GRAPH CRITICAL WITH RESPECT TO VARIANTS OF DOMINATION

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In

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<u>CERTIFICATE</u>

I certify that the thesis entitled **" GRAPH CRITICAL WITH RESPECT TO VARIANTS OF DOMINATION "** submitted to the Saurashtra University, Rajkot, by **Mr. JEEGNESH C. BOSAMIYA** for the award of **DOCTOR OF PHYLOSOPHY** degree in **MATHEMATICS** embodies the original work of the candidate himself, and the candidate has worked under my supervision during the year 2008-2011.

Date:

Place:

(Dr. D. K. Thakkar)

Guide

(Statement under O. Ph. D.7 of Saurashtra University-Rajkot)

D E C L A R A T I O N

I hereby declare that

(a) the research work embodied in this thesis on **" Graph Critical With Respect To Variants Of Domination"** submitted for Ph. D. degree has not been submitted for my other degree of this or any other university on any previous occasion.

(b) to the best of my knowledge no work of this type has been reported on the above subject. Since I have discovered new relations of facts, this work can be considered to be contributory of the advancement of knowledge on Graph Theory.

(c) all the work presented in the thesis is original and whenever references have been made to the work of others, it has been clearly indicated as such.

Signature of the Student

Date:

Acknowledgement

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(Jeegnesh C. Bosamiya)

CONTENTS

Chapter – 0

INTRODUCTION

Graph Theory is a branch of Mathematics which has become quite rich and interesting for several reasons. In last three decades hundreds of research article have been published in Graph Theory. There are several areas of Graph Theory which have received good attention from mathematicians. Some of these areas are Coloring of Graphs, Matching Theory, Domination Theory, Labeling of Graphs and areas related to Algebraic Graph Theory.

We found that the Theory of Domination in Graphs deserves further attention. Thus, we explode this area for our research work.

The present dissertation is a study of some variants of domination in graphs from a particular point of view. In fact we consider the numbers associated with these variants. More explicitly the study is about the change in these numbers when a vertex is removed from the graph. To do this we consider vertices having different effect when they are removed. Our purpose is also to characterize these vertices using so called "Minimum Sets".

The dissertation consists of five chapters incorporating the aspects describe above.

Now we give brief description of individual chapters of the dissertation.

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Chapter-0: Introduction.

This chapter provides an introduction to domination and its variants. Some fundamental results regarding domination and its variants have been given in this chapter. Some historical background of domination has also been given in this chapter.

The mathematical study of Domination Theory in graphs started around 1960. Its roots go back to 1862 when C.F. De Jaenisch studied the problem of determining the minimum number of queens necessary to cover an $n \times n$ chess board in such way that every square is attacked by one of the queens.

Domination Theory studied to solve basically three types of problem which are described as follows

(1) **Covering-** what is the minimum number of chess pieces of a given type which are necessary to cover/attack/dominate every square of an $n \times n$ board?. This is an example of the problem of finding a dominating set of minimum cardinality.

(2) **Independent Covering-** what is the minimum number of mutually non attacking chess pieces of a given type which are necessary to dominate every square of $n \times n$ board?. This is an example of the problem of finding a minimum cardinality of independent dominating set.

(3) **Independence Number-** what is the maximum number of chess pieces of a given type which can be placed on an $n \times n$ chess board in such a way that no two of them attack/dominate each other?. This is an example of the problem of finding the maximum cardinality of an independent set. When the chess piece is the queen, this problems known as the N-queen problem. It is known that for every positive integer $n \geq 4$, it is possible to place n non attacking (independent) queen on an $n \times n$ board.

For over a hundred years people have studied ways of doing this.
These problems were studied in detail by two brothers A. M. Yaglom and
I. M. Yaglom around 1964.
They have derived solutions of some of these kinds of problems for rooks, knights, kings and bishops.

Chapter-0: Introduction

C. Berge wrote a book on Graph Theory in which he defined the concept of the domination number in 1958.

He called this number the coefficient of external stability. Actually the names "Dominating Set" and " Domination Number " published in 1962. He used the notation d(G) for the domination number of a graph.

The notation $\gamma(G)$ was first used by E. J. Cockayne and S. T. Hedetniemi for the domination number of a graph which subsequently became the accepted notation.

We also give a brief description of the notations to be used in this dissertation.

Some conventions will be declared in individual chapter.

Chapter-1: Extended Total Domination.

Chapter-1 is about extended total domination in graphs. We define the concept of extended total domination and extended total domination number for any graph. We characterize those vertices whose removal increases, decreases of does not change the extended total domination number of a given graph. As consequence we prove that if there is a vertex whose removal increases the extended total domination number then there are at least two vertices such that removal of each one of them does not change the extended total domination number.

Chapter-2: Independent Domination and Vertex Covering.

Chapter-2 contains the concepts of independent domination and vertex covering of a graph.

First we have consider independent domination and characterized those vertices whose removal increases or decreases the independent domination number of a graph. It may be noted that every maximal independent dominating set is an independent set and vice-versa.

We have also considered vertex covering sets and vertex covering number of the graph. We prove that the vertex covering number of graph never increases when a vertex is removed from the graph. In particular we have proved that if a graph has at least one edge then it has vertices whose removal decreases the vertex covering number of the graph. Also we have proved that if a graph is vertex transitive and have at least one edge then removal of any vertex decreases the vertex covering number. Moreover we prove that if a graph is vertex transitive then the intersection of all minimum vertex covering sets is empty. As a consequence we prove that a vertex transitive graph with even number of vertices is bipartite if and only if it has exactly two minimum vertex covering sets.

Chapter-3: Total k – Domination and k - touple Domination, k-dependent k-dominating set.

In chapter-3, we consider total k-domination, k-tuple domination and k-dependent k-domination. Here also we define this concept and characterize those vertices whose removal increases or decreases the total k-domination number and k-touple domination number and k-dependent k-domination number.

Chapter-4: Perfect Domination.

In chapter-4, we consider the perfect domination and prove similar results.

DOMINATION

Here we give an introduction of basic concepts domination and total domination. Diagrams have been provided whenever they are required. Proofs have been given and omitted. This chapter also provides some notations and conventions.

Let G be a graph and S be a subset of the vertex set V(G) of G.

Definition-0.1 Dominating set:[44]

A subset S of V(G) is said to be dominating set if for every vertex v in V(G)-S, there is a vertex u in S such that u is adjacent to v.

That is a vertex v of G is in S or is adjacent to some vertex of S.

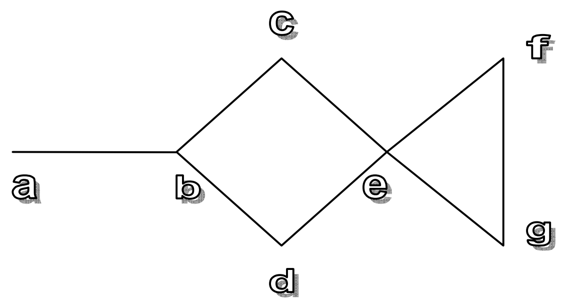


Figure-0.1

For instance the vertex set {b, g} is a dominating set in this Graph of Figure-0.1 The set {a, b ,c ,d , f} is a dominating set of the graph G.

For a graph G, G-{v} denote the graph obtain by removing vertex v and all edges incident to v.

Definition- 0.2: Minimal Dominating Set.[44]

A dominating set S of the graph G is said to be a minimal dominating set if for every vertex v in S, S-{v} is not a dominating set. That is no proper subset of S is a dominating set.

For example, in graph of Figure -0.1 {b, e}and{a, c, d, f} are minimal dominating sets. Every dominating set contains at least one minimal dominating set.

Definition -0.3: Minimum Dominating Set.[44]

A dominating set with least number of vertices is called minimum dominating set. It is denoted as γ set of the graph G.

Definition -0.4: Domination Number.[44]

The number of vertices in a minimum dominating set is called domination number of the graph G. It is denoted by $\gamma(G)$.

Theorem -0.5:[44] **A dominating set S of a graph G is a minimal dominating set of G if and only if every vertex v in S satisfies at least one of the following two conditions.**

(1) There exists a vertex w in V(G)-S such that $N(w) \cap S=\{v\}$

(2) v is adjacent to no vertex of S.

Proof:

First observe that if each vertex v in S has at least one of the conditions (1) and (2), then S-{v} is not a dominating set of G. Consequently, S is a minimal dominating set of G.

Conversely, assume that S is a minimal dominating set of G. Then certainly for each vertex $v \in S$, then set S-{v} not a dominating set of G. Hence there is a vertex w is no adjacent to any vertex of S-{v}.

If w = v, then v is adjacent to no vertex of S, suppose then that w ≠ v. Since S is a dominating set of G and w ∉ S, the vertex w is at least one vertex of S. However, w is adjacent to no vertex of S-{v}.Consequently N(w) ∩ S={v}. ■

Theorem- 0.6: [44]**Every graph G without isolated vertices contains a minimum dominating set S such that for every vertex v of S, there exists a vertex w of V(G) – S such that N(w) ∩ S ={v}.** ■

Examples-0.7: Now we Consider the following examples:

(1) Cycle Graph C_5 with vertices v_1, v_2, v_3, v_4, v_5 :

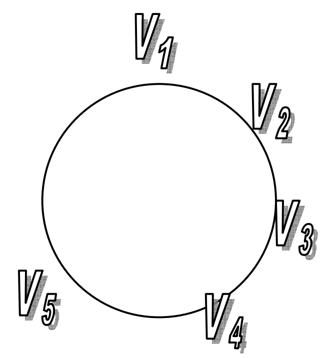


Figure-0.2

In this graph set { v_1, v_3} is minimal and minimum dominating set. Then $\gamma(C_5)$ =2.

(2) Consider the graph G = Petersen Graph :

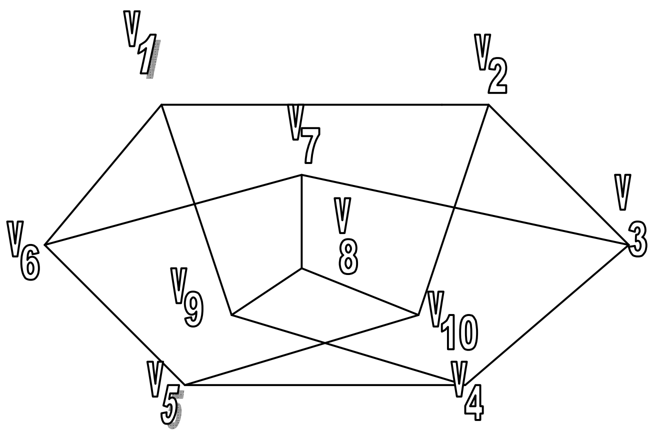


Figure-0.3

In this graph set {v_2, v_5 , v_8} is minimal and minimum dominating set. Then $\gamma(G) = 3$.

(3) P_5 : The path with five vertices v_1, v_2, v_3, v_4, v_5:

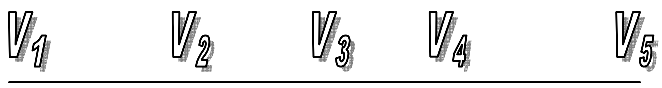


Figure-0.4

In this graph set {v_2,v_4 } is minimal and minimum dominating set. Then $\gamma(G) = 2$.

(4) Stra Graph $K_{1,8}$ with nine vertices1,2,3,4,5,6,7,8,9:

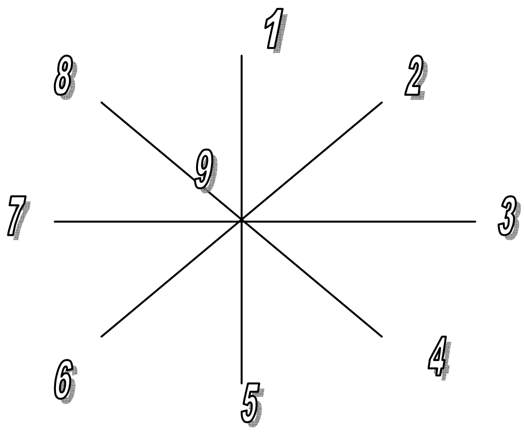


Figure-0.5

In this graph {9} is minimum dominating set and $\gamma(G) = 1$.

(5) Wheel Graph with nine vertices1, 2 ,3, 4, 5, 6,7, 8, 9:

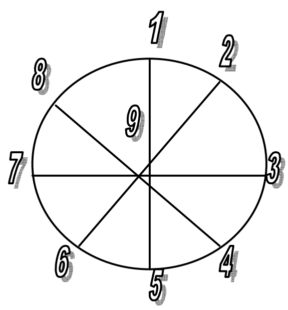


Figure-0.6

In this graph {9} is minimum dominating and $\gamma(G) = 1$.

(6) Complete Graph with K_5 vertices 1,2,3,4,5:

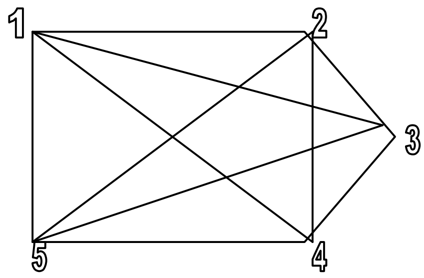


Figure-0.7

In this graph every singleton set is minimum dominating set and $\gamma(K_5) = 1$.

(7) Hyper Cube Graph with eight vertices v_1, v_2,….. v_8 :

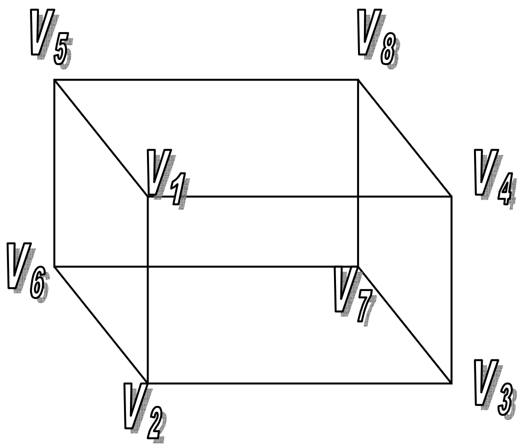


Figure-0.8

In this graph {v_1, v_7}is minimal and minimum dominating set and $\gamma(G) = 2$.

(8) Consider the Path P_n with n vertices.

Let P_n be a path with n vertices. It may be proved that the domination number of P_n is n/3 if n is divisiable by 3. And is [n/3] +1 .

We shall use the following notations. [44]

Let G be a graph and V(G) be the vertex set of the graph G.

$$\mathbf{V^0 = \{v \in V(G): \gamma(G\text{-}v) = \gamma(G)\}.}$$

$$\mathbf{V^+ = \{v \in V(G): \gamma(G\text{-}v) > \gamma(G)\}.}$$

$$\mathbf{V^- = \{v \in V(G): \gamma\ G\text{-}v) < \gamma(G)\}.}$$

Obviously $V(G) = V^- \cup V^+ \cup V^0$

Example-0.8:

(1)

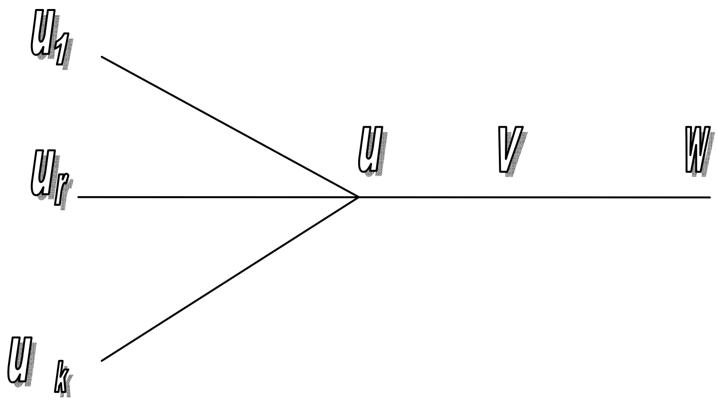


Figure -0.9

The graph in figure 2.1 with $k \geq 3$ has

$V^0 = \{ u_i : 1 \leq i \leq k\} \cup \{v\}$

$V^+ = \{u\}$

$V^- = \{w\}$

(2) Consider Star Graph with nine vertices: (see Figure-0.5)

In this graph

$V^+ = \{9\}$

$V^0 = \{1,2,3,4,5,6,7,8\}$

$V^- = \phi$.

(3) Consider the graph G =Petersen Graph: (see Figure-0.3)

In this graph

$V^0 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}\}$

$V^- = \phi$,

$V^+ = \phi$.

(4) Consider the graph G = C_5 with five vertices v_1, v_2, v_3, v_4, v_5: (see Figure-0.2)

In this graph

$V^+ = \phi$

$V^- = \phi$

$V^0 = \{v_1, v_2, v_3, v_4, v_5\}$

Remark -0.9:[44] **Note that removing a vertex from the graph G can increase the domination number by more than one, but can decrease it by at most one.**

Proof: Let S be a γ set of G. Suppose $\gamma(G-v)$ is less than $\gamma(G) - 1$.

i.e. $\gamma(G-v) < \gamma(G) - 1$.

Let S_1 be a γ set of G – {v}. So, $|S_1| < \gamma(G) - 1$. So, $S_1 \cup \{v\}$ is a dominating set in G. So $\gamma(G) \leq |S_1| \cup \{v\} = \gamma(G) - 1$.

$$\gamma(G) \leq \gamma(G) - 1.$$

This is a contradiction

$$. \text{ So, } \gamma(G\text{-}v) = \gamma(G) - 1 . \blacksquare$$

Theorem -0.10: [44]**A vertex $v \in V^+$ if and only if**

(a) v is not an isolate vertex.

(b) v is in every γ set of G.

(c) No subset $S \subseteq V(G) - N[v]$ with cardinality $\gamma(G)$ dominates $G - \{v\}$.

Proof: (a)

Suppose v is an isolate vertex in G and S is a γ set of G. Then $v \in S$. Then $S\text{-}\{v\}$ is a dominating set of $G\text{-}\{v\}$.

So,

$$\gamma(G\text{-}v) \leq |S\text{-}\{v\}| < |S| = \gamma(G)$$

So,

$$\gamma(G\text{-}v) < \gamma(G)$$

So,

$$v \in V^-$$

This is a contradiction.

($v \in V^+$ is given in hypothesis.) So, v is not an isolate vertex.

(b)

Suppose $v \notin S$ for some γ set S of G. So, $v \in V(G) - S$ and $S \subseteq G - \{v\}$. So, S is a dominating set in $G - \{v\}$.

So,

$$\gamma(G\text{-}v) \leq |S| = \gamma(G)$$

So

$$,\gamma(G\text{-}v) \leq \gamma(G)$$

So,

$$v \notin V^+$$

This is a contradiction.($v \in V^+$ is given in hypothesis.)

So, v is in every γ set of G.

(c)

Suppose (c) is not true. There is a set $S \subseteq V(G) - N[v]$ such that $|S| = \gamma(G)$ and S dominates $G - \{v\}$.

So,

$$\gamma(G\text{-}v) \leq |S| = \gamma(G)$$

So,

$$v \notin V^{+}$$

This is a contradiction. ($v \in V^{+}$ is given in hypothesis.)

Now we prove converse. i.e. we want to prove $v \in V^{+}$.

Case -1:

Suppose $\gamma(G\text{-}v) = \gamma(G)$

Let S be a minimum dominating set of $G - \{v\}$ such that $|S| = \gamma(G)$. If v is not adjacent to any vertex of S then S is subset of V(G)-N[v] with $|S| = \gamma(G)$ and S is a dominating set of G-{v} which contradicts (c).

If v is adjacent to some vertex of S then S is a minimum dominating set of G not containing v – which contradicts (b).

Thus, v can not be in V^{0}.

