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# **DISCUSSION ON SOME INTERESTING TOPICS IN GRAPH THEORY**

*a thesis submitted to*

**SAURASHTRA UNIVERSITY**

**RAJKOT**

*for the award of the degree of*

**DOCTOR OF PHILOSOPHY**

*in*

**MATHEMATICS**

*by*

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**(Reg. No.:3989/Date: 31-07-2008)**

*under the supervision of*

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(Reaccredited "B" Grade by NAAC)  
(CGPA 2.93)

August 2011

# Certificate

This is to certify that the thesis entitled **Discussion on Some Interesting Topics in Graph Theory** submitted by **Prakash L. Vihol** to **Saurashtra University, RAJKOT (GUJARAT)** for the award of the degree of **DOCTOR OF PHILOSOPHY** in Mathematics is a bonafide record of research work carried out by him under my supervision. The contents embodied in the thesis have not been submitted in part or full to any other Institution or University for the award of any degree or diploma.

Place: RAJKOT.

Date: 02/08/2011

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# Declaration

I hereby declare that the content embodied in the thesis is a bonafide record of investigations carried out by me under the supervision of **Prof. S. K. Vaidya** at the Department of Mathematics, **Saurashtra University, RAJKOT**. The investigations reported here have not been submitted in part or full for the award of any degree or diploma of any other Institution or University.

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# *Acknowledgement*

It is a matter of tremendous pleasure for me to submit my Ph. D. thesis entitled "**DISCUSSION ON SOME INTERESTING TOPICS IN GRAPH THEORY**" in the subject of Mathematics. My registration was done in the year 2008 for carrying out the work related to the subject. Which seemed to me to be herculean task ab initio but with the passing of time, everything seemed to be within the reach by god's grace.

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**Prakash L. Vihol**

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*Dedicated to my Parents .....*

# **Chapter 1**

## **Introduction**

Though human cognition appears to comprehend the continuum, human action by and large takes place in discrete epochs. Discrete mathematics is that part of mathematical sciences which deals with systematic treatment and understanding of discrete structures and process encountered in our day-to-day life, which are inherently quite complex in nature but seemingly understandable. Illustration of this character is abound with excitement, occasionally even to a common man. The graph theory is one such field of discrete mathematics which cuts across wide range of disciplines of human understanding not only in the areas of pure mathematics but also in variety of application areas ranging from computer science to social sciences and in engineering to mention a few.

The later part of last century has witnessed intense activity in graph theory. Development of computer science boost up the research work in the field. There are many interesting fields of research in graph theory. Some of them are domination in graphs, topological graph theory, fuzzy graph theory and labeling of discrete structures.

The labeling of graphs is one of the potential areas of research due to its vital applications. The problems related to labeling of graphs challenges to our mind for their eventual solutions. This field has become a field of multifaceted applications ranging from neural network to bio-technology and to coding theory to mention a few.

Graph labeling were first introduced by A.Rosa during 1960. At present couple of dozens labeling techniques as well as enormous amount of literature is available in printed and electronic form on various graph labeling problems. The present work is aimed to discuss some graph labeling problems. The content of the thesis is divided into seven chapters.

The Chapter - 1 is of introductory nature.

The immediate Chapter - 2 is aimed to provide basic terminology and preliminaries.

The Chapter - 3 is focused on cordial and 3-equitable labelings. We investigate some new families of cordial and 3-equitable graphs. We also discuss embedding and NP-complete problems for these two labelings.

The next Chapter - 4 is targeted to discuss total product cordial labeling and prime cordial labeling. Here we investigate several new results.

The graceful labeling is one of the popular and well explored labeling. Most of the labeling problems found their origin with it. Several attempts to settle the Ringel-Kotzig-Rosa tree conjecture provided the reason for many graph labeling problems. Some labeling techniques having graceful theme are also introduced. The Fibonacci graceful and super Fibonacci graceful are such labeling. In Chapter - 5 we prove that trees are Fibonacci graceful graph while wheels and helms are not Fibonacci graceful graphs. We also show that the joint sum of two fans and the graph obtained by switching of a vertex in cycle  $C_n$  admit Fibonacci graceful labeling. Moreover we show that the graph obtained by switching of a vertex in  $C_n$  is super Fibonacci graceful except for  $n \geq 6$ . We also show that the graph obtained by switching of a vertex in  $C_n$  is not super Fibonacci graceful graph for  $n \geq 6$  but it can be embedded as an induced subgraph of a super Fibonacci graceful graph.

The penultimate Chapter - 6 is focused on triangular sum labeling. We investigate some results on triangular sum graph.

The discussion carried out in above two chapters is a nice combination of graph theory and elementary number theory.

The last Chapter - 7 is intended to report the investigations concern to  $L(2,1)$ -labeling and Radio labeling of graphs.

Some of the results reported here are also published in scholarly, peer reviewed and indexed journals as well as presented in various conferences. The reprints of the published papers are given as an annexure.

Throughout this work we pose some open problems and throw some light on future scope of research which will provide motivation to any researcher.

The references and list of symbols are given alphabetically at the end.

## List of Publications Arising From the Thesis

---

1. Some Important Results on Triangular Sum Graphs., *Applied Mathematical Sciences*, 3, 2009, 1763-1772.  
(Available online: <http://www.m-hikari.com/ams/>)
2. Total Product Cordial Graphs Induced By Some Graph Operations On Cycle Related Graphs., *International Journal of Information Science and Computer Mathematics*, 1(2), 2010, 113-126.  
(Available online: <http://www.pphmj.com/journals/ijiscm.htm>)
3. Prime Cordial Labeling for Some Graphs., *Modern Applied Science*, 4(8), 2010, 119-126.  
(Available online: <http://ccsenet.org/journal/index.php/mas/>)
4. L(2,1)-Labeling in the Context of Some Graph Operations., *Journal of Mathematics Research*, 2(3), 2010, 109-119.  
(Available online: <http://ccsenet.org/journal/index.php/jmr/>)
5. Fibonacci and Super Fibonacci graceful labeling of some graphs., *Studies in Mathematical Sciences*, 2(2), 2011, 24-35.  
(Available online: <http://www.cscanada.net/index.php/sms/index>)
6. Cordial labeling for middle graph of some graphs., *Elixir Dis. Math.*, 34C, 2011, 2468-2476.  
(Available online: <http://elixirjournal.org>)
7. Embedding and NP-Complete problems for 3-equitable graphs., *International Journal of Contemporary Advanced Mathematics*, 2(1), 2011, 1-7.  
(Available online: <http://www.cscjournals.org/csc/description.php?JCode=IJCM>)

## **Details of the Work Presented in Conferences**

1. The paper entitled as "L(2,1)-labeling in the context of some graph operations" presented in *Science Excellence - 2010* at Gujarat University, Ahmedabad(Gujarat) on 9<sup>th</sup> January, 2010 and won the award of "Best Paper Presentation".
2. The paper entitled as "Total product cordial graphs induced by some graph operations on cycle related graphs" presented in 76<sup>th</sup> *Annual Instructional Conference of the INDIAN MATHEMATICAL SOCIETY* at S.V.National Institute of Technology, Surat(Gujarat), during 27-30 December, 2010.
3. The paper entitled as "Fibonacci and Super Fibonacci Graceful labeling of some graphs" presented in *State level Mathematics Meet - 2011* at Gujarat University, Ahmedabad(Gujarat), during 3-5 February, 2011.
4. The paper entitled as "Radio Labeling for some cycle related graphs" presented in *Seventh annual conference of Academy of Discrete Mathematics and Applications and graph theory day VII* at National Institute of Technology, Calicut(Kerala), during 9-11 June, 2011.

## **Chapter 2**

### **Preliminaries**

## 2.1 Introduction

This chapter is intended to provide all the fundamental terminology and notations which are needed for the present work.

## 2.2 Basic Definitions

**Definition 2.2.1.** A graph  $G = (V(G), E(G))$  consists of two sets,  $V(G) = \{v_1, v_2, \dots\}$  called *vertex set* of  $G$  and  $E(G) = \{e_1, e_2, \dots\}$  called *edge set* of  $G$ . Sometimes we denote vertex set of  $G$  as  $V(G)$  and edge set of  $G$  as  $E(G)$ . Elements of  $V(G)$  and  $E(G)$  are called *vertices* and *edges* respectively.

**Definition 2.2.2.** An edge of a graph that joins a vertex to itself is called a *loop*. A loop is an edge  $e = v_i v_i$ .

**Definition 2.2.3.** If two vertices of a graph are joined by more than one edge then these edges are called *multiple edges*.

**Definition 2.2.4.** A graph which has neither loops nor parallel edges is called a *simple graph*.

**Definition 2.2.5.** If two vertices of a graph are joined by an edge then these vertices are called *adjacent vertices*.

**Definition 2.2.6.** Two vertices of a graph which are adjacent are said to be *neighbours*. The set of all neighbours of a vertex  $v$  of  $G$  is called the *neighbourhood set* of  $v$ . It is denoted by  $N(v)$  or  $N[v]$  and they are respectively known as open and closed neighbourhood set.

$$N(v) = \{u \in V(G) / u \text{ adjacent to } v \text{ and } u \neq v\}$$

$$N[v] = N(v) \cup \{v\}$$

**Definition 2.2.7.** If two or more edges of a graph have a common vertex then these edges are called *incident edges*.

**Definition 2.2.8.** *Degree* of a vertex  $v$  of any graph  $G$  is defined as the number of edges incident on  $v$ , counting twice the number of loops. It is denoted by  $\deg(v)$  or  $d(v)$ .

**Definition 2.2.9.** The *eccentricity* of a vertex  $u$ , written  $e(u)$ , is  $\max_{v \in V(G)} d(u, v)$ .

**Definition 2.2.10.** The *middle graph*  $M(G)$  of a graph  $G$  is the graph whose vertex set is  $V(G) \cup E(G)$  and in which two vertices are adjacent if and only if either they are adjacent edges of  $G$  or one is a vertex of  $G$  and the other is an edge incident with it.

**Definition 2.2.11.** The *Crown*  $(C_n \odot K_1)$  is obtained by joining a pendant edge to each vertex of  $C_n$ .

**Definition 2.2.12.** *Tadpole*  $T(n, l)$  is a graph in which path  $P_l$  is attached to any one vertex of cycle  $C_n$ .

**Definition 2.2.13.** The *shadow graph*  $D_2(G)$  of a connected graph  $G$  is constructed by taking two copies of  $G$  say  $G'$  and  $G''$ . Join each vertex  $u'$  in  $G'$  to the neighbours of the corresponding vertex  $u''$  in  $G''$ .

**Definition 2.2.14.** The *total graph*  $T(G)$  of a graph  $G$  is the graph whose vertex set is  $V(G) \cup E(G)$  and two vertices are adjacent whenever they are either adjacent or incident in  $G$ .

**Definition 2.2.15.** A graph obtained by replacing each vertex of a star  $K_{1,n}$  by a graph  $G$  is called *star of  $G$*  denoted as  $G'$ . The central graph in  $G'$  we mean the graph which replaces the apex vertex of  $K_{1,n}$ .

**Definition 2.2.16.** A *one point union*  $C_n^{(k)}$  of  $k$  copies of cycles is the graph obtained by taking  $v$  as a common vertex such that any two cycles  $C_n^i$  and  $C_n^j$  ( $i \neq j$ ) are edge disjoint and do not have any vertex in common except  $v$ .

**Definition 2.2.17.** A *Friendship graph*  $F_n$  is a one point union of  $n$  copies of cycle  $C_3$ .

**Definition 2.2.18.** A *vertex switching*  $G_v$  of a graph  $G$  is the graph obtained by taking a vertex  $v$  of  $G$ , removing all the edges incident to  $v$  and adding edges joining  $v$  to every other vertex which are not adjacent to  $v$  in  $G$ .

**Definition 2.2.19.** For a graph  $G$  the *split graph* is obtained by adding to each vertex  $v$  a new vertex  $v'$  such that  $v'$  is adjacent to every vertex that is adjacent to  $v$  in  $G$ . We will denote it as  $spl(G)$ .

**Definition 2.2.20.** A *shell*  $S_n$  is the graph obtained by taking  $n - 3$  concurrent chords in cycle  $C_n$ . The vertex at which all the chords are concurrent is called the apex vertex. The shell is also called fan  $f_{n-1}$ .

i.e.  $S_n = f_{n-1} = P_{n-1} + K_1$

**Definition 2.2.21.** The *composition* of two graph  $G_1$  and  $G_2$  denoted by  $G_1[G_2]$  has vertex set  $V(G_1[G_2]) = V(G_1) \times V(G_2)$  and edge set  $E(G_1[G_2]) = \{(u_1, v_1), (u_2, v_2) / u_1 u_2 \in E(G_1) \text{ or } u_1 = u_2 \text{ and } v_1 v_2 \in E(G_2)\}$ .

**Definition 2.2.22.** *Duplication* of a vertex  $v_k$  by a new edge  $e = v'_k v''_k$  in a graph  $G$  produces a new graph  $G'$  such that  $N(v'_k) \cap N(v''_k) = v_k$ .

**Definition 2.2.23.** *Duplication* of an edge  $e = uv$  by a new vertex  $w$  in a graph  $G$  produces a new graph  $G'$  such that  $N(w) = \{u, v\}$ .

**Definition 2.2.24.** Let graphs  $G_1, G_2, \dots, G_n, n \geq 2$  be all copies of a fixed graph  $G$ . Adding an edge between  $G_i$  to  $G_{i+1}$  for  $i = 1, 2, \dots, n - 1$  is called the *path union* of  $G$ .

**Definition 2.2.25.** Consider two copies of a graph  $G$  and define a new graph known as *joint sum* is the graph obtained by connecting a vertex of first copy with a vertex of second copy.

**Definition 2.2.26.** A chord of a cycle  $C_n$  is an edge joining two non-adjacent vertices of cycle  $C_n$ .

**Definition 2.2.27.** Two chords of a cycle are said to be twin chords if they form a triangle with an edge of the cycle  $C_n$ .

**Definition 2.2.28.** Let  $G$  be a graph. A graph  $H$  is called a *supersubdivision* of  $G$  if  $H$  is obtained from  $G$  by replacing every edge  $e_i$  of  $G$  by a complete bipartite graph  $K_{2,m_i}$  (for some  $m_i$  and  $1 \leq i \leq q$ ) in such a way that the ends of each  $e_i$  are merged with the two vertices of 2-vertices part of  $K_{2,m_i}$  after removing the edge  $e_i$  from graph  $G$ .

**Definition 2.2.29.** A *petal graph* is a connected graph  $G$  with  $\Delta(G) = 3, \delta(G) = 2$  in which the set of vertices of degree three induces a 2-regular graph and the set of vertices of degree two induces an empty graph. In a petal graph  $G$  if  $w$  is a vertex of  $G$  with degree two, having neighbors  $v_1, v_2$  then the path  $P_w = v_1 w v_2$  is called *petal* of  $G$ . We name  $w$  the *center of the petal* and  $v_1, v_2$  the *basepoints*. If  $d(v_1, v_2) = k$ , we say that the size of the petal is  $k$ . If the size of each petal is  $k$  then it is called *k-petal graph*.

## 2.3 Concluding Remarks

This chapter provides basic definitions and terminology required for the advancement of the topic. For all other standard terminology and notations we refer to Harray[37], West[82], Gross and Yellen[35], Clark and Helton[18].

The next chapter is focused on the cordial and 3-equitable labeling of graphs.

# **Chapter 3**

## **Cordial Labeling**

**&**

## **3-Equitable Labeling**

### 3.1 Introduction

The discrete mathematics is that part of mathematics which deals with systematic treatment and understanding of discrete structures and process and encountered in our daily life, which are often inherently quite complex in nature but seemingly understandable. This character of discrete mathematics is abound with excitement, occasionally even to a common man. The labeling of discrete structures is also a field which possess the same characteristic. The problems arising from the study of a variety of labeling schemes of the elements of a graph, or of any discrete structure is a potential area of challenge as it cuts across wide range of disciplines of human understanding. Graph labeling problems are actually not of recent origin. For instance, coloring the vertices of a graph arose in connection with the four color theorem, which remained for a long time known by the name four color conjecture, took more than 150 years for its solution in 1976. In recent times, new contexts have emerged wherein the labeling of the vertices or edges of a given graph by elements of certain subsets of the set of real numbers  $\mathbb{R}$ . In the late of 1960's a problem in radio astronomy let to the assignment of the absolute differences of pairs of numbers occurring on the positions of radio antennae to the links of the lay-out plans of the antennae under constraints of the optimal lay-outs to scan the visible regions of the celestial dome quickly made its way to formulate more terse mathematical problems on graph labelings. The notion of  $\beta$ -valuation was introduced by Alexander Rosa[59] in 1967. In 1972 S.W.Golomb[32] independently discovered  $\beta$ -valuations and renamed them as graceful labeling. He also pointed out the importance of studying graceful graphs in trying to settle the complex problem of decomposing the complete graph by isomorphic copies of a given tree of the same order. The Ringle-Kotzig-Rosa[58] conjecture and many illustrious works on graceful graph provided the reason for different ways for labeling of graph structures.

## 3.2 Graceful labeling

### 3.2.1 Graph labeling

If the vertices of the graph are assigned values subject to certain condition(s) is known as *graph labeling*.

For detailed survey on graph labeling problems along with extensive bibliography we refer to Gallian[25]. A systematic study on variety of applications of graph labeling is carried out by Bloom and Golomb[9].

### 3.2.2 Graceful graph

A function  $f$  is called *graceful labeling* of a graph  $G$  if  $f : V(G) \rightarrow \{0, 1, 2, \dots, q\}$  is injective and the induced function  $f^* : E(G) \rightarrow \{1, 2, \dots, q\}$  defined as  $f^*(e = uv) = |f(u) - f(v)|$  is bijective.

A graph which admits graceful labeling is called *graceful graph*.

### 3.2.3 Illustration

In the following *Figure 3.1*  $K_{3,3}$  and its graceful labeling is shown.

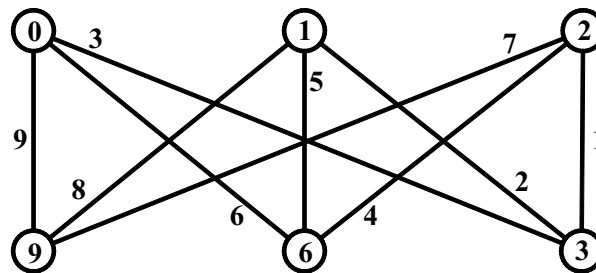


FIGURE 3.1:  $K_{3,3}$  and its graceful labeling

### 3.2.4 Some common families of graceful graphs

- Truszczyński[68] studied unicyclic graphs and conjectured that *All unicyclic graphs except  $C_n$ , for  $n \equiv 1, 2 \pmod{4}$  are graceful.* Because of the immense diversity of unicyclic graphs a proof of above conjecture seems to be out of reach in the near future.
- Delorme et al.[19] and Ma and Feng[55] proved that the cycle with one chord is graceful.
- Ma and Feng[56] proved that all gear graph are graceful.
- Gracefulness of cycle with  $k$  consecutive chords is discussed by Koh et al.[48] and Goh and Lim[31].
- Koh and Rogers[49] conjectured that cycle with triangle is graceful if and only if  $n \equiv 0, 1 \pmod{4}$ .
- Koh and Yap[48] defined and proved that a cycle with a  $P_k$ -chord are graceful when  $k = 3$ .
- In 1987 Punnim and Pabhapote[57] proved that a cycle with a  $P_k$ -chord are graceful for  $k \geq 4$ .
- Golomb[32] proved that the complete graph  $K_n$  is not graceful for  $n \geq 5$ .
- Frucht[23], Hoede and Kuiper[40] proved that all wheels  $W_n$  are graceful.
- Frucht[23] proved that crown are graceful.
- Bu et al.[10] have shown that any cycle with a fixed number of pendant edges adjoined to each vertex is graceful.
- Drake and Redl[20] enumerated the non graceful Eulerian graph with  $q \equiv 1, 2 \pmod{4}$  edges.
- Kathiresan[45] has investigated the graceful labeling for subdivisions of ladders.
- Sethuraman and Selvaraju[63] have discussed gracefulness of arbitrary super subdivisions of cycles.

- Chen et al.[16] proved that firecrackers are graceful and conjecture that banana trees are graceful.
- Kang et al.[44] proved that web graph are graceful.
- Vaidya et al.[78] have discussed gracefulness of union of two path graphs with grid graph and complete bipartite graph.
- Kaneria et al.[43] have discussed gracefulness of some classes of disconnected graphs.
- The conjecture of Ringel-Kotzig-Rosa[58] states that "*All the trees are graceful.*" has been the focus of many research papers. Kotzig called the efforts to prove gracefulness of trees a 'disease'. Among all the trees known to be graceful are caterpillars, paths, olive trees, banana trees etc.
- Bermond[8] conjectured that *Lobsters are graceful* (a lobster is a tree with the property that the removal of the endpoints leaves a caterpillar).

### 3.3 Harmonious labeling

#### 3.3.1 Harmonious graph

A function  $f$  is called *harmonious labeling* of a graph  $G$  if  $f : V(G) \rightarrow \{0, 1, 2, \dots, q-1\}$  is injective and the induced function  $f^* : E(G) \rightarrow \{0, 1, 2, \dots, q-1\}$  defined as  $f^*(e = uv) = (f(u) + f(v)) \pmod{q}$  is bijective.

A graph which admits harmonious labeling is called a *harmonious graph*.

### 3.3.2 Illustration

In the following Figure 3.2  $C_5$  and its harmonious labeling is shown.

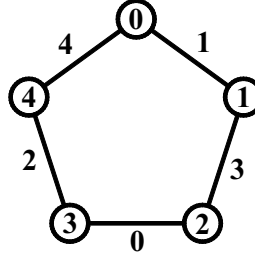


FIGURE 3.2:  $C_5$  and its harmonious labeling

### 3.3.3 Some Known Results

- Graham and Sloane[33] conjectured that *Every tree is harmonious*.
- Graham and Sloane[33] also proved that
  - ◇  $K_{m,n}$  is harmonious if and only if  $m$  or  $n = 1$ .
  - ◇  $W_n$  is harmonious  $\forall n$ .
  - ◇ Petersen graph is harmonious.
  - ◇ Cycle  $C_n$  is harmonious if and only if  $n$  is odd.
  - ◇ If a harmonious graph has even number of edges  $q$  and degree of every vertex is divisible by  $2^\alpha$  ( $\alpha \geq 1$ ) then  $q$  is divisible by  $2^{\alpha+1}$ .
  - ◇ All ladders except  $L_2$  are harmonious.
  - ◇ Friendship graph  $F_n$  is harmonious except  $n \equiv 2(mod 4)$ .
  - ◇ Fan  $f_n = P_n + K_1$  is harmonious.
  - ◇ For  $n \geq 2$  the graph  $g_n$  (the graph obtained by joining all the vertices of  $P_n$  to two additional vertices) is harmonious.
  - ◇  $C_3^{(n)}$  is harmonious if and only if  $n \not\equiv 2(mod 4)$

- Aldred and McKay[2] suggested an algorithm and use computer to show that all trees with at most 26 vertices are harmonious.
- Golomb[32] proved that complete graph is harmonious if and only if  $n \leq 4$ .
- Gnanajothi[30] has shown that webs with odd cycles are harmonious.
- Seoud and Youssef[62] have shown that the one point union of a triangle and  $C_n$  is harmonious if and only if  $n \equiv 1(mod 4)$
- Figueroa et al.[22] have shown that if  $G$  is harmonious then the one point union of an odd number of copies of  $G$  using the vertex labeled 0 as the shared point is harmonious.
- In 1992 Jungreis and Reid[42] showed that the grids  $P_m \times P_n$  are harmonious when  $(m, n) \neq (2, 2)$ .
- Gallian et al.[26] proved that all prisms  $C_m \times P_2$  with a single vertex deleted or single edge deleted are harmonious.
- In 1989 Gallian[24] showed that all Möbius ladders except  $M_3$  are harmonious.

### 3.4 Cordial labeling - A weaker version of graceful and harmonious labeling

In a seminal paper Cahit[11] introduced the concept of cordial labeling as a weaker version of graceful and harmonious labeling.

#### 3.4.1 Cordial graph

A function  $f : V(G) \rightarrow \{0, 1\}$  is called *binary vertex labeling* of a graph  $G$  and  $f(v)$  is called *label of the vertex  $v$*  of  $G$  under  $f$ . For an edge  $e = uv$ , the induced function  $f^* : E(G) \rightarrow \{0, 1\}$  is given by  $f^*(e = uv) = |f(u) - f(v)|$ . Let  $v_f(0), v_f(1)$  be number of vertices of  $G$  having labels 0 and 1 respectively under  $f$  and let  $e_f(0), e_f(1)$  be number

of edges of  $G$  having labels 0 and 1 respectively under  $f^*$ . A binary vertex labeling  $f$  of a graph  $G$  is called *cordial labeling* if  $|v_f(0) - v_f(1)| \leq 1$  and  $|e_f(0) - e_f(1)| \leq 1$ . A graph which admits cordial labeling is called a *cordial graph*.

### 3.4.2 Illustration

In the following Figure 3.3  $C_3^{(3)}$  and its cordial labeling is shown.

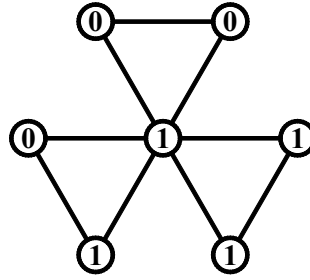


FIGURE 3.3:  $C_3^{(3)}$  and its cordial labeling

### 3.4.3 Some known results

- Lee and Liu[51], Du[21] proved that complete  $n$ -partite graph is cordial if and only if at most three of its partite sets have odd cardinality.
- Seoud and Maqsoud[61] proved that if  $G$  is a graph with  $p$  vertices and  $q$  edges and every vertex has odd degree then  $G$  is not cordial when  $p + q \equiv 2(mod 4)$ .
- Andar et al. in [3],[4],[5] and [6] proved that
  - ◊ Multiple shells are cordial.
  - ◊  $t$ -ply graph  $P_t(u, v)$  is cordial except when it is Eulerian and the number of edges is congruent to  $2(mod 4)$ .
  - ◊ Helms, closed helms and generalized helms are cordial.
- In [6], Andar et al. showed that a cordial labeling  $g$  of a graph  $G$  can be extended to a cordial labeling of the graph obtained from  $G$  by attaching  $2m$  pendant edges

at each vertex of  $G$ . They also proved that a cordial labeling  $g$  of a graph  $G$  with  $p$  vertices can be extended to a cordial labeling of the graph obtained from  $G$  by attaching  $2m + 1$  pendant edges at each vertex of  $G$  if and only if  $G$  does not satisfy either of the following conditions:

1.  $G$  has an even number of edges and  $p \equiv 2(mod 4)$ .
  2.  $G$  has an odd number of edges and either  $p \equiv 1(mod 4)$  with  $e_g(1) = e_g(0) + i(G)$  or  $p \equiv 3(mod 4)$  with  $e_g(0) = e_g(1) + i(G)$ , where  $i(G) = \min\{|e_g(0) - e_g(1)|\}$
- Vaidya et al.[76, 77, 80] have discussed cordial labeling for some cycle related graphs.
  - Vaidya et al.[81] have discussed some new cordial graphs.
  - Vaidya et al.[72] have discussed some shell related cordial graphs.
  - Vaidya and Dani[73] and Vaidya et al.[71] have discussed the cordial labeling for some star related graphs.

In the succeeding sections we will report the results investigated by us.

### 3.5 Cordial Labeling of middle graph of some graphs

**Theorem 3.5.1.** The middle graph  $M(G)$  of an Eulerian graph  $G$  is Eulerian and  $|E(M(G))| = \frac{\sum_{i=1}^n d(v_i)^2 + 2e}{2}$ .

*Proof.* Let  $G$  be an Eulerian graph. If  $v_1, v_2, v_3 \dots, v_n$  are vertices of  $G$  and  $e_1, e_2, e_3 \dots, e_q$  are edges of  $G$  then  $v_1, v_2, v_3 \dots, v_n, e_1, e_1, e_2 \dots, e_q$  are the vertices of  $M(G)$ . Then it is obvious that if  $d(v_i)$  is even in  $G$  then it remains even in  $M(G)$ . Now it remains to show that  $d(e_i)$  is even in  $M(G)$ . For that if  $v'$  and  $v''$  are the vertices adjacent to any vertex  $e_i$  then

$$\begin{aligned}
d(e_i) &= d(v') + d(v'') \\
&= \text{even as both } d(v') \text{ and } d(v'') \text{ are even for } 1 \leq i \leq q.
\end{aligned}$$

Therefore  $M(G)$  is an Eulerian graph.

It is also obvious that the  $d(v_i)$  number of edges are incident with each vertex  $v_i$  of  $G$  which forms a complete graph  $K_{d(v_i)}$  in  $M(G)$ .

Now if the total number of edges in  $M(G)$  be denoted as  $|E(M(G))|$  then

$$\begin{aligned}
|E(M(G))| &= d(v_1) + d(v_2) + d(v_3) + \dots + d(v_n) + |E(K_{d(v_1)})| + |E(K_{d(v_2)})| + |E(K_{d(v_3)})| + \\
&\quad \dots + |E(K_{d(v_n)})| \\
&= d(v_1) + d(v_2) + d(v_3) + \dots + d(v_n) + \frac{d(v_1)(d(v_1)-1)}{2} + \frac{d(v_2)(d(v_2)-1)}{2} + \dots + \frac{d(v_n)(d(v_n)-1)}{2} \\
&= \frac{d(v_1)^2}{2} + \frac{d(v_2)^2}{2} + \dots + \frac{d(v_n)^2}{2} + \frac{d(v_1)}{2} + \frac{d(v_2)}{2} + \dots + \frac{d(v_n)}{2} \\
&= \frac{\sum_{i=1}^n d(v_i)^2 + \sum_{i=1}^n d(v_i)}{2} \\
&= \frac{\sum_{i=1}^n d(v_i)^2 + 2e}{2}
\end{aligned}$$

But  $\sum_{i=1}^n d(v_i) = 2e$ , Hence  $|E(M(G))| = \frac{\sum_{i=1}^n d(v_i)^2 + 2e}{2}$ . □

**Corollary 3.5.2.** The middle graph  $M(G)$  of any graph  $G$  is not cordial when  $|E(M(G))| = \frac{\sum_{i=1}^n d(v_i)^2 + 2e}{2} \equiv 2 \pmod{4}$ .

*Proof.* By Theorem 3.5.1, for  $M(G)$  of any graph  $G$ ,  $|E(M(G))| = \frac{\sum_{i=1}^n d(v_i)^2 + 2e}{2}$ .

Then as proved by Cahit[11] an Eulerian graph with  $|E(G)| \equiv 2 \pmod{4}$  is not cordial. □

**Theorem 3.5.3.**  $M(P_n)$  is a cordial graph.

*Proof.* If  $v_1, v_2, \dots, v_n$  and  $e_1, e_2, \dots, e_n$  are respectively the vertices and edges of  $P_n$  then  $v_1, v_2, \dots, v_n, e_1, e_2, \dots, e_n$  are the vertices of  $M(P_n)$ .

To define  $f : V(M(P_n)) \rightarrow \{0, 1\}$ , we consider following four cases.

**Case 1:**  $n$  is odd,  $n = 2k + 1, k=1,3,5,7,\dots$

In this case  $|V(M(P_n))| = 2n - 1, |E(M(P_n))| = 2n + 2k - 3$

We label the vertices as follows.

$$f(v_{2i-1}) = 0 \text{ for } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor + 1$$

$$f(v_{2i}) = 1 \text{ for } 1 \leq i \leq \lfloor \frac{n}{2} \rfloor$$

$$\left. \begin{array}{l} f(e_{4i-3}) = 1 \\ f(e_{4i-2}) = 1 \end{array} \right\} 1 \leq i \leq \lfloor \frac{n}{4} \rfloor + 1$$

$$\left. \begin{array}{l} f(e_{4i-1}) = 0 \\ f(e_{4i}) = 0 \end{array} \right\} 1 \leq i \leq \lfloor \frac{n}{4} \rfloor$$

In view of the above defined labeling pattern we have

$$v_f(0) + 1 = v_f(1) = n, e_f(0) = e_f(1) + 1 = n + k - 1$$

**Case 2:**  $n$  odd,  $n = 2k + 1, k=2,4,6,\dots$

$$\text{In this case } |V(M(P_n))| = 2n - 1, |E(M(P_n))| = 2n + 2k - 3$$

We label the vertices as follows.

$$\begin{array}{l} f(v_{2i-1}) = 0 \text{ for } 1 \leq i \leq \lfloor \frac{n}{2} \rfloor + 1 \\ f(v_{2i}) = 1 \text{ for } 1 \leq i \leq \lfloor \frac{n}{2} \rfloor \end{array}$$

$$\left. \begin{array}{l} f(e_{4i-3}) = 0 \\ f(e_{4i-2}) = 0 \\ f(e_{4i-1}) = 1 \\ f(e_{4i}) = 1 \end{array} \right\} 1 \leq i \leq \lfloor \frac{n}{4} \rfloor$$

In view of the above defined labeling pattern we have

$$v_f(0) = v_f(1) + 1 = n, e_f(0) = e_f(1) + 1 = n + k - 1$$

**Case 3:**  $n$  even,  $n = 2k, k=1,3,5,7,\dots$

$$\text{In this case } |V(M(P_n))| = 2n - 1, |E(M(P_n))| = 2n + 2k - 4$$

We label the vertices as follows.

$$\left. \begin{array}{l} f(v_{2i-1}) = 0 \\ f(v_{2i}) = 1 \end{array} \right\} 1 \leq i \leq \frac{n}{2}$$

$$f(e_{4i-3}) = 0 \text{ for } 1 \leq i \leq \left\lfloor \frac{n}{4} \right\rfloor + 1$$

$$f(e_{4i-2}) = 0 \text{ for } 1 \leq i \leq \left\lfloor \frac{n}{4} \right\rfloor$$

$$\left. \begin{array}{l} f(e_{4i-1}) = 1 \\ f(e_{4i}) = 1 \end{array} \right\} 1 \leq i \leq \left\lfloor \frac{n}{4} \right\rfloor$$

**Case 4:**  $n$  even,  $n = 2k$   $k=2,4,6,\dots$

In this case  $|V(M(P_n))| = 2n - 1$ ,  $|E(M(P_n))| = 2n + 2k - 4$

We label the vertices as follows.

$$\left. \begin{array}{l} f(v_{2i-1}) = 0 \\ f(v_{2i}) = 1 \end{array} \right\} 1 \leq i \leq \frac{n}{2}$$

$$\left. \begin{array}{l} f(e_{4i-3}) = 0 \\ f(e_{4i-2}) = 0 \end{array} \right\} 1 \leq i \leq \frac{n}{4}$$

$$\begin{array}{l} f(e_{4i-1}) = 1 \text{ for } 1 \leq i \leq \frac{n}{4} \\ f(e_{4i}) = 1 \text{ for } 1 \leq i \leq \frac{n}{4} - 1 \end{array}$$

In above two cases we have

$$v_f(0) = v_f(1) + 1 = n, e_f(0) = e_f(1) = n + k - 2$$

Thus in all the four cases  $f$  satisfies the condition for cordial labeling. That is,  $M(P_n)$  is a cordial graph.  $\square$

**Illustration 3.5.4.** Consider the graph  $M(P_7)$ . The cordial labeling is as shown in Figure 3.4.

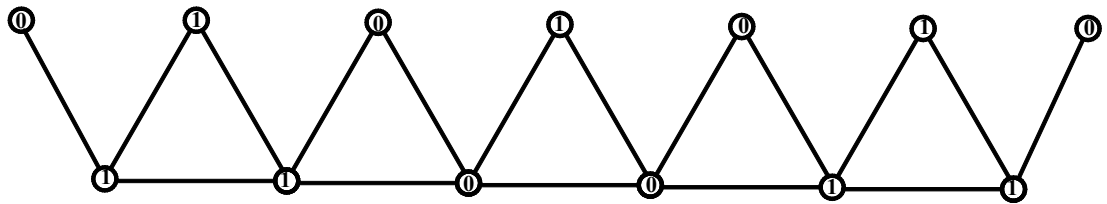


FIGURE 3.4:  $M(P_7)$  and its cordial labeling

**Theorem 3.5.5.**  $M(C_n \odot K_1)$  is a cordial graph.

*Proof.* Consider the crown  $C_n \odot K_1$  in which  $v_1, v_2, \dots, v_n$  be the vertices of cycle  $C_n$  and  $v'_1, v'_2, \dots, v'_n$  be the pendant vertices attached at each vertex of  $C_n$ . Let  $e_1, e_2, \dots, e_n$  and  $e'_1, e'_2, \dots, e'_n$  are vertices corresponding to edges of  $C_n$  and  $K_1$  respectively in  $M(C_n \odot K_1)$ .

To define  $f : V(M(C_n \odot K_1)) \longrightarrow \{0, 1\}$ , we consider following three cases.

**Case 1:**  $n$  is odd,  $n = 2k + 1, k=2,4,6,\dots$

In this case  $|V(M(C_n \odot K_1))| = 4n, |E(M(C_n \odot K_1))| = 6n + 2 \lfloor \frac{n}{2} \rfloor + 1$

We label the vertices as follows.

$$\left. \begin{array}{l} f(v_{2i-1}) = 0 \\ f(v_{2i}) = 1 \end{array} \right\} 1 \leq i \leq \lfloor \frac{n}{2} \rfloor$$

$$\begin{aligned} f(v_n) &= 1 \\ f(v'_{2i-1}) &= 1 \text{ for } 1 \leq i \leq \lfloor \frac{n}{4} \rfloor + 1 \\ f(v'_{2i}) &= 0 \text{ for } 1 \leq i \leq \lfloor \frac{n}{4} \rfloor \end{aligned}$$

$$\left. \begin{array}{l} f(v'_{2\lfloor \frac{n}{4} \rfloor + 2i}) = 1 \\ f(v'_{2\lfloor \frac{n}{4} \rfloor + 2i+1}) = 0 \end{array} \right\} 1 \leq i \leq \lfloor \frac{n}{4} \rfloor$$

$$\left. \begin{array}{l} f(e_{2i-1}) = 1 \\ f(e_{2i}) = 0 \end{array} \right\} 1 \leq i \leq \lfloor \frac{n}{2} \rfloor$$

$$\begin{aligned} f(e_n) &= 0 \\ f(e'_{2i-1}) &= 0 \text{ for } 1 \leq i \leq \lfloor \frac{n}{2} \rfloor + 1 \\ f(e'_{2i}) &= 1 \text{ for } 1 \leq i \leq \lfloor \frac{n}{2} \rfloor \end{aligned}$$

In view of the above defined pattern

$$v_f(0) = v_f(1) = 2n, e_f(0) + 1 = e_f(1) = 3n + \lfloor \frac{n}{2} \rfloor + 1$$

**Case 2:**  $n$  is odd,  $n = 2k + 1, k=1,3,5,7,\dots$

In this case  $|V(M(C_n \odot K_1))| = 4n, |E(M(C_n \odot K_1))| = 6n + 2 \lfloor \frac{n}{2} \rfloor + 1$

We label the vertices as follows.

$$\left. \begin{array}{l} f(v'_{2\lfloor \frac{n}{4} \rfloor + 2i}) = 1 \\ f(v'_{2\lfloor \frac{n}{4} \rfloor + 2i+1}) = 0 \end{array} \right\} 1 \leq i \leq \lfloor \frac{n}{4} \rfloor + 1$$

Now label the remaining vertices as in case 1.

In view of the above defined pattern we have

$$v_f(0) = v_f(1) = 2n, e_f(0) = e_f(1) + 1 = 3n + \lfloor \frac{n}{2} \rfloor + 1$$

**Case 3:**  $n$  is even,  $n = 2k$ ,  $k = 2, 3, \dots$

In this case  $|V(M(C_n \odot K_1))| = 3n$ ,  $|E(M(C_n \odot K_1))| = 7n$

We label the vertices as follows.

$$\left. \begin{array}{l} f(v_{2i-1}) = 0 \\ f(v_{2i}) = 1 \end{array} \right\} 1 \leq i \leq \frac{n}{2}$$

$$f(v'_i) = 1 \text{ for } 1 \leq i \leq n$$

$$f(e_i) = 0 \text{ for } 1 \leq i \leq n$$

$$\left. \begin{array}{l} f(e'_{2i-1}) = 1 \\ f(e'_{2i}) = 0 \end{array} \right\} 1 \leq i \leq \frac{n}{2}$$

In view of the above defined pattern we have

$$v_f(0) = v_f(1) = 2n, e_f(0) = e_f(1) = 3n + \frac{n}{2}$$

Thus  $f$  is a cordial labeling for  $M(C_n \odot K_1)$ . That is, middle graph of crown is a cordial graph. □

**Illustration 3.5.6.** Consider  $G = M(C_7 \odot K_1)$ . The cordial labeling is as shown in Figure 3.5.

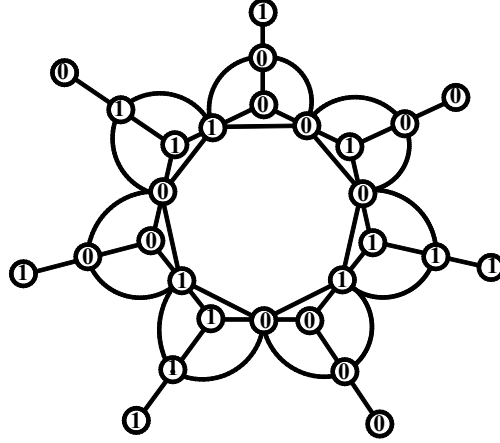


FIGURE 3.5:  $M(C_7 \odot K_1)$  and its cordial labeling

**Theorem 3.5.7.**  $M(K_{1,n})$  is a cordial graph.

*Proof.* Let  $v, v_1, v_2, \dots, v_n$  be the vertices of star  $K_{1,n}$  with  $v$  as an apex vertex and  $e_1, e_2, \dots, e_n$  be the vertices in  $M(K_{1,n})$  corresponding to the edges  $e_1, e_2, \dots, e_n$  in  $K_{1,n}$ . To define  $f : V(M(K_{1,n})) \rightarrow \{0, 1\}$ , we consider following two cases.

**Case 1:**  $n = 2k + 1, k = 1, 2, 3, 4, \dots$

In this case  $|V(M(K_{1,n}))| = 2n + 1$ ,  $|E(M(K_{1,n}))| = 2n(\lfloor \frac{k}{2} \rfloor + 1)$  or  $|E(M(K_{1,n}))| = 2n(\lfloor \frac{k}{2} \rfloor + 1) + 2k + 1$  depending upon  $k = 2, 4, 6, 8, \dots$  or  $k = 3, 5, 7, 9, \dots$

$$f(e_{2i-1}) = 0, \quad 1 \leq i \leq \lfloor \frac{n}{2} \rfloor + 1$$

$$f(e_{2i}) = 1, \quad 1 \leq i \leq \lfloor \frac{n}{2} \rfloor$$

$$f(v_{n-i}) = p_i, \text{ where } p_i = 1, \text{ if } i \text{ is even,}$$

$$= 0, \text{ if } i \text{ is odd, } 0 \leq i \leq \lfloor \frac{k}{2} \rfloor - 1$$

$$f(v_{n-\lfloor \frac{k}{2} \rfloor - i}) = f(e_{n-\lfloor \frac{k}{2} \rfloor - i}), \quad 0 \leq i \leq n - \lfloor \frac{k}{2} \rfloor - 1$$

$$f(v) = 1$$

Using above pattern if  $k = 2, 3, 6, 7, \dots$  then  $v_f(0) + 1 = v_f(1) = n + 1$  and if  $k = 1, 4, 5, 8, 9, \dots$  then  $v_f(0) = v_f(1) + 1 = n + 1$ .

