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Mathematical Modelling

Ph.D. Thesis

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Declaration

I here by declare that

a) The research work embodied in this thesis on A Mathematical Modelling submitted for Ph.D.degree has not been submitted for my other degree of this or any other university on any previous occasion.

b) To the best of my knowledge no work of this type has been reported on the above subject. Since I have discovered new relations of facts, this work can be considered to be contributory of the advancement of knowledge of Mathematical Modelling.

c) All the work presented in the thesis is original and wherever references have been made to the work of others,it has been clearly indicated as such.

Countersigned by the Guide

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Signature of Research Student

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Certificate Of Approval

This thesis directed by the Candidate's guide has been accepted by the Department of Mathematics, Saurashtra University, Rajkot in the fulfillment of the requirement for the degree of

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CHAPTER 1

Introduction

Graph theory has witnessed rapid growth because of its applications in other areas like Computer Sciences, Engineering, Psychology, Biological Sciences and other Social Sciences. Researchers in Mathematics and in other Sciences have successfully used this branch of Mathematics to solve their research problems.

Several areas of graph theory have been accepted by Mathematicians and other

Scientists. For example Trees have been extensively utilised in theoretical Computer Science. Labelling of graphs have been used in coding theory and Chemistry. Domination theory which was originated from Chess-Board problems has been used to solve some problems in Computer Networks, Communication Theory and other areas. Colouring of graphs has been a reach area of interest of Mathematician with many new directions coming up.

Domination theory in graphs has become a reach area of interest of Graph Theorists. This theory which was originated from Chess-Board problems has attracted many researchers in graph theory. Over 1500 research papers have been published so far and still it is an active area of

interest. Domination theory encompasses several other parameters along with domination number. This theory has provided many many variance of domination. Which have enriched this area to a great extent.

Graphs which are critical with respect to certain property P occupy an important place in Graph theory. A graph is said to be critical with respect to the property P if the graph G has property P but the sub graph obtain by removing every vertex or every edge does not have that property P. This area of graph theory has important place in communication network theory.

If we consider domination as property P then graphs which are domination critical have been studied by many another's. (see [28][51][52]).

They have studied various aspects related to a domination critical graphs, in particular a graph G is said to be domination critical if its domination number changes whenever a vertex or an edge is removed from the graph. However we will also regard a graph to be critical if its domination number changes whenever an edge is added to the graph.

Several authors have studied the effect of removing a vertex from the graph on the domination number of the graph. It may be noted that this number may increase or decrease or remains same. When a vertex is removed several authors have characterized the vertices of the above three types using so called minimum sets which are also called γ-sets.

It may be noted that total domination, k-domination, distance domination and connected domination have been studied by G.J.Vala. He has obtained in his Ph.D. thesis. Characterization of vertices whose removal increases, decreases or dose not change the corresponding numbers associated with the graphs. On the other hand J.C .Bosamia has considered extended total domination, independent domination, vertex covering and extended total k-domination in his Ph.D. thesis.

Like domination number there is associated that any graph a number called big domination number. This number is in general is bigger than the domination number of this graph.

Our study in this thesis is focused on these big numbers associated with some properties that is like total domination, independence, vertex covering and packing. We shall prove that the big number for all the first three properties decreases, or remains same when a vertex is remove from the graph. For packing this number may increase, decrease or remains same.

Our dissertation consists of four chapters.

In chapter 1 we give introduction, preliminaries and notations.

In chapter 2 we define so called Γ_t – sets which is infect a minimal totally dominating set with maximum cardinality. We define the big total domination number of the graph to be the

cardinality of any Γ_t –set. We denote this number by $\Gamma_t(G)$. We characterize those vertices whose removal dose not change the big total domination number and also characterize those vertices whose removal reduces the big total domination number. We also consider so called well totally dominated graphs. We prove some interesting results for well totally dominated graphs.

In chapter 3 we consider vertex covering sets and maximum independent sets. A minimal vertex covering set with maximum cardinality is called Γ_{cr}–set, the number of elements in Γ_{cr}–set is called big vertex covering number of the graph and is denoted as $\Gamma_{cr}(G)$. A vertex covering set with minimum cardinality is called γ_{cr}-set and a number of elements in such a set is

called the vertex covering number of the graph and is denoted as γ_{cr}(G). Minimum vertex covering sets have been considered by J.C .Bosmia in his Ph.D. thesis. We establish that the big vertex covering number of a graph does not increase when the vertex is removal from the graph. We give a characterization of a vertex whose removal does not change the big vertex covering number.

It may be noted that the complement of a vertex covering set is an independent set. Thus the complement of a Γ_{cr}–set is a maximal independent set with minimum cardinality. It is denoted as i(G). It may be noted that Γ_{cr}(G)+i(G)=n where n is the number of vertices in G. We have proved some related results. We have also proved some theorems related to maximum independent

sets. (I) The vertex covering number of a graph G is denoted as $\alpha_0(G)$ and (II) The maximum independence number of a graph G is denoted as $\beta_0(G)$. It may be noted that $\alpha_0(G)+\beta_0(G)=n$, where n is the number of vertices in G.

In chapter 4 we have considered perfect dominating sets and packing. We have defined so called γ_{pr}-sets and Γ_{pr}-sets for perfect domination. In particular we have proved that if S is a γ_{pr}-sets and T is a Γ_{pr}-set of G then S∩T for a graph for which γ_{pr}-sets < $\Gamma_{pr}(G)$. We have also given some examples.

Preliminaries

If G is a graph V(G) will denote the vertex set of the graph G. If S is a subset of V(G) then |S| will denote the number of elements in the set S. G-v will denote the sub graph obtain by removing a vertex v from the graph G. All graphs considered in this thesis are finite and simple. It is assumed that a totally dominating set contains at least two vertices.

Also P_n denotes the path graph with n vertices, W_n denotes the wheel graph with n vertices and C_n denotes the cycle graph with n vertices.

An automorphism of a graph G is an isomorphism from G to G.

CHAPTER 2

Total Domination In Graphs

In this chapter we consider minimal totally dominating sets with highest cardinality. They are called Γ_t sets and the cardinality of such set is called the big total domination number of the graph and is denoted as $\Gamma_t(G)$.

Through out this chapter we assume that graphs do not have isolated vertices.

DEFINITION 2.1 [51]

Let G be a graph and S be a subset of V(G). The set S is said to be a totally dominating set if for every vertex v of G , v is adjacent to some vertex of S.

Obviously , every totally dominating set is a dominating set . But every dominating set need not be a totally dominating set . We assume that every totally dominating set has at least two vertices.

DEFINITION 2.2 [51]

A totally dominating set S of G is said to be a minimal totally dominating set if for every vertex v of S , S – v is not a totally dominating set .

DEFINITION 2.3 [21]

Let S be a subset of V(G) and $v \in S$ then the total private neighbourhood of v with respect to the set S is defined as

$P_{rt}(v, S) = \{ w \in V(G) / N(w) \cap S = \{ v \} \}$.

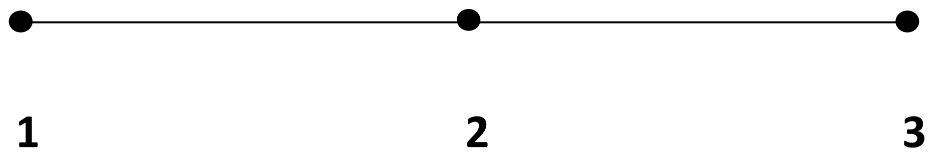


Fig. 2.1 : Path graph with three vertices.

In the above figure the set { 2 , 3 } is a minimal totally dominating set of the graph G = the path graph with three vertices. Also if S = { 2 , 3 } and v = 2 then $P_{rt}(2, S) = \{1, 3\}$ and $P_{rt}(3, S) = \{2\}$.

{1,3} is dominating set but not totally dominating set .

DEFINITION 2.4 [51]

A totally dominating set with minimum cardinality is called a minimum totally dominating set and is called a γ_t set of the graph.

The cardinality of a minimum totally dominating set is called the total domination number of the graph G and is denoted as γ_t(G).

In the above example of the path graph with three vertices the total domination number of the graph is 2.

REMARK 2.5

It may be noted that every minimum totally dominating set is a minimal totally dominating set but the converse may not be true.

However a minimal totally dominating set with smallest cardinality is a minimum totally dominating set.

We state the following theorem with out proof. The proof can be found in D.K.Thakkar and G.J.Vala [9]

THEOREM 2.6

A totally dominating set S of the graph G is a minimal totally dominating set if and only if for every vertex $v \in S$, $P_{rt}(v, S)$ is a non empty set.

DEFINITION 2.7 [51]

A minimal totally dominating set with maximum cardinality is called Γ_t set of the graph G.

The cardinality of a Γ_t set is called the big total domination number of the graph G and is denoted as $\Gamma_t(G)$.

EXAMPLE 2.8

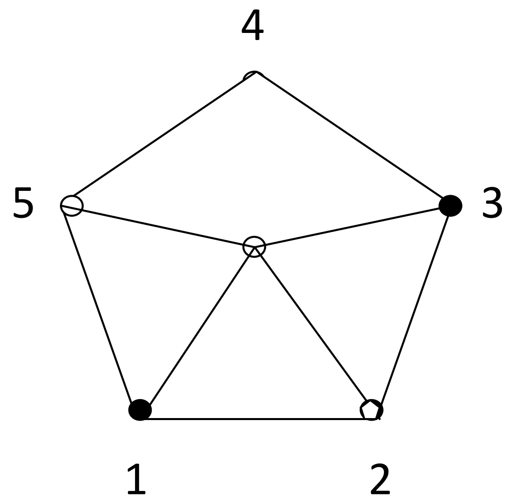


Fig. 2.2: W_6 = Wheel graph with six vertices

Consider the wheel graph W_6 with six vertices as mentioned in the above figure.

(i) The total domination number of this graph is 2.

(ii) The set S = { 1,2,3 } is a minimal totally dominating set with the highest cardinality. Hence S is a Γ_t set of the graph.

(iii) **The big total domination number of this graph $\Gamma_t(W_6) = 3$.**

It may happen that the total domination number of a graph is same as the big total domination number of the graph.(For example the path graph with three vertices.)

REMARK 2.9

It may be noted that a totally dominating set does not exist if the graph has an isolated vertex , also if v is a vertex of graph G such that G – v has an isolated vertex then a totally dominating set does not exist in G – v . Thus we consider only those graphs which do not have isolated vertices . Also we avoid those vertices whose removal creates isolated vertices .

We introduce the following notations.

V_t^i = { v ∈ V(G) / G – v has an isolated vertex }.

We now introduce the following sets.

(i) W_t^+ = { v ∈ V(G) / v ∉ V_t^i and Γ_t(G – v) > Γ_t(G) }.

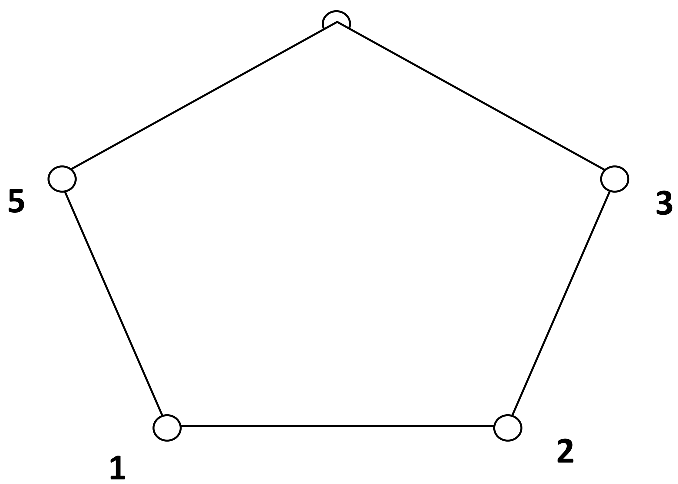
(ii) W_t^- = { v ∈ V(G) / v ∉ V_t^i and Γ_t(G – v) < Γ_t(G) }.

