution

Detail index is given in chapter-1

Chapter-2 deals with the study of some basic results and characterizations of Length Biased Exponential distribution. Length Based sampling was introduced by Cox (1962) (see Patil 2002). It has various applications in biomedical area such as family history and disease, survival and intermediate events and latency period of AIDS due to blood transfusion (Gupta and Akman 1995). Patil and Rao (1978) wrote an article on “The study of human families and wildlife populations” The most common forms of all weight function useful in scientific and statistical literature are some basic theorems for weighted distribution and size-biased. As special case they arrived at a conclusion that the length biased version of some discrete distribution arises as mixture of the length biased version of these distributions.

A lot of work has been done by Khatree (1989) to derive relationship between original distributions and their length biased versions. A very useful result giving a relationship between original random variable X and its length biased version Y when X is either inverse Gaussian or Gamma distribution. He also proved that length biased random variable Y can be written as a linear combination of the original random variable X and a chi-square random variable Z and inversely the original random variable can be characterized through this relationship.

Several authors such as Patil et al. (1986), Jain et al. (1989), Gupta and Kirmani (1990) and recently by Olyede and George (2002) treated relationships in the perspective of reliability. In these works the survival function, the failure rate, and the mean residual life function of the length-biased distribution were expressed in relation with the original distribution.

If a random variable X follows any distribution with probability density function f(x) then the probability density function of length biased distribution of X is defined as $g(x) = \frac{xf(x)}{E(X)}$.

We have considered estimation related to parameters of the length biased exponential distribution based on progressively Type-II censored samples. Maximum likelihood estimators as well as approximate Bayes estimators of the parameters are developed. A simulation study is considered for different patterns of censoring. The results based on this chapter are published by Bhimani, Arora and Patel (2008).

Chapter -3 is considered with the estimation of parameters of reciprocal exponential distribution based on progressive interval Type-I censored samples. Maximum likelihood estimator along with its asymptotic variance is derived and compared for different censoring patterns. Confidence interval estimation is considered based on bootstrap and r - level likelihood ratio, under the three censoring patterns. Non parametric as well as parametric estimate of the survival function are obtained with their asymptotic variances. Using the method suggested by Kendall and Anderson (1971) expected duration of life test is derived and computed for different choice of time intervals.

In most applications, the data may be interval-censored. By interval-censored data, we mean that a random variable of interest is known only to lie in an interval, instead of being observed exactly. In such cases, the only information we have for each individual is that their event time falls in an interval, but the exact time is unknown.

Generally statistician faces lot of problem in the analysis of time-to-event data such as failure time data, incubation time data etc. Such data arises in lot of fields such as medicine, engineering, economics. For example doctor may be interested to know the time of convergence to AIDS for HIV positive individual, the time to the death for cancer patients, lifetime of a device etc. The analysis of time-to-event later becomes more complicated on account of censoring.

Interval censoring also known as group censoring arises when observations occur in some interval of time a and b. Such data occurs in variety of circumstances but generally it is encountered in medical studies where patients are only monitored at regular intervals (e.g. weekly or quarterly checkup). Thus, the exact time of occurrence of some changed response may only be known to have some time between two visits.

Samuelson and Kongerud (1994); Kokasa et al (1993); Farrington (1996); Odell et al (1992), Sun (1997); Lindsey and Ryan (1998) and Scallan (1999) have discussed application of interval censoring in clinical, medical, biomedical and engineering studies. Rao (1998) gave standard methods for analyzing interval censored data and discussed efficiencies of estimators derived from censoring over conventional Type-I and Type-II censoring schemes.

Estimation related to the parameters of reciprocal exponential distribution is discussed for progressively Type-II censored samples. A maximum likelihood estimator for the parameters is developed. A simulation study is considered for different pattern of censoring. These results are presented in **Chapter-4**.

Chapter-5 deals with progressive Type-II censored sample for a Type-II generalized half logistic distribution. Classical inference is carried out using simulation of such a censored sample. Maximum likelihood estimator as well as approximate Bayes estimator of the parameter along with their asymptotic variances and MSE's are derived and compared for different censoring patterns. Confidence interval estimation is considered based on bootstrap and r- level likelihood ratio under the three censoring patterns.

Half logistic model obtained as the distribution of the absolute standard logistic variate is probability model considered by Balakrishnan (1985). Balakrishnan and Puthenpura (1986) obtained best linear unbiased estimator of location and scale parameters of the half logistic distribution through linear functions of order statistics. Balakrishnan and Wong (1991) obtained approximate maximum likelihood estimates for the location and scale parameters of the half logistic distribution with Type-II Right-Censoring. Olapade (2003) proved some theorems that characterized the half logistic distribution. The half logistic distribution has not received much attention from researchers in terms of generalization. A generalized version of half logistic distribution namely Type-I and Type–II generalized half logistic distributions are considered by Ramakrishna (2008).

In **chapter-6** we have discussed the maximum likelihood estimators of the generalized half logistic distribution under Type-I progressive censoring with changing failure rates is considered. The numerical evaluation of their relative performance is made for selected values of n and p. MLE and its asymptotic variance are obtained using a simulation study based on 1000 random samples. Further results including total expected waiting time are obtained in case of interval censoring schemes also.

CHAPTER 2*

Estimation under Progressive Type-II censoring for Length Biased Exponential Distribution

2.1 INTRODUCTION

Reliability studies frequently involve testing of items that are designed to last for a long period of time. In such studies constraints are in the form of truncation and / or censoring would be deemed essential as a mean of obtaining information within reasonable time limitations.

While there are several types of censorship, two are of common usage. These are commonly referred to as Type-I and Type-II censoring. Type-I censoring occurs when the researcher sets a time limit on terminating the life test even though some of the test items remain operational.

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Type-II censoring occurs when the life test is terminated at the particular (say, $r < n$) failure.

Progressive Type-II censoring defined by Cohen (1963) is as follows.
Before conducting a life test the experimenter fixes a sample size n, a number of complete observation m and a censoring scheme (R_1, R_2, ……… R_m), n = $m + \sum R_i$. The n units are placed on a life test. Immediately after the first failure, R_1 surviving units are randomly chosen and removed from the experiment. Then after second failure, R_2 units are withdrawn and so on. The procedure is continued until all R_m remaining units are removed after the m[th] failure.

If $R_1 = R_2 = ………… = R_m = 0$, then n = m which corresponds to a complete sample. If $R_1 = R_2 = ………… = R_{m-1} = 0$ then $R_m = n – m$ corresponds to conventional Type-II right censoring scheme.

Balakrishnan and Aggarwala (2000) provided a comprehensive reference on progressive censoring, its application and techniques for analyzing data from progressive Type-II censoring schemes.

2.2 Length Biased Exponential Distribution

Consider a group of subjects who experience some event (say, the onset of disease) at times [x.sub.i], followed by some other event (say, death) at endpoints [x.sub.y]. In epidemiology studies it is often the aim to estimate the distribution of the intervals from initiation to the endpoints or to compare the distributions of these survival times for two or more well-defined groups. When it is possible to follow all subjects in a group prospectively, standard techniques of survival analysis are applicable. Frequently, however, subjects are identified to have experienced initiation through a cross-sectional study at some fixed time point; hence those who have survived to that time are recruited into the study, whereas those who have not will not be included in this initial recruitment phase, and indeed will not even be identified.

Thereafter, the group of recruited subjects is followed until a second time point, corresponding to the end of the study. Of course, some of these subjects will have censored failure times for various reasons, including their survival until the end of the study. We assume that for every subject included, an initiation date is recorded. Therefore, the data on each subject include the dates of onset and failure/censoring (as well as censoring indicators) for those subjects who have been recruited.

The intervals from initiation to failure/censoring are well known to be "length biased," which means that those time intervals actually observed tend to be longer than those arising from the true underlying failure (censoring distributions). The phenomenon of length bias was systematically studied by McFadden (1962), Blumenthal (1967), and later by Cox (1969) in the context of estimating the distribution of fiber lengths in a fabric.

Length biased sampling has various applications in biomedical area such as family history and disease, survival and intermediate events and latency period of AIDS due to blood transfusion (Gupta and Akman 1995). Patil and Rao (1978) wrote an article on “The study of human families and wildlife populations” They arrived at a conclusion that the length biased version of some discrete distribution arises as a mixture of the length biased version of these distributions.

A lot of work has been done by Khatree (1989) to derive relationship between original distributions and their length biased versions. A very useful result giving a relationship between original random variable X and its length biased version Y when X is either inverse Gaussian or Gamma distribution. He also proved that length biased random variable Y can be written as a linear combination of the original random variable X and a chi-square random variable Z and inversely the original random variable can be characterized through this relationship.

Several authors such as Patil et al. (1986), Jain et al. (1989), Gupta and Kirmani (1990) and recently by Olyede and George (2002) treated relationships in the perspective of reliability. In these works the survival function, the failure rate, and the mean residual life function of the length-biased distribution were expressed in relation with the original distribution.

If a random variable X follows any distribution with probability density function f(x) then the probability density function of length biased distribution of X is defined as $g(x)=\frac{xf(x)}{E(X)}$.

2.3 Maximum Likelihood Estimation

The probability density function and cumulative density function of a length biased exponential distribution with parameter θ is given by,

$$g(x)=\frac{x}{\theta}\frac{1}{\theta}e^{\frac{-x}{\theta}},\ x>0,\ \theta>0. \tag{2.3.1}$$

and

$$G(x) = 1-\left[\frac{x}{\theta}e^{\frac{-x}{\theta}}+e^{\frac{-x}{\theta}}\right]. \tag{2.3.2}$$

If n item are put on test, then the likelihood function under Progressive Type-II censoring scheme as discussed in the section 2.1 is given by,

$$L = \text{constant} \prod_{i=1}^{m} g(x_i|\theta) \prod_{i=1}^{m} \left[1 - G(x_i|\theta)\right]^{R_i}.$$

Using (2.3.1) and (2.3.2) the likelihood function becomes

$$L = \text{constant} \prod_{i=1}^{m} \frac{x_i}{\theta^2} e^{\frac{-x_i}{\theta}} \prod_{i=1}^{m} \left[\frac{x_i}{\theta} e^{\frac{-x_i}{\theta}} + e^{\frac{-x_i}{\theta}} \right]^{R_i}. \tag{2.3.3}$$

The log likelihood function is given by

$$\begin{aligned} \ln L &= \ln c + \sum_{i=1}^{m} \ln x_i - 2m\ln\theta - \sum_{i=1}^{m} \frac{x_i}{\theta} + \sum_{i=1}^{m} R_i \ln\left\{ e^{\frac{-x_i}{\theta}} \left(\frac{x_i}{\theta} + 1 \right) \right\}. \\ &= \ln c + \sum_{i=1}^{m} \ln x_i - 2m\ln\theta - \sum_{i=1}^{m} \frac{x_i(1+R_i)}{\theta} + \sum_{i=1}^{m} R_i \ln\left(\frac{x_i}{\theta} + 1 \right). \end{aligned}$$

Differentiating ln L with respect to θ and equating to zero we obtain

$$-\frac{2m}{\theta}+\sum_{i=1}^{m}\frac{x_i(1+R_i)}{\theta^2}+\sum_{i=1}^{m}\frac{R_i}{\frac{x_i}{\theta}+1}\left(-\frac{x_i}{\theta^2}\right)=0\,. \tag{2.3.4}$$

Hence we obtain the mle $\hat{\theta}$ as

$$\hat{\theta}=\frac{\sum_{i=1}^{m}x_i(1+R_i)}{\left[2m+\sum_{i=1}^{m}\frac{x_iR_i}{x_i+\hat{\theta}}\right]}. \tag{2.3.5}$$

Solving the equation (2.3.5) by any iterative method like Newton-Raphson for $\hat{\theta}$, maximum likelihood estimate of θ can be obtained.

Now again differentiating (2.3.4) we get

$$\frac{\partial^2 \ln L}{\partial \theta^2} = \frac{2m}{\theta^2} - 2\sum_{i=1}^{m} \frac{x_i(1+R_i)}{\theta^3} - \sum_{i=1}^{m} \frac{x_i R_i (x_i + 2\theta)}{\left(\theta (x_i+\theta)^2\right)}$$

Hence observed asymptotic variance of $\hat{\theta}$ is given by (Due to Cohen1963)

$$V(\hat{\theta}) = \frac{-1}{\left.\frac{\partial^2 \ln L}{\partial \theta^2}\right|_{\theta=\hat{\theta}}}.$$

2.4 Bayes Estimate

Since last three decades lot of work has been developed in the field of reliability using Bayesian approach. Under certain limitations, the maximum likelihood estimators have a number of desirable properties and are extensively used in preference to other classical estimators. A Bayesian, however, interprets probability as a person's degree of belief in a certain proposition based on prior knowledge (or current knowledge) about parameter θ and this degree of belief is successively revised or updated as new information is accumulated about the proposition.

In Bayesian framework the parameter is justifiably regarded as a random variable and the data once obtained, is given or fixed. Also it is realistic to assume that life parameter is stochastically dynamic. Martz and Waller (1982) have done lot of work regarding Bayes estimation in the field of life testing and reliability.

In this section the Bayesian approach is used to derive estimate of the parameter θ, assuming we are in the situation where very less is known about a prior about the values of θ.

Prior distribution is an essential component of Bayesian inference. There is no single answer to the question, “What should be the right prior?” For much of the time the prior information is subjective and is based on a person’s own experience and judgement. Different types of priors like non-informative prior, uniform prior, Jeffreys’ prior, Hartigan’s prior, natural conjugate prior, minimal informative prior and Dirichlet’s prior are used in Bayesian inference.

To avoid the complexity involved in solving Bayes estimates. Here we consider prior distribution of θ as exponential distribution with mean β.