If  $k = 2, 4, 6, 8, \dots$  then  $e_f(0) = e_f(1) = n(\lfloor \frac{k}{2} \rfloor + 1)$  and if  $k = 1, 3, 5, 7, \dots$  then  $e_f(0) = e_f(1) + 1 = n(\lfloor \frac{k}{2} \rfloor + 1) + k + 1$

**Case 2:**  $n = 2k, k = 2, 3, 4, \dots$

In this case  $|V(M(K_{1,n}))| = 2n + 1$ ,  $|E(M(K_{1,n}))| = 2n(\frac{k}{2} + 1) - k$  or  $|E(M(K_{1,n}))| = 2n(\lfloor \frac{k}{2} \rfloor + 1) + 2\lfloor \frac{k}{2} \rfloor - 1$  depending upon  $k = 2, 4, 6, 8, \dots$  or  $k = 3, 5, 7, 9, \dots$

$$f(e_{2i-1}) = 0, \quad 1 \leq i \leq \frac{n}{2}$$

$$f(e_{2i}) = 1, \quad 1 \leq i \leq \frac{n}{2}$$

$$f(v_{n-i}) = p_i, \text{ where } p_i = 0, \text{ if } i \text{ is even,}$$

$$= 1, \text{ if } i \text{ is odd, } 0 \leq i \leq \lfloor \frac{k}{2} \rfloor - 1$$

$$f(v_{n-\lfloor \frac{k}{2} \rfloor - i}) = f(e_{n-\lfloor \frac{k}{2} \rfloor - i}), \quad 0 \leq i \leq n - \lfloor \frac{k}{2} \rfloor$$

$$f(v) = 1$$

Using above pattern if  $k = 2, 3, 6, 7, \dots$  then  $v_f(0) = v_f(1) + 1 = n + 1$  and if  $k = 4, 5, 8, 9, \dots$  then  $v_f(0) + 1 = v_f(1) = n + 1$ .

If  $k = 2, 4, 6, 8, \dots$  then  $e_f(0) = e_f(1) = n(\frac{k}{2} + 1) - \frac{k}{2}$  and if  $k = 3, 5, 7, \dots$  then  $e_f(0) = e_f(1) + 1 = n(\lfloor \frac{k}{2} \rfloor + 1) + \lfloor \frac{k}{2} \rfloor + 1$ .

Also note that for  $n = 2$  we have  $v_f(0) = v_f(1) + 1 = 3$  and  $e_f(0) + 1 = e_f(1) = 3$ .

Thus  $f$  is a cordial labeling for  $M(K_{1,n})$ . That is,  $M(K_{1,n})$  admits cordial labeling.  $\square$

**Illustration 3.5.8.** Consider a graph  $M(K_{1,6})$ . The cordial labeling is as shown in Figure 3.6.

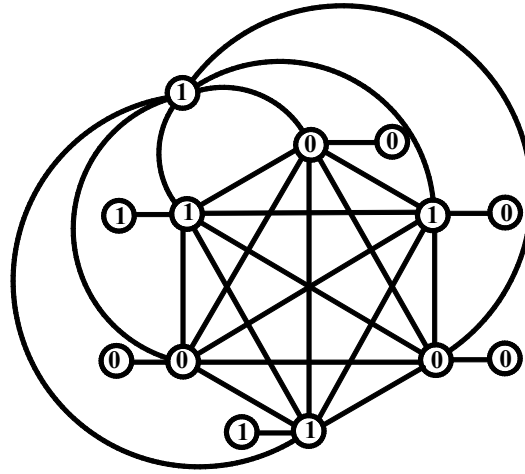


FIGURE 3.6:  $M(K_{1,6})$  and its cordial labeling

**Theorem 3.5.9.**  $M(T(n, l+1))$  is a cordial graph.

*Proof.* Consider the tadpole  $T(n, l+1)$  in which  $v_1, v_2, \dots, v_n$  be the vertices of cycle  $C_n$  and  $v'_1, v'_2, v'_3, \dots, v'_{l+1}$  be the vertices of the path attached to the cycle  $C_n$ . Also let  $e_1, e_2, \dots, e_n$  and  $e'_1, e'_2, \dots, e'_l$  be the vertices in  $M(T(n, l+1))$  corresponding to the edges of cycle  $C_n$  and path  $P_{l+1}$  respectively in  $T(n, l+1)$ .

To define  $f : V(M(T(n, l+1))) \longrightarrow \{0, 1\}$ , we consider the following cases.

**Case 1:**  $n$  is odd

**Subcase 1:**  $n = 2k + 1, k = 2, 4, 6, \dots$  and  $l = 2j, j = 2, 4, 6, \dots$

In this subcase  $|V(M(T(n, l+1)))| = 2n + 2l, |E(M(T(n, l+1)))| = 2 \lfloor \frac{n}{2} \rfloor + 2n + 2l + 6$

$$\left. \begin{array}{l} f(v_{2i-1}) = 1 \\ f(v_{2i}) = 0 \end{array} \right\} 1 \leq i \leq \lfloor \frac{n}{2} \rfloor$$

$$\begin{aligned} f(e_{4i-3}) &= 0, 1 \leq i \leq \lfloor \frac{n}{4} \rfloor + 1 \\ f(e_{4i-2}) &= 0, 1 \leq i \leq \lfloor \frac{n}{4} \rfloor \end{aligned}$$

$$\left. \begin{array}{l} f(e_{4i-1}) = 1 \\ f(e_{4i}) = 1 \end{array} \right\} 1 \leq i \leq \lfloor \frac{n}{4} \rfloor$$

$$f(v_n) = f(v'_1) = 1 \text{ (when } v'_1 \text{ is attached to } v_n)$$

$$\left. \begin{array}{l} f(v'_{2i}) = 0 \\ f(v'_{2i+1}) = 1 \end{array} \right\} 1 \leq i \leq \lfloor \frac{l}{2} \rfloor$$

$$\left. \begin{array}{l} f(e'_{4i-3}) = f(e'_{4i-2}) = 0 \\ f(e'_{4i-1}) = f(e'_{4i}) = 1 \end{array} \right\} 1 \leq i \leq \lfloor \frac{l}{4} \rfloor$$

In view of the above defined labeling pattern

$$v_f(0) = v_f(1) = n + l, e_f(0) = e_f(1) = \lfloor \frac{n}{2} \rfloor + n + l + 3$$

**Subcase 2:**  $n = 2k + 1, k = 2, 4, 6, \dots$  and  $l = 2j, j = 3, 5, 7, \dots$

In this subcase  $|V(M(T(n, l+1)))| = 2n + 2l, |E(M(T(n, l+1)))| = 2 \lfloor \frac{n}{2} \rfloor + 2n + 2l + 8$

$$f(e'_{n-1}) = 0, f(e'_n) = 1$$

$$f(v_n) = 1, f(v'_1) = 1 (\text{when } v'_1 \text{ is attached to } v_1)$$

remaining vertices are labeled as in subcase 1.

In view of the above defined labeling pattern

$$v_f(0) = v_f(1) = n + l, e_f(0) = e_f(1) = \lfloor \frac{n}{2} \rfloor + n + l + 4$$

For  $l = 2$  we have  $e_f(0) = e_f(1) = 11$ .

**Subcase 3:**  $n = 2k + 1, k = 2, 4, 6, \dots$  and  $l = 2j + 1, j = 1, 3, 5, 7, \dots$

In this subcase  $|V(M(T(n, l + 1)))| = 2n + 2l, |E(M(T(n, l + 1)))| = 2 \lfloor \frac{n}{2} \rfloor + 2n + 2l + 5$

$$f(v_n) = f(v'_1) = 1 (\text{when } v'_1 \text{ is attached to } v_n)$$

$$f(v'_{2i}) = 0, \quad 1 \leq i \leq \lfloor \frac{l}{2} \rfloor + 1$$

$$f(v'_{2i+1}) = 1, \quad 1 \leq i \leq \lfloor \frac{l}{2} \rfloor$$

$$\left. \begin{array}{l} f(e'_{4i-3}) = f(e'_{4i-2}) = 1 \\ f(e'_{4i-1}) = 0 \end{array} \right\} \quad 1 \leq i \leq \lfloor \frac{l}{4} \rfloor + 1$$

$$f(e'_{4i}) = 0, \quad 1 \leq i \leq \lfloor \frac{l}{4} \rfloor$$

remaining vertices are labeled as in subcase 1.

Using above pattern we have

$$v_f(0) = v_f(1) = n + l, e_f(0) + 1 = e_f(1) = \lfloor \frac{n}{2} \rfloor + n + l + 3$$

For  $l = 1$  we have  $e_f(0) + 1 = e_f(1) = 10$ .

**Subcase 4:**  $n = 2k + 1, k = 2, 4, 6, \dots$  and  $l = 2j + 1, j = 2, 4, 6, \dots$

In this subcase  $|V(M(T(n, l + 1)))| = 2n + 2l, |E(M(T(n, l + 1)))| = 2 \lfloor \frac{n}{2} \rfloor + 2n + 2l + 7$

$$f(v_n) = f(v'_1) = 1 (\text{when } v'_1 \text{ is attached to } v_n)$$

$$f(v'_{2i}) = 0, \quad 1 \leq i \leq \lfloor \frac{l}{2} \rfloor + 1$$

$$f(v'_{2i+1}) = 1, \quad 1 \leq i \leq \lfloor \frac{l}{2} \rfloor$$

$$f(e'_{4i-3}) = 1, \quad 1 \leq i \leq \lfloor \frac{l}{4} \rfloor + 1$$

$$\left. \begin{array}{l} f(e'_{4i-2}) = 1 \\ f(e'_{4i-1}) = f(e'_{4i}) = 0 \end{array} \right\} \quad 1 \leq i \leq \lfloor \frac{l}{4} \rfloor$$

remaining vertices are labeled as in subcase 1.

Using above pattern we have

$$v_f(0) = v_f(1) = n + l, e_f(0) + 1 = e_f(1) = \lfloor \frac{n}{2} \rfloor + n + l + 4$$

**Subcase 5:**  $n = 2k + 1$ ,  $k = 1, 3, 5, 7, \dots$  and  $l = 2j$ ,  $j = 2, 4, 6, \dots$

In this subcase  $|V(M(T(n, l + 1)))| = 2n + 2l$ ,  $|E(M(T(n, l + 1)))| = 2 \lfloor \frac{n}{2} \rfloor + 2n + 2l + 6$

$$\left. \begin{array}{l} f(v_{2i-1}) = 1 \\ f(v_{2i}) = 0 \end{array} \right\} 1 \leq i \leq \lfloor \frac{n}{2} \rfloor$$

$$\left. \begin{array}{l} f(e_{4i-3}) = f(e_{4i-2}) = 0 \\ f(e_{4i-1}) = 1 \end{array} \right\} 1 \leq i \leq \lfloor \frac{n}{4} \rfloor + 1$$

$$f(e_{4i}) = 1, 1 \leq i \leq \lfloor \frac{n}{4} \rfloor$$

$$f(v_n) = f(v'_1) = 1 \text{ (when } v'_1 \text{ is attached to } v_n)$$

$$\left. \begin{array}{l} f(v'_{2i}) = 1 \\ f(v'_{2i+1}) = 0 \end{array} \right\} 1 \leq i \leq \lfloor \frac{l}{2} \rfloor$$

$$\left. \begin{array}{l} f(e'_{4i-3}) = f(e'_{4i-2}) = 0 \\ f(e'_{4i-1}) = f(e'_{4i}) = 1 \end{array} \right\} 1 \leq i \leq \frac{l}{4}$$

Using above pattern we have

$$v_f(0) = v_f(1) = n + l, e_f(0) = e_f(1) = \lfloor \frac{n}{2} \rfloor + n + l + 3$$

**Subcase 6:**  $n = 2k + 1$ ,  $k = 1, 3, 5, 7, \dots$  and  $l = 2j$ ,  $j = 3, 5, 7, \dots$

In this subcase  $|V(M(T(n, l + 1)))| = 2n + 2l$ ,  $|E(M(T(n, l + 1)))| = 2 \lfloor \frac{n}{2} \rfloor + 2n + 2l + 8$

$$f(v_n) = f(v'_1) = 1 \text{ (when } v'_1 \text{ is attached to } v_n)$$

$$\left. \begin{array}{l} f(v'_{2i}) = 0 \\ f(v'_{2i+1}) = 1 \end{array} \right\} 1 \leq i \leq \frac{l}{2}$$

$$\left. \begin{array}{l} f(e'_{4i-3}) = f(e'_{4i-2}) = 0 \\ f(e'_{4i-1}) = f(e'_{4i}) = 1 \end{array} \right\} 1 \leq i \leq \lfloor \frac{l}{4} \rfloor$$

$$f(e'_{n-1}) = 0, f(e'_n) = 1$$

remaining vertices are labeled as in subcase 5.

Using above pattern we have

$$v_f(0) = v_f(1) = n + l, e_f(0) = e_f(1) = \lfloor \frac{n}{2} \rfloor + n + l + 4$$

For  $l = 2$  we have  $e_f(0) = e_f(1) = 8$ .

**Subcase 7:**  $n = 2k + 1, k = 1, 3, 5, 7, \dots$  and  $l = 2j + 1, j = 1, 3, 5, \dots$

In this subcase  $|V(M(T(n, l + 1)))| = 2n + 2l, |E(M(T(n, l + 1)))| = 2 \lfloor \frac{n}{2} \rfloor + 2n + 2l + 5$

$$f(v_n) = f(v'_1) = 1 (\text{when } v'_1 \text{ is attached to } v_n)$$

$$f(v'_{2i}) = 0, \quad 1 \leq i \leq \lfloor \frac{l}{2} \rfloor + 1$$

$$f(v'_{2i+1}) = 1, 1 \leq i \leq \lfloor \frac{l}{2} \rfloor$$

$$\left. \begin{array}{l} f(e'_{4i-3}) = f(v'_{4i-2}) = 1 \\ f(e'_{4i-1}) = 0 \end{array} \right\} 1 \leq i \leq \lfloor \frac{l}{4} \rfloor + 1$$

$$f(e'_{4i}) = 0, 1 \leq i \leq \lfloor \frac{l}{4} \rfloor$$

remaining vertices are labeled as in subcase 5.

Using above pattern we have

$$v_f(0) = v_f(1) = n + l, e_f(0) = e_f(1) + 1 = \lfloor \frac{n}{2} \rfloor + n + l + 3$$

For  $l = 1$  we have  $e_f(0) = e_f(1) + 1 = 7$ .

**Subcase 8:**  $n = 2k + 1, k = 1, 3, 5, 7, \dots$  and  $l = 2j + 1, j = 2, 4, 6, \dots$

In this subcase  $|V(M(T(n, l + 1)))| = 2n + 2l, |E(M(T(n, l + 1)))| = 2 \lfloor \frac{n}{2} \rfloor + 2n + 2l + 7$

$$f(v_n) = f(v'_1) = 1 (\text{when } v'_1 \text{ is attached to } v_n)$$

$$f(v'_{2i}) = 0, \quad 1 \leq i \leq \lfloor \frac{l}{2} \rfloor + 1$$

$$f(v'_{2i+1}) = 1, 1 \leq i \leq \lfloor \frac{l}{2} \rfloor$$

$$f(e'_{4i-3}) = 1, 1 \leq i \leq \lfloor \frac{l}{4} \rfloor + 1$$

$$\left. \begin{array}{l} f(e'_{4i-2}) = 1 \\ f(e'_{4i-1}) = f(e'_{4i}) = 0 \end{array} \right\} 1 \leq i \leq \lfloor \frac{l}{4} \rfloor$$

remaining vertices are labeled as in subcase 5.

Using above pattern we have

$$v_f(0) = v_f(1) = n + l, e_f(0) = e_f(1) + 1 = \lfloor \frac{n}{2} \rfloor + n + l + 4$$

**Case 2:**  $n$  is even

**Subcase 1:**  $n = 2k, k = 2, 4, 6, \dots$  and  $l = 2j, j = 2, 4, 6, \dots$

In this subcase  $|V(M(T(n, l+1)))| = 2n + 2l, |E(M(T(n, l+1)))| = 3n + 2l + 5$

$$\left. \begin{array}{l} f(v_{2i-1}) = 1 \\ f(v_{2i}) = 0 \end{array} \right\} 1 \leq i \leq \frac{n}{2}$$

$$\left. \begin{array}{l} f(e_{4i-3}) = f(e_{4i-2}) = 0 \\ f(e_{4i-1}) = f(e_{4i}) = 1 \end{array} \right\} 1 \leq i \leq \frac{n}{4}$$

$$f(v'_1) = 1 \text{ (when } v'_1 \text{ is attached to } v_1)$$

$$\left. \begin{array}{l} f(v'_{2i}) = 0 \\ f(v'_{2i+1}) = 1 \end{array} \right\} 1 \leq i \leq \frac{l}{2}$$

$$\left. \begin{array}{l} f(e'_{4i-3}) = f(e'_{4i-2}) = 0 \\ f(e'_{4i-1}) = f(e'_{4i}) = 1 \end{array} \right\} 1 \leq i \leq \frac{l}{4}$$

Using above pattern we have

$$v_f(0) = v_f(1) = n + l, e_f(0) = e_f(1) + 1 = \frac{3n}{2} + l + 3$$

**Subcase 2:**  $n = 2k, k = 2, 4, 6, \dots$  and  $l = 2j, j = 3, 5, 7, \dots$

In this subcase  $|V(M(T(n, l+1)))| = 2n + 2l, |E(M(T(n, l+1)))| = 3n + 2l + 7$

$$f(v'_1) = 1 \text{ (when } v'_1 \text{ is attached to } v_1)$$

$$\left. \begin{array}{l} f(e'_{4i-3}) = f(e'_{4i-2}) = 0 \\ f(e'_{4i-1}) = f(e'_{4i}) = 1 \end{array} \right\} 1 \leq i \leq \lfloor \frac{l}{4} \rfloor$$

$$f(e'_{n-1}) = 0, f(e'_n) = 1$$

remaining vertices are labeled as in subcase 1 of case(2).

Using above pattern we have

$$v_f(0) = v_f(1) = n + l, e_f(0) + 1 = e_f(1) = \frac{3n}{2} + l + 4$$

For  $l = 2$  we have  $e_f(0) + 1 = e_f(1) = 10$ .

**Subcase 3:**  $n = 2k$ ,  $k = 2, 4, 6, \dots$  and  $l = 2j + 1$ ,  $j = 1, 3, 5, 7, \dots$

In this subcase  $|V(M(T(n, l + 1)))| = 2n + 2l$ ,  $|E(M(T(n, l + 1)))| = 3n + 2l + 4$

$$f(v'_1) = 1 (\text{when } v'_1 \text{ is attached to } v_1)$$

$$f(v'_{2i}) = 0, \quad 1 \leq i \leq \lfloor \frac{l}{2} \rfloor + 1$$

$$f(v'_{2i+1}) = 1, \quad 1 \leq i \leq \lfloor \frac{l}{2} \rfloor$$

$$\left. \begin{array}{l} f(e'_{4i-3}) = f(e'_{4i-2}) = 1 \\ f(e'_{4i-1}) = 0 \end{array} \right\} \quad 1 \leq i \leq \lfloor \frac{l}{4} \rfloor + 1$$

$$f(e'_{4i}) = 0, \quad 1 \leq i \leq \lfloor \frac{l}{4} \rfloor$$

remaining vertices are labeled as in subcase 1 of case(2).

Using above pattern we have

$$v_f(0) = v_f(1) = n + l, \quad e_f(0) = e_f(1) = \frac{3n}{2} + l + 2$$

For  $l = 1$  we have  $e_f(0) = e_f(1) = 8$ .

**Subcase 4:**  $n = 2k$ ,  $k = 2, 4, 6, \dots$  and  $l = 2j + 1$ ,  $j = 2, 4, 6, \dots$

In this subcase  $|V(M(T(n, l + 1)))| = 2n + 2l$ ,  $|E(M(T(n, l + 1)))| = 3n + 2l + 6$

$$f(v'_1) = 1 (\text{when } v'_1 \text{ is attached to } v_1)$$

$$f(v'_{2i}) = 0, \quad 1 \leq i \leq \lfloor \frac{l}{2} \rfloor + 1$$

$$f(v'_{2i+1}) = 1, \quad 1 \leq i \leq \lfloor \frac{l}{2} \rfloor$$

$$f(e'_{4i-3}) = 1, \quad 1 \leq i \leq \lfloor \frac{l}{4} \rfloor + 1$$

$$\left. \begin{array}{l} f(e'_{4i-2}) = 1 \\ f(e'_{4i-1}) = f(e'_{4i}) = 0 \end{array} \right\} \quad 1 \leq i \leq \lfloor \frac{l}{4} \rfloor$$

remaining vertices are labeled as in subcase 1 of case(2).

Using above pattern we have

$$v_f(0) = v_f(1) = n + l, \quad e_f(0) = e_f(1) = \frac{3n}{2} + l + 3.$$

**Subcase 5:**  $n = 2k$ ,  $k = 3, 5, 7, \dots$  and  $l = 2j$ ,  $j = 2, 4, 6, \dots$

In this subcase  $|V(M(T(n, l + 1)))| = 2n + 2l$ ,  $|E(M(T(n, l + 1)))| = 3n + 2l + 5$

$$\left. \begin{array}{l} f(v_{2i-1}) = 1 \\ f(v_{2i}) = 0 \end{array} \right\} \quad 1 \leq i \leq \frac{n}{2}$$

$$\left. \begin{array}{l} f(e_{4i-3}) = f(e_{4i-2}) = 0 \\ f(e_{4i-1}) = f(e_{4i}) = 1 \end{array} \right\} 1 \leq i \leq \lfloor \frac{n}{4} \rfloor$$

$$\begin{array}{l} f(e_{n-1}) = 0, f(e_n) = 1 \\ f(v'_1) = 1 \text{ (when } v'_1 \text{ is attached to } v_1) \end{array}$$

$$\left. \begin{array}{l} f(v'_{2i}) = 0 \\ f(v'_{2i+1}) = 1 \end{array} \right\} 1 \leq i \leq \frac{l}{2}$$

$$\left. \begin{array}{l} f(e'_{4i-3}) = f(e'_{4i-2}) = 0 \\ f(e'_{4i-1}) = f(e'_{4i}) = 1 \end{array} \right\} 1 \leq i \leq \frac{l}{4}$$

Using above pattern we have

$$v_f(0) = v_f(1) = n + l, e_f(0) + 1 = e_f(1) = \frac{3n}{2} + l + 3$$

**Subcase 6:**  $n = 2k, k = 3, 5, 7, \dots$  and  $l = 2j, j = 3, 5, 7, \dots$

In this subcase  $|V(M(T(n, l+1)))| = 2n + 2l, |E(M(T(n, l+1)))| = 3n + 2l + 7$

$$f(v'_1) = 1 \text{ (when } v'_1 \text{ is attached to } v_1)$$

$$\left. \begin{array}{l} f(e'_{4i-3}) = f(e'_{4i-2}) = 0 \\ f(e'_{4i-1}) = f(e'_{4i}) = 1 \end{array} \right\} 1 \leq i \leq \lfloor \frac{l}{4} \rfloor$$

$$f(e'_{n-1}) = 1, f(e'_n) = 0$$

remaining vertices are labeled as in subcase 5 of case(2).

Using above pattern we have

$$v_f(0) = v_f(1) = n + l, e_f(0) + 1 = e_f(1) = \frac{3n}{2} + l + 4$$

For  $l = 2$  we have  $e_f(0) + 1 = e_f(1) = 13$

**Subcase 7:**  $n = 2k, k = 3, 5, 7, \dots$  and  $l = 2j + 1, j = 1, 3, 5, 7, \dots$

In this subcase  $|V(M(T(n, l+1)))| = 2n + 2l, |E(M(T(n, l+1)))| = 3n + 2l + 4 + 4\lfloor \frac{j}{2} \rfloor$

$$f(v'_1) = 0 \text{ (when } v'_1 \text{ is attached to } v_2)$$

$$f(v'_{2i}) = 1, 1 \leq i \leq \lfloor \frac{l}{2} \rfloor + 1$$

$$f(v'_{2i+1}) = 0, 1 \leq i \leq \lfloor \frac{l}{2} \rfloor$$

$$\left. \begin{array}{l} f(e'_{4i-3}) = f(e'_{4i-2}) = 0 \\ f(e'_{4i-1}) = 1 \end{array} \right\} 1 \leq i \leq \lfloor \frac{l}{4} \rfloor + 1$$

$$f(e'_{4i}) = 1, 1 \leq i \leq \lfloor \frac{l}{4} \rfloor$$

remaining vertices are labeled as in subcase 5 of case(2).

Using above pattern we have

$$v_f(0) = v_f(1) = n + l, e_f(0) = e_f(1) = \frac{3n}{2} + l + 2 + 2 \lfloor \frac{j}{2} \rfloor$$

For  $l = 1$  we have  $e_f(0) = e_f(1) = 11$

**Subcase 8:**  $n = 2k, k = 3, 5, 7, \dots$  and  $l = 2j + 1, j = 2, 4, 6, \dots$

In this subcase  $|V(M(T(n, l + 1)))| = 2n + 2l, |E(M(T(n, l + 1)))| = 3n + 2l + 2 + 4 \lfloor \frac{j}{2} \rfloor$

$$f(v'_1) = 0 (\text{when } v'_1 \text{ is attached to } v_2)$$

$$f(v'_{2i}) = 1, 1 \leq i \leq \lfloor \frac{l}{2} \rfloor + 1$$

$$f(v'_{2i+1}) = 0, 1 \leq i \leq \lfloor \frac{l}{2} \rfloor$$

$$f(e'_{4i-3}) = 0, 1 \leq i \leq \lfloor \frac{l}{4} \rfloor + 1$$

$$\left. \begin{array}{l} f(e'_{4i-2}) = 0 \\ f(e'_{4i-1}) = f(e'_{4i}) = 1 \end{array} \right\} 1 \leq i \leq \lfloor \frac{l}{4} \rfloor$$

remaining vertices are labeled as in subcase 5 of case(2).

Using above pattern we have

$$v_f(0) = v_f(1) = n + l, e_f(0) = e_f(1) = \frac{3n}{2} + l + 1 + 2 \lfloor \frac{j}{2} \rfloor$$

Thus in all the cases described above  $f$  admits cordial labeling for  $M(T(n, l + 1))$ .

That is,  $M(T(n, l + 1))$  admits cordial labeling.  $\square$

**Illustration 3.5.10.** Consider  $G = M(T(6,5))$ . The cordial labeling is as shown in Figure 3.7.

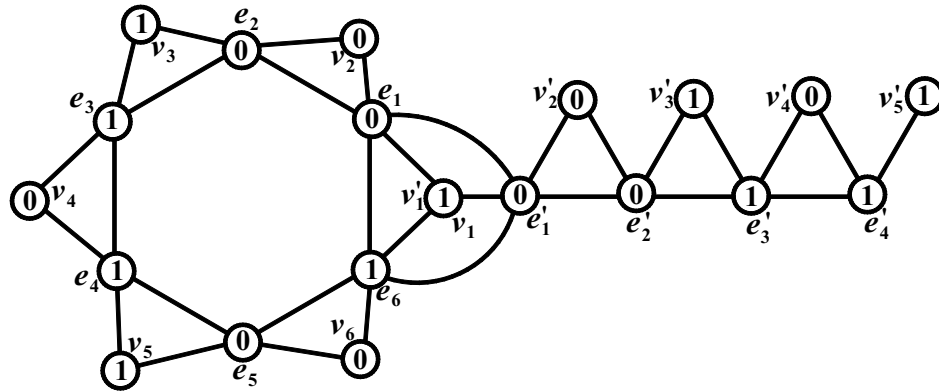


FIGURE 3.7:  $M(T(6,5))$  and its cordial labeling

### 3.6 NP-Complete problems

The detailed discussion on this concept is reported in [38](pp 30-34) as follows. Let  $\mathbf{P}$  denote the class of all problems that can be solved by a polynomial time algorithm, that is, polynomial in the length of the inputs for an instance of the problem. We can think of these algorithms as running on a relatively simple computer, for example a Turing machine, named after the British mathematician/logician Alan Turing. Briefly, a Turing machine is a computer with (i) a two-way infinite storage tape, divided into cells, in each of which can be written one symbol chosen from a finite alphabet, and (ii) a finite-state control. The finite control can be thought of as a random access machine or RAM. The execution time of such a RAM is usually measured by the number of operations it performs in solving an instance of a problem. Each operation can be assumed to require a constant amount of time, say  $C$ . Typical operations include addition, subtraction, multiplication, and division of two numbers, storing a number in a random access memory, and comparing two numbers.

Turing postulated the thesis that what we think of as an effective algorithm is precisely what can be done by a Turing machine or, equivalently, by a RAM with an infinite amount of auxiliary memory. Thus, we can say that a computational problem is in class

**P** if there exists an algorithm for solving any instance of the problem in time  $O(n^k)$  for some fixed positive integer  $k$ , where  $n$  is the length of the input for the given instance. Typical examples of problems which can be solved in polynomial time, and are therefore in the class **P**, include:

- sorting  $n$  integers.
- finding a shortest path between two vertices  $u$  and  $v$  in a graph  $G$ .
- finding a maximum matching in a graph  $G$ .
- determining whether two trees  $T_1$  and  $T_2$  with  $n$  vertices are isomorphic.
- deciding whether a given graph is bipartite or connected.
- computing the convex hull of a set of  $n$  points in the plane.

In the theory of NP-completeness, we restrict our attention to the class of problems called *decision problems*. These are problems, every instance of which can be stated in such a way that the answer is either ‘yes’ or ‘no’. Thus, for example, we do not seek an algorithm for finding the minimum cardinality of a dominating set in a graph. Instead we seek an algorithm which, given a graph  $G$  and a positive integer  $k$ , can decide whether  $G$  has a dominating set of size  $\leq k$ .

Let **NP** denote the class of all decision problems which can be solved in polynomial time by a nondeterministic Turing machine. Thus, **NP** stands for Nondeterministic Polynomial time. Again, wishing to avoid the extended discussion required to give a technical definition of a nondeterministic Turing machine, suffice it to say that such a machine has the ability to make guesses at certain points in a computation. Some of these guesses may be correct, some may be incorrect.

Instead of using the notion of nondeterminism, we can define the class **NP** in terms of the concept of *polynomial-time verification*. A verification algorithm is an algorithm  $A$  which takes as input an instance of a problem and a candidate solution to the problem, called a *certificate*, and verifies in polynomial time whether the certificate is a solution to the given problem instance. Thus, the class **NP** is the class of problems which can be verified in polynomial time.

If  $P_1$  is polynomial-time reducible to  $P_2$ , we can say that any algorithm for solving  $P_2$  can be used to solve  $P_1$ . Intuitively, problem  $P_1$  is ‘no harder’ to solve than problem  $P_2$ . We define a problem  $P$  to be *NP-complete* if (i)  $P \in \mathbf{NP}$ , and (ii) for every problem  $P' \in \mathbf{NP}$ ,  $P' \leq_p P$ . If a problem  $P$  can be shown to satisfy condition (ii), but not necessarily condition (i), then we say that it is NP-hard.

### 3.7 Embedding and NP-Complete problem for cordial Graphs

Embedding problems related to graph structures are of great importance. We will discuss such problems in the context of labeling. The common problem is *Given a graph  $G$  having the graph theoretic property  $P$ , is it possible to embed  $G$  as an induced subgraph of  $G$ , having the property  $P$ ?*

Such problems are extensively investigated recently by Acharya et al.[1] in the context of graceful graphs. We present here an affirmative answer for planar graphs, trianglefree graphs and graphs with given chromatic number in the context of cordial graphs. As a consequence we deduce that deciding whether the chromatic number is less than or equal to  $k$ , where  $k \geq 3$ , is NP-complete even for cordial graphs. We obtain similar result for clique number also. The similar discussion will be held in section 3.10 for 3-equitable graph.

**Theorem 3.7.1.** Any graph  $G$  can be embedded as an induced subgraph of a cordial graph.

*Proof.* Without loss of generality we assume that it is always possible to label the vertices of any graph  $G$  such that the vertex condition for cordial graph is satisfied. i.e.  $|v_f(0) - v_f(1)| \leq 1$ . Let  $V_0$  and  $V_1$  be the set of vertices with label zero and one respectively. Let  $E_0$  and  $E_1$  be the set of edges with label zero and one respectively. Let  $n(V_0)$  and  $n(V_1)$  be the number of element in set  $V_0$  and  $V_1$ . Let  $n(E_0)$  and  $n(E_1)$  be the number of element in set  $E_0$  and  $E_1$ . Let  $|n(E_0) - n(E_1)| = r > 1$  (for  $r = 0$  or  $1$  graph  $G$  will become cordial). Graph  $H$  can be obtained by adding  $r$  vertices to the graph  $G$

with following condition given in the different cases reported below.

**Case 1:**  $n(V_0) = n(V_1)$  and  $n(E_0) > n(E_1)$

Let  $r = s + t$  with  $|s - t| \leq 1$ . Out of new  $r$  vertices label  $s$  vertices with 0 and  $t$  vertices with 1.i.e. label the vertices  $v_1, v_2, v_3 \dots, v_s$  with 0 and label the vertices  $u_1, u_2, u_3 \dots, u_t$  with 1. Join each  $v_i$  with unique element of set  $V_1$  and join each  $u_i$  with unique element of set  $V_0$ . Therefore all the new edges will have label 1. For the graph  $H$  number of vertices with label 0 and 1 are  $n(V_0) + s$  and  $n(V_1) + t$  respectively. Therefore  $|v_f(0) - v_f(1)| = |n(V_0) + s - n(V_1) - t| \leq 1$ , Hence vertex condition is satisfied. For the graph  $H$  number of edges with label 0 and 1 are  $n(E_0)$  and  $n(E_1) + r$  respectively. Therefore  $|e_f(0) - e_f(1)| = |n(E_0) - n(E_1) - r| = 0$ , as  $n(E_0) > n(E_1)$ . Hence edge condition is satisfied.

**Case 2:**  $n(V_0) = n(V_1)$  and  $n(E_0) < n(E_1)$

Let  $r = s + t$  with  $|s - t| \leq 1$ . Out of new  $r$  vertices label  $s$  vertices with 0 and  $t$  vertices with 1.i.e. label the vertices  $v_1, v_2, v_3 \dots, v_s$  with 0 and label the vertices  $u_1, u_2, u_3 \dots, u_t$  with 1. Join each  $v_i$  with unique element of set  $V_0$  and join each  $u_i$  with unique element of set  $V_1$ . Therefore all the new edges will have label 0. For the graph  $H$  number of vertices with label 0 and 1 are  $n(V_0) + s$  and  $n(V_1) + t$  respectively. Therefore  $|v_f(0) - v_f(1)| = |n(V_0) + s - n(V_1) - t| \leq 1$ , Hence vertex condition is satisfied. For Graph  $H$  number of edges with label 0 and 1 are  $n(E_0) + r$  and  $n(E_1)$  respectively. Therefore  $|e_f(0) - e_f(1)| = |n(E_0) + r - n(E_1)| = 0$ , as  $n(E_0) < n(E_1)$ . Hence edge condition is satisfied.

**Case 3:**  $|n(V_0) - n(V_1)| = 1$  and  $n(E_0) > n(E_1)$

Let  $r = s + t$  with  $|n(V_0) + s - n(V_1) - t| \leq 1$ . Out of new  $r$  vertices label  $s$  vertices with 0 and  $t$  vertices with 1.i.e. label the vertices  $v_1, v_2, v_3 \dots, v_s$  with 0 and label the vertices  $u_1, u_2, u_3 \dots, u_t$  with 1. Join each  $v_i$  with unique element of set  $V_1$  and join each  $u_i$  with unique element of set  $V_0$ . Therefore all the new edges will have label 1. For the graph  $H$  number of vertices with label 0 and 1 are  $n(V_0) + s$  and  $n(V_1) + t$  respectively. Therefore,  $|v_f(0) - v_f(1)| = |n(V_0) + s - n(V_1) - t| \leq 1$ , Hence vertex condition is satisfied. For

the graph  $H$  number of edges with label 0 and 1 are  $n(E_0)$  and  $n(E_1) + r$  respectively. Therefore,  $|e_f(0) - e_f(1)| = |n(E_0) - n(E_1) - r| = 0$ , as  $n(E_0) > n(E_1)$ . Hence edge condition is satisfied.

**Case 4:**  $|n(V_0) - n(V_1)| = 1$  and  $n(E_0) < n(E_1)$

Let  $r = s + t$  with  $|n(V_0) + s - n(V_1) - t| \leq 1$ . Out of new  $r$  vertices label  $s$  vertices with 0 and  $t$  vertices with 1. i.e. label the vertices  $v_1, v_2, v_3, \dots, v_s$  with 0 and label the vertices  $u_1, u_2, u_3, \dots, u_t$  with 1. Join each  $v_i$  with unique element of set  $V_0$  and join each  $u_i$  with unique element of set  $V_1$ . Therefore all the new edges will have label 0. For the graph  $H$  number of vertices with label 0 and 1 are  $n(V_0) + s$  and  $n(V_1) + t$  respectively. Therefore,  $|v_f(0) - v_f(1)| = |n(V_0) + s - n(V_1) - t| \leq 1$ . Hence vertex condition is satisfied. For Graph  $H$  number of edges with label 0 and 1 are  $n(E_0) + r$  and  $n(E_1)$  respectively. Therefore,  $|e_f(0) - e_f(1)| = |n(E_0) + r - n(E_1)| = 0$ , as  $n(E_0) < n(E_1)$ . Hence edge condition is satisfied.

Thus in all the possibilities the graph  $H$  resulted due to above construction satisfies the condition for cordial graph. i.e. Any graph  $G$  can be embedded as an induced subgraph of a cordial graph.  $\square$

**Corollary 3.7.2.** Any planar graph  $G$  can be embedded as an induced subgraph of a planar cordial graph.

*Proof.* Let  $G$  be a planar graph. Then the graph  $H$  obtained by Theorem 3.7.1 is a planar graph.  $\square$

**Corollary 3.7.3.** Any triangle-free graph  $G$  can be embedded as an induced subgraph of a triangle free cordial graph.

*Proof.* Let  $G$  be a triangle-free graph. Then the graph  $H$  obtained by Theorem 3.7.1 is a triangle-free graph.  $\square$

**Corollary 3.7.4.** The problem of deciding whether the chromatic number  $\chi \leq k$ , where  $k \geq 3$  is NP-complete even for cordial graphs.

*Proof.* Let  $G$  be a graph with chromatic number  $\chi(G) \geq 3$ . Let  $H$  be the cordial graph constructed in Theorem 3.7.1, which contains  $G$  as an induced subgraph. Since  $H$  is constructed by adding pendant vertices only to  $G$ . We have  $\chi(H) = \chi(G)$ .

Since the problem of deciding whether the chromatic number  $\chi \leq k$ , where  $k \geq 3$  is NP-complete [27]. It follows that deciding whether the chromatic number  $\chi \leq k$ , where  $k \geq 3$ , is NP-complete even for cordial graphs.  $\square$

**Corollary 3.7.5.** The problem of deciding whether the clique number  $\omega(G) \geq k$  is NP-complete even when restricted to cordial graphs.

*Proof.* Since the problem of deciding whether the clique number of a graph  $\omega(G) \geq k$ , is NP-complete [27] and  $\omega(H) = \omega(G)$  for the cordial graph  $H$  constructed in Theorem 3.7.1, the result follows.  $\square$

## 3.8 3-equitable labeling of graphs

In 1990 Cahit[13] proposed the idea of distributing the vertex and the edge labels among  $\{0, 1, 2, \dots, k-1\}$  as evenly as possible to obtain a generalization of graceful labeling and named it as  $k$ -equitable labeling which is defined as follows.

### 3.8.1 $k$ -equitable labeling

A vertex labeling of a graph  $G = (V(G), E(G))$  is a function  $f : V(G) \rightarrow \{0, 1, 2, \dots, k-1\}$  and the value  $f(u)$  is called label of vertex  $u$ . For the vertex labeling function  $f : V(G) \rightarrow \{0, 1, \dots, k-1\}$ , the induced function  $f^* : E(G) \rightarrow \{0, 1, \dots, k-1\}$  defined as  $f^*(e = uv) = |f(u) - f(v)|$  which satisfies the conditions

$$\left. \begin{array}{l} |v_f(i) - v_f(j)| \leq 1 \\ |e_f(i) - e_f(j)| \leq 1 \end{array} \right\} \text{ where } 0 \leq i, j \leq k-1$$

where  $v_f(i)$  and  $e_f(i)$  denotes the number of vertices and the number of edges having label  $i$  under  $f$  and  $f^*$  respectively. Such labeling  $f$  is called  $k$ -equitable labeling for

the graph  $G$ . A graph which admits  $k$ -equitable labeling is called  $k$ -equitable graph. Obviously 2-equitable labeling is *cordial* labeling which is already discussed in section 3.4. When  $k = 3$  the labeling is called *3-equitable labeling*. The remaining part of this chapter is devoted to the discussion of 3-equitable labeling of graphs.

### 3.8.2 Illustration

In the following Figure 3.8  $H_4$  and its 3-equitable labeling is shown.

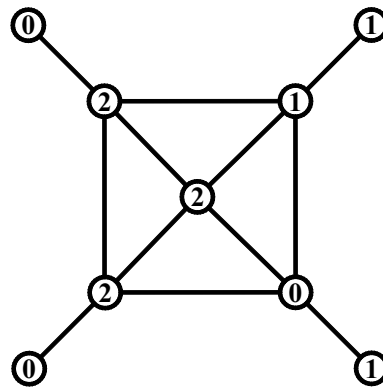


FIGURE 3.8:  $H_4$  and its 3-equitable labeling

### 3.8.3 Some known results

- Cahit[12],[13] proved that
  - ◇  $C_n$  is 3-equitable if and only if  $n \not\equiv 3 \pmod{6}$ .
  - ◇ An Eulerian graph with  $q \equiv 3 \pmod{6}$  is not 3-equitable where  $q$  is the number of edges.
  - ◇ All caterpillars are 3-equitable.
  - ◇ A triangular cactus with  $n$  blocks is 3-equitable if and only if  $n$  is even. (Conjecture)
  - ◇ Every tree with fewer than five end vertices has a 3-equitable labeling.

- Seoud and Abdel Maqsoud[60] proved that
  - ◊ A graph with  $p$  vertices and  $q$  edges in which every vertex has odd degree is not 3-equitable if  $p \equiv 0(mod 3)$  and  $q \equiv 3(mod 6)$ .
  - ◊ All fans except  $P_2 + K_1$  are 3-equitable.
  - ◊  $P_n^2$  is 3-equitable for all  $n$  except 3.
  - ◊  $K_{m,n}$ (where  $3 \leq m \leq n$ ) is 3-equitable if and only if  $(m, n) = (4, 4)$ .
- Bapat and Limaye[7] proved that Helms  $H_n$ (where  $n \geq 4$ ) are 3-equitable.
- Youssef[84] proved that  $W_n = C_n + K_1$  is 3-equitable for all  $n \geq 4$ .
- Vaidya et al.[74, 75] have discussed wheel related 3-equitable graphs.
- Vaidya et al.[72] have discussed some shell related 3-equitable graphs.
- Vaidya et al.[71, 73] have discussed some star related 3-equitable graphs.