(iii) W_t^0 = { v ∈ V(G) / v ∉ V_t^i and Γ_t(G – v) = Γ_t(G) }.

Note that the above sets are mutually disjoint and their union is v(G) - V_t^i.

EXAMPLE 2.10 Consider the Cycle C_5. Fig.2.3:

(i)



It may be noted that $\Gamma_t(C_5) = 3$. If we remove the vertex 5 then the resulting graph is the path graph P_4.

Fig. 2.4: Path graph with four vertices.

$\Gamma_t(P_4) = 2$, $5 \in W_t^-$. It may be noted that every vertex of C_5 is a member of W_t^- . That is W_t^- = The vertex set of C_5.

(Note that V_t^i is the empty set for this graph C_5).

(ii) Consider the wheel graph W_6. If we remove the vertex 0 from the graph W_6 then the resulting graph is C_5. $\Gamma_t(W_6) = 3$ and $\Gamma_t(C_5) = 3$.

Thus $0 \in W_t^0$.

Now we prove that when a vertex is remove the big total domination number does not increase.

THEOREM 2.11

Suppose G is a graph $v \in V(G)$ such that

$v \notin V_t^i$ then $\Gamma_t(G - v) \leq \Gamma_t(G)$.

PROOF : Suppose S is a Γ_t set of $G - v$ there are three possibilities for the vertex v

(i) v is not adjacent to any vertex of S. Let w be a vertex adjacent to the vertex v

Since S is a totally dominating set in $G - v$,

$S_1 = S \cup \{ w \}$ is a totally dominating set in G. In fact S_1 is a minimal totally dominating set in G.

Therefore $\Gamma_t(G) \geq$ Cardinality of $S_1 = |S_1| >$ Cardinality of $S = \Gamma_t(G - v)$.

Thus $\Gamma_t(G - v) \leq \Gamma_t(G)$.

(ii) v is adjacent to exactly one vertex w of S.

Thus S is a minimal totally dominating set in G.

Therefore $\Gamma_t(G) \geq$ Cardinality of $S = \Gamma_t(G - v)$.

(iii) v is adjacent to at least two vertices of S. Then $S_1 = S \cup \{ v \}$ is a minimal totally dominating set of G.

Therefore $\Gamma_t(G) \geq |S_1| > |S| = \Gamma_t(G - v)$.

Therefore $\Gamma_t(G) \geq \Gamma_t(G - v)$.

Thus in all cases $\Gamma_t(G - v) \leq \Gamma_t(G)$.■

THEOREM 2.12

Let G be a graph and v be a vertex of V(G) such that $v \notin V_t^i$ then $v \in W_t^0$ if and only if either there is a Γ_t set S of G such that $v \notin S$ and v is adjacent to at least two vertices of S, or there is a Γ_t set S_1 of G such that $v \notin S_1$ and there is a

vertex w in S_1 such that the total private neighbourhood of w with respect to S_1 contains at least two vertices including v.

PROOF :

Suppose v $\in W_t^0$ .Let S be a Γ_t set of G – v . If v is not adjacent to any vertex of S then let w be any vertex adjacent to v then T = S U { w } is a minimal totally dominating set of graph G and

| T | > | S | .

Therefore Γ_t (G) ≥ | T | > | S | = Γ_t (G - v) .

That is Γ_t (G - v) < Γ_t (G) . This means that v $\in W_t^-$ which contradicts with our assumption. Therefore v must be adjacent to some vertex of S.

Suppose there is a vertex $w \in S$ such that v is adjacent to only w in S , Therefore $v \in P_{rt}(w , S)$ also S is a minimal totally dominating set in $G - v$. Therefore total private neighbourhood of w with respect to S in $G - v$ contains a vertex v' . Thus $P_{rt}(w , S)$ contains at least two vertices and one of them is v.

In the other case , that is v is adjacent to at least two vertices of S then S is a minimal totally dominating set of G not containing v and v is adjacent to at least two vertices of S .

<u>**CONVERSE**</u>

Suppose S is a Γ_t set of G not containing v such that v is adjacent to at least two vertices of S , then for every vertex w in S. $P_{rt}(w , S)$ can not contain v . Therefore S is a minimal totally dominating set in $G - v$. Therefore $\Gamma_t(G - v) \geq |S| = \Gamma_t(G)$. Since $\Gamma_t(G - v) > \Gamma_t(G)$ is not possible, we have $\Gamma_t(G - v) = \Gamma_t(G)$. Hence $v \in W_t^0$.

Suppose S is a $\Gamma_t(G)$ such that $v \notin S$ and there is a vertex $w \in S$ such that $P_{rt}(w, S)$ contains at least two vertices and one of them is v, therefore $P_{rt}(w, S)$ contains a vertex of $G - v$. Also for other vertices w' in S. w can not be a member of $P_{rt}(w', S)$, $P_{rt}(w', S)$ contains a vertex of $G - v$. Thus, S is a minimal totally dominating set in $G - v$.

By similar argument of S in above case, we have $\Gamma_t(G - v) = \Gamma_t(G)$. ■

Now we characterize the vertices of the set W_t^-.

<u>THEOREM 2.13</u>

Let G be a graph and v be a vertex of G such that $v \notin V_t^i$ then $v \in W_t^-$ if and only if , whenever S is a Γ_t set of G not containing v then there is a vertex w in S such that $P_{rt}(w, S) = \{v\}$.

<u>PROOF :</u>

Suppose $v \in W_t^-$. Let S be a Γ_t set of G such that $v \notin S$. Now v is adjacent to some vertex of S if v is adjacent to at least two vertices of S then by previous theorem $v \in W_t^0$. Which contradicts our assumption with $v \in W_t^-$, there fore there is a vertex w in S such that v is adjacent to w and v is not adjacent to any other vertex

of S , this implies that if there is a another vertex v' in G such that v' $\in P_{rt}(w, S)$ then again by previos theorem v $\in W_t^0$. Which is a contradiction. Hence

$P_{rt}(w, S) = \{v\}$.

<u>CONVERSE :</u>

Suppose v $\in W_t^0$ then there is a Γ_t set S of G not containing v such that one of the following two conditions hold

(a) There is a vertex w in S such that $P_{rt}(w, S)$ Contains at least two vertices including v.

(b) Now v is adjacent to at least two vertices of S

Now , suppose condition (a) holds.

There is a vertex w' in s such that

Prt(w' , S) = {v}.

If w = w' then our condition is violated . Suppose w ‡ w' , then v is adjacent to two vertices of S, it implies that v $\notin$ P_{rt}(w' , S).

If v is adjacent to at least two vertices of S then v $\notin$ P_{rt}(w' , S) for any w' in S . This again violate with our condition .

Thus v $\in$ W_t^0 gives rise to a contradiction in either case , thus v $\in$ W_t^- .Hence the theorem is proved. ■

EXAMPLE 2.14

Consider the Cycle C_6. Let { 1,2,3,4,5,6 } be its vertex set.

(i)

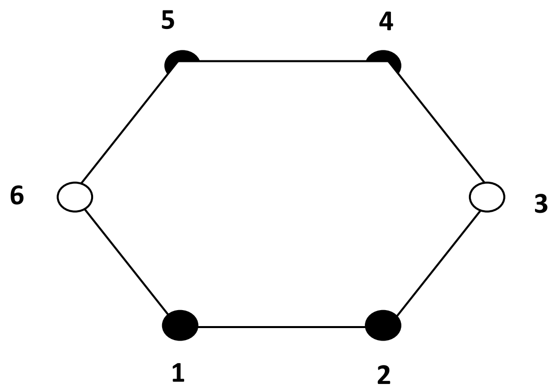


Fig.2.5: Cycle graph with six vertices.

The big total domination number of this graph is 4, and S = { 1,2,4,5 } is a Γ_t set of C_6. Now $6 \notin S$, also 6 is adjacent to two vertices of S namely 1 and 5 .

Therefore by above theorem $6 \in W_t^0$

Similarly it can be proved that every other vertex of $C_6 \in W_t^0$.

(ii)

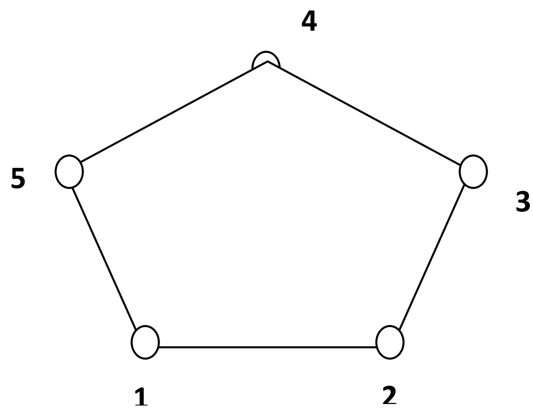


Fig.2.6: Cycle graph with five vertices.

Consider the Cycle C_5 with vertex set { 1,2,3,4,5 }.

Its big total domination number is 3. Consider the vertex 5. Consider the set S = { 2,3,4 } which is a Γ_t set of G not containing 5 . P_{rt}(4 , S) = { 5 } and 4 ∈ S. Similarly if we consider the set S_1 = { 1,2,3 } then S_1 is a

Γ_t set not containing 5 and P_{rt}(1 , S_1) = { 5 }.

Therefore 5 ∈ W_t^-.

THEOREM 2.15

Let G be a graph for which W_t^- is an empty set if the set { $S_1,S_2,S_3,\ldots,S_K$ } is the set of all Γ_t sets of the graph G then $S_1 \cap S_2 \cap S_3 \cap \ldots \cap S_K = V_t^i$.

PROOF :

Suppose v $\in S_1 \cap S_2 \cap S_3 \cap \ldots \cap S_K$. Suppose v $\notin V_t^i$,

Then v $\in W_t^0$. Therefore there is a Γ_t set S_j which does not contain v , by theorem 2.12 that is v $\notin$ $S_1 \cap S_2 \cap S_3 \cap \ldots \cap S_K$, and this is a contradiction.

Hence v $\in V_t^i$

Therefore $S_1 \cap S_2 \cap S_3 \cap \ldots \cap S_K \subset V_t^i$.

Suppose v $\in V_t^i$

If $v \notin S_1 \cap S_2 \cap S_3 \cap \ldots \cap S_K$ then for some j , $v \notin S_j$. If v is adjacent to at least two vertices of S_j then by theorem 2.12 $v \in W_t^0$, this is a contradiction.

If there is a vertex w in S_j such that $P_{rt}(w, S_j)$ contains at least two vertices including v then also by theorem 2.12 $v \in W_t^0$. This is a contradiction.

If there is a vertex w in S_j such that $P_{rt}(w, S_j) = \{ v \}$ then $v \in W_t^-$, but W_t^- is empty and so this possibility is ruled out .

Hence $v \in S_1 \cap S_2 \cap S_3 \cap \ldots \cap S_K$.

There fore $V_t^i \subset S_1 \cap S_2 \cap S_3 \cap \ldots \cap S_K$.