Suppose $v \in V^{-}$. Let S_1 be a γ set of G-{v}. Then $|S_1| = \gamma(G) - 1$. If v is adjacent to any vertex of S_1 then S_1 is a dominating set of G-{v}. Therefore $\gamma(G) \leq |S_1| = \gamma(G) - 1$,which is impossible. So, v is not adjacent to any vertex of S_1. Let $S = S_1 \cup \{v\}$. Then S is a minimum dominating set not containing v which contradicts (b).Thus, v can not be in V^{-}.Therefore $v \in V^{+}$. █

Definition- 0. 11:[44] **Private Neighborhood of v with respect to S set. i.e. Pn[v,S]**

Let S be a subset of V(G) and v∈ S. Then the private neighborhood of v with respect to S set = Pn[v,S] = {w ∈ V(G): N[w] ∩ S = {v}.}

Example-0.12:

(1) Consider the given graph:

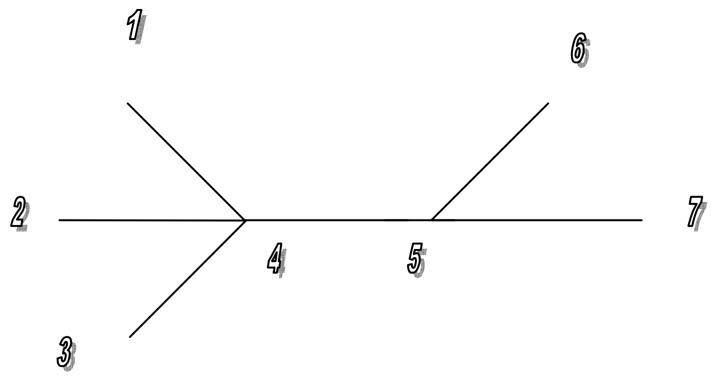


Figure -0.10

For the given graph, S is a any subset of G.

S = {4,5}

Pn[4,S] = {1,2,3}

Pn[5,S] = {6,7}

Theorem -0.13:[44] **A vertex v is in V^- if and only if Pn[v,S] = {v} for some γ set S containing v in G.**

Proof:

We are given $v \in V^-$, then $\gamma(G-v) < \gamma(G)$. So, $\gamma(G-v) = \gamma(G) - 1$.Let S be a γ set of G – {v} and $|S| = \gamma(G) - 1$ Now $v \notin S$, let $S_1 = S \cup \{v\}$, So, $|S_1| = \gamma(G)$ S_1 is a γ set of G and $v \in S_1$. Note that v can not be adjacent to any vertex of S, So, $v \in Pn[v,S_1]$.

Suppose $w \in V(G) - S_1$ and w is adjacent to only v, then w is not adjacent to any vertex of S, i.e. S is not a dominating set in G – {v}. This is a contradiction. This implies that either w is not adjacent v or w is adjacent v and some other vertex of S_1. i.e. $w \notin Pn[v, S_1]$. So, $Pn[v, S_1] = \{v\}$.

Now we prove converse.

Suppose we have γ set S containing v such that Pn[v,S] = {v}. Note that $|S| = \gamma(G)$. We prove that S – {v} is a dominating set in G – { v}. Let $w \in V(G-\{v\})-(S-\{v\})$. We have S is a γ set and w is adjacent to some vertex t of S.

Case -1:

Suppose t = v ,then w is adjacent to v. Since $w \notin Pn[v,S]$, So w must be adjacent to some vertex x of S such that $x \neq v$. So, $x \in S - \{v\}$ and x is adjacent to w.

Case- 2:

Suppose $t \neq v$ then $t \in S - \{v\}$ and t is adjacent to w which is required. So, S – {v} is a dominating set in G – {v}. So, $\gamma(G-v) < \gamma(G)$. So, $v \in V^-$ ∎

Theorem -0.14:[4]**For any graph G.**

(a) If $v \in V^+$, then for every γ set S of G, $v \in$ S and Pn[v,S] contains at least two non-adjacent vertices.

(b) If $x \in V^+$ and $y \in V^-$ then x and y are not adjacent.

(c) $|V^0| \geq 2|V^+|$

(d) $\gamma(G) \neq \gamma(G\text{-}v)$ for all $v \in V(G)$ if and only if $V(G) = V^-$.

(e) If $v \in V^-$ and v is not an isolated in G, then there exist a γ set S of G Such that $v \notin$ S.

Proof:

(a)

We know that by Theorem -0.10, each $v \in V^+$ is not an isolated vertex and is in every γ set S. If Pn[v,S] = {v}.We prove S – {v} U {u} is γ set, where $u \in N(v)$.

Let $t \in V(G) – (S - \{v\} \cup \{u\})$. If t = v then t is adjacent u.

If $t \neq v$ then it has two cases.

Case -1: t is adjacent to v,

Then t must be adjacent to some other vertices of S because Pn[v,S] = {v}. Thus, t is adjacent to some vertex of S – {v}. Hence t is adjacent to some vertex of S–{v}U {u}.

Case -2: t is not adjacent to v.

Then t is adjacent to some vertex of S – {v}. So, t is adjacent to some vertex of S – {v} U {u}. So, S – {v} U {u} is a dominating set. So, | S – {v} U {u} | = |S|.

S – {v} U {u} is a γ set of G. Thus, we have γ set which does not contain v. This is a contradiction. So, Pn[v,S] $\neq$ {v}. i.e. Pn[v,S] is contains at least two vertices.

Now we prove these two vertices are non adjacent.

Suppose u_1 and u_2 are adjacent vertices in Pn[v,S]. If u_1 and u_2 are adjacent, then S– {v}U{u_1} or S – {v} U {u_2} is a γ set not contain v. This is a contradiction.
u_1 and u_2 must be non adjacent.

(b)

We have $x \in V^+$ and $y \in V^-$. By Theorem-0.13 If $y \in V^-$ there is γ set S containing y such that Pn[y,S] = {y}. Now $x \in V^+$ .So x is in every γ set of G. Thus $x \in$ S. Since Pn[y,S] = {y}. i.e. y can not be adjacent to any vertex of S. So, $x \in V^+$ and $y \in V^-$ then x and y are not adjacent.

(c)

Let $v \in V^+$. There is a γ set S such that $v \in$ S and Pn[v,S] contains at least two non adjacent vertices u_1 and u_2. Note that $u_1 \notin$ S, $u_2 \notin$ S. So, u_1 and u_2 are not belongs to S. So, u_1 and u_2 are not belongs to V^+.If $u_1 \in V^-$, then v and u_1 must be non adjacent.
(by above result-b) which is not true. So, u_1 and u_2 are not belongs to V^-. So,
u_1 ,u_2 $\in V^0$. Thus, for every vertex in V^+, we get two distinct vertices in V^0.So,
$| V^0 | \geq 2| V^+ |$.

(d)

Suppose we have $V(G) = V^-$, then $\gamma(G) \neq \gamma(G\text{-}v)$.
Now we prove converse part.

If $\gamma(G) \neq \gamma(G\text{-}v)$ for all $v \in V(G)$ then V^0 must be empty. So, V^+ is also empty because $| V^0 | \geq 2| V^+|$ then all vertices are in V^- .So, $V(G) = V^-$.

(e)

Let S_0 be γ set containing v such that Pn[v,S_0] ={v}. Since v is not an isolated vertex there is a vertex $w \in V(G) - S_0$ such that w is adjacent to v. Let
$S = S_0 - \{v\} \cup \{w\}$ then S is a γ set of G which does not contain v, i.e. $v \notin$ S. $\blacksquare$

Result -0.15:[44] **Let G be a graph for every vertex $v \in V(G)$ and $\gamma(G\text{-}v) \neq \gamma(G)$ then $V(G) = V^-$. i.e. V^0 and V^+ are empty set.**

Proof :

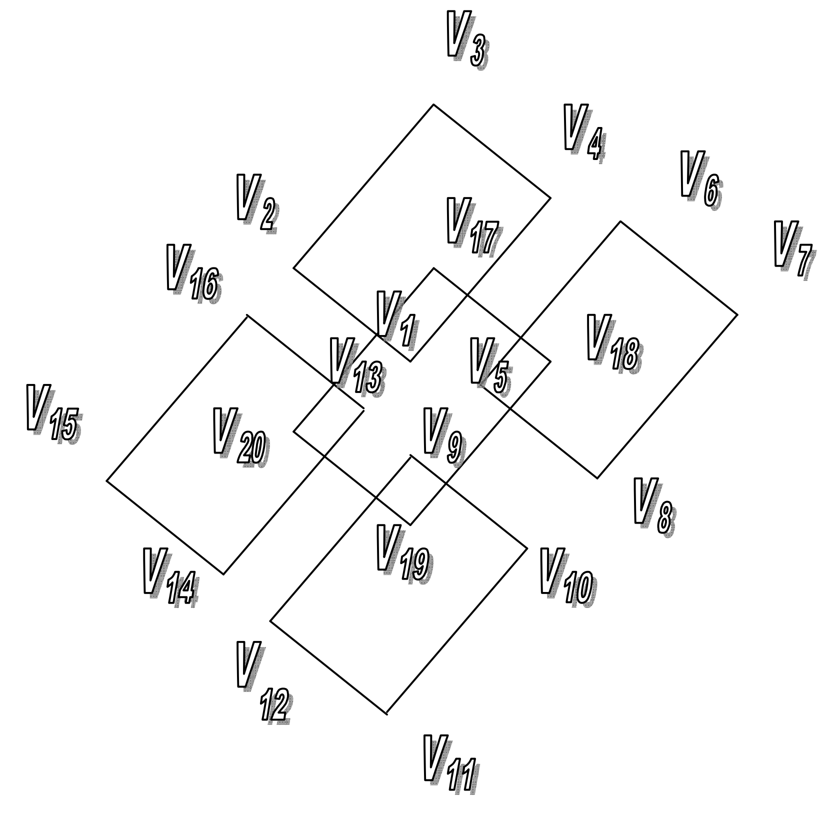
Suppose V^+ is not empty then V^0 is also non empty. i.e. then there is a vertex v such that $\gamma(G) = \gamma(G\text{-}v)$. This is a contradiction. So, V^+ is empty. ……(1)

Now if V^0 is non empty then there is vertex $w \in V^0$ such that $\gamma(G\text{-}w)=\gamma(G)$.This is a contradiction. So, V^0 is also empty. …(2)

So. By 1 and 2 ,$V(G) = V^-$.■

Result-0.16: The following graph which has all vertices are in V^0. Thus, it may happen that V^+ is the empty set.

We have proved that V^+ is empty set for a vertex transitive graph.



Fogure-0.11

γ -set = {$v_1, v_3, v_5, v_7, v_9, v_{11}, v_{13}, v_{15}, v_{17}, v_{19}$} and = $\gamma(G) = 10$.

Corollary-0.17:[44] **For a graph G, $V(G) = V^-$ if and only if for each vertex $v \in V(G)$. $P_n[v,S]=\{v\}$ for some γ set S containing v.**

Proof :

Let $V(G) = V^-$ i.e. $v \in V^-$. Then by an earlier Theorem-0.13for every vertex if $v \in V^-$ then $P_n[v,S] =\{v\}$ for some γ set S containing v.

Now we prove converse.

Suppose for every vertex v, let $P_n[v,S]=\{v\}$ for some γ set S containing v. Then (by an earlier Theorem-0.13) $V=V^-$. So, $v \in V^-$ for every vertex $v \in V(G)$. So $V=V^-$. So, $V(G) = V^-$.█

Theorem-0.18:[44] **If a graph G has a non isolated vertex v such that the sub graph induced by N(v) is complete, then $V(G) \neq V^-$.**

Proof:

Let v be a vertex which is non isolated and N(v) is complete.

Suppose $v \in V^-$ then there is a γ set S such that $P_n[v,S]=\{v\}$. Let w be a vertex adjacent to v, then $w \notin S$. Let T be a γ set containing w if $v \in T$, then $w \notin P_n[w,T]$ (because of w is adjacent to v, $v \in T$). If $v \notin T$ then either $v \in P_n[w,T]$ or $P_n[w,T] =\phi$. So, $w \notin V^-$. Thus, we have obtained a vertex $w \notin V^-$. So, $V(G) \neq V^-$. █

Next we consider the concept of total domination which will be used in the next chapter. The concept of total domination is stronger than domination. Also the total domination number of a graph is, in general, bigger than the domination number.

It will be observed that a graph having an isolated vertex can not have a totally dominating set.

TOTAL DOMINATION

Definition -0.19: Totally Dominating set. [2]

A set $T \subset V(G)$ is said to be a totally dominating set if for every vertex $v \in V(G)$, v is adjacent to some vertex of T.

Note that a graph with an isolated vertex can not have a totally dominating set and we assume that a totally dominating set has at least two vertices.

Definition -0.20: Minimal Totally Dominating set.[2]

A totally dominating set S of G is said to be a minimal totally dominating set if S-{v} is not a totally dominating set for every vertex v in S.

Definition -0.21 : Minimum Totally Dominating set.[2]

A totally dominating set with least number of vertices is called minimum totally dominating set. It is called a γ_T set of graph G.

Definition -0.22: Total Domination number. [2]

The number of vertices in a minimum totally dominating set is called total domination number of the graph G and it is denoted by $\gamma_T(G)$.

Example-0.23:

(1) Consider the G =Petersen Graph :(See Figure-0.3)

For Petersen graph minimum totally dominating set is $\{v_2, v_5, v_8, v_{10}\}$

and $\gamma_T(G) = 4$

(2) Consider the graph G = C_9 with nine vertices $v_1, v_2, \ldots v_9$:

For this graph minimum totally dominating set is $\{v_2, v_3, v_6, v_7, v_9\}$ and $\gamma_T(G) = 5$.

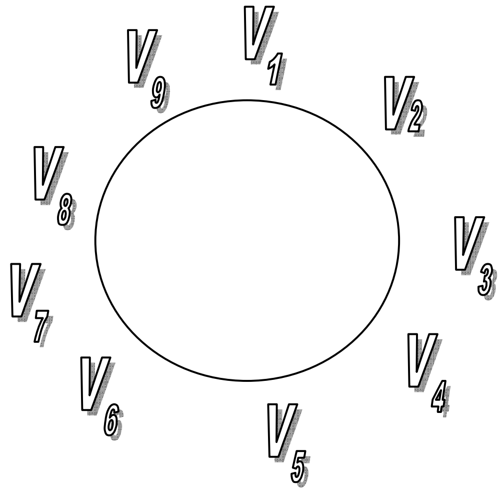


Figure-0.12

(3) Consider the G =Petersen Graph :(See Figure-0.3)

For Petersen graph minimum totally dominating set is $\{v_2, v_5, v_8, v_{10}\}$ and $\gamma_T(G) = 4$

(4) Consider the Wheel Graph with nine vertices 1, 2, 3, 4, 5, 6, 7, 8, 9: (See Figure-0.6)

For wheel graph minimum totally dominating set is {1, 9} and $\gamma_T(G) = 2$.

(5) Consider the Star Graph with nine vertices 1, 2, 3, 4, 5, 6, 7, 8, 9:

(See Figure-0.5)

For wheel graph minimum totally dominating set is {1, 9} and $\gamma_T(G) = 2$.

(6)Consider the Path Graph G = P_6 with six vertices $v_1, v_2, v_3, v_4, v_5, v_6$:

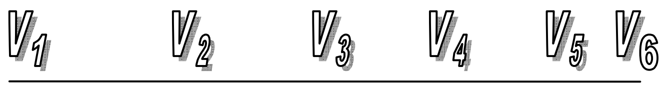


Figure-0.13

For Path graph P_6 minimum totally dominating set is $\{v_2, v_3, v_4, v_5\}$ and $\gamma_T(G) = 4$.

(7) Consider the following graph

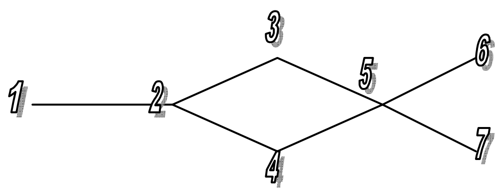


Figure-0.14

For this graph minimum totally dominating set is {2, 3, 5} and $\gamma_T(G) = 3$.

Definition-0.24: V_T^0, V_T^+, V_T^-, V_T^i .

$$V^0_T = \{ v \in V(G) : \gamma_T(G-v) = \gamma_T(G) \}.$$

$$V^+_T = \{ v \in V(G) : \gamma_T(G-v) > \gamma_T(G) \}.$$

$$V^-_T = \{ v \in V(G): \gamma_T(G-v) < \gamma_T(G) \}.$$

$$V^i_T = \{ v \in V(G) : G - \{v\} \text{ has an isolated vertices} \}.$$

Example-0.25:

(1) Consider the graph G = C_9 with nine vertices $v_1, v_2, v_3, v_4, v_5, v_6\ v_7, v_8, v_9$: (See Figure-0.11)

$V^0_T = \{ v_1, v_2, v_3, v_4, v_5, v_6\ v_7, v_8, v_9 \}$

$V_T^+ = \phi$,

$V_T^- = \phi$,

$V_T^i = \phi$.

(2) Consider the graph G = Petersen Graph : (See Figure-0.3)

$V^0_T = \{ v_1, v_3, v_4, v_6\ v_7, v_9 \}$

$V^+_T = \{ v_2, v_5, v_8, v_{10} \}$

$V^-_T = \phi$

$V^i_T = \phi$

(3) Consider graph W_9 Wheel Graph with vertices 1, 2, 3, 4, 5, 6, 7, 8, 9: (See Figure-0.6)

$V^0_T = \{1,2,3,4,5,6,7,8\}$

$V^+_T = \{9\}$

$V^-_T = \phi$

$V^i_T = \phi$

(4) Consider graph $K_{1,8}$ Star Graph with vertices 1, 2, 3, 4, 5, 6, 7, 8, 9: (See Figure-0.5)

V^0_T = {1,2,3,4,5,6,7,8}

V^+_T = ϕ

V^-_T = ϕ

V^i_T = {9}

(5) Consider Path Graph P_6 Graph with vertices v_1, v_2, v_3, v_4, v_5, v_6 : (See Figure-0.13)

V^0_T = { v_3, v_4}

V^+_T = ϕ

V^-_T = { v_1, v_6 }

V^i_T = { v_2, v_5 }

(6) Consider the following graph (See Figure-0.14)

V^0_T = {1,3,4,5,7}

V^+_T= ϕ

V^-_T = ϕ

V^i_T = {2,5}

(7) Consider the following graph.

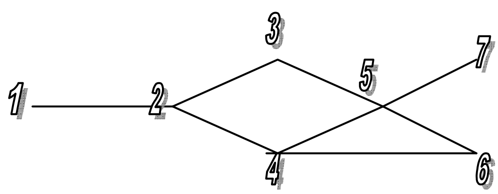


Figure-0.15

γ_T = {2,3,5}, {2,4,5} ,{2,5,6}, (2,5,7) all vertices are in V^0_T. So,

$V(G) \in V^0_T$.

Definition -0.26: Totally Private Neighborhood.[2]

Let $S \subset V(G)$ and $v \in S$ then total private neighborhood of v with respect to S is $T_{pn}[v,S] = \{w \in V(G) : N(w) \cap S = \{v\}\}$.

Example-0.27:

(1) Consider the Cycle Graph G = C_9 with nine vertices v_1, v_2, v_3, v_4, v_5, v_6 v_7, v_8, v_9 : (See Figure-0.12)

For cycle graph minimum totally dominating set is {2, 3, 6, 7, 9}.

$T_{pn}[2,T] = \{v_3\}$

$T_{pn}[3,T] = \{v_3, v_4\}$

$T_{pn}[6,T] = \{v_5, v_7\}$

$T_{pn}[7,T] = \{v_6\}$

$T_{pn}[9,T] = \phi$

(2) Consider the graph G = Petersen Graph : (See Figure-0.3)

For Petersen graph minimum totally dominating set is $\{v_2, v_5, v_8, v_{10}\}$

$T_{pn}[2,T] = \{v_1, v_3\}$

$T_{pn}[5,T] = \{v_4, v_6\}$

$T_{pn}[8,T] = \{v_7, v_9\}$

$T_{pn}[10,T] = \{v_2, v_5, v_8\}$

(3) Consider the Wheel Graph G = W_9 with nine vertices 1, 2, 3, 4, 5, 6, 7, 8, 9 : (See Figure-0.6)

For wheel graph minimum totally dominating set is {1, 9}.