### 3.9 3-equitable graphs in the context of some graph operations

**Theorem 3.9.1.** The graph  $D_2(C_n)$  is 3 - equitable except for  $n = 3, 5$ .

*Proof.* If  $D_2(C_n)$  be the shadow graph of cycle  $C_n$  then let  $v_1, v_2, \dots, v_n$  be the vertices of cycle  $C_n$  and  $v'_1, v'_2, \dots, v'_n$  be the vertices added corresponding to the vertices  $v_1, v_2, \dots, v_n$  in order to obtain  $D_2(C_n)$ .

Define  $f : V(D_2(C_n)) \rightarrow \{0, 1, 2\}$ , we consider following six cases.

**Case 1:** When  $n = 3, 5$

In order to satisfy the vertex condition for 3-equitable graph when  $n = 3$  it is essential to assign label 0 to two vertices, label 1 to two vertices and label 2 to two vertices. The vertices with label 1 will give rise to six edges with label 1 out of total twelve edges of  $D_2(C_3)$ . Then obviously the graph is not 3-equitable.

In order to satisfy the vertex condition for 3-equitable graph when  $n = 5$  it is essential

to assign label 1 to at least three vertices. The vertices with label 1 will give rise to at least eight edges with label 1 out of total twenty edges of  $D_2(C_5)$ . Then obviously the graph is not 3-equitable.

**Case 2:** For  $n = 4, 6, 8$  and the respective graphs and their 3-equitable labeling is shown in following Figure 3.9.

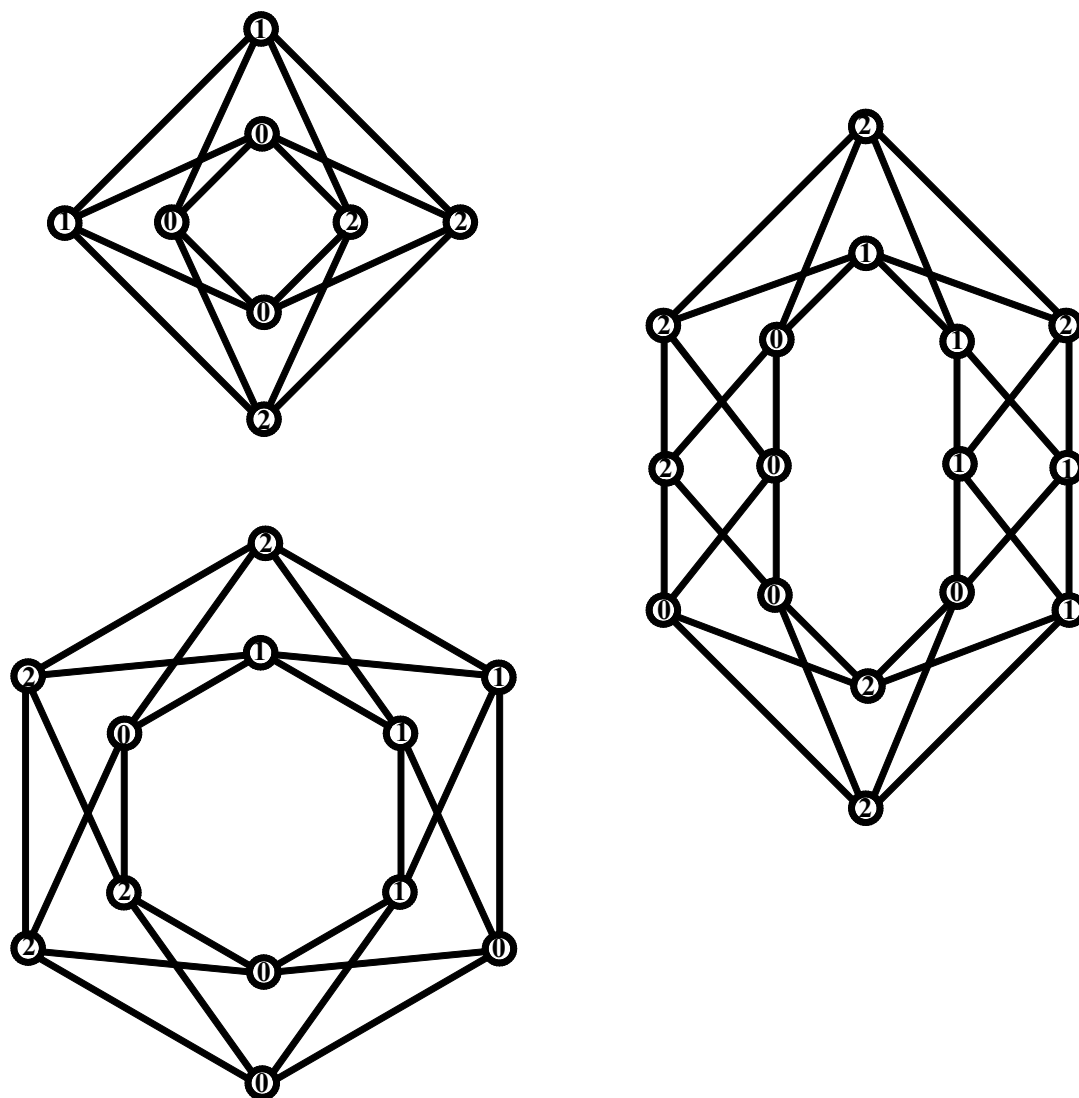


FIGURE 3.9: 3-equitable labeling of  $D_2(C_4)$ ,  $D_2(C_6)$  and  $D_2(C_8)$

**Case 3:** For  $n = 14$  the 3-equitable labeling is shown in following Figure 3.10.

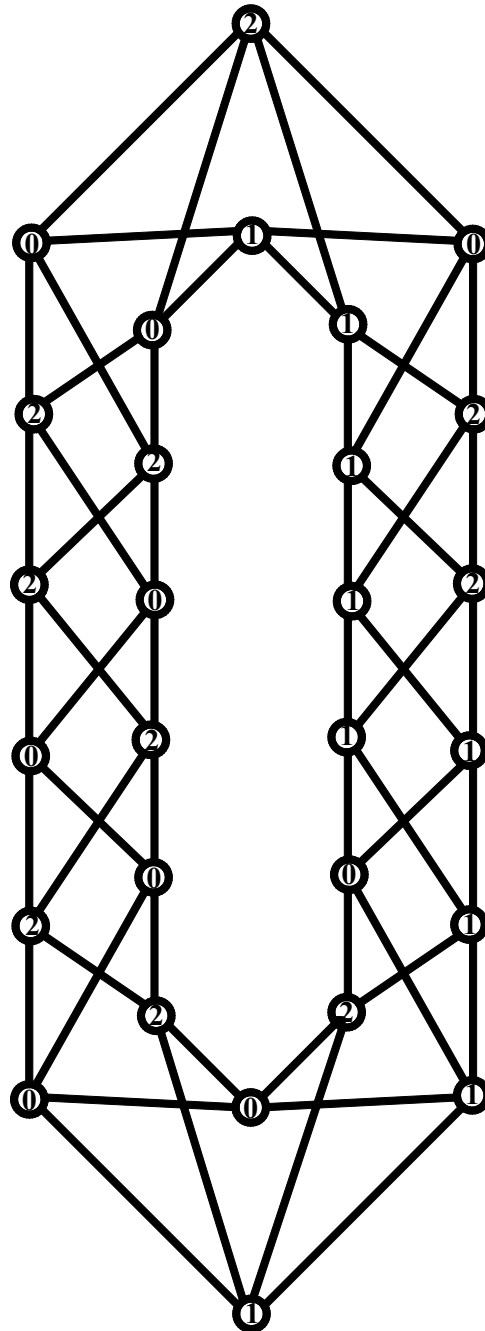


FIGURE 3.10: 3-equitable labeling of  $D_2(C_{14})$

**Case 4:** When  $n \equiv 0(mod 3)$ .

**Sub Case 1:** When  $n = 3k, k = 4, 6, \dots$

$$\begin{aligned}
 f(v_{4i-3}) &= f(v_{4i-2}) = f(v_{4i-1}) = 1, & 1 \leq i \leq \frac{k}{2} \\
 f(v_{4i}) &= 0, & 1 \leq i \leq \frac{k}{2} - 1 \\
 f(v_{2k}) &= 2, \\
 f(v_i) &= 0, & \text{otherwise;} \\
 f(v'_1) &= 2, \\
 f(v'_{4i-2}) &= 1, & 1 \leq i \leq \frac{k}{2} \\
 f(v'_{4i-1}) &= 2, & 1 \leq i \leq \frac{k}{2} - 1 \\
 f(v'_5) &= 2, \\
 f(v'_{4i+5}) &= 2, & 1 \leq i \leq \frac{k}{2} - 2 \\
 f(v'_{4i}) &= 0, & 1 \leq i \leq \frac{k}{2} - 1 \\
 f(v'_{2k-1}) &= f(v'_n) = 0, \\
 f(v'_i) &= 2, & \text{otherwise.}
 \end{aligned}$$

**Sub Case 2:**  $n = 3k, k = 3, 5, \dots$

$$\begin{aligned}
 f(v_i) &= 1, & 1 \leq i \leq k \\
 f(v_{k+2i-1}) &= 2, & 1 \leq i \leq \lfloor \frac{k}{2} \rfloor + 1 \\
 f(v_i) &= 0, & \text{otherwise;} \\
 f(v'_{2i-1}) &= 2, & 1 \leq i \leq \lfloor \frac{k}{2} \rfloor \\
 f(v'_{2i}) &= 0, & 1 \leq i \leq \lfloor \frac{k}{2} \rfloor \\
 f(v'_{2\lfloor \frac{k}{2} \rfloor + i}) &= 1, & 1 \leq i \leq k \\
 f(v'_{2\lfloor \frac{k}{2} \rfloor + k + i}) &= 2, & 1 \leq i \leq k \\
 f(v'_n) &= 0
 \end{aligned}$$

In view of the above labeling pattern we have

$$v_f(0) = v_f(1) = v_f(2) = 2k$$

$$e_f(0) = e_f(1) = e_f(2) = 4k$$

**Case 5:** When  $n \equiv 1 \pmod{3}$ .

**Sub Case 1:** When  $n = 3k + 1, k = 3, 5, \dots$

$$\begin{aligned}
 f(v_1) &= 2 \\
 f(v_{1+i}) &= 0, & 1 \leq i \leq k-1 \\
 f(v_{k+i}) &= 1, & 1 \leq i \leq k \\
 f(v_{2k+2i-1}) &= 2, & 1 \leq i \leq k-1 \\
 f(v_{2k+2i}) &= 0, & 1 \leq i \leq k-1 \\
 f(v'_i) &= 1, & 1 \leq i \leq k \\
 f(v'_{k+i}) &= 2, & 1 \leq i \leq k-1 \\
 f(v'_{2k+2i-2}) &= 0, & 1 \leq i \leq \lfloor \frac{k}{2} \rfloor \\
 f(v'_{2k+2i-1}) &= 2, & 1 \leq i \leq \lfloor \frac{k}{2} \rfloor \\
 f(v'_{n-2}) &= 2, & f(v'_{n-1}) = f(v'_n) = 0
 \end{aligned}$$

**Sub Case 2:**  $n = 3k + 1, k = 2, 4, \dots$

$$\begin{aligned}
 f(v_{2i-1}) &= 2, & 1 \leq i \leq k \\
 f(v_i) &= 0, & \text{otherwise;} \\
 f(v'_i) &= 1, & 1 \leq i \leq 2k \\
 f(v'_i) &= 2, & \text{otherwise.}
 \end{aligned}$$

In view of the above labeling pattern we have

$$\begin{aligned}
 v_f(0) &= v_f(1) + 1 = v_f(2) = n - k \\
 e_f(0) + 1 &= e_f(1) = e_f(2) + 1 = n + k + 1
 \end{aligned}$$

**Case 6:**  $n \equiv 2 \pmod{3}$ .

**Sub Case 1:**  $n = 3k + 2, k = 3, 5, \dots$

$$\begin{aligned}
 f(v_i) &= 1, & 1 \leq i \leq k+1 \\
 f(v_{k+2i}) &= 0, & 1 \leq i \leq k+1 \\
 f(v_{k+2i+1}) &= 2, & 1 \leq i \leq k \\
 f(v'_1) &= 2, \\
 f(v'_{2i}) &= 0, & 1 \leq i \leq \lfloor \frac{k}{2} \rfloor \\
 f(v'_{2i+1}) &= 2, & 1 \leq i \leq \lfloor \frac{k}{2} \rfloor \\
 f(v'_{2\lfloor \frac{k}{2} \rfloor + i + 1}) &= 1, & 1 \leq i \leq k
 \end{aligned}$$

$$\begin{aligned}
f(v'_{2\lfloor \frac{k}{2} \rfloor + k + 2i}) &= 2, & 1 \leq i \leq \lfloor \frac{k}{2} \rfloor \\
f(v'_{n-2}) &= f(v'_{n-1}) = 2, \\
f(v'_i) &= 0, & \text{otherwise.}
\end{aligned}$$

**Sub Case 2:**  $n = 3k + 2, k = 6, 8, \dots$

$$\begin{aligned}
f(v_i) &= 1, & 1 \leq i \leq k + 1 \\
f(v_{k+2i}) &= 0, & 1 \leq i \leq k + 1 \\
f(v_{k+2i+1}) &= 2, & 1 \leq i \leq k \\
f(v'_{2i-1}) &= 2, & 1 \leq i \leq \frac{k}{2} - 2 \\
f(v'_{2i}) &= 0, & 1 \leq i \leq \frac{k}{2} - 2 \\
f(v'_{k+i-4}) &= 2, & 1 \leq i \leq 4 \\
f(v'_{k+i}) &= 1, & 1 \leq i \leq k \\
f(v'_{2k+2i-1}) &= 0, & 1 \leq i \leq \frac{k}{2} - 1 \\
f(v'_{2k+2i}) &= 2, & 1 \leq i \leq \frac{k}{2} - 1 \\
f(v'_{n-3}) &= f(v'_{n-2}) = f(v'_n) = 0, f(v'_{n-1}) = 2
\end{aligned}$$

In view of the above labeling pattern we have

$$\begin{aligned}
v_f(0) + 1 &= v_f(1) + 1 = v_f(2) = n - k \\
e_f(0) &= e_f(1) + 1 = e_f(2) = n + k + 1
\end{aligned}$$

Thus in each cases we have  $|v_f(i) - v_f(j)| \leq 1$  and  $|e_f(i) - e_f(j)| \leq 1$ , for all  $0 \leq i, j \leq 2$ .

Hence  $D_2(C_n)$  is 3 - equitable graph except for  $n = 3, 5$ .  $\square$

**Illustration 3.9.2.** Consider the graph  $D_2(C_{12})$ . The 3-equitable labeling is as shown in Figure 3.11.

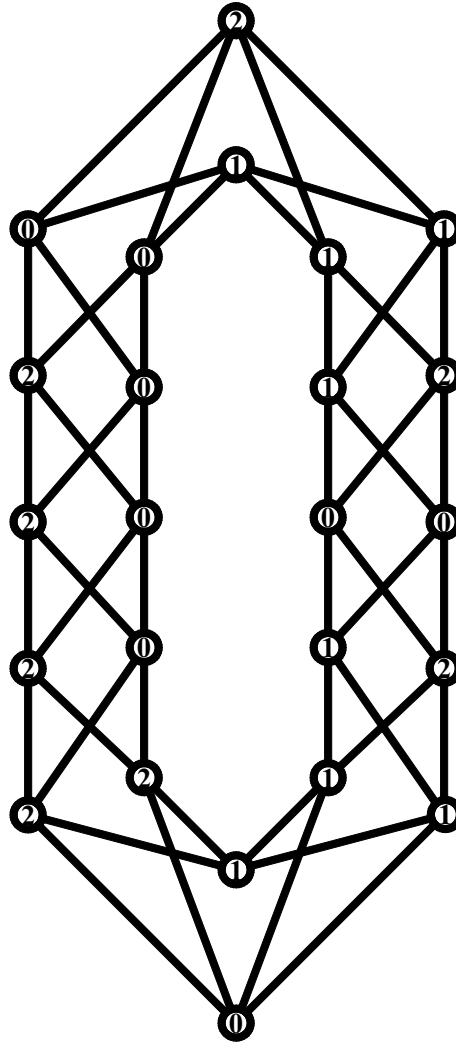


FIGURE 3.11: 3 - equitable labeling of  $D_2(C_{12})$

**Theorem 3.9.3.** The graph  $D_2(P_n)$  is 3 - equitable except for  $n = 3$ .

*Proof.* If  $D_2(P_n)$  be the shadow graph of path  $P_n$  then let  $v_1, v_2, \dots, v_n$  be the vertices of path  $P_n$  and  $v'_1, v'_2, \dots, v'_n$  be the vertices added corresponding to the vertices  $v_1, v_2, \dots, v_n$  in order to obtain  $D_2(P_n)$ .

Define  $f : V(D_2(P_n)) \rightarrow \{0, 1, 2\}$ , we consider following five cases.

**Case 1:** when  $n = 3$ .

In order to satisfy the vertex condition for 3-equitable graph when  $n = 3$  it is essential to assign label 0 to two vertices, label 1 to two vertices and label 2 to two vertices. The vertices with label 1 will give rise to four edges with label 1 out of total eight edges of  $D_2(P_3)$ . Then obviously the graph is not 3-equitable.

**Case 2:** For  $n = 2$  the respective graph and its 3-equitable labeling is shown in Figure 3.12.

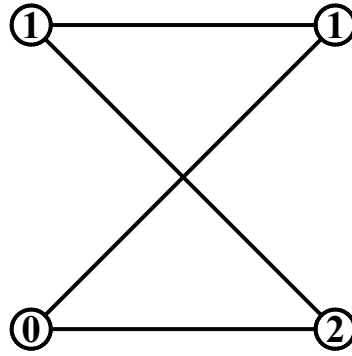


FIGURE 3.12: 3-equitable labeling of  $D_2(P_2)$

**Case 3:** When  $n \equiv 0 \pmod{3}$ , ( $n = 3k, k = 2, 3, 4, \dots$ ).

$$\begin{aligned}
 f(v_i) &= 1, & 1 \leq i \leq \lfloor \frac{n}{2} \rfloor \text{ if } n \text{ is odd and } 1 \leq i \leq \frac{n}{2} - 1 \text{ if } n \text{ is even} \\
 f(v_{\lfloor \frac{n}{2} \rfloor + 1 + i}) &= 0, & 1 \leq i \leq \lfloor \frac{n}{2} \rfloor \text{ if } n \text{ is odd} \\
 f(v_{\frac{n}{2} + i}) &= 0, & 1 \leq i \leq \frac{n}{2} \text{ if } n \text{ is even} \\
 f(v_n) &= 1, f(v'_1) = 1, \\
 f(v'_{2i}) &= 2, & 1 \leq i \leq k - 1 \\
 f(v'_{2i+1}) &= 0, & 1 \leq i \leq k - 1 \\
 f(v'_i) &= 2, & \text{otherwise}
 \end{aligned}$$

In view of the above labeling pattern we have

$$v_f(0) = v_f(1) = v_f(2) = 2k$$

$$e_f(0) = e_f(1) + 1 = e_f(2) = n + k - 1$$

**Case 4:** When  $n \equiv 1 \pmod{3}$ , ( $n = 3k + 1, k = 1, 2, 3, \dots$ ).

$$f(v_i) = 1, \quad 1 \leq i \leq \lfloor \frac{n}{2} \rfloor + 1 \text{ if } n \text{ is odd and } 1 \leq i \leq \frac{n}{2} \text{ if } n \text{ is even}$$

$$f(v_{\lfloor \frac{n}{2} \rfloor + 1 + i}) = 0, \quad 1 \leq i \leq \lfloor \frac{n}{2} \rfloor \text{ if } n \text{ is odd}$$

$$f(v_{\frac{n}{2} + i}) = 0, \quad 1 \leq i \leq \frac{n}{2} \text{ if } n \text{ is even}$$

$$f(v'_{2i}) = 0, \quad 1 \leq i \leq k$$

$$f(v'_i) = 2, \quad \text{otherwise}$$

In view of the above labeling pattern we have

$$v_f(0) = v_f(1) + 1 = v_f(2) = n - k$$

$$e_f(0) = e_f(1) = e_f(2) = 2(n - k - 1)$$

**Case 5:** When  $n \equiv 2 \pmod{3}$ , ( $n = 3k + 2, k = 1, 2, 3, \dots$ )

$$f(v_i) = 1, \quad 1 \leq i \leq 2k + 1$$

$$f(v_i) = 0, \quad \text{otherwise;}$$

$$f(v'_{2i-1}) = 2, \quad 1 \leq i \leq k$$

$$f(v'_{2i}) = 0, \quad 1 \leq i \leq k$$

$$f(v'_{2k+i}) = 2, \quad 1 \leq i \leq n - 2k$$

In view of the above labeling pattern we have

$$v_f(0) + 1 = v_f(1) + 1 = v_f(2) = n - k$$

$$e_f(0) + 1 = e_f(1) = e_f(2) + 1 = n + k$$

Thus in each cases we have  $|v_f(i) - v_f(j)| \leq 1$  and  $|e_f(i) - e_f(j)| \leq 1$ , for all  $0 \leq i, j \leq 2$ .

Hence  $D_2(P_n)$  is 3 - equitable graph except for  $n = 3$ .  $\square$

**Illustration 3.9.4.** Consider the graph  $D_2(P_7)$ . The 3-equitable labeling is as shown in Figure 3.13.

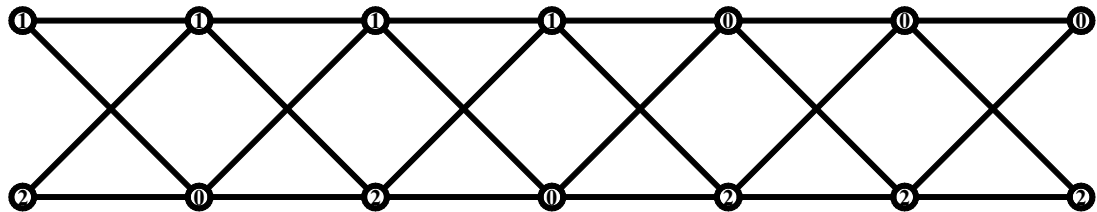


FIGURE 3.13: 3-equitable labeling of  $D_2(P_7)$

**Theorem 3.9.5.**  $M(C_n)$  is 3-equitable for  $n$  even and not 3-equitable for  $n$  odd.

*Proof.* If  $M(C_n)$  be the middle graph of cycle  $C_n$  then let  $v_1, v_2, \dots, v_n$  be the vertices of cycle  $C_n$  and  $v'_1, v'_2, \dots, v'_n$  be the vertices added corresponding to the edges  $e_n, e_1, \dots, e_{n-1}$  in order to obtain  $M(C_n)$ .

Define  $f : V(M(C_n)) \rightarrow \{0, 1, 2\}$ , we consider following two cases.

**Case 1:** When  $n$  is even.

**Sub Case 1:**  $n = 6k, k = 1, 2, \dots$

$$f(v_{3i-2}) = 1, \quad 1 \leq i \leq k$$

$$f(v_{3k-1+i}) = 1, \quad 1 \leq i \leq k$$

$$f(v_i) = 2, \quad \text{otherwise}$$

$$f(v'_{3i-2}) = 1, \quad 1 \leq i \leq k$$

$$f(v'_{3i-1}) = 1, \quad 1 \leq i \leq k$$

$$f(v'_i) = 0, \quad \text{otherwise}$$

In view of the above labeling pattern we have

$$v_f(0) = v_f(1) = v_f(2) = 4k$$

$$e_f(0) = e_f(1) = e_f(2) = 6k$$

**Sub Case 2:**  $n = 6k - 2, k = 2, 3, \dots$

$$f(v_{3i-2}) = 1, \quad 1 \leq i \leq k$$

$$f(v_{3i-1}) = 1, \quad 1 \leq i \leq k-1$$

$$f(v_n) = 1, \quad i = n$$

$$f(v_i) = 0, \quad \text{otherwise}$$

$$f(v'_{3i-2}) = 1, \quad 1 \leq i \leq k$$

$$f(v'_{3i-1}) = 1, \quad 1 \leq i \leq k-1$$

$$f(v'_{3k-1+i}) = 0, \quad 1 \leq i \leq k-1$$

$$f(v'_i) = 2, \quad \text{otherwise}$$

In view of the above labeling pattern we have

$$v_f(0) = v_f(1) = v_f(2) + 1 = 4k - 1$$

$$e_f(0) = e_f(1) = e_f(2) = 6k - 2$$

**Sub Case 3:**  $n = 6k + 2, k = 1, 2, \dots$

$$f(v_{3i-2}) = 1, \quad 1 \leq i \leq k$$

$$f(v_{3k-1+i}) = 1, \quad 1 \leq i \leq k+1$$

$$f(v_i) = 2, \quad \text{otherwise}$$

$$f(v'_{3i-2}) = 1, \quad 1 \leq i \leq k$$

$$f(v'_{3i-1}) = 1, \quad 1 \leq i \leq k$$

$$f(v'_i) = 0, \quad \text{otherwise}$$

In view of the above labeling pattern we have

$$v_f(0) = v_f(1) + 1 = v_f(2) + 1 = 4k + 2$$

$$e_f(0) = e_f(1) = e_f(2) = 6k + 2$$

**Case 2:** When  $n$  is odd.

In this case  $M(C_n)$  is an Eulerian graph with  $|E(M(C_n))| \equiv 3 \pmod{6}$ . So it is not 3-equitable as we stated earlier.

Thus in each cases we have  $|v_f(i) - v_f(j)| \leq 1$  and  $|e_f(i) - e_f(j)| \leq 1$ , for all  $0 \leq i, j \leq 2$ .

Hence  $M(C_n)$  is 3-equitable for  $n$  even and not 3-equitable for  $n$  odd.  $\square$

**Illustration 3.9.6.** Consider the graph  $M(C_{14})$ . The 3-equitable labeling is as shown in Figure 3.14.

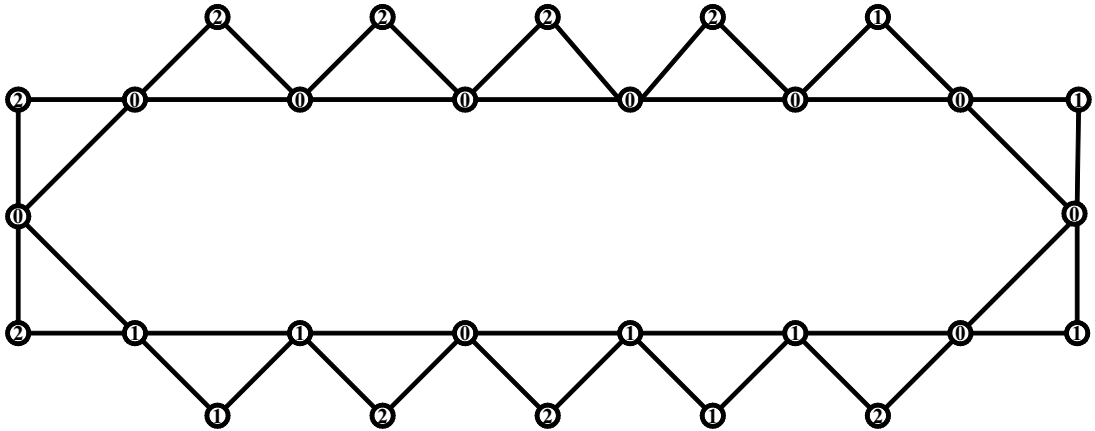


FIGURE 3.14: 3-equitable labeling of  $M(C_{14})$

**Theorem 3.9.7.** The graph  $M(P_n)$  is 3 - equitable.

*Proof.* If  $M(P_n)$  be the middle graph of path  $P_n$  then let  $v_1, v_2, \dots, v_n$  be the vertices of path  $P_n$  and  $v'_1, v'_2, \dots, v'_{n-1}$  be the vertices added corresponding to the edges  $e_1, e_2, \dots, e_{n-1}$  in order to obtain  $M(P_n)$ .

Define  $f : V(M(P_n)) \rightarrow \{0, 1, 2\}$ , we consider following four cases.

**Case 1:** when  $n = 3, 5, 7$  and  $9$  the 3-equitable labeling of the corresponding graphs are given in Figure 3.15.

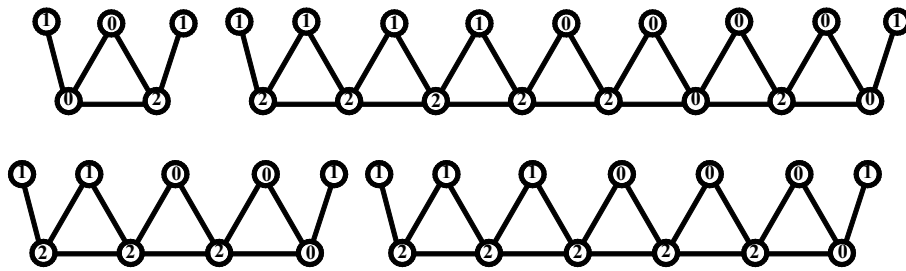


FIGURE 3.15: 3-equitable labeling of  $M(P_3)$ ,  $M(P_5)$ ,  $M(P_7)$  and  $M(P_9)$

**Case 2:** when  $n = 2, 4$  and  $6$  the 3-equitable labeling of the corresponding graphs are given in Figure 3.16.

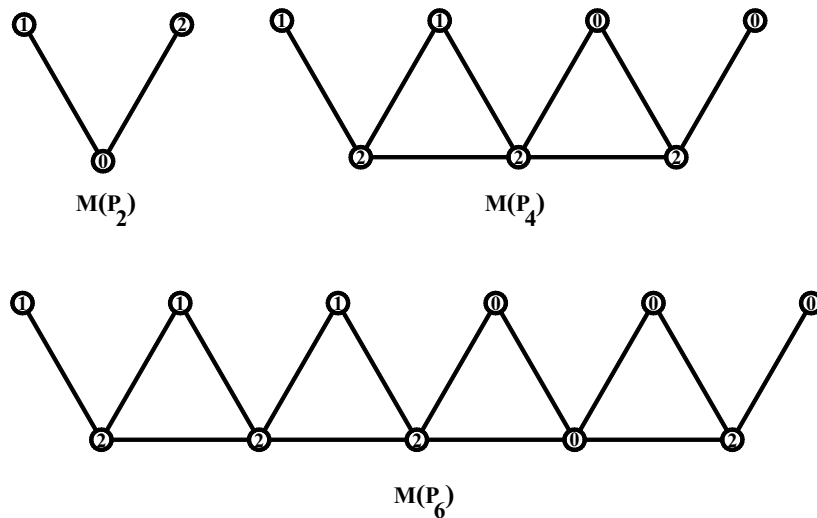


FIGURE 3.16: 3-equitable labeling of  $M(P_2)$ ,  $M(P_4)$ , and  $M(P_6)$

**Case 3:** When  $n$  is even ( $n = 2k, k = 4, 5, 6, \dots$ ).

$$\begin{aligned}
 f(v_i) &= 1, & 1 \leq i \leq \frac{n}{2} \\
 f(v_{\frac{n}{2}+i}) &= 0, & 1 \leq i \leq \frac{n}{2} \\
 f(v'_{2i-1}) &= 1, & 1 \leq i \leq \lfloor \frac{k}{3} \rfloor \text{ and} \\
 f(v'_{2i-1}) &= 1, & 1 \leq i \leq \lfloor \frac{k}{3} \rfloor - 1, \text{ for } k = 3j, j = 2, 3, \dots \\
 f(v'_{n-2i}) &= 0, & 1 \leq i \leq \lfloor \frac{k}{3} \rfloor \\
 f(v'_i) &= 2, & \text{otherwise;}
 \end{aligned}$$

In view of the above labeling pattern we have

$$\begin{aligned}
 v_f(0) &= v_f(1) = v_f(2) = \frac{2n-1}{3} \\
 e_f(0) &= e_f(1) + 1 = e_f(2) = n - 1, \text{ when } n \equiv 2(\text{mod } 3)
 \end{aligned}$$

$$\begin{aligned}
 v_f(0) + 1 &= v_f(1) + 1 = v_f(2) = \lfloor \frac{2n-1}{3} \rfloor + 1 \\
 e_f(0) &= e_f(1) + 1 = e_f(2) = n - 1, \text{ when } n \equiv 1(\text{mod } 3)
 \end{aligned}$$

$$\begin{aligned}
 v_f(0) &= v_f(1) + 1 = v_f(2) = \lfloor \frac{2n-1}{3} \rfloor + 1 \\
 e_f(0) &= e_f(1) + 1 = e_f(2) = n - 1, \text{ when } n \equiv 0(\text{mod } 3)
 \end{aligned}$$

**Case 4:** When  $n$  is odd ( $n = 2k + 1, k = 5, 6, \dots$ ).

$$\begin{aligned}
 f(v_i) &= 1, & 1 \leq i \leq \lfloor \frac{n}{2} \rfloor, i = n \\
 f(v_{\lfloor \frac{n}{2} \rfloor + i}) &= 0, & 1 \leq i \leq \lfloor \frac{n}{2} \rfloor \\
 f(v'_{2i-1}) &= 1, & 1 \leq i \leq \lfloor \frac{k}{3} \rfloor - 1 \text{ and} \\
 f(v'_{2i-1}) &= 1, & 1 \leq i \leq \lfloor \frac{k}{3} \rfloor, \text{ for } k = 3j + 2, j = 1, 2, 3, \dots \\
 f(v'_{n-2i+1}) &= 0, & 1 \leq i \leq \lfloor \frac{k}{3} \rfloor + 1 \text{ and} \\
 f(v'_{n-2i+1}) &= 0, & 1 \leq i \leq \lfloor \frac{k}{3} \rfloor \text{ for } k = 3j, j = 2, 3, 4, \dots \\
 f(v'_i) &= 2, & \text{otherwise}
 \end{aligned}$$

In view of the above labeling pattern we have

$$\begin{aligned}
 v_f(0) &= v_f(1) = v_f(2) = \frac{2n-1}{3} \\
 e_f(0) &= e_f(1) + 1 = e_f(2) = n - 1, \text{ when } n \equiv 2(\text{mod } 3)
 \end{aligned}$$

$$\begin{aligned}
 v_f(0) + 1 &= v_f(1) + 1 = v_f(2) = \lfloor \frac{2n-1}{3} \rfloor + 1 \\
 e_f(0) &= e_f(1) + 1 = e_f(2) = n - 1, \text{ when } n \equiv 1(\text{mod } 3)
 \end{aligned}$$

$$v_f(0) + 1 = v_f(1) = v_f(2) = \left\lfloor \frac{2n-1}{3} \right\rfloor + 1$$

$$e_f(0) = e_f(1) = e_f(2) + 1 = n - 1, \text{ when } n \equiv 0 \pmod{3}$$

Thus in each cases we have  $|v_f(i) - v_f(j)| \leq 1$  and  $|e_f(i) - e_f(j)| \leq 1$ , for all  $0 \leq i, j \leq 2$ .

Hence  $M(P_n)$  is 3 - equitable graph.  $\square$

**Illustration 3.9.8.** Consider the graph  $M(P_{10})$ . The 3-equitable labeling is as shown in Figure 3.17.

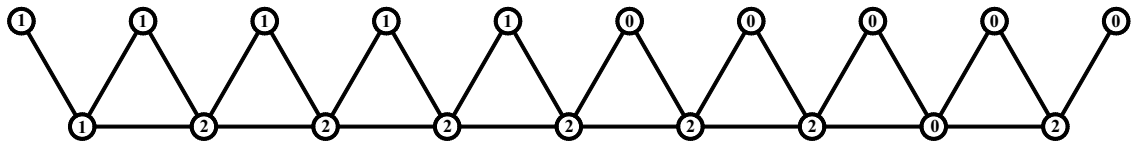


FIGURE 3.17: 3-equitable labeling of  $M(P_{10})$

### 3.10 Embedding and NP-complete problems for 3-equitable graphs

The embedding and NP-complete problems in the context of cordial labeling is discussed briefly in section 3.7 while this section is aimed to discuss such problems for 3-equitable graphs.

**Theorem 3.10.1.** Any graph  $G$  can be embedded as an induced subgraph of a 3-equitable graph.

*Proof.* Let  $G$  be the graph with  $n$  vertices. Without loss of generality we assume that it is always possible to label the vertices of any graph  $G$  such that the vertex conditions for 3-equitable graphs are satisfied. i.e.  $|v_f(i) - v_f(j)| \leq 1$ ,  $0 \leq i, j \leq 2$ . Let  $V_0, V_1$  and  $V_2$  be the set of vertices with label 0, 1 and 2 respectively. Let  $E_0, E_1$  and  $E_2$  be the set of edges with label 0, 1 and 2 respectively. Let  $n(V_0)$ ,  $n(V_1)$  and  $n(V_2)$  be the number of elements in sets  $V_0, V_1$  and  $V_2$  respectively. Let  $n(E_0)$ ,  $n(E_1)$  and  $n(E_2)$  be the number

of elements in sets  $E_0$ ,  $E_1$  and  $E_2$  respectively.

**Case 1:**  $n \equiv 0 \pmod{3}$

**Subcase 1:**  $n(E_0) \neq n(E_1) \neq n(E_2)$ .

Suppose  $n(E_0) < n(E_1) < n(E_2)$ . Let  $|n(E_2) - n(E_0)| = r > 1$  and  $|n(E_2) - n(E_1)| = s > 1$ . The new graph  $H$  can be obtained by adding  $r + s$  vertices to the graph  $G$ .

Define  $r + s = p$  and consider a partition of  $p$  as  $p = a + b + c$  with  $|a - b| \leq 1$ ,  $|b - c| \leq 1$  and  $|c - a| \leq 1$ .

Now out of new  $p$  vertices label  $a$  vertices with 0,  $b$  vertices with 1 and  $c$  vertices with 2. i.e. label the vertices  $u_1, u_2, \dots, u_a$  with 0,  $v_1, v_2, \dots, v_b$  with 1 and  $w_1, w_2, \dots, w_c$  with 2.

Now we adapt the following procedure.

**Step 1:** To obtain required number of edges with label 1.

- Join  $s$  number of elements  $v_i$  to the arbitrary element of  $V_0$ .
- If  $b < s$  then join  $(s - b)$  number of elements  $u_1, u_2, \dots, u_{s-b}$  to the arbitrary element of  $V_1$ .
- If  $a < s - b$  then join  $(s - a - b)$  number of vertices  $w_1, w_2, \dots, w_{s-b-a}$  to the arbitrary element of  $V_1$ .

Above construction will give rise to required number of edges with label 1.

**Step 2:** To obtain required number of edges with label 0.

- Join remaining number of  $u_i$ 's (which are left at the end of step 1) to the arbitrary element of  $V_0$ .
- Join the remaining number of  $v_i$ 's (which are left at the end of step 1) to the arbitrary element of  $V_1$ .
- Join the remaining number of  $w_i$ 's (which are left at the end of step 1) to the arbitrary element of  $V_2$ .

As a result of above procedure we have following vertex conditions and edge conditions.

$$|v_f(0) - v_f(1)| = |n(V_0) + a - n(V_1) - b| \leq 1,$$

$$|v_f(1) - v_f(2)| = |n(V_1) + b - n(V_2) - c| \leq 1,$$

$$|v_f(2) - v_f(0)| = |n(V_2) + c - n(V_0) - a| \leq 1$$

and

$$|e_f(0) - e_f(1)| = |n(E_0) + n(E_2) - n(E_0) - n(E_1) - n(E_2) + n(E_1)| = 0,$$

$$|e_f(1) - e_f(2)| = |n(E_1) + n(E_2) - n(E_1) - n(E_2)| = 0,$$

$$|e_f(2) - e_f(0)| = |n(E_2) - n(E_0) - n(E_2) + n(E_0)| = 0.$$

Similarly one can handle the following cases.

$$n(E_0) < n(E_2) < n(E_1),$$

$$n(E_2) < n(E_0) < n(E_1),$$

$$n(E_1) < n(E_2) < n(E_0),$$

$$n(E_2) < n(E_1) < n(E_0),$$

$$n(E_1) < n(E_0) < n(E_2).$$

**Subcase 2:**  $n(E_i) = n(E_j) < n(E_k), i \neq j \neq k, 0 \leq i, j, k \leq 2$

Suppose  $n(E_0) = n(E_1) < n(E_2)$

$$|n(E_2) - n(E_0)| = r$$

$$|n(E_2) - n(E_1)| = r$$

The new graph  $H$  can be obtained by adding  $2r$  vertices to the graph  $G$ .

Define  $2r = p$  and consider a partition of  $p$  as  $p = a + b + c$  with  $|a - b| \leq 1, |b - c| \leq 1$  and  $|c - a| \leq 1$ .

Now out of new  $p$  vertices, label  $a$  vertices with 0,  $b$  vertices with 1 and  $c$  vertices with 2. i.e. label the vertices  $u_1, u_2, \dots, u_a$  with 0,  $v_1, v_2, \dots, v_b$  with 1 and  $w_1, w_2, \dots, w_c$  with 2.

Now we adapt the following procedure.

**Step 1:** To obtain required number of edges with label 0.

- Join  $r$  number of elements  $u_i$ 's to the arbitrary element of  $V_0$ .
- If  $a < r$  then join  $(r - a)$  number of elements  $v_1, v_2, \dots, v_{r-a}$  to the arbitrary element of  $V_1$ .
- If  $b < r - a$  then join  $(r - a - b)$  number of vertices  $w_1, w_2, \dots, w_{r-a-b}$  to the arbitrary element of  $V_2$ .

Above construction will give rise to required number of edges with label 0.

**Step 2:** To obtain required number of edges with label 1.

- Join remaining number of  $w_i$ 's (which are not used at the end of step 1) to the arbitrary element of  $V_1$ .
- Join the remaining number of  $v_i$ 's (which are not used at the end of step 1) to the arbitrary element of  $V_0$ .
- Join the remaining number of  $u_i$ 's (which are not used at the end of step 1) to the arbitrary element of  $V_1$ .

Similarly we can handle the following possibilities.

$$n(E_1) = n(E_2) < n(E_0)$$

$$n(E_0) = n(E_2) < n(E_1)$$

**Subcase 3 :**  $n(E_i) < n(E_j) = n(E_k), i \neq j \neq k, 0 \leq i, j, k \leq 2$

Suppose  $n(E_2) < n(E_0) = n(E_1)$

Define  $|n(E_2) - n(E_0)| = r$

The new graph  $H$  can be obtained by adding  $r$  vertices to the graph  $G$  as follows .

Consider a partition of  $r$  as  $r = a + b + c$  with  $|a - b| \leq 1, |b - c| \leq 1$  and  $|c - a| \leq 1$ .

Now out of new  $r$  vertices label  $a$  vertices with 0,  $b$  vertices with 1 and  $c$  vertices with 2. i.e. label the vertices  $u_1, u_2, \dots, u_a$  with 0,  $v_1, v_2, \dots, v_b$  with 1 and  $w_1, w_2, \dots, w_c$  with 2.

Now we adapt the following procedure.

**Step 1:** To obtain required number of edges with label 2.

- Join  $r$  number of vertices  $w_i$ 's to the arbitrary element of  $V_0$ .
- If  $c < r$  then join  $r - c$  number of elements  $u_1, u_2, \dots, u_{r-c}$  to the arbitrary element of  $V_2$ .