Hence $S_1 \cap S_2 \cap S_3 \cap \ldots \cap S_K = V_t^i$. ■

Well Totally Dominated Graphs

DEFINITION 2.16

Let G be a graph then G is said to be well totally dominated graph if all minimal totally dominating sets of G have the same cardinality . Equivalently ,

$\gamma_t(G) = \Gamma_t(G)$.

EXAMPLE 2.17 **Consider the Cycle C_5 .**

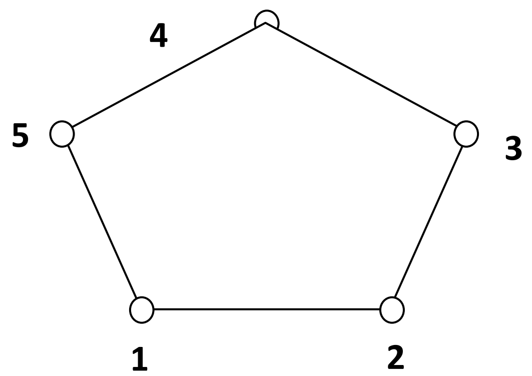


Fig.2.7: Cyclic graph with five vertices.

$\gamma_t(C_5) = \Gamma_t(C_5) = 3$.

Thus C_5 is a well totally dominated graph.

EXAMPLE 2.18

Consider the wheel graph W_6 then $\gamma_t(W_6) = 2$, and $\Gamma_t(W_6) = 3$. Thus W_6 is not a well totally dominated graph.

THEOREM 2.19

Suppose G is a well totally dominated graph and v $\in$ V(G) such that v $\notin V_t^i$ then the following statements are true.

(i) v $\notin V_t^+$ (That is V_t^+ is empty)

(ii) If v $\in V_t^0$ then G – v is well totally dominated graph .

(iii) If v $\in V_t^0$ then v $\in W_t^0$.

PROOF :

(i) If v $\in V_t^+$ then γ_t (G)<γ_t (G - v) $\leq \Gamma_t$ (G - v) $\leq \Gamma_t$ (G).

Since γ_t (G) = Γ_t (G). This implies that γ_t (G - v) = γ_t (G)

Which is a contradiction . Thus v $\notin V_t^+$.

(ii) If v $\in V_t^0$ then

γ_t (G) = γ_t (G - v) $\leq \Gamma_t$ (G - v) $\leq \Gamma_t$ (G).

This implies that γ_t (G - v) $= \Gamma_t$ (G - v) .

Thus G – v is well totally dominated graph .

(iii) From (ii) γ_t (G) = γ_t (G - v) = Γ_t (G - v) = Γ_t (G).

Therefore v $\in W_t^0$. Hence the theorem . ■

DEFINITION 2.20

A graph G is said to be an approximately well totally dominated graph , if Γ_t (G) = γ_t (G) + 1.

For example P_5 is an approximately well totally dominated graph .

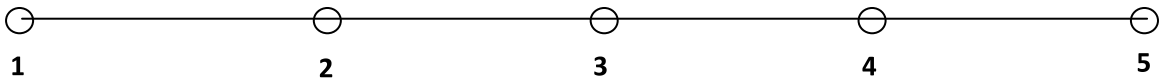


Fig.2.8: Path graph with five vertices.

THEOREM 2.21

Let G be a well totally dominated graph and v be a vertex of G such that v $\notin V_t^i$ then either G – v is well totally dominated graph or it is an approximately well totally dominated graph .

PROOF :

Suppose v $\in V_t^-$ therefore γ_t (G – v) = γ_t (G) – 1 .

γ_t (G – v) $\leq$ Γ_t (G – v) $\leq$ Γ_t (G) .

Case : (i) If Γ_t (G – v) = γ_t (G – v) then G – v is well totally dominated graph .

Case : (ii) γ_t (G)$\leq$ Γ_t (G – v)$\leq$ Γ_t (G) .

Since G is well totally dominated graph .

$\gamma_t(G) = \Gamma_t(G)$ and thus $\Gamma_t(G - v) = \gamma_t(G)$. Which is equal to $\gamma_t(G - v) + 1$.

Thus G – v is an approximately well totally dominated graph .

If $v \in V_t^0$, then $\gamma_t(G) = \gamma_t(G - v) \leq \Gamma_t(G - v) \leq \Gamma_t(G)$.

Therefore $\Gamma_t(G - v) = \gamma_t(G - v)$ and hence G – v is a well totally dominated graph. ■

<u>EXAMPLE 2.22</u>

Consider the path graph P_5 for this graph $\gamma_t(P_5) = 3$ and $\Gamma_t(P_5) = 4$. Thus P_5 is not well totally dominated graph .

However if v is any vertex in P_5 such that $v \notin V_t^i$.

Then γ_t (P_5 – v) = Γ_t (P_5 – v). Thus it is well totally dominated graph .

The above two theorems can be summarized as follows.

For any vertex v which is not in V_t^i . G – v is either well totally dominated graph or an approximately well totally dominated graph provided the given graph G is well totally dominated graph .

THEOREM 2.23

Suppose G is an approximately well totally dominated graph and v is a vertex such that v $\notin$ V_t^i, then if

v $\in$ V_t^+ then G – v is well totally dominated graph ,

v $\in W_t^0$ **and** $\Gamma_t(G - v) = \gamma_t(G) + 1$.

PROOF :

Since v $\in V_t^+$,

$\gamma_t(G) < \gamma_t(G - v) \leq \Gamma_t(G - v) \leq \Gamma_t(G) = \gamma_t(G) + 1$.

Therefore $\gamma_t(G - v) = \gamma_t(G) + 1$.

Therefore $\gamma_t(G) + 1 \leq \Gamma_t(G - v) \leq \gamma_t(G) + 1$.

Hence $\Gamma_t(G - v) = \gamma_t(G) + 1 = \gamma_t(G - v)$.

That is G – v is well totally dominated graph .

Also , $\Gamma_t(G - v) = \gamma_t(G) + 1 = \Gamma_t(G)$.

Thus v $\in W_t^0$. ■

THEOREM 2.24

Suppose G is an approximately well totally dominated graph and v is a vertex such that v $\notin V_t^i$.

If $v \in V_t^0$, then G – v is either an approximately well totally dominated graph or it is well totally dominated graph .

In the first case $v \in W_t^0$ and in the second case $v \in W_t^-$.

PROOF :

Since $v \in V_t^0$,

$\gamma_t(G) = \gamma_t(G - v) \le \Gamma_t(G - v) \le \Gamma_t(G)$.

Case : (i)

$\Gamma_t(G - v) = \Gamma_t(G)$,

then $\Gamma_t(G - v) = \Gamma_t(G) = \gamma_t(G) + 1 = \gamma_t(G - v) + 1$.

Thus G – v is an approximately well totally dominated graph.

Since $\Gamma_t(G - v) = \Gamma_t(G)$.

That is v $\in W_t^0$.

Case : (ii) Γ_t (G – v) = γ_t (G – v) , then obviously G – v is well totally dominated graph.

Since Γ_t (G – v) = γ_t (G – v)= γ_t (G)< Γ_t (G),

Therefore v $\in W_t^-$.■

<u>THEOREM 2.25</u>

Let G be an approximately well totally dominated graph and v is a vertex such that v $\notin V_t^i$. If v $\in V_t^-$ then exactly one of the following three possibilities holds.

(i) **G – v is well totally dominated graph .**

(ii) **G – v is an approximately well totally dominated graph .**

(iii) **v $\in W_t^0$.**

PROOF :

Since , v $\in V_t^-$.

$\gamma_t(G - v) = \gamma_t(G) - 1$.

Now , $\gamma_t(G - v) \leq \Gamma_t(G - v) \leq \Gamma_t(G)$.

Therefore $\Gamma_t(G - v) = \gamma_t(G) - 1$ **, or** $\Gamma_t(G - v) = \gamma_t(G)$ **or**

$\Gamma_t(G - v) = \Gamma_t(G)$.

If the first equality holds then $\Gamma_t(G - v) = \gamma_t(G - v)$ **and the graph G – v is well totally dominated graph.**

If the second equality is true then

$\Gamma_t(G - v) = \gamma_t(G) = \gamma_t(G - v) + 1$ **, and so the graph is an approximately well totally dominated graph .**

If the third equality holds then v $\in W_t^0$. ■

CHAPTER 3

Vertex Covering Sets

In this chapter we consider vertex covering sets in graphs. We will define so called Γ_{cr} sets which are infect minimal vertex covering sets with maximum cardinality. We will find conditions under which the big vertex covering number of a graph decreases or remains same. Before that we will prove that this number never increases when a vertex is remove.

DEFINITION 3.1 [51]

A subset S of V(G) is said to be a vertex covering set if for every edge of the graph at least one end vertex is S.

DEFINITION 3.2 [51]

A vertex covering set S is said to be a minimal vertex covering set if $S - v$ is not a vertex covering set for every v in S.

DEFINITION 3.3

A vertex covering set S with minimum cardinality is called a minimum vertex covering set and is denoted as γ_{cr} set. Note that every minimum vertex covering sets is a minimal vertex covering sets but converse is not true(see example 3.5).

The cardinality of a minimum vertex covering set of a graph G is called the vertex covering number of G and is denoted as α_0(G).

DEFINITION 3.4 [51]

A minimal vertex covering set with maximum cardinality is called Γ_{cr} set.

The cardinality of Γ_{cr} set is called a big vertex covering number of the graph G and is denoted as Γ_{cr}(G). Obviously $\alpha_0(G) \leq \Gamma_{cr}(G)$.

EXAMPLE 3.5

Consider P_5. The path graph with 5 vertices.

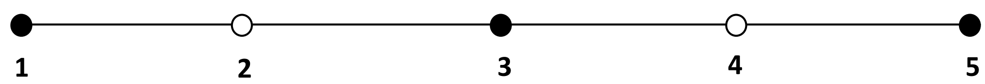


Fig. 3.1: Path graph with five vertices.

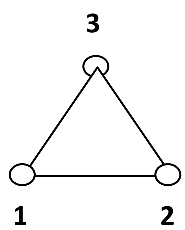
Let denote this graph P_5 = G. In this graph the set S = { 2,4 } is a minimal vertex covering set and hence α_0(G) = 2.

Also, T = { 1,3,5 } is a minimal vertex covering set with maximum cardinality. That is T is a Γ_{cr} set, and hence Γ_{cr}(G) = 3. Here T is a minimal vertex covering set but not a minimum vertex covering set.

Note that every vertex covering set is a dominating set and hence $\gamma(G) \leq \alpha_0(G)$ for any graph G without isolated vertices.

In P_5 = G , $\gamma(G) = \alpha_0(G) = 2$.

In C_3 = G , $\gamma(G) = 1$ and $\alpha_0(G) = 2$.



Therefore $\gamma(G) < \alpha_0(G)$.

EXAMPLE 3.6

Consider the Peterson graph G as shown in the figure

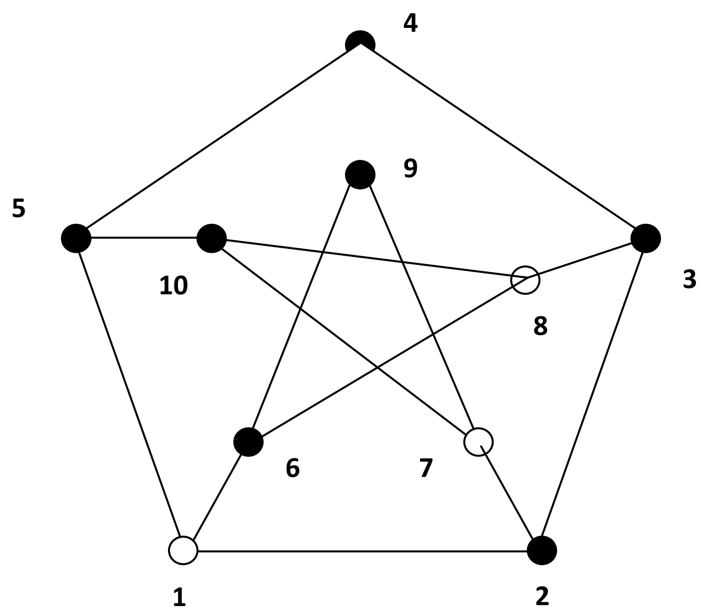


Fig. 3.2: Peterson graph

For this graph G the set

S = { 2,3,4,5,6,9,10 } is a Γ_{cr} set and the big vertex covering number of this graph is 7.