$T_{pn}[1,T] = \{9\}$

$T_{pn}[9,T] = \{1,3,4,5,6,7\}$

(4) Consider Star Graph $K_{1,8}$ with nine vertices 1, 2, 3, 4, 5, 6, 7, 8, 9 : (See Figure-0.5)

For wheel graph minimum totally dominating set is {1, 9} .

$T_{pn}[1,T] = \{9\}$

$T_{pn}[9,T] = \{1,2,3,4,5,6,7,8\}$

(5) Consider the Path Graph G = P_6 with six vertices $v_1, v_2, v_3, v_4, v_5, v_6$: (See Figure-0.13)

For Path graph G = P_6 , minimum totally dominating set is $\{ v_2, v_3, v_4, v_5 \}$

$T_{pn}[2,T] = \{ v_1\}$

$T_{pn}[3,T] = \{ v_2\}$

$T_{pn}[4,T] = \{v_5\}$

$T_{pn}[5,T] = \{ v_6\}$

(6) Consider the following graph . (See Figure-0.14)

For this graph minimum totally dominating set is {2, 3, 5}.

$T_{pn}[2,T] = \{1\}$

$T_{pn}[3,T] = \{2,5\}$

$T_{pn}[5,T] = \{6,7\}$

Theorem-0.28:[2] **A Totally dominating set T of a graph G is a minimal totally dominating set of G if and if there exist a vertex w in V(G) such that N(w) ∩ T = {v}.i.e. A Totally dominating set T is minimal totally dominating set if and only if for every vertex v in T, T_{pn}[v,T] ≠ ϕ.**

Proof :

First we assume T is a minimal totally dominating set.

To prove: For every vertex v in T there exist a vertex w in V(G) such N(w)∩T = {v}.

Let v ∈ T. Now we know that T is a minimal totally dominating set. So, T-{v} is not a totally dominating set. So, there exist a vertex w ∈ V(G) such that w is not adjacent to any vertex of T-{v}.But we have T is totally dominating set in G. So, w is adjacent to some vertex of T. So, w is adjacent to only v in T. So, N(w) ∩ T = {v}.

Now we prove converse.

We assume that for every vertex v in T there exist a vertex w in V(G) such that N(w) ∩ T = {v}.

To prove: T is a minimal totally dominating set.

Let v ∈ T, now there exist a vertex w in V(G) which is adjacent to only v in T.So, w is not adjacent to any vertex of T-{v}.So T-{v} is not a minimal totally dominating set. So, T is a minimal totally dominating set.■

Example-0.29: Note that removing a vertex can increase the total domination number more than one but will decrease at most one.

Proof:

Wheel Graph with 11 vertices is example for removing a vertex can increase the total domination number more than one.

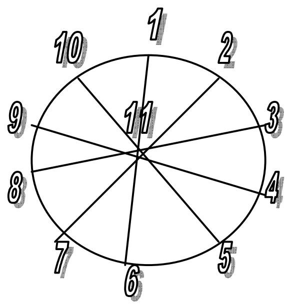


Figure-0.16

For the wheel graph with eleven vertices γ_T set is {1,11} and $\gamma_T(G)$: 2. Now for the graph G –{11} γ_T set is {2,3,6,7,9,10} and γ_T (G-11) : 6.

Theorem-0.30: [2] **If $v \in V^-_T$ then $\gamma_T(G-v) = \gamma_T(G) - 1$.**

Proof :

let T be a γ_T set of G. Suppose $\gamma_T(G-v) < \gamma_T(G) - 1$. Let T_1 be a γ_T set of G-{v}. So, $| T_1 | < \gamma_T(G) - 1$.

Option -1: Suppose v is adjacent to some vertex of T_1, then T_1 is totally dom inating set in G.

So, $\gamma_T(G) \leq | T_1 | \leq \gamma_T(G-v)$.

So,

$$\gamma_T(G) \leq \ | \ T_1 \ | \leq \gamma_T(G) - 1.$$

So,

$$\gamma_T(G) \leq \gamma_T(G) - 1.$$

This is a contradiction.

Option 2 :

If v is not an isolated vertex so, v is adjacent to some vertex w of G – {v}. We have T_1 is totally dominating set in G – {v}. Let T = T_1 U {w} and we want to prove T is a totally dominating set in G. For this we have to show that every vertex of G is adjacent to some vertex of T = T_1 U {w}, where $w \notin T_1$, i.e. w is adjacent to v then T = T_1 U {w} is totally dominating set in G.

So,

$$\gamma_T(G) \leq | \ T \ | \leq \gamma_T(G) - 1.$$

So,

$$\gamma_T(G) \leq \gamma_T(G) - 1.$$

This is a contradiction and we have

$$\gamma_T(G) < \gamma_T(G) - 1.$$

So,

$$\gamma_T(G) = \gamma_T(G) - 1. \blacksquare$$

Theorem-0.31:[2] **A vertex $v \in V^{+}_{T}$ if and only**

(a) v is not an isolated vertex.

(b) v is in every γ_T set of G.

(c) No subset T $\subseteq$ V(G) – N[v] with cardinality of T = γ_T (G) can totally dominate G-{v}.

Proof :

First we prove (a)

We are given $v \in V^{+}_{T}$ and T is a totally dominating set. So , by definition of totally dominating set it has no any isolated vertex.

Now we prove (b)

We prove v is in every γ_T set. i.e. all minimum totally dominating set contains v.

Now we assume v is in every γ_T set is not true. Therefore some γ_T set in T such that v does not belongs to T. i.e. $v \in G - T$. Let $w \in G - \{v\}$, and we have T is a totally dominating set in G. So, w is adjacent to some vertex of T. So, T is totally dominating set in $G - \{v\}$.

So,

$$\gamma_T(G\text{-}v) \leq |T| = \gamma_T(G).$$

So,

$$\gamma_T(G\text{-}v) \leq \gamma_T(G).$$

This is a contradiction. (because $v \in V_T^+$)

So, v is in every γ_T set.

(c)

Suppose we have subset $T \subseteq V(G) - N[v]$ with cardinality of T = γ_T (G) can totally dominate $G - \{v\}$.

Let $w \in G - \{v\}$ and T is a totally dominating set in G. So, w is adjacent to some vertex of T. So, T is a totally dominating set in $G - \{v\}$.

$$\gamma_T(G\text{-}v) \leq |T| = \gamma_T(G).$$

$$\gamma_T(G\text{-}v) \leq \gamma_T(G).$$

This is a contradiction (because $v \in V_T^+$)

So, no subset $T \subseteq V(G) - N[v]$ with cardinality of T = γ_T (G) can totally dominate $G\text{-}\{v\}$.

Now, we prove converse .

Case-1: $v \in V^0{}_T$ i.e. $\gamma_T(G\text{-}v) = \gamma_T(G)$.

Let T be a minimum totally dominating set in G – {v}.

Case -(a):

v is adjacent to some vertex of T. i.e. T is a minimum totally dominating set in G which does not contain v. So, condition (b) is violated. So, our assumption is wrong.

Case- (b):

v is not adjacent to any vertex of T.

i.e. $T \subseteq V(G) - N[v]$ and $| T | = \gamma_T(G\text{-}v) = \gamma_T(G)$. So, $| T | = \gamma_T(G)$

So, T is a totally dominating set in G – {v}. This violate condition (c). So, our assumption is wrong. Thus, v does not belong to $V_T{}^0$. So, $v \in V_T{}^+$.

Case -2 : $v \in V^-{}_T$.

$$\text{i.e. } \gamma_T(G\text{-}v) < \gamma_T(G)$$

So,

$$\gamma_T(G\text{-}v) = \gamma_T(G) - 1.$$

Let T be a γ_T set in G – {v}. Let $w \in G - \{v\}$ which is adjacent to v, i.e. w is adjacent v. Let $T_1 = T \cup \{w\}$ is a γ_T set in G but T_1 does not contains v. So, condition (b) is violated. So, our assumption is wrong. So, $v \in V^-{}_T$ is not possible. So, $v \in V_T{}^+$.

Now by case 1 and 2 we prove that $v \in V^0{}_T$ and $v \in V^-{}_T$ is not possible. So, $v \in V^+{}_T$ ∎

Theorem-0.32:[2] **$v \in V^-_T$ if and only if there is a γ_T set T such that $v \notin T$ and $v \in Tpn[w,T]$, for some $w \in T$.**

Proof :

Suppose $v \in V^-_T$. i.e. $\gamma_T(G\text{-}v) = \gamma_T(G) - 1$.

Let T be a minimum totally dominating set in G – {v}, then obviously T can not be a totally dominating set in G. Let w be a vertex not in T which is adjacent to v.
Let $T_1 = T \cup \{w\}$, then v is adjacent to only one vertex w of T_1. So, T_1 is a γ_T set of G. So, $v \in Tpn[w,T_1]$. Suppose $z \in Tpn[w,T_1]$ then $z \neq v$ implies that z is not adjacent any vertex of T. So, z is a vertex of G – {v} which is not adjacent to any vertex of T. This contradicts the fact that T is a totally dominating set in G – {v}.Thus, if $z \neq v$ then $z \notin Tpn[w,T_1]$.

Now we prove converse.

Conversely suppose there is a γ_T set S of G such that $v \notin S$ and $v \in Tpn[w, S]$ for some w in S.

Let T = S – {w} then v is not adjacent to any vertex of T. Also $z \neq v$, then z is not adjacent to at least one vertex of S different from w. So, z is adjacent to some vertex of T. Thus, S – {w} which is equal to T is a totally dominating set in G – {v}.

So,

$$\gamma_T(G\text{-}v) \leq |S - \{w\}| = |T| < |S| = \gamma_T(G)$$

So,

$$\gamma_T(G\text{-}v) < \gamma_T(G)$$

So,

$v \in V_T^-$. ■

Theorem-0.33[2]: **If $v \in V^+_T$ then for every γ_T set T, $v \in T$ and one of the following condition is satisties.**

(1) If $T_{pn}[v,T] = \{w\}$, then $w \notin T$.

(2) If $w_1, w_2 \in T_{pn}[v,T]$ and both are in T then they are non adjacent.

Proof :

(1)

Let T is γ_T set of G and $v \in V_T^+$ and we have $T_{pn}[v,T] = \{w\}$.We prove $w \notin T$.

Suppose $w \in T$.

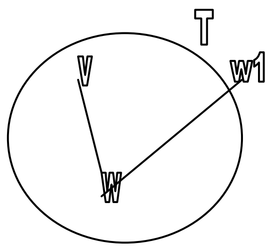


Figure-0.17

Let w^1 is any vertex is adjacent to w and $w^1 \notin T$. Let $T_1 = T - \{v\} \cup \{w\}$, but w is adjacent w^1 .So, all vertices of G are adjacent to some vertex of T_1 and $|T_1| = |T| = \gamma_T(G)$. So, T_1 is γ_T set of G but $v \notin T_1$.This is a contradiction .So, $w \notin T$.

(2)

Let T is a γ_T set and $w_1, w_2 \in T_{pn}[v,T]$. $w_1, w_2 \in T$.

We prove: w_1, w_2 are non adjacent.

We assume w_1 is adjacent to w_2.

Now, w_1 is adjacent to v and w_1 is adjacent w_2, it is not possible.

(By the definition of $T_{pn}[v,T]$.) So, w_1 and w_2 are in T then they are non adjacent. $\blacksquare$

Chapter:-1

EXTENDED TOTAL DOMINATION

It is clear from the definition of domination that a dominating set exists in any graph. However it is not true for total domination. A totally dominating set does not exist in a graph having isolated vertices. Moreover when a vertex is removed from the graph the resulting graph may have the isolated vertices. Considering this fact it is desirable to have the concept which is approximately same as total domination and it can be define for any graph. To accomplish this we introduce the concept of so called extended total domination.

In this chapter we introduce the concept of extended total domination and relevant concepts. In particular we define minimum extended totally dominating set and extended total domination number. This chapter is devoted to characterize those vertices whose removal increases, decreases or does not change the extended total domination number of a graph. We prove that if the extended total domination number changes whenever any vertex is removed then it decreases when any vertex is removed.

We may mention that a totally dominating set is assumed to have at least two vertices and all our graphs are simple.

In this chapter I will denote the set of all isolated vertices of G.

Definition-1.1: Totally Dominating Set.[2]

Let G be graph and S be a set of vertices. Then S is said to be a totally dominating set if every v in V(G) is adjacent to some vertex of S.

We introduced the following definition.

Definition-1.2: Extended Totally Dominating Set.

A set S is said to be a extended totally dominating set if $S = S_1 \cup I$, where S_1 is totally dominating set in G – I.

Definition-1.3: Minimal Extended Totally Dominating Set.

An extended totally dominating set S is said to be minimal extended totally dominating set if every $v \in S$ then S- {v} is not an extended totally dominating set.

Definition-1.4: Total Private Neighborhood.

Let $S \subset V(G)$. Then the total private neighborhood of v with respect to S is $T_{pn}[v,S] = \{w \in V(G) : N(w) \cap S = \{v\}.\}$

Definition-1.5: Minimum Extended Totally Dominating Set.

An extended totally dominating set with smallest cardinality is said to be minimum extended totally dominating set. It is denoted by γ_{Te} set of graph the G.

Definition-1.6: Extended Total Domination Number.

The number of vertices in a minimum extended totally dominating set is called extended total domination number of the graph G. It is denoted by $\gamma_{Te}(G)$.

Example-1.7 We Give an example of graph whose domination number and extended total domination number are different.

(1)

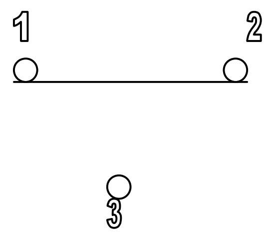


Figure-1.1

γ set = {2,3} and $\gamma(G) = 2$

$\gamma_{Te} = \{1,2,3\}$ and $\gamma_{Te}(G) = 3$.

(2)

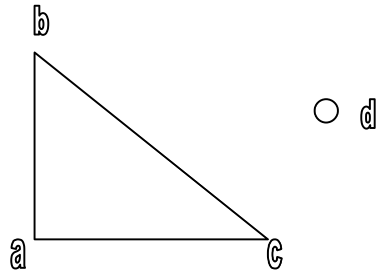


Figure-1.2

γ_{Te} set = {a, c, d} , $\gamma_{Te}(G) = 3$

γ set = {a, d} $\gamma(G) = 2$.

Note: It may be noted that a set S is an extended totally dominating set if and only if every vertex v of G is either isolated or adjacent to some vertex of S.

We characterizes those vertices whose removal increases, decreases or does not change the extended total domination number of the graph. For this purpose we will define three types of sets as follows. V^+_{Te} , V^-_{Te} , and V^0_{Te} .

$$\mathbf{V^+_{Te} = \{v \in V(G): \gamma_{Te}(G-v) > \gamma_{Te}(G)\}.}$$

$$\mathbf{V^-_{Te} = \{v \in V(G): \gamma_{Te}(G-v) < \gamma_{Te}(G)\}.}$$

$$\mathbf{V^0_{Te} = \{v \in V(G): \gamma_{Te}(G-v) = \gamma_{Te}(G)\}.}$$

Obviously all the three sets are mutually disjoint and their union is V(G).

First we characterize minimal extended totally dominating sets in the following theorem.

Theorem -1.8: An extended totally dominating set S of graph G is minimal extended totally dominating set if and only if every vertex v in S satisfies only one of the following two conditions.

(1) $T_{pn}[v,S] \neq \phi$.

(2) v is an isolated vertex of G.

Proof:

We are given S is minimal extended totally dominating set of G. Suppose $v \in S$, if v is an isolated vertex then second condition will be satisfied.

Suppose v is not an isolated vertex. Now $S_1 = S - I$,(because $S = S_1 \cup I$), and S_1 is minimal totally dominating set in G – I. So, $S_1 - \{v\}$ is not a totally dominating set in G – I. So, there is at least one vertex w which is not adjacent to any vertex of $S_1 - \{v\}$, where $w \in G$. i.e. suppose w = v, then v is not adjacent to any vertex of $S_1 - \{v\}$. This contradicts fact that S_1 is a totally dominating set in G – I. So, $w \neq v$.

We know that S_1 is totally dominating set in G – I. So, w is adjacent to some vertex of S_1 and we also know that w is not adjacent to any vertex of $S_1 - \{v\}$.So, w is adjacent to only v. So, $N(w) \cap S = \{v\}$.

Now we prove converse.

Let $v \in S$. If v is an isolated vertex then $S - \{v\}$is not an extended totally dominating set in G .

Now let v is not an isolated vertex. So, there is only one vertex w is adjacent to only v in S_1 and w is not adjacent to any vertex of $S_1 - \{v\}$.So, $S_1 - \{v\}$ is not a totally dominating set in G – I. Hence S-{v} is not an extended totally dominating set of G. $\blacksquare$

Lemma -1.9: Suppose v is a vertex of G such that $\gamma_{Te}(G\text{-}v) < \gamma_{Te}(G)$. Then $\gamma_{Te}(G\text{-}v) = \gamma_{Te}(G) - 1$.

Proof: Suppose $v \in V^{-}_{Te}$. If v is an isolated vertex of G then v belongs to every γ_{Te} set of G. So, suppose v is not an isolated in G.

Let S_1 be a minimum extended totally dominating set of G-{v}.

Suppose v is adjacent to some vertex of S_1.Then w must be unique because if w_1 is any other vertex of S_1 such that v is adjacent to w_1 then T = S_1-{w}U{v} is an extended totally dominating set of G with $|T| < \gamma_{Te}(G)$ – a contradiction. Thus, w is unique. Let S = S_1 U {v}. Then S is a minimum extended totally dominating set of G. Therefore $\gamma_{Te}(G) = |S| = |S_1 + 1| = \gamma_{Te}(G\text{-}v) + 1$.Thus, $\gamma_{Te}(G\text{-}v) = \gamma_{Te}(G)-1$.

Suppose v is not adjacent to any vertex of S_1. Let w be a vertex of G which is adjacent to v. Let S = S_1U{w}. Then S is a minimum extended totally dominating set of G. Thus, $\gamma_{Te}(G\text{-}v) = \gamma_{Te}(G)-1$. █

Now we prove necessary and sufficient conditions under which the extended total domination number increases when a vertex v is removed. Note that these conditions are similar to those for domination.(Theorem– 0.31)

Theorem- 1.10: $v \in V^{+}_{Te}$ if and only if the following three conditions are satisfies.

(1) v is not an isolated vertex of G.

(2) v is in every minimum extended totally dominating set of G.

(3) There is no set S which satisfies any one of the following two conditions.

(a)S is a minimum extended totally dominating set of G-{v} with $|S| \leq \gamma_{Te}(G)$ such that $N[v] \cap S$ is an empty set.

(b) S is a minimum extended totally dominating set of G-{v} with $|S| \leq \gamma_{Te}(G)$ and there is a neighbor of v in S which is an isolated vertex in G-{v}.

Proof: (1)

We assume v is an isolated vertex of G. Let S be an extended totally dominating set in G, then $v \in S$. Let w be any vertex of G –{v}.

If w is an isolated vertex in G-{v} then w is an isolated also in G.(because v is an isolated.). Hence $w \in S-\{v\}$.

If w is not an isolated vertex in G-{v} then w is adjacent to some vertex t of S. Since v is an isolated vertex and $t \neq v$. Thus, w is adjacent to some vertex of S-{v}. Thus, S-{v} is an extended totally dominating set in G-{v}. Hence $\gamma_{Te}(G-v) < \gamma_{Te}(G)$. This is a contradiction.

(2)

Suppose there a γ_{Te} set T of G such that $v \notin T$. Now we prove that T is an extended totally dominating set in G – {v}.Now, $T = T_1 \cup I$ where I is the set of isolated vertices of G. Since $v \notin T$. Now, we prove that T_1 is totally dominating set of G- {v}.Let w be any vertex of G – {v} which is not an isolated vertex. Since T is an extended totally dominating set of G. So, w is adjacent to some vertex z of T. Since $v \notin T$, and $z \neq v$. So, w is adjacent to z, for some z in T_1. Hence T is an extended totally dominating set in G – {v}. Thus, $\gamma_{Te}(G-v) \leq |T| = \gamma_{Te}(G)$, a contradiction (Because we are given $v \in V^{+}_{Te}$). So, v must be in every γ_{Te} set T of G.