Above construction will give rise to required number of edges with label 2.

At the end of this step if the required number of 2 as edge labels are generated then we have done. If not then move to step 2. This procedure should be followed in all the situations described earlier when  $n(E_2) < n(E_0)$  or  $n(E_2) < n(E_1)$ .

**Step 2:** To obtain the remaining (at the end of step 1) number of edges with label 2.

- If  $k$  number of edges are required after joining all the vertices with label 0 and 2 then add  $k$  number of vertices labeled with 0,  $k$  number of vertices labeled with 1 and  $k$  number of vertices labeled with 2. Then vertex conditions are satisfied.
- Now we have  $k$  number of new vertices with label 2,  $k$  number of new vertices with label 0 and  $2k$  number of new vertices with label 1.
- Join  $k$  new vertices with label 2 to the arbitrary element of the set  $V_0$ .
- Join  $k$  new vertices with label 0 to the arbitrary element of the set  $V_2$ .
- Join  $k$  new vertices with label 1 to the arbitrary element of set  $V_0$ .
- Join  $k$  new vertices with label 1 to the arbitrary element of the set  $V_1$ .

**Case 2:**  $n \equiv 1 \pmod{3}$ .

**Subcase 1:**  $n(E_i) \neq n(E_j) \neq n(E_k), i \neq j \neq k, 0 \leq i, j, k \leq 2$ .

Suppose  $n(E_0) < n(E_1) < n(E_2)$  Let  $|n(E_2) - n(E_0)| = r > 1$  and  $|n(E_2) - n(E_1)| = s > 1$ .

Define  $r + s = p$  and consider a partition of  $p$  such that  $p = a + b + c$  with

$$|n(V_0) + a - n(V_1) - b| \leq 1$$

$$|n(V_1) + b - n(V_2) - c| \leq 1$$

$$|n(V_0) + a - n(V_2) - c| \leq 1.$$

Now we can follow the procedure which we have discussed in case 1.

**Case 3:**  $n \equiv 2 \pmod{3}$

We can proceed as case 1 and case 2.

Thus in all the possibilities the graph  $H$  resulted due to above construction satisfies the conditions for 3-equitable graph. That is, any graph  $G$  can be embedded as an induced subgraph of a 3-equitable graph.  $\square$

For the better understanding of result derived above consider following illustrations.

**Illustration 3.10.2.** For a graph  $G = C_9$  we have  $n(E_0) = 0$ ,  $n(E_1) = 6$ ,  $n(E_2) = 3$ .

Now  $|n(E_1) - n(E_0)| = 6 = r$ ,  $|n(E_1) - n(E_2)| = 3 = s$ .

This is the case related to subcase(1) of case(1).

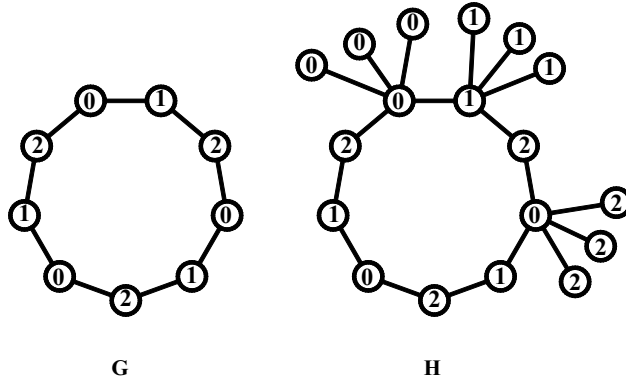


FIGURE 3.18:  $C_9$  and its 3-equitable embedding

**Procedure to construct  $H$  :**

**Step 1:**

- Add  $p = r + s = 6 + 3 = 9$  vertices in  $G$  and partition  $p$  as  $p = a + b + c = 3 + 3 + 3$ .
- Label 3 vertices with 0 as  $a = 3$ .
- Label 3 vertices with 1 as  $b = 3$ .
- Label 3 vertices with 2 as  $c = 3$ .

**Step 2:**

- Join the vertices with 0 and 1 to the arbitrary element of the set  $V_0$  and  $V_1$  respectively.
- Join the vertices with label 2 to the arbitrary element of set  $V_0$ .

The resultant graph  $H$  is shown in Figure 3.18 is 3-equitable.

**Illustration 3.10.3.** Consider a graph  $G = K_4$  as shown in following Figure 3.19 for which  $n(E_0) = 1, n(E_1) = 4, n(E_2) = 1$ .

Here  $|n(E_1) - n(E_0)| = 3 = r, |n(E_1) - n(E_2)| = 3 = s$  i.e.  $r = s$ .

This is the case related to subcase(2) of case(2).

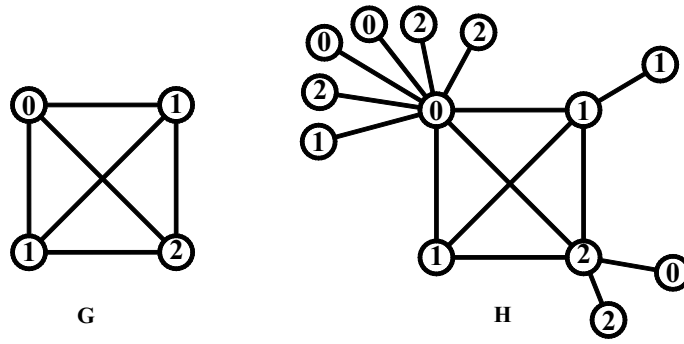


FIGURE 3.19:  $K_4$  and its 3-equitable embedding

**Procedure to construct  $H$  :**

**Step 1:**

- Add  $p = 2r = 3 + 3 = 6$  vertices in  $G$  and partition  $p$  as  $p = a + b + c = 2 + 1 + 3$ .
- Label 2 vertices with 0 as  $a = 2$ .
- Label 1 vertex with 1 as  $b = 1$ .
- Label 3 vertices with 2 as  $c = 2$ .

**Step 2:**

- Join the vertices with label 0 to the arbitrary element of the set  $V_0$  and join one vertex with label 2 to the arbitrary element of  $V_2$ .
- Join the remaining vertices with label 2 with the arbitrary element of set  $V_0$ .

**Step 3:**

- Now add three more vertices and label them as 0,1 and 2 respectively.
- Now join the vertices with label 0 and 2 with the arbitrary elements of  $V_2$  and  $V_0$  respectively.
- Now out of the remaining two vertices with label 1 join one vertex with arbitrary element of set  $V_0$  and the other with the arbitrary element of set  $V_1$ .

The resultant graph  $H$  shown in *Figure 3.19* is 3-equitable.

**Corollary 3.10.4.** Any planar graph  $G$  can be embedded as an induced subgraph of a planar 3-equitable graph.

*Proof.* If  $G$  is planar graph. Then the graph  $H$  obtained by Theorem 3.10.1 is a planar graph. □

**Corollary 3.10.5.** Any triangle free graph  $G$  can be embedded as an induced subgraph of a triangle free 3-equitable graph.

*Proof.* If  $G$  is triangle free graph. Then the graph  $H$  obtained by Theorem 3.10.1 is a triangle free graph. □

**Corollary 3.10.6.** The problem of deciding whether the chromatic number  $\chi \leq k$ , where  $k \geq 3$  is NP-complete even for 3-equitable graphs.

*Proof.* Let  $G$  be a graph with chromatic number  $\chi(G) \geq 3$ . Let  $H$  be the 3-equitable graph constructed in Theorem 3.10.1, which contains  $G$  as an induced subgraph. Since  $H$  is constructed by adding only pendant vertices to  $G$ . We have  $\chi(H) = \chi(G)$ . Since the problem of deciding whether the chromatic number  $\chi \leq k$ , where  $k \geq 3$  is NP-complete [27]. It follows that deciding whether the chromatic number  $\chi \leq k$ , where  $k \geq 3$ , is NP-complete even for 3-equitable graphs. □

**Corollary 3.10.7.** The problem of deciding whether the clique number  $\omega(G) \geq k$  is NP-complete even when restricted to 3-equitable graphs.

*Proof.* Since the problem of deciding whether the clique number of a graph  $\omega(G) \geq k$  is NP-complete [27] and  $\omega(H) = \omega(G)$  for the 3-equitable graph  $H$  constructed in Theorem 3.10.1, the above result follows.  $\square$

### 3.11 Concluding Remarks and Scope of Further Research

We have contributed four new results for cordial labeling and 3-equitable labeling each. Some new families of cordial and 3-equitable graphs are also obtained. We have also discussed embedding and NP-complete problems in the context of both the labelings. To investigate some more cordial and 3-equitable graphs which remains invariant under various graph operations is a potential area of research.

The next chapter is intended to discuss the total product cordial and the prime cordial labelings of graphs.

## **Chapter 4**

### **Total Product Cordial Labeling**

**&**

### **Prime Cordial Labeling**

## 4.1 Introduction

The previous chapter was focused on cordial and 3-equitable labeling of graphs while the present chapter is aimed to discuss two labelings which are having cordial theme.

## 4.2 Total Product cordial labeling

### 4.2.1 Total product cordial graph

A *total product cordial labeling* of a graph  $G$  is a function

$f : (V(G) \cup E(G)) \longrightarrow \{0, 1\}$  such that  $f(xy) = f(x)f(y)$  where  $x, y \in V(G)$ ,  $xy \in E(G)$  and the total number of 0 and 1 are balanced. That is, if  $v_f(i)$  and  $e_f(i)$  denote the set of vertices and edges which are labeled as  $i$  for  $i = 0, 1$  respectively, then  $|(v_f(0) + e_f(0)) - (v_f(1) + e_f(1))| \leq 1$ . If there exists a total product cordial labeling of a graph  $G$  then it is called a *total product cordial graph*.

### 4.2.2 Some existing results

Sundaram, Ponraj and Somasundaram in [66, 67] have shown that the following graphs are total product cordial.

- Every product cordial graph of even order or odd order and even size.
- Trees.
- All cycles except  $C_4$ .
- The graph  $K_{n, 2n-1}$ .
- $C_n$  with  $m$  edges appended at each vertex.
- fans; double fans; wheels; helms.
- The graph  $C_2 \times P_2$ .

- $P_m \times P_n$  if and only if  $(m, n) \neq (2, 2)$ .
- $K_{2,n}$  if and only if  $n \equiv 2 \pmod{4}$ .
- $C_n + 2K_1$  if and only if  $n$  is even or  $n \equiv 1 \pmod{3}$ .
- $\overline{K_n} \times 2K_2$  if  $n$  is odd, or  $n \equiv 0$  or  $2 \pmod{6}$ ;  $n \equiv 2 \pmod{8}$ .

### 4.3 Total product cordial graphs induced by some graph operations on cycle related graphs

**Theorem 4.3.1.**  $T(C_n)$  is a total product cordial graph.

*Proof.* Let  $v_1, v_2, \dots, v_n$  be the vertices of cycle  $C_n$  and  $e'_1, e'_2, \dots, e'_n$  be the vertices in  $T(C_n)$  corresponding to the edges  $e_1, e_2, \dots, e_n$  in  $C_n$ .

To define  $f : V(T(C_n)) \cup E(T(C_n)) \rightarrow \{0, 1\}$ , we consider following two cases.

**Case 1:**  $n \geq 6$  is even

We label the vertices as follows.

$$f(e'_i) = 0, \quad 1 \leq i \leq \frac{n}{2}$$

$$f(e'_i) = 1, \quad \frac{n}{2} + 1 \leq i \leq n$$

$$f(v_i) = 0, \quad 1 \leq i \leq \frac{n}{2} - 1$$

$$f(v_i) = 1, \quad \frac{n}{2} \leq i \leq n$$

**Case 2:**  $n$  is odd

We label the vertices as follows.

$$f(e'_i) = 1, \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor + 1$$

$$f(e'_i) = 0, \quad \left\lfloor \frac{n}{2} \right\rfloor + 2 \leq i \leq n$$

$$f(v_i) = 1, \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor + 1$$

$$f(v_i) = 0, \quad \left\lfloor \frac{n}{2} \right\rfloor + 2 \leq i \leq n$$

In view of the labeling pattern defined above we have

$$v_f(0) + e_f(0) = v_f(1) + e_f(1) = 3n$$

Thus we conclude that the graph  $f$  satisfies the condition

$$|(v_f(0) + e_f(0)) - (v_f(1) + e_f(1))| \leq 1.$$

That is,  $T(C_n)$  is a total product cordial graph.  $\square$

**Illustration 4.3.2.** Consider a graph  $T(C_6)$ . The corresponding total product cordial labeling is as shown in Figure 4.1.

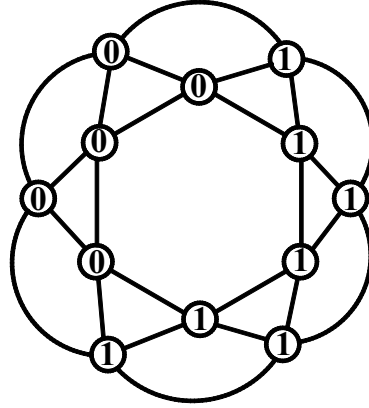


FIGURE 4.1:  $T(C_6)$  and its total product cordial labeling

**Theorem 4.3.3.** The star of cycle  $C_n$  admits total product cordial labeling.

*Proof.* Let  $v_1, v_2, \dots, v_n$  be the vertices of central cycle  $C_n$  and  $v_{ij}$  be the vertices of cycle  $C_n^j$ , where  $1 \leq j \leq n$ , which are adjacent to the  $i^{th}$  vertex of central cycle  $C_n$  to obtain  $C_n'$ .

To define  $f : (V(C_n') \cup E(C_n')) \longrightarrow \{0, 1\}$ , we consider following two cases.

**Case 1:**  $n$  is even

We label the vertices as

$$f(v_i) = 1, \quad 1 \leq i \leq n$$

$$f(v_{1j}) = 1, \quad 1 \leq j \leq \frac{n}{2}$$

$$f(v_{1j}) = 0, \quad \frac{n}{2} + 1 \leq j \leq n$$

$$f(v_{ij}) = 0, \quad 1 \leq j \leq \frac{n}{2}, 2 \leq i \leq n$$

$$f(v_{ij}) = 1, \quad \frac{n}{2} + 1 \leq j \leq n, 2 \leq i \leq n$$

In view of the labeling pattern defined above we have

$$v_f(0) + e_f(0) = v_f(1) + e_f(1) = n^2 + \frac{3n}{2}$$

**Case 2:**  $n$  is odd

We label the vertices as

$$f(v_i) = 1 \quad 1 \leq i \leq n$$

$$f(v_{ij}) = 0, \quad 1 \leq j \leq \lfloor \frac{n}{2} \rfloor + 1, 1 \leq i \leq n$$

$$f(v_{ij}) = 1, \quad \lfloor \frac{n}{2} \rfloor + 2 \leq j \leq n, 1 \leq i \leq n$$

In view of the labeling pattern defined above we have

$$v_f(0) + e_f(0) = v_f(1) + e_f(1) + 1 = 2n \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{2} \rfloor + 2n + 1$$

$$\text{Thus } |(v_f(0) + e_f(0)) - (v_f(1) + e_f(1))| \leq 1.$$

Hence  $f$  is a total product cordial labeling for the graph star of cycle.

□

**Illustration 4.3.4.** Consider the graph star of cycle  $C_7$ . The corresponding total product cordial labeling is shown in Figure 4.2.

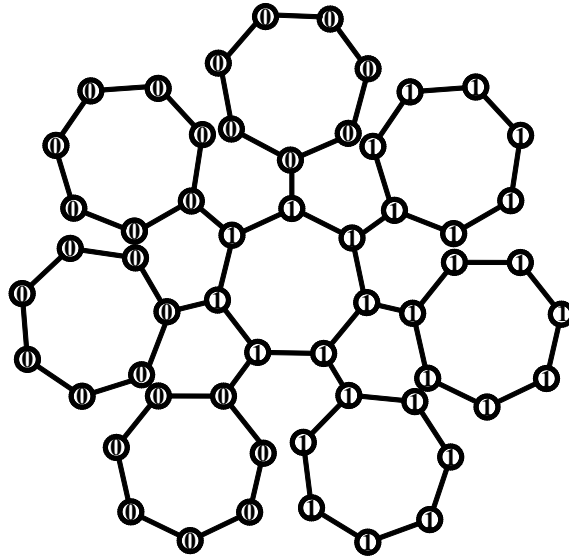


FIGURE 4.2: Total product cordial labeling for star of cycle  $C_7$

**Theorem 4.3.5.**  $C_n^{(k)}$  admits total product cordial labeling.

*Proof.* Let  $v_{ij}$  be the  $i^{th}$  vertex of  $j^{th}$  copy of cycle  $C_j$ . Let  $v_1$  be the common vertex of all the cycles. Without loss of generality we start the label assignment from  $v_1$ .

To define  $f : V(C_n^{(k)}) \cup E(C_n^{(k)}) \longrightarrow \{0, 1\}$  we consider following three cases.

**Case 1:**  $n \in N(n \geq 3)$  and  $k$  is even

$$f(v_1) = 1$$

$$f(v_{ij}) = 1, \quad 1 \leq j \leq \frac{k}{2}, 2 \leq i \leq n$$

$$f(v_{ij}) = 0, \quad \frac{k}{2} + 1 \leq j \leq k, 2 \leq i \leq n$$

In view of the labeling pattern defined above we have

$$v_f(0) + e_f(0) + 1 = v_f(1) + e_f(1) = 2n$$

**Case 2:**  $n > 3$  is even,  $k$  is odd

$$f(v_1) = 1$$

$$f(v_{ij}) = 1, \quad 1 \leq j \leq \lfloor \frac{k}{2} \rfloor, 2 \leq i \leq n$$

$$f(v_{ij}) = 0, \quad \lfloor \frac{k}{2} \rfloor + 1 \leq j \leq k - 2, 2 \leq i \leq n$$

$$f(v_{ij}) = 1, \quad j = k - 1, 2 \leq i \leq \frac{n}{2}$$

$$f(v_{ij}) = 0, \quad j = k - 1, \frac{n}{2} + 1 \leq i \leq n$$

$$f(v_{3j}) = 1, \quad j = k$$

$$f(v_{ij}) = 0 \quad j = k, 2 \leq i \leq n, i \neq 3$$

In view of the labeling pattern defined above we have

$$v_f(0) + e_f(0) = v_f(1) + e_f(1) = 2n \lfloor \frac{k}{2} \rfloor - \lfloor \frac{k}{2} \rfloor + n$$

**Case 3:**  $n \geq 3$  is odd,  $k$  is odd

$$f(v_1) = 1$$

$$f(v_{ij}) = 1, \quad 1 \leq j \leq \lfloor \frac{k}{2} \rfloor, 2 \leq i \leq n$$

$$f(v_{ij}) = 0, \quad \lfloor \frac{k}{2} \rfloor + 1 \leq j \leq k - 1, 2 \leq i \leq n$$

$$f(v_{ij}) = 1, \quad j = k, 2 \leq i \leq \lfloor \frac{n}{2} \rfloor + 1$$

$$f(v_{ij}) = 0, \quad j = k, \lfloor \frac{n}{2} \rfloor + 2 \leq i \leq n$$

In view of the labeling pattern defined above we have

$$v_f(0) + e_f(0) + 1 = v_f(1) + e_f(1) = 2n \lfloor \frac{k}{2} \rfloor - \lfloor \frac{k}{2} \rfloor + \lfloor \frac{n}{2} \rfloor + \frac{n+1}{2}$$

Thus in all the four cases  $|(v_f(0) + e_f(0)) - (v_f(1) + e_f(1))| \leq 1$ .

Hence  $C_n^{(k)}$  admits total product cordial labeling. □

**Illustration 4.3.6.** Consider a graph  $C_6^{(3)}$ . The corresponding total product cordial labeling is as shown in Figure 4.3.

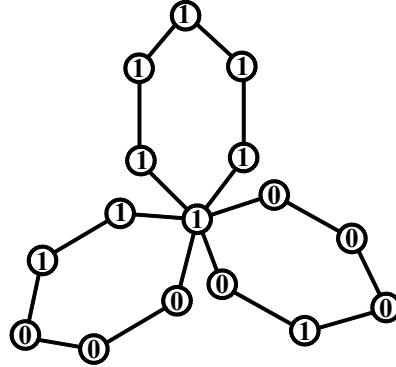


FIGURE 4.3:  $C_6^{(3)}$  and its total product cordial labeling

**Corollary 4.3.7.** Friendship graph  $F_n$  admits total product cordial labeling.

*Proof.* The proof is obvious from the case 1 and 3 of the above Theorem 4.3.5. □

**Theorem 4.3.8.**  $M(C_n)$  is a total product cordial graph.

*Proof.* Let  $v_1, v_2, \dots, v_n$  be the vertices of cycle  $C_n$  and  $e'_1, e'_2, \dots, e'_n$  be the vertices in  $M(C_n)$  corresponding to the edges  $e_1, e_2, \dots, e_n$  of  $C_n$ .

To define  $f : V(M(C_n)) \cup E(M(C_n)) \longrightarrow \{0, 1\}$ , we consider following two cases.

**Case 1:**  $n$  is even

$$f(v_i) = 1, \quad 1 \leq i \leq n$$

$$f(e'_{2i-1}) = 0, \quad 1 \leq i \leq \frac{n}{2}$$

$$f(e'_{2i}) = 1, \quad 1 \leq i \leq \frac{n}{2}$$

In view of the labeling pattern defined above we have

$$v_f(0) + e_f(0) = v_f(1) + e_f(1) = \frac{5n}{2}$$

**Case 2:**  $n$  is odd

$$f(v_i) = 1, \quad 1 \leq i \leq n-1$$

$$f(v_i) = 0, \quad i = n$$

$$f(e'_{2i-1}) = 1, \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor + 1$$

$$f(e'_{2i}) = 0, \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$$

In view of the labeling pattern defined above we have

$$v_f(0) + e_f(0) + 1 = v_f(1) + e_f(1) = 2n + \lfloor \frac{n}{2} \rfloor + 1$$

In view of the above defined pattern  $f$  satisfies the condition

$$|(v_f(0) + e_f(0)) - (v_f(1) + e_f(1))| \leq 1.$$

Hence  $M(C_n)$  admits total product cordial labeling.  $\square$

**Illustration 4.3.9.** Consider a graph  $M(C_9)$ . The total product cordial labeling is as shown in *Figure 4.4*.

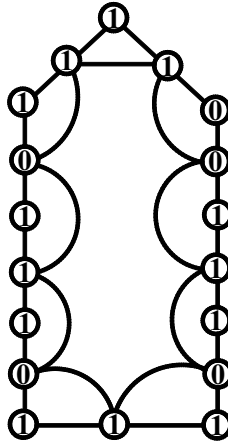


FIGURE 4.4: Total product cordial labeling for  $M(C_9)$

**Theorem 4.3.10.** The graph obtained by switching of an arbitrary vertex in cycle  $C_n$  admits total product cordial labeling.

*Proof.* Let  $v_1, v_2, \dots, v_n$  be the successive vertices of  $C_n$  and  $G_v$  denotes the graph obtained by switching of vertex  $v$  of  $G$ . Without loss of generality let the switched vertex be  $v_1$  and we initiate the labeling from this switched vertex  $v_1$ .

To define  $f : (V(G_{v_1}) \cup E(G_{v_1})) \longrightarrow \{0, 1\}$  we consider following four cases.

**Case 1:** The graph obtained by vertex switching in cycle  $C_4$  is an acyclic graph and its total product cordial labeling is given in following *Figure 4.5*.

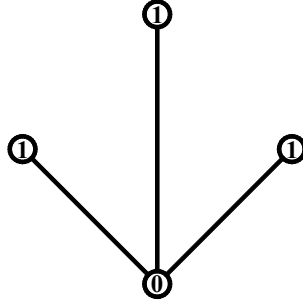


FIGURE 4.5: Total product cordial labeling for the graph obtained by switching of a vertex in  $C_4$

**Case 2:**  $n$  is even,  $n = 2k$ ,  $k = 3, 5, 7, 9, \dots$

$$f(v_1) = 0$$

$$f(v_i) = 1, \quad 2 \leq i \leq \frac{n}{2} + 1$$

$$f(v_{\frac{n}{2}+1+i}) = 0, \quad 1 \leq i \leq \lfloor \frac{n}{4} \rfloor$$

$$f(v_{\frac{n}{2}+\lfloor \frac{n}{4} \rfloor+1+i}) = 1, \quad 1 \leq i \leq \lfloor \frac{n}{4} \rfloor$$

In view of the labeling pattern defined above we have

$$v_f(0) + e_f(0) = v_f(1) + e_f(1) + 1 = n + 2 \lfloor \frac{n}{4} \rfloor - 1$$

**Case 3:**  $n$  is even,  $n = 2k$ ,  $k = 4, 6, 8, \dots$

$$f(v_1) = 0$$

$$f(v_i) = 1, \quad 2 \leq i \leq \frac{n}{2} + 1$$

$$f(v_{\frac{n}{2}+1+i}) = 0, \quad 1 \leq i \leq \frac{n}{4} - 1$$

$$f(v_{\frac{n}{2}+\frac{n}{4}-1+i}) = 1, \quad 1 \leq i \leq \frac{n}{4}$$

In view of the labeling pattern defined above we have

$$v_f(0) + e_f(0) + 1 = v_f(1) + e_f(1) = \frac{3n}{2} - 2$$

**Case 4:**  $n$  is odd

$$f(v_1) = 1$$

$$f(v_i) = 0, \quad 2 \leq i \leq \lfloor \frac{n}{2} \rfloor + 1$$

$$f(v_{\lfloor \frac{n}{2} \rfloor+1+i}) = 1, \quad 1 \leq i \leq \lfloor \frac{n}{2} \rfloor$$

In view of the labeling pattern defined above we have

$$v_f(0) + e_f(0) = v_f(1) + e_f(1) = 3 \lfloor \frac{n}{2} \rfloor - 1$$

Thus  $f$  satisfies the condition  $|(v_f(0) + e_f(0)) - (v_f(1) + e_f(1))| \leq 1$ .

Hence  $G_{v_1}$  admits total product cordial labeling. □

**Illustration 4.3.11.** Consider the graph obtained by switching of a vertex in cycle  $C_{10}$ . The total product cordial labeling is as shown in *Figure 4.6*.

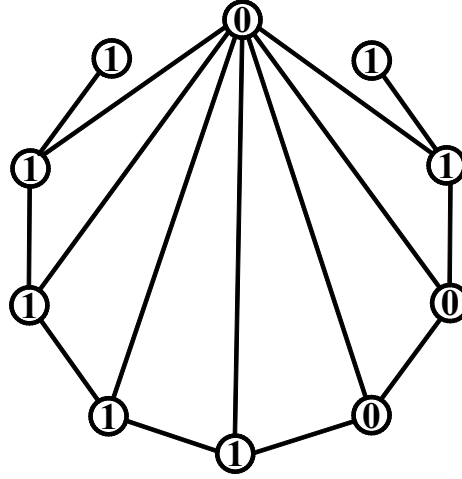


FIGURE 4.6: Vertex switching in  $C_{10}$  and its total product cordial labeling

## 4.4 Total Product cordial labeling for split graph of some graphs

**Theorem 4.4.1.**  $spl(C_n)$  is total product cordial graph.

*Proof.* Let  $v'_1, v'_2, \dots, v'_n$  be the added vertices corresponding to  $v_1, v_2, \dots, v_n$  of cycle  $C_n$ .

To define  $f : V(spl(C_n)) \cup E(spl(C_n)) \longrightarrow \{0, 1\}$ , we consider following two cases.

**Case 1:**  $n$  is even

We label the vertices as follows.

$$f(v_{2i-1}) = 1, \quad 1 \leq i \leq \frac{n}{2}$$

$$f(v_{2i}) = 0, \quad 1 \leq i \leq \frac{n}{2}$$

$$f(v'_i) = 1, \quad 1 \leq i \leq n$$

Using above pattern we have

$$v_f(0) + e_f(0) = v_f(1) + e_f(1) = \frac{5n}{2}$$

**Case 2:**  $n$  is odd

We label the vertices as follows.

$$f(v_{2i-1}) = 0, \quad 1 \leq i \leq \frac{n-1}{2}$$

$$f(v_{2i}) = 1, \quad 1 \leq i \leq \frac{n-1}{2}$$

$$f(v_n) = 1,$$

$$f(v'_i) = 1, \quad 1 \leq i \leq n-1$$

$$f(v'_n) = 0$$

Using above pattern we have

$$v_f(0) + e_f(0) + 1 = v_f(1) + e_f(1) = \frac{5n-1}{2}$$

Thus  $f$  satisfies the condition  $|(v_f(0) + e_f(0)) - (v_f(1) + e_f(1))| \leq 1$ .

That is,  $spl(C_n)$  is total product cordial graph. □

**Illustration 4.4.2.** Consider a graph  $spl(C_7)$ . The corresponding total product cordial labeling is shown in Figure 4.7.

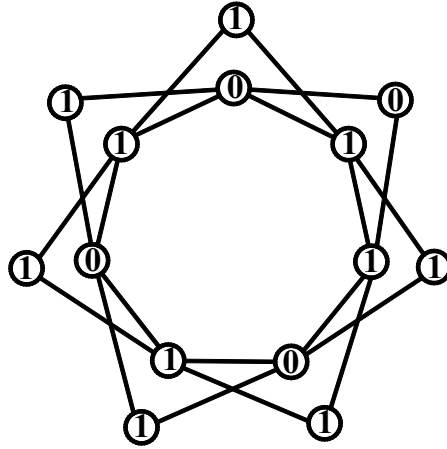


FIGURE 4.7:  $spl(C_7)$  and its total product cordial labeling

**Theorem 4.4.3.**  $spl(P_n)$  is total product cordial graph.

*Proof.* Let  $u_1, u_2, u_3, \dots, u_n$  be the vertices corresponding to  $v_1, v_2, v_3, \dots, v_n$  of  $P_n$  which are added to obtain  $spl(P_n)$ .

We define vertex labeling  $f : V(spl(P_n)) \cup E(spl(P_n)) \rightarrow \{0, 1\}$  as follows. We consider following two cases.

**Case 1:**  $n$  is even

$$f(v_i) = 0, \quad 1 \leq i \leq \frac{n}{2}$$

$$f(v_{\frac{n}{2}+i}) = 1, \quad 1 \leq i \leq \frac{n}{2}$$

$$f(u_i) = 0, \quad 1 \leq i \leq \frac{n}{2} - 1$$

$$f(u_{\frac{n}{2}+i}) = 1, \quad 0 \leq i \leq \frac{n}{2}$$

Using above pattern we have

$$v_f(0) + e_f(0) + 1 = v_f(1) + e_f(1) = 2n + 1$$

**Case 2:**  $n$  is odd

$$f(v_i) = 0, \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$$

$$f(v_{\frac{n}{2}+i}) = 1, \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$$

$$f(v_n) = 0,$$

$$f(u_i) = 0, \quad 1 \leq i \leq \frac{n-1}{2} - 1$$

$$f(u_{\frac{n-1}{2}+i}) = 1, \quad 0 \leq i \leq \frac{n+1}{2}$$

Using above pattern we have

$$v_f(0) + e_f(0) = v_f(1) + e_f(1) = \frac{5n-3}{2}$$

Thus  $f$  satisfies the condition  $|(v_f(0) + e_f(0)) - v_f(1) + e_f(1)| \leq 1$ .

That is,  $spl(P_n)$  is total product cordial graph. □

**Illustration 4.4.4.** Consider a graph  $spl(P_7)$ . The total product cordial labeling is shown in Figure 4.8.

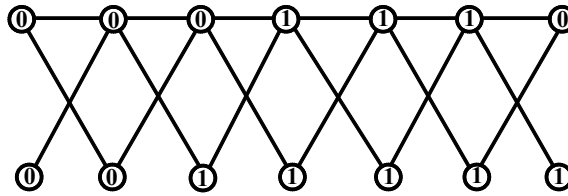


FIGURE 4.8:  $spl(P_7)$  and its total product cordial labeling

**Theorem 4.4.5.**  $spl(K_{1,n})$  is total product cordial graph.

*Proof.* Let  $u, u_1, u_2, u_3, \dots, u_n$  be the vertices corresponding to  $v, v_1, v_2, v_3, \dots, v_n$  of  $K_{1,n}$  which are added to obtain  $spl(K_{1,n})$ , where  $v$  be the apex vertex. We define vertex labeling  $f : V(spl(K_{1,n})) \cup E(spl(K_{1,n})) \rightarrow \{0, 1\}$  as follows. We consider following two cases.

**Case 1:**  $n$  is even

$$f(v) = 0, f(u) = 1$$

$$f(v_i) = 1, \quad 1 \leq i \leq n$$

$$f(u_{2i-1}) = 0, \quad 1 \leq i \leq \frac{n}{2}$$

$$f(u_{2i}) = 1, \quad 1 \leq i \leq \frac{n}{2}$$

In view of the labeling pattern defined above we have

$$v_f(0) + e_f(0) = v_f(1) + e_f(1) = \frac{5n}{2} + 1$$

**Case 2:**  $n$  is odd

$$f(v) = 0, f(u) = 1$$

$$f(v_i) = 1, \quad 1 \leq i \leq n$$

$$f(u_{2i-i}) = 0, \quad 1 \leq i \leq \lfloor \frac{n}{2} \rfloor + 1$$

$$f(u_{2i}) = 1, \quad 1 \leq i \leq \lfloor \frac{n}{2} \rfloor$$

In view of the labeling pattern defined above we have

$$v_f(0) + e_f(0) = v_f(1) + e_f(1) + 1 = \frac{5n+1}{2} + 1$$

Thus  $f$  satisfies the condition  $|(v_f(0) + e_f(0)) - v_f(1) + e_f(1)| \leq 1$ .

That is,  $spl(K_{1,n})$  is a total product cordial graph. □

**Illustration 4.4.6.** Consider a graph  $spl(K_{1,6})$ . The total product cordial labeling is as shown in Figure 4.9.

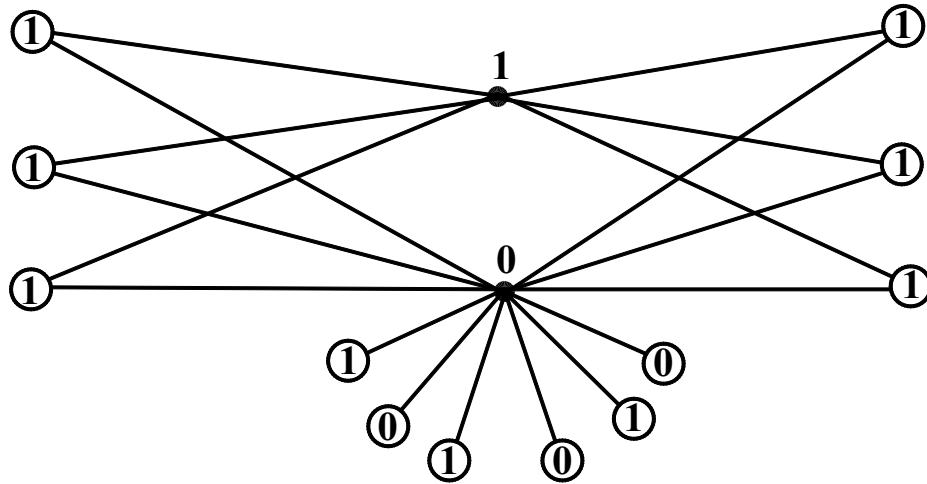


FIGURE 4.9:  $spl(K_{1,6})$  and its total product cordial labeling

**Theorem 4.4.7.**  $spl(C_n \odot K_1)$  is total product cordial graph.

*Proof.* Consider the crown  $C_n \odot K_1$  in which  $v_1, v_2, v_3, \dots, v_n$  be the vertices of cycle  $C_n$  and  $u_1, u_2, u_3, \dots, u_n$  be the pendant vertices attached at each vertex of  $C_n$ . Let  $v'_1, v'_2, v'_3, \dots, v'_n$  and  $u'_1, u'_2, u'_3, \dots, u'_n$  be the vertices corresponding to the vertices of  $C_n$  and  $K_1$  which are added to obtain  $spl(C_n \odot K_1)$ .

We define vertex labeling  $f : V(spl(C_n \odot K_1)) \cup E(spl(C_n \odot K_1)) \rightarrow \{0, 1\}$  as follows.

$$f(v_i) = 1, \quad 1 \leq i \leq n$$

$$f(v'_i) = 1, \quad 1 \leq i \leq n$$

$$f(u_i) = 0, \quad 1 \leq i \leq n$$

$$f(u'_i) = 0, \quad 1 \leq i \leq n$$

using above pattern we have

$$v_f(0) + e_f(0) = v_f(1) + e_f(1) = 5n$$

Thus  $f$  satisfies the condition  $|(v_f(0) + e_f(0)) - (v_f(1) + e_f(1))| \leq 1$ .

That is,  $spl(C_n \odot K_1)$  is total product cordial graph.  $\square$

**Illustration 4.4.8.** Consider a graph  $spl(C_5 \odot K_1)$ . The total product cordial labeling is as shown in Figure 4.10.

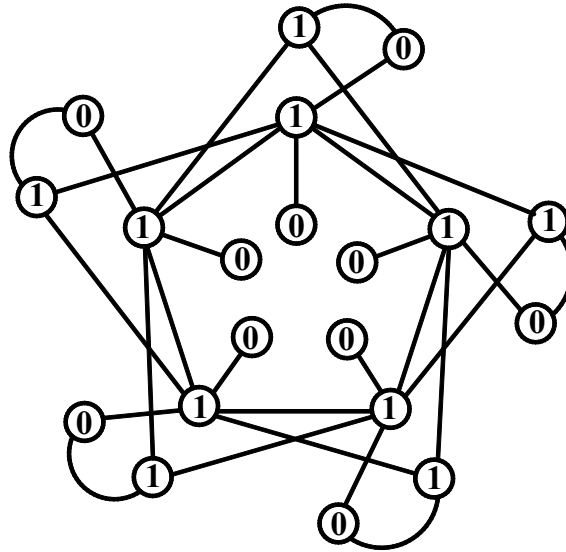


FIGURE 4.10:  $spl(C_5 \odot K_1)$  and its total product cordial labeling

**Theorem 4.4.9.**  $spl(F_n)$  is total product cordial graph.

*Proof.* Let  $v_1, v_2, v_3, \dots, v_{2n}$  be the vertices of  $F_n$  and  $v$  be the apex vertex. Let  $v', v'_1, v'_2, v'_3, \dots, v'_{2n}$  be the vertices corresponding to the vertices of  $F_n$  which are added to obtain  $spl(F_n)$ .

We define vertex labeling  $f : V(spl(F_n)) \cup E(spl(F_n)) \rightarrow \{0, 1\}$  as follows. We consider following two cases.

**Case 1:  $n$  even**

$$\begin{aligned} f(v) &= 1, f(v') = 0, f(v_i) = 0, & 1 \leq i \leq n \\ f(v_i) &= 1, & n+1 \leq i \leq 2n \\ f(v'_{2i-1}) &= 0, & 1 \leq i \leq \frac{n}{2} \\ f(v'_{2i}) &= 1, & 1 \leq i \leq \frac{n}{2} \\ f(v'_i) &= 1, & n+1 \leq i \leq 2n \end{aligned}$$

In view of the labeling pattern defined above we have

$$v_f(0) + e_f(0) = v_f(1) + e_f(1) = \frac{13n}{2} + 1$$

**Case 2:  $n$  odd**

$$\begin{aligned} f(v) &= 1, f(v') = 1, f(v_1) = 0 \\ f(v_i) &= 1, & \text{otherwise} \\ f(v'_2) &= 1 \\ f(v'_i) &= 0, & \text{otherwise} \end{aligned}$$

In view of the labeling pattern defined above we have

$$v_f(0) + e_f(0) + 1 = v_f(1) + e_f(1) = \frac{13n+1}{2} + 1$$

Thus we conclude that the graph  $f$  satisfies the condition

$$|(v_f(0) + e_f(0)) - (v_f(1) + e_f(1))| \leq 1$$

That is,  $spl(F_n)$  is a total product cordial graph. □

**Illustration 4.4.10.** Consider a graph  $spl(F_4)$ . The total product cordial labeling is as shown in Figure 4.11.

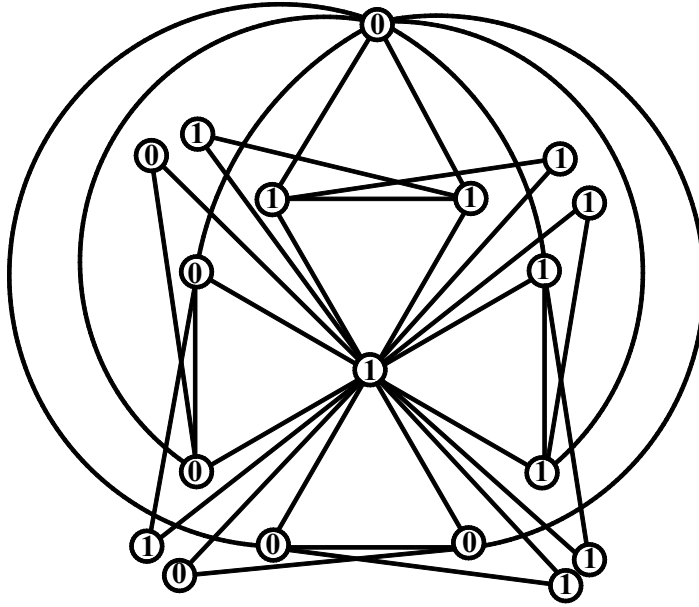


FIGURE 4.11:  $spl(F_4)$  and its total product cordial labeling

**Theorem 4.4.11.**  $spl(S_n)$  is total product cordial graph.

*Proof.* Let  $v_1, v_2, v_3, \dots, v_n$  be the vertices of  $S_n$  and  $v_1$  be the apex vertex of  $S_n$ . Let  $v'_1, v'_2, v'_3, \dots, v'_n$  be the vertices corresponding to the vertices of  $S_n$  which are added to obtain  $Spl(S_n)$ .

We define vertex labeling  $f : V(spl(S_n)) \cup E(spl(S_n)) \rightarrow \{0, 1\}$  as follows.

$$f(v_i) = 1, \quad 1 \leq i \leq n-1$$

$$f(v_i) = 0, \quad i = n$$

$$f(v'_i) = 1, \quad i = 1, n$$

$$f(v'_i) = 0, \quad 2 \leq i \leq n-1$$

Using above pattern we have

$$v_f(0) + e_f(0) + 1 = v_f(1) + e_f(1) = 4n - 4$$

Thus  $f$  satisfies the condition  $|(v_f(0) + e_f(0)) - (v_f(1) + e_f(1))| \leq 1$

That is,  $spl(S_n)$  is total product cordial graph. □

**Illustration 4.4.12.** Consider a graph  $spl(S_6)$ . The total product cordial labeling is as shown in Figure 4.12.

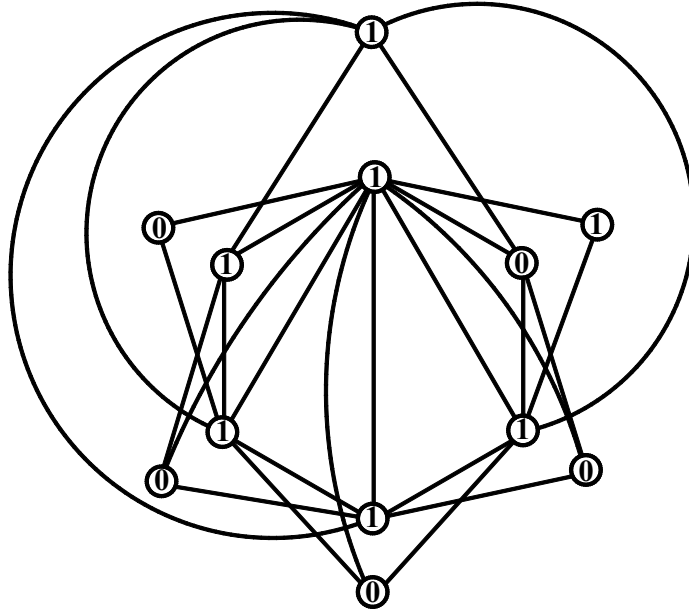


FIGURE 4.12:  $spl(S_6)$  and its total product cordial labeling

## 4.5 Prime Cordial Labeling

Any investigation related to prime numbers is much interesting as the number of prime is infinite and there are arbitrarily large gaps in the series of primes. When this important characteristic is take-up in the frame work of graph theory then it becomes more appealing. In the present section we will investigate some results on prime cordial labeling of graphs.