That is $\Gamma_{cr}(G) = 7$. Also T = { 1,3,4,6,7,10 } is a γ_{cr} set and the vertex covering number of graph

G = 6 = α_0(G) = γ_{cr} (G).

Note that α_0(G) < Γ_{cr}(G).

First we introduce the following notations.

V_{cr}^{+} **= { v ∈ V(G) / α_0(G - v) > α_0(G) }.**

V_{cr}^{-} **= { v ∈ V(G) / α_0(G - v) < α_0(G) }.**

V_{cr}^{0} **= { v ∈ V(G) / α_0(G - v) = α_0(G) }.**

The above sets are mutually disjoint and their union = V(G).

First we prove that for any graph G , V_{cr}^{+} is empty.

THEOREM 3.7

Let G be a graph and v ∈ V(G) then α_0(G - v) $\leq$ α_0(G).

PROOF :

Let S be a minimum vertex covering set of graph G. Then every edge of G has at least one end point in S. Now every edge of G – v is also an edge of G. Therefore every edge of G – v has at least one end vertex in S. Thus S is a vertex covering set of G – v , if v $\notin$ S.

Therefore α_0(G - v) $\leq$ | S | = α_0(G).

If v $\in$ S then S – v is a vertex covering set in G – v .Therefore α_0(G - v) $\leq$ | S-v | < | S | = α_0(G).

Thus α_0(G - v) $\leq$ α_0(G).■

THEOREM 3.8

Let G be a graph, v be a vertex of G such that

v $\in V_{cr}^{-}$ then α_0(G - v) $=$ α_0(G) - 1 .

PROOF :

Let S be a minimum vertex covering set of G – v . If all the neighbours of v are in S then S is a vertex covering set of G and hence

$\alpha_0(G) \leq |S| = \alpha_0(G - v) \leq \alpha_0(G)$.

Therefore $\alpha_0(G - v) = \alpha_0(G)$, and hence $v \in V_{cr}^0$.

Which is not true.

Therefore there is some neighbours v' of v such that $v' \notin S$. Let $S_1 = S \cup \{v\}$. Then S_1 is a vertex covering set of G .

Therefore $\alpha_0(G) \leq |S_1| = |S| + 1$.

Therefore $|S| < \alpha_0(G) \leq |S| + 1$.

Hence $\alpha_0(G) = |S| + 1$.

Therefore $\alpha_0(G) = \alpha_0(G - v) + 1$.

Therefore $\alpha_0(G - v) = \alpha_0(G) - 1$.■

THEOREM 3.9

Let G be a graph and $v \in V(G)$.

Then $\Gamma_{cr}(G - v) \leq \Gamma_{cr}(G)$.

PROOF :

Let S be a Γ_{cr} set of $G - v$. If all the neighbours of v are in S then S is a minimal vertex covering set of G and therefore $|S| \leq \Gamma_{cr}(G)$ and thus $\Gamma_{cr}(G - v) \leq \Gamma_{cr}(G)$.

If some neighbour of v is not in S then $S \cup \{v\}$ is a minimal vertex covering set of graph G. Therefore $|S| < |S \cup \{v\}| \leq \Gamma_{cr}(G)$.

That is $\Gamma_{cr}(G - v) < \Gamma_{cr}(G)$. ■

We define the following symbols.

W_{cr}^{+} = **$\{ v \in V(G) / \Gamma_{cr}(G - v) > \Gamma_{cr}(G) \}$.**

W_{cr}^{-} = **$\{ v \in V(G) / \Gamma_{cr}(G - v) < \Gamma_{cr}(G) \}$.**

W_{cr}^{0} = { v ∈ V(G) / Γ_{cr}(G – v) = Γ_{cr} (G) }.

We now prove the following theorem.

THEOREM 3.10

Let G be a graph and v ∈ V(G). Then v ∈ W_{cr}^{0} If and only if there is a Γ_{cr} set S of G not containing v such that S is also Γ_{cr} set of (G - v).

PROOF :

Suppose that v ∈ W_{cr}^{0}.

Let S_1 be any Γ_{cr} set of G – v . If some neighbour of v is not in S_1 then S = S_1 U {V} is a minimal vertex covering set of G and hence ∣S1∣<∣S∣ ≤ Γ_{cr} (G). That is Γ_{cr} (G – v) < Γ_{cr} (G). Which implies that v ∈ W_{cr}^{-}.

Which is not true. Thus all neighbours of v must in S_1. Let $S = S_1$ then as proved in previous theorem S is a minimal vertex covering set of G . If S is not a Γ_{cr} set of G then $|S| < \Gamma_{cr}(G)$.

That is $\Gamma_{cr}(G - v) < \Gamma_{cr}(G)$. Which is a contradiction . Hence S is a Γ_{cr} set of G. Also, thus S is the required Γ_{cr} set .

Conversely, suppose S is a Γ_{cr} set of G not containing v such that S is also a Γ_{cr} set of G – v then $\Gamma_{cr}(G) = |S| = \Gamma_{cr}(G - v)$. Thus $v \in W_{cr}^{0}$. ■

<u>COROLLARY 3.11</u>

Let G be a graph and $v \in V(G)$. Then $v \in W_{cr}^{-}$

If and only if whenever S is a Γ_{cr} set of G not containing v then S is not a Γ_{cr} set of G – v .

EXAMPLE 3.12

Let G = C_5.

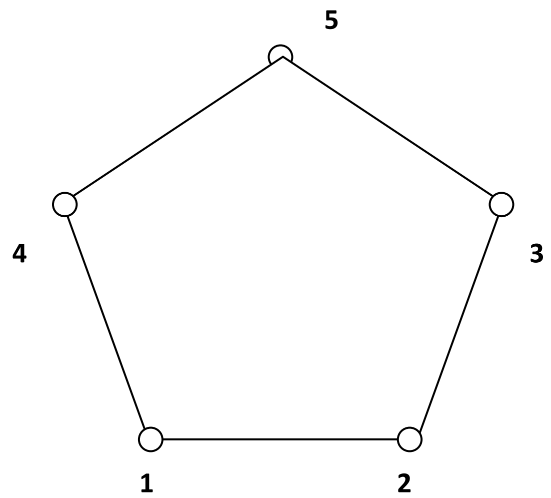


Fig. 3.3: Cycle graph with five vertices.

Then $\Gamma_{cr}(G) = 3$. Let v = 5. Then G – v = The path Graph P_4. $\Gamma_{cr}(G - v) = 2$.

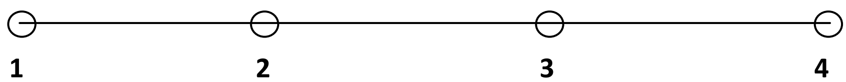


Fig. 3.4: Path graph with four vertices.

Thus v $\in W_{cr}^{-}$. In G there are two Γ_{cr} set not containing S

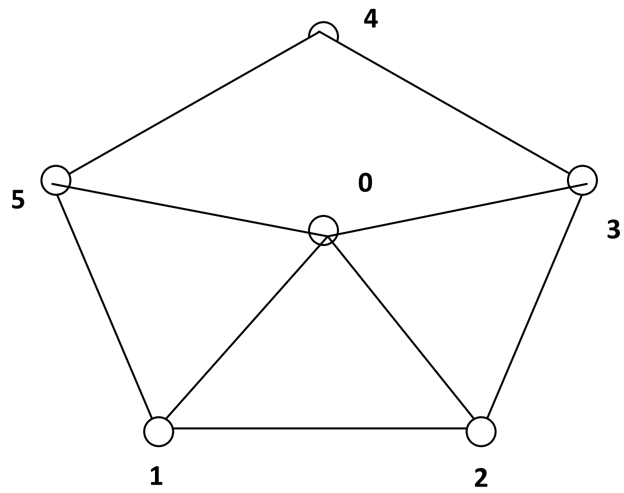
(i) $S_1 = \{ 1,3,4 \}$

(ii) $S_2 = \{ 1,2,4 \}$

Note that neither S_1 nor S_2 is a Γ_{cr} set in $G - v$.

EXAMPLE 3.13

Fig. : 3.5: Wheel graph with six vertices.



The $\Gamma_{cr}(G) = 3$. Let $v = 0$ then $G - v = C_5$ and

$\Gamma_{cr}(G - v) = 3$. Thus $v \in W_{cr}^{0}$.

In fact $S = \{ 1,3,4 \}$ is a Γ_{cr} set of G not containing 0 such that S is a Γ_{cr} set of $G - v$.

Well Covered Graphs

DEFINITION 3.14 [51]

A graph G is said to be a well covered if any two minimal vertex covering sets have the same cardinality.

Equivalently a graph G is well covered if $\alpha_0(G) = \Gamma_{cr}(G)$.

For example C_4 and P_4 are well covered graphs.

On the other hand the Peterson graph is not a well covered graph(see example 3.6)

THEOREM 3.15

Let G be a well covered graph and $v \in V(G)$

(i) G – v is well covered or $v \in W_{cr}^{0}$

(ii) If v $\in V_{cr}^{0}$ then v $\in W_{cr}^{0}$ and G – v is well covered.

(iii) If v $\in V_{cr}^{-}$ then either G – v is well covered and

$\Gamma_{cr}(G - v) = \Gamma_{cr}(G) - 1$ or v $\in W_{cr}^{0}$.

<u>PROOF :</u>

(i) $\alpha_0(G - v) \leq \alpha_0(G) \leq \Gamma_{cr}(G)$.

Also $\alpha_0(G - v) \leq \Gamma_{cr}(G - v) \leq \Gamma_{cr}(G)$

Hence if $\Gamma_{cr}(G - v) = \alpha_0(G - v)$ then G – v is well covered or if $\Gamma_{cr}(G - v) = \Gamma_{cr}(G)$ then v $\in W_{cr}^{0}$.

(ii) In this case

$\alpha_0(G - v) = \alpha_0(G) \leq \Gamma_{cr}(G - v) \leq \Gamma_{cr}(G)$.

Therefore $\alpha_0(G - v) = \alpha_0(G) = \Gamma_{cr}(G - v) = \Gamma_{cr}(G)$.

Thus G – v is well covered and v $\in W_{cr}^{0}$.

(iii) $\alpha_0(G - v) = \alpha_0(G) - 1 \leq \Gamma_{cr}(G - v) \leq \Gamma_{cr}(G)$.

Therefore $\Gamma_{cr}(G - v) = \alpha_0(G - v)$ or

$\Gamma_{cr}(G - v) = \alpha_0(G) = \Gamma_{cr}(G)$.

Thus either $G - v$ is well covered and

$\Gamma_{cr}(G - v) = \Gamma_{cr}(G) - 1$ or $v \in W_{cr}^0$. ■

We introduce the following concept.