That is $$\pi(\theta)=\frac{1}{\beta}e^{-\frac{\theta}{\beta}},\ \ \theta>0,\ \ \beta>0. \qquad (2.4.1)$$

Using the likelihood function given in (2.3.3) and the prior defined in (2.4.1), the posterior distribution of θ is given by:

$$h(\theta|x) \alpha\, L\pi(\theta)$$

$$\alpha \prod_{i=1}^{m} \frac{x_i}{\theta^{2m}} e^{-\sum_{i=1}^{m}\frac{x_i}{\theta}} \prod_{i=1}^{m}\left(\frac{x_i}{\theta} e^{-\frac{x_i}{\theta}} + e^{-\frac{x_i}{\theta}}\right)^{R_i} \frac{1}{\beta} e^{-\frac{\theta}{\beta}}$$

$$\alpha \prod_{i=1}^{m} \frac{x_i}{\beta\theta^{2m}} e^{-\left(\sum_{i=1}^{m}\frac{x_i}{\theta}+\frac{\theta}{\beta}\right)} \prod_{i=1}^{m}\left(\frac{x_i}{\theta} e^{-\frac{x_i}{\theta}} + e^{-\frac{x_i}{\theta}}\right)^{R_i}. \qquad (2.4.2)$$

Now under squared error loss function, the Bayes estimator of θ can be obtained as

$$\theta^* = E(\theta|x)$$

$$= c \int_0^{\infty} \theta \prod_{i=1}^{m} \frac{x_i}{\beta\theta^{2m}} e^{-\left(\sum_{i=1}^{m}\frac{x_i}{\theta}+\frac{\theta}{\beta}\right)} \prod_{i=1}^{m}\left(\frac{x_i}{\theta} e^{-\frac{x_i}{\theta}} + e^{-\frac{x_i}{\theta}}\right)^{R_i} d\theta$$

where c is normalizing constant. Here it is not possible to get θ^* in closed form, so we refer to numerical integration to find a solution. Lindley (1980) gave an alternative method to approximate the integrals that occur in Bayesian statistics. According to Lindley (1980), the Bayes estimator θ^* is approximated as

$$\theta^* = E(\theta|x) \approx \hat{\theta} + \frac{1}{2}\left(u_2 + 2u_1\rho_1\right)\sigma^2 + \frac{1}{2}l_3 u_1 \sigma^4\Big|_{\theta=\hat{\theta}} . \tag{2.4.3}$$

where

$$u_1 = \frac{d\theta}{d\theta} = 1$$

$$u_2 = \frac{\partial^2\theta}{\partial\theta^2} = 0$$

$$\rho = \ln\pi\left(\theta\right) \ = \text{-}\ln\beta\text{-}\frac{\theta}{\beta}$$

$$\rho_1 = \frac{d\rho}{d\theta} = \frac{-1}{\beta}$$

$$l = \ln L$$

$$l_2 = \frac{\partial^2 l}{\partial\theta^2}$$

$$l_3 = \frac{\partial^3 l}{\partial\theta^3}$$

$$\sigma^2 = \left(-l_2\right)^{-1} .$$

2.5 Simulation Study

In this section we consider a simulation study to observe behavior of ML and Bayes estimate of θ under different censoring patterns. Here we generate 8000 random samples of size 15, 25 and 50 from length biased distribution defined in (2.3.1) for $\theta = 0.2, 0.8$ and 1. To generate a sample (x) under progressive Type-II censoring with m = 5 we have used the following method as discussed by Aggarwala and Balakrishnan (2002).

Step 1:- Generate U_i, where U_i is a set of random number i = 1, 2, 3, 4, 5

Step 2:- $Z_i = -\ln(1-U_i)$

Step 3:- $Y_i = \frac{Z_1}{n} + \frac{Z_2}{n-R_1-1} + \ldots\ldots\ldots\ldots\ldots + \frac{Z_i}{n-\sum_{j=1}^{i-1} R_j - i + 1}$

Step 4:- $G(x_i) = 1 - \exp(-Y_i)$

i.e. $$1-\left[\frac{x_i}{\theta} e^{\frac{-x_i}{\theta}} + e^{\frac{-x_i}{\theta}}\right] = 1-\exp(-Y_i)$$

By solving the equation in Step 4 we will get the values of x_i. On the basis of simulated samples ML estimates of θ, as given (2.3.5) along with its asymptotic variance are demonstrated in Table-1, where as Table-2 represents Bayes estimates of θ, as given in (2.4.3) with its simulated variance for the three censoring patterns.

Here we have considered the following three censoring patterns for simulation.

n = 15	n= 25	n = 50
R_1: (3, 3, 2, 0, 2)	R_1: (3, 3, 2, 0, 12)	R_1: (3, 3, 2, 0, 37)
R_2 : (1, 2, 3, 3, 1)	R_2 : (1, 2, 3, 3, 11)	R_2 : (1, 2, 3, 3, 36)
R_3 : (0, 0, 0, 0, 10)	R_3 : (0, 0, 0, 0, 20)	R_3 : (0, 0, 0, 0, 45)

For n=15, for 8000 iterations.

Table-1 Estimator of θ under Maximum Likelihood Estimation

		$\hat{\theta}$ arithmetic mean	Min θ	Max θ	Asy $V(\hat{\theta})$	Sim $V(\hat{\theta})$
$\theta = 0.2$	R_1	0.566	0.1495	5.6925	0.1489	0.0411
	R_2	0.5457	0.1614	8.1831	0.0906	0.0379
	R_3	0.3082	0.0917	0.7361	0.0072	0.009
$\theta = 0.8$	R_1	2.4793	0.5718	13.8227	2.8925	0.7022
	R_2	2.3761	0.5012	6.4057	1.6432	0.6356
	R_3	1.3184	0.2759	20.9738	0.138	0.1707
$\theta = 1$	R_1	3.0341	0.6918	7.6316	4.163	1.8975
	R_2	2.9665	0.5518	47.5621	2.7197	1.0741
	R_3	1.6836	0.412	26.7325	0.2268	1.0526

Table-2 Estimator of θ under Bayesian Analysis

		$\beta=3$		$\beta=6$		$\beta=10$	
		θ^* arithmetic mean	$SimV(\theta^*)$	θ^* arithmetic mean	$SimV(\theta^*)$	θ^* arithmetic mean	$SimV(\theta^*)$
θ= 0.2	R_1	0.1683	0.4887	0.1931	0.4782	0.203	0.4751
	R_2	0.3992	0.043	0.4143	0.0486	0.4203	0.0542
	R_3	0.3084	0.0095	0.3096	0.0097	0.3101	0.0098
θ =0.8	R_1	2.1406	0.725	2.6225	0.8511	2.8154	1.0436
	R_2	2.2728	0.6483	2.5465	0.6783	2.6561	0.6941
	R_3	1.355	0.1729	1.378	0.2054	1.3872	0.2227
θ = 1	R_1	2.563	1.8237	3.2566	2.1097	3.5342	2.4257
	R_2	2.6834	1.6742	3.1365	1.4811	3.3179	1.2338
	R_3	1.721	1.2644	1.7588	0.3238	1.7739	0.3595

For n=25, for 8000 iterations.

Table-3 Estimator of θ under Maximum Likelihood Estimation

		$\hat{\theta}$ arithmetic mean	Min θ	Max θ	Asy $V(\hat{\theta})$	Sim $V(\hat{\theta})$
θ = 0.2	R_1	0.4736	0.1261	2.4111	0.0223	0.0239
	R_2	0.4629	0.1527	1.1386	0.0188	0.0202
	R_3	0.2862	0.1012	0.7749	0.0035	0.007
θ =0.8	R_1	2.056	0.5121	31.6839	0.4376	0.6438
	R_2	1.9946	0.516	34.1653	0.3696	0.5664
	R_3	1.1905	0.3176	11.1648	0.0617	0.1592
θ = 1	R_1	2.608	0.5089	14.1227	0.6835	1.6775
	R_2	2.5394	0.6083	43.7947	0.6129	0.9857
	R_3	1.5437	0.4084	32.6237	0.1106	0.7625

Table-4 Estimator of θ under Bayesian Analysis

		β=3		β=6		β=10	
		θ^* arithmetic mean	$SimV(\theta^*)$	θ^* arithmetic mean	θ^* arithmetic mean	$SimV(\theta^*)$	θ^* arithmetic mean
θ= 0.2	R_1	0.4449	0.023	0.4486	0.0238	0.4501	0.0232
	R_2	0.4396	0.02	0.4427	0.0206	0.444	0.0208
	R_3	0.2851	0.007	0.2857	0.007	0.286	0.007
θ =0.8	R_1	2.0163	0.3742	2.0893	0.4713	2.1184	0.5309
	R_2	1.9653	0.3304	2.0269	0.4564	2.0516	0.5254
	R_3	1.2023	0.1556	1.2125	0.1535	1.2167	0.1567
θ = 1	R_1	2.5318	0.5354	2.6457	0.6598	2.6913	0.614
	R_2	2.4703	0.5166	2.5724	0.6266	2.6133	0.5941
	R_3	1.5537	0.2474	1.5721	0.3101	1.5795	0.3419

For n=50, for 8000 iterations.

Table-6 Estimator of θ under Maximum Likelihood Estimation

		$\hat{\theta}$ arithmetic mean	Min θ	Max θ	Asy $V(\hat{\theta})$	Sim $V(\hat{\theta})$
θ = 0.2	R_1	0.4135	0.1352	1.032	0.0071	0.0161
	R_2	0.4063	0.1314	1.029	0.0064	0.014
	R_3	0.2654	0.0812	0.642	0.0017	0.0058
θ =0.8	R_1	1.7522	0.4264	4.5736	0.1296	0.2864
	R_2	1.7103	0.4021	33.2933	0.1198	0.425
	R_3	1.0881	0.2614	23.1429	0.0291	0.1493
θ = 1	R_1	2.2318	0.6165	34.6223	0.2127	0.8761
	R_2	2.1906	0.6222	57.393	0.2089	0.8393
	R_3	1.4055	0.3517	43.7734	0.0546	0.4856

Table-7 Estimator of θ under Bayesian Analysis

		β=3		β=6		β=10	
		θ^* arithmetic mean	$SimV(\theta^*)$	θ^* arithmetic mean	θ^* arithmetic mean	$SimV(\theta^*)$	θ^* arithmetic mean
θ= 0.2	R_1	0.4066	0.015	0.4078	0.0152	0.4083	0.0153
	R_2	0.4007	0.0132	0.4018	0.0133	0.4022	0.0134
	R_3	0.2649	0.0058	0.2652	0.0058	0.2653	0.0058
θ =0.8	R_1	1.7414	0.2742	1.763	0.2858	1.7717	0.2862
	R_2	1.7004	0.3302	1.7204	0.1878	1.7283	0.1117
	R_3	1.0903	0.1347	1.0951	0.1446	1.097	0.1087
θ = 1	R_1	2.2088	0.4801	2.2442	0.543	2.2584	0.5703
	R_2	2.1657	0.4623	2.2005	0.4695	2.2145	0.4865
	R_3	1.4056	0.2238	1.4147	0.3053	1.4184	0.2435

2.6 Conclusions and Suggestions:-

1) For a small sample size (n = 15) $\text{Sim V}(\hat{\theta}) < \text{Sim V}(\theta^*)$ i.e. the MLE is better than the Bayes estimator for a given θ and β in the case of all the three censoring schemes.

2) For any sample size (n= 15, n=25, n=50) simulated variance of MLE and Bayes estimator decreases in case of all the three censoring schemes.

3) For the fixed values of θ and β simulated variance of MLE and Bayes estimator decreases according to the selection of the censoring schemes R_1, R_2 and R_3 respectively.

4) As n increases $\text{Sim V}(\hat{\theta})$ as well as $\text{Sim V}(\theta^*)$ decreases for fixed values of θ and β for the three censoring schemes.

CHAPTER 3

Estimation under Progressive Interval Type-I Censoring for Reciprocal Exponential Distribution

3.1 INTRODUCTION

In most applications, the data may be interval-censored. By interval-censored data, we mean that a random variable of interest is known only to lie in an interval, instead of being observed exactly. In such cases, the only information we have for each individual is that their event time falls in an interval, but the exact time is unknown.

Generally statistician faces lot of problem in the analysis of time-to-event data such as failure time data, incubation time data etc. Such data arises in lot of fields such as medicine, engineering, economics. For example doctor may be interested to know the time of convergence to AIDS for HIV positive individual, the time to the death for cancer patients, lifetime of a device etc. The analysis to time-to-event later becomes more complicated on account of censoring.

Interval censoring also known as group censoring arises when observations occur in some interval of time a and b. Such data occurs in variety of circumstances but generally it is encountered in medical studies where patients are only monitored at regular intervals (e.g. weekly or quarterly checkup). Thus, the exact time of occurrence of some changed response may only be known to have some time between two visits.

Samuelson and Kongerud (1994); Kokasa et al (1993); Farrington (1996); Odell et al (1992), Sun (1997); Lindsey and Ryan (1998) and Scallan (1999) have discussed application of interval censoring in clinical, medical, biomedical and engineering studies. Rao (1998) gave standard methods for analyzing interval censored data and discussed efficiencies of estimators derived from censoring over conventional Type-I and Type-II censoring schemes.

In many life test studies, it is common that the lifetimes of test units may not be recorded exactly. An experimenter may terminate the life test before all n products fail in order to save time or cost. Hence, the test is said to be censored in which data collected are the exact failure times on those functional (none failed) units. Moreover, some of the test units may have to be removed at different stage(s) of censoring related study for various other reasons; which leads to progressive censoring. For example some products are withdrawn for more thorough inspection or are saved so that it can be used as test specimens in other studies, or patients who for some reasons do not turn up in a clinical study would also result in progressive removal.