### 4.5.1 Prime Cordial graph

A *prime cordial labeling* of a graph  $G$  with vertex set  $V(G)$  is a bijection  $f : V(G) \longrightarrow \{1, 2, 3, \dots, p\}$  such that

$$f^*(e = uv) = \begin{cases} 1; & \text{if } \gcd(f(u), f(v)) = 1 \\ = 0; & \text{otherwise} \end{cases}$$

and  $|e_f(0) - e_f(1)| \leq 1$ . A graph which admits prime cordial labeling is called a *prime cordial graph*.

### 4.5.2 Some existing results

Sundaram, Ponraj and Somasundaram[65] have shown that following graphs are prime cordial.

- $C_n$  if and only if  $n \geq 6$ .
- $P_n$  if and only if  $n \neq 3$ .
- The graph  $K_{1,n}$  ( $n$  odd).
- The graph obtained by subdividing each edge of  $K_{1,n}$  if and only if  $n \geq 3$ .
- bistars; dragons; crowns.
- Triangular snakes  $T_n$  if and only if  $n \geq 3$ .
- The ladder graphs  $L_n$ .
- $K_{1,n}$  if  $n$  is even and there exists a prime  $p$  such that  $2p < n + 1 < 3p$ .
- $K_{2,n}$  if  $n$  is even and there exists a prime  $p$  such that  $3p < n + 2 < 4p$ .
- $K_{3,n}$  if  $n$  is odd and there exists a prime  $p$  such that  $5p < n + 3 < 6p$ .
- If  $G$  is a prime cordial graph of even size, then the graph obtained by identifying the central vertex of  $K_{1,n}$  with the vertex of  $G$  labeled with 2 is prime cordial and if  $G$  is a prime cordial graph of odd size, then the graph obtained by identifying the central vertex of  $K_{1,2n}$  with the vertex of  $G$  labeled with 2 is prime cordial.

In the same paper they have also shown that  $K_{m,n}$  is not prime cordial for a number of special cases of  $m$  and  $n$ .

## 4.6 Prime Cordial Labeling For Some Graphs

**Theorem 4.6.1.**  $T(P_3)$  is not a prime cordial graph.

*Proof.* For the graph  $T(P_3)$  the possible pairs of labels of adjacent vertices are (1,2), (1,3), (1,4), (1,5), (2,3), (2,4), (2,5), (3,4), (3,5), (4,5). Then obviously  $e_f(0) = 1$ ,  $e_f(1) = 6$ . That is,  $|e_f(0) - e_f(1)| = 5$  and in all other possible arrangement of vertex labels  $|e_f(0) - e_f(1)| > 5$ . Therefore  $T(P_3)$  is not a prime cordial graph.  $\square$

**Theorem 4.6.2.**  $T(P_n)$  is not a prime cordial graph, for  $n = 2, 4$ .

*Proof.* For the graph  $T(P_2)$  the possible pairs of labels of adjacent vertices are (1,2), (1,3), (2,3) then obviously  $e_f(0) = 0$ ,  $e_f(1) = 3$ . Therefore  $T(P_2)$  is not a prime cordial graph.

For the graph  $T(P_4)$  the possible pairs of labels of adjacent vertices are (1,2), (1,3), (1,4), (1,5), (1,6), (1,7), (2,3), (2,4), (2,5), (2,6), (2,7), (3,4), (3,5), (3,6), (3,7), (4,5), (4,6), (4,7), (5,6), (5,7), (6,7). Then obviously  $e_f(0) = 4$ ,  $e_f(1) = 7$ . That is  $|e_f(0) - e_f(1)| = 3$  and in all other possible arrangement of vertex labels  $|e_f(0) - e_f(1)| > 3$ . Thus  $T(P_4)$  is not a prime cordial graph.  $\square$

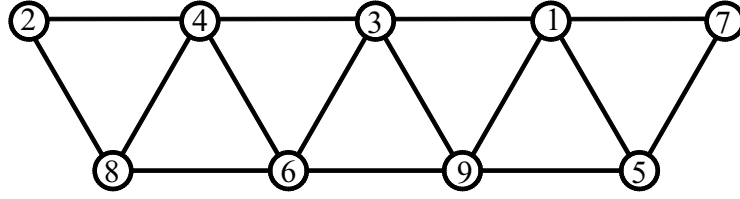
**Theorem 4.6.3.**  $T(P_n)$  is prime cordial graph, for all  $n \geq 5$ .

*Proof.* If  $v_1, v_2, \dots, v_n$  and  $e_1, e_2, \dots, e_n$  be the vertices and edges of  $P_n$  then  $v_1, v_2, \dots, v_n, e_1, e_2, \dots, e_n$  are vertices of  $T(P_n)$ .

We define vertex labeling  $f : V(T(P_n)) \longrightarrow \{1, 2, 3, \dots, |V(G)|\}$  we consider following four cases.

**Case 1:**  $n = 5$

The case when  $n = 5$  is to be dealt separately. The graph  $T(P_5)$  and its prime cordial labeling is shown in Figure 4.13.

FIGURE 4.13:  $T(P_5)$  and its prime cordial labeling

**Case 2:**  $n$  odd,  $n \geq 7$

$$f(v_1) = 2, f(v_2) = 4$$

$$f(v_{i+2}) = 2(i+3), \quad 1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 2,$$

$$f(v_{\lfloor \frac{n}{2} \rfloor + 1}) = 3, f(v_{\lfloor \frac{n}{2} \rfloor + 2}) = 1, f(v_{\lfloor \frac{n}{2} \rfloor + 3}) = 7,$$

$$f(v_{\lfloor \frac{n}{2} \rfloor + 3 + i}) = 4i + 9, \quad 1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 2,$$

$$f(e_i) = f(v_{\lfloor \frac{n}{2} \rfloor}) + 2i, \quad 1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1,$$

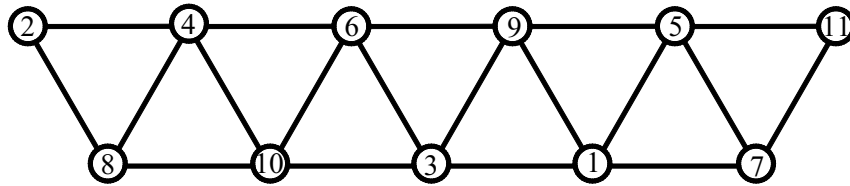
$$f(e_{\lfloor \frac{n}{2} \rfloor}) = 6, f(e_{\lfloor \frac{n}{2} \rfloor + 1}) = 9, f(e_{\lfloor \frac{n}{2} \rfloor + 2}) = 5,$$

$$f(e_{\lfloor \frac{n}{2} \rfloor + i + 2}) = 4i + 7, \quad 1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 2,$$

Using above pattern we have  $e_f(0) = e_f(1) + 1 = 2(n-1)$

**Case 3:**  $n = 6$

The case when  $n = 6$  is to be dealt separately. The graph  $T(P_6)$  and its prime cordial labeling is shown in Figure 4.14.

FIGURE 4.14:  $T(P_6)$  and its prime cordial labeling

**Case 4:**  $n$  even,  $n \geq 8$

$$f(v_1) = 2, f(v_2) = 4$$

$$f(v_{i+2}) = 2(i+3), \quad 1 \leq i \leq \frac{n}{2} - 3,$$

$$f(v_{\frac{n}{2}}) = 6, f(v_{\frac{n}{2}+1}) = 9, f(v_{\frac{n}{2}+2}) = 5,$$

$$f(v_{\frac{n}{2}+2+i}) = 4i + 7, \quad 1 \leq i \leq \frac{n}{2} - 2,$$

$$f(e_i) = f(v_{\frac{n}{2}-1}) + 2i, \quad 1 \leq i \leq \frac{n}{2} - 1,$$

$$f(e_{\frac{n}{2}}) = 3, f(e_{\frac{n}{2}+1}) = 1, f(e_{\frac{n}{2}+2}) = 7,$$

$$f(e_{\frac{n}{2}+i+2}) = 4i + 9, \quad 1 \leq i \leq \frac{n}{2} - 3,$$

Usinf above pattern we have  $e_f(0) = e_f(1) + 1 = 2(n - 1)$

That is,  $T(P_n)$  is a prime cordial graph, for every  $n \geq 5$ .  $\square$

**Illustration 4.6.4.** The graph  $T(P_7)$  and its prime cordial labeling is as shown in Figure 4.15.

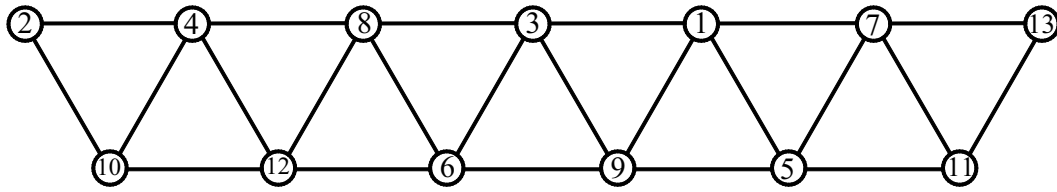


FIGURE 4.15:  $T(P_7)$  and its prime cordial labeling

**Theorem 4.6.5.**  $T(C_3)$  is not a prime cordial graph.

*Proof.* For the graph  $T(C_3)$  the possible pairs of labels of adjacent vertices are (1,2), (1,3), (1,4), (1,5), (1,6), (2,3), (2,4), (2,5), (2,6), (3,4), (3,5), (3,6), (4,5), (4,6), (5,6). Then obviously  $e_f(0) = 4$ ,  $e_f(1) = 8$ . That is,  $|e_f(0) - e_f(1)| = 4$  and in all other possible arrangement of vertex labels  $|e_f(0) - e_f(1)| > 4$ . Therefore  $T(C_3)$  is not a prime cordial graph.  $\square$

**Theorem 4.6.6.**  $T(C_4)$  is not a prime cordial graph.

*Proof.* For the graph  $T(C_4)$  the possible pairs of labels of adjacent vertices are (1,2), (1,3), (1,4), (1,5), (1,6), (1,7), (1,8), (2,3), (2,4), (2,5), (2,6), (2,7), (2,8), (3,4), (3,5), (3,6), (3,7), (3,8), (4,5), (4,6), (4,7), (4,8), (5,6), (5,7), (5,8), (6,7), (6,8), (7,8). Then obviously  $e_f(0) = 6$ ,  $e_f(1) = 10$ . That is,  $|e_f(0) - e_f(1)| = 4$  and in all other possible arrangement of vertex labels  $|e_f(0) - e_f(1)| > 4$ . Therefore  $T(C_4)$  is not a prime cordial graph.  $\square$

**Theorem 4.6.7.**  $T(C_n)$  is a prime cordial graph, for every  $n \geq 5$ .

*Proof.* If  $v_1, v_2, \dots, v_n$  and  $e_1, e_2, \dots, e_n$  be the vertices and edges of  $C_n$  then  $v_1, v_2, \dots, v_n, e_1, e_2, \dots, e_n$  are vertices of  $T(C_n)$ .

To define vertex labeling  $f : V(T(C_n)) \longrightarrow \{1, 2, 3, \dots, |V(G)|\}$  we consider following two cases.

**Case 1:**  $n$  even,  $n \geq 6$

$$f(v_1) = 2, f(v_2) = 8$$

$$f(v_{i+2}) = 4i + 10, \quad 1 \leq i \leq \frac{n}{2} - 3,$$

$$f(v_{\frac{n}{2}}) = 12, f(v_{\frac{n}{2}+1}) = 3, f(v_{\frac{n}{2}+2}) = 9, f(v_{\frac{n}{2}+3}) = 7,$$

$$f(v_{\frac{n}{2}+3+i}) = 4i + 9, \quad 1 \leq i \leq \frac{n}{2} - 3,$$

$$f(e_1) = 4, f(e_2) = 10$$

$$f(e_{i+2}) = 4(i + 3), \quad 1 \leq i \leq \frac{n}{2} - 3,$$

$$f(e_{\frac{n}{2}}) = 6, f(e_{\frac{n}{2}+1}) = 1, f(e_{\frac{n}{2}+2}) = 5,$$

$$f(e_{\frac{n}{2}+i+2}) = 4i + 7, \quad 1 \leq i \leq \frac{n}{2} - 2,$$

**Case 2:**  $n$  odd,  $n \geq 5$

$$f(v_1) = 2,$$

$$f(v_{i+1}) = 4(i + 1), \quad 1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1,$$

$$f(v_{\lfloor \frac{n}{2} \rfloor + 1}) = 6, f(v_{\lfloor \frac{n}{2} \rfloor + 2}) = 9, f(v_{\lfloor \frac{n}{2} \rfloor + 3}) = 5,$$

$$f(v_{\lfloor \frac{n}{2} \rfloor + 3+i}) = 4i + 7, \quad 1 \leq i \leq n - \lfloor \frac{n}{2} \rfloor - 3,$$

$$f(e_1) = 4,$$

$$f(e_{i+1}) = 4i + 6, \quad 1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1,$$

$$f(e_{\lfloor \frac{n}{2} \rfloor + 1}) = 3, f(e_{\lfloor \frac{n}{2} \rfloor + 2}) = 1, f(e_{\lfloor \frac{n}{2} \rfloor + 3}) = 7,$$

$$f(e_{\lfloor \frac{n}{2} \rfloor + i + 3}) = 4i + 9, \quad 1 \leq i \leq n - \lfloor \frac{n}{2} \rfloor - 3,$$

In view of the labeling pattern defined above we have  $e_f(0) = e_f(1) = 2n$

Thus  $f$  is a prime cordial labeling of  $T(C_n)$ . □

**Illustration 4.6.8.** The graph  $T(C_6)$  and its prime cordial labeling is shown in Figure 4.16.

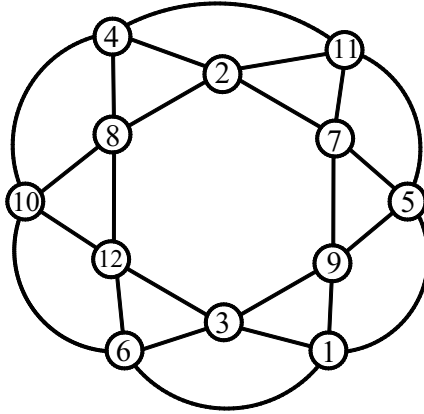


FIGURE 4.16:  $T(C_6)$  and its prime cordial labeling

**Theorem 4.6.9.**  $P_2[P_m]$  is not a prime cordial graph, for  $m = 2, 4$ .

*Proof.* For the graph  $P_2[P_2]$  the possible pairs of labels of adjacent vertices are (1,2), (1,3), (1,4), (2,3), (2,4), (3,4). Then obviously  $e_f(0) = 1$ ,  $e_f(1) = 5$ . That is,  $|e_f(0) - e_f(1)| = 4$  and in all other possible arrangement of vertex labels  $|e_f(0) - e_f(1)| > 4$ . Therefore  $P_2[P_2]$  is not a prime cordial graph.

For the graph  $P_2[P_4]$  the possible pairs of labels of adjacent vertices are (1,2), (1,3), (1,4), (1,5), (1,6), (1,7), (1,8), (2,3), (2,4), (2,5), (2,6), (2,7), (2,8), (3,4), (3,5), (3,6), (3,7), (3,8), (4,5), (4,6), (4,7), (4,8), (5,6), (5,7), (5,8), (6,7), (6,8), (7,8). Then obviously  $e_f(0) = 7$ ,  $e_f(1) = 9$ . That is,  $|e_f(0) - e_f(1)| = 2$  and in all other possible arrangement of vertex labels  $|e_f(0) - e_f(1)| > 2$ . Therefore  $P_2[P_4]$  is not a prime cordial graph.  $\square$

**Theorem 4.6.10.**  $P_2[P_3]$  is not a prime cordial graph.

*Proof.* For the graph  $P_2[P_3]$  the possible pairs of labels of adjacent vertices are (1,2), (1,3), (1,4), (1,5), (1,6), (2,3), (2,4), (2,5), (2,6), (3,4), (3,5), (3,6), (4,5), (4,6), (5,6). Then obviously  $e_f(0) = 4$ ,  $e_f(1) = 7$ . That is,  $|e_f(0) - e_f(1)| = 3$  and in all other possible arrangement of vertex labels  $|e_f(0) - e_f(1)| > 3$ . Therefore  $P_2[P_3]$  is not a prime cordial graph.  $\square$

**Theorem 4.6.11.**  $P_2[P_m]$  is prime cordial graph, for all  $m \geq 5$ .

*Proof.* If  $u_1, u_2, \dots, u_m$  be the vertices of  $P_m$  and  $v_1, v_2$  be the vertices  $P_2$ .

We define vertex labeling  $f : V(P_2[P_m]) \longrightarrow \{1, 2, 3, \dots, |V(G)|\}$  we consider following two cases.

**Case 1:**  $m$  even,  $m \geq 6$

$$\begin{aligned} f(u_1, v_1) &= 2, f(u_2, v_1) = 8 \\ f(u_{2+i}, v_1) &= 4i + 10, & 1 \leq i \leq \frac{m}{2} - 3, \\ f(u_{\frac{m}{2}}, v_1) &= 12, \\ f(u_{\frac{m}{2}+i}, v_1) &= 4i - 3, & 1 \leq i \leq \frac{m}{2}, \\ f(u_1, v_2) &= 4, f(u_2, v_2) = 10 \\ f(u_{2+i}, v_2) &= 4(i + 3), & 1 \leq i \leq \frac{m}{2} - 3, \\ f(u_{\frac{m}{2}}, v_2) &= 6, f(u_{\frac{m}{2}+1}, v_2) = 3, \\ f(u_{\frac{m}{2}+1+i}, v_2) &= 4i + 3, & 1 \leq i \leq \frac{m}{2} - 1, \end{aligned}$$

Using above pattern we have  $e_f(0) = e_f(1) = \frac{5n-4}{2}$

**Case 2:**  $m$  odd,  $m \geq 5$

$$\begin{aligned} f(u_i, v_1) &= 4(i + 1), & 1 \leq i \leq \lfloor \frac{m}{2} \rfloor - 1, \\ f(u_{\lfloor \frac{m}{2} \rfloor}, v_1) &= 2, \\ f(u_{\lfloor \frac{m}{2} \rfloor+1}, v_1) &= 6, f(u_{\lfloor \frac{m}{2} \rfloor+2}, v_1) = 9, f(u_{\lfloor \frac{m}{2} \rfloor+3}, v_1) = 5, \\ f(u_{\lfloor \frac{m}{2} \rfloor+3+i}, v_1) &= 4i + 7, & 1 \leq i \leq \lfloor \frac{m}{2} \rfloor - 2, \\ f(u_1, v_2) &= 4, \\ f(u_{1+i}, v_2) &= 4i + 6, & 1 \leq i \leq \lfloor \frac{m}{2} \rfloor - 1, \\ f(u_{\lfloor \frac{m}{2} \rfloor+1}, v_2) &= 3, f(u_{\lfloor \frac{m}{2} \rfloor+2}, v_2) = 1, f(u_{\lfloor \frac{m}{2} \rfloor+3}, v_2) = 7, \\ f(u_{\lfloor \frac{m}{2} \rfloor+2+i}, v_2) &= 4i + 9, & 1 \leq i \leq \lfloor \frac{m}{2} \rfloor - 2, \end{aligned}$$

Using above pattern we have  $e_f(0) = e_f(1) + 1 = 2n + \lfloor \frac{n}{2} \rfloor - 1$ .

Thus in case 1 and case 2 the graph  $f$  satisfies the condition  $|e_f(0) - e_f(1)| \leq 1$ .

That is,  $P_2[P_m]$  is a prime cordial graph for all  $m \geq 5$ . □

**Illustration 4.6.12.** The graph  $P_2[P_5]$  and its prime cordial labeling is shown in Figure 4.17.

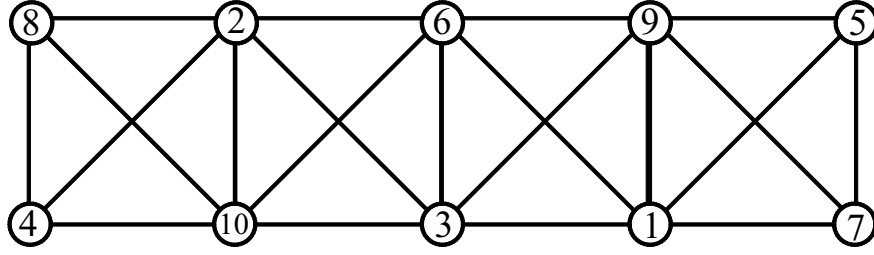


FIGURE 4.17:  $P_2[P_5]$  and its prime cordial labeling

**Theorem 4.6.13.** The graph obtained by joining two cycles by a path  $P_m$  admits prime cordial labeling.

*Proof.* Let  $G$  be the graph obtained by joining two cycles  $C_n$  and  $C'_n$  by a path  $P_m$ . Let  $v_1, v_2, \dots, v_n, v'_1, v'_2, \dots, v'_n$  be the vertices of  $C_n$  and  $C'_n$  respectively. Here  $u_1, u_2, \dots, u_m$  are the vertices of  $P_m$ .

We define vertex labeling  $f : V(G) \longrightarrow \{1, 2, 3, \dots, |V(G)|\}$  we consider following four cases.

**Case 1:**  $m$  odd,  $m \geq 5$

$$f(v_1) = f(u_1) = 2, f(v_2) = 4$$

$$f(v_{i+2}) = 2(i+3), \quad 1 \leq i \leq n-2,$$

$$f(u_{i+1}) = f(v_n) + 2i, \quad 1 \leq i \leq \lfloor \frac{m}{2} \rfloor - 2,$$

$$f(u_{\lfloor \frac{m}{2} \rfloor}) = 6, f(u_{\lfloor \frac{m}{2} \rfloor + 1}) = 3, f(u_{\lfloor \frac{m}{2} \rfloor + 2}) = 5,$$

$$f(u_{\lfloor \frac{m}{2} \rfloor + 2 + i}) = 2i + 5, \quad 1 \leq i \leq \lfloor \frac{m}{2} \rfloor - 2,$$

$$f(v'_1) = f(u_m) = 1, f(v'_{i+1}) = f(u_{m-1}) + 2i, \quad 1 \leq i \leq n-1,$$

In view of the labeling pattern defined above we have  $e_f(0) = e_f(1) = n + \lfloor \frac{m}{2} \rfloor$

**Case 2:**  $m = 3$

$$f(v_1) = f(u_1) = 6, f(v_2) = 2, f(v_3) = 4$$

$$f(v_{i+3}) = 2(i+3), \quad 1 \leq i \leq n-3,$$

$$f(u_2) = 3, f(v'_1) = f(u_3) = 1, f(v'_2) = 5,$$

$$f(v'_{2+i}) = 2i + 5, \quad 1 \leq i \leq n-2$$

In view of the labeling pattern defined above we have  $e_f(0) = e_f(1) = n + 1$

**Case 3:**  $m$  even,  $m \geq 4$

$$f(v_1) = f(u_1) = 2, f(v_2) = 4$$

$$f(v_{i+2}) = 2(i+3), \quad 1 \leq i \leq n-2,$$

$$f(u_{i+1}) = f(v_n) + 2i, \quad 1 \leq i \leq \frac{m}{2} - 2,$$

$$f(u_{\frac{m}{2}}) = 6, f(u_{\frac{m}{2}+1}) = 3, f(u_{\frac{m}{2}+2}) = 5,$$

$$f(u_{\frac{m}{2}+2+i}) = 2i + 5, \quad 1 \leq i \leq \frac{m}{2} - 3,$$

$$f(v'_1) = f(u_m) = 1, f(v'_{i+1}) = f(u_{m-1}) + 2i, \quad 1 \leq i \leq n-1,$$

In view of the labeling pattern defined above we have

$$e_f(0) = e_f(1) + 1 = n + \frac{m}{2}$$

**Case 4:**  $m = 2$

$$f(v_1) = f(u_1) = 2,$$

$$f(v_{i+1}) = 2(i+1), \quad 1 \leq i \leq n-1,$$

$$f(v'_1) = f(u_2) = 1,$$

$$f(v'_{1+i}) = 2i + 1, \quad 1 \leq i \leq n-1$$

In view of the labeling pattern defined above we have  $e_f(0) + 1 = e_f(1) = n + 1$

Thus in all the cases described above  $f$  satisfies the condition  $|e_f(0) - e_f(1)| \leq 1$ .

That is,  $G$  is a prime cordial graph. □

**Illustration 4.6.14.** The graph obtained by joining two copies of  $C_5$  by the path  $P_7$  and its prime cordial labeling is shown in Figure 4.18.

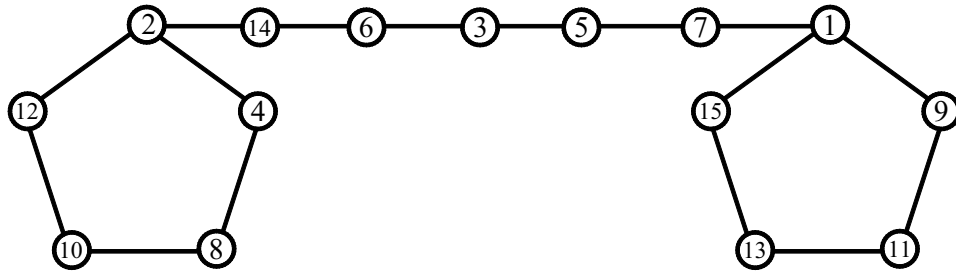


FIGURE 4.18: The graph obtained by joining two copies of  $C_5$  and its prime cordial labeling

**Theorem 4.6.15.** The graph obtained by switching of an arbitrary vertex in cycle  $C_n$  admits prime cordial labeling except for  $n = 5$ .

*Proof.* Let  $v_1, v_2, \dots, v_n$  be the successive vertices of  $C_n$  and  $G_v$  denotes the graph obtained by switching of a vertex  $v$ . Without loss of generality let the switched vertex be  $v_1$  and we initiate the labeling from the switched vertex  $v_1$ .

We define vertex labeling  $f : V(G_{v_1}) \rightarrow \{1, 2, 3, \dots, |V(G_{v_1})|\}$  we consider following four cases.

**Case 1:  $n = 4$**

The case when  $n = 4$  is to be dealt separately. The graph  $G_{v_1}$  and its prime cordial labeling is shown in Figure 4.19.

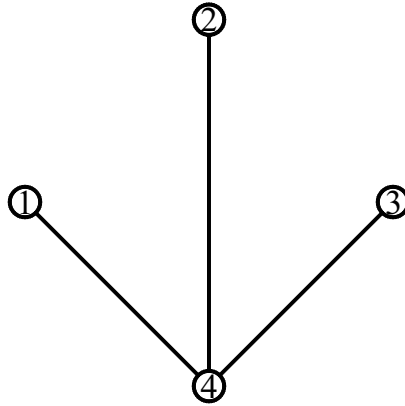


FIGURE 4.19: The graph obtained by vertex switching in  $C_4$  and its prime cordial labeling

**Case 2:  $n$  even,  $n \geq 6$**

$$f(v_1) = 2, f(v_2) = 1, f(v_3) = 4,$$

$$f(v_{i+3}) = 2(i+3), \quad 1 \leq i \leq \frac{n}{2} - 3,$$

$$f(v_{\frac{n}{2}+1}) = 6, f(v_{\frac{n}{2}+2}) = 3,$$

$$f(v_{\frac{n}{2}+2+i}) = 2i+3, \quad 1 \leq i \leq \frac{n}{2} - 2$$

Using above pattern we have  $e_f(0) = e_f(1) + 1 = n - 2$

**Case 3:  $n = 5$**

For the graph  $G_{v_1}$  the possible pairs of labels of adjacent vertices are (1,2), (1,3), (1,4), (1,5), (2,3), (2,4), (2,5), (3,4), (3,5), (4,5). Then obviously  $e_f(0) = 1$ ,  $e_f(1) = 4$ .

That is,  $|e_f(0) - e_f(1)| = 3$  and in all other possible arrangement of vertex labels  $|e_f(0) - e_f(1)| > 3$ . Therefore  $G_{v_1}$  is not a prime cordial graph.

**Case 4:**  $n$  odd,  $n \geq 7$

$$f(v_1) = 2, f(v_2) = 1, f(v_3) = 4$$

$$f(v_{i+3}) = 2(i+3), \quad 1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 3,$$

$$f(v_{\lfloor \frac{n}{2} \rfloor + 1}) = 6, f(v_{\lfloor \frac{n}{2} \rfloor + 2}) = 3,$$

$$f(v_{\lfloor \frac{n}{2} \rfloor + 2 + i}) = 2i + 3, \quad 1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1,$$

Using above pattern we have  $e_f(0) + 1 = e_f(1) = n - 2$

Thus in cases 1,2 and 4  $f$  satisfies the condition for prime cordial labeling. That is,  $G_{v_1}$  is a prime cordial graph.  $\square$

**Illustration 4.6.16.** Consider the graph obtained by switching the vertex in  $C_7$ . The prime cordial labeling is as shown in *Figure 4.20*.

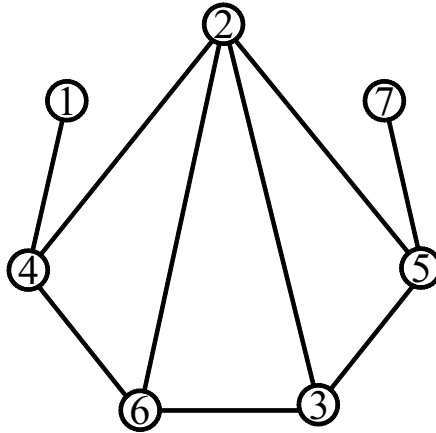


FIGURE 4.20: The graph obtained by vertex switching in  $C_7$  and its prime cordial labeling

## 4.7 Prime Cordial Labeling For Some Cycle Related Graphs

### Graphs

**Theorem 4.7.1.** The graph obtained by duplicating each edge by a vertex in cycle  $C_n$  admits prime cordial labeling except for  $n = 4$ .

*Proof.* Let  $C'_n$  be the graph obtained by duplicating an edge by a vertex in a cycle  $C_n$  then let  $v_1, v_2, \dots, v_n$  be the vertices of cycle  $C_n$  and  $v'_1, v'_2, \dots, v'_n$  be the added vertices to obtain  $C'_n$  corresponding to the vertices  $v_1, v_2, \dots, v_n$  in  $C_n$ .

Define  $f : V(C'_n) \longrightarrow \{1, 2, 3, \dots, 2p\}$ , we consider following two cases.

**Case 1:**  $n$  is odd

**Sub Case 1:**  $n = 3, 5$

The prime cordial labeling of  $C'_n$  for  $n = 3, 5$  is as shown in Figure 4.21.

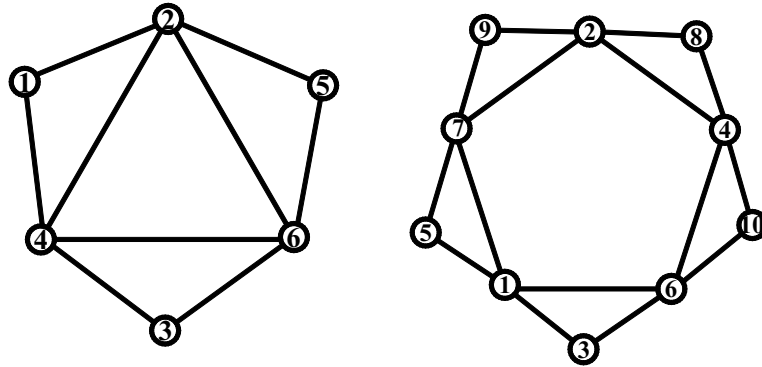


FIGURE 4.21: Prime cordial labeling of  $C'_3$  and  $C'_5$

**Sub Case 2:**  $n \geq 7$

$$f(v_1) = 2, f(v_2) = 4,$$

$$f(v_{2+i}) = 6 + 2i; \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 2$$

$$f(v_{\frac{n+1}{2}}) = 6,$$

$$f(v_{\frac{n+1}{2}+1}) = 1,$$

$$f(v_{\left\lfloor \frac{n}{2} \right\rfloor + 2 + i}) = 4i + 3; \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$$

$$f(v'_i) = f(v_{\left\lfloor \frac{n}{2} \right\rfloor}) + 2i; \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$$

$$f(v'_{\left\lfloor \frac{n}{2} \right\rfloor + 1}) = 3,$$

$$f(v'_{\left\lfloor \frac{n}{2} \right\rfloor + 1 + i}) = 4i + 1; \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$$

In the view of the labeling pattern defined above we have

$$e_f(0) + 1 = e_f(1) = 3 \left\lfloor \frac{n}{2} \right\rfloor + 2$$

**Case 2:**  $n$  is even

**Sub Case 1:**  $n = 4$

For the graph  $C'_4$  the possible pairs of labels of adjacent vertices are  $(1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (1, 7), (1, 8), (2, 3), (2, 4), (2, 5), (2, 6), (2, 7), (2, 8), (3, 4), (3, 5), (3, 6), (3, 7), (3, 8), (4, 5), (4, 6), (4, 7), (4, 8), (5, 6), (5, 7), (5, 8), (6, 7), (6, 8), (7, 8)$ .

Then obviously  $e_f(0) = 5, e_f(1) = 7$ . That is,  $e_f(1) - e_f(0) = 2$  and in all other possible arrangement of vertex labels  $|e_f(0) - e_f(1)| \geq 2$ . Thus  $C'_4$  is not a prime cordial graph.

**Sub Case 2:**  $n = 6, 8, 10$

The prime cordial labeling of  $C'_6, C'_8$  and  $C'_{10}$  is as shown in Figure 4.22.

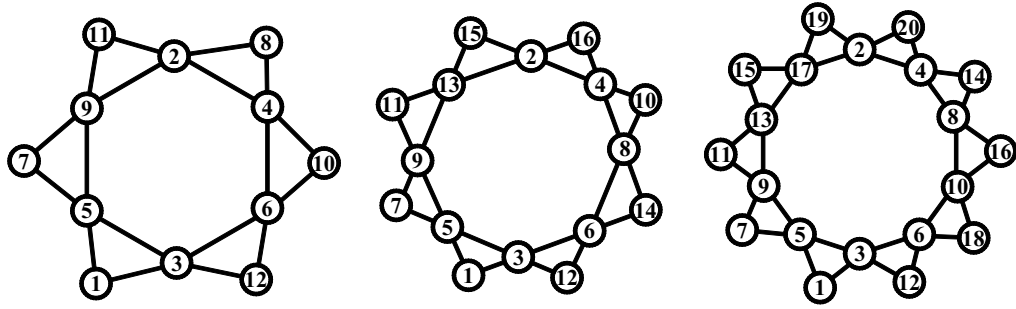


FIGURE 4.22: Prime cordial labeling of  $C'_6, C'_8$  and  $C'_{10}$

**Sub Case 3:**  $n \geq 12$

$$f(v_1) = 2, f(v_2) = 4, f(v_3) = 8, f(v_4) = 10, f(v_5) = 14,$$

$$f(v_{5+i}) = 14 + 2i; \quad 1 \leq i \leq \frac{n}{2} - 6$$

$$f(v_{\frac{n}{2}}) = 6,$$

$$f(v_{\frac{n}{2}+1}) = 3,$$

$$f(v_{\frac{n}{2}+1+i}) = 4i + 1; \quad 1 \leq i \leq \frac{n}{2} - 1$$

$$f(v'_1) = 2n$$

$$f(v'_{1+i}) = f(v_{\frac{n}{2}-1}) + 2i; \quad 1 \leq i \leq \frac{n}{2} - 2$$

$$f(v'_{\frac{n}{2}}) = 12$$

$$f(v'_{\frac{n}{2}+1}) = 1$$

$$f(v'_{\frac{n}{2}+1+i}) = 4i + 3; \quad 1 \leq i \leq \frac{n}{2} - 1$$

In the view of the labeling above defined we have

$$e_f(0) = e_f(1) = \frac{3n}{2}$$

Thus in above two cases we have  $|e_f(0) - e_f(1)| \leq 1$  Hence the graph obtained by duplicating each edge by a vertex in a cycle  $C_n$  admits prime cordial labeling except for  $n = 4$ .  $\square$

**Illustration 4.7.2.** Consider a graph  $C'_{12}$ . The prime cordial labeling is as shown in Figure 4.23.

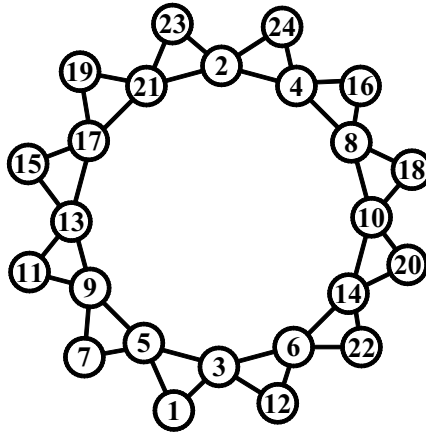


FIGURE 4.23: Prime cordial labeling of  $C'_{12}$

**Theorem 4.7.3.** The graph obtained by duplicating a vertex by an edge in cycle  $C_n$  is prime cordial graph.

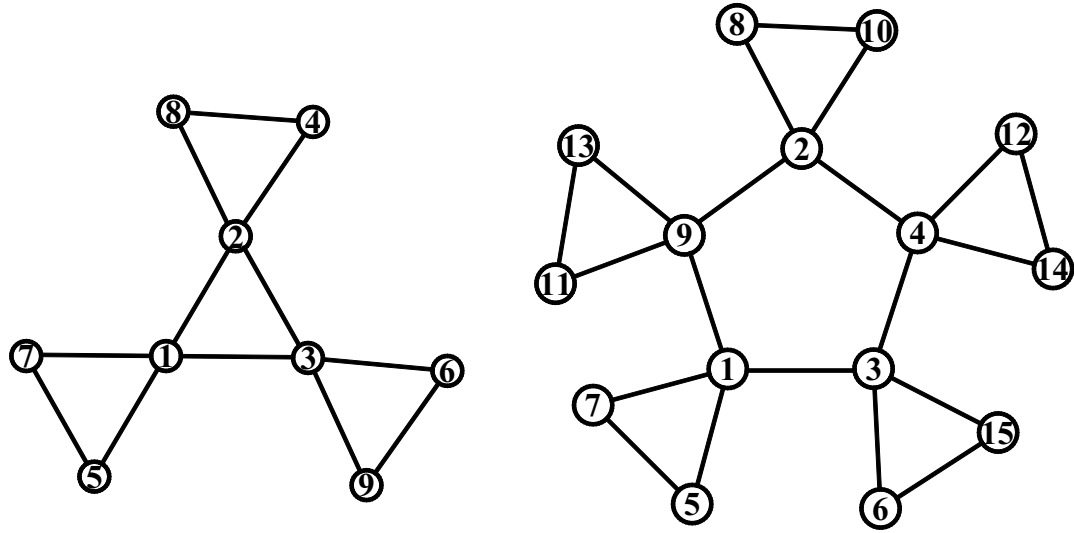
*Proof.* Let  $C'_n$  be the graph obtained by duplicating a vertex by an edge in cycle  $C_n$  then let  $v_1, v_2, \dots, v_n$  be the vertices of cycle  $C_n$  and  $v'_1, v'_2, \dots, v'_{2n}$  be the added vertices to obtain  $C'_n$  corresponding to the vertices  $v_1, v_2, \dots, v_n$  in  $C_n$ .

To define  $f : V(C'_n) \longrightarrow \{1, 2, 3, \dots, 3p\}$  we consider following two cases.

**Case 1:**  $n$  is odd

**Sub Case 1:**  $n = 3, 5$

The prime cordial labeling of  $C'_n$  for  $n = 3, 5$  is shown in Figure 4.24.

FIGURE 4.24: Prime cordial labeling of  $C'_3$  and  $C'_5$ 

**Sub Case 2:**  $n \geq 7$

$$f(v_1) = 2, f(v_2) = 4,$$

$$f(v_{2+i}) = 6 + 2i; \quad 1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 2$$

$$f(v_{\frac{n+1}{2}}) = 3,$$

$$f(v_{\frac{n+1}{2}+1}) = 1,$$

$$f(v_{\lfloor \frac{n}{2} \rfloor + 2 + i}) = 6i + 5; \quad 1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1$$

$$f(v'_i) = f(v_{\lfloor \frac{n}{2} \rfloor}) + 2i; \quad 1 \leq i \leq 2 \lfloor \frac{n}{2} \rfloor$$

$$f(v'_{2\lfloor \frac{n}{2} \rfloor + 1}) = 6, f(v'_{2\lfloor \frac{n}{2} \rfloor + 2}) = 9$$

$$f(v'_{2\lfloor \frac{n}{2} \rfloor + 3}) = 5, f(v'_{2\lfloor \frac{n}{2} \rfloor + 4}) = 7$$

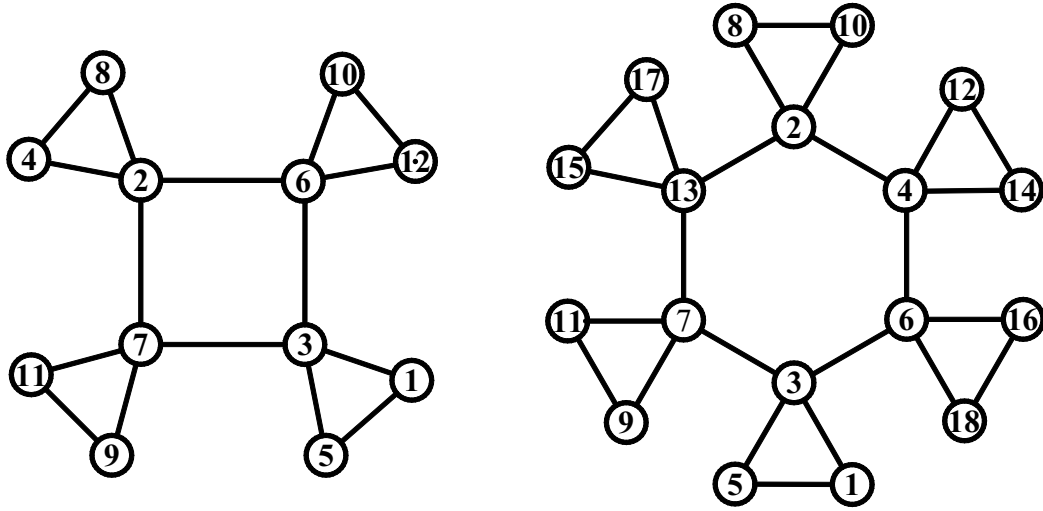
$$f(v'_{2\lfloor \frac{n}{2} \rfloor + 4 + 2i - 1}) = 6i + 7; \quad 1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1$$

$$f(v'_{2\lfloor \frac{n}{2} \rfloor + 4 + 2i}) = 6i + 9; \quad 1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1$$

**Case 2:**  $n$  is even

**Sub Case 1:**  $n = 4, 6$

The prime cordial labeling of  $C'_n$  for  $n = 4, 6$  is shown in Figure 4.25.