DEFINITION 3.16

A graph G is said to be approximately well covered if $\alpha_0(G) = \Gamma_{cr}(G) - 1$.

For example Peterson graph is an approximately well covered graph.

THEOREM 3.17

Let G be an approximately well covered graph and $v \in V(G)$.

(i) If $v \in V_{cr}^0$ then either $G - v$ is well covered or approximately well covered .

(ii) If v $\in V_{cr}^{-}$ then either G – v is well covered or approximately well covered or v $\in W_{cr}^{0}$.

PROOF :

(i) α_0**(G – v) =** α_0**(G)** $\leq$ **Γ_{cr}(G).**

Also α_0**(G – v)** $\leq$ **Γ_{cr}(G – v)** $\leq$ **Γ_{cr}(G).**

Thus if v $\in V_{cr}^{0}$ then Γ_{cr}(G – v) = α_0**(G – v) and in this case G – v is well covered or if Γ_{cr}(G – v) = Γ_{cr}(G). Then** α_0**(G – v) =** α_0**(G) = Γ_{cr}(G) – 1 = Γ_{cr}(G – v) – 1 and hence G – v is an approximately well covered.**

(ii) If v $\in V_{cr}^{-}$ then if Γ_{cr}(G – v) = α_0**(G – v) then G – v is well covered. If Γ_{cr}(G – v) =** α_0**(G) then Γ_{cr}(G – v) =** α_0**(G) = Γ_{cr}(G) – 1. Then G – v is an approximately well covered. If Γ_{cr}(G – v) = Γ_{cr}(G) then v $\in W_{cr}^{0}$.** ■

We introduce the following concept.

DEFINITION 3.18

A graph G is said to be approximately well dominated if γ(G) = Γ(G) – 1.

THEOREM 3.19

If graph G is approximately well dominated then either G is well covered or G is approximately well covered.

PROOF :

Since G is approximately well dominated γ(G) = Γ(G) – 1. Now every maximal independent set is a minimal dominating set. Therefore cardinality of every maximal independent set is equal to Γ(G) – 1 or Γ(G).

Therefore $i(G) = \Gamma(G)$ or $i(G) = \Gamma(G) - 1$.

Now $i(G) \leq \beta_0(G) \leq \Gamma(G)$.(Because a maximum independent set is a minimal dominating set).

Case(i) $i(G) = \Gamma(G)$. Then from the above inequality $\beta_0(G) = \Gamma(G) = i(G)$. Therefore $n - \beta_0(G) = n - i(G)$.

Now $\alpha_0(G) + \beta_0(G) = n$ and $i(G) + \Gamma_{cr}(G) = n$.

Thus $\alpha_0(G) = \Gamma_{cr}(G)$. Therefore the graph is well covered.

Case(ii) $i(G) = \Gamma(G) - 1$.

Now again $i(G) \leq \beta_0(G) \leq \Gamma(G)$. If $\beta_0(G) = \Gamma(G) - 1$. Then $\beta_0(G) = i(G)$. Therefore by the argument in Case(i) G is well covered.

Suppose $\beta_0(G) = \Gamma(G)$ then $i(G) = \beta_0(G) - 1$.

Therefore $n - i(G) = (n - \beta_0(G)) + 1$.

Therefore $\Gamma_{cr}(G) = \alpha_0(G) + 1$. Therefore $\alpha_0(G) = \Gamma_{cr}(G) - 1$

Therefore $\alpha_0(G) = \Gamma_{cr}(G) - 1$.

Therefore the graph is an approximately well covered.■

<u>THEOREM 3.20</u>

Suppose G is a graph and $v \in V(G)$ such that $\Gamma_{cr}(G - v) = \Gamma_{cr}(G) - 1$ and if $G - v$ is well covered then G is also well covered or an approximately well covered.

<u>PROOF :</u>

$\alpha_0(G - v) = \Gamma_{cr}(G - v) = \Gamma_{cr}(G) - 1$.

Now $\alpha_0(G - v) = \alpha_0(G)$ or $\alpha_0(G) - 1$.

Case(i) If $\alpha_0(G - v) = \alpha_0(G)$ then $\alpha_0(G) = \Gamma_{cr}(G) - 1$.

Thus G is an approximately well covered.

Case(ii) If $\alpha_0(G - v) = \alpha_0(G) - 1$ then from above $\alpha_0(G) - 1 = \Gamma_{cr}(G) - 1$. Hence $\alpha_0(G) = \Gamma_{cr}(G)$.

Thus the graph is well covered.■

Maximum Independence

DEFINITION 3.21 [52]

Let G be a graph and S be a subset of V(G), then S is said to be an independent set, if any two distinct vertices of S are non adjacent.

DEFINITION 3.22 [52]

An independent set S is said to be a maximal independent if it is not properly contain in any independent set.

DEFINITION 3.23 [52]

An independent set of maximum size is called a maximum independent set.

REMARK 3.24

Note that every maximum independent set is a maximal independent set but the converse is not true.

Also note that every maximal independent set is an independent dominating set

Also note that the complement of an independent set is a vertex covering set. Therefore a set S is a maximal independent if and only if V(G) – S is a minimal vertex covering set. Also a set S is maximum independent if and only if V(G) – S is a minimum vertex covering set.

DEFINITION 3.25 [52]

The cardinality of a maximum independent set is called the independence number of the graph G and it is denoted as $\beta_0(G)$.

Thus if a graph has n vertices then $\alpha_0(G)+\beta_0(G)=n$.

Now we prove the following theorem.

THEOREM 3.26

If G is a graph and $v \in V(G)$ then $\beta_0(G-v) \leq \beta_0(G)$.

PROOF :

Let S be a maximum independent set of $G-v$.

If v is not adjacent to any vertex of S then $S \cup \{v\}$ is an independent set in the graph G. Therefore $\beta_0(G-v) < |S \cup \{v\}| \leq \beta_0(G)$.

If v is adjacent to same vertex of S, Then S is a maximal independent set in the graph G. Therefore $\beta_0(G-v) = |S| \leq \beta_0(G)$.

Thus in all the cases $\beta_0(G-v) \leq \beta_0(G)$.■

Now, we introduce the following notations.

$I^- = \{ v \in V(G) \,/\, \beta_0(G-v) < \beta_0(G) \}$.

I^{0} = { v $\in$ V(G) / β_0(G – v) = β_0(G) }.

First we prove the following Lemma.

LEMMA 3.27

If v $\in I^{-}$ then β_0(G – v) = β_0(G) - 1 .

PROOF :

We know that β_0(G – v) < β_0(G). Let T be a maximum independent set of G, then

β_0(G – v) < | T | .

If v $\notin$ T then T is a maximum independent set of G – v , Which is not possible. Therefore v $\in$ T. Now T – v is an independent set in G – v and its size must be maximum, because v $\in I^{-}$.

Therefore β_0(G – v) < | T – v | = | T | - 1 = β_0(G) – 1. ■

THEOREM 3.28

Let G be a graph then v $\in I^0$ if and only if there is a maximum independent set S of G such that v $\notin$ S.

PROOF :

Suppose v $\in I^0$. Let S be a maximum independent set of G – v , then S is also a maximum independent set of G, because v $\in I^0$. Obviously , v $\notin$ S

Conversely , suppose there is a maximum independent set S of G such that v $\notin$ S. Then S is also a maximum independent set of G – v . Therefore v $\in I^0$.■

COROLLARY 3.29

If $v \in I^-$ if and only if v belongs to every maximum independent set of G. ■

COROLLARY 3.30

I^- is equal to the intersection of all maximum independent sets of G. ■

COROLLARY 3.31

Let G be a graph then I^- = V(G) if and only if the graph G is a null graph.

PROOF :

Suppose G is a null graph then G has only one maximum independent set. Therefore I^- = V(G).

Conversely, suppose I^- = V(G) .

Since, I^- is contained in every maximum independent set, V(G) is contained in every

maximum independent set. Therefore G has only one maximum independent set. Namely V(G). Hence G is a null Graph.■

<u>COROLLARY 3.32</u>

A graph G has at least one edge if and only if I^0 is a non empty.■

<u>COROLLARY 3.33</u>

If u and v belongs to I^- then u and v are non adjacent.

<u>PROOF :</u>

Let S be a maximum independent set of G then u and v belongs to S. Since S is an independent, u and v are non adjacent.■

<u>COROLLARY 3.34</u>

Let G be a graph and v be a vertex of G then if

$v \in I^-$ then all its neighbours are in I^0.

<u>PROOF :</u>

Suppose u is a neighbour of v. Let S be a maximum independent set of G, then $v \in S$ and $u \notin S$. Because u and v are adjacent and $v \in I^-$, and $u \notin S$. Thus by previous theorem $u \in I^0$. ■

In other words N(v) is a subset of I^0.

<u>COROLLARY 3.35</u>

For any graph G, $\delta(G) \leq |I^0|$. ■

Now we consider so called vertex transitive graphs. In these graphs there are enough automorphism.

DEFINITION 3.36 [7]

Let G be a graph then G is said to be a vertex transitive graph if for any two vertices u and v of G, there is an automorphism $f:V(G)\rightarrow V(G)$ such that $f(v) = u$.

The complete graph K_n, the cycle C_n and the Peterson graphs are some examples of vertex transitive graphs. However a tree with at least three vertices is not vertex transitive graph. In fact every vertex transitive graph is regular graph.

We now prove the following theorem.

THEOREM 3.37

Let G be a vertex transitive graph and v $\in$ V(G) such that v $\in$ I^0 then every vertex of G is a member of I^0. That is I^0 = V(G).

<u>PROOF:</u>

Since v $\in$ I^0 there is a maximum independent set S such that v does not belongs to S.

Let u be any vertex of G, then there is an automorphism f : V(G)$\rightarrow$V(G) $\ni$ f(v) = u.

Now, f(S) is a maximum independent set because f is an automorphism. Since v does not belongs to S, f(v) does not belongs to f(S). That is u does not belongs to f(S). Thus there is a maximum

independent set namely f(S) which does not contains u. There fore u $\in I^0$. ■

CORROLLARY 3.38

let G be a vertex transitive graph and v $\in$ V(G). If v $\in I^-$ then every vertex of G belongs to I^-.

That is I^- = V(G).

PROOF :

Let u be any vertex of G, then there is an automorphisum f such that f(v) = u. Since v $\in$ every maximum independent set of G, f(v) $\in$ every maximum independent set of G. That is u $\in$ every maximum independent set of G. Hence u $\in I^-$. ■

THEOREM 3.39

Let G be a vertex transitive graph then the union of all maximum independent sets of G is V(G).

PROOF :

Suppose u is a vertex, which does not belongs to any maximum independent set of G and $v \in S$. Now there is an automorphisum f of G such that f(v) = u. Since $v \in S$, $f(v) \in f(S)$ and f(S) is a maximum independent set, which contains u. This contradicts our assumption. There fore union of all maximum independent sets is V(G).■

EXAMPLE 3.40

Consider the path graph $G = P_5$. Whose vertices are 1,2,3,4,5.

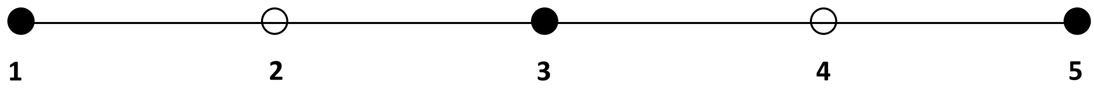


Fig.3.6: Path graph with five vertices.

This graph has only one maximum independent set S = { 1,3,5 } and the union of maximum independent set is not V(G).