(3)

Suppose there is a subset S of the graph G-{v} with $|S| \leq \gamma_{Te}(G)$ and suppose S satisfies either (a) or (b). Then $\gamma_{Te}(G-v) \leq |S| \leq \gamma_{Te}(G)$. Thus, $v \notin V^{+}_{Te}$. This is a contradiction. Thus, S can not satisfies (a) or (b).

Now we prove converse.

Conversely assume conditions (1), (2) and (3) hold for the graph G.

Suppose $v \in V^0_{Te}$. Let S_1 be a minimum extended totally dominating set of $G-\{v\}$. Then $|S_1| \leq \gamma_{Te}(G)$. If v is not adjacent to any vertex of S_1 then $N[v] \cap S_1 = \phi$ This is not possible.

Let w be any vertex of S_1 which is adjacent to v. By 3-(b), w must be non isolated in $G-\{v\}$. Since S_1 is an extended totally dominating set of $G-\{v\}$, w must be adjacent to some vertex of S_1. Thus, S_1 is a minimum extended totally dominating set of G not containing v which contradicts condition (2).

Suppose $v \in V^-_{Te}$. Let S_1 be minimum extended totally dominating set of $G-\{v\}$. Then $|S_1| = \gamma_{Te}(G)-1$. Suppose v is not adjacent to any vertex of S_1. Let w be neighbor of v (which is not in S_1). Let $S = S_1 U \{w\}$. Then S is a minimum extended totally dominating set of G not containing v – which contradicts (2).

If v is adjacent to some vertex z of S_1 then z must be non isolated in $G-\{v\}$ and therefore z be adjacent to some vertex of S_1. This is true for any such vertex z of S_1 which is adjacent to v.

Thus, S_1 is an extended totally dominating set of G with $|S_1| \leq \gamma_{Te}(G)$. This is again a contradiction.

Thus, v does not belongs to V^-_{Te} or V^0_{Te}. Hence $v \in V^+_{Te}$. This complete the theorem. $\blacksquare$

Example-1.11: Consider the path graph $G = P_5$ with vertices v_1, v_2, v_3, v_4, v_5 : (See Figure-0.4)

Note that $\gamma_{Te}(G) = 3$, and $S = \{ v_2, v_3, v_4 \}$ is the unique γ_{Te} set of G. Also note that $v_3 \in V^+_{Te}$ and $\gamma_{Te}(G- v_3) = 4$.

Consider the graph $G-\{ v_4\}$. Note that $\gamma_{Te}(G- v_4) =3$. The sets $S_1 =\{v_1, v_2, v_5\}$ and $S_2 =\{ v_2, v_3, v_5\}$ are γ_{Te} sets of $G-\{v_4\}$. Also note that there is a neighbor of v_4 ((namely v_5) which is an isolated vertex of $G-\{ v_4\}$.Also note that $|S_1| = |S_2| = \gamma_{Te}(G)$.

Theorem -1.12: Let G be a graph and v be a vertex of G. v $\in$ V^{-}_{Te} if and only if one of the following conditions is satisfied.

(1) v is an isolated vertex of G and v is in every γ_{Te} set.

(2) There is γ_{Te} set S not containing v and a vertex w in S such that $T_{pn}[w,S]$ ={v}.

Proof:

Suppose v $\in$ V^{-}_{Te} . If v is an isolated vertex then v is in every γ_{Te} set S.

Suppose v is not an isolated vertex in G. Since v $\in V^{-}_{Te}$, then there is a γ_{Te} set S of G-{v} with |S| = $\gamma_{Te}(G) - 1$.If v is adjacent to some vertex w of S then w must be unique. Because if w_1 is any other vertex of S adjacent to v then (S-{w_1}) $\cup$ {v} is an extended totally dominating set of G with cardinality less than $\gamma_{Te}(G)$. This is a contradiction. Thus, w is unique. Also w is not adjacent to any other vertex of S because otherwise S would be an extended totally dominating set in G with cardinality less than $\gamma_{Te}(G)$. Let S_1 = S $\cup$ {v} then S_1 is a γ_{Te} set of G and $T_{pn}[w,S_1]$ contains v. Since v is not an isolated vertex in G then there is a vertex w in G-{v} which is adjacent to v. Note that w can not be an isolated vertex in G-{v}, because w $\notin$ S and S is an extended totally dominating set of G-{v}. Let w_1 be vertex of S which is adjacent to w. Let S_1=S U {w}, then S_1 is a γ_{Te} set in G. Since v is not adjacent to any vertex of S and v $\in$ $T_{pn}[w,S_1]$.

Thus, in both the cases we have proved that there is a vertex w in S_1 such that $T_{pn}[w,S_1]$ contains v.

Let t be a vertex of G where t $\neq$ v. If t is an isolated vertex of G-{v}then t can not be adjacent to w because w $\neq$ v. If t is not an isolated in G – {v}then t is adjacent to some vertex z of S. If t is adjacent w implies that t is adjacent to two distinct vertices of S_1. i.e. t does not belongs to $T_{pn}[w,S_1]$. So, $T_{pn}[w,S_1]$ = {v}.

Now we prove converse.

Let S be an extended totally dominating set of G such that condition (1) or (2) holds.

If v is an isolated vertex of G then S – {v} is an extended totally dominating set of G – {v} and we have $v \in V^{-}_{Te}$.

If v is not an isolated vertex of G then there is a vertex w in S such that $T_{pn}[w,S] = \{v\}$. We prove that S – {w} is an extended totally dominating set of G – {v}.

Let x be any vertex of G-{v}.If x is an isolated vertex in G – {v} and also isolated in G then $x \in S$ and obviously $x \in S$-{v}.

If x is an isolated in G – {v}but not isolated in G then since S is an extended totally dominating set in G, x must be adjacent to some vertex z of S. Since $x \notin T_{pn}[w,S]$, we may assume that $z \neq w$, thus x is adjacent to some vertex of S-{w}.Hence S-{w} is an extended totally dominating set of G-{v}.This implies that $v \in V^{-}_{Te}$.

If x is not an isolated vertex in G-{v} then x is not an isolated in G also and since S is an extended totally dominating set in G, there is a vertex z in S different from w which is adjacent to x. Thus, x is adjacent to some vertex of S-{w}.This proves that S-{w} is an extended totally dominating set of G-{v}.

Hence $v \in V^{-}_{Te}$. ■

Theorem-1.13: Let G be graph and v be a vertex which belongs to V^{+}_{Te}, then for any γ_{Te} set S, v $\in$ S and T_{pn}[v,S] contains at least two vertices.

Proof:

First we prove that if $T_{pn}[v,S] = \{w\}$then w $\notin$ S.

Suppose w $\in$ S, then w is not adjacent to any other vertex of S. Suppose also that w is not adjacent to any vertex outside S, then w is an isolated vertex in G-{v}. Now we prove that S_1 = S-{v} is an extended totally dominating set in G- {v}.For this let z be any vertex of G – {v}.If z is an isolated in G then z $\in$ S-{v}. Suppose z is an isolated in G-{v} but is not isolated in G. Since S is an extended totally dominating set in G, z is adjacent to some vertex t of S. Since z $\notin$ T_{pn}[v,S], we may assume that t $\neq$ v. Thus, z is adjacent to at least two vertices of S which is a contradiction. Therefore this possibility does not arise.

If z is not an isolated in G-{v} then z is adjacent to some vertex of S different from v. Therefore z is adjacent to some vertex of S-{v}. Therefore S_1 = S-{v} is an extended totally dominating set in G- {v}. So, $\gamma_{Te}(G-v) < \gamma_{Te}(G)$. So, v $\in$ V^{-}_{Te} , This is a contradiction. So, w must be adjacent to some vertex w_1 outside S. Now w_1 is adjacent to some vertex w_{11} of S. Now let S_1= S-{v}U{w_1}.Then S_1 is a γ_{Te} set of G not containing v. This contradict the fact that v $\in$ V^{+}_{Te}. Thus, w can not be in S. So, w$\in$V{G}-S.

Now suppose w is an isolated in G – {v} then S_1= S-{v}U{w} is an extended totally dominating set in G-{v}. So, $\gamma_{Te}(G-v) \leq \gamma_{Te}(G)$. So, v $\notin$ V^{+}_{Te}. This is a contradiction. Thus, w can not be an isolated vertex in G-{v}. So, there is a vertex z $\notin$ S such that z is adjacent to w. Now, z is adjacent to z_1 of S. Let S_1= S-{v}U{z} is a minimum extended totally dominating set in G not containing v. This contradicts the fact that v $\in$ V^{+}_{Te}. Thus, if we assume that T_{pn}[v,S] = {w} then we have a contradiction. Thus, T_{pn}[v,S] must contain at least two vertices..█

Remark-1.14:

The above Theorem 1.13 implies that if $v \in V^{+}_{Te}$ then $d(v) \geq 2$. Thus, any vertex of degree one is either in V^{-}_{Te} or V^{0}_{Te}. Of course we know that any vertex of degree zero is always in $V^{-}_{Te.}$

Example 1.15: It may be noted that if w_1 and w_2 belongs to $T_{pn}[v,S]$ and w_1 and w_2 does not belongs to S then w_1 and w_2 may or may not be adjacent.

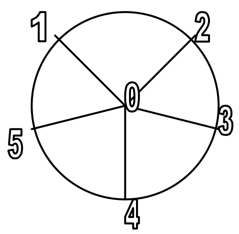


Figure -1.3

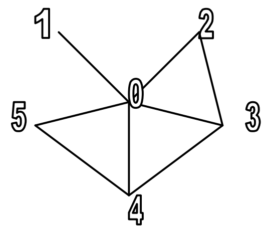


Figure -1.4

In Figure-1.31, the graph has vertices 0,1,2,3,4,5. And $0 \in V^{+}_{Te}$ T ={0,1}then $T_{pn}[0,T]$ = {4,3}, where 4 and 3 adjacent

In Figure-1.4, $0 \in V^{+}_{Te}$ T = {0,2} $T_{pn}[0,T]$ = {2,5}, where 2 and 5 are non adjacent.

Theorem -1.16: Let G be a graph and v and w are distinct vertices of G such that $v \in V^{+}_{Te}$ and $w \in V^{-}_{Te}$ then v and w are non adjacent vertices.

Proof:

If w is an isolated vertex of G then v and w are non adjacent.

Suppose w is non isolated vertex of G then by above Theorem-1.12 there is a γ_{Te} set S and vertex w_1 in S such that $T_{pn}[w_1,S]$ = {w}. Now, $v \in$ S because $v \in V^{+}_{Te}$. If v and w are adjacent and v and w_1 are the same vertices then it implies that $T_{pn}[v,S]$= {w}.which contradicts the statement of above (Theorem-1.13).

If v and w_1 are distinct vertices then w is adjacent to two vertices of S and one of them is v which is implies that $w \notin T_{pn}[w_1,S]$. This is a contradiction. Therefore v and w must be non adjacent. ■

Theorem-1.17: Let G be a graph then $|V^0_{Te}| \geq 2|V^+_{Te}|$.

Proof:

We will prove that every $v \in V^+_{Te}$ give rise at least two vertices v_1 and v_2 in V^0_{Te}. Let S be a γ_{Te} set containing v then (by above Theorem-1.13), $T_{pn}[v,S]$ contains at least two vertices w_1 and w_2.

Case 1 : Suppose w_1 and $w_2 \in S$. If $w_1,w_2 \in V^0_{Te}$. Let $v_1 = w_1$ and $v_2 = w_2$.

Suppose $w_1 \notin V^0_{Te}$ then $w_1 \in V^-_{Te}$ or $w_1 \in V^+_{Te}$. Since w_1 and v are adjacent, $w_1 \notin V^-_{Te}$ then $w_1 \in V^+_{Te}$. Since $w_1 \in V^+_{Te}$, $T_{pn}[w_1,S]$ contains a vertex z different from v (by above theorem 1.13) so, $z \notin S$, again by similar above argument $z \notin V^-_{Te}$. So, $z \in V^0_{Te}$.

Let $v_1 = z$ .If $w_1 \in V^0_{Te}$ then $v_1 = w_1$.
If $w_2 \in V^+_{Te}$ then by similar above arguments there is a vertex z_1 not is S such that $z_1 \in T_{pn}[w_2,S]$.Let $v_2 = z_1$

If $w_2 \in V^0_{Te}$, then $v_2 = w_2$.

Case 2 : If $w_1,w_2 \notin S$ then $v_1 = w_1$ and $v_2 = w_2$.

Case 3 : Suppose $w_1 \in S$ and $w_2 \notin S$ if $w_1 \in V^0_{Te}$ then $v_1 = w_1$ and $v_2 = w_2$.
If $w_1 \in V^+_{Te}$ then as in case (1) there is a vertex z not in S such that z is adjacent to w_1 and $z \in V^0_{Te}$.Let in this case let $v_1 = z$ and $v_2 = w_2$.

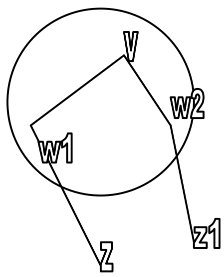


Figure-1.5

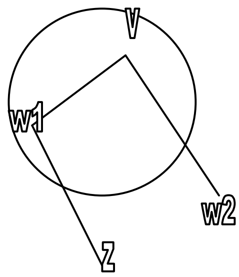


Figure-1.6

Case 4 :

If $w_1 \notin S$ and $w_2 \in S$. The proof of this part is as in above case. So, in all cases we get two vertices v_1 and v_{11} in V^0_{Te} corresponding to vertex $v \in V^+_{Te}$. It can be proved if v_1 and v_2 are distinct vertices of V^+_{Te} . Then the sets $\{v_{11,} v_{12}\}$ and $\{v_{21,} v_{22}\}$ are disjoint. So, $|V^0_{Te}| \geq 2|V^+_{Te}|$. ■

Corollary-1.18: If G is graph such that $\gamma_{Te}(G\text{-}v) \neq \gamma_{Te}(G)$ then $\gamma_{Te}(G\text{-}v) < \gamma_{Te}(G)$ for every $v \in V(G)$.

Proof:

Suppose for every vertex v of G, $\gamma_{Te}(G\text{-}v) \neq \gamma_{Te}(G)$ then $v \in V^+_{Te}$ or $v \in V^-_{Te}$. If for some vertex $v \in V^+_{Te}$, then there are two vertices v_1 and v_2 such that v_1 and v_2 belongs to V^0_{Te}. This contradicts the hypothesis of corollary. Hence $V(G) = V^-_{Te}$. ■

Theorem-1.19: Let G be a graph and v be a non isolated vertex of G. If for every vertex $w \in N(v)$ and N(w) is complete then $v \notin V^-_{Te.}$

Proof :

Suppose $v \in V^-_{Te}$ then there is a γ_{Te} set S not containing v and a vertex w in S such that $T_{pn}[w,S] = \{v\}$. Now, w is adjacent to some vertex w^1 in S. Since N(w) is complete it implies that v is adjacent to w^1.This contradicts the fact that $v \in T_{pn}[w,S]$.

Hence $v \notin V^-_{Te.}$ ■

Chapter :-2

INDEPENDENT DOMINATION

AND

VERTEX COVERING

Independent sets play an important role in Graph Theory and other areas like discrete optimization. They appear in matching theory, coloring of graphs and in trees. Our aim in this chapter is to consider independent domination and characterize those vertices whose removal increases, decreases or does not change independent domination number. Further we also characterize those vertices whose removal decreases the independent domination number in terms of maximal independent sets with minimum cardinality (i set).

Further we consider vertex covering sets of graphs. We define the vertex covering number of a graph and prove that this number does not increase when a vertex is removed from the graph. We prove the characterization for those vertices whose removal reduces the vertex covering number of a graph. We further prove that when the vertex covering number decreases the independence number remain same and conversely when the independence number decreases the vertex covering number remain same. We also prove that if G is a vertex transitive graph then either removal of every vertex reduces the vertex covering number or removal of any vertex does not change the vertex covering number. We also characterize vertex transitive graphs which are bipartite.

Definition-2.1: Independent Set. [44]

A set of vertices in a graph G is said to be an independent set or an internally stable set if no two vertices in the set are adjacent.

Definition-2.2: Maximal independent set.[44]

An independent set S is said to be maximal independent set if S U {v} is not an independent set for every vertex v not in S.

Definition-2.3: Independence number. [44]

The independence number is the, maximum cardinality of an independent set in G. It is denoted by $\beta_0(G)$.

Definition-2.4: Minimum independent dominating set. [44]

A set with minimum cardinality among all the maximal independent set of G is called minimum independent dominating set of G or just i set of G.

Definition-2.5: Independent domination number. [44]

The cardinality of a minimum independent dominating set is called independent domination number of the graph G and it is denoted by i(G).

Note: A maximal independent set is a dominating set of G.

Definition 2.6: (Vertex Transitive Graph).[1]

Let G be a graph then G is said to be vertex transitive if for every u, v, $\in$ V(G) there is an automorphism f(G) such that f(u) = v.

We introduce the following sets.

$$\mathbf{V_i^0 = \{ v \in V(G) : i(G - v) = i(G) \}}$$

$$\mathbf{V_i^+ = \{ v \in V(G) : i(G - v) > i(G) \}}$$

$$\mathbf{V_i^- = \{ v \in V(G) : i(G - v) < i(G) \}}$$

These three sets are mutually disjoint and its union is V(G).

Note: If S is an independent dominating set of graph G then for every vertex v in S, Pn[v,S] contains v.

Example-2.7:

(1) Consider the graph G = C_5 (See Figure -0.2)

For this graph minimum independent dominating set is $\{V_2, V_5\}$ and independence domination number is 2. and $V_i^0 = \{V_1,V_2,V_3,V_4,V_5\}$, $V_i^+ = \phi$, $V_i^- = \phi$.

(2)Consider the graph G = P_7 :

Figure-2.1

For Path Graph P_7, minimum independent dominating set is $\{V_2, V_4, V_6\}$ and independence domination number is 3. and $V_i^0 = \{V_1, V_2, V_3, V_5, V_6, V_7\}$, $V_i^+ = \phi$, and $V_i^- = \{V_4\}$.

(3) Consider the graph G = Patersen Graph : (See Figure -0.3)

For Peterson Graph minimum independent dominating set is $\{V_2, V_5, V_8\}$ and independence domination number is 3.

And $V_i^0 = \{V_1, V_2, V_3, V_4, V_5, V_6, V_7, V_8, V_9, V_{10}\}$, $V_i^+ = \phi$ and $V_i^- = \phi$.

(4) Consider the Graph G = Hyper Cube Graph : (See Figure -0.8)

For Hyper Cube Graph minimum independent dominating set is $\{V_1, V_6\}$ and independence domination number is 2.

And $V_i^+ = \phi$, $V_i^0 = \{V_1, V_2, V_3, V_4, V_5, V_6, V_7, V_8\}$, and $V_i^- = \phi$.

Theorem-2.8: Let G be a graph i(G-v) < i(G) then i(G-v) = i(G) – 1.

Proof:

Suppose i(G-v) < i(G) -1. Let S is a i set of G – {v}.

Case-1: v is adjacent to some vertex of S.

Then S is an independent dominating set in G with |S| < i(G). This is contradiction. (Because a set with |S| < i(G) is not an independent dominating set of G.)

Case-2: v is not adjacent to any vertex of S.

So, S_1 = S U {v} is an independent dominating set in G.

So, $|S_1| = |S| + 1 \leq i(G\text{-}v) < i(G)$. So, $|S_1| < i(G)$.This is a contradiction.

(because $i(G\text{-}v) < i(G)-1$). Thus, $i(G\text{-}v) = i(G)-1$.■

Theorem -2.9: A vertex v ∈ V_i^+ if and only if

(a) v is not an isolated vertex.

(b) v is in every i set of G.

(c) No independent subset S of V(G)– N[v] with |S| = i(G) or i(G) -1 can dominates G-{v}.

Proof: (a)

suppose v is an isolated vertex of G. Let S be a i set of G then v ∈ S then S – {v} is an independent set in G – {v}.Let w ∈ (G – {v}) – (S – {v}) ,So w ≠ v. So, w ∉ S. (i.e. G-S), and S is i set of G. So, w must be adjacent to some vertex t of S where t ≠ v because v is an isolated vertex of G. So, t ∈ S – {v}.So, w is adjacent to some vertex t of S – {v}. So, S – {v} is an independent dominating set in G – {v}.