According to the current trend Type-I and Type-II progressive censoring schemes are becoming quite popular for analyzing highly reliable data. Cohen (1963) had introduced progressive Type-II censoring. Mahmond et al (2006) considered progressive Type-II censoring samples for many continuous life time models. Balakrishnan and Aggarwala (2000) give an insight on this method and the applications of this scheme.

Aggarwala (2001) introduced progressive Type-I interval censoring scheme for exponential life time model. In this type of censoring n units are put on test at time 0 and each unit is kept on life test until the unit fails or is censored. All the units are observed during pre-set times T_1, T_2,…, Tm where m is a fixed integer. Thus the time axis is partitioned into interval $I_i = (T_{i-1}, T_i]$ where i = 1, 2,…, m+1 and $T_0 = 0$, $T_{m+1} = \infty$, T_m is the time at which we will terminate the experiment. Let n_i denote the number of units which fail in the interval I_i. The values R_1, R_2… R_m may be specified as positive integers or percentages p_1, p_2… p_m with $p_m = 100$ of remaining functional units and the number of units which are functioning at time T_1, T_2, …,T_m are random variables.

In case when R_1, R_2, …, R_m are pre- specified positive integers, the number of units removed at time T_i is R_i^{obs} = min(R_i, no. of units remaining) i = 1, 2, …, m-1 and R_m^{obs} = all the remaining units at time Tm, when life test experiment is terminated.

In this chapter we have considered reciprocal exponential distribution as a continuous lifetime model and apply progressively Type-I interval censoring without changing the parameters at different stages of censoring. Section 2 states the properties and applications of Reciprocal Exponential distribution; in Section 3 the method of maximum likelihood estimation is described. Simulation of progressive Type-I interval censored samples is carried out in Section 4.

Section 5 deals with interval estimation. Expected duration of the life test is discussed in Section 6. Comparison between Non-parametric and Parametric estimation of survival function and its confidence interval are considered in Section 7. The methods are illustrated using numerical examples for different censoring pattern.

3.2 Reciprocal Exponential Distribution

A random variable X follows a Reciprocal Exponential distribution if its reciprocal 1/X follows an Exponential distribution with scale parameter θ, θ>0.

The probability density function (pdf) and cumulative distribution function (cdf) of reciprocal exponential distribution are as follows,

$$g(x,\theta)=\frac{\theta}{x^2}e^{-\theta/x} \quad , x>0, \theta>0. \tag{3.2.1}$$

and

$$G(x,\theta)=e^{-\theta/x} \ . \tag{3.2.2}$$

Reciprocal Exponential distribution is a special case of Inverted Gamma distribution, having pdf $f(x;\alpha,\beta) = \frac{\beta^{\alpha} x^{\alpha-1} e^{-\beta/x}}{\Gamma\alpha} \quad x \geq 0, \alpha,\beta \geq 0$ with β = θ and α = 1.

The Reciprocal Exponential distribution appears in Bayesian inference in a natural way as the posterior distribution of the variance in normal sampling when reference or conjugate distributions on the parameters are used. Reciprocal Exponential distribution is especially used in reliability applications (see Barlow and Proschan (1981)). It is also hidden among the Pearson curves, specifically Pearson V and Vinci (1921) should be credited for his income distribution applications. In actuarial literature, Cummis et al. (1990) used the Inverse Gamma distribution for approximating the fire loss experiences of a major university. The distribution turns out to be one of the best two parameter models; in fact the data are approximately modeled by one parameter special case where α = 1, an Inverse Exponential distribution.

3.3 Maximum Likelihood Estimation

Suppose a progressive Type-I interval censored sample is collected as described in Section1, beginning with a random sample of n units having probability density distribution function given by (3.2.1). Based on the observed data, the likelihood function L is proportional to the expression.

$$L \,\alpha\, \left[G_x(T_1)-G_x(T_0)\right]^{n_1}\left[1-G_x(T_1)\right]^{R_1}\times$$
$$\prod_{i=2}^{m}\left[G_x(T_i)-G_x(T_{i-1})\right]^{n_i}\left[1-G_x(T_i)\right]^{R_i}.$$

$$L \,\alpha\, e^{-n_1\theta/T_i}\left[1-e^{-\theta/T_1}\right]^{R_1}\prod_{i=2}^{m}\left[e^{-\theta/T_i}-e^{-\theta/T_{i-1}}\right]^{n_i}\left[1-e^{-\theta/T_i}\right]^{R_i}.$$

Here for the sake of simplicity we consider equal length time interval

$$\text{i.e. } T_i - T_{i-1} = t$$

$$\text{Thus } T_i = it,\ \ i=1,2,........,m$$

Thus the likelihood function reduces to,

$$L\ \alpha\ e^{-n_1\theta/t}\left[1-e^{-\theta/t}\right]^{R_1}\prod_{i=2}^{m}\left[e^{-\theta/(i-1)t}\left(e^{-\theta/i(i-1)t}-1\right)\right]^{n_i}\left[1-e^{-\theta/it}\right]^{R_i}.$$

(3.3.1)

The likelihood equation for estimating θ is obtained by

$$\frac{\partial \ln L}{\partial \theta} = 0.$$

which gives

$$\frac{-n_1}{t}+\frac{R_1}{1-e^{-\theta/t}}\left(-e^{-\theta/t}\right)\left(\frac{-1}{t}\right)-\sum_{i=2}^{m}\frac{n_i}{(i-1)t}$$
$$+\sum_{i=2}^{m}\frac{n_i}{e^{-\theta/i(i-1)t}-1}e^{-\theta/i(i-1)t}\left(\frac{-1}{ti(i-1)}\right)+\sum_{i=2}^{m}\frac{R_i}{1-e^{-\theta/it}}\left(-e^{-\theta/it}\right)\left(\frac{-1}{it}\right)=0.$$

(3.3.2)

Under this situation the MLE of θ can be obtained by using any iterative procedure like Newton-Raphson and solving the equation (3.3.2). Hence we get maximum likelihood estimator of θ, denoted by $\hat{\theta}$.

Now again differentiating (3.3.2) we get,

$$\frac{\partial^2 \ln L}{\partial \theta^2} = \frac{-R_1 e^{-\theta/t}}{t^2\left(1-e^{-\theta/t}\right)^2} - \sum_{i=2}^{m} \frac{R_i e^{-\theta/it}}{i^2 t^2\left(1-e^{-\theta/it}\right)^2} - \sum_{i=2}^{m} \frac{n_i e^{-\theta/i(i-1)t}}{t^2 i^2 (i-1)^2\left(e^{-\theta/i(i-1)t}-1\right)^2}.$$

Hence observed asymptotic variance of $\hat{\theta}$ is given by (Due to Cohen 1963).

$$V(\hat{\theta}) \approx \frac{-1}{\left.\frac{\partial^2 \ln L}{\partial \theta^2}\right|_{\theta=\hat{\theta}}}.$$

3.4 Comparison of censoring patterns via simulation

In this section considering equal interval length, the Reciprocal Exponential distribution defined in (3.2) as the lifetime model from which 1000 samples were generated using the values θ = 3 and 5, t = 2, m = 5 and sample size n = 20 and 50 respectively, under the following censoring patterns.

n = 20	n = 50
S_1: (3, 3, 2, 1, $n-n_1-n_2-n_3-n_4-R_1-R_2-R_3-R_4$)	S_1: (12, 10, 8, 6, $n-n_1-n_2-n_3-n_4-R_1-R_2-R_3-R_4$)
S_2: (1, 2, 3, 3, $n-n_1-n_2-n_3-n_4-R_1-R_2-R_3-R_4$)	S_2: (6, 8, 10, 12, $n-n_1-n_2-n_3-n_4-R_1-R_2-R_3-R_4$)
S_3: (0, 0, 0, 0, $n-n_1-n_2-n_3-n_4-R_1-R_2-R_3-R_4$)	S_3: (0, 0, 0, 0, $n-n_1-n_2-n_3-n_4-R_1-R_2-R_3-R_4$)

We have used the simulation algorithm given by Aggarwala (2001) to generate samples from progressive Type-I intervals censoring scheme. Here we have specified the fixed number of units instead of proportion of surviving units to be removed at five monitoring and censoring points. The removing units from the surviving units at five stages are decreasing in pattern S_1 while increasing in pattern S_2. In pattern S_3, a convectional Type-I interval censoring scheme is employed.

Steps for Simulation:-

Consider $n_1 \sim$ Binomial $(n, G(T_1))$.

and

$$n_i \mid n_{i-1},........,R_{i-1},.........,R_1 \sim \text{Binomial}\left(n-\sum_{j=1}^{i-1}\left(n_j+R_j\right), \frac{G\left(T_i\right)-G\left(T_{i-1}\right)}{1-G\left(T_{i-1}\right)} \right).$$

Table-1 gives the summary statistics of the maximum likelihood estimators for the three censoring patterns; with its observed asymptotic variance $AV(\hat{\theta})$ and simulated variance $SV(\hat{\theta})$, in case of 1000 random samples generated for n = 20 and 50, θ = 3 and 5, m= 5 and t = 2. Here $\hat{\theta}$ is the average of simulated MLE.

Table-1: Summary Statistics

For θ = 3, n = 20					
Scheme	**Min** $\hat{\theta}$	**Max** $\hat{\theta}$	$\hat{\theta}$	$AV(\hat{\theta})$	$SV(\hat{\theta})$
S_1	1.6994	8.6723	2.8066	0.5811	0.8779
S_2	1.9003	19.7141	2.9473	0.5757	1.0764
S_3	0.8838	10.5502	2.6735	0.4813	1.0975
For θ =5, n = 20					
Scheme	**Min** $\hat{\theta}$	**Max** $\hat{\theta}$	$\hat{\theta}$	$AV(\hat{\theta})$	$SV(\hat{\theta})$
S_1	2.099	26.0886	4.5302	1.5908	4.6609
S_2	2.1848	17.8871	4.4387	1.3212	3.5148
S_3	1.707	15.7642	4.3839	1.2028	3.8289
For θ = 3, n = 50					
Scheme	**Min** $\hat{\theta}$	**Max** $\hat{\theta}$	$\hat{\theta}$	$AV(\hat{\theta})$	$SV(\hat{\theta})$
S_1	2.8121	5.2529	3.4482	0.2431	0.0662
S_2	3.1273	4.7801	3.6856	0.2383	0.0605

S_3	1.3171	5.2174	2.5494	0.1627	0.3636
For θ = 5, n = 50					
Scheme	**Min** $\hat{\theta}$	**Max** $\hat{\theta}$	$\hat{\theta}$	$AV(\hat{\theta})$	$SV(\hat{\theta})$
S_1	3.3257	12.6134	4.4603	0.4786	0.6969
S_2	3.563	18.6403	4.6533	0.4549	0.5998
S_3	2.0822	8.446	4.0116	0.3466	0.8636

From results of Table-1 we observe from the asymptotic variance of three schemes that the censoring pattern S_3 produces the most precise estimate of θ followed by S_2 and then S_1. This is due to the fact that more units are kept in the experiment for a longer period of time in S_3 followed by S_2 and then S_1.

3.5 Confidence Interval Estimation

In this section we consider interval estimation of unknown parameter θ using the method of parametric bootstrap confidence interval and the method of r-level likelihood. According to Davison and Hinkley (1997) a 100(1-α) % parametric bootstrap confidence interval for θ is given by

$$\left(\frac{\hat{\theta}^2}{\hat{\theta}_{boot}(1-\alpha/2)}, \frac{\hat{\theta}^2}{\hat{\theta}_{boot}(\alpha/2)}\right) \qquad (3.5.1)$$

where $\hat{\theta}_{boot}(p)$ is the p^{th} percentile of the simulated sample of 1000 estimates simulated using the observed value of $\hat{\theta}$ of the given sample.

The shape and magnitude of $L(\theta)$ relative to $L(\hat{\theta})$ over all possible values of θ describe the information on θ that is considered in $data_i$, i = 1, 2, …,n this suggest importance of the relative likelihood function (RLF), $R(\theta) = \frac{L(\hat{\theta})}{L(\theta)}$.

Considering R(θ) ≤ r, where r is the desired level of RLF, it is observed that larger values of r will result in wider intervals for variation in θ. The inequality R(θ) ≤ r to be solved to construct a r - level likelihood interval for θ. From a graph of likelihood ratio= $L(\hat{\theta})/L(\theta)$ plotted against various values of θ, the r - level likelihood interval for θ can be obtained for given level r, by drawing a horizontal line at $L(\hat{\theta})/L(\theta)$ = r and the corresponding r - level likelihood interval will contain all values of θ below this line.

For bootstrapping, we again have simulated 1000 samples using the value of $\hat{\theta}$ as a true value of θ and calculated $\hat{\theta}_{boot}(p)$ for p = 0.025 and p = 0.975 for all the three censoring patterns and the values are as follows:

Table-2: Values of $\hat{\theta}_{boot}(p)$ under the three censoring patterns

For θ = 3, n = 20	S_1	S_2	S_3
$\hat{\theta}_{boot}(0.025)$	1.8149	2.0993	1.1563
$\hat{\theta}_{boot}(0.975)$	5.1884	5.0289	4.9493
For θ = 3, n = 50	S_1	S_2	S_3
$\hat{\theta}_{boot}(0.025)$	3.1529	3.4876	1.4108
$\hat{\theta}_{boot}(0.975)$	4.3971	4.8167	3.3716
For θ = 5, n = 20	S_1	S_2	S_3
$\hat{\theta}_{boot}(0.025)$	2.3107	2.4725	1.8652
$\hat{\theta}_{boot}(0.975)$	8.6723	9.18	7.7237
For θ = 5, n = 50	S_1	S_2	S_3
$\hat{\theta}_{boot}(0.025)$	3.474	3.7621	2.171
$\hat{\theta}_{boot}(0.975)$	5.7877	5.7845	5.0382

Using the result given in (3.5.1), parametric bootstrap confidence interval and likelihood level r = 5 confidence intervals for θ in case of all the three censoring patterns S_1, S_2 and S_3 is given in the Table-3. The advantage of likelihood level confidence interval estimation is that it does not require large amounts of simulation as required in bootstrapping.