This is because the path graph P_5 is not vertex transitive. In fact it is not even regular graph.

<u>EXAMPLE 3.41</u>

Consider the cycle $G = C_5$ with vertices 1,2,3,4,5.

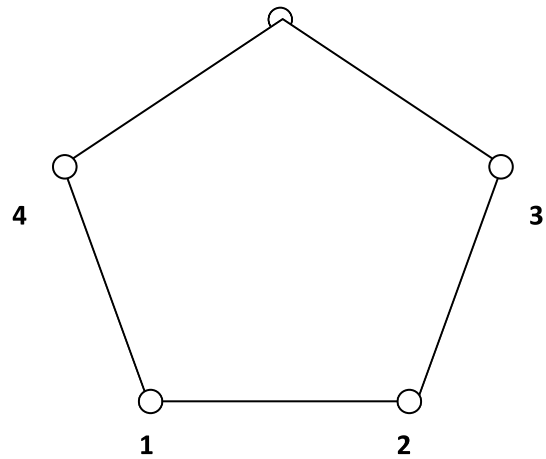


Fig.3.7: Cycle graph with five vertices.

Note that any maximum independent set of C_5 has size 2. Also note that { 1,3 }, { 2,4 }, { 3,5 }, { 4,1 }, { 2,5 } are all maximum independent sets of C_5, and the union of these sets is { 1,2,3,4,5 } = V(G).

Note that this graph is a vertex transitive graph.

Well Covered Graph Again

Here again we consider well covered graphs. We recall the following notations.

α_0(G) = The size of the smallest vertex covering set Of G.

i(G) = The independent domination number of G.

= The size of the smallest maximal independent Set.

Γ_{cr}(G) = The size of the largest minimal vertex covering set of G.

V_i^+ = { v ∈ V(G) / i(G – v) > i(G) }.

V_i^- = { v ∈ V(G) / i(G – v) < i(G) }.

V_i^0 = { v ∈ V(G) / i(G – v) = i(G) }.

Note that,

(I) **A set is a vertex covering set if and only if its compliment is an independent set.**

(II) **A set is a minimal vertex covering set if and only if its compliment is a maximal independent set.**

(III) **G is well covered if and only if i(G) = β_0(G).**

We will denote maximal independent set with minimum cardinality as an i-set of G.

Note that a graph G is well covered if and only if all maximal independent sets have the same cardinality, equivalently all independent dominating sets have same cardinality.

<u>THEOREM 3.42</u>

Let G be a graph and v $\in$ V(G).

(I) If G is well covered then V_i^+ is empty.

(II) If v $\in V_i^-$ and G is well covered then either

v $\in I^0$ or G – v is well covered.

(III) If G is well covered and v $\in V_i^0$ then G – v is

well covered and v $\in I^0$.

<u>PROOF :</u>

(I) If there is a vertex v in V_i^+ then

i(G) < i(G-v) $\leq \beta_0$(G-v) $\leq \beta_0$(G).

Since i(G) = β_0(G) , i(G-v) = β_0(G) = i(G).

Which contradicts the fact that i(G-v) > i(G).

Hence V_i^+ is empty.

(II) Now i(G-v) < i(G) = β_0(G).

Also, i(G-v) $\leq$ β_0(G-v) $\leq$ β_0(G). If β_0(G-v)$=$ β_0(G), then v $\in I^0$.

Otherwise β_0(G-v) = i(G-v). Which implies that G – v is well covered.

(III) Now, $i(G-v) = i(G) = \beta_0(G)$.

Also $i(G-v) \leq \beta_0(G-v) \leq \beta_0(G) = i(G)$.

Since $i(G-v) = i(G)$. Which implies that

$\beta_0(G-v) = i(G-v) = i(G) = \beta_0(G)$. Thus $G - v$ is well

covered and $v \in I^0$. ■

Also we consider so called an approximately well covered graphs which have been already defined earlier.

Note that a graph G is approximately well covered if and only if ,

$i(G) = \beta_0(G) - 1$. (Because $\alpha_0(G) = \Gamma_{cr}(G) - 1$ implies $n - \alpha_0(G) = n - \Gamma_{cr}(G) + 1$. That is $\beta_0(G) = i(G) + 1$.)

<u>THEOREM 3.43</u>

(I) Suppose G is approximately well covered

and $v \in V_i^+$ then $i(G-v) = i(G) + 1$, and $v \in I^0$, and

$G - v$ is well covered.

(II) If G is approximately well covered and

$v \in V_i^-$ then either β_0(G-v) = β_0(G) – 2, and

$v \in I^-$, or β_0(G-v) = β_0(G) – 1, and

$v \in I^-$, or $v \in I^0$.

(III) If G is approximately well covered and

$v \in V_i^0$ then either $v \in I^-$,and G – v is well covered or $v \in I^0$.

<u>PROOF : </u>(I)

Suppose G is approximately well covered and $v \in V_i^+$. Then i(G) < i(G – v) $\leq$ β_0(G-v) $\leq$ i(G) + 1. This implies that i(G – v) = i(G) + 1, and β_0(G-v) = i(G – v) and thus G – v is well covered and since β_0(G-v) = i(G) +1 = β_0(G), $v \in I^0$.

(II)

$i(G - v) = i(G) - 1$. $i(G) - 1 \leq \beta_0(G\text{-}v) \leq i(G) + 1$.

If $\beta_0(G\text{-}v) = i(G) + 1$. Then $\beta_0(G\text{-}v) = \beta_0(G) - 2$, and hence $v \in I^-$. If $\beta_0(G\text{-}v) = i(G) = \beta_0(G) - 1$, then $v \in I^-$. If $\beta_0(G\text{-}v) = i(G) + 1$, then $\beta_0(G\text{-}v) = \beta_0(G)$ and hence $v \in I^0$.

(III)

$i(G - v) = i(G) \leq \beta_0(G\text{-}v) \leq \beta_0(G)$.

If $\beta_0(G\text{-}v) = i(G - v)$, then $G - v$ is well covered and $v \in I^-$. Otherwise $\beta_0(G\text{-}v) = \beta_0(G)$ and $v \in I^0$.■

CHAPTER 4

Perfect Domination

In this chapter we consider so called perfect dominating sets. Perfect dominating sets are closely related to perfect codes which have applications in coding theory.

In this chapter we consider minimal perfect dominating sets with maximum cardinality. The cardinality of any such set is called the big perfect domination number of the graph. As we did in earlier chapter we prove necessary and sufficient conditions under which this number decreases or remains same.

DEFINITION 4.1 [51]

A subset S of V(G) is said to be a perfect dominating set if for every vertex v not in S, v is adjacent to exactly one vertex of S.

Note that every perfect dominating set is a dominating set.

DEFINITION 4.2

A perfect dominating set S is said to be a minimal perfect dominating set if for every vertex v in S, S – v is not a perfect dominating set.

DEFINITION 4.3 [51]

A perfect dominating set with smallest cardinality is called a minimum perfect dominating set. It is

also called γ_{pr}-set of G. The cardinality of a γ_{pr}-set is called a perfect domination number of the graph G and is denoted as γ_{pr}(G).

EXAMPLE 4.4

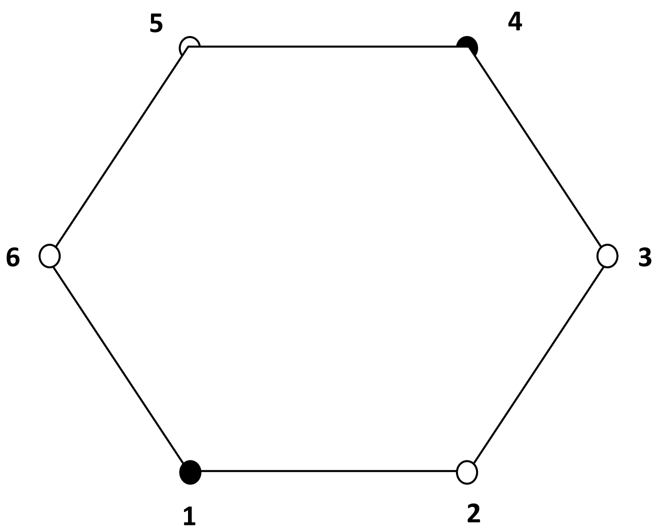


Fig.4.1: Cycle graph with six vertices.

Let G be the graph C_6, with vertices 1,2,3,4,5,6. Let S = { 1,4 } then S is a minimal perfect dominating set and in fact it is a minimum perfect dominating

set, and perfect domination number of G is 2. That is $\gamma_{pr}(G) = 2$.

DEFINITION 4.5

Let G be a graph, S be a subset of V(G) and $v \in S$, then the perfect private neighbourhood of v with respect to S is pprn(v,s) = $\{ w \in V(G) \,/\, w$ does not belongs to S and $n[w] \cap S = \{v\} \} \cup \{v\}$. If v is adjacent to no vertex of S or at least two vertices of S.

EXAMPLE 4.6

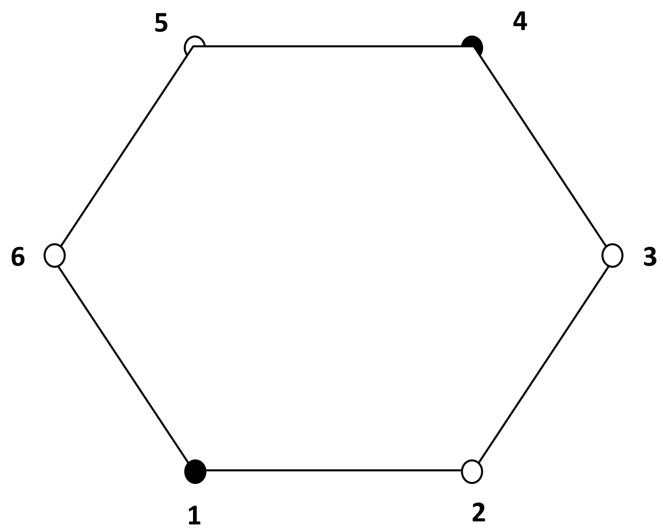


Fig.4.2: Cycle graph with six vertices.

Consider the graph C_6 , $S = \{1,4\}$ and $v = 1$,

Then $pprn(1,S) = \{ 1,2,6 \}$

THEOREM 4.7

A perfect dominating set S is a minimal perfect dominating set if and only if for every $v \in S$ $pprn(v,S)$ is non empty.

PROOF:

Suppose S is a minimal perfect dominating set and $v \in S$. Now $S - v$ is not a perfect dominating set. There fore there is a vertex w not in $S - v$ which is either adjacent to at least two vertices of $S - v$ or is adjacent to no vertex of $S - v$.

If $w \neq v$ and w is adjacent to at least two vertices of $S - v$ then $N[w] \cap S$ contains at

least two vertices. Thus w does not belongs to S and w is adjacent to at least two vertices of S. Which is a contradiction.

Thus w is adjacent to no vertices of S – v . since w is not in S , and S is a perfect dominating set, w must be adjacent to v only in S. That is $N[w] \cap S = \{v\}$. Hence $w \in$ pprn(v,S).

If w = v and if w is adjacent to at least two vertices of S – v, then $w = v \in$ pprn(v,S).

If w = v and w is non adjacent to any vertex of S – v then it means that v is not adjacent to any vertex of S. Thus $w = v \in$ pprn(v,S).

Thus in all cases pprn(v,S) is non empty.

<u>CONVERSELY</u>

Suppose pprn(v,S) is non empty for every v in S.