FIGURE 4.25: Prime cordial labeling of  $C'_4$  and  $C'_6$ 

**Sub Case 2:**  $n \geq 8$

$$f(v_1) = 2, f(v_2) = 4,$$

$$f(v_{2+i}) = 6 + 2i; \quad 1 \leq i \leq \frac{n}{2} - 3$$

$$f(v_{\frac{n}{2}}) = 6,$$

$$f(v_{\frac{n}{2}+1}) = 3,$$

$$f(v_{\frac{n}{2}+1+i}) = 6i + 1; \quad 1 \leq i \leq \frac{n}{2} - 1$$

$$f(v'_i) = f(v_{\frac{n}{2}-1}) + 2i; \quad 1 \leq i \leq n$$

$$f(v'_{n+1}) = 1, f(v'_{n+2}) = 5$$

$$f(v'_{n+1+2i}) = 6i + 3; \quad 1 \leq i \leq \frac{n}{2} - 1$$

$$f(v'_{n+2+2i}) = 6i + 5; \quad 1 \leq i \leq \frac{n}{2} - 1$$

Thus in both the cases defined above we have

$$e_f(0) = e_f(1) = 2n$$

Hence  $C'_n$  admits prime cordial labeling. □

**Illustration 4.7.4.** The graph  $C_7'$  and its prime cordial labeling is shown in Figure 4.26.

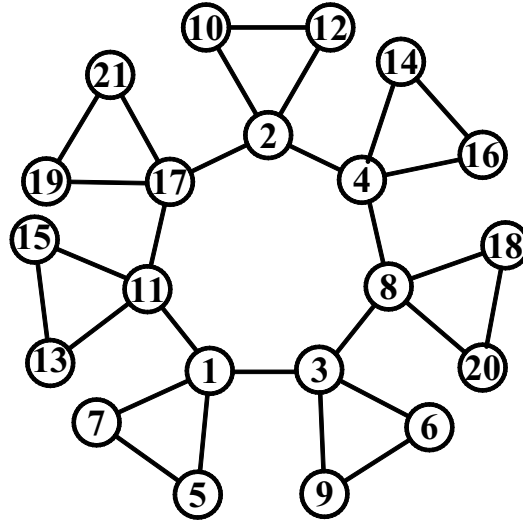


FIGURE 4.26: Prime cordial labeling of  $C_7'$

**Theorem 4.7.5.** The path union of  $m$  copies of cycle  $C_n$  is a prime cordial graph.

*Proof.* Let  $G'$  be the path union of  $m$  copies of cycle  $C_n$  and  $v_1, v_2, v_3, v_4, \dots, v_{mn}$  be the vertices of  $G'$ .

To define  $f : V(G') \longrightarrow \{1, 2, 3, \dots, mn\}$  we consider following four cases.

**Case 1:**  $n$  even,  $m$  even

$$f(v_i) = 2i; \quad 1 \leq i \leq \frac{mn}{2}$$

$$f(v_{\frac{mn}{2}+1}) = 1,$$

$$f(v_{\frac{mn}{2}+1+i}) = 4i - 1; \quad 1 \leq i \leq \frac{n}{2}$$

$$f(v_{\frac{mn}{2}+\frac{n}{2}+2}) = f(v_{\frac{mn}{2}+\frac{n}{2}+1}) - 2,$$

$$f(v_{\frac{mn}{2}+\frac{n}{2}+2+i}) = f(v_{\frac{mn}{2}+\frac{n}{2}+2}) - 4i; \quad 1 \leq i \leq \frac{n}{2} - 2$$

$$f(v_{\frac{mn}{2}+jn+i}) = f(v_{\frac{mn}{2}+(j-1)n+i}) + 2n; \quad 1 \leq j \leq \frac{m}{2} - 1, 1 \leq i \leq n$$

**Case 2:**  $n$  odd,  $m$  even

$$f(v_i) = 2i; \quad 1 \leq i \leq \frac{mn}{2}$$

$$f(v_{\frac{mn}{2}+1}) = 1,$$

$$f(v_{\frac{mn}{2}+1+i}) = 4i - 1; \quad 1 \leq i \leq \lfloor \frac{n}{2} \rfloor$$

$$f(v_{\frac{mn}{2}+\lfloor \frac{n}{2} \rfloor+2}) = f(v_{\frac{mn}{2}+\frac{n}{2}+1}) + 2,$$

$$f(v_{\frac{mn}{2} + \lfloor \frac{n}{2} \rfloor + 2 + i}) = f(v_{\frac{mn}{2} + \lfloor \frac{n}{2} \rfloor + 2}) - 4i; 1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1$$

$$f(v_{\frac{mn}{2} + jn + i}) = f(v_{\frac{mn}{2} + (j-1)n + i}) + 2n; 1 \leq j \leq \frac{m}{2} - 1, 1 \leq i \leq n$$

using above pattern we have  $e_f(0) + 1 = e_f(1) = \frac{m(n+1)}{2}$

**Case 3:**  $n$  even,  $m$  odd

$$f(v_1) = 4, f(v_2) = 8,$$

$$f(v_{2+i}) = 8 + 2i; 1 \leq i \leq n \lfloor \frac{m}{2} \rfloor - 2$$

$$f(v_{n \lfloor \frac{m}{2} \rfloor + 1}) = 2$$

$$f(v_{n \lfloor \frac{m}{2} \rfloor + 1 + i}) = f(v_{n \lfloor \frac{m}{2} \rfloor}) + 2i; 1 \leq i \leq \frac{n}{2} - 2$$

$$f(v_{n \lfloor \frac{m}{2} \rfloor + \frac{n}{2}}) = 6,$$

$$f(v_{n \lfloor \frac{m}{2} \rfloor + \frac{n}{2} + 1}) = 3, f(v_{n \lfloor \frac{m}{2} \rfloor + \frac{n}{2} + 2}) = 1$$

$$f(v_{n \lfloor \frac{m}{2} \rfloor + \frac{n}{2} + 2 + i}) = 2i + 3 1 \leq i \leq \frac{n}{2} - 2$$

$$f(v_{n \lfloor \frac{m}{2} \rfloor + n + 1}) = f(v_{n \lfloor \frac{m}{2} \rfloor + n}) + 2 \text{ or } f(v_{n \lfloor \frac{m}{2} \rfloor + n + 1}) = f(v_{n \lfloor \frac{m}{2} \rfloor + n}) + 4 \text{ for } n = 4$$

$$f(v_{n \lfloor \frac{m}{2} \rfloor + n + 2}) = f(v_{n \lfloor \frac{m}{2} \rfloor + n + 1}) + 2,$$

$$f(v_{n \lfloor \frac{m}{2} \rfloor + n + 2 + i}) = f(v_{n \lfloor \frac{m}{2} \rfloor + n + 1}) + 4i, 1 \leq i \leq \frac{n}{2} - 1$$

$$f(v_{n \lfloor \frac{m}{2} \rfloor + n + \frac{n}{2} + 2}) = f(v_{n \lfloor \frac{m}{2} \rfloor + n + \frac{n}{2} + 1}) + 2$$

$$f(v_{n \lfloor \frac{m}{2} \rfloor + n + \frac{n}{2} + 2 + i}) = f(v_{n \lfloor \frac{m}{2} \rfloor + n + \frac{n}{2} + 2}) - 4i; 1 \leq i \leq \frac{n}{2} - 2$$

$$f(v_{n \lfloor \frac{m}{2} \rfloor + (j+1)n + i}) = f(v_{n \frac{m}{2} + (j)n + i}) + 2n; 1 \leq j \leq \lfloor \frac{m}{2} \rfloor - 1, 1 \leq i \leq n$$

using above pattern we have  $e_f(0) = e_f(1) = \lfloor \frac{m}{2} \rfloor (n+1) + \frac{n}{2}$

**Case 4:**  $n$  odd,  $m$  odd

**Sub Case 1:**  $n = 3$

$$f(v_1) = 2, f(v_2) = 4,$$

$$f(v_{2+i}) = 6 + 2i; 1 \leq i \leq n \lfloor \frac{m}{2} \rfloor - 2$$

$$f(v_{n \lfloor \frac{m}{2} \rfloor + 1}) = 6,$$

$$f(v_{n \lfloor \frac{m}{2} \rfloor + 2}) = 3,$$

$$f(v_{n \lfloor \frac{m}{2} \rfloor + 3}) = 5,$$

$$f(v_{n \lfloor \frac{m}{2} \rfloor + 4}) = 1;$$

$$f(v_{n \lfloor \frac{m}{2} \rfloor + 3 + i}) = 2i + 5; 1 \leq i \leq n \lfloor \frac{m}{2} \rfloor - 1$$

using above pattern we have  $e_f(0) + 1 = e_f(1) = \lfloor \frac{m}{2} \rfloor (n+1) + 2$

**Sub Case 2:**  $n \geq 5$

$$f(v_1) = 4, f(v_2) = 8,$$

$$f(v_{2+i}) = 8 + 2i; 1 \leq i \leq n \lfloor \frac{m}{2} \rfloor - 2$$

$$\begin{aligned}
f(v_{n\lfloor \frac{m}{2} \rfloor + 1}) &= 2, \\
f(v_{n\lfloor \frac{m}{2} \rfloor + 1 + i}) &= f(v_{n\lfloor \frac{m}{2} \rfloor}) + 2i; \quad 1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 2 \\
f(v_{n\lfloor \frac{m}{2} \rfloor + \lfloor \frac{n}{2} \rfloor}) &= 6, \\
f(v_{n\lfloor \frac{m}{2} \rfloor + \lfloor \frac{n}{2} \rfloor + 1}) &= 3, \\
f(v_{n\lfloor \frac{m}{2} \rfloor + \lfloor \frac{n}{2} \rfloor + 2}) &= 1, \\
f(v_{n\lfloor \frac{m}{2} \rfloor + \lfloor \frac{n}{2} \rfloor + 2 + i}) &= 2i + 3, \quad 1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1 \\
f(v_{n\lfloor \frac{m}{2} \rfloor + n + 1}) &= f(v_{n\lfloor \frac{m}{2} \rfloor + n}) + 2, \\
f(v_{n\lfloor \frac{m}{2} \rfloor + n + 1 + i}) &= f(v_{n\lfloor \frac{m}{2} \rfloor + n + 1}) + 2i, \quad 1 \leq i \leq n - 1 \\
f(v_{n\lfloor \frac{m}{2} \rfloor + (j+1)n + i}) &= f(v_{n\frac{m}{2} + (j)n + i}) + 2n; \quad 1 \leq j \leq \lfloor \frac{m}{2} \rfloor - 1, 1 \leq i \leq n
\end{aligned}$$

using above pattern we have  $e_f(0) + 1 = e_f(1) = \lfloor \frac{m}{2} \rfloor (n + 1) + \lfloor \frac{n}{2} \rfloor + 1$

Thus in all the above cases we have  $|e_f(0) - e_f(1)| \leq 1$ .

Hence  $G'$  admits prime cordial labeling.  $\square$

**Illustration 4.7.6.** Consider a path union of three copies of  $C_7$ . The prime cordial labeling is as shown in Figure 4.27.

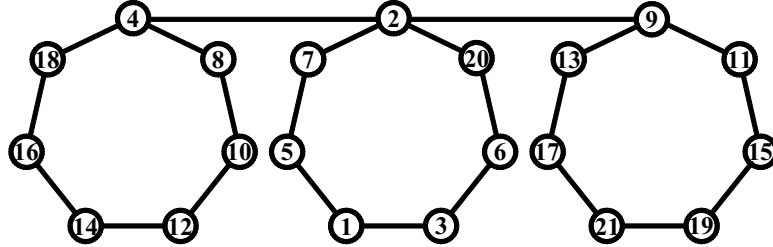


FIGURE 4.27: Prime cordial labeling of  $C'_7$

**Theorem 4.7.7.** The friendship graph  $F_n$  is a prime cordial graph for  $n \geq 3$ .

*Proof.* Let  $v_1$  be the vertex common to all the cycles. Without loss of generality we start the label assignment from  $v_1$ .

To define  $f : V(F_n) \longrightarrow \{1, 2, 3, \dots, 2n + 1\}$ , we consider following two cases.

**Case 1:**  $n$  even

let  $p$  be the highest prime such that  $3p \leq 2n + 1$ ,

$$f(v_1) = 2p,$$

now label the remaining vertices from 1 to  $2n + 1$  first even and then odd except  $2p$ .

In view of the labeling pattern defined above we have

$$e_f(0) = e_f(1) = \frac{3n}{2}$$

**Case 2:**  $n$  odd

let  $p$  be the highest prime such that  $2p \leq 2n + 1$ ,

$$f(v_1) = 2p,$$

now label the remaining vertices from 1 to  $2n + 1$  first even and then odd except  $2p$ .

In view of the labeling above defined we have

$$e_f(0) + 1 = e_f(1) = 3\lfloor \frac{n}{2} \rfloor + 2$$

Thus in above two cases  $|e_f(0) - e_f(1)| \leq 1$

Hence friendship graph  $F_n$  admits prime cordial labeling.  $\square$

**Illustration 4.7.8.** Consider the friendship graph  $F_8$ . The prime cordial labeling is as shown in *Figure 4.28*.

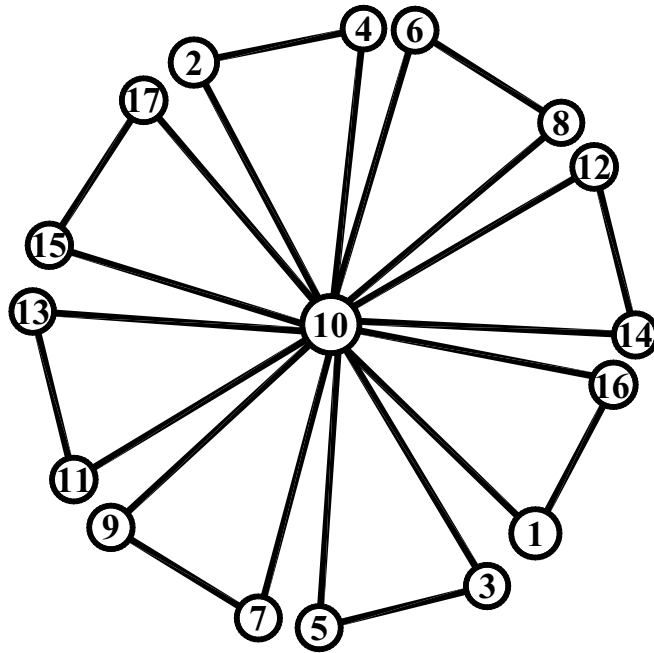


FIGURE 4.28: Prime cordial labeling of  $F_8$

## **4.8 Concluding Remarks and Scope of Further Research**

This chapter was targeted to discuss two labelings with cordial theme. We have investigated several results for total product cordial labeling and prime cordial labeling. To derive similar results in the context of different graph labeling problems and for various graph families is an open area of research.

The next chapter is focused on Fibonacci graceful labeling of graphs.

## **Chapter 5**

# **Fibonacci Graceful Labeling of Some Graphs**

## 5.1 Introduction

The brief account of graceful labeling is given in chapter 3. As we mention there the Ringel conjecture and many efforts to settle it provided the reason for various graph labeling problems. Some labeling with variations in graceful theme are also introduced. Some of them are edge graceful labeling, odd graceful labeling, Fibonacci graceful labeling etc. The present chapter is intended to discuss Fibonacci graceful labeling and its extension.

## 5.2 Fibonacci and Super Fibonacci Graceful Labeling

### 5.2.1 Fibonacci numbers

The *Fibonacci numbers*  $F_0, F_1, F_2 \dots$  are defined by  $F_0 = 0, F_1 = 1, F_2 = 1$  and  $F_{n+1} = F_n + F_{n-1}$ .

### 5.2.2 Fibonacci graceful labeling

The function  $f : V(G) \rightarrow \{0, 1, 2, \dots, F_q\}$  (where  $F_q$  is the  $q^{th}$  Fibonacci number) is said to be *Fibonacci graceful* if  $f^* : E(G) \rightarrow \{F_1, F_2, \dots, F_q\}$  defined by  $f^*(uv) = |f(u) - f(v)|$  is bijective.

### 5.2.3 Super Fibonacci graceful labeling

The function  $f : V(G) \rightarrow \{0, F_1, F_2, \dots, F_q\}$  (where  $F_q$  is the  $q^{th}$  Fibonacci number) is said to be *Super Fibonacci graceful* if the induced edge labeling  $f^* : E(G) \rightarrow \{F_1, F_2, \dots, F_q\}$  defined by  $f^*(uv) = |f(u) - f(v)|$  is bijective.

Above two concepts were introduced by Kathiresan and Amutha [46]. Deviating from the standard definition of Fibonacci numbers they assumed that  $F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, \dots$  which also avoid repetition of 1 as vertex(edge) label.

### 5.2.4 Some existing results

Kathiresen and Amutha [46] have proved that

- $K_n$  is Fibonacci graceful if and only if  $n \leq 3$ .
- If  $G$  is Eulerian and Fibonacci graceful then  $q \equiv 0(mod 3)$ .
- Every path  $P_n$  of length  $n$  is Fibonacci graceful.
- $P_n^2$  is a Fibonacci graceful graph.
- Caterpillars are Fibonacci graceful.
- The bistar  $B_{m,n}$  is Fibonacci graceful but not Super Fibonacci graceful for  $n \geq 5$ .
- $C_n$  is Super Fibonacci graceful if and only if  $n \equiv 0(mod 3)$ .
- Every fan  $F_n$  is Super Fibonacci graceful.
- If  $G$  is Fibonacci or Super Fibonacci graceful then its pendant edge extension  $G'$  is Fibonacci graceful.
- If  $G_1$  and  $G_2$  are Super Fibonacci graceful in which no two adjacent vertices have the labeling 1 and 2, then their union  $G_1 \cup G_2$  is Fibonacci graceful.
- If  $G_1, G_2, \dots, G_n$  are super Fibonacci graceful graphs in which no two adjacent vertices are labeled with 1 and 2 then amalgamation of  $G_1, G_2, \dots, G_n$  obtained by identifying the vertices having labels 0 is also a super Fibonacci graceful.

### 5.3 Fibonacci and Super Fibonacci Graceful Labeling of Some Graphs

**Theorem 5.3.1.** Trees are Fibonacci graceful.

*Proof.* Consider a vertex with minimum eccentricity as the root of tree  $T$ . Let this vertex be  $v$ . Without loss of generality at each level of tree  $T$  we initiate the labeling from left to right. Let  $P^1, P^2, P^3, \dots, P^n$  be the children of  $v$ .

Define  $f : V(T) \longrightarrow \{0, 1, 2, \dots, F_q\}$  in the following manner.

$$f(v) = 0, f(P^1) = F_1$$

Now if  $P_{1i}^1 (1 \leq i \leq t)$  are children of  $P^1$  then

$$f(P_{1i}^1) = f(P^1) + F_{i+1}, 1 \leq i \leq t$$

If there are  $r$  vertices at level two of  $P^1$  and out of these  $r$  vertices,  $r_1$  be the children of  $P_{11}^1$  then label them as follows,

$$f(P_{11i}^1) = f(P_{11}^1) + F_{t+1+i}, 1 \leq i \leq r_1$$

Let there are  $r_2$  vertices, which are children of  $P_{12}^1$  then label them as follows,

$$f(P_{12i}^1) = f(P_{12}^1) + F_{t+1+r_1+i}, 1 \leq i \leq r_2$$

Following the same procedure to label all the vertices of a subtree with root as  $P^1$ .

we can assign label to each vertex of the subtree with roots as  $P^2, P^3, \dots, P^{n-1}$  and define  $f(P^{i+1}) = F_{f_i+1}$ , where  $F_{f_i}$  is the  $f_i^{th}$  Fibonacci number assign to the last edge of the tree rooted at  $P^i$ .

Now for the vertex  $P^n$ . Define  $f(P^n) = F_q$

Let us denote  $P_{ij}^n$ , where  $i$  is the level of vertex and  $j$  is number of vertices at  $i^{th}$  level. At this stage one has to be cautious to avoid the repetition of vertex labels in right most branch. For that we first assign vertex label to that vertex which is adjacent to  $F_q$  and is a internal vertex of the path whose length is largest among all the paths whose origin is  $F_q$  (That is,  $F_q$  is a root). Without loss of generality we consider this path to be a left most path to  $F_q$  and continue label assignment from left to right as stated earlier.

If  $P_{1i}^n (1 \leq i \leq s)$  be the children of  $P^n$  then define

$$f(P_{1i}^n) = f(P^n) - F_{q-i}, 1 \leq i \leq s$$

If there are  $P_{2i}^n (1 \leq i \leq b)$  vertices at level two of  $P^n$  and out of these  $b$  vertices,  $b_1$  be

the children of  $P_{11}$  then label them follows.

$$f(P_{2i}^n) = f(P_{11}^n) - F_{q-s-i}, 1 \leq i \leq b_1$$

If there are  $b_2$  vertices, which are children of  $P_{12}^n$  then label them as follows.

$$f(P_{2(b_1+i)}^n) = f(P_{12}^n) - F_{q-s-b_1-i}, 1 \leq i \leq b_2$$

We will also consider the situation when all the vertices of subtree rooted at  $F_q$  is having all the vertices of degree two after  $i^{th}$  level then we define labeling as follows.

$f(P_{(i+1)1}^n) = f(P_{i1}^n) + (-1)^{(i)} F_{q-(\text{labeled vertices in the branch rooted at } P^n)}$  Continue this labeling scheme unless all the vertices of a subtree with root as  $P^n$  are labeled.

Thus we have labeled all the vertices at each level. That is,  $T$  admits Fibonacci graceful labeling and accordingly trees are Fibonacci graceful graph.  $\square$

**Illustration 5.3.2.** Consider the tree with 12 edges then the Fibonacci graceful labeling is as follows.

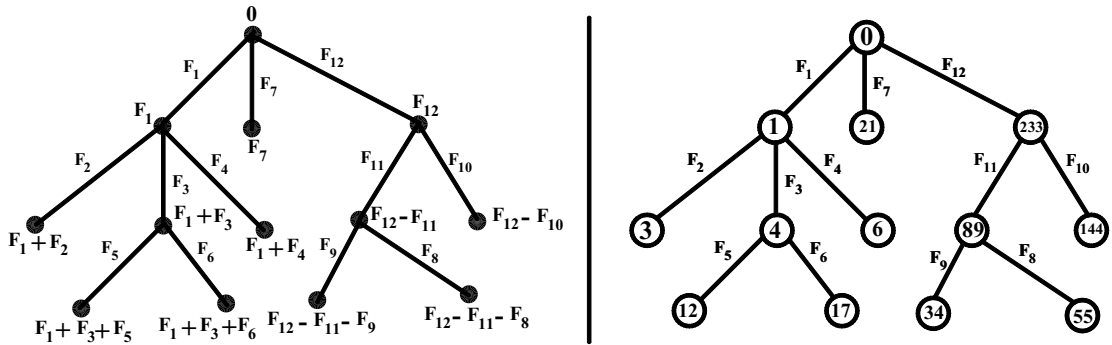


FIGURE 5.1: Fibonacci graceful labeling of a tree with 12 edges

**Theorem 5.3.3.** Wheels are not Fibonacci graceful.

*Proof.* Let  $v$  be the apex vertex of the wheel  $W_n$  and  $v_1, v_2, \dots, v_n$  be the rim vertices.

Define  $f : V(W_n) \rightarrow \{0, 1, 2, \dots, F_q\}$

We consider following cases.

**Case 1:** Let  $f(v) = 0$

so, the vertices  $v_1, v_2, \dots, v_n$  must be label with Fibonacci numbers.

Let  $f(v_1) = F_q$  then  $f(v_2) = F_{q-1}$  or  $f(v_2) = F_{q-2}$ .

If  $f(v_2) = F_{q-2}$  then  $f(v_n) = F_{q-1}$  is not possible as  $f(v_1 v_n) = f(v v_2) = F_{q-2}$ .

If  $f(v_2) = F_{q-1}$  then  $f(v_n) \neq F_{q-2}$  otherwise  $f(v_1v_n) = f(vv_2) = F_{q-1}$ .

If  $f(v_n) = F_p$  be the Fibonacci number other than  $F_{q-1}$  and  $F_{q-2}$  then  $|f(v_n) - f(v_1)| = |F_p - F_q|$  can not be Fibonacci number for  $|p - q| > 2$

**Case 2:** If  $v_1$  is a rim vertex then define  $f(v_1) = 0$

If  $f(v_2) = F_q$  then the apex vertex must be labeled with  $F_{q-1}$  or  $F_{q-2}$ .

**Sub Case 1:** Let  $f(v) = F_{q-1}$

Now  $f(v_n)$  must be labeled with either by  $F_{q-2}$  or by  $F_{q-3}$ .

If  $f(v_n) = F_{q-2}$  then  $f(v_1v_n) = f(vv_2) = F_{q-2}$

and if  $f(v_n) = F_{q-3}$  then  $f(vv_n) = f(vv_2) = F_{q-2}$

**Sub Case 2:** Let  $f(v) = F_{q-2}$

Now  $f(v_n)$  must be label with either by  $F_{q-1}$  or by  $F_{q-3}$  or by  $F_{q-4}$ .

if  $f(v_n) = F_{q-1}$  then  $f(v_1v_n) = f(vv_2) = F_{q-1}$

if  $f(v_n) = F_{q-3}$  then

$$f(v_1v_2) = F_q$$

$$f(vv_1) = F_{q-2}$$

$$f(vv_2) = F_{q-1}$$

$$f(v_nv_1) = F_{q-3}$$

$$f(vv_n) = F_{q-4}$$

For  $W_3$ ,  $f(v_2v_3)$  can not be Fibonacci number. Now for  $n > 3$  let us assume that

$f(v_3) = k$  which is not Fibonacci number because for  $f(v_3) = F_{q-1}$ , we have  $f(vv_1) =$

$$f(v_2v_3) = F_{q-2}.$$

now we have following cases. (1)  $F_{q-2} < k < F_q$ , (2)  $k < F_{q-2} < F_q$

In (1) we have.....

$$F_q - k = F_s$$

$$k - F_{q-2} = F_{s'}$$

$F_q - F_{q-2} = F_s + F_{s'} \implies F_{q-1} = F_s + F_{s'}$  is possible only when  $s = q - 2$  and  $s' = q - 3$ ,

then  $f(v_2v_3) = f(vv_1)$  and  $f(vv_3) = f(v_1v_n)$

In (2) we have.....

$$F_q - k = F_s$$

$$F_{q-2} - k = F_{s'}$$

$F_q - F_{q-2} = F_s + F_{s'} \implies F_{q-1} = F_s + F_{s'}$  is possible only when  $s = q - 2$  and  $s' = q - 3$ , then  $f(v_2v_3) = f(vv_1)$  and  $f(vv_3) = f(v_1v_n)$

so, we can not find a number  $f(v_3) = k$  such that  $f(v_2v_3)$  and  $f(vv_3)$  have the distinct Fibonacci numbers.

For  $f(v_n) = F_{q-4}$  we can argue as above.

**Sub Case 3:** If  $f(v) = F_q$

Then we do not have two Fibonacci numbers corresponding to  $f(v_1)$  and  $f(v_n)$  such that the edges will receive distinct Fibonacci numbers.

Thus we conclude that wheels are not Fibonacci graceful.  $\square$

**Theorem 5.3.4.** Helms are not Fibonacci graceful.

*Proof.* Let  $H_n$  be the helm and  $v'_1, v'_2, v'_3, \dots, v'_n$  be the pendant vertices corresponding to it. If 0 is the label of any of the rim vertices of wheel corresponding to  $H_n$  then all the possibilities to admit Fibonacci graceful labeling is ruled out as we argued in above Theorem 5.3.3. Thus possibilities of 0 being the label of any of the pendant vertices is remained at our disposal.

Define  $f : V(H_n) \longrightarrow \{0, 1, 2, \dots, F_q\}$

Without loss of generality we assume  $f(v'_1) = 0$  then  $f(v_1) = F_q$

Let  $f(v_2) = p$  and  $f(v) = r$

In the following *Figures 5.2(1) to 5.2(3)* the possible labeling is demonstrated. In first two arrangements the possibility of  $H_3$  being Fibonacci graceful is washed out by the similar arrangements for wheels are not Fibonacci graceful held in Theorem 5.3.3. For the remaining arrangement as shown in *Figure 5.2* we have to consider following two possibilities.

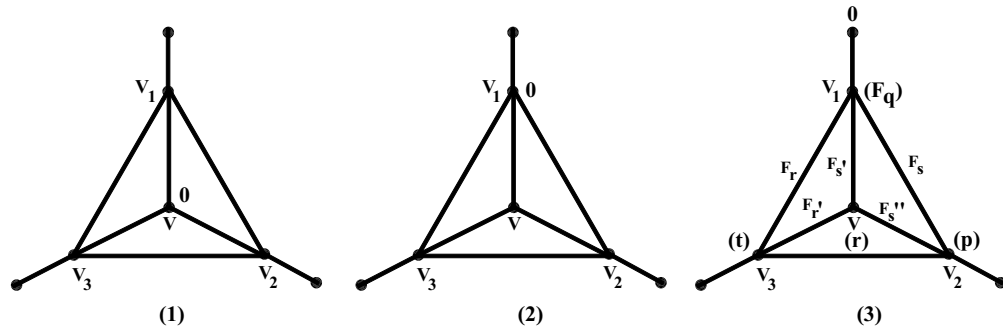


FIGURE 5.2

**Case 1:**  $p < r < F_q$

$$F_q - p = F_s$$

$$F_q - r = F_{s'}$$

$$r - p = F_{s''} \text{ then}$$

$$F_{s'} + F_{s''} - F_s = 0 \implies F_s = F_{s'} + F_{s''}$$

**Case 2:**  $r < p < F_q$

$$F_q - p = F_s$$

$$F_q - r = F_{s'}$$

$$p - r = F_{s''} \text{ then}$$

$$F_s + F_{s''} - F_{s'} = 0 \implies F_{s'} = F_s + F_{s''}$$

Now let  $f(v_3) = t$  then consider the case  $p < r < t < F_q$ ,

$$F_s = F_{s'} + F_{s''}$$

$$F_{s'} = F_r + F_{r'}$$

From these two equations we have...

$$F_{s'} = F_r + F_{r'} = F_s - F_{s''}$$

so we have  $F_r < F_{r'} < F_{s'} < F_{s''} < F_s$  and they are consecutive Fibonacci numbers.

For  $r \geq p, t$  we have  $F_s = F_{s'} + F_{s''}$  and  $F_r = F_{s'} + F_{r'}$  so we have

$$F_{s'} = F_s - F_{s''} \text{ and } F_{s'} = F_r - F_{r'} \text{ which is not possible.}$$

similar argument can be made for  $r \leq p, t$ .

i.e. we have either  $p < r < t$  or  $t < r < p$ .

As  $F_{s'} < F_{s''} < F_s$ , so we can say that with  $f(vv_2) = F_{s''}$  the edges of the triangle with vertices  $f(v), f(v_2)$  and  $f(v_3)$  will not have Fibonacci numbers such that  $F_{s''} = \text{sum of}$

two Fibonacci numbers.

Similar arguments can also be made for  $t < r < p < F_q$ .

Hence Helms are not Fibonacci graceful graphs.  $\square$

**Theorem 5.3.5.** The graph obtained by switching of a vertex in cycle  $C_n$  admits Fibonacci graceful labeling.

*Proof.* Let  $v_1, v_2, v_3, \dots, v_n$  be the vertices of cycle  $C_n$  and  $C'_n$  be the graph resulted from switching of the vertex  $v_1$ .

Define  $f : V(C'_n) \rightarrow \{0, 1, 2, \dots, F_q\}$  as follows.

$$f(v_1) = 0$$

$$f(v_2) = F_q - 1$$

$$f(v_3) = F_q$$

$$f(v_{i+3}) = F_{q-2i}, 1 \leq i \leq n-3$$

Above defined function  $f$  admits Fibonacci graceful labeling.

Hence we have the result.  $\square$

**Illustration 5.3.6.** Consider  $C'_8$ . The corresponding Fibonacci graceful labeling is as shown in Figure 5.3

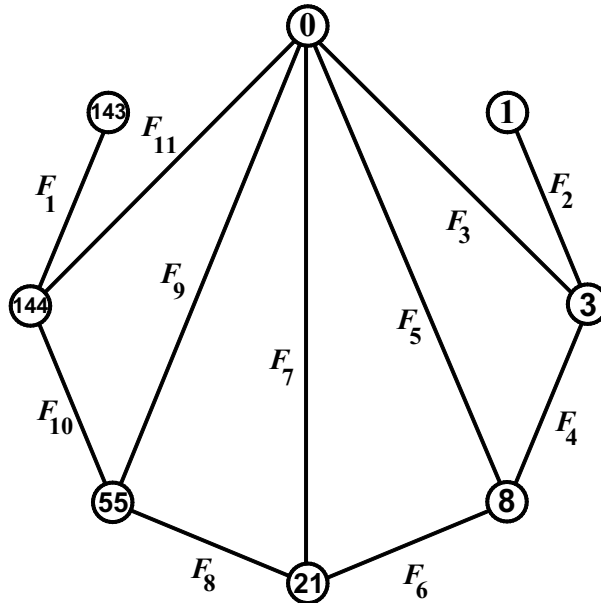


FIGURE 5.3: Fibonacci graceful labeling of  $C'_8$

**Theorem 5.3.7.** Joint sum of two copies of fans( $f_n = P_n + K_1$ ) is Fibonacci graceful.

*Proof.* Let  $v_1, v_2, \dots, v_n$  and  $v'_1, v'_2, \dots, v'_n$  be the vertices of two copies  $f_n^1$  and  $f_m^2$  respectively. Let  $v$  be the apex vertex of  $F_n^1$  and  $v'$  be the apex vertex of  $f_m^2$  and let  $G$  be the joint sum of two fans.

Define  $f : V(G) \longrightarrow \{0, 1, 2, \dots, F_q\}$  as follows.

$$f(v) = 0$$

$$f(v') = F_q$$

$$f(v_i) = F_{2i-1}, 1 \leq i \leq n$$

$$f(v'_1) = F_q - F_{2n+1}$$

$$f(v'_2) = F_q - F_{2n+2}$$

$$f(v'_{2+i}) = F_q - F_{2n+2+2i}, 1 \leq i \leq m-2$$

In view of the above defined pattern the graph  $G$  admits Fibonacci graceful labeling.  $\square$

**Illustration 5.3.8.** Consider the Joint Sum of two copies of  $F_4$ . The corresponding Fibonacci graceful labeling is as shown in Figure 5.4

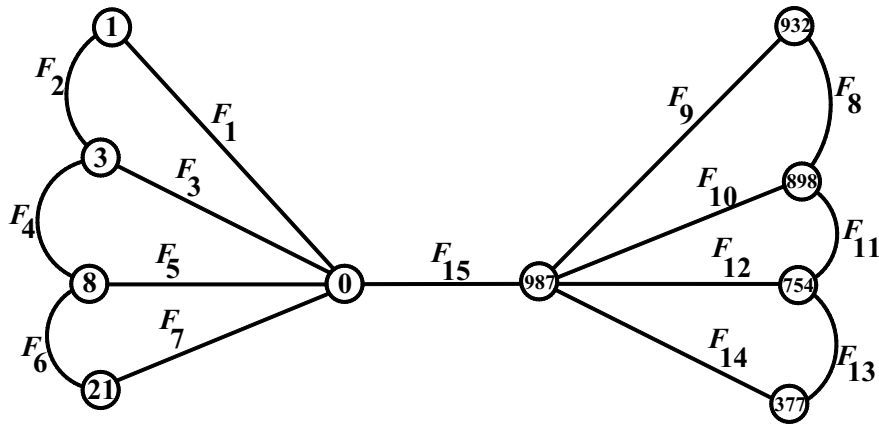


FIGURE 5.4: Joint Sum of two copies of  $F_4$  and its Fibonacci graceful labeling

**Theorem 5.3.9.** The graph obtained by switching of a vertex in a cycle  $C_n$  is super Fibonacci graceful except for  $n \geq 6$ .

*Proof.* We consider here two cases.

**Case 1:**  $n = 3, 4, 5$

For  $n = 3$  the graph obtained by switching of a vertex is a disconnected graph which is

not desirable for the Fibonacci graceful labeling.

Super Fibonacci graceful labeling of switching of a vertex in  $C_n$  for  $n = 4, 5$  is as shown in *Figure 5.5*.

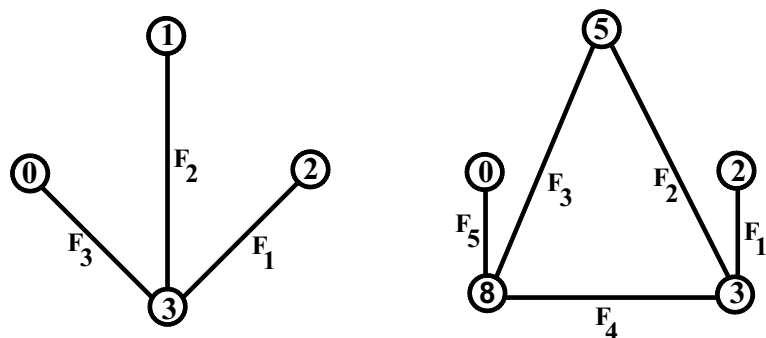


FIGURE 5.5: Super Fibonacci graceful labeling of switching of a vertex in  $C_4$  and  $C_5$

**Case 2:**  $n \geq 6$  The graph shown in *Figure 5.6* will be the subgraph of all the graphs obtained by switching of a vertex in  $C_n (n \geq 6)$ .

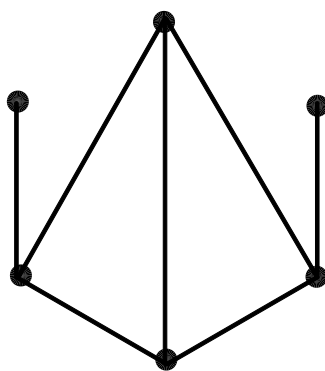


FIGURE 5.6

In *Figure 5.7* all the possible assignment of vertex labels is shown which demonstrates the repetition of edge labels.

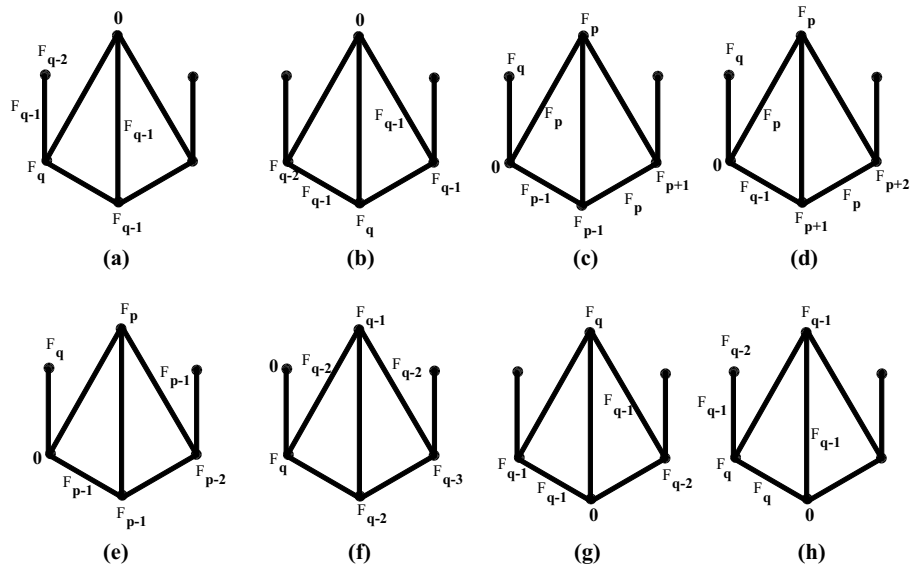


FIGURE 5.7

(1) In *Fig8(a)* edge label  $F_{q-1}$  is repeated as

$$|F_q - F_{q-2}| = F_{q-1} \text{ \& } |F_{q-1} - 0| = F_{q-1}$$

(2) In *Fig8(b)* edge label  $F_{q-1}$  is repeated as

$$|F_q - F_{q-2}| = F_{q-1} \text{ \& } |F_{q-1} - 0| = F_{q-1}$$

(3) In *Fig8(c)* edge label  $F_p$  is repeated as  $|F_{p+1} - F_{p-1}| = F_p$  \&  $|F_p - 0| = F_p$ , where  $F_p$  is any Fibonacci number.

(4) In *Fig8(d)* edge label  $F_p$  is repeated as  $|F_{p+2} - F_{p+1}| = F_p$  \&  $|F_p - 0| = F_p$ , where  $F_p$  is any Fibonacci number.

(5) In *Fig8(e)* edge label  $F_{p-1}$  is repeated as  $|F_p - F_{p-2}| = F_{p-1}$  \&  $|F_{p-1} - 0| = F_{p-1}$ , where  $F_p$  is any Fibonacci number.

(6) In *Fig8(f)* edge label  $F_{q-2}$  is repeated as

$$|F_{q-1} - F_{q-3}| = F_{q-2} \text{ \& } |F_q - F_{q-1}| = F_{q-2}$$

(7) In *Fig8(g)* edge label  $F_{q-1}$  is repeated as

$$|F_q - F_{q-2}| = F_{q-1} \text{ \& } |F_{q-1} - 0| = F_{q-1}$$

(8) In *Fig8(h)* edge label  $F_{q-1}$  is repeated as

$$|F_q - F_{q-2}| = F_{q-1} \text{ \& } |F_{q-1} - 0| = F_{q-1}$$

□

**Theorem 5.3.10.** Switching of a vertex in cycle  $C_n$  for  $n \geq 6$  can be embedded as an induced subgraph of a super Fibonacci graceful graph.

*Proof.* Let  $v_1, v_2, v_3, \dots, v_n$  be the vertices of  $C_n$  and  $v_1$  be the switched vertex.

Define  $f : V(G_{v_1}) \longrightarrow \{0, F_1, F_2, \dots, F_{q+3}\}$

$$f(v_1) = 0$$

$$f(v_{i+1}) = F_{2i-1}, 1 \leq i \leq n-1$$

Now it remains to assign Fibonacci numbers  $F_1, F_{q+2}$  and  $F_{q+3}$ . Put 3 vertices in the graph. Join first vertex  $v'$  labeled with  $F_2$  to the vertex  $v_3$ . Now join second vertex  $v''$  labeled with  $F_{q+3}$  to the vertex  $v_1$  and vertex  $v'''$  labeled with  $F_{q+2}$  to the vertex  $v''$ .