Table-3: Values of bootstrap and r-level likelihood interval for different n and θ.

		Bootstrap Confidence Interval	r-level likelihood interval, where r = 5
For θ = 3, n = 20	S_1	(1.5182, 4.3402)	(1.7095, 3.23449)
	S_2	(1.7273, 4.1378)	(1.7842, 4.7749)
	S_3	(1.4442, 6.1814)	(2.1459, 5.1862)
For θ = 3, n = 50	S_1	(2.7041, 3.7712)	(2.7195, 3.6354)
	S_2	(2.8201, 3.8948)	(2.9814, 3.7543)
	S_3	(1.9277, 4.6069)	(2.0187,4.5518)
For θ = 5, n = 20	S_1	(2.3665, 8.8816)	(2.6709, 8.4437)
	S_2	(2.1462, 7.9685)	(2.2784, 7.9877)
	S_3	(2.4883, 10.3038)	(2.5685, 10.1621)
For θ = 5, n = 50	S_1	(3.4373, 5.7266)	(3.3755, 5.4234)
	S_2	(3.7433, 5.7556)	(3.7842, 5.5793)
	S_3	(3.1942, 7.4127)	(3.0459, 7.1742)

From Table – 3 we observe that 5- level likelihood gives smaller length confidence interval rather than the 95% bootstrap confidence interval.

3.6 Expected Duration of Life Test (EDLT)

In case of a life test with m-stage interval Type-I progressive censoring the expected duration of the test can be obtained using the method suggested by Kendall and Anderson (1971).

The expected duration of the life test (EDLT) is given by,

EDLT = E [D({t_i}, T_m, θ]

$$=T_1 p_1^n + \sum_{i=2}^{m-1} T_i \left[\left(p_1 + \ldots.. + p_i\right)^{n-\sum_{j=1}^{i-1} R_j} - \left(p_1 + \ldots.. + p_{i-1}\right)^{n-\sum_{j=1}^{i-2} R_j} \right]$$

$$+T_m \left[1 - \left(p_1 + \ldots.. + p_{m-1}\right)^{n-\sum_{j=1}^{m-2} R_j} \right],$$

where $p_i = G_i - G_{i-1}$.

$$EDLT = T_1 \left(e^{-\theta/T_i}\right)^n + \sum_{i=2}^{m-1} T_i \left[\left(e^{-\theta/T_i}\right)^{n-\sum_{j=1}^{i-1} R_j} - \left(e^{-\theta/T_{i-1}}\right)^{n-\sum_{j=1}^{i-2} R_j} \right]$$

$$+T_m \left[1 - \left(e^{-\theta/T_{m-1}}\right)^{n-\sum_{j=1}^{m-2} R_j} \right].$$

For equal length intervals i.e. (T_i = it, i= 1, 2, ..,m), the EDLT reduces to

$$EDLT = mt - t\sum_{i=1}^{m-1}\left[1-\left(e^{-\theta/it}\right)\right]^{n-\sum_{j=1}^{i-1}R_j}. \quad (3.6.1)$$

If the sample size (n = 20) and intervals (m = 5) are fixed for the three censoring patterns as discussed in Section 3 the values of EDLT are calculated for different value of time interval t keeping θ fixed as shown in Table-4. In Table-5 the values are tabulated for different θ keeping t fixed.

Table-4: Expected duration of life test

n = 20, m = 5, θ = 3			
T	S_1	S_2	S_3
2	5.6701	6.481	7.1192
3	9.7215	12.6913	14.3452
4	12.962	18.9565	19.8974
5	16.2025	24.531	24.9846
6	19.443	29.7816	29.9975
7	22.6835	34.8935	34.9996
8	25.924	39.9455	39.9999
9	29.1645	44.9709	44.5
10	32.405	49.9838	49.5

Table-5: Expected duration of life test

N = 20, m = 5, t = 2			
θ	S_1	S_2	S_3
0.5	9.9999	9.9991	10
1	9.9918	9.9272	9.9992
1.5	9.8394	9.4782	9.9487
2.5	7.9628	7.0733	8.5574
3.5	5.1242	4.4972	5.6673
4	4.069	3.6291	4.4725
4.5	3.3253	3.033	3.6018
5	2.8312	2.6449	3.0118
8.5	2.0264	2.0203	2.0325

We observe that as t increase the EDLT increases for all three censoring patterns, even EDLT is smaller for scheme S_1 than that of scheme S_2 followed by scheme S_3. Also as θ increase EDLT decreases for all the three censoring patterns. Here EDLT for fixed t and θ variable is smaller for scheme S_2 than that of scheme S_1 followed by scheme S_3.

Chapter 4

Estimation under Progressive Type-II censoring for Reciprocal Exponential Distribution

4.1 Introduction

The times to the occurrences of events are termed as “lifetimes”. i.e. the actual length of an individual is termed as lifetime. When we buy any item or device such as television, computer, electric bulb etc, we expect it to function properly for a reasonable period of time, i.e. we would like to know the average life or warranty period of an item. Thus reliability function is nothing but the survival function of an item.

In a life testing experiment, items are subjected to test and failed times of items are observed. From practical point of view it is just not possible to examine the sample fully. A complete examination of a sample involves considerable amount of time and money. In addition one requires sufficient space for conducting the experiment. This further adds to the costs of life-test experiment.

Hence on account of time and cost consideration a sample has to be truncated. Truncation of the sample is known as censoring.

There are many types of censoring schemes, but Type-I and Type-II censoring schemes are generally used. If we terminate the experiment when a pre assigned time is observed, such an experiment is known as time censored sampling or Type-I censoring. This kind of censoring is used when cost of experiment increases heavily with time. In Type-II censoring a life test is terminated as soon as fixed number if items (say r) have failed. Such an experiment is known as failure censored sampling which is related with very high cost sophisticated items such as color television tubes.

Generally Type-I and Type-II censoring schemes do not allow removal of units at points other than the terminal point of experiment. A generalized censoring scheme, defined by Cohen (1963) which is known as progressive Type-II censoring scheme is described below.

Before conducting, a life experiment the experimenter fixes a sample size n, a number of complete observation m and a censoring scheme (R_1, R_2, …… R_m), n = $m+\sum R_i$. The n units are placed on a life test. Immediately after first failure, R_1 surviving units are randomly chosen and removed from the experiment. Then after second failure, R_2 units are withdrawn and so on. The procedure is continued until all R_m remaining units are removed after this m^{th} failure.

If $R_1 = R_2 = \ldots\ldots\ldots\ldots = R_m = 0$, then n = m which corresponds to complete sample. If $R_1 = R_2 = \ldots\ldots\ldots\ldots = R_{m-1} = 0$ them $R_m = n - m$ corresponds to conventional Type-II right censoring scheme.

Balakrishnan and Aggarwala (2000) has provided a comprehensive reference in the subject of progressive censoring, its application and techniques for analyzing data from the employment of progressive Type-II censoring schemes.

In this chapter we have considered reciprocal exponential distribution as a continuous lifetime model and apply progressively Type-II censoring without changing the parameters at different stages of censoring. In section 2 the method of maximum likelihood estimation described. Simulation of progressive Type-II censored samples is carried out in section 3. Section 4 deals with confidence interval under three different methods. The methods are illustrated using numerical examples for different censoring pattern.

4.2 Maximum Likelihood Estimation

The probability density function and cumulative distribution function of a reciprocal exponential distribution with parameter θ is given by,

$$g(x) = \frac{\theta}{x^2} e^{\frac{-\theta}{x}}, x > 0, \ \theta > 0$$

$$G(x) = e^{\frac{-\theta}{x}}. \qquad (4.2.1)$$

Let n items are kept on test, then the likelihood function under Progressive Type–II censoring scheme as discussed in section 1 is given by

$$L = \text{constant} \prod_{i=1}^{m} g(x_i/\theta) \prod_{i=1}^{m} \left[1 - G(x_i/\theta)\right]^{R_i}.$$

Using (4.2.1) the likelihood function becomes

$$L = \text{constant} \ \frac{\theta^m}{\prod_{i=1}^{m} x_i^2} e^{-\theta \sum_{i=1}^{m} \frac{1}{x_i}} \prod_{i=1}^{m} \left(1 - e^{-\theta/x_i}\right)^{R_i}. \qquad (4.2.2)$$

The log likelihood function is given by,

$$\ln L = \ln c + m \ln\theta - \theta\sum_{i=1}^{m}\frac{1}{x_i} - \sum_{i=1}^{m}\ln x_i^2 + \sum_{i=1}^{m} R_i \ln\left(1 - e^{-\theta/x_i}\right)$$

Differentiating ln L with respect to θ and equating to zero we obtain,

$$\frac{m}{\theta} - \sum_{i=1}^{m}\frac{1}{x_i} + \sum_{i=1}^{m}\frac{R_i}{1 - e^{-\theta/x_i}}\left(-e^{-\theta/x_i}\right)\left(\frac{-1}{x_i}\right) = 0. \tag{4.2.3}$$

Hence we obtain the maximum likelihood estimating equation as,

$$\hat{\theta} = \frac{m}{\sum_{i=1}^{m}\frac{1}{x_i} - \sum_{i=1}^{m}\frac{R_i e^{-\theta/x_i}}{x_i\left(1 - e^{-\theta/x_i}\right)}}. \tag{4.2.4}$$

Using any iterative procedure like Newton Raphson method one can solve the equation (4.2.4) to obtain maximum likelihood estimator of θ, denoted by $\hat{\theta}$.

Now again differentiating (4.2.3) we get,

$$\frac{\partial^2 \ln L}{\partial \theta^2} = \frac{-m}{\theta^2} - \sum_{i=1}^{m} \frac{R_i}{x_i^2} \frac{e^{-\theta/x_i}}{\left(1 - e^{-\theta/x_i}\right)^2}. \tag{4.2.5}$$

Hence observed asymptotic variance of $\hat{\theta}$ is given by (Due to Cohen 1963)

$$V(\hat{\theta}) = \frac{-1}{\left.\frac{\partial^2 \ln L}{\partial \theta^2}\right|_{\theta=\hat{\theta}}}. \tag{4.2.6}$$

4.3 Comparison of censoring patterns via simulation

In this section considering the reciprocal exponential distribution defined in (2.1) as a life time model from which 1000 samples were generated using the value θ = 3, m = 5, sample size 20 and 50 for each of the following progressive Type-II censoring patterns.

R_1 :(25%, 25%, 50%, 50%, 100%) (ascending)

R_2 :(50%, 50%, 25%, 25%, 100%) (descending)

R_3 :(0, 0, 0, 0, 100%). (regular Type-II)

Here we have used the simulation algorithm given by Aggarwala (2001) to generate samples. Here we have specified the proportion of surviving units to be removed at five monitoring and censoring point. The percentages of removing units from the surviving units at five stages are increasing in pattern R_1 while decreasing in pattern R_2. In pattern R_3, a conventional Type-II censoring scheme is employed.

The simulation scheme is as follows:-

1) Generate U_i, where U_i is a set of random number i= 1, 2, 3, 4, 5

2) $Z_i = -\ln(1-U_i)$

3) $$Y_i = \frac{Z_1}{n} + \frac{Z_2}{n-R_1-1} + \ldots\ldots\ldots\ldots\ldots + \frac{Z_i}{n-\sum_{j=1}^{i-1} R_j - i + 1}$$

4) $G(x_i) = 1 - \exp(-Y_i)$

$$\text{i.e. } \exp(-\theta/x_i) = 1 - \exp(-Y_i)$$

By solving the equation in step 4 we will get the values of x_i. On the basis of simulated samples Maximum Likelihood estimates of θ as given in (4.2.4) along with its asymptotic variance as given in (4.2.6) with simulated variance are demonstrated in Table-1.

For n = 20 the three censoring patterns as discussed earlier the comes out as

R_1 : (5, 3, 5, 2, 0)

R_2 : (10, 4, 1, 0, 0)

R_3 : (0, 0, 0, 0, 15).

Table-1 gives the summary statistics of the maximum likelihood estimators for the three censoring patterns; with its observed asymptotic variance and simulated variance, in case of 1000 random samples generated for n = 20, θ = 3, m= 5

Table-1

Scheme	**Min** $\hat{\theta}$	**Max** $\hat{\theta}$	$\hat{\theta}$	Asy $V(\hat{\theta})$	Sim $V(\hat{\theta})$
R_1	1.5945	7.2547	3.3064	0.8109	0.8631
R_2	1.3139	9.481	3.4278	1.0163	1.0735
R_3	1.6171	7.7873	3.3125	0.6656	0.6236

From the result of Table-1 we observe that the censoring pattern R_3 produces the most precise estimate of θ followed by R_1 and then R_2. This is due to the fact that more units are kept in the experiment for a longer period of time in R_3 followed by R_1 and then R_2.

For n = 50 the three censoring patterns as discussed earlier comes out as

R_1 :(12, 9, 13, 6, 5)

R_2 :(25, 12, 2, 2, 4)

R_3 :(0, 0, 0, 0, 45).