So,

$$i(G\text{-}v) \leq |S - \{v\}| < |S|.$$

So,

$$i(G\text{-}v) < |S| = i(G).$$

So, $i(G\text{-}v) < i(G)$.

So, $v \notin V_i^+$.

This is a contradiction.

So, v is not an isolated vertex of G.

(b)

Suppose there is some i set S of G which does not contains v. Now, S is an independent dominating set in G – {v}. So, $i(G-v) \leq |S| = i(G)$. So, $v \notin V_i^+$. This is a contradiction. Thus, v is in every i set of G.

(c)

Suppose there is an independent subset of V(G) – N[v] with |S|= i(G) or i(G) -1 can dominates G – {v}. Then $i(G-v) \leq |S| \leq i(G)$. So, $v \notin V_i^+$. This is a contradiction. Thus, no independent set $S \subseteq V(G) - N[v]$ with |S| = i(G) or i(G) -1 can dominates G – {v}.

Now we prove converse.

Case 1: Suppose $v \in V_i^-$

Let S be a i set of G – {v}.Suppose v is adjacent to some vertex w of S in G. Then S is an independent dominating set in G such that $|S| = i(G-v) < i(G)$, which is not possible. So, v is not adjacent to any vertex of S. Then $S \subseteq V(G) - N[v]$ such that |S| = i(G) -1 and S dominates G – {v}. This is a contradiction. So, $v \notin V_i^-$.

Case 2: Suppose $v \in V_i^0$.

Let S be a i set of G – {v}. Therefore |S| = i(G). Suppose v is adjacent to some vertex w of S. Then S is a i set in G and $v \notin S$. This is contradiction. So, v is not adjacent to any vertex of S. Then $S \subseteq V(G) - N[v]$ which is independent and dominates G – {v}. This is a contradiction. So, $v \notin V_i^0$. Thus, $v \in V_i^+$. ■

We introduce the following definition.

Definition-2.10: External private neighborhood.

Let v be a vertex of the graph G and $S \subset V(G)$ containing v then external private neighborhood of v with respect to S,

i.e. $E_{pex}[v, S] = \{ w \in V(G) - S: N(w) \cap S = \{v\} \}$.

Note: $E_{pex}[v, S] \subset P_n[v, S]$.

Theorem -2.11: The following conditions are equivalent for a graph G and a vertex $v \in V(G)$

(1) $v \in V_i^-$.

(2) There is a i set S containing v such that $E_{pex}[v, S] = \phi$.

(3) There is a i set S containing v such that S – {v} is an independent dominating set in G – {v}.

Proof: Now (1) => (2).

Let S_1 be a i set in G – {v} then $|S_1| = i(G) - 1$. Now v can not be adjacent to any vertex of S_1 (because otherwise S would be an independent dominating set in G with cardinality less than i(G)). Let $S = S_1 \cup \{v\}$, then obviously S is a i set in G and $v \in S$. If $w \in E_{pex}[v, S]$ then w is not adjacent to any vertex of S_1. This contradicts that S_1 is an independent dominating set G–{v}. Hence $E_{pex}[v, S] = \phi$. So, (1) => (2) is proved.

Now, (2) => (3).

Let S be the given set in statement (2). Suppose S – {v} is not an independent dominating set in G – {v}. So, there is a vertex w in (G – {v}) – (S – {v}) which is not adjacent to any vertex of S – {v} implies that $w \neq v$. Now S is an independent dominating set in G. Therefore w is adjacent to v only in S. This is a contradiction. So, $E_{pex}[v, S] = \phi$. So, (2) => (3) is proved.

Now, (3) => (1).

It follows that $i(G-v) < i(G)$ implies that $v \in V_i^-$. So, (3) => (1) is proved. ■

Remark – 2.12:

From the above Theorem-2.11 it follows that there is an one –one correspondence between the minimum independent set of G containing v and minimum independent sets of G-{v} if v $\in V^-_i$. It also follows that there are at least as many minimum independent dominating sets of G as that of G-{v}.

Now we consider vertex transitive graphs. We prove the following theorem.

Theorem-2.13: Let G be a vertex transitive graph and v $\in$ V(G). If i(G-v) < i(G) then i(G-w) < i(G) for all w $\in$ V(G).

Proof:

We use the statement (2) of Theorem -2.11. Let w be any vertex different from v and f be an automorphism of the graph G such that f(v) = w. Since v $\in V^-_i$ then there is i set S containing v such that $E_{pex}[v,S] = \phi$.

Now consider the set f(S) which is a i set because f is an automorphism of G. Since v $\in$ S and f(v) = w $\in$ f(S).

Now suppose $w^1 \in E_{pex}[w,f(S)]$ .Let $v^1 \in$ V(G) such that $f(v^1) = w^1$. Since $w^1 \notin$ S, also w^1 is adjacent to w implies that v^1 is adjacent to v. Since w^1 is not adjacent to any other vertex of f(S), and v^1 is not adjacent to any other vertex of S that is $v^1 \in E_{pex}[v,S]$. This is a contradiction.

Thus, $E_{pex}[w,f(S)] = \phi$. This is equivalent to say that w $\in V^-_i$.
(by Theorem-2.11.)■

Theorem -2.14: For any graph G

(a) If $v \in V_i^+$ then for every i set S of G, $v \in S$ and $E_{pex}[v, S]$ contains at least two non adjacent vertices.

(b) If $x \in V_i^+$ and $y \in V_i^-$ then x and y are not adjacent.

(c) $|V_i^0| \geq 2|V_i^+|$.

(d) $i(G) \neq i(G\text{-}v)$ for all $v \in V(G)$ if and only if $V = V_i^-$.

Proof:

(a)

Let S be a i set of G. Since $v \in V_i^+$ and $v \in S$. If $E_{pex}[v, S] = \phi$ then it implies that $v \in V_i^-$ (by Theorem -2.11). Therefore if w is the only vertex such that $w \in E_{pex}[v, S]$, then $S_1 = S - \{v\} \cup \{w\}$ is a i set not containing v which contradicts the Theorem – 2.9.

Suppose any two vertices in the $E_{pex}[v, S]$ are adjacent. Select any two vertices say w_1 and w_2 in the $E_{pex}[v, S]$. Now let $S_1 = (S - \{v\}) \cup \{w_1\}$ then S_1 is a i set not containing v. This is again contradiction.

(b)

There is a i set S is containing y such that $S - \{y\}$ is an independent dominating set in $G - \{y\}$. (by Theorem – 2.11). Since $x \in V_i^+$ and $x \in S$. Since S is an independent set. So, x and y are non adjacent.

(c)

Let $x \in V_i^+$ and S be a i set containing x (by - a). Therefore vertices x_1 and x_2 in the $E_{pex}[v, S]$ (which are possibly non adjacent). Since x and x_1 are adjacent and $x_1 \notin V_i^-$. (by –b).Therefore $x_1 \in V^0_i$.Similarly $x_2 \in V_i^0$. Thus, every vertex $x \in V_i^+$ gives rise two distinct vertices in V_i^0. It can be verified. If x and x^1 are distinct vertices in V_i^+ then the sets $\{x_1,x_2\}$ $\{x_1^1,x_2^1\}$are disjoint. Thus, it follows that $|V_i^0| \geq 2|V_i^+|$.

(d)

Suppose $i(G\text{-}v) \neq i(G)$ for all $v \in V(G)$. If some $v \in V_i^+$ then it implies that $V_i^0 \neq \phi$.(by- c).That is there is vertex w such that $i(G\text{-}w) = i(G)$ which contradicts our hypothesis. Its converse is obvious. █

VERTEX COVERING

Definition -2.15: Vertex Covering Set.[44]

Let G be a graph. A set $S \subset V(G)$ is said to be a vertex covering set of the graph G if every edge has at least one end point in S.

Definition -2.16: Minimal Vertex Covering Set.[44]

If S is a vertex covering set such that no proper subset of S is a vertex covering set then S is called minimal vertex covering set.

Definition-2.17: Minimum Vertex Covering Set.[44]

A vertex covering set with minimum cardinality is called minimum vertex covering set. It is also called γ_{cr} set.

Note that every minimum vertex covering set is minimal vertex covering set.

Definition -2.18: Vertex Covering Number.[44]

The vertex covering number of the graph G is the cardinality of any minimum vertex covering set of the graph G. It is denoted by $\alpha_0(G)$ or simply α_0.

Definition -2.19 : Independent Set.[44]

A set $S \subset V(G)$ is said to be independent set if any two distinct vertices of S are nonadjacent.

We will regard a single tone set as an independent set.

Definition -2.20: Maximum independent Set.[44]

A independent set with maximum cardinality is called maximum independent set.

Definition -2.21: Independence Number.[44]

The cardinality of a maximum independent set is called independence number of the graph G and it is denoted by $\beta_0(G)$ or simply β_0.

Now we see that vertex covering number of a graph does not increase when a vertex is removed from the graph.

Lemma -2.22: If $v \in V(G)$ then

(1) $\alpha_0(G\text{-}v) \leq \alpha_0(G)$.

(2) If $\alpha_0(G\text{-}v) < \alpha_0(G)$ then $\alpha_0(G\text{-}v) = \alpha_0(G) - 1$.

Proof : (1)

Case- a:

Let S be a γ_{cr} set in G and $v \in S$. Consider the set S – {v} of G –{v}. If e = xy is an edge of G – {v} then at least one end vertex x or y lies in S. Since e is an edge of G – {v}, $x \neq v$ and $y \neq v$. Thus, the end vertex of e which lies in S actually a vertex of S - {v}. Thus, S- {v} is a vertex covering set of G –{v}.Therefore $\alpha_0(G\text{-}v) \leq \alpha_0(G)$.

Case –b: $v \notin S$.

Here also by similar argument S is γ_{cr} set of G – {v}.Thus, $\alpha_0(G\text{-}v) \leq \alpha_0(G)$.

(2) Suppose $\alpha_0(G\text{-}v) < \alpha_0(G) - 1$.

Let S be a minimum vertex covering set of G – {v}.

Case -1: Suppose v is not adjacent to any vertex of S.

Let S_1 = S U {v}, then S_1 is a minimal vertex covering set of the graph G.

So,

$$\alpha_0(G) \leq \alpha_0(G - v) + 1 < \alpha_0(G) - 1 + 1$$

So,

$$\alpha_0(G) < \alpha_0(G).$$

This is a contradiction.

Case -2: Suppose v is adjacent to some vertex of S.

Let S_1 = S U {v}, then S_1 is a vertex covering set of the graph G.

So,

$$\alpha_0(G) \leq |S_1| = |S| + 1 = \alpha_0(G - v) + 1 < \alpha_0(G) - 1 + 1$$

So,

$$\alpha_0(G) < \alpha_0(G).$$

This is a contradiction.

So, by above case we have contradiction. Thus, $\alpha_0(G - v) = \alpha_0(G) - 1.$ ■

Theorem -2.23: Let G be a graph and v ∈ V(G) then v ∈ V^-_{cr} if and only if there is a γ_{cr} set S_1 such that v ∈ S_1.

Proof:

Suppose that v ∈ V^-_{cr}. Let S be a minimum vertex covering set of G-{v} and let S_1 = S U {v}. Then since $\alpha_0(G\text{-}v) = \alpha_0(G) - 1$.So, S_1 is a minimum vertex covering set of the graph G. and v ∈ S_1.

Now we prove converse.

Let S_1 be a minimum vertex covering set of the graph G containing the vertex v. Let S = S_1 – {v} then |S| < |S_1|. We now prove that S is a vertex covering set of the graph G – {v}. Let e = xy be an edge of the G-{v} then x ≠ v and y ≠ v. Since S_1 is a vertex covering set of the graph G so, x ∈ S_1 or y ∈ S_1. In fact (by above Theorem-2.22), x ∈ S or y ∈ S. Thus, S is a vertex covering set of G – {v}.

So,

$$\alpha_0(G - v) \leq |S| < |S_1| = \alpha_0(G)$$

So,

$$\alpha_0(G - v) < \alpha_0(G)$$

So,

$$v \in V^-_{cr}.$$ ■

Corollary -2.24: Let G be a graph and $v \in V(G)$ then $v \in V^0_{cr}$ if and only if v does not belongs to any minimum vertex covering set of the graph G. ∎

Corollary -2.25 : Suppose $S_1, S_2, \ldots S_k$ are all γ_{cr} set of the graph G and $v \in V^0_{cr}$ then N(v) is subset of $S_1 \cap S_2 \cap \ldots \cap S_k$.

Proof:

If $N(v) = \phi$ then the result is obvious.

If $w \in N(v)$ then w adjacent to v and since $v \notin S_i$ for any i (i = 1,2, 3, …..k) $w \in S_i$ for every i . Hence $w \in S_1 \cap S_2 \cap \ldots \cap S_k$. ∎

Corollary -2.26 : Let G be a graph and $v \in V^0_{cr}$ such that v is not an isolated vertex in G then $S_1 \cap S_2 \cap \ldots \cap S_k$ is non empty. ∎

Corollary -2.27: The set V^0_{cr} is an independent set.

Proof :

If u and v belongs to V^0_{cr} and if u and v adjacent then either u or v belongs to some minimum vertex covering set of the graph. In other words $u \in V^-_{cr}$ or $v \in V^-_{cr}$ (by Theorem -2.23). This is a contradiction. Hence u and v are non adjacent. ∎

$\delta(G)$ denote minimum degree of the graph G.

Corollary -2.28: Let G be a graph then $|V^-_{cr}| \geq \delta(G)$.

Proof :

If $V^0_{cr} = \phi$ then $V^-_{cr} = V(G)$. Hence the result is true.

Suppose $V^0_{cr} \neq \phi$. Let $v \in V^0_{cr}$. If v is an isolated vertex then also the result is true.

Suppose v is not an isolated vertex then N(v) is a subset of $S_1 \cap S_2 \cap \ldots \cap S_k$. which is a subset of $S_1 \cup S_2 \cup \ldots \cup S_k$. Hence

$$\delta(G) \leq |N(v)| \leq |S_1 \cup S_2 \cup \ldots \cup S_k| = |V^-_{cr}|$$

Thus, $\delta(G) \leq |V^-_{cr}|$. ∎

Corollary 2.29: If G is any graph then $|V^0_{cr}| \leq |V(G)| - \delta(G)$. ■

Corollary -2.30: For any graph G

(1) $|V^-_{cr}| \geq \alpha_0(G)$.

(2) $|V^0_{cr}| \leq \beta_0(G)$.

Proof:

Note that the union of all minimum vertex covering sets = V^-_{cr} and $\alpha_0(G)$ is the cardinality of the minimum vertex covering set. It follows that $|V^-_{cr}| \geq \alpha_0(G)$ and similarly $|V^0_{cr}| \leq \beta_0(G)$. ■

● Note that a graph having at least one edge has at least one non empty vertex covering set.

● We make following convention.

The graph with no edges has only one vertex covering set namely the empty set. So, vertex covering number of such a graph is zero.

Theorem -2.31: Let G be a graph then $v \in V^0_{cr}$ for every vertex $v \in V(G)$ if and only if the graph is a null graph.

Proof:

If G is a null graph then its vertex covering number is zero and it can not decrease further when any vertex is removed. Hence every vertex belongs to V^0_{cr}.

Conversely suppose there is at least one edge in the graph G. Then it has a non empty vertex covering set. Hence any vertex x of this set belongs to V^-_{cr} by above Theorem 2.24 This contradicts the assumption. Thus, the graph has no edges. ■

Remark -2.32:

(1) Note that the compliment of a minimum vertex covering set is a maximum independent set. Hence $\alpha_0(G) + \beta_0(G) = n$.

(2) Let v be a vertex of the graph G.

Now $\alpha_0(G\text{-}v) + \beta_0(G\text{-}v) = n - 1$. If $v \in V^-_{cr}$, then $\alpha_0(G\text{-}v) = \alpha_0(G) - 1$, then by above equation,

$$\alpha_0(G) - 1 + \beta_0(G\text{-}v) = n - 1.$$

So,

$$\alpha_0 (G) + \beta_0(G\text{-}v) = n.$$

So,

$$\beta_0(G\text{-}v) = n - \alpha_0 (G) .$$

So,

$$\beta_0(G\text{-}v) = \beta_0(G) \text{ (because } \alpha_0(G) + \beta_0(G) = n.$$

Thus, we conclude that if the vertex covering number decrease (when a vertex is removed.) then the independence number of the graph G does not change (when a vertex is removed.).

Similarly if the vertex covering number does not change when a vertex is removed then the vertex independent number of G is decrease (when that vertex is removed).

Example-2.33:

(1) Consider the complete graph K_n for $n \geq 2$. Its vertex covering number is n-1.

For any v of K_n , $K_n - \{v\} = K_n -1$, and its vertex covering number is n -2. Thus, every vertex of K_n belongs to V^-_{cr}.

(2) Consider the cycle C_n, $n \geq 3$ then every vertex of C_n belongs to V^-_{cr} . Similarly every vertex of the Hyper Qube Graph –Q_3 belongs to V^-_{cr}.

Theorem -2.34: If G is vertex transitive graph with at least one edge then every vertex v $\in$ V^{-}_{cr}.

Proof:

Let S be a non empty minimum vertex covering set in G and v be a vertex of the graph G. If v $\in$ S then v $\in$ V^{-}_{cr} by Theorem 2.23.

If v $\notin$ S then, let u $\in$ S. Let f be an automorphism of the graph G such that f(u) = v (because G is vertex transitive graph). Now consider the set f(S) which is minimum vertex covering set of the graph G and it contains f(u) = v that is v $\in$ f(S). Thus, f(S) is a minimum vertex covering set of G such that v $\in$ f(S). So, again by Theorem - 2.23 , v $\in$ V^{-}_{cr}. Thus, every vertex of the graph G belongs to V^{-}_{cr}.■

Theorem 2.35: If G is a graph without isolated vertices and if S_1 and S_2 are disjoint vertex covering set of the graph G then,

(1) G is a bipartite graph.

(2)S_1 and S_2 are minimal vertex covering set of the graph G.

Proof:

(1)

Let e = uv be an edge of graph G then either u $\in$ S_1 and v $\in$ S_2 or u $\in$ S_2 and v $\in$ S_1. Thus, every edge joins a vertex of S_1 to a vertex of S_2 (No edge can join two vertices of the same set of S_1 or S_2).

Moreover if x is any vertex of graph G and if e is an edge whose one end vertex is x then x $\in$ S_1 or x $\in$ S_2. Thus every vertex of the graph G belongs to either S_1 or S_2. Thus, G is a bipartite graph.

(2)

Now let v be any vertex of S_1 and if e is an edge whose end vertex is v then S_1-{v} does not contain the end vertex v of the edge e. Thus, S_1-{v} is not a vertex covering set of the graph G. Hence S_1 is a minimal vertex covering set of the graph G. Similarly S_2 is a minimal vertex covering set of the graph G. ∎

Corollary-2.36: If G is a graph without isolated vertices and if G has an odd number of vertices then any two minimum vertex covering set have non empty intersection.

Proof :

Suppose S_1 and S_2 are disjoint minimum vertex covering set of the graph G. Then by Theorem-2.35 graph G is a bipartite. Hence $|V(G)| = |S_1| + |S_2|$ .Since
$|S_1| = |S_2|$, $|V(G)|$ is an even number which is not true. Thus, $S_1 \cap S_2 \neq$ ¢ . ∎

Definition-2.37: Co-vertex covering set.

A set S of vertices is said to be co-vertex covering set if u and v are non adjacent then $u \in S$ or $v \in S$.

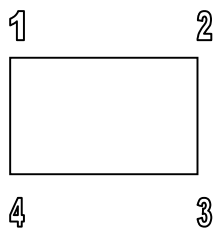


Figure-2.2

S = {1, 3} is not co-vertex covering set because 2 and 4 are not adjacent and
2 , 4 $\notin$ {1,3 }.