Let w be a vertex in pprn(v,S). If w = v then w is not adjacent to any vertex of S. Thus w does not belongs to S – v and w is not adjacent to S – v . if w = v and w is adjacent to at least two vertices of S then w does not belongs to S – v and w is adjacent to at least two vertices of S – v .

If w $\neq$ v then w does not belongs to S. Since w $\in$ pprn(v,S), N[w] $\cap$ S = {v}. Thus w is not adjacent to any vertex of S – v . Thus in all cases either w is adjacent to no vertex of S – v or adjacent to at least two vertices of S – v . Hence S – v is not a perfect dominating set. Hence S is a minimal perfect dominating set.■

Now we introduce the following notations.

V^{+}_{pr} **= { v $\in$ V(G) / γ_{pr}(G - v) > γ_{pr}(G) }.**

V^{-}_{pr} **= { v $\in$ V(G) / γ_{pr}(G - v) < γ_{pr}(G) }.**

V_{pr}^{0} **= { v $\in$ V(G) / γ_{pr}(G - v) = γ_{pr}(G) }.**

We note that the above three sets are mutually disjoint and their union is V(G).

DEFINITION 4.8

A minimal perfect dominating set with higest cardinality is called Γ_{pr} – set. The number of elements of such a set is called the big perfect domination number of G, and is denoted as Γ_{pr}(G).

LEMMA 4.9

Let G be a graph and v $\in$ V(G), then

Γ_{pr}(G – v) $\leq$ Γ_{pr}(G).

PROOF:

Let S be a Γ_{pr} – set of G – v . if v is adjacent to exactly one vertex w of S then S is a minimal perfect dominating set of G.

There fore $\Gamma_{pr}(G) \geq |S| = \Gamma_{pr}(G - v)$.

If v is adjacent to no vertex of S or is adjacent to at least two vertices of S then $S_1 = S \cup \{v\}$ is a minimal perfect dominating set of G. There fore $\Gamma_{pr}(G) > |S_1| > |S| = \Gamma_{pr}(G - v)$.

Thus $\Gamma_{pr}(G - v) < \Gamma_{pr}(G)$. ■

Now we define the following notations.

W_{pr}^{-} **= $\{ v \in V(G) / \Gamma_{pr}(G - v) < \Gamma_{pr}(G) \}$.**

W_{pr}^{0} **= $\{ v \in V(G) / \Gamma_{pr}(G - v) = \Gamma_{pr}(G) \}$.**

Note that the above sets are disjoint and their union is the vertex set V(G).

THEOREM 4.10

Let G be a graph and $v \in V(G)$ then $v \in W_{pr}^{0}$ if and only if there is a Γ_{pr} – set S of G not containing v

and a vertex w in S such that pprn(w,S) contains at least two vertices and one of them is v.

PROOF:

Suppose v $\in W_{pr}^{0}$. Let S be a Γ_{pr} – set of G – v.

Claim : v is adjacent to exactly one vertex of S.

Proof of the claim: If v is adjacent to no vertex of S or at least two vertices of S then $S_1 = S \cup \{v\}$ is a minimal perfect dominating set of G and hence $\Gamma_{pr}(G – v) < \Gamma_{pr}(G)$. Which contradicts our assumption then v $\in W_{pr}^{0}$. Thus v is adjacent to exactly one vertex of S.

Let w be the only vertex of S to which v is adjacent. There fore v $\in$ pprn(w,S). Also S is a minimal perfect dominating set of G – v. There fore pprn(w,S) contains a vertex v' of G – v.

Thus pprn(w,S) contains at least two vertices and one of them is v.

Converse: Let S be a Γ_{pr} – set of G not containing v such that for some vertex w in S, pprn(w,S) contains at least two vertices and one of them is v. Thus pprn(w,S) contains a vertex of $G - v$, also for any other vertex p of S, pprn(p,S) contains a vertex v' of G. This vertex v' cannot be equal to v, because otherwise v would be adjacent to two distinct vertices w and p of S. Which contradicts that S is a perfect dominating set in G.

Thus v' is different from v. Thus for every point z of S pprn(z,S) is non empty in $G - v$. Hence S is a minimal perfect dominating set of $G - v$. There fore $\Gamma_{pr}(G - v) \geq |S| = \Gamma_{pr}(G)$. But it is impossible that $\Gamma_{pr}(G - v) > \Gamma_{pr}(G)$, because of Lemma 4.9. There fore

$\Gamma_{pr}(G - v) = \Gamma_{pr}(G)$. Hence $v \in W^0_{pr}$. ■

COROLLARY 4.11

Let G be a graph and $v \in V(G)$ then $v \in W^-_{pr}$ if and only if for every Γ_{pr} – set S of G either $v \in S$ or there is a unique vertex w in S such that pprn(w,S) is equal to v.

PROOF:

Suppose $v \in W^-_{pr}$ then v does not belongs to W^0_{pr}.

Let S be a Γ_{pr} – set of G. If $v \in S$ then the corollary is proved.

Suppose v does not belongs to S. Let w be the unique vertex of S which is adjacent to v(S is a perfect dominating set in G). Then $v \in$ pprn(w,S). If there is another vertex $w \neq v$ such that $v' \in$ pprn(w,S) then it means that pprn(w,S)

contains at least two vertices and one of them is v. This implies that v $\in W_{pr}^{0}$ by above theorem and we have a contradiction. Thus pprn(w,S) = {v}.

CONVERSE

Suppose v $\in W_{pr}^{0}$ then by above theorem there is a Γ_{pr} – set S of G not containing v and a vertex w of S such that pprn(w,S) contains at least two vertices and one of them is v. This contradicts our assumption, and hence v $\in W_{pr}^{-}$.■

EXAMPLE 4.12

Consider the cycle C_5 with vertices 1,2,3,4,5. Note that the vertex set V(C_5) it self is a perfect dominating set. Also if we remove any vertex i from the graph the remaining set is not a perfect dominating set. There fore V(C_5) is a minimal

perfect dominating set of the graph C_5. In fact $\Gamma_{pr}(C_5) = 5$.

If we remove any vertex i from C_5. The remaining graph is a path graph with four vertices and its big perfect dominating number is 2. There fore $\Gamma_{pr}(C_5 - i) = 2$. Thus every vertex belongs to W^{-}_{pr} .

REMARKS 4.13

It may be noted that a set S is a minimal dominating set if and only if S – v is not a dominating set for every vertex v in V(G) if and only if no proper subset of S is a dominating set.

However for perfect domination the situation is not exactly similar. That is we cannot say that if a set S is a minimal perfect dominating

set then no proper subset of S is a perfect dominating set.

For example consider the cycle C_5 with vertex set { 1,2,3,4,5 } = V(G). Then V(G) is a minimal perfect dominating set. However the set S_1 = { 1,2,3 } which is a proper subset of S is also a minimal perfect dominating set of C_5.

Next we prove the following lemma.

LEMMA 4.14

If S_1 and S_2 are minimal perfect dominating sets of G which are disjoint then $|S_1| = |S_2|$.

PROOF:

Every vertex v of S_1 is adjacent to a unique vertex v' of S_2. Also every vertex v' of S_2 is adjacent to a unique vertex u of S_1. Since S_1 and S_2 are perfect dominating sets, and $v \neq u$ if and only if

v' $\neq$ u'. There fore $|S_1| = |S_2|$.■

THEOREM 4.15

Let G be a graph for which $\gamma_{pr}(G) < \Gamma_{pr}(G)$. If S is a γ_{pr}-set of G and T is a Γ_{pr}-set of G then $S \cap T$ is non empty.

PROOF:

Note that $|S| = |T|$. If S and T are disjoint then since they are minimal perfect dominating sets. Their cardinality will be same if they are disjoint. Hence the theorem.■

DEFINITION 4.16

Let G be a graph and S be a proper subset of V(G) then S is said to be a maximal perfect dominating set if for every vertex v not in S, $S \cup \{v\}$ is not a perfect dominating set.

THEOREM 4.17

Let G be a graph and S be a proper subset of V(G) and S is a perfect dominating set, then S is a maximal perfect dominating set if and only if it contains all pendent vertices of the graph G.

PROOF:

Suppose S is a maximal perfect dominating set and suppose that there is some pendent vertex v of G such that v does not belongs to S. Then it is easily verified that $S \cup \{v\}$ is a perfect dominating set. Which is a contradiction. Thus $v \in S$.

CONVERSE

Suppose S is not a maximal perfect dominating set. Then there is some vertex v does not belongs to S such that $S \cup \{v\}$ is a perfect dominating set.

Claim v is a pendent vertex of G.

<u>Proof of the claim:</u> If v is not a pendent vertex of G, then let w_1 and w_2 be two neighbours of v.

If w_1 and w_2 belongs to S then we have a contradiction because S is a perfect dominating set.

When either $w_1 \in S$ or $w_2 \in S$. Suppose $w_1 \in S$ and w_2 does not belongs to S. Now $S \cup \{w_2\}$ is a perfect dominating set(by assumption). However v is adjacent to two distinct vertices w_1 and w_2 of $S \cup \{w_2\}$. This is a contradiction. Thus v must be a pendent vertex of G.

Thus we have proved that if S is not maximal perfect dominating set then there is a pendent vertex out side of S. ■

REMARK 4.18

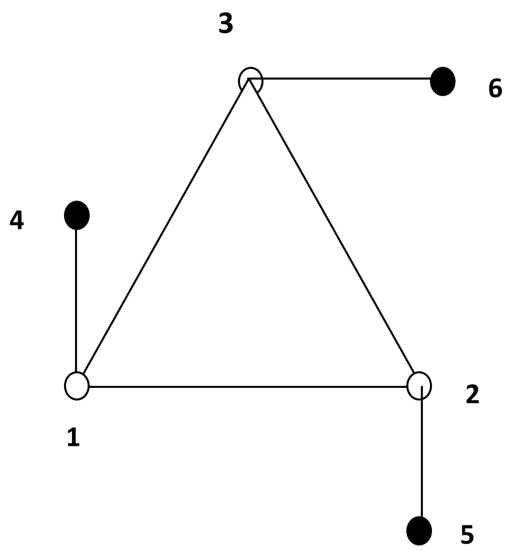
It is usual to expect that a minimal set is not a maximal set and vice versa.

However this does not happen in the case of perfect domination.

EXAMPLE 4.19

Consider the below graph G.

Fig.2.3:



Then the set S = { 4,5,6 } is a maximal perfect dominating set, because it contains all pendent vertices, and also it is a minimal perfect dominating set.

THEOREM 4.20

Let G be a graph which has no pendent vertices then every minimal perfect dominating set of G is a maximal perfect dominating set.

PROOF:

A minimal perfect dominating set contains the set of all pendent vertices (because it is empty) and there fore by above theorem it is a maximal perfect dominating set.■

Maximum Packing

Let G be a graph and , u and v be two vertices of G. then the distance between u and v, denoted as d(u,v), is the length of the shortest path in G joining u and v. If there is no path joining u and v. We write $d(u,v) = \infty$ and we accept that

$d(u,v) > k$, for all positive integer k.

DEFINITION 4.21[51]

A subset S of V(G) is said to be packing of G if $d(u,v) > 2$, for all distinct vertices u and v of S.

REMARK 4.22

It may be easily verified that a subset S of V(G) is a packing if and only if for every vertex $v \in V(G)$, $N[v] \cap S$ is either empty or a singleton set.

EXAMPLE 4.23

Consider the path graph $G = P_4$ with vertices 1,2,3,4. Then $S = \{1,4\}$ is a packing of G. It may be noted that no set with cardinality higer then 2 is a packing of G.