Table-2 gives the summary statistics of the maximum likelihood estimators for the three censoring patterns; with its observed asymptotic variance and simulated variance, in case of 1000 random samples generated for n = 50, θ = 3, m= 5

Table - 2

Scheme	**Min** $\hat{\theta}$	**Max** $\hat{\theta}$	$\hat{\theta}$	Asy $V(\hat{\theta})$	Sim $V(\hat{\theta})$
R_1	1.5742	5.5017	3.0504	0.3436	0.3259
R_2	1.3285	5.7127	3.0504	0.4069	0.4103
R_3	1.6867	4.9825	3.0386	0.2837	0.2702

From the result of Table-2 we observe that the censoring pattern R_3 produces the most precise estimate of θ followed by R_1 and then R_2. This is due to the fact that more units are kept in the experiment for a longer period of time in R_3 followed by R_1 and then R_2.

This shows that result obtained for small sample is same as the one obtained with large sample.

4.4 Confidence Interval Estimation

In this section we consider interval estimation of unknown parameter θ using the method of parametric bootstrap confidence interval and the method of r-level likelihood. According to Davison and Hinkley (1997) a 100(1-α) % parametric bootstrap confidence interval for θ is given by $\left(\frac{\hat{\theta}^2}{\hat{\theta}_{boot}(1-\alpha/2)}, \frac{\hat{\theta}^2}{\hat{\theta}_{boot}(\alpha/2)} \right)$.

(4.4.1)

where $\hat{\theta}_{boot}(p)$ is the p[th] percentile of the simulated sample of 1000 estimates simulated using the observed value of $\hat{\theta}$ of the given sample.

Using the likelihood level r, the likelihood inequality can be solved in order to construct a likelihood interval for θ. From a graph of likelihood ratio= $L(\hat{\theta})/L(\theta)$ plotted against various values of θ, the likelihood interval for θ can be obtained for given level r, by drawing a horizontal line at $L(\hat{\theta})/L(\theta)$ = r and the corresponding likelihood interval will contain all values of θ below this line. For bootstrapping, we again have simulated 1000 samples using the value of

$\hat{\theta}$ as a true value of θ and calculated $\hat{\theta}_{boot}(p)$ for p = 0.025 and p = 0.975 to obtain the 95% confidence interval in case of all the three censoring patterns and the values are as follows:

	R_1	R_2	R_3
$\hat{\theta}_{boot}(0.025)$	1.8607	1.9545	2.077
$\hat{\theta}_{boot}(0.975)$	5.456	5.872	5.0326

Using the result given in (4.4.1), parametric bootstrap confidence interval for θ in case of all the three censoring patterns $R_{1,}$ R_2 and R_3 is given by (1.269692, 4.849047), (1.414528, 5.635088), (1.042647, 4.394512) respectively whereas likelihood level r = 5 confidence intervals of θ for the schemes $R_{1,}$ R_2 and R_3 are obtained as (1.7095, 3.23449), (1.7842, 4.7749) and (2.14599,5.1862) respectively. The advantage of likelihood level confidence interval estimation is that it does not require large amounts of simulation as required in bootstrapping.

Now instead of taking 1000 samples we took single sample and computed the values for different values of likelihood level for each censoring pattern to check the effect it creates on the confidence interval. We found that as the likelihood level is increased the confidence interval also increases for the three censoring patterns, i.e. the difference between the lower limit and upper limit increases. The result is shown in the table given below.

For n = 20, θ = 3, m = 5

Likelihood level (r)	R_1	R_2	R_3
3	(2.63635, 5.66161) Diff : 3.02526	(2.72244, 6.0743) Diff : 3.35186	(2.50933, 5.1987) Diff : 2.68937
5	(2.41711, 6.102605) Diff : 3.685495	(2.48913, 6.5807) Diff : 4.09157	(2.30711, 5.57799) Diff : 3.27088
7	(2.29828, 6.36793) Diff : 4.06965	(2.36323, 6.88705) Diff : 4.52382	(2.197025, 5.805) Diff : 3.607975

Sprott (1973) has indicated that the distribution $\hat{\varphi} = \hat{\theta}^{-1/3}$ in small samples is much more closely approximated by a normal distribution than the distribution of $\hat{\theta}$. The distribution of $\hat{\varphi}$ is approximately normal with mean $\varphi = \theta^{-1/3}$ and variance

$$V(\hat{\varphi}) = \left(\frac{d\varphi}{d\theta}\right)^2 .Asy(\hat{\theta})$$

Thus $\dfrac{\hat{\varphi}-\varphi}{\sqrt{V(\hat{\varphi})\big|_{\theta=\hat{\theta}}}} \sim N(0,1).$ (4.4.2)

	$\hat{\Phi}=\hat{\theta}^{-1/3}$	$V(\hat{\theta})=\frac{1}{9}\hat{\theta}^{-8/3}\times AsyV(\hat{\theta})$	$1.96\times V(\hat{\theta})$
R_1	0.6712	0.0037	0.0073
R_2	0.6632	0.0042	0.0082
R_3	0.6708	0.0030	0.0059

Using the result given in (4.4.2), confidence interval for θ (given by Sprott) in case of all the three censoring patterns R_1, R_2 and R_3 is given by (1.1404, 1.1463), (1.1420, 1.1515), and (1.1390, 1.1457) respectively.

Chapter 5

Estimation under Progressive Type-II Censoring for Type-II Generalized Half Logistic Distribution

5.1 Introduction

Half logistic model obtained as the distribution of the absolute standard logistic variate is probability model considered by Balakrishnan (1985). Balakrishnan and Puthenpura (1986) obtained best linear unbiased estimator of location and scale parameters of the half logistic distribution through linear functions of order statistics. Balakrishnan and Wong (1991) obtained approximate maximum likelihood estimates for the location and scale parameters of the half logistic distribution with Type-II Right-Censoring. Olapade (2003) proved some theorems that characterized the half logistic distribution. The half logistic distribution has not received much attention from researchers in terms of generalization. A generalized version of half logistic distribution namely Type-I and Type–II generalized half logistic distributions are considered by Ramakrishna (2008)

Here we consider Type–II generalized half logistic distribution as a life time model with probability density function.

$$g(x) = \frac{\theta\left(2e^{-x}\right)^{\theta}}{\left(1+e^{-x}\right)^{\theta+1}}, \quad x>0, \ \theta>0. \tag{5.1.1}$$

and cumulative distribution function

$$G(x) = 1-\left[\frac{2e^{-x}}{1+e^{-x}}\right]^{\theta}. \tag{5.1.2}$$

In this chapter we have considered estimation of the parameter θ under progressive Type-II censoring described in the chapter 4 using Maximum likelihood and Bayes estimation under squared error as well as linex loss functions. A simulation study is also carried out and confidence interval estimation is discussed in the last section.

5.2 Maximum Likelihood Estimation

If n item are put on test, then the likelihood function under Progressive Type-II censoring scheme is given by,

$$L = c\prod_{i=1}^{m} g(x_i|\theta)\prod_{i=1}^{m}\left[1-G(x_i|\theta)\right]^{R_i}.$$

Using (5.1.1) and (5.1.2) the likelihood function becomes

$$L = c \prod_{i=1}^{m} \left[\frac{\theta\left(2e^{-x_i}\right)^{\theta}}{\left(1+e^{-x_i}\right)^{\theta+1}} \right] \prod_{i=1}^{m} \left[\frac{2e^{-x_i}}{1+e^{-x_i}} \right]^{\theta R_i} .$$

$$L = c \; \frac{2^{m\theta}\, \theta^{m} e^{-\theta \sum_{i=1}^{m} x_i}}{\prod_{i=1}^{m}\left(1+e^{-x_i}\right)^{\theta+1}} \times \frac{2^{\theta \sum_{i=1}^{m} R_i}\; e^{-\theta \sum_{i=1}^{m} R_i x_i}}{\prod_{i=1}^{m}\left(1+e^{-x_i}\right)^{\theta R_i}} . \qquad (5.2.1)$$

where c is constant

The log likelihood function is given by

$$\ln L = \theta\left[m \ln 2 - \sum_{i=1}^{m} x_i - \sum_{i=1}^{m} \ln\left(1 + e^{-x_i}\right) + \ln 2 \sum_{i=1}^{m} R_i - \sum_{i=1}^{m} R_i x_i - \sum_{i=1}^{m} R_i \ln\left(1 + e^{-x_i}\right) \right]$$
$$\ln c + m \ln\theta - \sum_{i=1}^{m} \ln\left(1 + e^{-x_i}\right).$$

Differentiating ln L with respect to θ and equating to zero we obtain

$$\frac{\partial \ln L}{\partial \theta} = 0 \Rightarrow \text{m} \ln 2 + \frac{\text{m}}{\theta} - \sum_{i=1}^{m} x_i (1+R_i) - \sum_{i=1}^{m} \ln\left(1+ e^{-x_i}\right)(1+R_i) + \ln 2 \sum_{i=1}^{m} R_i = 0.$$

$$\Rightarrow \frac{\text{m}}{\theta} = \sum_{i=1}^{m} x_i (1+R_i) + \sum_{i=1}^{m} \ln\left(1+ e^{-x_i}\right)(1+R_i) - \ln 2 \sum_{i=1}^{m} R_i - \text{m} \ln 2. \tag{5.2.2}$$

Hence we obtain the mle $\hat{\theta}$ as

$$\hat{\theta} = \frac{\text{m}}{\sum_{i=1}^{m} x_i (1+R_i) + \sum_{i=1}^{m} \ln\left(1+ e^{-x_i}\right)(1+R_i) - \ln 2 \sum_{i=1}^{m} R_i - \text{m} \ln 2}. \tag{5.2.3}$$

Solving the equation (5.2.3) for $\hat{\theta}$, maximum likelihood estimate of θ can be obtained.

Now again differentiating (5.2.2) we get

$$\frac{\partial^2 \ln L}{\partial \theta^2} = - \frac{\text{m}}{\theta^2}\Bigg|_{\theta = \hat{\theta}}.$$

Hence estimated asymptotic variance of $\hat{\theta}$ is given by (Due to Cohen1963)

$$V\left(\hat{\theta}\right) = \frac{-1}{\frac{\partial^2 \ln L}{\partial \theta^2}} = \frac{\hat{\theta}^2}{m}.$$

5.3 Bayes Estimate

Since last three decades lot of work has been developed in the field of reliability using Bayesian approach. Also it is realistic to assume that life parameter is stochastically dynamic. Martz and Waller (1982) have done lot of work regarding Bayes estimation in the field of life testing and reliability.

In this section the Bayesian approach is used to derive estimate of the parameter θ, assuming we are in the situation where very less is known about a prior about the values of θ.

To avoid the complexity involved in solving Bayes estimates here we consider prior distribution of θ as exponential distribution with mean β.

That is $$\pi\left(\theta\right)=\frac{1}{\beta}e^{-\frac{\theta}{\beta}}, \quad \theta>0, \ \beta>0 \tag{5.3.1}$$

Using likelihood function given in (5.3.3) and the prior defined in (5.4.1), the posterior distribution of the parameter θ is given by

$$h(\theta|x)\ \alpha\ L\pi(\theta)$$

$$\alpha\ \frac{2^{m\theta}\ \theta^{m} e^{-\theta\sum_{i=1}^{m}x_i}}{\prod_{i=1}^{m}\left(1+e^{-x_i}\right)^{\theta+1}} \times \frac{2^{\theta\sum_{i=1}^{m}R_i}\ e^{-\theta\sum_{i=1}^{m}R_i x_i}}{\prod_{i=1}^{m}\left(1+e^{-x_i}\right)^{\theta R_i}} \times \frac{1}{\beta}e^{-\frac{\theta}{\beta}}$$

$$\alpha\ \frac{2^{\theta\left(m+\sum_{i=1}^{m}R_i\right)}\ \theta^{m}}{\beta\prod_{i=1}^{m}\left(1+e^{-x_i}\right)^{\theta+1}} \times \frac{e^{-\theta\left(\sum_{i=1}^{m}x_i+\sum_{i=1}^{m}R_i x_i\ +\ 1/\beta\right)}}{\prod_{i=1}^{m}\left(1+e^{-x_i}\right)^{\theta R_i}}$$

Now under squared error loss function, the Bayes estimator of parameter θ is nothing but the posterior mean, which can be obtained as

$$\theta^{*}_{BS}=E(\theta|x)$$

$$= c\int_{0}^{\infty}\theta\ \frac{2^{\theta\left(m+\sum_{i=1}^{m}R_i\right)}\ \theta^{m}}{\beta\prod_{i=1}^{m}\left(1+e^{-x_i}\right)^{\theta+1}} \times \frac{e^{-\theta\left(\sum_{i=1}^{m}x_i+\sum_{i=1}^{m}R_i x_i\ +\ 1/\beta\right)}}{\prod_{i=1}^{m}\left(1+e^{-x_i}\right)^{\theta R_i}}\ d\theta$$

$$= c\int_{0}^{\infty}\theta^{m+1}\ \frac{2^{\theta\left(m+\sum_{i=1}^{m}R_i\right)}}{\beta\prod_{i=1}^{m}\left(1+e^{-x_i}\right)^{\theta+1}} \times \frac{e^{-\theta\left(\sum_{i=1}^{m}x_i+\sum_{i=1}^{m}R_i x_i\ +\ 1/\beta\right)}}{\prod_{i=1}^{m}\left(1+e^{-x_i}\right)^{\theta R_i}}\ d\theta \qquad (5.3.2)$$

where c is normalizing constant. Here it is not possible to get θ^*_{BS} in closed form, so we refer to numerical integration to find a solution. Lindley (1980) gave an alternative method to approximate the integrals that occur in Bayesian statistics. According to Lindley (1980), the Bayes estimator θ^*_{BS} is approximated as

$$\theta^*_{BS} = E(\theta|x) \approx \hat{\theta} + \frac{1}{2}\left(u_2 + 2u_1\rho_1\right)\sigma^2 + \frac{1}{2}l_3 u_1 \sigma^4 \Big|_{\theta=\hat{\theta}}$$