Thus the resultant graph is a super Fibonacci graceful graph.  $\square$

**Illustration 5.3.11.** In the following Figure 5.8 the graph obtained by switching of a vertex in cycle  $C_6$  and its super Fibonacci graceful labeling of its embedding is shown.

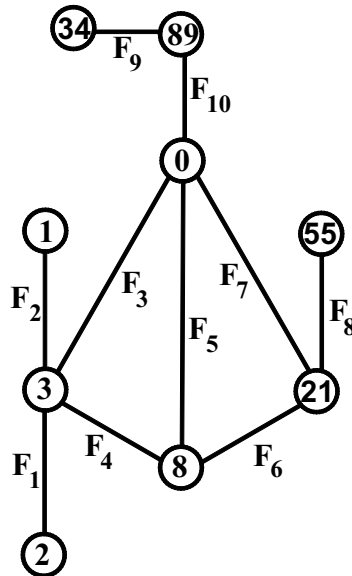


FIGURE 5.8: Super Fibonacci graceful labeling of embedding of switching of a vertex in cycle  $C_6$

## 5.4 Concluding Remarks

The work reported here is a nice combination of graph theory and elementary number theory. To investigate some more graphs or graph families of Fibonacci graceful graphs as well as to derive some characterizations for Fibonacci graceful graph is an open area or research.

The pen ultimate chapter also possess the same flavour.

## **Chapter 6**

# **Triangular Sum Labeling of Graphs**

## 6.1 Introduction

This chapter is intended to discuss triangular sum labeling of graphs. We will show that some classes of graph can be embedded as an induced subgraphs of a triangular sum graph. In the succeeding section we will provide brief summary of definitions which are necessary for the subsequent development.

## 6.2 Triangular sum labeling

### 6.2.1 Triangular number

A *triangular number* is a number obtained by adding all positive integers less than or equal to a given positive integer  $n$ . If  $n^{\text{th}}$  *triangular number* is denoted by  $T_n$  then  $T_n = \frac{1}{2}n(n+1)$ . It is easy to observe that there does not exist consecutive integers which are *triangular numbers*.

### 6.2.2 Triangular sum graph

A *triangular sum labeling* of a graph  $G$  is a one-to-one function

$f : V \rightarrow N$  ( where  $N$  is the set of all non-negative integers) that induces a bijection

$f^+ : E(G) \rightarrow \{T_1, T_2, \dots, T_q\}$  of the edges of  $G$  defined by  $f^+(uv) = f(u) + f(v)$ ,

$\forall e = uv \in E(G)$ .

The graph which admits such labeling is called a *triangular sum graph*.

### 6.2.3 Some existing results

This concept was introduced by Hegde and Shankaran [39] and they proved that

- Path  $P_n$ , Star  $K_{1,n}$  are triangular sum graphs.

- Any tree obtained from the star  $K_{1,n}$  by replacing each edge by a path is a triangular sum graph.
- The lobster  $T$  obtained by joining the centers of  $k$  copies of a star to a new vertex  $w$  is a triangular sum graph.
- The complete  $n$ -ary tree  $T_m$  of level  $m$  is a triangular sum graph.
- The complete graph  $K_n$  is triangular sum if and only if  $n \leq 2$ .

They also shown that

- If  $G$  is an Eulerian  $(p, q)$ -graph admitting a triangular sum labeling then  $q \not\equiv 1 \pmod{12}$ .
- The dutch windmill  $DW(n)$  ( $n$  copies of  $K_3$  sharing a common vertex) is not a triangular sum graph.
- The complete graph  $K_4$  can be embedded as an induced subgraph of a triangular sum graph.

In a paper by Vaidya et al.[79] it has been shown that

- In any triangular sum graph  $G$  the vertices with labels 0 and 1 are always adjacent.
- In any triangular sum graph  $G$ , 0 and 1 cannot be the vertex labels in the same triangle contained in  $G$ .
- In any triangular sum graph  $G$ , 1 and 2 cannot be the vertex labels of the same triangle contained in  $G$ .
- The helm graph  $H_n$  is not a triangular sum graph.
- If every edge of a graph  $G$  is an edge of a triangle then  $G$  is not a triangular sum graph.

### 6.3 Some important results on triangular sum graphs

**Theorem 6.3.1.** Every cycle can be embedded as an induced subgraph of a triangular sum graph.

*Proof.* Let  $G = C_n$  be a cycle with  $n$  vertices. We define labeling  $f : V(G) \rightarrow N$  as follows such that the induced function  $f^+ : E(G) \rightarrow \{T_1, T_2, \dots, T_q\}$  is bijective.

$$f(v_1) = 0$$

$$f(v_2) = 6$$

$$f(v_i) = T_{i+2} - f(v_{i-1}); \quad 3 \leq i \leq n-1$$

$$f(v_n) = T_{f(v_{n-1})-1}$$

Now let  $A = \{T_1, T_2, \dots, T_r\}$  be the set of missing edge labels. That is, elements of set  $A$  are the missing triangular numbers between 1 and  $T_{f(v_{n-1})-1}$ . Now add  $r$  pendant vertices which are adjacent to the vertex with label 0 and label these new vertices with labels  $T_1, T_2, \dots, T_r$ . This construction will give rise to edges with labels  $T_1, T_2, \dots, T_r$  such that the resultant supergraph  $H$  admits triangular sum labeling. Thus we proved that every cycle can be embedded as an induced subgraph of a triangular sum graph.  $\square$

**Illustration 6.3.2.** In the following Figure 6.1 embedding of  $C_5$  as an induced subgraph of a triangular sum graph is shown.

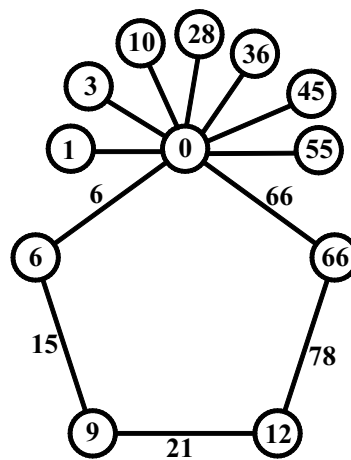


FIGURE 6.1: Embedding of  $C_5$  as an induced subgraph of a triangular sum graph

**Theorem 6.3.3.** Every cycle with one chord can be embedded as an induced subgraph of a triangular sum graph.

*Proof.* Let  $G$  be the cycle with one chord and  $e = v_1v_k$  be the chord of cycle  $C_n$ .

We define labeling  $f : V(G) \rightarrow N$  as follows such that the induced function

$f^+ : E(G) \rightarrow \{T_1, T_2, \dots, T_q\}$  is bijective.

$$f(v_1) = 0$$

$$f(v_2) = 6$$

$$f(v_i) = T_{i+2} - f(v_{i-1}); \quad 3 \leq i \leq k-1$$

$$f(v_k) = T_{f(v_{k-1})-1}$$

$$f(v_{k+i-1}) = T_{f(v_{k-1})-1+i} - f(v_{k+i-2}); \quad 2 \leq i \leq n-k$$

$$f(v_n) = T_{f(v_{n-1})-1}$$

Now following the procedure described in Theorem 6.3.1 and the resultant supergraph  $H$  admits triangular sum labeling. Thus we proved that every cycle with one chord can be embedded as an induced subgraph of a triangular sum graph.  $\square$

**Illustration 6.3.4.** In the following Figure 6.2 embedding of  $C_4$  with one chord as an induced subgraph of a triangular sum graph is shown.

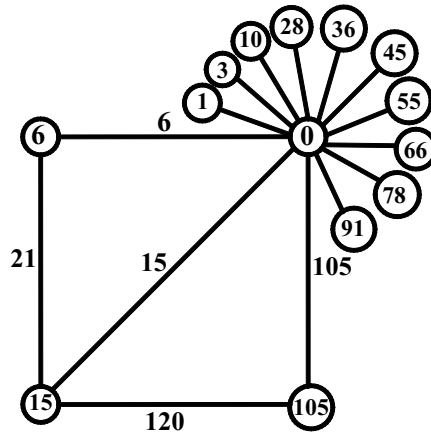


FIGURE 6.2: Embedding of  $C_4$  with one chord as an induced subgraph of a triangular sum graph

**Theorem 6.3.5.** Every cycle with twin chords can be embedded as an induced subgraph of a triangular sum graph.

*Proof.* Let  $G$  be the cycle with twin chords and  $e_1 = v_1v_k$  and  $e_2 = v_1v_{k+1}$  be its chords.

We define labeling  $f : V(G) \rightarrow N$  such that the induced function

$f^+ : E(G) \rightarrow \{T_1, T_2, \dots, T_q\}$  is bijective.

$$f(v_1) = 0$$

$$f(v_2) = 6$$

$$f(v_i) = T_{i+2} - f(v_{i-1}); \quad 3 \leq i \leq k-1$$

$$f(v_k) = T_{f(v_{k-1})-1}$$

$$f(v_{k+1}) = T_{f(v_k)-1}$$

$$f(v_{k+i}) = T_{f(v_k)-1+i} - f(v_{k+i-1}); \quad 2 \leq i \leq n-k-1$$

$$f(v_n) = T_{f(v_{n-1})-1}$$

Now following the procedure adapted in Theorem 6.3.1 the resulting supergraph  $H$  admits triangular sum labeling. That is, every cycle with twin chords can be embedded as an induced subgraph of a triangular sum graph.  $\square$

**Illustration 6.3.6.** In the following *Figure 6.3* embedding of  $C_6$  with twin chord as an induced subgraph of a triangular sum graph is shown.

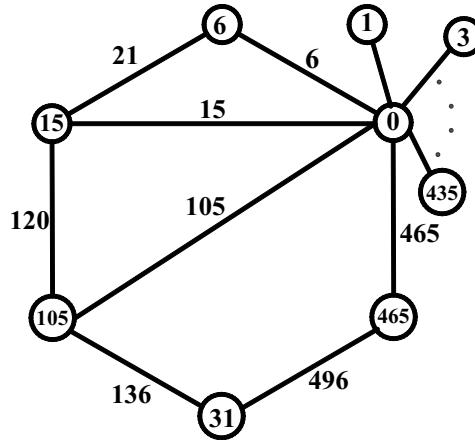


FIGURE 6.3: Embedding of  $C_6$  with twin chord as an induced subgraph of a triangular sum graph

## 6.4 Concluding Remarks

As every graph is not a triangular sum graph it is very interesting to investigate graphs or graph families which are not triangular sum graphs but they can be embedded as an induced subgraph of a triangular sum graph. We show that cycle, cycle with one chord and cycle with twin chords can be embedded as an induced subgraph of a triangular sum graph.

The next chapter is focused on  $L(2,1)$  and Radio labeling of graphs.

# **Chapter 7**

## **L(2,1)-Labeling**

**&**

## **Radio Labeling**

## 7.1 Introduction

The unprecedented growth of different modes of communication provided the reason for many real life problems. The allocation of radio channels or frequencies to radio transmitters network is one such problem and it is the focus of our investigations. The present chapter is aimed to discuss  $L(2,1)$ -labeling and Radio labeling of graphs.

## 7.2 Channel assignment problem

The channel assignment problem is the problem to assign a channel (non negative integer) to each TV or radio transmitters located at various places such that communication do not interfere. This problem was first formulated as a graph coloring problem by Hale[36] who introduced the notion of T-coloring of a graph.

In a graph model of this problem, the transmitters are represented by the vertices of a graph; two vertices are *very close* if they are adjacent in the graph and *close* if they are at distance two apart in the graph.

In a private communication with Griggs during 1988 Roberts proposed a variation of the channel assignment problem in which *close* transmitters must receive different channels and *very close* transmitters must receive channels that are at least two apart. Motivated by this problem Griggs and Yeh[34] introduced  $L(2,1)$ -labeling which is defined as follows.

## 7.3 $L(2,1)$ - Labeling and $L'(2,1)$ - Labeling

### 7.3.1 $L(2,1)$ - Labeling and $\lambda$ -number

For a graph  $G$ ,  $L(2,1)$ -labeling (or *distance two labeling*) with span  $k$  is a function  $f : V(G) \longrightarrow \{0, 1, \dots, k\}$  such that the following conditions are satisfied:

$$(1) |f(x) - f(y)| \geq 2 \text{ if } d(x, y) = 1$$

$$(2) |f(x) - f(y)| \geq 1 \text{ if } d(x, y) = 2$$

In other words the  $L(2, 1)$ -labeling of a graph is an abstraction of assigning integer frequencies to radio transmitters such that (1) Transmitters that are one unit of distance apart receive frequencies that differ by at least two and (2) Transmitters that are two units of distance apart receive frequencies that differ by at least one. The *span* of  $f$  is the largest number in  $f(V)$ . The minimum span taken over all  $L(2, 1)$ -labeling of  $G$ , denoted as  $\lambda(G)$  is called the  $\lambda$ -number of  $G$ . The minimum label in  $L(2, 1)$ -labeling of  $G$  is assumed to be 0.

### 7.3.2 $L'(2, 1)$ -labeling and $\lambda'$ -number

An injective  $L(2, 1)$ -labeling is called an  $L'(2, 1)$ -labeling and the minimum span taken over all such  $L'(2, 1)$ -labeling is called  $\lambda'$ -number of the graph.

### 7.3.3 Some existing results

- In [34] Griggs and Yeh have discussed  $L(2,1)$ -labeling for path, cycle, tree and cube. They also derived the relation between  $\lambda$ -number and other graph invariants of  $G$  such as chromatic number and the maximum degree. They have also shown that determining  $\lambda$ -number of a graph is an NP-Complete problem, even for graphs with diameter 2.
- Chang and Kuo [14] provided an algorithm to obtain  $\lambda(T)$ .
- Georges et al.[28, 83] have discussed  $L(2,1)$ -labeling of cartesian product of paths and  $n$ -cube.
- Georges and Mauro[29] proved that the  $\lambda$ -number of every generalized Petersen graph is bounded from above by 9.
- Kuo and Yan [50] have discussed  $L(2,1)$ -labeling of cartesian product of paths and cycles.

- Vaidya and Bantva[69] have discussed  $L(2,1)$ -labeling of middle graphs.
- Vaidya and Bantava[70] have discussed  $L(2,1)$ -labeling of cacti.
- Jha et al.[41] have discussed  $L(2,1)$ -labeling of direct product of paths and cycles.
- Chiang [17] studied  $L(d,1)$ -labeling for  $d \geq 2$  on the cartesian product of cycle and a path.

## 7.4 $L(2,1)$ -Labeling in the Context of Some Graph Operations

**Theorem 7.4.1.**  $\lambda(spl(C_n)) = 7$ . (where  $n > 3$ )

*Proof.* Let  $v'_1, v'_2, \dots, v'_n$  be the duplicated vertices corresponding to  $v_1, v_2, \dots, v_n$  of cycle  $C_n$ .

To define  $f : V(spl(C_n)) \rightarrow N \cup \{0\}$  we consider following four cases.

**Case 1:**  $n \equiv 0(mod 3)$  (where  $n > 5$ )

We label the vertices as follows.

$$f(v_i) = 0, i = 3j - 2, \quad 1 \leq j \leq \frac{n}{3}$$

$$f(v_i) = 2, i = 3j - 1, \quad 1 \leq j \leq \frac{n}{3}$$

$$f(v_i) = 4, i = 3j, \quad 1 \leq j \leq \frac{n}{3}$$

$$f(v'_i) = 7, i = 3j - 2, \quad 1 \leq j \leq \frac{n}{3}$$

$$f(v'_i) = 6, i = 3j - 1, \quad 1 \leq j \leq \frac{n}{3}$$

$$f(v'_i) = 5, i = 3j, \quad 1 \leq j \leq \frac{n}{3}$$

**Case 2:**  $n \equiv 1(mod 3)$  (where  $n > 5$ )

We label the vertices as follows.

$$f(v_i) = 0, i = 3j - 2, \quad 1 \leq j \leq \left\lfloor \frac{n}{3} \right\rfloor - 1$$

$$f(v_i) = 2, i = 3j - 1, \quad 1 \leq j \leq \left\lfloor \frac{n}{3} \right\rfloor - 1$$

$$\begin{aligned}
f(v_i) &= 4, i = 3j, & 1 \leq j \leq \lfloor \frac{n}{3} \rfloor - 1 \\
f(v_{n-3}) &= 0, f(v_{n-2}) = 3, f(v_{n-1}) = 1, f(v_n) = 4 \\
f(v'_i) &= 7, i = 3j - 2, & 1 \leq j \leq \lfloor \frac{n}{3} \rfloor - 1 \\
f(v'_i) &= 6, i = 3j - 1, & 1 \leq j \leq \lfloor \frac{n}{3} \rfloor - 1 \\
f(v'_i) &= 5, i = 3j, & 1 \leq j \leq \lfloor \frac{n}{3} \rfloor - 1 \\
f(v'_{n-3}) &= 7, f(v'_{n-2}) = 7, f(v'_{n-1}) = 6, f(v'_n) = 5
\end{aligned}$$

**Case 3:**  $n \equiv 2(mod 3)$  (where  $n > 5$ )

We label the vertices as follows.

$$\begin{aligned}
f(v_i) &= 0, i = 3j - 2, & 1 \leq j \leq \lfloor \frac{n}{3} \rfloor \\
f(v_i) &= 2, i = 3j - 1, & 1 \leq j \leq \lfloor \frac{n}{3} \rfloor \\
f(v_i) &= 4, i = 3j, & 1 \leq j \leq \lfloor \frac{n}{3} \rfloor \\
f(v_{n-1}) &= 1, f(v_n) = 3 \\
f(v'_1) &= 6, f(v'_n) = 7 \\
f(v'_i) &= 6, i = 3j - 1, & 1 \leq j \leq \lfloor \frac{n}{3} \rfloor \\
f(v'_i) &= 5, i = 3j, & 1 \leq j \leq \lfloor \frac{n}{3} \rfloor \\
f(v'_i) &= 7, i = 3j + 1, & 1 \leq j \leq \lfloor \frac{n}{3} \rfloor
\end{aligned}$$

**Case 4:**  $n = 4, 5$

These cases are to be dealt separately. The  $L(2,1)$ -labeling for  $spl(C_n)$  when  $n = 4, 5$  are as shown in *Figure 7.1*

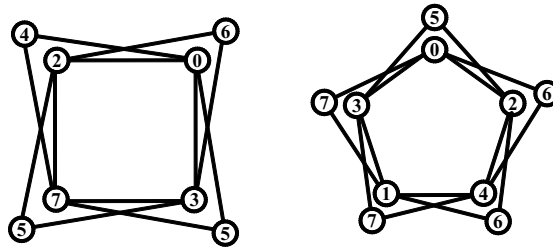


FIGURE 7.1:  $spl(C_4), spl(C_5)$  and its  $L(2,1)$ -labeling

Thus in all the possibilities  $R_f = \{0, 1, 2, \dots, 7\} \subset N \cup \{0\}$ .

i.e.  $\lambda(spl(C_n)) = 7$ .

□

**Remark :** The  $L(2,1)$ -labeling for  $spl(C_3)$  is shown in Figure 7.2

Thus  $R_f = \{0, 1, 2, \dots, 6\} \subset \mathbb{N} \cup \{0\}$ .

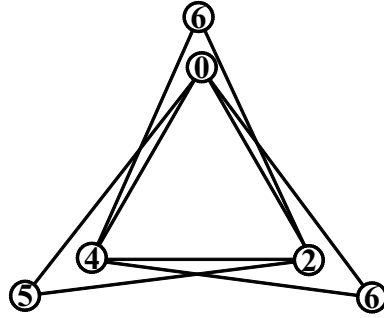


FIGURE 7.2:  $spl(C_3)$  and its  $L(2,1)$ -labeling

**Illustration 7.4.2.** Consider the graph  $spl(C_6)$ . The  $L(2,1)$ -labeling is as shown in Figure 7.3.

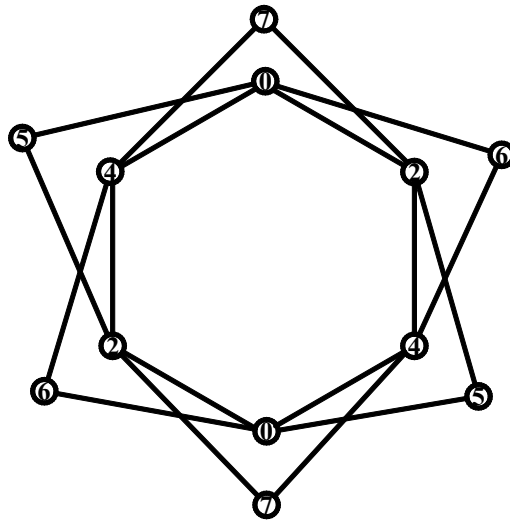


FIGURE 7.3:  $spl(C_6)$  and its  $L(2,1)$ -labeling

**Theorem 7.4.3.**  $\lambda'(spl(C_n)) = p - 1$ , where  $p$  is a total number vertices in  $spl(C_n)$  (where  $n > 3$ ).

*Proof.* Let  $v'_1, v'_2, \dots, v'_n$  be the duplicated vertices corresponding to  $v_1, v_2, \dots, v_n$  of cycle  $C_n$ .

To define  $f : V(spl(C_n)) \rightarrow \mathbb{N} \cup \{0\}$ , we consider following two cases.

**Case 1:**  $n > 5$

$$f(v_i) = 2i - 7, \quad 4 \leq i \leq n$$

$$f(v_i) = f(v_n) + 2i, \quad 1 \leq i \leq 3$$

$$f(v'_i) = 2i - 2, \quad 1 \leq i \leq n$$

Now label the vertices of  $C'_n$  using the above defined pattern we have

$$R_f = \{0, 1, 2, \dots, p-1\} \subset N \cup \{0\}$$

This implies that  $\lambda'(spl(C_n)) = p-1$ .

**Case 2:**  $n = 4, 5$  These cases to be dealt separately. The  $L'(2,1)$ -labeling for  $spl(C_n)$  when  $n = 4, 5$  are as shown in the following Figure 7.4.  $\square$

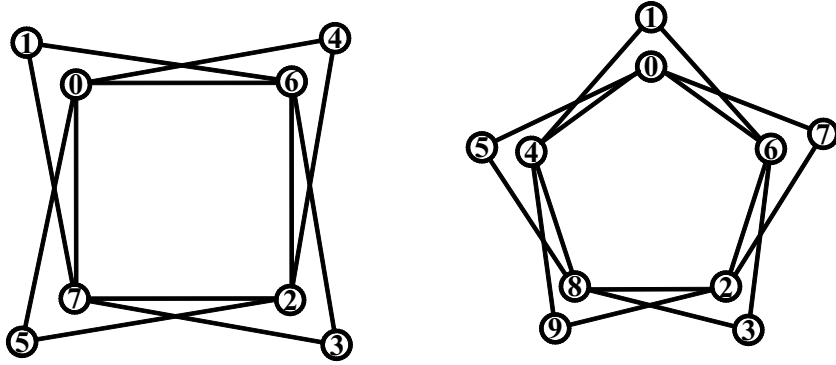


FIGURE 7.4:  $spl(C_4), spl(C_5)$  and its  $L'(2,1)$ -labeling

**Remark** The  $L'(2,1)$ -labeling for  $spl(C_3)$  is shown in the following Figure 7.5.

Thus  $R_f = \{0, 1, 2, \dots, 6\} \subset N \cup \{0\}$ .

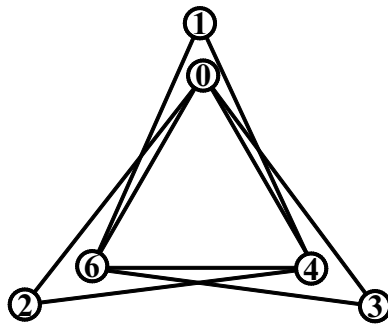


FIGURE 7.5:  $spl(C_3)$  and its  $L'(2,1)$ -labeling

**Illustration 7.4.4.** Consider the graph  $spl(C_6)$ . The  $L'(2,1)$ -labeling is as shown in Figure 7.6.

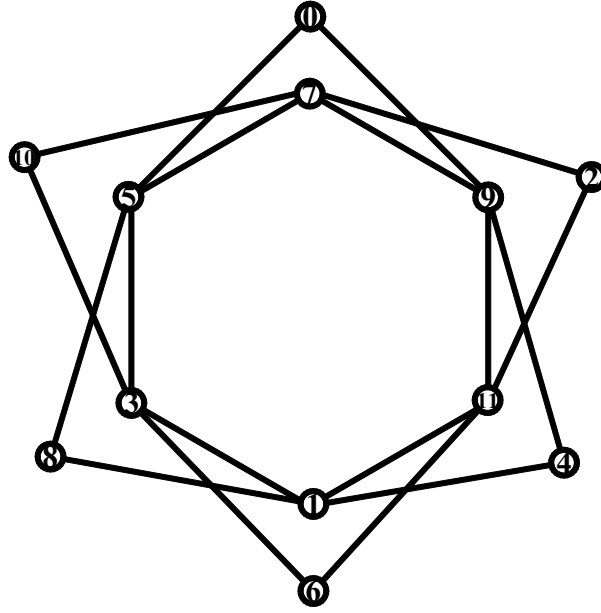


FIGURE 7.6:  $spl(C_6)$  and its  $L'(2,1)$ -labeling

**Theorem 7.4.5.** Let  $C'_n$  be the graph obtained by taking arbitrary supersubdivision of each edge of cycle  $C_n$  then

1 For  $n$  even

$$\lambda(C'_n) = \Delta + 2$$

2 For  $n$  odd

$$\lambda(C'_n) = \begin{cases} \Delta + 2; & \text{if } s + t + r < \Delta, \\ \Delta + 3; & \text{if } s + t + r = \Delta, \\ s + t + r + 2; & \text{if } s + t + r > \Delta \end{cases}$$

where  $v_k$  is a vertex with label 2,

$s$  is number of subdivision between  $v_{k-2}$  and  $v_{k-1}$ ,

$t$  is number of subdivision between  $v_{k-1}$  and  $v_k$ ,

$r$  is number of subdivision between  $v_k$  and  $v_{k+1}$ ,

$\Delta$  is the maximum degree of  $C'_n$ .

*Proof.* Let  $v_1, v_2, \dots, v_n$  be the vertices of cycle  $C_n$ . Let  $C'_n$  be the graph obtained by arbitrary super subdivision of cycle  $C_n$ .

It is obvious that for any two vertices  $v_i$  and  $v_{i+2}$ ,  $N(v_i) \cap N(v_{i+2}) = \emptyset$

To define  $f : V(C'_n) \longrightarrow N \cup \{0\}$ , we consider following two cases.

**Case 1:**  $n$  is even

$$f(v_{2i-1}) = 0, \quad 1 \leq i \leq \frac{n}{2}$$

$$f(v_{2i}) = 1, \quad 1 \leq i \leq \frac{n}{2}$$

If  $P_{ij}$  is the number of supersubdivisions between  $v_i$  and  $v_j$  then for the vertex  $v_1$ ,  $|N(v_1)| = P_{12} + P_{n1}$ . Without loss of generality we assume that  $v_1$  is the vertex with maximum degree i.e.  $d(v_1) = \Delta$ . suppose  $u_1, u_2, \dots, u_\Delta$  be the members of  $N(v_1)$ . We label the vertices of  $N(v_1)$  as follows.

$$f(u_i) = 2 + i, \quad 1 \leq i \leq \Delta$$

As  $N(v_1) \cap N(v_3) = \emptyset$  then it is possible to label the vertices of  $N(v_3)$  using the vertex labels of the members of  $N(v_1)$  in accordance with the requirement for  $L(2,1)$ -labeling. Extending this argument recursively upto  $N(v_{n-1})$  it is possible to label all the vertices of  $C'_n$  using the distinct numbers between 0 and  $\Delta + 2$ .

$$\text{i.e. } R_f = \{0, 1, 2, \dots, \Delta + 2\} \subset N \cup \{0\}$$

Consequently  $\lambda(C'_n) = \Delta + 2$ .

**Case 2:**  $n$  is odd

Let  $v_1, v_2, \dots, v_n$  be the vertices of cycle  $C_n$ .

Without loss of generality we assume that  $v_1$  is a vertex with maximum degree and  $v_k$  be the vertex with minimum degree.

Define  $f(v_k) = 2$  and label the remaining vertices alternatively with labels 0 and 1 such that  $f(v_1) = 0$ . Then either  $f(v_{k-1}) = 1$  ;  $f(v_{k+1}) = 0$  OR  $f(v_{k-1}) = 0$  ;  $f(v_{k+1}) = 1$ . We assign labeling in such a way that  $f(v_{k-1}) = 1$  ;  $f(v_{k+1}) = 0$ .

Now following the procedure adapted in case (1) it is possible to label all the vertices except the vertices between  $v_{k-1}$  and  $v_k$ . Label the vertices between  $v_{k-1}$  and  $v_k$  using

the vertex labels of  $N(v_1)$  except the labels which are used earlier to label the vertices between  $v_{k-2}$ ,  $v_{k-1}$  and between  $v_k$ ,  $v_{k+1}$ .

If there are  $p$  vertices  $u_1, u_2 \dots u_p$  are left unlabeled between  $v_{k-1}$  and  $v_k$  then label them as follows,

$$f(u_i) = \max\{\text{labels of the vertices between } v_{k-2} \text{ and } v_{k-1}, \text{ labels of the vertices between } v_k \text{ and } v_{k+1}\} + i, 1 \leq i \leq p$$

Now if  $s$  is the number of subdivisions between  $v_{k-2}$  and  $v_{k-1}$

$t$  is the number of subdivisions between  $v_{k-1}$  and  $v_k$

$r$  is the number of subdivisions between  $v_k$  and  $v_{k+1}$

then (1)  $R_f = \{0, 1, 2, \dots, \Delta + 2\} \subset N \cup \{0\}$ , when  $s + t + r < \Delta$

i.e.  $\lambda(C'_n) = \Delta + 2$

(2)  $R_f = \{0, 1, 2, \dots, \Delta + 3\} \subset N \cup \{0\}$ , when  $s + t + r = \Delta$

i.e.  $\lambda(C'_n) = \Delta + 3$

(3)  $R_f = \{0, 1, 2, \dots, s + t + r + 2\} \subset N \cup \{0\}$ , when  $s + t + r > \Delta$

i.e.  $\lambda(C'_n) = s + t + r + 2$

□

**Illustration 7.4.6.** Consider the graph  $C_8$ . The  $L(2,1)$ -labeling of  $C'_8$  is shown in Figure 7.7.

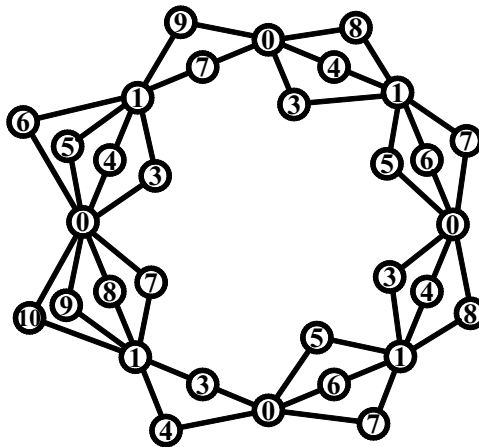


FIGURE 7.7:  $L(2,1)$ -labeling of  $C'_8$

**Theorem 7.4.7.** Let  $G'$  be the graph obtained by taking arbitrary supersubdivision of each edge of graph  $G$  with number of vertices  $n \geq 3$  then  $\lambda'(G') = p - 1$ , where  $p$  is the total number of vertices in  $G'$ .

*Proof.* Let  $v_1, v_2, \dots, v_n$  be the vertices of any connected graph  $G$  and let  $G'$  be the graph obtained by taking arbitrary supersubdivision of  $G$ . Let  $u_k$  be the vertices which are used for arbitrary supersubdivision of the edge  $v_i v_j$  where  $1 \leq i \leq n, 1 \leq j \leq n$  and  $i < j$ . Here  $k$  is a total number of vertices used for arbitrary supersubdivision.

We define  $f : V(G') \rightarrow N \cup \{0\}$  as

$$f(v_i) = i - 1, \text{ where } 1 \leq i \leq n$$

Now we label the vertices  $u_i$  in the following order.

First we label the vertices between  $v_1$  and  $v_{1+j}$ ,  $1 \leq j \leq n$  then following the same procedure for  $v_2, v_3, \dots, v_n$

$$f(u_i) = f(v_n) + i, 1 \leq i \leq k$$

Now label the vertices of  $G'$  using the above defined pattern we have  $R_f = \{0, 1, 2, \dots, p - 1\} \subset N \cup \{0\}$

This implies that  $\lambda'(G') = p - 1$ . □

**Illustration 7.4.8.** Consider the graph  $P_4$  and its supersubdivision. The  $L'(2,1)$ -labeling is as shown in Figure 7.8.

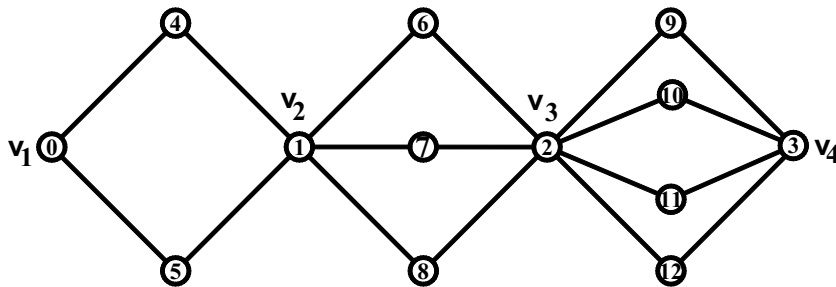


FIGURE 7.8:  $L'(2,1)$ -labeling of  $P'_4$

**Theorem 7.4.9.** Let  $C'_n$  be the graph obtained by taking star of a cycle  $C_n$  then  $\lambda(C'_n) = 5$ .

*Proof.* Let  $v_1, v_2, \dots, v_n$  be the vertices of cycle  $C_n$  and  $v_{ij}$  be the vertices of cycle  $C_n$  which are adjacent to the  $i^{th}$  vertex of cycle  $C_n$ .

To define  $f : V(C'_n) \longrightarrow N \cup \{0\}$ , we consider following four cases.

**Case 1:**  $n \equiv 0(\text{mod } 3)$

$$f(v_i) = 0, \quad i = 3j - 2, \quad 1 \leq j \leq \frac{n}{3}$$

$$f(v_i) = 2, \quad i = 3j - 1, \quad 1 \leq j \leq \frac{n}{3}$$

$$f(v_i) = 4, \quad i = 3j, \quad 1 \leq j \leq \frac{n}{3}$$

Now we label the vertices  $v_{ij}$  of star of a cycle according to the label of  $f(v_i)$ .

(1) when  $f(v_i) = 0, i = 3j - 2, 1 \leq j \leq \frac{n}{3}$

$$f(v_{ik}) = 3, \quad k = 3p - 2, \quad 1 \leq p \leq \frac{n}{3}$$

$$f(v_{ik}) = 5, \quad k = 3p - 1, \quad 1 \leq p \leq \frac{n}{3}$$

$$f(v_{ik}) = 1, \quad k = 3p, \quad 1 \leq p \leq \frac{n}{3}$$

(2) when  $f(v_i) = 2, i = 3j - 1, 1 \leq j \leq \frac{n}{3}$

$$f(v_{ik}) = 5, \quad k = 3p - 2, \quad 1 \leq p \leq \frac{n}{3}$$

$$f(v_{ik}) = 3, \quad k = 3p - 1, \quad 1 \leq p \leq \frac{n}{3}$$

$$f(v_{ik}) = 1, \quad k = 3p, \quad 1 \leq p \leq \frac{n}{3}$$

(3) when  $f(v_i) = 4, i = 3j, 1 \leq j \leq \frac{n}{3}$

$$f(v_{ik}) = 1, \quad k = 3p - 2, \quad 1 \leq p \leq \frac{n}{3}$$

$$f(v_{ik}) = 3, \quad k = 3p - 1, \quad 1 \leq p \leq \frac{n}{3}$$

$$f(v_{ik}) = 5, \quad k = 3p, \quad 1 \leq p \leq \frac{n}{3}$$

**Case 2:**  $n \equiv 1(\text{mod } 3)$

$$f(v_i) = 0, \quad i = 3j - 2, \quad 1 \leq j \leq \lfloor \frac{n}{3} \rfloor$$

$$f(v_i) = 2, \quad i = 3j - 1, \quad 1 \leq j \leq \lfloor \frac{n}{3} \rfloor$$

$$f(v_i) = 5, \quad i = 3j, \quad 1 \leq j \leq \lfloor \frac{n}{3} \rfloor$$

$$f(v_n) = 3$$

Now we label the vertices of star of a cycle  $v_{ij}$  according to label of  $f(v_i)$ .

(1) when  $f(v_i) = 0, i = 3j - 2, 1 \leq j \leq \lfloor \frac{n}{3} \rfloor$

$$\begin{aligned} f(v_{ik}) &= 4, & k &= 3p - 2, & 1 \leq p &\leq \lfloor \frac{n}{3} \rfloor \\ f(v_{ik}) &= 2, & k &= 3p - 1, & 1 \leq p &\leq \lfloor \frac{n}{3} \rfloor \\ f(v_{ik}) &= 0, & k &= 3p, & 1 \leq p &\leq \lfloor \frac{n}{3} \rfloor - 1 \\ f(v_{i(n-1)}) &= 5, \\ f(v_{in}) &= 1 \end{aligned}$$

(2) when  $f(v_i) = 2, i = 3j - 1, 1 \leq j \leq \lfloor \frac{n}{3} \rfloor$

$$\begin{aligned} f(v_{ik}) &= 4, & k &= 3p - 2, & 1 \leq p &\leq \lfloor \frac{n}{3} \rfloor \\ f(v_{ik}) &= 0, & k &= 3p - 1, & 1 \leq p &\leq \lfloor \frac{n}{3} \rfloor \\ f(v_{ik}) &= 2, & k &= 3p, & 1 \leq p &\leq \lfloor \frac{n}{3} \rfloor - 1 \\ f(v_{i(n-1)}) &= 3, \\ f(v_{in}) &= 1 \end{aligned}$$

(3) when  $f(v_i) = 5, i = 3j, 1 \leq j \leq \lfloor \frac{n}{3} \rfloor$

$$\begin{aligned} f(v_{ik}) &= 1, & k &= 3p - 2, & 1 \leq p &\leq \lfloor \frac{n}{3} \rfloor \\ f(v_{ik}) &= 3, & k &= 3p - 1, & 1 \leq p &\leq \lfloor \frac{n}{3} \rfloor \\ f(v_{ik}) &= 5, & k &= 3p, & 1 \leq p &\leq \lfloor \frac{n}{3} \rfloor - 1 \\ f(v_{i(n-1)}) &= 0, \\ f(v_{in}) &= 4 \end{aligned}$$

(4) when  $f(v_i) = 3, i = n$

$$\begin{aligned} f(v_{ik}) &= 1, & k &= 3p - 2, & 1 \leq p &\leq \lfloor \frac{n}{3} \rfloor \\ f(v_{ik}) &= 5, & k &= 3p - 1, & 1 \leq p &\leq \lfloor \frac{n}{3} \rfloor \\ f(v_{ik}) &= 3, & k &= 3p, & 1 \leq p &\leq \lfloor \frac{n}{3} \rfloor - 1 \\ f(v_{i(n-1)}) &= 0 \\ f(v_{in}) &= 4 \end{aligned}$$

**Case 3:**  $n \equiv 2(\text{mod } 3)$ ,  $n \neq 5$

$$\begin{aligned}
 f(v_i) &= 1, & i &= 3j - 2, \quad 1 \leq j \leq \lfloor \frac{n}{3} \rfloor - 1 \\
 f(v_i) &= 3, & i &= 3j - 1, \quad 1 \leq j \leq \lfloor \frac{n}{3} \rfloor - 1 \\
 f(v_i) &= 5, & i &= 3j, \quad 1 \leq j \leq \lfloor \frac{n}{3} \rfloor - 1 \\
 f(v_{n-4}) &= 0, f(v_{n-3}) = 2, \\
 f(v_{n-2}) &= 5, f(v_{n-1}) = 0, f(v_n) = 4
 \end{aligned}$$

Now we label the vertices  $v_{ij}$  of star of a cycle according to the label of  $f(v_i)$ .