If S = {1, 2} is co-vertex covering set because 1 and 3 are not adjacent and
$1 \in$ {1, 2}.Similarly 2 and 4 are not adjacent and $2 \in$ {1, 2}.

Theorem-2.38: If S is a vertex covering set and its compliment has at least two vertices and any two of them are not adjacent then S is not a co-vertex covering set. ■

Theorem-2.39: S ⊆ V(G) is a vertex covering set and co-vertex covering set of G if and only if S = V(G) or V(G) – S is a single tone set.

Proof:

Suppose S is both vertex and co-vertex covering set. If S = V(G) then the condition is satisfies.

If S ≠ V(G) and if V(G) – S has at least two vertices then any two of them are adjacent or non adjacent.

If they are adjacent it implies that S is not a vertex covering set.

If they are nonadjacent it implies that S is not a co- vertex covering set. This is a contradiction. Hence V(G) – S must be a single tone set.

Now we prove converse.

If S = V(G) then S is both vertex and co-vertex covering set.

If V(G) – S is a single tone set and u and v are two vertices of the graph G then at least one of them must belongs to S.

Hence S is both vertex and co-vertex covering set of the graph G. ■

Corollary-2.40: If G is a graph and |V(G)| = n, then there are exactly n + 1 sets which are both vertex and co-vertex covering sets. ■

Suppose G is a graph and $\delta(G) = k$. Let S be a minimum vertex covering set and v be a vertex such that $d(v) \geq k$ and $v \notin S$ then all the neighbors of v are in S. Thus, S is a k-dominating set of the graph G. Therefore the k-domination number of the graph G is less than of equal to $\alpha_0(G)$.

Definition-2.41: K-perfect dominating set.

Let G be a graph and $S \subset V(G)$ then S is said to be a k- perfect dominating set if for every vertex v not in S, v is adjacent to exactly k vertices of S.

The minimum cardinality of a perfect k-dominating set is called perfect k-domination number of the graph G. It is denoted by $\gamma_{pk}(G)$.

Theorem-2.42: Let G be a k-regular graph then $\alpha_0(G) = \gamma_{pk}(G)$.

Proof:

Let S be a minimum vertex covering set of the graph G. If $v \notin S$ then v is adjacent to exactly k vertices of S because d(v) = k and S is vertex covering set. Thus, S is a perfect k-dominating set of G .Hence $\gamma_{pk}(G) \leq \alpha_0(G)$.

Let T be a minimum perfect k-dominating set of the graph G. We prove that T is vertex covering set of the graph G.

Let e = uv be an edge of the graph G. Suppose $u \notin T$. Since d(u) = k and T is a perfect k-dominating set, u is adjacent to exactly k vertices of T and therefore v must be in T. Thus, T is a vertex covering set of the graph G. Therefore $\alpha_0(G) \leq \gamma_{pk}(G)$ .This proves that $\alpha_0(G) = \gamma_{pk}(G)$ █

Theorem-2.43: If G is a vertex transitive graph which is not null graph then

(1) There are at least two distinct minimum vertex covering sets in the graph G.

(2) The intersection of all minimum vertex covering sets of G is empty set.

Proof:

(1)

Since G is not a null graph there is a proper vertex covering set of the graph G, therefore there is a proper subset S of V(G) which is a minimum vertex covering set. Now let $y \in S$ and $x \notin S$. Since G is a vertex transitive graph. So, there is an automorphism f: V(G) → V(G) such that f(y) = x then f(s) is a minimum vertex covering set containing f(y) = x. Note that S ≠ f(S) because x ∈ f(S) but x ∉ S. Thus, S and f(S) are two distinct minimum vertex covering sets of G.

(2)

Let $S_1, S_2, S_3, \ldots S_k$ be all the minimum vertex covering sets of the graph G and we assume that $S_i \neq S_j$, if $i \neq j$.

Now suppose $S_1 \cap S_2 \cap S_3 \cap \ldots \cap S_k \neq \phi$ Let $y \in S_1 \cap S_2 \cap S_3 \cap \ldots \cap S_k$. Note that this intersection is a proper subset of S_i for every i.

Let $x \in S_1$ such that $x \neq y$. Now since G is vertex transitive then there is an automorphism f such that f(y) = x. Now the set

$\{S_1, S_2, S_3, \ldots S_k\} = \{ f(S_1), f(S_2) f(S_3), \ldots f(S_k) \}$

Now $f(y) \in f(S_1) \cap f(S_2) \cap f(S_3) \cap, \ldots \cap f(S_k)$

$x \in S_1 \cap S_2 \cap S_3 \cap \ldots \cap .S_k$ but $x \notin S_1 \cap S_2 \cap S_3 \cap \ldots \cap .S_k$. This is a contradiction.

Hence $S_1 \cap S_2 \cap S_3 \cap \ldots \cap .S_k = \phi$ ■

Theorem-2.44: Suppose G is a vertex transitive graph which is not a null graph

(1) If G has exactly two minimum vertex covering sets then they are disjoint, the graph is bipartite graph and the $\gamma_{cr}(G) = n/2$. (i.e. n is an even number of vertices of G.)

(2) If G is a bipartite graph and if G has n (even) vertices then G has exactly two disjoint minimum vertex covering sets and $\gamma_{cr}(G) = n/2$.

Proof :

(1)

Suppose S_1 and S_2 are the only minimum vertex covering sets of the graph G then S_1 and S_2 are disjoint.(by last theorem-2.43).

Now since every edge has one end point in S_1 and the other end point in S_2 then the graph is bipartite. Also by (Theorem-2.35) n must be even. Also note that S_1 or S_2 does not contain any isolated vertex.(In fact all vertices are of the graph G have the same degree.Since the graph is vertex transitive and therefore regular.) Thus, $\{S_1,S_2\}$ is partition of V(G) and since $|S_1| = |S_2|$. So, $\gamma_{cr}(G) = n/2$.

(2)

Let m = n\2. We will prove that $\beta_0(G) = m$. First we note that if V_1 and V_2 is the partition of the graph G then $|V_1| = |V_2| = m$. Since the graph G is vertex transitive it is k-regular for some $k \geq 1$. The number of edges incident with vertices of $V_1 = k|V_1|$ and the same edges are incident with vertices of V_2 and the number of such edges $=k|V_2|$. Hence $k|V_1| = k|V_2|$ So, $|V_1| = |V_2| = m$ = n\2.The set of vertices of V_1 is an independent set. Hence $\beta_0(G) \geq m$. Now we prove that any set with m+1 vertices can not be an (maximum) independent set.

Let S be any set with m+1 vertices. S has at least one vertex fromV_1 and at least one vertex from V_2. Let t be the number of vertices in S which in V_1 then m+1-t vertices of S are in $V_{2.}$

Suppose S is an independent set. Consider the edges which are incident with those vertices of S which are in V_1. The number of such edges = kt. The other end point of these edges are those vertices of V_2 which are not in S. There are exactly m-(m+1-t) = t-1 such vertices. The number of edges incident with these vertices is k(t -1).Thus, $k(t-1) \geq kt$. i.e. $t - 1 \geq t$ which is not true. Thus, S can not be an independent set. This implies that $\beta_0(G) = m$. Hence $\alpha_0(G) = m$ = n\2. ∎

Chapter :- 3

TOTAL k-DOMINATION ,

K-TUPLE DOMINATION

AND

K-DEPENDENT K-DOMINATION

Chapter-3 :Total k-Domination and k-tuple
Domination and k-dependent k-Domination

In this chapter we consider the notions of total k–domination, k-tuple domination and k-dependent k-domination for graphs.(k ≥2) It may be noted that if a graph has a vertex of degree less than k then there does not exist a totally k-dominating set in the graph. Similarly if a graph has a vertex of degree less than k -1 then a k-tuple dominating set does not exist. In this chapter we consider and characterize those vertices whose removal increases or decreases total k-domination number of the graph. We prove similar result for k-tuple domination and k-dependent k-domination.

TOTAL k-DOMINATION

In this section we introduced a totally k-dominating sets. We prove theorems similar to those of domination.

Definition-3.1: Totally k-dominating set.

Let k be an integer k≥1. Let G be a graph and S ⊂ V(G). The set S is said to be totally k-dominating set if for every vertex v ∈ V(G) , v is adjacent to at least k vertices of S.

Note that every totally k – dominating set is a k-dominating set. However the converse is not true.

Example-3.2:

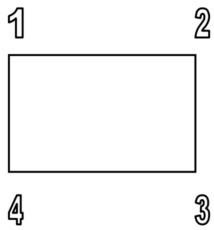


Figure -3.1

Consider the above graph G with vertices 1,2,3,4. Let S = {1,3} if k =2 then S is a 2- dominating set but it is not a totally 2 –dominating set.

Remark-3.3:

Note that if a graph G contains a vertex v with degree less than k then no subset of V(G) can be totally k-dominating set. (Although it may be k-dominating set.)

k-dominating set: A set S is k-dominating set if for every vertex $v \in V(G)- S$, v is adjacent to at least k vertices of S. i.e.$| N(v) \cap S| \geq k$.

Definition -3.4: Minimal totally k-dominating set.

Let S be a totally k-dominating set then S is said to be minimal totally k-dominating set if for every vertex v in S, S-{v}is not a totally k-dominating set.

Definition -3.5: Minimum totally k-dominating set.

A totally k-dominating set with minimum cardinality is called a minimum totally k- dominating set. It is called a γ_{Tk} set.

Definition-3.6: Total k- Domination Number.

The cardinality of a minimum totally k-dominating set is called total k-domination number of the graph G and it is denoted as $\gamma_{Tk}(G)$.

Note that any totally k-dominating set must contain at least k+1 vertices therefore total k-domination number of any graph, if it is define is grater than or equal to k+1.

Definition -3.7: Total k- private neighborhood.

Let G be a graph and $S \subset V(G)$ and $v \in S$ then total k-private neighborhood of v with respect to the set S.

$P_{Tk}[v,S] = \{ w \in V(G)$: w is adjacent to exactly k vertices of S including v. $\}$

Example -3.8: Consider the cycle C_5 with five vertices v_1, v_2, v_3, v_4, v_5 :

(See Figure-0.2)

$S = \{ v_1, v_3, v_4\}$. We consider the cycle C_5 with vertices v_1, v_2, v_3, v_4, v_5. Let $v = v_1$ then $P_{T2}[v_1, S] = \{ v_2 , v_5 \}$

Theorem-3.9: Let G be a graph. $k \geq 1$(k is a positive integer.) A totally k-dominating set S is minimal if and only if for every vertex v of S, $P_{Tk}[v,S] \neq \phi$.

Proof:

Suppose S is a minimal totally k-dominating set. Let $v \in S$ then S-{v} is not a totally k-dominating set. Hence there is a vertex w in V(G) which is adjacent to at most k-1 vertices of S-{v}.

If w = v then we have a contradiction because v is adjacent to at least k vertices of S. So, $w \neq v$.

Now w is adjacent to at least k vertices of S and is adjacent to at most k-1 vertices of S-{v}. This means that w is adjacent to exactly k vertices of S including v. Hence $w \in P_{Tk}[v,S]$.

Now we prove converse.

Suppose $v \in S$. Let $w \in P_{Tk}[v,S]$. Now w is adjacent to exactly k vertices of S including v therefore w is adjacent to k-1 vertices of S-{v}. i.e. S-{v} is not a totally k-dominating set. This implies that S is a minimal totally k-dominating set. ∎

Comments-3.10:

As we have noted earlier a graph having vertices with degree less than k can not have totally k-dominating set. Also it may happen that when a vertex is removed the resulting graph may have vertices having degree less than k.

Let G be a graph. Let I_k denote the set of vertices whose degree is less than k.

Notations: We define the following notations.

$\mathbf{V^i_{Tk}}$ **: {v ∈ V(G) : G-{v} has vertex of degree less than k in (G-{v})}**

$\mathbf{V^+_{Tk}}$ **: {v ∈ V(G) :** $\boldsymbol{\gamma_{Tk}(G\text{-}v) > \gamma_{Tk}(G)}$**.}**

$\mathbf{V^-_{Tk}}$ **: {v ∈ V(G) :** $\boldsymbol{\gamma_{Tk}(G\text{-}v) < \gamma_{Tk}(G)}$**.}**

$\mathbf{V^0_{Tk}}$ **: {v ∈ V(G) :** $\boldsymbol{\gamma_{Tk}(G\text{-}v) = \gamma_{Tk}(G)}$**.}**

Theorem -3.11: Let v ∈ V(G) such that d(v) ≥ k and v ∉ $\mathbf{V^i_{Tk}}$**. If v ∈** $\mathbf{V^-_{Tk}}$ **then** $\boldsymbol{\gamma_{Tk}(G)\text{-}k \le \gamma_{Tk}(G\text{-}v) \le \gamma_{Tk}(G)\text{-}1}$**.**

Proof:

Let S_1 be a minimum totally k-dominating set of G-{v}. Since v ∈ V^-_{Tk} , | S_1 | < γ_{Tk}(G) and v is adjacent to at most k-1 vertices of S_1. Suppose v is not adjacent to any vertex of S_1. Let $z_1, z_2, \ldots\ldots z_k$ be k neighbor of v.

Let S = S_1 U { $z_1, z_2, \ldots\ldots z_k$ }, then S is a totally k- dominating set in G. Therefore $\gamma_{Tk}(G) \le |S| = |S_1| + k = \gamma_{Tk}(G\text{-}v) + k$. Therefore $\gamma_{Tk}(G) - k \le \gamma_{Tk}(G\text{-}v)$.

If v is adjacent to m vertices say $z_1, z_2, \ldots\ldots z_m$ (m<k). Let z_{m+1} , z_{m+2} , .. z_k be the vertices adjacent to v and not in S_1.

Let S = S_1 U { $z_{m+1}, z_{m+2}, \ldots\ldots z_k$ }, then as above S is a totally k- dominating set in G and by similar argument $\gamma_{Tk}(G) - k \le \gamma_{Tk}(G) -(k\text{-}m) \le \gamma_{Tk}(G\text{-}v)$.

Thus in both the cases the inequality holds. ∎

Theorem -3.12: Suppose $v \in V(G)$, $d(v) \geq k$ and $v \notin V^{i}_{Tk}$ then $v \in V^{+}_{Tk}$ if and only if the following conditions hold.

(1) v is contained in every γ_{Tk} set of G.

(2) No subset S of V(G) which intersects N[v] in at most k-1 vertices of N[v] and with $|S| \leq \gamma_{Tk}(G)$ can be a totally k-dominating set of G-{v}.

Proof:

(1)

Suppose $v \in V^{+}_{Tk}$. Suppose S_0 is a γ_{Tk} of G such that $v \notin S_0$. Let v_1 be any vertex of G-{v}. Since $v \notin V^{i}_{Tk}$, $d(v_1) \geq k$ in G-{v} and hence G also. Thus, v_1 is adjacent to at least k vertices of S_0. Thus, S_0 is a totally k – dominating set of G-{v}. Thus, $\gamma_{Tk}(G-v) \leq |S_0| = \gamma_{Tk}(G)$. That is $v \notin V^{+}_{Tk}$. a contradiction.

(2)

Suppose there is a set S_0 which intersects N[v] in at most k-1 vertices, and $|S_0| \leq \gamma_{Tk}(G)$ and S_0 is a totally k-dominating set of G-{v}. Then $\gamma_{Tk}(G-v) \leq |S_0| \leq \gamma_{Tk}(G)$. This is again a contradiction. Therefore condition (2) holds.

Now we prove converse.

Suppose $v \in V^{0}_{Tk}$. Let S be a minimum totally k-dominating set of G-{v}.If v is adjacent to at least k vertices of S then S is a minimum totally k-dominating set of G not containing v, which contradict (1).

Suppose v is adjacent to m vertices of S where $0 \leq m < k$. Then S is a set which intersects N[v] in at most k -1 vertices, $|S| \leq \gamma_{Tk}(G)$ and S is a totally k-dominating set of G-{v} which contradicts (2).

Suppose $v \in V^{-}_{Tk}$. Then $\gamma_{Tk}(G) - k \leq \gamma_{Tk}(G-v) \leq \gamma_{Tk}(G) - 1$.

Let S be a minimum totally k-dominating set of G-{v}. If v is adjacent to at least k vertices of S then S is a totally k-dominating set of G with $|S| < \gamma_{Tk}(G)$. That is

$\gamma_{Tk}(G) < \gamma_{Tk}(G)$ - a contradiction. So v is adjacent to at most k-1 vertices of S then also S is a set which intersects N[v] in at most k-1 vertices and $|S| \leq \gamma_{Tk}(G)$ is a totally k-dominating set of G-{v}, which contradicts (2).

Thus, v can not be in V^-_{Tk} or V^0_{Tk}. Hence $v \in V^+_{Tk}$ ■

Next we prove the following theorem.

Theorem -3.13: Suppose $d(v) \geq k$ and $v \in V^+_{Tk}$. Then for any γ_{Tk} set S, $v \in S$ and $P_{Tk}[v,S]$ contains at least two vertices.

Proof :

Let S be any γ_{Tk} set of G. Since $v \in V^+_{Tk}$, $v \in S$. Since S is a minimum set, $P_{Tk}[v,S]$ contains at least one vertex.

Suppose $P_{Tk}[v,S]$ contains only one vertex say w.

Claim: $w \notin S$.

Proof of the Claim: Suppose $w \in S$. If w is not adjacent to any vertex out side S then $d(w) < k$ in G-{v}which contradicts that $v \notin V^i_{Tk}$. Thus, there is a vertex w_1 outside S which is adjacent to w.

Now let $S_1 = S - \{v\} \cup \{w_1\}$. Then S_1 is a minimum totally k-dominating set of G not containing v. which contradicts that $v \in V^+_{Tk}$.

This proves that $w \notin S$. Since $d(w) \geq k$, in G-{v}, w is adjacent to some vertex w_1 which is out side S.

Now let $S_1 = S - \{v\} \cup \{w_1\}$. Then S_1 is a minimum totally k-dominating set of G not containing v , which is a contradiction.

Thus, in any case we get a contradiction. Hence $P_{Tk}[v,S]$ contains at least two vertices. ■

Theorem -3.14: Let v be a vertex of G such that $d(v) \geq k$ and $v \notin V^{i}_{Tk}$. Then $v \in V^{-}_{Tk}$ if and only if there is a minimum totally k-dominating set S and k vertices $w_1, w_2, \ldots w_k$ in S such that $P_{Tk}[w_i, S] = \{v\}$ for every i.

Proof:

Suppose $v \in V^{-}_{Tk}$. Let S_1 be a minimum totally k-dominating set of G-{v}.

If v is not adjacent to any vertex of S_1 then let $w_1, w_2, \ldots w_k$ be k vertices adjacent to v. Let $S = S_1 U \{ w_1, w_2, \ldots w_k \}$ then S is a minimum totally k-dominating set of G. For each i v is adjacent to exactly k vertices of S including w_i (other vertices to which v is adjacent are $w_1, w_2, \ldots w_{i-1}, w_{i+1}, \ldots w_k$) Thus, $v \in P_{Tk}[w_i, S]$ for every i.

Let v_1 be a vertex different from v. Since S_1 is a totally k-dominating set of G-{v}, v_1 is adjacent to at least k-vertices of S_1 and no w_i is member of S_1. Therefore $v_1 \notin P_{Tk}[w_i, S]$.

Hence $P_{Tk}[w_i, S] = \{v\}$ for each i.

To prove converse suppose S is a minimum totally k-dominating set of G and $w_1, w_2, \ldots w_k$ are vertices of S such that $P_{Tk}[w_i, S] = \{v\}$ for each i.

Let $S_1 = S - \{w_1\}$. We will prove that S_1 is a totally k-dominating set of G-{v}. Let z be any vertex of G-{v}.First suppose that $z = w_1$. Since S is a totally k-dominating set in G, $z = w_1$ is adjacent to at least k vertices of S_1.

Suppose $z \neq w_1$. Since $z \neq v$, $z \notin P_{Tk}[w_i, S]$. Hence if z is adjacent to w_1 in G then z must be adjacent to at least k other vertices of S. This means that z is adjacent to at least k vertices of S_1. If z is not adjacent to w_1 then since S is a totally k-dominating set of G, z is adjacent to at least k vertices of S_1.