(5.3.3)

where

$$u_1 = \frac{d\theta}{d\theta} = 1$$

$$u_2 = \frac{\partial^2\theta}{\partial\theta^2} = 0$$

$$\rho = \ln\pi(\theta) = -\ln\beta - \frac{\theta}{\beta}$$

$$\rho_1 = \frac{d\rho}{d\theta} = \frac{-1}{\beta}$$

$$l = \ln L$$

$$l_2 = \frac{\partial^2 l}{\partial \theta^2} = \frac{-m}{\theta^2}$$

$$l_3 = \frac{\partial^3 l}{\partial \theta^3} = \frac{2m}{\theta^3}$$

$$\sigma^2 = \left(-l_2\right)^{-1} = \frac{\theta^2}{m}$$

$$\sigma^4 = \left(\sigma^2\right)^2 .$$

Bayes Estimation under the Linex Loss Function (LLF)

A symmetric loss function assumes that positive and negative errors are equally serious. However, in some estimation problems such assumptions may be inappropriate. A positive error may be more serious than a negative error or vice-versa. In this situation, asymmetric linex loss function is appropriate. The linex loss function is defined as

$$L\left(\hat{\theta},\theta\right) = k\left\{e^{c_1\left(\hat{\theta}-\theta\right)} - c_1\left(\hat{\theta}-\theta\right) - 1\right\}.$$

For the above linex loss function, the Bayes estimator of parameter θ is given by,

$$\theta^*_{BL} = -\frac{1}{c_1}\ln E(e^{-c_1\theta}|x) \tag{5.3.5}$$

$$= c^* \int_0^\infty e^{-c_1\theta} \frac{2^{\theta\left(m+\sum_{i=1}^{m}R_i\right)}\theta^m}{\beta\prod_{i=1}^{m}\left(1+e^{-x_i}\right)^{\theta+1}} \times \frac{e^{-\theta\left(\sum_{i=1}^{m}x_i+\sum_{i=1}^{m}R_i x_i + 1/\beta\right)}}{\prod_{i=1}^{m}\left(1+e^{-x_i}\right)^{\theta R_i}}\, d\theta$$

$$= c^* \int_0^\infty \frac{2^{\theta\left(m+\sum_{i=1}^{m}R_i\right)}\theta^m}{\beta\prod_{i=1}^{m}\left(1+e^{-x_i}\right)^{\theta+1}} \times \frac{e^{-\theta\left(\sum_{i=1}^{m}x_i+\sum_{i=1}^{m}R_i x_i + 1/\beta\right)}}{\prod_{i=1}^{m}\left(1+e^{-x_i}\right)^{\theta R_i}}\, d\theta$$

where c^* is normalizing constant. Here it is not possible to get θ^*_{BL} in closed form, so we refer to numerical integration to find a solution. Again using the approximation given by Lindley (1980) the Bayes estimator θ^*_{BL} is approximated as follows

$$\text{Here } E(e^{-c_1\theta}|x) \approx u + \frac{1}{2}\left(u_2 + 2u_1\rho_1\right)\sigma^2 + \frac{1}{2} l_3 u_1 \sigma^4 \Big|_{\theta=\hat{\theta}}$$

(5.3.6)

where

$$u_1 = \frac{de^{-c_1\theta}}{d\theta} = -c_1 e^{-c_1\theta}$$

$$u_2 = \frac{\partial^2 e^{-c_1\theta}}{\partial\theta^2} = c_1^2 e^{-c_1\theta}$$

$$\rho = \ln\pi(\theta) = -\ln\beta - \frac{\theta}{\beta}$$

$$\rho_1 = \frac{d\rho}{d\theta} = \frac{-1}{\beta}$$

$$l = \ln L$$

$$l_2 = \frac{\partial^2 l}{\partial\theta^2} = \frac{-m}{\theta^2}$$

$$l_3 = \frac{\partial^3 l}{\partial\theta^3} = \frac{2m}{\theta^3}$$

$$\sigma^2 = \left(-l_2\right)^{-1} = \theta^2 / m$$

$$\sigma^4 = \left(\sigma^2\right)^2.$$

Using equation (5.2.3) in (5.3.6) Bayes estimate of θ under LLF can be approximately obtained.

5.4 Simulation Study

In this section we consider a simulation study to observe behavior of ML and Bayes estimate of θ under different censoring patterns. Here we generate 1000 random samples of size 40, 80 from generalized half logistic distribution defined in (3.1) for θ = 3 and 5.

To generate a sample (x) under progressive Type-II censoring with m = 5 we have used the following method as discussed by Aggarwala and Balakrishnan (2002).

Step 1:- Generate U_i, where U_i is a set of random number i = 1, 2, 3, 4, 5

Step 2:- $Z_i = -\ln(1-U_i)$

Step 3:- $Y_i = \frac{Z_1}{n} + \frac{Z_2}{n-R_1-1} + \ldots\ldots\ldots\ldots\ldots + \frac{Z_i}{n-\sum_{j=1}^{i-1} R_j - i + 1}$

Step 4:- $G(x_i) = 1 - \exp(-Y_i)$

i.e. $1 - \left[\frac{2e^{-x}}{1+e^{-x}}\right]^{\theta} = 1 - \exp(-Y_i)$

By solving the equation in Step 4 we will get the values of x_i as

$$\left[\frac{2e^{-x_i}}{1+e^{-x_i}}\right]^{\theta} = \exp(-Y_i)$$

$$\frac{e^{-x_i}}{1+e^{-x_i}} = \frac{e^{-Y_i/\theta}}{2}$$

$$\frac{e^{-x_i}}{1+e^{-x_i} - e^{-x_i}} = \frac{e^{-Y_i/\theta}}{2- e^{-Y_i/\theta}}$$

$$e^{-x_i} = \frac{e^{-Y_i/\theta}}{2- e^{-Y_i/\theta}}$$

$$-x_i = \ln\left(\frac{e^{-Y_i/\theta}}{2- e^{-Y_i/\theta}}\right)$$

$$x_i = -\ln\left(\frac{e^{-Y_i/\theta}}{2- e^{-Y_i/\theta}}\right)$$

On the basis of simulated samples ML estimates of θ, as given (2.5) along with its asymptotic variance are demonstrated in Table-1, where as Table-2 represents Bayes estimates of θ, as given in (3.3) with its simulated variance for the three censoring patterns.

Here we have consider the following three censoring patterns

n = 40	n= 80
S_1: (10, 7, 10, 5, 3)	S_1: (10, 7, 10, 5, 43)
S_2 : (20, 9, 2, 1, 3)	S_2 : (20, 9, 2, 1, 43)
S_3 : (0, 0, 0, 0, 35)	S_3 : (0, 0, 0, 0, 75)

Table-1: Summary Statistics

For $\theta = 3$, n = 40					
Scheme	**Min** $\hat{\theta}$	**Max** $\hat{\theta}$	$\hat{\theta}$	$AV(\hat{\theta})$	$SV(\hat{\theta})$
S_1	0.8404	17.7134	3.7535	3.6796	4.3119
S_2	0.8404	17.6999	3.7533	3.6801	4.3107
S_3	0.8404	17.6497	3.7529	3.678	4.3054
For $\theta = 5$, n = 40					
Scheme	**Min** $\hat{\theta}$	**Max** $\hat{\theta}$	$\hat{\theta}$	$AV(\hat{\theta})$	$SV(\hat{\theta})$
S_1	1.4009	29.545	6.2555	10.2211	11.974
S_2	1.4009	29.5648	6.2556	10.2219	11.9774
S_3	1.4007	29.3284	6.2548	10.2155	11.9552
For $\theta = 3$, n = 80					
Scheme	**Min** $\hat{\theta}$	**Max** $\hat{\theta}$	$\hat{\theta}$	$AV(\hat{\theta})$	$SV(\hat{\theta})$
S_1	0.8406	17.6712	3.7531	3.6779	4.3082
S_2	0.8404	17.5783	3.7529	3.6788	4.3052
S_3	0.8407	17.667	3.7528	3.6774	4.3033
For $\theta = 5$, n = 80					
Scheme	**Min** $\hat{\theta}$	**Max** $\hat{\theta}$	$\hat{\theta}$	$AV(\hat{\theta})$	$SV(\hat{\theta})$
S_1	1.4009	29.4548	6.2553	10.2186	11.9642
S_2	1.4011	29.5436	6.2552	10.2207	11.9763
S_3	1.401	29.3278	6.2542	10.2118	11.9442

From results of Table-1 we observe from the asymptotic variance of three schemes that the censoring pattern S_3 produces the most precise estimate of θ followed by S_1 and then S_2. This is due to the fact that more units are kept in the experiment for a longer period of time in S_3 followed by S_1 and then S_2.

Table- 2: Simulated Variance and MSE of MLE and Bayes Estimate under SELF and LLF

n = 40, θ = 3, β = 3											
MLE		Bayes Estimate									
		SELF		LLF							
$SV(\hat{\theta})$	MSE	$SV(\hat{\theta}_{BS})$	MSE	$SV(\hat{\theta}_{BL})$ C = -1	MSE	$SV(\hat{\theta}_{BL})$ c = -0.5	MSE	$SV(\hat{\theta}_{BL})$ c = 0.5	MSE	$SV(\hat{\theta}_{BL})$ c = 1	MSE
4.3119	4.8797	1.0928	1.1699	6.0015	8.3375	4.2649	5.5844	1.4388	1.4402	1.9961	2.0162
4.3107	4.8782	1.0927	1.1698	5.9998	8.3352	4.2627	5.5817	1.4383	1.4397	1.9954	2.0155
4.3054	4.8723	1.0923	1.1694	5.9928	8.3267	4.2561	5.5740	1.4359	1.4373	1.9921	2.0123

n = 40, θ = 3, β = 7											
MLE		Bayes									
		SELF		LLF							
$SV(\hat{\theta})$	MSE	$SV(\hat{\theta}_{BS})$	MSE	$SV(\hat{\theta}_{BL})$ c = -1	MSE	$SV(\hat{\theta}_{BL})$ c = -0.5	MSE	$SV(\hat{\theta}_{BL})$ c = 0.5	MSE	$SV(\hat{\theta}_{BL})$ c = 1	MSE
4.3119	4.8797	3.2753	4.2324	6.5136	9.5058	6.4756	9.0187	1.9087	2.0165	2.1250	2.1284
4.3107	4.8782	3.2743	4.2312	6.5118	9.5033	6.4736	9.0160	1.908	2.0158	2.1242	2.1276
4.3054	4.8723	3.2713	4.2276	6.5045	9.4946	6.4656	9.0064	1.9054	2.0130	2.1208	2.1242

n = 40, θ = 3, β = 12												
MLE		Bayes									
		SELF		LLF							
$SV(\hat{\theta})$	MSE	$SV(\hat{\theta}_{BS})$	MSE	$SV(\hat{\theta}_{BL})$ c = -1	MSE	$SV(\hat{\theta}_{BL})$ c = -0.5	MSE	$SV(\hat{\theta}_{BL})$ c = 0.5	MSE	$SV(\hat{\theta}_{BL})$ c = 1	MSE
4.3119	4.8797	4.3646	5.7991	6.6331	9.8147	6.8950	9.7952	2.1312	2.3463	2.1799	2.1972
4.3107	4.8782	4.3630	5.7970	6.6314	9.8123	6.8930	9.7925	2.1305	2.3455	2.1791	2.1964
4.3054	4.8723	4.3577	5.7907	6.6240	9.8034	6.8847	9.7825	2.1278	2.3426	2.1757	2.1929

n = 80, θ = 3, β = 3												
MLE		Bayes Estimate									
		SELF		LLF							
$SV(\hat{\theta})$	MSE	$SV(\hat{\theta}_{BS})$	MSE	$SV(\hat{\theta}_{BL})$ c = -1	MSE	$SV(\hat{\theta}_{BL})$ c = -0.5	MSE	$SV(\hat{\theta}_{BL})$ c = 0.5	MSE	$SV(\hat{\theta}_{BL})$ c = 1	MSE
4.3082	4.8754	1.0937	1.1707	5.9965	8.3313	4.2532	5.5717	1.4370	1.4384	1.9937	2.0138
4.3052	4.8721	1.0932	1.1702	5.9922	8.3261	4.2473	5.5711	1.4358	1.4373	1.9920	2.0122
4.3033	4.8700	1.0929	1.1700	5.9902	8.3238	4.2540	5.5643	1.4351	1.4366	1.9907	2.0109

n = 80, θ = 3, β = 7											
MLE		Bayes Estimate									
		SELF		LLF							
$SV(\hat{\theta})$	MSE	$SV(\hat{\theta}_{BS})$	MSE	$SV(\hat{\theta}_{BL})$ c = -1	MSE	$SV(\hat{\theta}_{BL})$ c = -0.5	MSE	$SV(\hat{\theta}_{BL})$ c = 0.5	MSE	$SV(\hat{\theta}_{BL})$ c = 1	MSE
4.3082	4.8754	3.2720	4.2285	6.5085	9.4993	6.4696	9.0111	1.9067	2.0143	2.1225	2.1259
4.3052	4.8721	3.2689	4.2267	6.5041	9.4939	6.4646	9.0054	1.9052	2.0128	2.1207	2.1241
4.3033	4.8700	3.2706	4.2250	6.5019	9.4913	6.4627	9.0032	1.9045	2.0120	2.1195	2.1229

n = 80, θ = 3, β = 12											
MLE		Bayes Estimate									
		SELF		LLF							
$SV(\hat{\theta})$	MSE	$SV(\hat{\theta}_{BS})$	MSE	$SV(\hat{\theta}_{BL})$ c = -1	MSE	$SV(\hat{\theta}_{BL})$ c = -0.5	MSE	$SV(\hat{\theta}_{BL})$ c = 0.5	MSE	$SV(\hat{\theta}_{BL})$ c = 1	MSE
4.3082	4.8754	4.3590	5.7925	6.6280	9.8092	6.8889	9.7877	2.1291	2.3440	2.1774	2.1947
4.3052	4.8721	4.3545	5.7888	6.6236	9.8030	6.8838	9.7816	2.1275	2.3422	2.1756	2.1928
4.3033	4.8700	4.3560	5.7873	6.6214	9.8001	6.8816	9.7791	2.1269	2.3416	2.1744	2.1916

From the Table-2 we observe that MSE of MLE as well as Bayes estimators becomes smaller in the censoring pattern S_3 than S_2 followed by S_1 for various choice of n and β.