DEFINITION 4.24

A packing with largest cardinality is called a maximum packing of G. A cadinality of such a set is denoted as $\rho(G)$.

It may be noted that a subset S of V(G) is a packing if and only if for any two distinct vertices u and v of S, $N[u] \cap N[v] = \emptyset$.

We now consider a operation of removing a vertex from the graph and its effect on the number ρ . it may be noted that if $v \in V(G)$

and a and b are distinct vertices of G – v , then d(a,b) in G – v $\geq$ d(a,b) in G.

EXAMPLE 4.25

(I) Consider the following graph G.

Fig.2.4:

The set S = {4,5} is a maximum packing of this graph G. Thus ρ(G) = 2. Let v = 1 then G – v is

Fig.2.5:

In this graph $T = \{3,5,6\}$ is a maximum packing and thus $\rho(G\text{-}v) > \rho(G)$. That is $3 > 2$.

(II) Consider the cycle graph $G = C_6$ with vertices 1,2,3,4,5,6. Then the set $\{1,4\}$ is a maximum packing of G. There fore $\rho(G) = 2$. Now consider the graph $G – v$, where $v = 6$ then $G – v$ is the path graph with vertices 1,2,3,4,5. Here also the set $\{1,4\}$ is a maximum packing in $G – v$. there fore $\rho(G\text{-}v)=2$. There fore $\rho(G\text{-}v)= \rho(G)$. That is $2=2$.

(III) Consider the cycle $G = C_5$ with vertices 1,2,3,4,5. In this graph distance between any two vertices is less than or equal to 2. Thus $\rho(G)=1$. Now let $v = 5$ then $G - v$ is the path graph P_4 with vertices 1,2,3,4. Note that the

set {1,4} is maximum packing of P_4. Thus $\rho(G\text{-}v) > \rho(G)$. that is 2 >1.

(IV) Consider the path graph $G = P_4$ with vertices 1,2,3,4. Then $\rho(G) = 2$. If we remove any end vertex say $v = 1$ then the resulting graph is P_3 and $\rho(P_3) = 1$. There fore $\rho(G\text{-}v) < \rho(G)$. That is $1 < 2$.

(V) Consider the path graph $G = P_7$ with vertices 1,2,3,4,5,6,7. Then $\rho(G) = 3$. If we remove any end vertex say $v = 7$, then the resulting graph is P_6 and $\rho(P_6) = 2$. There fore $\rho(G\text{-}v) < \rho(G)$. That is $2 < 3$ Consider the path graph $G = P_4$ with vertices 1,2,3,4. Then $\rho(G) = 2$. If we remove any end vertex say $v = 1$ then the resulting graph is P_3 and $\rho(P_3) = 1$. There fore $\rho(G\text{-}v) < \rho(G)$. That is $1 < 2$.

Notation: **Let $k \geq 1$ then**

$N_k(v) = \{ w \in V(G) \ni 1 \leq d(v,w) \leq k \}$.

THEOREM 4.26

Let G be a graph and $v \in V(G)$ then the following statements are equivalent.

(I) $\rho(G\text{-}v) < \rho(G)$.

(II) There is a maximum packing S in $G - v$ such that $N_2(v) \cap S = \emptyset$.

(III) Every maximum packing T contains v of G and $T - v$ is a maximum packing in $G - v$.

PROOF:

(I) Implies (II).

Let S_1 be a maximum packing of G. If v does not belongs to S_1 then s_1 is a packing of

$G - v$ also and hence $\rho(G) \leq \rho(G\text{-}v)$. Which is a contradiction to our assumption. Thus v must

belongs to S_1. Since S_1 is a packing in G, $d(v,x) > 2$ for all x in S_1 with $x \neq v$. Now let $S = S_1 - \{v\}$ then S is a maximum packing in $G - v$ and since $d(v,x) > 2$ for all x in S, and $N_2(v) \cap S = \emptyset$.

(II) Implies (I).

Let S be a maximum packing in $G - v$ such that $N_2(v) \cap S = \emptyset$. Let $S_1 = S \cup \{v\}$ then S_1 is a packing in G with $|S_1| > |S|$. There fore $\rho(G) > \rho(G\text{-}v)$.

(I) Implies (III)

Suppose there is a maximum packing T in G such that v does not belongs to T then T is a packing in $G - v$. there fore $\rho(G\text{-}v) \geq \rho(G)$. which contradicts (I). There fore every maximum packing T of G contains v. Since T is a packing in G, $T - v$ is also a packing in G, and hence a packing in $G - v$ .Since

$\rho(G-v) < \rho(G)$, $T - v$ must be a maximum packing in $G - v$.

(III) Implies (I)

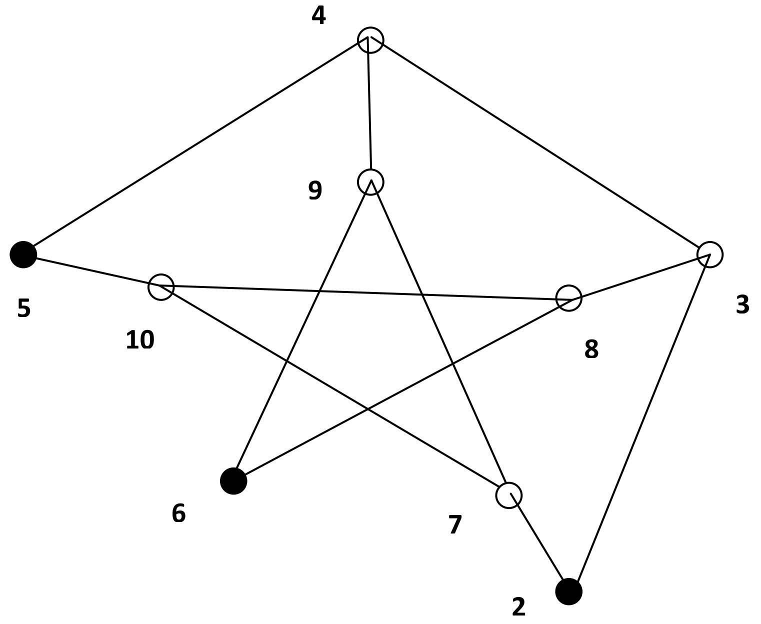
Let T be a maximum packing in G then $v \in T$ and $T - v$ is a maximum packing of $G - v$. There fore $\rho(G-v) < \rho(G)$.■

EXAMPLE 4.27

Consider the Peterson graph G with $V(G) = \{1,2,3,4,5,6,7,8,9,10\}$. It may be noted that the distance between any two non-adjacent vertices is 2. There fore a set with at least two element cannot be a packing. Thus the packing number of this graph is 1.

Now consider the graph obtain by removing vertex 1.

Fig.2.6:



ρ(G) = 1. ρ(G - 1) = 3.

It may observed that this graph has a maximum packing consisting of three vertices namely S = {2,5,6}. Thus ρ(G - 1) = 3. Thus maximum packing number is increases whenever any vertex is removed from the graph.

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Symbols

Chapter-2

C_n Cycle graph with n vertices.

G – v The sub graph obtain by removing the vertex v and all edges incident to v.

P_n Path graph with n vertices.

$P_{rt}(v,S)$ Total Private neighbourhood of v with respect to a set S.

V(G) Set of all vertices in G.

W_n Wheel graph with n vertices.

V_t^i { v ∈ V(G) / G – v has an isolated vertex }.

V_t^+ $\{ v \in V(G) / \gamma_t (G - v) > \gamma_t (G) \}$.

V_t^- $\{ v \in V(G) / \gamma_t (G - v) < \gamma_t (G) \}$.

V_t^0 $\{ v \in V(G) / \gamma_t (G - v) = \gamma_t (G) \}$.

W_t^+	{ v ∈ V(G) / v $\notin V_t^i$ and Γ_t(G – v) > Γ_t(G) }.
W_t^-	{ v ∈ V(G) / v $\notin V_t^i$ and Γ_t(G – v) < Γ_t(G) }.
W_t^0	{ v ∈ V(G) / v $\notin V_t^i$ and Γ_t(G – v) = Γ_t(G) }.
| S |	Cardinality of a set S.
γ_t (G)	Total domination number of a graph G.
γ_t – set	Minimum totally dominating set in a graph.
Γ_t-set	Minimum totally dominating set with maximum cardinality.

Chapter-3

i –set **Maximum independent set with lowest cardinality.**

i(G) **The independent domination number of the graph G.**

I^- **{ v ∈ V(G) / β_0(G – v) < β_0(G) }.**

I^0 **{ v ∈ V(G) / β_0(G – v) = β_0(G) }.**

N(v) **The set of vertices adjacent to the vertex v.**

S – v **The set obtain by removing the element v from S.**

δ(G) **The minimum degree of the graph G.**

V_t^+ **{ v ∈ V(G) / i(G – v) > i(G) }.**

V_t^- **{ v ∈ V(G) / i(G – v) < i(G) }.**

V_t^0 **{ v ∈ V(G) / i(G – v) = i(G) }.**

V_{cr}^{+} **{ v ∈ V(G) / α_0(G - v) > α_0(G) }.**

V_{cr}^{-} **{ v ∈ V(G) / α_0(G - v) < α_0(G) }.**

V_{cr}^{0} **{ v ∈ V(G) / α_0(G - v) = α_0(G) }.**

W_{cr}^{+} **{ v ∈ V(G) / Γ_{cr}(G – v) > Γ_{cr} (G) }.**

W_{cr}^{-} **{ v ∈ V(G) / Γ_{cr}(G – v) < Γ_{cr} (G) }.**

W_{cr}^{0} **{ v ∈ V(G) / Γ_{cr}(G – v) = Γ_{cr} (G) }.**

γ(G) **Domination number of a graph G.**

γ_{cr} – set **Minimum vertex covering set in a graph.**

γ_{cr}(G) **Vertex covering number of a graph G.**

Γ_{cr}-set **Minimum vertex covering set with maximum cardinality.**

Γ_{cr}(G) **Big vertex covering number of a graph G.**

Γ(G)	**Upper domination number of a G.**
α_0(G)	**Vertex covering number of a graph G.**
β_0(G)	**Independence number of a graph G.**

Chapter-4

d(u,v)	**Distance between u and v in a graph.**
N[v]	**N(v) ∪ {v}.**
N_k(v)	**{ w ∈ V(G) ∋ 1 ≤ d(u,w) ≤ k }.**
pprn(v,S)	**{ w ∈ V(G) / w does not belongs to S and n[w] ∩ S = {v} } ∪ {v}.**
V_{pr}^{+}	**{ v ∈ V(G) / γ_{pr}(G - v) > γ_{pr}(G) }.**
V_{pr}^{-}	**{ v ∈ V(G) / γ_{pr}(G - v) < γ_{pr}(G) }.**
V_{pr}^{0}	**{ v ∈ V(G) / γ_{pr}(G - v) = γ_{pr}(G) }.**
W_{pr}^{-}	**{ v ∈ V(G) / Γ_{pr}(G – v) < Γ_{pr}(G) }.**
W_{pr}^{0}	**{ v ∈ V(G) / Γ_{pr}(G – v) = Γ_{pr}(G) }.**

γ_{pr}-set	**Minimum perfect dominating set of a graph G.**
$\gamma_{pr}(G)$	**Perfect domination number of a graph G.**
Γ_{pr}-set	**Minimum perfect dominating set with highest cardinality.**
$\Gamma_{pr}(G)$	**Big perfect domination number of a graph G.**
$\rho(G)$	**The size of a maximum packing.**
	= The packing number of G.