Thus in any case z is adjacent to at least k vertices of S_1. This proves that S_1 is a totally k-dominating set of G-{v}. and hence $\gamma_{Tk}(G-v) \leq |S_1| < |S| = \gamma_{Tk}(G)$. This means that $v \in V^{-}_{Tk}$. ■

Corollary -3.15: Suppose v is a vertex in V^{+}_{Tk} and w is a vertex in V^{-}_{Tk} then v and w are non adjacent.

Proof:

There is a minimum totally k-dominating set S and k vertices $w_1, w_2, \ldots w_k$ in S such that $P_{Tk}[w_i, S] = \{w\}$for each i. Since $v \in V^{+}_{Tk}$, $v \in S$. (Theorem -3.3). Note that $v \neq w_i$ for any i, because $P_{Tk}[v,S]$ contains at least two vertices while $P_{Tk}[w_i,S]$ contains only w. Now if v and w are adjacent then w is adjacent to k+1 vertices of S including w_1 which contradicts the fact that $P_{Tk}[w_1,S] = \{w\}$. Thus, v and w can not be adjacent. █

K-TUPLE DOMINATION

The concept of k-tuple domination can be found in [44]. Note that every totally k-dominating set is a k-tuple dominating set but converse is not true. We begin with the definition of a k-tuple dominating set.

Definition -3.16: k-tuple dominating set.[44]

Let G be a graph and k be an integer greater than or equal to two. A subset S of V(G) is said to be a k-tuple dominating set if following conditions satisfied.

(1) If $v \in S$ then v is adjacent to at least k-1 vertices of S.

(2) If $v \notin S$ then v is adjacent to at least k vertices of S.

Definition -3.17: Minimal k-tuple dominating set.

A k-tuple dominating set S of G is said to be a minimal k-tuple dominating set if for each vertex v of S, S-{v} is not a k-tuple dominating set.

Definition -3.18: Minimum k-tuple dominating set.

A k-tuple dominating set with minimum cardinality is called minimum k-tuple dominating set which also called γ_{ku} set of G.

Definition -3.19: k-tuple domination number.

The cardinality of a minimum k-tuple dominating set is called k-tuple domination number of the graph G. It is denoted by $\gamma_{ku}(G)$.

Remark-3.20: Note that any minimum totally k-dominating set is a k-tuple dominating set, but converse is not true. This means that $\gamma_{ku}(G) \leq \gamma_{Tk}(G)$.

Example -3.21: Consider the cycle C_5 with vertices v_1, v_2, v_3, v_4, v_5. Let k=2 then 2-tuple domination number of C_5 is 4 and total 2-domination number is 5.
(See Figure –0.2)

Now we define so called k-tuple private neighborhood of a vertex v with respect to a set containing it.

Definition -3.22: k-tuple private neighborhood.

Let S be a subset of V(G) and $v \in S$. Then the k-tuple private neighborhood of v with respect to S. i.e. $P_{ku}[v,S] = S_1 U S_2 U S_3$
Where $S_1 = \{w \in S: w \neq v$ and w is adjacent to exactly k-1 vertices of S including v.$\}$,
$S_2 = \{w \in S: w = v$ and w is adjacent to exactly k-1 vertices of S$\}$,
$S_3 = \{w \notin S:$ w is adjacent to exactly k vertices of S including v.$\}$

For example if we consider the cycle graph C_5, (See Figure –0.2)

$S = \{ v_1, v_2, v_3, v_4 \}$, $v = v_1$ then $P_{ku}[v_1, S] = \{ v_1, v_5 \}$.

Note that in the above definition any one of S_1, S_2, S_3 can be an empty set.

Also note that every minimum k-tuple dominating set is a minimal k-tuple dominating set.

We state the following theorem without proof as it is similar to that of Theorem -3.9

Theorem-3.23: A subset S of V(G) is a minimal k-tuple dominating set if and only if for each vertex v of S $P_{ku}[v,S] \neq \phi$. █

Now we introduced the following symbols.

$$V^{+}_{ku} = \{ v \in V(G): \gamma_{ku}(G-v) > \gamma_{ku}(G) \}.$$

$$V^{-}_{ku} = \{ v \in V(G): \gamma_{ku}(G-v) < \gamma_{ku}(G) \}.$$

$$V^{0}_{ku} = \{ v \in V(G): \gamma_{ku}(G-v) = \gamma_{ku}(G) \}.$$

Theorem-3.24: Let $v \in V(G)$ such that $d(v) \geq k$ and $v \notin V^{i}_{Tk}$. Then $v \in V^{-}_{ku}$ if and only if $\gamma_{ku}(G) - k \leq \gamma_{ku}(G-v) < \gamma_{ku}(G)$.

Proof:

Suppose $v \in V^{-}_{ku}$. Let S_1 be a minimum k-tuple dominating set of G-{v}. Obviously v is adjacent to at most k-1 vertices of S_1.

If v is adjacent to exactly k-1 vertices of S_1 and in this case let $S = S_1 U\{v\}$. Then S is a minimum k-tuple dominating set of G and $|S| = |S_1| + 1$. This means that $\gamma_{ku}(G-v) = \gamma_{ku}(G) -1$.

If v is adjacent to no vertex of S_1 then let $w_1, w_2, \ldots w_k$ be vertices adjacent to v. Let $S = S_1 U\{ w_1, w_2, \ldots w_k \}$, then S is a k-tuple dominating set of G. Therefore $\gamma_{ku}(G) \leq | S | = | S_1 | + k = \gamma_{ku}(G-v) + k$. This proves that $\gamma_{ku}(G) - k \leq \gamma_{ku}(G-v) < \gamma_{ku}(G)$.

Suppose v is adjacent to m vertices of S_1 say $w_1, w_2, \ldots w_m$. ($1 \leq m < k$). Let w_{m+1} , w_{m+2} ……w_k be vertices adjacent to v and not in S_1. Let $S = S_1 U \{ w_{m+1} , w_{m+2} \ldots w_k \}$. Then S is a k-tuple dominating set of G and $| S | = | S_1 | + k$.

Therefore $\gamma_{ku}(G) \leq \gamma_{ku}(G-v) + k-m < \gamma_{ku}(G-v) +k$.

Hence

$$\gamma_{ku}(G) - k \leq \gamma_{ku}(G-v) < \gamma_{ku}(G).$$

This proves the theorem. █

We state the following theorem without proof as it is similar to that of Theorem-3.12

Thoerem-3.25: Let $v \in V(G)$ such that $d(v) \geq k$ and $v \notin V^{i}_{Tk}$. Then $v \in V^{+}_{ku}$ if and only if each of the following two conditions is satisfied.

(1) v is contained in every minimum k-tuple dominating set.

(2) No subset S of V(G-v) which intersects N[v] in at most k-1 vertices and with $|S| \leq \gamma_{ku}(G)$ can be a tuple dominating set of G-{v}. $\blacksquare$

Theorem-3.26: Let $v \in V(G)$ such that $d(v) \geq k$ and $v \notin V^{i}_{Tk}$. If $v \in V^{+}_{ku}$ and S is a minimum k-tuple dominating set then $v \in S$ and $P_{ku}[v,S]$ contains at least two vertices.

Proof:

By Theorem-3.23, $v \in S$. Since S is a minimal k-tuple dominating set, $P_{ku}[v,S]$ is non empty.

First suppose that $P_{ku}[v,S]$ consists only one vertex w.

Let $w \in P_{ku}[v,S]$. If w = v then S-{v} is a k-tuple dominating set of G-{v}.This means that $v \in V^{-}_{ku}$ and this is a contradiction. If $w \neq v$ then there are two cases:

Case-1:

$w \in S$. Then w is adjacent to exactly k-1 vertices including v of S. Since $d(w) \geq k$, there is a vertex w_1 outside S which is adjacent to w. Let $S_1 = S - \{v\} \cup \{w_1\}$, then S_1 is a minimum k-tuple dominating set of G not containing v. This contradicts the assumption that $v \in V^{+}_{ku}$.

Case-2:

$w \notin S$. Let $S_1 = S - \{v\} \cup \{w\}$, then S_1 is a minimum k-tuple dominating set of G not containing v ,which is again a contradiction . as $v \in V^{+}_{ku}$.

Thus, the assumption that the $P_{ku}[v, S]$ contains only one vertex leads to a contradiction. Therefore it must contain at least two vertices. $\blacksquare$

Example-3.27: Consider the following graph to understand for the Theorem-3.26.

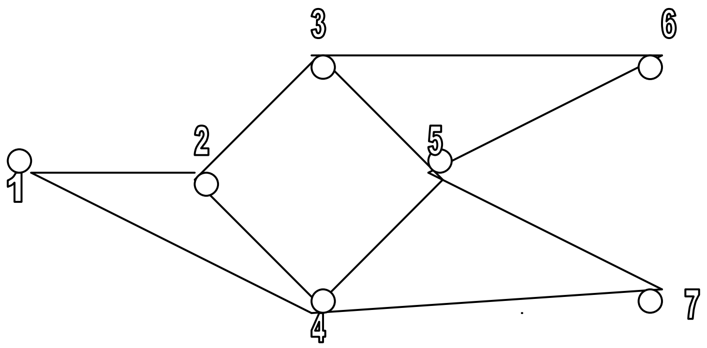


Figure-3.2

γ_{2u} = {2,3,4,5} and k =2, So, (G) = 4. ,5 ∈ V^{+}_{ku} Now for the graph G –{5}

γ_{2u} = {2,3,4,6,7} = S , and k = 2 , So, P_{2u}[5,S] = {6,7}

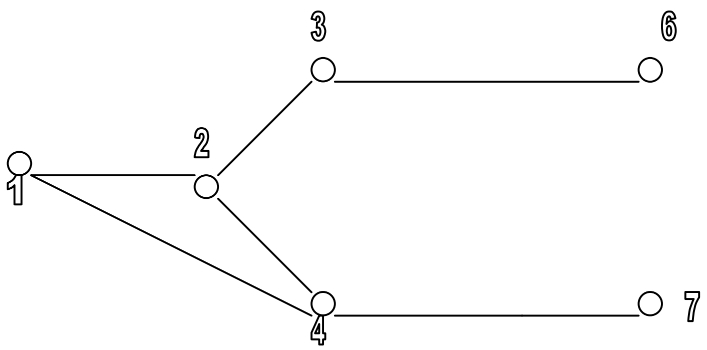


Figure-3.3

Now for the graph G –{5}

γ_{2u} = {2,3,4,6,7} = S , and k = 2 , So, P_{2u}[5,S] = {6,7}

Theorem-3.28: Let $v \in V(G)$, $d(v) \geq k$, and $v \notin V^{i}_{Tk}$,

(1) If $v \in V^{-}_{ku}$ then there is a minimal k-tuple dominating set S containing v such that $P_{ku}[v,S] = \{v\}$.

(2) If there is a minimum k-tuple dominating set S containing v such that $P_{ku}[v,S] =\{v\}$ then $v \in V^{-}_{ku}$.

Proof: (1)

Suppose $v \in V^{-}_{ku}$. Let S_1 is a minimum k-tuple dominating set of G-{v}. Then v is adjacent to at most k-1 vertices of S_1.

Case-1: v is adjacent to no vertex of S_1.

Let $w_1, w_2, \ldots\ldots w_{k-1}$ be vertices not in S_1 such that w_i is adjacent to v for every i. Let $S = S_1 \cup \{ w_1, w_2, \ldots\ldots w_{k-1} , v \}$. Then S is a minimal k-tuple dominating set of G containing v.

Suppose v_1 is a vertex different from v.

If $v_1 \in S_1$ then v_1 is adjacent to at least k-1 vertices of S_1. Thus, if v_1 is adjacent to v then v_1 is adjacent to at least k vertices of S. Therefore $v_1 \notin P_{ku}[v,S]$.

Suppose $v_1 = w_i$ for some i. Now $w_i \notin S_1$ and therefore w_i is adjacent to at least k vertices of S_1. Therefore if w_i is adjacent to v then w_i is adjacent to k+1 vertices of S. Therefore $w_i \notin P_{ku}[v,S]$.

Suppose $v_1 \notin S$ then v_1 is adjacent to at least k vertices of S_1 therefore if v_1 is adjacent to v then v_1 is adjacent to k+1 vertices of S. Therefore $v_1 \notin P_{ku}[v,S]$.

Case-2: v is adjacent to m vertices $w_1, w_2, \ldots\ldots w_m$ of S_1 where $1 \leq m < k$

Let $w_{m+1}, w_{m+2}, \ldots\ldots w_{k-1}$ be vertices not in $S_1$and adjacent to v. Let $S = S_1 \cup \{ w_{m+1}, w_{m+2}, \ldots\ldots w_{k-1}, v \}$ then S is a minimal k-tuple dominating set of G containing v.

Let v_1 be a vertex different from v.

If $v_1 = w_i$ for some $i \in \{1,2,3\ldots\ldots m\}$ then if w_i is adjacent to exactly k-1 vertices of S_1 then w_i is adjacent to k vertices of S if w_i is adjacent to v. Therefore
$v_1 = w_i \notin P_{ku}[v,S]$.

If $v_1 = w_i$ for some $i \in \{m+1, m+2, \ldots\ldots k-1\}$ then since w_i is adjacent to at least k vertices of S_1. w_i is adjacent to at least k+1 vertices of S, if w_i is adjacent to v. Therefore $w_i \notin P_{ku}[v,S]$.

Case-3: v is adjacent to exactly k-1 vertices of S_1.

Let $S = S_1 \cup \{v\}$, then S is a minimal k-tuple dominating set of G containing v. Let v_1 be a vertex different from v.

If $v_1 = w_i$ for some i, then since w_i is adjacent to at least k-1 vertices of S_1, w_i is adjacent to at least k vertices of S including v, if w_i is adjacent to v. Therefore
$v_1 = w_i \notin P_{ku}[v,S]$.

If $v_1 \in S_1$ then v_1 is adjacent to at least k-1 vertices of S_1. Therefore v_1 is adjacent to at least k vertices of S if v_1 is adjacent to v. Therefore $v_1 \notin P_{ku}[v,S]$.

If $v_1 \notin S_1$ then v_1 is adjacent to at least k vertices of S_1 and therefore adjacent to at least k+1 vertices of S if v_1 is adjacent to v. Therefore $v_1 \notin P_{ku}[v,S]$.

Note that $v \in P_{ku}[v,S]$. Hence $P_{ku}[v,S] = \{v\}$.

(2)

Suppose there is a minimum k-tuple dominating set S of G containing v such that $P_{ku}[v,S] = \{v\}$.

Let $S_1 = S-\{v\}$. We will prove that S_1 is a k-tuple dominating set of G-{v}.

Let v_1 be any vertex of G-{v}.

Case-1: $v_1 \in S_1$.

Since S is a k-tuple dominating set of G. v_1 is adjacent to at least k-1 vertices of S. Suppose v_1 is adjacent to v in G and v_1 is adjacent to exactly k-1 vertices of S then v_1 vertex different from v and $v_1 \in P_{ku}[v,S]$ which is not true. Since $P_{ku}[v,S] = \{v\}$. Therefore if v_1 is adjacent to v. Then $v_1$is adjacent to at least k-1 other vertices of S. Thus, v_1 is adjacent to at least k-1 vertices of S_1 = S-{v}. If v_1 is not adjacent to v then v_1 is adjacent to at least k-1 vertices of S different from v. Therefore v_1 is adjacent to at least k-1 vertices of S_1.

Suppose $v_1 \notin S_1$. Now $v_1 \neq v$. Therefore $v_1 \notin S$. Now since S is a k-tuple dominating set of G. v_1 is adjacent to at least k vertices of S different from v. Therefore v_1 is adjacent to at least k vertices of S_1.Thus, S_1 is a k-tuple dominating set of G-{v}.

Therefore,

$$\gamma_{ku}(G\text{-}v) \leq | S_1 | < | S | \leq \gamma_{ku}(G)$$

Therefore,

$$\gamma_{ku}(G\text{-}v) < \gamma_{ku}(G)$$

Therefore,

$$v \in V^{-}_{ku}. \blacksquare$$

The following definition of k-dependent set is due to J. F. Fink and M.S. Jacobson [21]

K-DEPENDENT K-DOMINATION

Definition -3.29: k-dependent set.[21]

Suppose $k \geq 1$. A set S subset of V(G) is said to be k-dependent set if for every vertex v in S, v is adjacent to at most k-1 vertices of S.

Note that if k=1 then 1-dependent set is just an independent set.

Definition -3.30: Maximal k-dependent set.

Let $k \geq 1$ and S be a subset of V(G). Then S is said to be a maximal k-dependent set if

(1) S is a k-dependent set.

(2) For every vertex v not in S, S U {v} is not a k-dependent set.

Note that every maximum k-dependent set is a maximal k-dependent set.

If S is a maximal k-dependent set then obviously for every vertex v not in S v is adjacent to at least k vertices of S. Thus, S is a k-dominating set. Hence every maximal k-dependent set is a k-dominating set.

Also if S is a k-dependent set and $v \in S$ then v is adjacent to at most k-1 vertices of S. Therefore v belongs to private k-neighborhood of v with respect to S, which is denoted as $P_k[v,S]$. That is $P_k[v,S]$ is non empty.

Therefore S is a minimal k-dominating set of G. [2] Thus, every maximal k-dependent set is a minimal k-dominating set.

Definition -3.31: k-dependent k-dominating set.

Let $k \geq 1$ and S is subset of V(G). Then S is said to be k-dependent k-dominating set if

(1) S is a k-dependent set.
(2) S is a k-dominating set.

Definition -3.32: Minimal k-dependent k-dominating set.

Let S be a k-dependent k-dominating set then S is said to be minimal k-dependent k-dominating set if for each vertex $v \in S$, S-{v} is not a (k-dependent) k-dominating set.

Definition -3.33: Minimum k-dependent k-dominating set.

A k-dependent k-dominating set S with minimum cardinality is called a minimum k-dependent k-dominating set. It is denoted by i_k set.

Definition -3.34: k-dependent k-domination number.

The cardinality of a minimum k-dependent k-dominating set is called k-dependent k-domination number of the graph G. It is denoted as $i_k(G)$.

Thus, by above remark every maximal k-dependent set is a minimal k-dependent k-dominating set.

Conserve is also true. That is every minimal k-dependent k-dominating set is also a maximal k-dependent set.

Thus, the minimum cardinality of a k-dependent k-dominating set = the minimum cardinality of a maximal k-dependent set. That is $i_k(G)$.

We define the following symbols.

$$\mathbf{V^{+}_{Ik} = \{v \in V(G) : \gamma_{ik}(G) < \gamma_{ik}(G\text{-}v)\}.}$$

$$\mathbf{V^{-}_{Ik} = \{v \in V(G) : \gamma_{ik}(G) > \gamma_{ik}(G\text{-}v)\}.}$$

$$\mathbf{V^{0}_{Ik} = \{v \in V(G) : \gamma_{ik}(G) = \gamma_{ik}(G\text{-}v)\}.}$$

$$\mathbf{V^{i}_{Tk} = \{ G\text{-}\{v\} \text{ has a vertex which degree is less than k.}\}}$$

Note that the above sets are mutually disjoint and their union is V(G).

We state the following theorem without proof.

Theorem-3.35: Let $v \in V(G)$, $d(v) \geq k$ and $v \notin V^{i}_{Tk}$ then $v \in V^{+}_{Ik}$ if and only if the following conditions holds.

(1) v belongs to every minimum k-dependent k-dominating set of G.

(2) No subset S of G-{v} which intersects N[v] in at most k-1 vertices and $|S| \leq i_k(G)$ can be a k-dependent k-dominating set of G-{v}.

Proof: The proof of this theorem is similar to that of corresponding theorem for total k-domination. ■

Example-3.36:

(1) Consider the graph G = Petersen Graph (See Figure- 0.3)

For the Petersen Graph i_3 set is {2, 4, 6, 7, 9, 10} and $i_3(G) = 6$.

and i_2 set is {1, 3, 6, 9, 10} and $i_2(G) = 5$.