5.5 Confidence Interval Estimation

In this section we consider interval estimation of unknown parameter θ using the method of parametric bootstrap confidence interval and the method of r-level likelihood. According to Davison and Hinkley (1997) a 100(1-α) % parametric bootstrap confidence interval for θ is given by $\left(\frac{\hat{\theta}^2}{\hat{\theta}_{boot}(1-\alpha/2)}, \frac{\hat{\theta}^2}{\hat{\theta}_{boot}(\alpha/2)}\right)$

(5.5.1)

where $\hat{\theta}_{boot}(p)$ is the p^{th} percentile of the simulated sample of 1000 estimates simulated using the observed value of $\hat{\theta}$ of the given sample.

The shape and magnitude of $L(\theta)$ relative to $L(\hat{\theta})$ over all possible values of θ describe the information on θ that is considered in $data_i$, i = 1, 2, ..., n this suggest importance of the relative likelihood function (RLF), $R(\theta) = \frac{L(\hat{\theta})}{L(\theta)}$.

Considering R(θ) ≤ r, where r is the desired level of RLF, it is observed that larger values of r will result in wider intervals for variation in θ.

The inequality R (θ) ≤ r to be solved to construct a r- level likelihood interval for θ. From a graph of likelihood ratio= $L(\hat{\theta})/L(\theta)$ plotted against various values of θ, the r- level likelihood interval for θ can be obtained for given level r, by drawing a horizontal line at $L(\hat{\theta})/L(\theta)$ = r and the corresponding r- level likelihood interval will contain all values of θ below this line. The advantage of likelihood level confidence interval estimation is that it does not require large amounts of simulation as required in bootstrapping.

For bootstrapping, we again have simulated 1000 samples using the value of $\hat{\theta}$ as a true value of θ and calculated $\hat{\theta}_{boot}(p)$ for p = 0.025 and p = 0.975 for all the three censoring patterns and the values are as follows:

Table-3: Values of $\hat{\theta}_{boot}(p)$ under the three censoring patterns

For θ = 0.2, n = 40	S_1	S_2	S_3
$\hat{\theta}_{boot}(0.025)$	0.1227	0.1165	0.1232
$\hat{\theta}_{boot}(0.975)$	0.7981	0.7880	0.7306
For θ = 3, n = 40	S_1	S_2	S_3
$\hat{\theta}_{boot}(0.025)$	1.8405	1.8483	1.7970
$\hat{\theta}_{boot}(0.975)$	13.0324	12.0725	10.9089
For θ = 0.2, n = 80	S_1	S_2	S_3
$\hat{\theta}_{boot}(0.025)$	0.1235	0.122	0.1119
$\hat{\theta}_{boot}(0.975)$	0.8467	0.734	0.7607
For θ = 3, n = 80	S_1	S_2	S_3
$\hat{\theta}_{boot}(0.025)$	1.8098	1.8095	1.8952
$\hat{\theta}_{boot}(0.975)$	11.4896	11.4475	11.3348

Using the result given in (5.5.1), parametric bootstrap confidence interval for θ in case of all the three censoring patterns $S_{1,}$ S_2 and S_3 is given by

Table-4: Values of bootstrap and r-level likelihood interval for different n and θ.

		Bootstrap Confidence interval	r-level likelihood interval, where r = 5
For θ = 0.2, n = 40	S_1	(0.0800, 0.5204)	(0.0951, 0.4255)
	S_2	(0.0739, 0.4998)	(0.1027, 0.4258)
	S_3	(0.0842, 0.4992)	(0.1342, 0.4432)
For θ = 3, n = 40	S_1	(1.0818, 7.6602)	(1.7095, 3.2345)
	S_2	(1.1726, 7.6591)	(1.7842, 4.7749)
	S_3	(1.2774, 7.7548)	(2.1460, 5.1862)
For θ = 0.2, n = 80	S_1	(0.0726, 0.4980)	(0.0851, 0.4010)
	S_2	(0.0819, 0.4928)	(0.0997, 0.4050)
	S_3	(0.0764, 0.5195)	(0.1199, 0.4289)
For θ = 3, n = 80	S_1	(1.2306, 7.8125)	(1.6956, 3.2388)
	S_2	(1.1952, 7.5615)	(1.7082, 4.3794)
	S_3	(1.3085, 7.8259)	(2.1138, 5.1136)

From Table-4 we observe that 5- level likelihood gives smaller length confidence interval rather than the 95% bootstrap confidence interval.

Chapter 6

Some Results on Maximum Likelihood Estimators of Parameters of Generalized Half Logistic Distribution under Type-I Progressive Censoring with Changing Failure Rate

6.1 Introduction

In most applications, the data may be interval-censored. By interval-censored data, we mean that a random variable of interest is known only to lie in an interval, instead of being observed exactly. In such cases, the only information we have for each individual is that their event time falls in an interval, but the exact time is unknown.

Generally statistician faces lot of problem in the analysis of time-to-event data such as failure time data, incubation time data etc. Such data arises in lot of fields such as medicine, engineering, economics. For example doctor may be interested to know the time of convergence to AIDS for HIV positive individual, the time to the death for cancer patients, lifetime of a device etc. The analysis to time-to-event later becomes more complicated on account of censoring.

In many life test studies, it is common that the lifetimes of test units may not be recorded exactly. An experimenter may terminate the life test before all n products fail in order to save time or cost. Hence, the test is said to be censored in which data collected are the exact failure times on those functional (none failed) units. Moreover, some of the test units may have to be removed at different stage(s) of censoring related study for various other reasons; which leads to progressive censoring. For example some products are withdrawn for more thorough inspection or are saved so that it can be used as test specimens in other studies, or patients who for some reasons do not turn up in a clinical study would also result in progressive removal.

According to the current trend Type-I and Type-II progressive censoring schemes are becoming quite popular for analyzing highly reliable data. Cohen (1963) had introduced progressive Type-II censoring. Mahmond et al (2006) considered progressive Type-II censoring samples for many continuous life time models. Balakrishnan and Aggarwala (2000) give an insight on this method and the applications of this scheme.

In this chapter we have considered Type-I progressive censoring scheme to obtain Maximum likelihood estimate of the parameter of the generalized half logistic distribution. Here we have also assumed that the parameter changes under different stages of censoring. Estimation is also carried out in the case of progressive Type-I interval censoring with changing parameters at each stage. Under this scheme expected duration of the life test is also derived.

6.2 Generalized Half Logistic Distribution

Half logistic model obtained as the distribution of the absolute standard logistic variate is probability model considered by Balakrishnan (1985). Balakrishnan and Puthenpura (1986) obtained best linear unbiased estimator of location and scale parameters of the half logistic distribution through linear functions of order statistics.

Balakrishnan and Wong (1991) obtained approximate maximum likelihood estimates for the location and scale parameters of the half logistic distribution with Type-II Right-Censoring. Olapade (2003) proved some theorems that characterized the half logistic distribution. The half logistic distribution has not received much attention from researchers in terms of generalization. A generalized (Type-II) version of logistic distribution is considered and some interesting properties of the distribution were derived by Balakrsihnan and Hossain (2007). The generalized versions of half logistic distribution namely Type-I and Type–II were considered along with point estimation of scale parameters and estimation of stress strength reliability based on complete sample by Ramakrishna (2008).

6.3 Maximum Likelihood Estimation

Let the life X of an item have the generalized half logistic distribution with cdf $F(x,\theta) = 1-\left[\frac{2e^{-x}}{1+e^{-x}}\right]^{\theta}$ and density function $f(x,\theta) = \frac{\theta\left(2e^{-x}\right)^{\theta}}{\left(1+e^{-x}\right)^{\theta+1}}, \quad x > 0, \theta > 0$.

In services certain stores and equipments are subjected to regular check up even though they are functioning normally. When such items are placed on life test at some stages the items that have not failed are checked up and over-hauled, repairing the minor defects. This, naturally, changes the life time distribution of the items and consequently the failure rate also changes. It has been assumed that the times of censoring coincide with the times of regular check ups and, thus, are predetermined so that the failure rate changes at the time of censoring.

Suppose that the times of censoring are T_i , i = 1, 2, …, k-1 and the experiment is finally terminated at T_k, $T_k = \infty$ where $T_i < T_{i+1}$ for i = 1, 2, …, k-1. Suppose that the parameter θ of the distribution changes at T_1, T_2, …,T_k. If θ_i is the parameter in the interval $[T_{i-1}, T_i)$ for i = 1, 2, …, k-1 with $T_0 = 0$, using the lemma given by Patel and Gajjar (1995), the composite density is given by

$$
f(x)=\begin{cases}
f_1(x)=\dfrac{\theta_1\left(2e^{-x}\right)^{\theta_1}}{\left(1+e^{-x}\right)^{\theta_1+1}}, \quad 0 \le x < T_1 \\[2ex]
f_2(x)=\dfrac{\left(2e^{-T_1}/1+e^{-T_1}\right)^{\theta_1}}{\left(2e^{-T_1}/1+e^{-T_1}\right)^{\theta_2}}\dfrac{\theta_2\left(2e^{-x}\right)^{\theta_2}}{\left(1+e^{-x}\right)^{\theta_2+1}}, \quad T_1 \le x < T_2 \\[2ex]
f_k(x)=\dfrac{\left(2e^{-T_1}/1+e^{-T_1}\right)^{\theta_1}}{\left(2e^{-T_{i-1}}/1+e^{-T_{i-1}}\right)^{\theta_i}}\prod_{j=2}^{k-1}\dfrac{\left(2e^{-T_j}/1+e^{-T_j}\right)^{\theta_j}}{\left(2e^{-T_{j-1}}/1+e^{-T_{j-1}}\right)^{\theta_j}}\dfrac{\theta_2\left(2e^{-x}\right)^{\theta_2}}{\left(1+e^{-x}\right)^{\theta_2+1}}, \quad T_{k-1} \le x < T_k = \infty \\
\qquad k=3,4,5,........
\end{cases}
$$

(6.3.1)

The corresponding distribution function is given by,

$$
F(x)=\begin{cases}
F_1(x)=1-\left(\dfrac{2e^{-x}}{1+e^{-x}}\right)^{\theta_1}, 0 \le x < T_1 \\[2ex]
F_2(x)=1-\left(\dfrac{2e^{-T_1}}{1+e^{-T_1}}\right)^{\theta_1}\left\{\dfrac{2e^{-x}/1+e^{-x}}{2e^{-T_1}/1+e^{-T_1}}\right\}^{\theta_2}, T_1 \le x < T_2 \\[2ex]
F_k(x)=1-\left(\dfrac{2e^{-T_1}}{1+e^{-T_1}}\right)^{\theta_1}\prod_{j=2}^{k-1}\left\{\dfrac{2e^{-T_j}/1+e^{-T_j}}{2e^{-T_{j-1}}/1+e^{-T_{j-1}}}\right\}^{\theta_j}\left\{\dfrac{2e^{-x}/1+e^{-x}}{2e^{-T_{k-1}}/1+e^{-T_{k-1}}}\right\}^{\theta_k}, T_{k-1} \le x < T_k = \infty \\
\qquad k=3,4,5,........
\end{cases}
$$

(6.3.2)

Suppose n items are placed on a life test without replacement and that n_i be the number of items that fail during i^{th} stage and let $x_1^{(i)} \le x_2^{(i)} \le .... \le x_{n_i}^{(i)}$ be the times of failure for i = 1, 2, …, k-1 (k>1).