(2) Consider the graph G = Hyper Qube (See Figure – 0.8)

For the Hyper Qube Graph i_3 set is {2, 3, 4, 5, 6, 8} and $i_3(G) = 6$.

and i_2 set is {2, 4, 6, 8} and $i_2(G) = 4$.

Definition -3.37: External private k-neighborhood.

Let S be subset of V(G) and v $\in$ S, then the external private k-neighborhood of v with respect to S. i.e $E_x[v,S]$

$E_x[v,S]$ = {w $\in$ V(G)-S : w is adjacent to exactly k vertices of S including v.}

Now we state and prove the equivalent conditions for vertex v to be in $V^-_{Ik.}$

Theorem-3.38: Let v $\in$ V(G), d(v) $\geq$ k, and v $\notin$ V^i_{Tk} then the following statements are equivalent.

(1) v $\in$ V^-_{Ik}.

(2) There is a minimum k-dependent k-dominating set S containing v such that $E_{xk}[v,S]$ is empty.

(3) There is a minimum k-dependent k-dominating set S of G containing v such that S-{v} is a k-dependent k-dominating set of G-{v}.

Proof:

(1) => (2)

Let S_1 be a k-dependent k-dominating set of G – {v}. Then $|S_1| < i_k(G)$. If v is adjacent to at least k vertices of S_1 then S_1 is a k-dependent k-dominating set of G and therefore $i_k(G) \leq |S_1| < i_k(G)$. This is a contradiction. Therefore v is adjacent to at most k-1 vertices of S_1.

Let S = S_1 U {v} then S is a minimum k-dependent k-dominating set of G containing v.

Suppose w $\in E_{xk}[v,S]$ then w is adjacent to exactly k-vertices of S including v therefore w is a vertex of G-{v} such that w $\notin$ S_1 and w is adjacent to exactly k-1 vertices of S_1. This is a contradiction because S_1 is a maximal k-dependent set in G-{v}. Therefore $E_{xk}[v,S]$ is empty. Hence (1) => (2) is proved.

Now (2) => (3).

Let S be a minimum k-dependent k-dominating set of G containing v such that $E_{xk}[v,S]$ is empty.

Consider the set S_1 = S-{v}. We prove that S_1 is a k-dependent k-dominating set of G-{v}.

Let w be a vertex of G-{v} such that $w \notin S-\{v\}$. Then w is a vertex of G with $w \notin S$. If w is adjacent to v in G then w must be adjacent to at least k other vertices of S (because $w \notin E_{xk}[v,S]$) Therefore w is adjacent to at least k vertices of S –{v}.

Since S is a k-dependent set in G, S-{v} is also k-dependent set in G-{v}. Thus, S-{v} is a k-dependent k-dominating set of G-{v}. Hence (2) => (3) is proved.

Now (3) => (1)

Let S be a minimum k-dependent k-dominating set of G containing v such that S-{v} is a k-dependent k-dominating set of G-{v}. Then

$$i_k(G-v) \leq |S-\{v\}| < |S| = i_k(G)$$

Therefore,

$$i_k(G-v) < i_k(G).$$

Hence $v \in V^{-}_{Ik}$. Thus, (3) => (1) is proved. $\blacksquare$

Chapter-4:

PERFECT DOMINATION

Perfect Domination is closely related to Perfect Codes and Perfect Codes have been used in Coding Theory. In this chapter we study the effect of removing a vertex from the graph on perfect domination.

Definition-4.1: Perfect dominating set.[42]

A subset S of V(G) is said to be a perfect dominating set if for each vertex v not in S, v is adjacent to exactly one vertex of S.

Consider the path P_4 with four vertices 1,2,3,4. The set S= {2, 3} is perfect dominating set in this graph.

It may be noted that if G is a graph then V(G) is always a perfect dominating set of G.

Definition-4.2: Minimal perfect dominating set.

A perfect dominating set S of the graph G is said to be minimal perfect dominating set if for each vertex v in S, S-{v} is not a perfect dominating set.

It may be noted that it is not necessary that a proper subset of minimal perfect dominating set is not a perfect dominating set.

Example-4.3:

Consider the cycle graph G = C_6 with six vertices 1, 2, 3, 4, 5, 6. Then obviously V(G) is a minimal perfect dominating set of G.

However the set {1, 4} is proper subset of V(G) and is a perfect dominating set in the graph G.

Definition-4.4: Minimum perfect dominating set.

A perfect dominating set with smallest cardinality is called minimum perfect dominating set. It is called γ_{pf} set of the graph G.

Definition-4.5: Perfect domination number.

The cardinality of a minimum perfect dominating set is called the perfect domination number of the graph G. It is denoted as $\gamma_{pf}(G)$.

The perfect domination number of cycle C_6 is 2 and that of the path P_3 is also 1.

Definition-4.6: Perfect private neighborhood.

Let S be a subset of V(G) and v $\in$ S. Then the perfect private neighborhood of v with respect to S =

$P_{pf}[v,S]$ = { w $\in$ V(G)-S: N(w) $\cap$ S ={v}} U {v, if v is adjacent to no vertex of S or at least vertices of S }.

Theorem-4.7: A perfect dominating set S of G is minimal perfect dominating set if and only if for each vertex v in S $P_{pf}[v,S]$ is non- empty.

Proof:

Suppose S is minimal and v $\in$ S. Therefore there is a vertex w not in S-{v} such that either w is adjacent to no vertex of S-{v} or w is adjacent to at least two vertices of S-{v}.

If w = v then this implies that v$\in$ $P_{pf}[v,S]$.

If w $\neq$ v then it is impossible that w is adjacent to at least two vertices of S-{v} because S is a perfect dominating set. Therefore w is not adjacent to any vertex of S-{v}. Since S is a perfect dominating set w is adjacent to only v in S. That is N(w) $\cap$ S = {v}. Thus, w $\in$ $P_{pf}[v,S]$.

Conversely suppose $v \in S$ and $P_{pf}[v,S]$ contains some vertex w of G.

If w = v then w is either adjacent to at least two vertices of S-{v} or w is adjacent to no vertex of S-{v}. Thus, S-{v} is not a perfect dominating set.

If $w \neq v$ then $N(w) \cap S = \{v\}$ implies that w is not adjacent to any vertex of S-{v}.

Thus, in all cases S-{v} is not a perfect dominating set if $v \in S$. Thus, S is minimal. ■

Example-4.8:

Consider the path $G = P_5$ (See Figure-0.4) with five vertices v_1, v_2, v_3, v_4, v_5 . Note that $S = \{v_2,v_5\}$ is minimum and therefore minimal perfect dominating set. $P_{pf}[v_2,,S] = \{ v_1, v_2, v_3\}$.

Now we define the following symbols.

$$\mathbf{V^{+}_{pf} = \{v \in V(G): \gamma_{pf}(G) < \gamma_{pf}(G\text{-}v)\}.}$$

$$\mathbf{V^{-}_{pf} = \{v \in V(G): \gamma_{pf}(G) > \gamma_{pf}(G\text{-}v)\}.}$$

$$\mathbf{V^{0}_{pf} = \{v \in V(G): \gamma_{pf}(G) = \gamma_{pf}(G\text{-}v)\}.}$$

Note that the above sets are mutually disjoint and their union is V(G).

Now we prove the following lemma.

Lemma-4.9: Let $v \in V(G)$ and suppose v is a pendent vertex and has a neighbor w of degree at least two. If $v \in V^{-}_{pf}$ then $\gamma_{pf}(G\text{-}v) = \gamma_{pf}(G) - 1$.

Proof:

Let S_1 be a minimum perfect dominating set of G-{v}. If $w \in S_1$ then S_1 is a perfect dominating set of G with $|S_1| < \gamma_{pf}(G)$. That is $\gamma_{pf}(G) \leq |S_1| < \gamma_{pf}(G)$, this is a contradiction.Therefore $w \notin S_1$. Let $S = S_1 \cup \{w\}$. Then S is a minimum perfect dominating set of G. Therefore $\gamma_{pf}(G) = |S| = |S_1| + 1 = \gamma_{pf}(G\text{-}v) + 1$.

This proves the lemma. ■

Next we prove the necessary and sufficient conditions for a pendent vertex (with a neighbor of degree at least two) to be in V^{+}_{pf}.

Theorem-4.10: Let v be a vertex of G Then v $\in$ V^{+}_{pf} if and only if the following conditions are satisfies.

(1) v belongs to every γ_{pf} set of G.

(2) No subset S of G-{v} which is either disjoint from N[v] or intersects N[v] in at least two vertices and |S| $\leq \gamma_{pf}(G)$ can be a perfectly dominating set of G-{v}.

Proof:

(1)

Suppose v $\in$ V^{+}_{pf} .

Suppose S is a γ_{pf} set of G which does not contain v then S is a perfect dominating set of G-{v}.Therefore $\gamma_{pf}(G\text{-}v) \leq |S| = \gamma_{pf}(G)$.Thus, v $\notin$ V^{+}_{pf}. This is a contradiction. Thus, v must belong to every γ_{pf} set of G.

(2)

If there is set S which satisfies the condition stated in (2). Then S is a perfect dominating set of G-{v} and therefore $\gamma_{pf}(G\text{-}v) \leq \gamma_{pf}(G)$. – This is a contradiction.

Conversely assume that (1) and (2) hold.

Suppose v $\in$ V^{0}_{pf}. Let S be a minimum perfect dominating set of G-{v}.Then |S| = $\gamma_{pf}(G)$.

Suppose v is not adjacent to any vertex of S. Then S is disjoint from N[v], |S|= $\gamma_{pf}(G)$ and S is a perfectly dominating set of G-{v}. This violates (2).

Suppose v is adjacent to exactly one vertex of S then S is a minimum perfect dominating set of G not containing v which violates (1).

Suppose v is adjacent to at least two vertices of S. Then S ∩N[v] in at least two vertices and S is a perfectly dominating set of G-{v} with $|S| = \gamma_{pf}(G)$, which again violate (2).

Thus, $v \in V^0_{pf}$ implies (1) or (2) violated.

Suppose $v \in V^-_{pf}$. Let S_1 be a minimum perfect dominating set of G-{v}.
Then $|S_1| < \gamma_{pf}(G)$. If v is not adjacent to any vertex of S_1 then as above (2) is violated.
If v is adjacent to exactly one vertex of S_1 then S_1 is a perfect dominating set of G with $|S_1| < \gamma_{pf}(G)$ – which is a contradiction.

If v is adjacent to at least two vertices of S_1 then $S_1 \cap N[v]$ in at least two vertices, $|S_1| \leq \gamma_{pf}(G)$ and S_1 is a perfect dominating set of G-{v} – which again violates (2).

Thus, $v \in V^-_{pf}$ implies that (2) is violated.

Thus, v does not belongs to V^0_{pf} or V^-_{pf}. Hence $v \in V^+_{pf}$. █

Theorem-4.11: Let v be a pendent vertex which has the neighbor w of degree at least two then $v \in V^-_{pf}$ if and only if there is γ_{pf} set S containing w and not containing v such that $P_{pf}[w, S] = \{v\}$.

Proof:

Suppose $v \in V^-_{pf}$. Let S_1 be a minimum perfect dominating set of G-{v}.Then as proved Lemma -4.9, $w \notin S_1$. Let $S = S_1 \cup \{w\}$. Then S is γ_{pf} containing w.

Since S_1 is a perfect dominating set of G-{v}, w is adjacent to some vertex of S_1. Therefore $w \notin P_{pf}[w,S]$. If x is any vertex different from v such that x is adjacent to w then x is also adjacent to some vertex of S_1 because S_1 is a perfect dominating set of G-{v}. Thus, $x \notin P_{pf}[w,S]$. Further v is adjacent to only w of S therefore $P_{pf}[w,S] = \{v\}$.

Conversely suppose there is a γ_{pf} set S containing w such that $P_{pf}[w,S] = \{v\}$. Let S_1 = S- {w}. Let x be any vertex of G-{v} which is not in S-{v}. Since $x \notin P_{pf}[w,S]$, x must be adjacent to some unique vertex S_1. Thus, S_1 is a minimum perfect dominating set of G-{v} with $|S_1| < \gamma_{pf}(G)$. Thus, $v \in V^{-}_{pf}$. ■

Example-4.12:

Consider the path G= P_4 with vertices 1,2,3,4. Then $\gamma_{pf}(G) = 2$. Let v = 1 and w = 2.

Now $\gamma_{pf}(G\text{-}1) = 1$. Thus, $1 \in V^{-}_{pf}$ also S = (2, 3) is γ_{pf} set of G , containing w = 2 and $P_{pf}[2,S] = \{1\}$.

Theorem-4.13: Let S_1 and S_2 be two disjoint perfect dominating sets of G. Then $|S_1| = |S_2|$

Proof:

For every vertex x in S_1 there is a unique vertex v(x) in S_2 which is adjacent to x. Also for every vertex y in S_2 there is a unique vertex u(y) in S_1 which is adjacent to y. It may be noted that these functions are inverses of each other. Therefore $|S_1|= |S_1|$. ■

Corolary-4.14: If in a graph G there are perfect dominating sets S_1 and S_2 such that $|S_1| \neq |S_2|$ then $S_1 \cap S_2 \neq \phi$. ■

Corolary-4.15: Let G be a graph with n vertices. If there is a perfect dominating set S with $|S| < n/2$ or $\geq n/2$ then V(G) - S is not a perfect dominating set. ■.

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LIST OF SYMBOLS

List of Symbols

$V(G)$	: Set of all vertices of the graph G.
$G - \{v\}$	: A sub graph removing a vertex from the graph G.
γ set	: Dominating set with minimum cardinality of the graph G.
$\gamma(G)$	: Cardinality of minimum dominating set of the graph G.
$N(v)$	: Open neighborhood of vertex v in the graph G.
C_5	: Cycle Graph with five vertices.
P_5	: Path Graph with five vertices.
P_n	: Path Graph with n vertices.
K_5	: Complete Graph with five vertices.
W_9	: Wheel Graph with nine vertices.
$K_{1,8}$	: Star Graph with nine vertices.
V^0	: $\{v \in V(G): \gamma(G) = \gamma(G-v)\}$.
V^+	: $\{v \in V(G): \gamma(G) < \gamma(G-v)\}$.
V^-	: $\{v \in V(G): \gamma(G) > \gamma(G-v)\}$.
$N[v]$	: Closed neighborhood of a vertex v in the graph G.
$\|S\|$	: Cardinality of the set S.
$P_n[v,S]$	: $\{w \in V(G): N[w] \cap S = \{v\}\}$.
γ_T set	: Totally dominating set with minimum cardinality of the graph G.
$\gamma_T(G)$	: Cardinality of a minimum totally dominating set of the graph G.
$V^0{}_T$	: $\{v \in V : \gamma_T(G-v) = \gamma_T(G) \}$.
$V^+{}_T$	: $\{ v \in V: \gamma_T(G-v) > \gamma_T(G) \}$.

List of Symbols

V^{-}_{T}	:{ v ∈ V: γ_T(G-v) < γ_T(G) }.
V^{i}_{T}	:{ v ∈ V: G – {v} has isolated vertices}.
T_{pn}[v,S]	:{w ∈ V(G) : N(w) ∩ S = {v}}.
I	: Set of all isolated vertices of the graph G.
G-I	: The sub graph of removing all isolated vertices of the graph G.
γ_{Te} set	:Extended totally dominating set with minimum cardinality of graph G.
γ_{Te}(G)	: Cardinality of minimum extended totally dominating set of the graph G.
V^{+}_{Te}	: {v ∈ V(G): γ_{Te}(G) < γ_{Te}(G-v)}.
V^{-}_{Te}	: {v ∈ V(G): γ_{Te}(G) > γ_{Te}(G-v)}.
V^{0}_{Te}	: {v ∈ V(G): γ_{Te}(G) = γ_{Te}(G-v)}..
β_0(G)	: Cardinality of a maximum independent set of the graph G.
i set	: Independent dominating set with minimum cardinality of the graph G.
i(G)	: Cardinality of a minimum independent dominating set of the graph G.
V^{0}_{i}	: {v ∈ V(G): i(G-v) = i(G)}.
V^{+}_{i}	: {v ∈ V(G): i(G-v) > i(G)}.
V^{-}_{i}	: {v ∈ V(G): i(G-v) < i(G)}.
E_{pex}[v,S]	: { w ∈ V(G) - S: N(w) ∩ S = {v} }.
γ_{cr} set	: Vertex covering set with minimum cardinality of the graph G.
α_0(G)	: Cardinality of a minimum vertex covering set of the graph G.
δ(G)	: Minimum degree of the graph G.
K_n	: Complete Graph with n vertices.
C_n	: Cycle Graph with n vertices.

List of Symbols

Q_3 : Hyper Qube Graph.

γ_{pk} set : Perfect k-dominating set with minimum cardinality of the graph G.

$\gamma_{pk}(G)$: Cardinality of a minimum perfect k-dominating set of the graph G.

γ_{Tk} set : Totally k-dominating set with minimum cardinality of the graph G.

$\gamma_{Tk}(G)$: Cardinality of a minimum k-dominating set of the graph G.

$P_{Tk}[v,S]$:$\{w \in V(G)$: w is adjacent to exactly k vertices of S including v$\}$.

I_k : Set of all vertices whose degree less than k of the graph G.

V^i_{Tk} : $\{v\in V(G)$: G-$\{v\}$ has vertices of degree less than k $\}$.

V^+_{Tk} : $\{v \in V(G): \gamma_{Tk}(G) < \gamma_{Tk}(G\text{-}v)\}$.

V^-_{Tk} : $\{v \in V(G): \gamma_{Tk}(G) > \gamma_{Tk}(G\text{-}v)\}$.

V^0_{Tk} : $\{v \in V(G): \gamma_{Tk}(G) = \gamma_{Tk}(G\text{-}v)\}$.

d(v) : degree of vertex v in the graph G.

γ_{ku} set : k-tuple dominating set with minimum cardinality of the graph G.

$\gamma_{ku}(G)$: Cardinality of a minimum k-tuple dominating set of the graph G.

$P_{ku}[v,S]$: $S_1US_2US_3$, Where $S_1=\{ w\in S: w \neq v$ and w is adjacent to exactly k-1 vertices of S including v.$\}$, $S_2 = \{w\in S: w = v$ and w is adjacent to exactly k-1 vertices of S$\}$,$S_3 = \{w\notin S$: w is adjacent to exactly k vertices of S including v.$\}$

V^+_{ku} : $\{v \in V(G): \gamma_{ku}(G) < \gamma_{ku}(G\text{-}v)\}$.

V^-_{ku} : $\{v \in V(G): \gamma_{ku}(G) > \gamma_{ku}(G\text{-}v)\}$.

V^0_{ku} : $\{v \in V(G): \gamma_{ku}(G) = \gamma_{ku}(G\text{-}v)\}$.

V^i_{Tk} : $\{v\in V(G)$: G-$\{v\}$ has vertex of degree less than k$\}$.

List of Symbols

i_k : k-dependent k-dominating set with minimum cardinality of the graph G.

$i_k(G)$: Cardinality of a minimum k-dependent k-dominating set of the graph G.

V^+_{Ik} : $\{v \in V(G): i_k(G) < i_k(G-v)\}$.

V^-_{Ik} : $\{v \in V(G): i_k(G) > i_k(G-v)\}$.

V^0_{Ik} : $\{v \in V(G): i_k(G) = i_k(G-v)\}$.

V^i_{Tk} : {G-{v} has vertex which degree less than k}.

$E_{xk}[v,S]$: $\{w \in V(G)-S$: w is adjacent to exactly k vertices of S including v}.

γ_{pf} set : Perfect dominating set with smallest cardinality.

$\gamma_{pf}(G)$: Cardinality of a minimum perfect dominating set.

$P_{pf}[v,S]$: $\{ w \in V(G)-S : N(w) \cap S = \{v\}\}$

U { v, if v is adjacent to no vertex of S or at least two vertices of S} .

V^+_{pf} : $\{ v \in V(G): \gamma_{pf}(G) < \gamma_{pf}(G-v)\}$.

V^-_{pf} : $\{ v \in V(G): \gamma_{pf}(G) > \gamma_{pf}(G-v)\}$.

V^0_{pf} : $\{ v \in V(G): \gamma_{pf}(G) = \gamma_{pf}(G-v)\}$.