Let r_i be the number of items removed or censored from the test immediately after time $T_{i-1,}$ i= 2, 3,…, k. Then the likelihood function from k-stage Type-I progressive censoring is given by,

$$L \propto \prod_{i=1}^{k}\left\{\prod_{j=1}^{n_i} f_i\left(x_j^{(i)}\right)\right\}\prod_{i=1}^{k}\left[1-F_i\left(T_i\right)\right]^{r_i}. \tag{6.3.3}$$

Using (6.3.1) and (6.3.2) it is easy to verify that the likelihood function L can be written as,

$$L \propto \prod_{i=1}^{k} L_i.$$

Where in the case of generalized Type-II half logistic distribution

$$L_i \propto \frac{\theta_i^{n_i} e^{-\sum_{j=i}^{n_i} x_j^{(i)}\theta_i}}{\prod_{j=1}^{n_i}\left\{1+e^{-x_j^{(i)}}\right\}^{\theta_i}}\left\{\frac{e^{-T_{i-1}}}{1+e^{-T_{i-1}}}\right\}^{-\theta_i n_i}\left\{\frac{e^{-\left(T_i-T_{i-1}\right)}\left(1+e^{-T_{i-1}}\right)}{1+e^{-T_i}}\right\}^{\theta_i\left(n^{(i)}-n_i\right)}.$$

for i = 2, 3, ..., k, with $T_0 = 0$

(6.3.4)

where $n^{(1)} = n$, $n^{(i)} = n^{(i-1)} - n_i - r_i$ for i = 1, 2, …, k

The log likelihood function is given by

$$\begin{aligned}\ln L_i &= \ln c + n_i \ln\theta_i - \sum_{j=1}^{n_i} x_j^{(i)}\theta_i - (\theta_i+1)\sum_{j=1}^{n_i}\ln\left(1+e^{-x_j^{(i)}}\right) \\ &\quad + n_i\theta_i T_{i-1} + n_i\theta_i \ln\left(1+e^{-T_{i-1}}\right) - \theta_i\left(n^{(i)}-n_i\right)\left(T_i - T_{i-1}\right) \\ &\quad + \theta_i\left(n^{(i)}-n_i\right)\ln\left(1+e^{-T_{i-1}}\right) - \theta_i\left(n^{(i)}-n_i\right)\ln\left(1+e^{-T_i}\right).\end{aligned} \tag{6.3.5}$$

Differentiating ln L with respect to θ_i and equating to zero we obtain

$$\begin{aligned}\frac{\partial \ln L_i}{\partial\theta_i} = 0 \Rightarrow \frac{n_i}{\theta_i} &- \sum_{j=1}^{n_i} x_j^{(i)} - \sum_{j=1}^{n_i}\ln\left(1+e^{-x_j^{(i)}}\right) + n_i T_{i-1} \\ &+ n_i \ln\left(1+e^{-T_{i-1}}\right) - \left(n^{(i)}-n_i\right)\left(T_i - T_{i-1}\right) \\ &+ \left(n^{(i)}-n_i\right)\ln\left(1+e^{-T_{i-1}}\right) - \left(n^{(i)}-n_i\right)\ln\left(1+e^{-T_i}\right) = 0.\end{aligned} \tag{6.3.6}$$

$$\begin{aligned}\frac{n_i}{\theta_i} &= \sum_{j=1}^{n_i} x_j^{(i)} + \sum_{j=1}^{n_i}\ln\left(1+e^{-x_j^{(i)}}\right) - n_i T_{i-1} - n_i \ln\left(1+e^{-T_{i-1}}\right) \\ &\quad + \left(n^{(i)}-n_i\right)\left(T_i - T_{i-1}\right) - \left(n^{(i)}-n_i\right)\ln\left(1+e^{-T_{i-1}}\right) + \left(n^{(i)}-n_i\right)\ln\left(1+e^{-T_i}\right).\end{aligned}$$

This implies the MLE $\hat{\theta}_i$ of θ_i as

$$\hat{\theta}_i = \frac{n_i}{\sum_{j=1}^{n_i} x_j^{(i)} + \sum_{j=1}^{n_i} \ln\left(1+e^{-x_j^{(i)}}\right) + \left(n^{(i)}\text{-}n_i\right)\left\{T_i + \ln\left(1+e^{-T_i}\right)\right\} - n^{(i)}\left\{T_{i\text{-}1} + \ln\left(1+e^{-T_{i\text{-}1}}\right)\right\}}. \tag{6.3.7}$$

Now again differentiating (6.3.6) we get

$$\frac{\partial^2 \ln L_i}{\partial \theta_i^2} = -\frac{n_i}{\theta_i^2}$$

Hence observed asymptotic variance of $\hat{\theta}$ is given by (Due to Cohen1963)

$$V(\hat{\theta}_i) = \frac{-1}{\left.\frac{\partial^2 \ln L}{\partial \theta_i^2}\right|_{\theta_i = \hat{\theta}_i}}.$$

Illustrative example

Here we generate 1000 random samples under progressive Type-I censoring scheme for the distribution given in (3.2). We have considered the following parameters. Based on simulated samples MLEs and their asymptotic variances are obtained

$n = 35$, $k=3, \theta_1 = 2$, $\theta_2 = 2.4$, $\theta_3 = 2.8$
$R_1=3$, $R_2 = 4$, $R_3 = 10, T_1 = 2$, $T_2 = 3$, and finally the experiment is terminated at $T_3 = 4$
$n_1 = 4$, $n_2 = 3$, $n_3 = 5$

Table1: MLE and its asymptotic variance for the parameters.

Parameters	MLE	Asymptotic Variance
θ_1	0.0868	0.0019
θ_2	0.1253	0.0052
θ_3	0.3248	0.0211

6.4 Estimation based on Interval-Censoring with different parameter at each stage

It is often the practice in life testing to examine the life test periodically and the number of items that have failed in each stage of censoring (T_{i-1}, T_i) are counted and some fixed number of surviving items are eliminated immediately after time T_i for i = 1, 2, …, k being the times of censoring. This kind of experimentation stems from economic or practical considerations where it may not be appropriate to collect exact failure times of the items on test. Kendell and Anderson (1971) have considered the ML estimators of the scale parameter θ of the exponential distribution when items placed on test are subjected to a stress condition for a predetermined time T and the test is periodically inspected at time t_i for i = 1, 2, …, k such that t_k = T. Where as Patel and Gajjar (1995) have considered estimation in case of exponential life time model for k-stage progressive Type-I interval censoring scheme with changing parameters at each stage.

Maximum Likelihood Estimation under Interval-Censoring

In this sub-section we consider the ML estimation of the parameters of the generalized half logistic distribution based on k-stage Type-I progressively interval censored samples under the assumption that the parameter θ changes at each stage of censoring. Under this scheme the likelihood function becomes

$$L \propto \prod_{i=1}^{k} \left\{ \int_{T_{i-1}}^{T_i} f_i(x)\,dx \right\}^{n_i} \prod_{i=1}^{k} \left(1 - F_i(T_i)\right)^{r_i} \qquad (6.4.1)$$

Using the probability model (6.3.2), the likelihood L given by (6.4.1) can be written as

$$L \propto \prod_{i=1}^{k} L_i.$$

where $L_i = \left[1 - \left\{ \dfrac{\dfrac{2e^{-T_i}}{1+e^{-T_i}}}{\dfrac{2e^{-T_{i-1}}}{1+e^{-T_{i-1}}}} \right\}^{\theta_i} \right]^{n_i} \times \left\{ \dfrac{\dfrac{2e^{-T_i}}{1+e^{-T_i}}}{\dfrac{2e^{-T_{i-1}}}{1+e^{-T_{i-1}}}} \right\}^{\theta_i\left(n^{(i)} - n_i\right)}$

Let $w_i = \left\{ \dfrac{\dfrac{2e^{-T_i}}{1+e^{-T_i}}}{\dfrac{2e^{-T_{i-1}}}{1+e^{-T_{i-1}}}} \right\}$, and $p_i = 1 - w_i^{\theta_i}$

Hence L_i can be rewritten as

$$L_i = (p_i)^{n_i} (1-p_i)^{n^{(i)}-n_i}$$

The log likelihood function is given by

$$\ln L_i = n_i \ln p_i + \left(n^{(i)}-n_i\right)\ln\left(1-p_i\right)$$

Differentiating ln L with respect to θ and equating to zero we obtain

$$\frac{\partial \ln L_i}{\partial p_i} = 0 \Rightarrow \frac{n_i}{p_i} - \frac{\left(n^{(i)}-n_i\right)}{1-p_i} = 0$$

$$p_i = \frac{n_i}{n^{(i)}}$$

Using $p_i = 1-w_i^{\theta_i}$, we get $\hat{\theta}_i = \dfrac{\ln\left(n^{(i)}-n_i / n^{(i)}\right)}{\ln w_i}$ (6.4.2)

The equation (6.4.2) can be rewritten as,

$$\frac{\hat{\theta}_i}{\theta_i} = \frac{\ln\left(\frac{n^{(i)}-n_i}{n^{(i)}}\right)}{\ln(1-p_i)}$$

Differentiating $\ln L_i$ again with respect to θ_i we get,

$$\frac{\partial^2 \ln L_i}{\partial \theta_i{}^2} = \frac{\partial}{\partial p_i}\left(\frac{\partial \ln L_i}{\partial \theta_i}\right)\frac{\partial p_i}{\partial \theta_i}$$

$$= \left[\left\{-\frac{n_i}{p_i^2} - \frac{\left(n^{(i)}-n_i\right)}{(1-p_i)^2}\right\}\{(p_i-1)\ln w_i\} + \left\{\frac{n_i}{p_i} - \frac{\left(n^{(i)}-n_i\right)}{1-p_i}\right\}\ln w_i\right]\frac{\partial p_i}{\partial \theta_i}$$

$$= \frac{-(\ln w_i)^2\, w_i^{\theta_i} n_i}{p_i^2}$$

Since $n_i \sim b\left(n^{(i)}, p_i\right)$

$$E\left(\frac{\partial^2 \ln L_i}{\partial \theta_i{}^2}\right) = \frac{-(\ln w_i)^2 (1-p_i)\, E\left(n^{(i)}\right)}{p_i}$$

Hence the asymptotic variance is given by,

$$\text{AsyV}\left(\hat{\theta}_i\right) = \frac{-1}{E\left(\frac{\partial^2 \ln L_i}{\partial \theta_i^2}\right)}$$

$$= \frac{p_i}{(\ln w_i)^2 (1-p_i) E\left(n^{(i)}\right)}$$

$$\frac{\text{AsyV}\left(\hat{\theta}_i\right)}{\theta_i^2} = \frac{p_i}{(\ln p_i)^2 (1-p_i) E\left(n^{(i)}\right)}, \text{ i= 1,2,...k}$$

Table 2: $\frac{\text{AsyV}\left(\hat{\theta}_i\right)}{\theta_i^2}$ for different values of $n^{(i)}$ and p_i

$n^{(i)}$ p_i	10	15	20	30	40	50	80	100
0.1	0.002096	0.00139	0.00104	0.00069	0.00524	0.00419	0.00026	0.00021
0.2	0.009651	0.00643	0.00482	0.00321	0.00241	0.00193	0.00120	0.00096
0.3	0.029566	0.01971	0.01478	0.00985	0.00739	0.00591	0.00369	0.00295
0.4	0.079404	0.05293	0.03970	0.02646	0.01985	0.01588	0.00992	0.00794
0.5	0.208137	0.13875	0.10406	0.06937	0.05203	0.04162	0.02601	0.02081
0.6	0.574839	0.38322	0.28741	0.19161	0.14371	0.11496	0.07185	0.05748
0.7	1.834136	1.22275	0.91706	0.61137	0.45853	0.36682	0.22926	0.18341

0.8	8.033251	5.3555	4.01662	2.67775	2.00831	1.60665	1.00415	0.80332
0.9	81.07496	54.0499	40.5374	27.0249	20.2687	16.2149	10.1343	8.10749
0.99	98010.82	65340.5	49005.4	32670.2	24502.7	19602.1	12251.3	9801.08

From above table we conclude that for fixed p_i as $n^{(i)}$ increases $\frac{AsyV\left(\hat{\theta}_i\right)}{\theta_i^2}$ decreases and for fixed $n^{(i)}$ $\frac{AsyV\left(\hat{\theta}_i\right)}{\theta_i^2}$ increases with p_i.

6.5 Expected Duration of the life test (EDLT)

In case of a life test with k-stage interval Type-I progressive censoring the expected duration of the test can be obtained using the method suggested by Kendall and Anderson (1971).

The expected duration of the life test (EDLT) is given by, EDLT = E [D({t_i}, T_k, θ]

$$=T_1 p_1^n + \sum_{i=2}^{k-1} T_i \left[\left(p_1 + \ldots + p_i\right)^{n-\sum_{j=1}^{i-1} R_j} - \left(p_1 + \ldots + p_{i-1}\right)^{n-\sum_{j=1}^{i-2} R_j} \right]$$

$$+T_k \left[1 - \left(p_1 + \ldots + p_{k-1}\right)^{n-\sum_{j=1}^{k-2} R_j} \right],$$

where $p_i = F_i - F_{i-1}$

$$=T_1 \left[1 - \left(\frac{2e^{-T_1}}{1+e^{-T_1}} \right)^{\theta_1} \right]^n + \sum_{i=2}^{k-1} T_i \left(1 - \left(\frac{2e^{-T_i}}{1+e^{-T_i}} \right)^{\theta_i} \right)^{n-\sum_{j=1}^{i-1} R_j}$$

$$- \left(1 - \left(\frac{2e^{-T_{i-1}}}{1+e^{-T_{i-1}}} \right)^{\theta_{i-1}} \right)^{n-\sum_{j=1}^{i-2} R_j} + T_m \left[1 - \left(1 - \left(\frac{2e^{-T_{k-1}}}{1+e^{-T_{k-1}}} \right)^{\theta_{k-1}} \right)^{n-\sum_{j=1}^{k-2} R_j} \right]$$

$$= T_k - \left(T_2 - T_1\right) \left[1 - \left(\frac{2e^{-T_1}}{1+e^{-T_1}} \right)^{\theta_1} \right]^n - \sum_{i=2}^{k-1} \left(T_{i+1} - T_i\right) \left[1 - \left(\frac{2e^{-T_i}}{1+e^{-T_i}} \right)^{\theta_i} \right]^{n-\sum_{j=1}^{i-1} R_j} .$$

For equal length intervals $T_i = it$, i.e $T_i - T_{i-1} = t$, i=1,2,...,k-1

EDLT reduces to

$$EDLT_1 = kt - t\left[1-\left(\frac{2e^{-t}}{1+e^{-t}}\right)^{\theta_1}\right]^{n} - \sum_{i=2}^{k-1} t\left[1-\left(\frac{2e^{-t}}{1+e^{-t}}\right)^{\theta_i}\right]^{n-\sum_{j=1}^{i-1} R_j}$$

Illustrative example

n = 35, k = 3

$\theta_1 = 2$, $\theta_2 = 2.4$, $\theta_3 = 2.8$

$R_1 = 3$, $R_2 = 4$, $R_3 = 10$

$T_1 = 2$, $T_2 = 3$,

And finally the experiment is terminated at $T_3 = 4$

$n_1 = 4$, $n_2 = 3$, $n_3 = 5$

t	2	3	4	5	6	7	8
EDLT	5.03635	4.132488	4.220808	5.036364	6.005695	7.000873	8.00013

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