**(1)** when  $f(v_i) = 1$ ,  $i = 3j - 2$ ,  $1 \leq j \leq \lfloor \frac{n}{3} \rfloor - 1$

$$\begin{aligned}
 f(v_{ik}) &= 5, & k &= 3p - 2, \quad 1 \leq p \leq \lfloor \frac{n}{3} \rfloor \\
 f(v_{ik}) &= 3, & k &= 3p - 1, \quad 1 \leq p \leq \lfloor \frac{n}{3} \rfloor \\
 f(v_{ik}) &= 1, & k &= 3p, \quad 1 \leq p \leq \lfloor \frac{n}{3} \rfloor \\
 f(v_{i(n-1)}) &= 4, \\
 f(v_{in}) &= 0
 \end{aligned}$$

**(2)** when  $f(v_i) = 3$ ,  $i = 3j - 1$ ,  $1 \leq j \leq \lfloor \frac{n}{3} \rfloor - 1$

$$\begin{aligned}
 f(v_{ik}) &= 0, & k &= 3p - 2, \quad 1 \leq p \leq \lfloor \frac{n}{3} \rfloor \\
 f(v_{ik}) &= 2, & k &= 3p - 1, \quad 1 \leq p \leq \lfloor \frac{n}{3} \rfloor \\
 f(v_{ik}) &= 4, & k &= 3p, \quad 1 \leq p \leq \lfloor \frac{n}{3} \rfloor \\
 f(v_{i(n-1)}) &= 1, \\
 f(v_{in}) &= 5
 \end{aligned}$$

**(3)** when  $f(v_i) = 5$ ,  $i = 3j$ ,  $1 \leq j \leq \lfloor \frac{n}{3} \rfloor - 1$  and  $i = n - 2$

$$\begin{aligned}
 f(v_{ik}) &= 1, & k &= 3p - 2, \quad 1 \leq p \leq \lfloor \frac{n}{3} \rfloor \\
 f(v_{ik}) &= 3, & k &= 3p - 1, \quad 1 \leq p \leq \lfloor \frac{n}{3} \rfloor \\
 f(v_{ik}) &= 5, & k &= 3p, \quad 1 \leq p \leq \lfloor \frac{n}{3} \rfloor \\
 f(v_{i(n-1)}) &= 2, \\
 f(v_{in}) &= 4
 \end{aligned}$$

(4) when  $f(v_i) = 0, i = n-4, n-1$

$$\begin{aligned} f(v_{ik}) &= 3, & k &= 3p-2, & 1 \leq p \leq \lfloor \frac{n}{3} \rfloor \\ f(v_{ik}) &= 5, & k &= 3p-1, & 1 \leq p \leq \lfloor \frac{n}{3} \rfloor \\ f(v_{ik}) &= 1, & k &= 3p, & 1 \leq p \leq \lfloor \frac{n}{3} \rfloor - 1 \\ f(v_{i(n-2)}) &= 2, \\ f(v_{i(n-1)}) &= 4, \\ f(v_{in}) &= 1 \end{aligned}$$

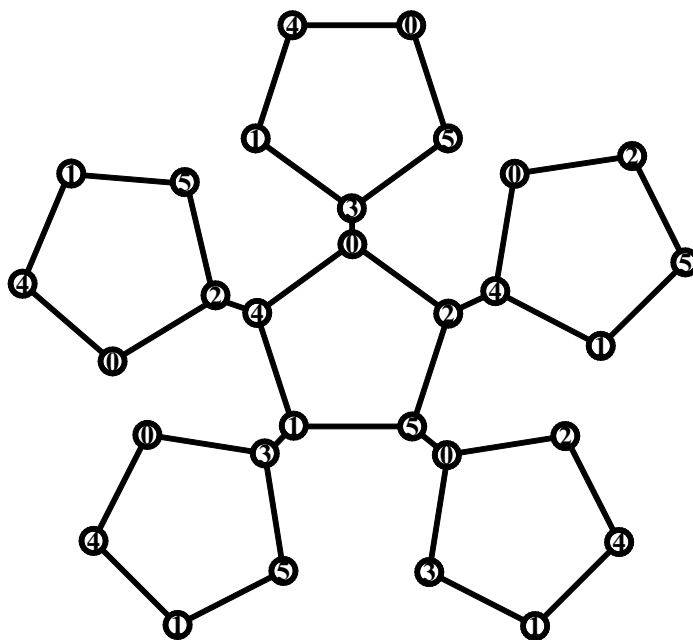
(5) when  $f(v_i) = 2, i = n-3$

$$\begin{aligned} f(v_{ik}) &= 4, & k &= 3p-2, & 1 \leq p \leq \lfloor \frac{n}{3} \rfloor \\ f(v_{ik}) &= 0, & k &= 3p-1, & 1 \leq p \leq \lfloor \frac{n}{3} \rfloor \\ f(v_{ik}) &= 2, & k &= 3p, & 1 \leq p \leq \lfloor \frac{n}{3} \rfloor \\ f(v_{i(n-1)}) &= 5, \\ f(v_{in}) &= 1 \end{aligned}$$

(6) when  $f(v_i) = 4, i = n$

$$\begin{aligned} f(v_{ik}) &= 2, & k &= 3p-2, & 1 \leq p \leq \lfloor \frac{n}{3} \rfloor \\ f(v_{ik}) &= 0, & k &= 3p-1, & 1 \leq p \leq \lfloor \frac{n}{3} \rfloor \\ f(v_{ik}) &= 4, & k &= 3p, & 1 \leq p \leq \lfloor \frac{n}{3} \rfloor \\ f(v_{i(n-1)}) &= 1, \\ f(v_{in}) &= 5 \end{aligned}$$

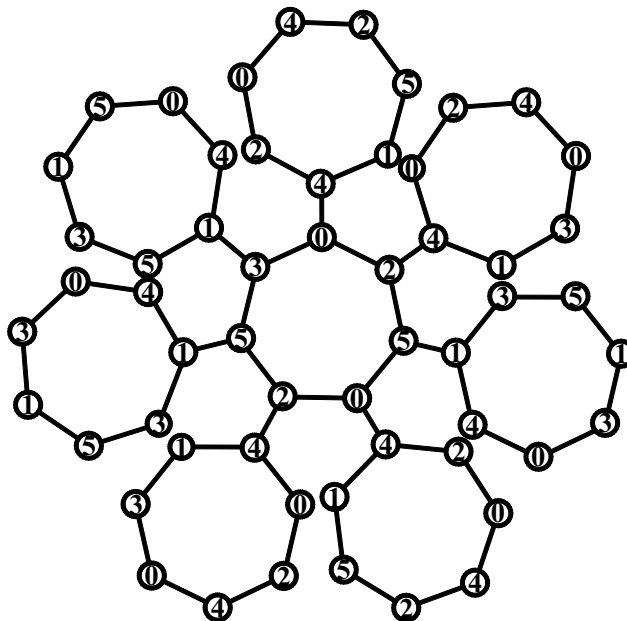
**Case 4:**  $n = 5$  This case is to be dealt separately. The  $L(2,1)$ -labeling for the graph obtained by taking star of the cycle  $C_5$  is shown in *Figure 7.9*.

FIGURE 7.9:  $L(2,1)$ -labeling for star of cycle  $C_5$ 

Thus in all the possibilities we have  $\lambda(C'_n) = 5$

□

**Illustration 7.4.10.** Consider the graph  $C_7$ , the  $L(2,1)$ -labeling is as shown in Figure 7.10.

FIGURE 7.10:  $L(2,1)$ -labeling for star of cycle  $C_7$

**Theorem 7.4.11.** Let  $G'$  be the graph obtained by taking star of a graph  $G$  then  $\lambda'(G') = p - 1$ , where  $p$  be the total number of vertices of  $G'$ .

*Proof.* Let  $v_1, v_2, \dots, v_n$  be the vertices of any connected graph  $G$ . Let  $v_{ij}$  be the vertices of a graph which is adjacent to the  $i^{th}$  vertex of graph  $G$ . By the definition of a star of a graph the total number of vertices in a graph  $G'$  are  $n(n+1)$ .

To define  $f: V(G') \longrightarrow N \cup \{0\}$

$$f(v_{i1}) = i - 1, \quad 1 \leq i \leq n$$

for  $1 \leq i \leq n$  do the labeling as follows:

$$f(v_i) = f(v_{ni}) + 1$$

$$f(v_{1(i+1)}) = f(v_i) + 1$$

$$f(v_{(j+1)(i+1)}) = f(v_{j(i+1)}) + 1, \quad 1 \leq j \leq n - 1$$

$$\text{Thus } \lambda'(G') = p - 1 = n^2 + n - 1$$

□

**Illustration 7.4.12.** Consider the star of a graph  $K_4$ , the  $L'(2,1)$ -labeling is as shown in Figure 7.11.

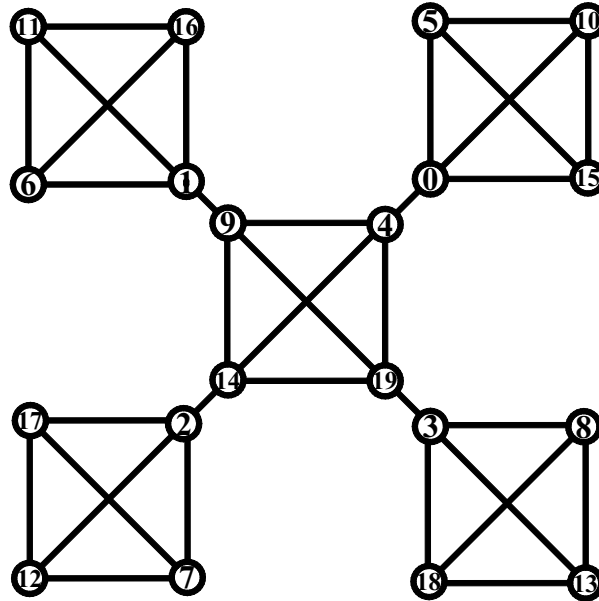


FIGURE 7.11:  $L'(2,1)$ -labeling for star of a complete graph  $K_4$

## 7.5 Radio labeling of graph

We have discussed the  $L(2,1)$ -labeling in previous section. Practically it has been observed that the interference among channels might go beyond two levels. Radio labeling extends the number of interference level considered for  $L(2,1)$ -labeling from two to the largest possible - the diameter of  $G$ . Motivated through the problem of channel assignments to FM radio stations Chartrand et al.[15] introduced the concept of radio labeling of graph as follows.

### 7.5.1 Radio labeling

A *radio labeling* of a graph  $G$  is an injective function  $f : V(G) \longrightarrow N \cup \{0\}$  such that for every  $u, v \in V$

$$|f(u) - f(v)| \geq \text{diam}(G) - d(u, v) + 1$$

The *span* of  $f$  is the difference of the largest and the smallest channels used. That is,  $\max(f(u), f(v))$ , for every  $u, v \in V$

The *radio number* of  $G$  is defined as the minimum span of a radio labeling of  $G$  and denoted as  $r_n(G)$ . In the survey of literature available on radio labeling we found that only two types of problems are considered in this area till this date.

- To investigate bounds for the radio number of a graph.
- To completely determine the radio number of a graph.

### 7.5.2 Some existing results

- Chartrand et al.[15] and Liu and Zhu[52] have discussed the radio number for paths and cycles.
- D.D-F Liu [53] has discuss radio number for trees.
- D.D-F.Liu and M.Xie[54] have discussed radio number of square of cycles.

- Mustapha et al.[47] have discussed radio  $k$ -labeling for cartesian products of graphs.
- Sooryanarayana et al.[64] have discussed radio number of cube of a path.

## 7.6 Radio labeling for some cycle related graphs

**Theorem 7.6.1.** Let  $G$  be the cycle with chords then.

$$r_n(G) = \begin{cases} (k+2)(2k-1)+1+\sum_i (d_i - d_i') & n \equiv 0 \pmod{4} \\ 2k(k+2)+1+\sum_i (d_i - d_i') & n \equiv 2 \pmod{4} \\ 2k(k+1)+\sum_i (d_i - d_i') & n \equiv 1 \pmod{4} \\ (k+2)(2k+1)+\sum_i (d_i - d_i') & n \equiv 3 \pmod{4} \end{cases}$$

*Proof.* Let  $C_n$  denote the cycle on  $n$  vertices and  $V(C_n) = \{v_0, v_1 \dots v_{n-1}\}$  be such that where  $v_i$  is adjacent to  $v_{i+1}$  and  $v_{n-1}$  is adjacent to  $v_0$ . We denote  $d = \text{diam}(C_n)$ .

Here the radio labeling of a cycle  $C_n$  is given by the following two sequences.

- the distance gap sequence  $D = (d_0, d_1 \dots d_{n-2})$
- the color gap sequence  $F = (f_0, f_1 \dots f_{n-2})$

The distance gap sequence in which each  $d_i \leq d$  is a positive integer is used to generate an ordering of the vertices of  $C_n$ . Let  $\tau : \{0, 1, \dots, n-1\} \longrightarrow \{0, 1, \dots, n-1\}$  be defined as  $\tau(0) = 0$  and  $\tau(i+1) = \tau(i) + d_i \pmod{n}$ . Here  $\tau$  is a corresponding permutation. Let  $x_i = v_{\tau(i)}$  for  $i = 0, 1, 2, \dots, n-1$  then  $\{x_0, x_1 \dots x_{n-1}\}$  is an ordering of the vertices of  $C_n$ . Let us denote  $d(x_i, x_{i+1}) = d_i$ .

The color gap sequence is used to assign labels to the vertices of  $C_n$ . Let  $f$  be the labeling defined by  $f(x_0) = 0$  and for  $i \geq 1, f(x_{i+1}) = f(x_i) + f_i$ . By definition of radio labeling  $f_i \geq d - d_i + 1$  for all  $i$ . We adopt the scheme for distance gap sequence and color gap sequence reported in [52] and proceed as follows.

**Case 1:**  $n = 4k$  in this case  $\text{diam}(G) = 2k$

Using the sequences given below we can generate the radio labeling of cycle  $C_n$  for  $n \equiv 0 \pmod{4}$  with minimum span.

The distance gap sequence is given by

$$\begin{aligned} d_i &= 2k && \text{if } i \text{ is even} \\ &= k && \text{if } i \equiv 1 \pmod{4} \\ &= k+1 && \text{if } i \equiv 3 \pmod{4} \end{aligned}$$

and the color gap sequence is given by

$$\begin{aligned} f_i &= 1 && \text{if } i \text{ is even} \\ &= k+1 && \text{if } i \text{ is odd} \end{aligned}$$

Then for  $i = 0, 1, 2, \dots, k-1$  we have the following permutation,

$$\begin{aligned} \tau(4i) &= 2ik + i \pmod{n} \\ \tau(4i+1) &= (2i+2)k + i \pmod{n} \\ \tau(4i+2) &= (2i+3)k + i \pmod{n} \\ \tau(4i+3) &= (2i+1)k + i \pmod{n} \end{aligned}$$

Now we add chords in cycle  $C_n$  such that diameter of cycle remain unchanged. Label the vertices of this newly obtained graph using above permutation. Suppose the new distance between  $x_i$  and  $x_{i+1}$  is  $d'_i(x_i, x_{i+1})$  then due to chords in the cycle it is obvious that  $d_i \geq d'_i$ .

We define the color gap sequence as

$$f'_i = f_i + (d_i - d'_i), 0 \leq i \leq n-2$$

So, that span  $f'$  for cycle with chords is

$$\begin{aligned} f'_0 + f'_1 + \dots + f'_{n-2} &= f_0 + f_1 + f_2 + \dots + f_{n-2} + \sum (d_i - d'_i) \\ &= r_n(C_n) + \sum (d_i - d'_i) \\ &= (k+2)(2k-1) + 1 + \sum (d_i - d'_i), 0 \leq i \leq n-2 \end{aligned}$$

which is an upper bound for radio number for cycle with arbitrary number of chords when  $n = 4k$ .

**Case 2:**  $n = 4k+2$  in this case  $\text{diam}(G) = 2k+1$

Using the sequences given below we can generate radio labeling of cycle  $C_n$  for

$n \equiv 2(\text{mod } 4)$  with minimum span.

The distance gap sequence is given by

$$\begin{aligned} d_i &= 2k + 1 & \text{if } i \text{ is even} \\ &= k + 1 & \text{if } i \text{ is odd} \end{aligned}$$

and the color gap sequence is given by

$$\begin{aligned} f_i &= 1 & \text{if } i \text{ is even} \\ &= k + 1 & \text{if } i \text{ is odd} \end{aligned}$$

Hence for  $i = 0, 1, \dots, 2k$ , we have the following permutation,

$$\begin{aligned} \tau(2i) &= i(3k + 2) \pmod{n} \\ \tau(2i + 1) &= i(3k + 2) + 2k + 1 \pmod{n} \end{aligned}$$

Now we add chords in cycle  $C_n$  such that diameter of cycle remain unchanged. Label the vertices of this newly obtained graph by using the above permutation. So, that span  $f'$  for cycle with chords is

$$\begin{aligned} f'_0 + f'_1 \dots + f'_{n-2} &= f_0 + f_1 + f_2 \dots + f_{n-2} + \sum (d_i - d'_i) \\ &= r_n(C_n) + \sum (d_i - d'_i) \\ &= 2k(k + 2) + 1 + \sum (d_i - d'_i). \quad 0 \leq i \leq n - 2 \end{aligned}$$

which is an upper bound for radio number for cycle with arbitrary number of chords.

when  $n = 4k + 2$

**Case 3:**  $n = 4k + 1$  in this case  $\text{diam}(G) = 2k$

Using the sequences given below we can generate radio labeling of cycle  $C_n$  for  $n \equiv 1(\text{mod } 4)$  with minimum span.

The distance gap sequence is given by

$$\begin{aligned} d_{4i} &= d_{4i+2} = 2k - i \\ d_{4i+1} &= d_{4i+3} = k + 1 + i \end{aligned}$$

and the color gap sequence is given by

$$f_i = 2k - d_i + 1$$

Then we have

$$\begin{aligned}\tau(2i) &= i(3k+1) \pmod{n}, 0 \leq i \leq 2k \\ \tau(4i+1) &= 2(i+1)k \pmod{n}, 0 \leq i \leq k-1 \\ \tau(4i+3) &= (2i+1)k \pmod{n}, 0 \leq i \leq k-1\end{aligned}$$

Label the vertices of this newly obtained graph by using the above permutation. So, that span of  $f'$  for cycle with chords is

$$\begin{aligned}f'_0 + f'_1 \dots + f'_{n-2} &= f_0 + f_1 + f_2 \dots + f_{n-2} + \sum (d_i - d'_i) \\ &= r_n(C_n) + \sum (d_i - d'_i) \\ &= 2k(k+1) + \sum (d_i - d'_i), 0 \leq i \leq n-2\end{aligned}$$

which is an upper bound for radio number for cycle with arbitrary number of chords.  
when  $n = 4k + 1$

**Case 4:**  $n = 4k + 3$  in this case  $\text{diam}(G) = 2k + 1$

Using the sequences below we can give radio labeling of cycle  $C_n$  for  $n \equiv 3 \pmod{4}$  with minimum span.

The distance gap sequence is given by

$$\begin{aligned}d_{4i} &= d_{4i+2} = 2k + 1 - i \\ d_{4i+1} &= k + 1 + i \\ d_{4i+3} &= k + 2 + i\end{aligned}$$

and the color gap sequence is given by

$$\begin{aligned}f_i &= 2k - d_i + 2 \quad \text{if } i \not\equiv 3 \pmod{4} \\ &= 2k - d_i + 3 \quad \text{otherwise}\end{aligned}$$

Then we have the following permutation,

$$\begin{aligned}\tau(4i) &= 2i(k+1) \pmod{n}, \quad 0 \leq i \leq k \\ \tau(4i+1) &= (i+1)(2k+1) \pmod{n}, \quad 0 \leq i \leq k \\ \tau(4i+2) &= (2i-1)(k+1) \pmod{n}, \quad 0 \leq i \leq k \\ \tau(4i+3) &= i(2k+1) + k \pmod{n}, \quad 0 \leq i \leq k-1, 1 \leq i \leq n-2\end{aligned}$$

Label the vertices of this newly obtained graph by following the above permutation. So, that span of  $f'$  for cycle with chords is

$$\begin{aligned}
f'_0 + f'_1 \dots + f'_{n-2} &= f_0 + f_1 + f_2 \dots + f_{n-2} + \sum (d_i - d'_i) \\
&= r_n(C_n) + \sum (d_i - d'_i) \\
&= (k+2)(2k+1) + \sum (d_i - d'_i)
\end{aligned}$$

which is an upper bound for radio number for cycle with arbitrary number of chords for the cycle with chords.

Thus in all the possibilities we have obtained the upper bounds of the radio numbers.  $\square$

**Illustration 7.6.2.** Consider the graph  $C_{12}$  with 5 chords, the radio labeling is as shown in Figure 7.12.

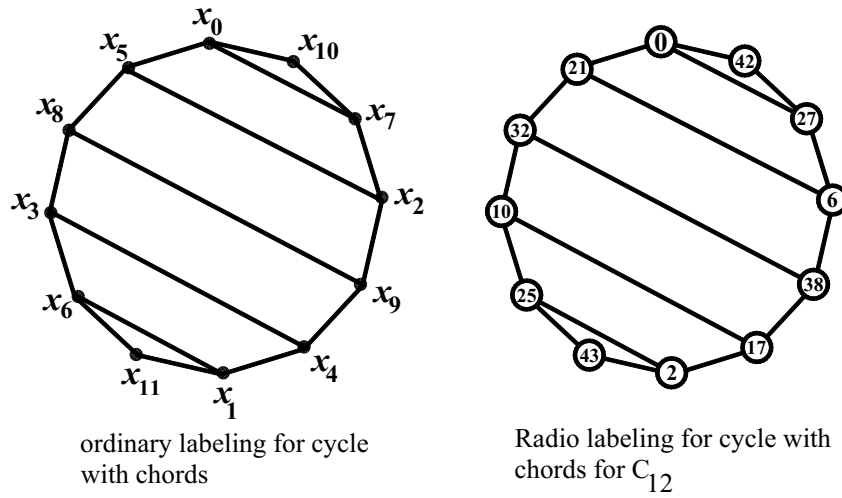


FIGURE 7.12: ordinary and radio labeling for cycle with chords for  $C_{12}$

**Theorem 7.6.3.** Let  $G$  be  $n/2$ -petal graph then

$$r_n(G) \leq \begin{cases} \frac{3p}{2} + n \left\lfloor \frac{n}{4} \right\rfloor - \left\lfloor \frac{n}{8} \right\rfloor - 2n - 2 \\ (p-1) + n \left\lfloor \frac{n}{4} \right\rfloor - \left\lfloor \frac{n}{8} \right\rfloor + 2n \end{cases}$$

*Proof.* Let  $G$  be  $n/2$ -petal graph with  $v_0, v_1 \dots v_{n-1}$  vertices of degree 3 and  $v'_1, v'_2 \dots v'_p$  vertices of degree 2. Here  $v_i$  is adjacent to  $v_{i+1}$  and  $v_{n-1}$  is adjacent to  $v_0$ .

**Case 1:**  $n \equiv 0(mod 4)$  and  $diam(G) = \lfloor n/4 \rfloor + 2$

First we label the vertices of degree 2. Let  $v'_1, v'_2 \dots v'_p$  be the vertices on the petals satisfying the order define by following distance sequence.

$$\begin{aligned} d'_i &= \lfloor n/4 \rfloor + 2 \text{ if } i \text{ is even} \\ &= \lfloor n/4 \rfloor + 1 \text{ if } i \text{ is odd} \end{aligned}$$

The color gap sequence for vertices on petals is defined as

$$\begin{aligned} f'_i &= 1 \text{ if } i \text{ is even} \\ &= 2 \text{ if } i \text{ is odd} \end{aligned}$$

Let  $v_1$  be the vertex on the cycle  $C_n$  such that  $d(v'_p, v_1) = \lfloor n/8 \rfloor + 1 = d(v'_{p-1}, v_1)$

then label  $v_1$  as  $f(v_1) = f(v'_p) + diam(G) - \lfloor n/8 \rfloor$

Now for the remaining vertices of degree 3 we use the permutation defined for the cycle  $C_n$  in case 1 of Theorem 7.6.1.

and the color gap sequence for the same vertices is defined as

$$f_i = \lfloor n/4 \rfloor + 2, 0 \leq i \leq n-2$$

Then span of  $f = 3p/2 + n\lfloor n/4 \rfloor - \lfloor n/8 \rfloor - 2n - 2$ .

Which is an upperbound for the radio number of  $n/2$ -petal graph when  $n \equiv 0(mod 4)$ .

**Case 2:**  $n \equiv 2(mod 4)$  and  $diam(G) = \lfloor n/4 \rfloor + 2$

First we label the vertices of degree 2. Let  $v'_1, v'_2 \dots v'_p$  be the vertices on the petals satisfying the order define by following distance sequence.

$$d(v'_i, v'_{i+1}) = \lfloor n/4 \rfloor + 2$$

The color gap sequence for the vertices on the petals is defined as

$$f'_i = 1, 1 \leq i \leq p$$

Let  $v_1$  be the vertex on the cycle  $C_n$  such that  $d(v'_p, v_1) = \lfloor n/8 \rfloor + 1 = d(v'_{p-1}, v_1)$

then label  $v_1$  as  $f(v_1) = f(v'_p) + diam(G) - \lfloor n/8 \rfloor$

Now for the remaining vertices of degree 3 we use the permutation defined for the cycle  $C_n$  in case 2 of Theorem 7.6.1.

and the color gap sequence for the same vertices is defined as

$$f_i = \lfloor n/4 \rfloor + 2, 0 \leq i \leq n-2$$

Then span of  $f = p - 1 + n\lfloor n/4 \rfloor - \lfloor n/8 \rfloor + 2n$ .

Which is an upperbound for the radio number of  $n/2$ -petal graph when  $n \equiv 2(mod 4)$ .

□

**Illustration 7.6.4.** Consider the  $n/2$ -petal graph of  $C_8$ . The radio labeling is shown in Figure 7.13.

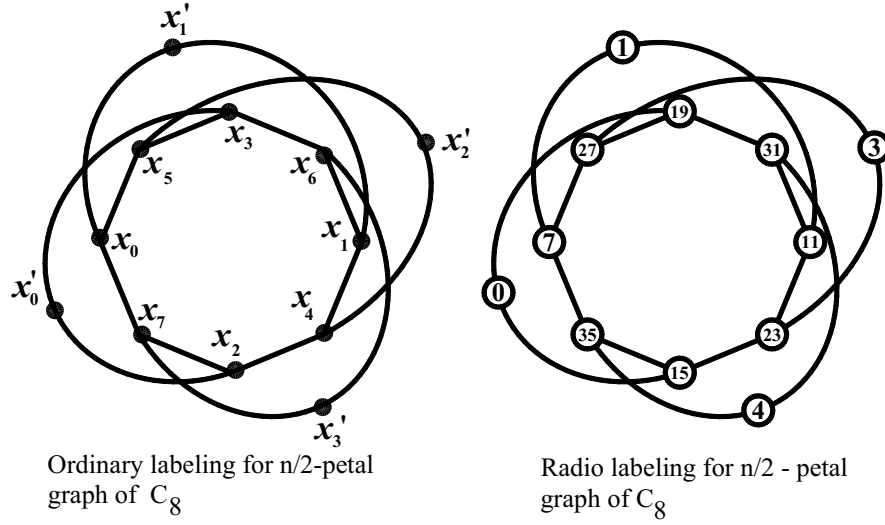


FIGURE 7.13: ordinary and radio labeling for a  $n/2$ -petal graph of  $C_8$

**Theorem 7.6.5.**

$$r_n(spl(C_n)) = \begin{cases} 2[(k+2)(2k-1)+1]+k+1 & n \equiv 0 \pmod{4} \\ 2[2k(k+2)+1]+k+1 & n \equiv 2 \pmod{4} \\ 2[2k(k+1)]+k+1 & n \equiv 1 \pmod{4} \\ 2[(k+2)(2k+1)]+k+1 & n \equiv 3 \pmod{4} \end{cases}$$

*Proof.* Let  $v'_1, v'_2, v'_3 \dots v'_n$  be the duplicated vertices corresponding to  $v_1, v_2, v_3 \dots v_n$ . As  $d(v_i, v_j) = d(v_i, v'_j)$  and in order to obtain the labeling with minimum span we employ twice the distance gap sequence, the color gap sequence and the permutation scheme used in [52]. First we label the vertices of cycle and then their duplicated vertices.

**Case 1:**  $n \equiv 4k (n > 4)$  then  $diam(G) = 2k$

We first label the vertices  $v_1, v_2, v_3 \dots v_n$  as according to Case 1 of Theorem 7.6.1 and then we label the vertices  $v'_1, v'_2, v'_3 \dots v'_n$  as follows.

Define  $f(v'_j) = f(v_{n-1}) + k + 1$  where  $v'_j$  be the vertex such that  $d(v_{n-1}, v'_j) = k$

Now with permutation scheme used in Case 1 of Theorem 7.6.1 for cycle  $n \equiv 0 \pmod{4}$  label the duplicated vertices starting from  $v'_j$ .

Then  $r_n(spl(C_n)) = f_0 + f_1 + f_2 \dots + f_{n-2} + 2k - k + 1 + f'_0 + f'_1 + \dots + f'_{n-2}$

As  $f_i = f'_i$ , for  $i = 0, 1, 2 \dots n-2$ , then

$$\begin{aligned} r_n(spl(C_n)) &= 2f_0 + 2f_1 + 2f_2 \dots + 2f_{n-2} + k + 1 \\ &= 2[(k+2)(2k-1) + 1] + k + 1 \end{aligned}$$

**Case 2:**  $n \equiv 4k + 2 (n > 6)$  then  $diam(G) = 2k + 1$

We first label the vertices  $v_1, v_2, v_3 \dots v_n$  according to Case 2 of Theorem 7.6.1 and then we label the vertices  $v'_1, v'_2, v'_3 \dots v'_n$  as follows.

Define  $f(v'_j) = f(v_{n-1}) + 1$  where  $v'_j$  be the vertex such that  $d(v_{n-1}, v'_j) = k + 1$

Now with permutation for cycle  $n \equiv 2(mod 4)$  used in Case 2 of Theorem 7.6.1 label the duplicated vertices starting from  $v'_j$ .

Then  $r_n(spl(C_n)) = f_0 + f_1 + f_2 \dots + f_{n-2} + 2k + 1 - k - 1 + 1 + f'_0 + f'_1 + \dots + f'_{n-2}$

As  $f_i = f'_i$ , for  $i = 0, 1, 2 \dots n-2$ , then

$$\begin{aligned} r_n(spl(C_n)) &= 2f_0 + 2f_1 + 2f_2 \dots + 2f_{n-2} + k + 1 \\ &= 2[2k(k+2) + 1] + k + 1 \end{aligned}$$

**Case 3:**  $n \equiv 4k + 1 (n > 5)$  then  $diam(G) = 2k$

We first label the vertices  $v_1, v_2, v_3 \dots v_n$  as defined in Case 3 of Theorem 7.6.1. Now we label the vertices  $v'_1, v'_2, v'_3 \dots v'_n$  as follows.

Define  $f(v'_j) = f(v_{n-1}) + k$  where  $v'_j$  be the vertex such that  $d(v_{n-1}, v'_j) = k + 1$

Now with permutation for cycle  $n \equiv 1(mod 4)$  used in Case 3 of Theorem 7.6.1 label the duplicated vertices starting from  $v'_j$ .

Then  $r_n(spl(C_n)) = f_0 + f_1 + f_2 \dots + f_{n-2} + 2k - k - 1 + 1 + f'_0 + f'_1 + \dots + f'_{n-2}$

As  $f_i = f'_i$ , for  $i = 0, 1, 2 \dots n-2$ , then

$$\begin{aligned} r_n(spl(C_n)) &= 2f_0 + 2f_1 + 2f_2 \dots + 2f_{n-2} + k + 1 \\ &= 2[2k(k+1)] + k + 1 \end{aligned}$$

**Case 4:**  $n \equiv 4k + 3 (n > 3)$  then  $diam(G) = 2k + 1$

We first label the vertices  $v_1, v_2, v_3 \dots v_n$  as defined in Case 4 of Theorem 7.6.1. Now we label the vertices  $v'_1, v'_2, v'_3 \dots v'_n$  as follows.

$v'_j$  be the vertex such that  $d(v_{n-1}, v'_j) = k + 1$

Define  $f(v'_j) = f(v_{n-1}) + k + 1$  where  $v'_j$  be the vertex such that  $d(v_{n-1}, v'_j) = k + 1$

Now with permutation for cycle  $n \equiv 3(mod 4)$  used in Case 4 of Theorem 7.6.1 label

the duplicated vertices starting from  $v'_j$ .

Then  $r_n(spl(C_n)) = f_0 + f_1 + f_2 \dots + f_{n-2} + 2k + 1 - k - 1 + 1 + f'_0 + f'_1 + \dots + f'_{n-2}$

As  $f_i = f'_i$ , for  $i = 0, 1, 2 \dots n - 2$ , then

$$\begin{aligned} r_n(spl(C_n)) &= 2f_0 + 2f_1 + 2f_2 \dots + 2f_{n-2} + k + 1 \\ &= 2[(k+2)(2k+1)] + k + 1 \end{aligned}$$

Thus in all the four possibilities we have determined radio number of graph  $G$  under consideration.  $\square$

**Illustration 7.6.6.** Consider the graph  $spl(C_{10})$ . The radio labeling is shown in Figure 7.14.

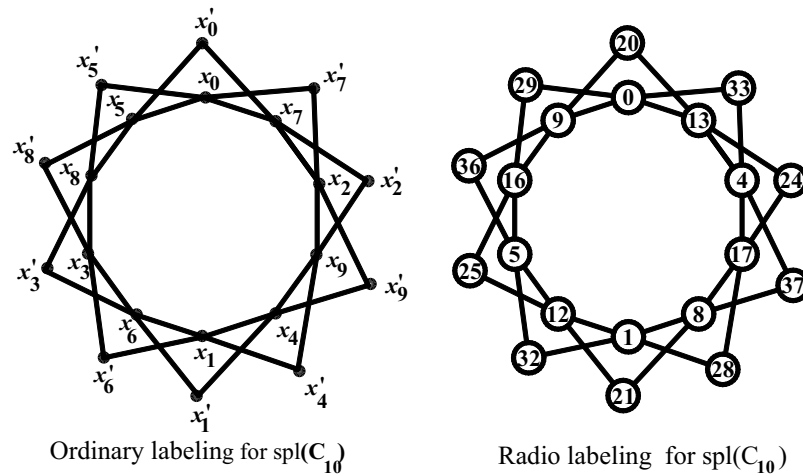


FIGURE 7.14: ordinary and radio labeling for  $spl(C_{10})$

**Application 7.6.7.** Above result can be applied for the purpose of expansion of existing circular network of radio transmitters. By applying the concept of duplication of vertex the number of radio transmitters are doubled and separation of the channels assigned to the stations is such that interference can be avoided. Thus our result can play vital role for the expansion of radio transmitter network without disturbing the existing one. In the expanded network the distance between any two transmitters is large enough to avoid the interference. Thus our result is useful for network expansion.

$$\textbf{Theorem 7.6.8. } r_n(M(C_n)) = \begin{cases} 2(k+2)(2k-1) + n + 3 & n \equiv 0 \pmod{4} \\ 4k(k+2) + k + n + 3 & n \equiv 2 \pmod{4} \\ 4k(k+1) + k + n & n \equiv 1 \pmod{4} \\ 2(k+2)(2k+1) + k + n + 1 & n \equiv 3 \pmod{4} \end{cases}$$

*Proof.* Let  $u_1, u_2, \dots, u_n$  be the vertices of the cycle  $C_n$  and  $u'_1, u'_2, \dots, u'_n$  be the newly inserted vertices corresponding to the edges of  $C_n$  to obtain  $M(C_n)$ . In  $M(C_n)$  the diameter is increased by 1.

Here  $d(u_i, u_j) \geq d(u_i, u'_j)$  for  $n \equiv 0, 2 \pmod{4}$  and  $d(u_i, u_j) = d(u_i, u'_j)$  for  $n \equiv 1, 3 \pmod{4}$ .

Through out the discussion first we label the vertices  $u_1, u_2, \dots, u_n$  and then newly inserted vertices  $u'_1, u'_2, \dots, u'_n$ . For this purpose we will employ twice the permutation scheme for respective cycle as in considered in Theorem 7.6.1.

**Case 1:**  $n \equiv 4k$  in this case  $\text{diam}(M(C_n)) = 2k + 1$

The distance gap sequence to order the vertices of original cycle  $C_n$  is defined as follows because  $f_i + f_{i+1} \leq f'_i + f'_{i+1}$ , for all  $i$

$$\begin{aligned} d_i &= 2k + 1 & \text{if } i \text{ is even} \\ &= k + 1 & \text{if } i \equiv 1 \pmod{4} \\ &= k + 2 & \text{if } i \equiv 3 \pmod{4} \end{aligned}$$

The color gap sequence is defined as follows

$$\begin{aligned} f_i &= 1 & \text{if } i \text{ is even} \\ &= k + 1 & \text{if } i \text{ is odd} \end{aligned}$$

Let  $u'_1$  be the vertex on the inscribed cycle such that  $d(u_n, u'_1) = k + 1$  and  $f = k + 1$

The distance gap sequence to order the vertices of the inscribed cycle  $C_n$  is defined as follows

$$\begin{aligned} d_i &= 2k & \text{if } i \text{ is even} \\ &= k & \text{if } i \equiv 1 \pmod{4} \\ &= k + 1 & \text{if } i \equiv 3 \pmod{4} \end{aligned}$$

The color gap sequence is defined as follows

$$\begin{aligned} f'_i &= 2 && \text{if } i \text{ is even} \\ &= k+2 && \text{if } i \equiv 1 \pmod{4} \\ &= k+1 && \text{if } i \equiv 3 \pmod{4} \end{aligned}$$

Thus in this case  $r_n(M(C_n)) = 2(k+2)(2k-1) + n + 3$

**Case 2:**  $n = 4k + 2$  in this case  $\text{diam}(M(C_n)) = 2k + 2$

The distance gap sequence to order the vertices of original cycle  $C_n$  is defined as follows

because  $f_i + f_{i+1} \leq f'_i + f'_{i+1}$ , for all  $i$

$$\begin{aligned} d_i &= 2k+2 && \text{if } i \text{ is even} \\ &= k+3 && \text{if } i \text{ is odd} \end{aligned}$$

and the color gap sequence is given by

$$\begin{aligned} f_i &= 1 && \text{if } i \text{ is even} \\ &= k+1 && \text{if } i \text{ is odd} \end{aligned}$$

Let  $u'_1$  be the vertex on the inscribed cycle such that  $d(u_n, u'_1) = k+1$  and  $f = k+2$

The distance gap sequence to order the vertices of the inscribed cycle  $C_n$  is defined as follows

$$\begin{aligned} d_i &= 2k+1 && \text{if } i \text{ is even} \\ &= k+1 && \text{if } i \text{ is odd} \end{aligned}$$

and the color gap sequence is given by

$$\begin{aligned} f'_i &= 2 && \text{if } i \text{ is even} \\ &= k+2 && \text{if } i \text{ is odd} \end{aligned}$$

Thus  $r_n(M(C_n)) = 4k(k+2) + k + n + 3$

**Case 3:**  $n = 4k + 1$  in this case  $\text{diam}(M(C_n)) = 2k + 1$

The distance gap sequence to order the vertices of original cycle  $C_n$  is defined as follows

because  $f_i + f_{i+1} \leq f'_i + f'_{i+1}$ , for all  $i$

$$\begin{aligned} d_{4i} &= d_{4i+2} = 2k+1-i \\ d_{4i+1} &= d_{4i+3} = k+2+i \end{aligned}$$

and the color gap sequence is given by

$$f_i = (2k+1) - d_i + 1$$

Let  $u'_1$  be the vertex on the inscribed cycle such that  $d(u_n, u'_1) = k+1$  and  $f = k+1$

The distance gap sequence to order the vertices of the inscribed cycle  $C_n$  is defined as follows

$$\begin{aligned} d_{4i} &= d_{4i+2} = 2k - i \\ d_{4i+1} &= d_{4i+3} = k + 1 + i \end{aligned}$$

and the color gap sequence is given by

$$f'_i = 2k - d_i + 2$$

Thus in this case  $r_n(M(C_n)) = 4k(k+1) + k + n$

**Case 4:**  $n = 4k + 3$  in this case  $\text{diam}(M(C_n)) = 2k + 2$

The distance gap sequence to order the vertices of original cycle  $C_n$  is defined as follows

because  $f_i + f_{i+1} \leq f'_i + f'_{i+1}$ , for all  $i$

$$\begin{aligned} d_{4i} &= d_{4i+2} = 2k + 2 - i \\ d_{4i+1} &= d_{4i+3} = k + 2 + i \end{aligned}$$

and the color gap sequence is given by

$$f_i = 2k - d_i + 3$$

Let  $u'_1$  be the vertex on the inscribed cycle such that  $d(u_n, u'_1) = k + 2$  and  $f = k + 1$

The distance gap sequence to order the vertices of inscribed cycle  $C_n$  is defined as follows

$$\begin{aligned} d_{4i} &= d_{4i+2} = 2k + 1 - i \\ d_{4i+1} &= k + 1 + i \\ d_{4i+3} &= k + 2 + i \end{aligned}$$

and the color gap sequence is given by

$$f'_i = 2k - d_i + 3$$

Thus in this case  $r_n(M(C_n)) = 2(k+2)(2k+1) + n$

Thus the radio number is completely determined for the graph  $M(C_n)$ .  $\square$

**Illustration 7.6.9.** consider the graph  $M(C_8)$ . The radio labeling is shown in Figure 7.15.

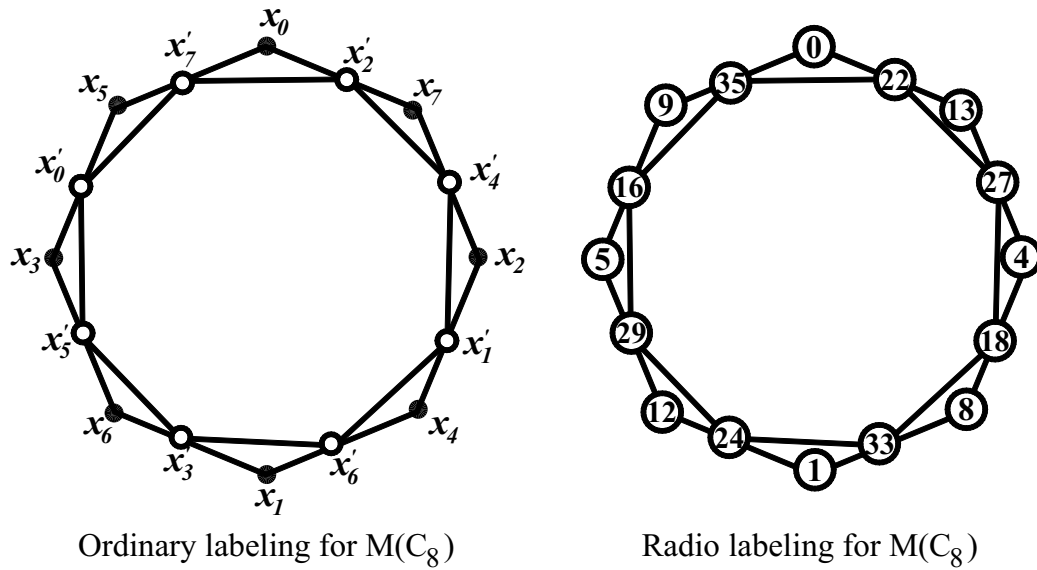


FIGURE 7.15: ordinary and radio labeling of  $M(C_8)$

**Application 7.6.10.** Above result is possibly useful for the expansion of existing radio transmitters network. In the expanded network two newly installed nearby transmitters are connected and interference is also avoided between them. Thus the radio labeling described in above Theorem 7.6.8 is rigourously applicable to accomplish the task of channel assignment for the feasible network.

**7.6.11.** The comparison between Radio numbers of  $C_n$ ,  $spl(C_n)$  and  $M(C_n)$  is tabulated in the following Table 1.

Table 1 Comparison of radio numbers of  $C_n$ ,  $spl(C_n)$  and  $M(C_n)$

n	Radio number of $C_n$	Radio number of $spl(C_n)$	Radio number of $M(C_n)$
$0(\text{mod } 4)$	$(k+2)(2k-1)+1$	$2[(k+2)(2k-1)+1]+k+1$	$2(k+2)(2k-1)+n+3$
$2(\text{mod } 4)$	$2k(k+2)+1$	$2[2k(k+2)+1]+k+1$	$4k(k+2)+k+n+3$
$1(\text{mod } 4)$	$2k(k+1)$	$2[2k(k+1)]+k+1$	$4k(k+1)+k+n$
$3(\text{mod } 4)$	$(k+2)(2k+1)$	$2[(k+2)(2k+1)]+k+1$	$2(k+2)(2k+1)+n$

## 7.7 Concluding Remarks and Scope of Further Research

The  $L(2,1)$ -labeling and the Radio labeling are the labelings which concern to channel assignment problem. The lowest level of interference will make the entertainment meaningful and enjoyable. The expansion of network is also demand of the recent time. The investigations reported in this chapter will serve both of these purposes. Our investigations can be applied for network expansion without disturbing the existing network. To develop such results for various graph families is an open area of research.

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# List of Symbols

$ B $	Cardinality of set B.
$C_n$	Cycle with $n$ vertices.
$C_n \odot K_1$	Crown graph.
$C_n^{(k)}$	One point union of $k$ copies of Cycle $C_n$ .
$D_2(G)$	Shadow graph of $G$ .
$diam(G)$	Diameter of $G$ .
$E(G)$ or $E$	Edge set of graph $G$ .
$e_f(n)$	Number of edges with edge label $n$ .
$f_n$	Fan on $n$ vertices.
$F_n$	Friendship graph with $2n + 1$ vertices.
$F_n$	$n^{th}$ Fibonacci number with special reference to chapter - 5.
$G = (V(G), E(G))$	A graph $G$ with vertex set $V(G)$ and edge set $E(G)$ .
$G_1[G_2]$	Composition of $G_1$ and $G_2$ .
$G_v$	The graph obtained by switching of a vertex $v$ in $G$ .
$H_n$	Helm on $n$ vertices.
$K_n$	Complete graph on $n$ vertices.
$K_{m,n}$	Complete bipartite graph.
$K_{1,n}$	Star graph.
$M(G)$	Middle graph of $G$
$N(v)$	Open neighbourhood of vertex $v$ .
$N[v]$	Closed neighbourhood of vertex $v$ .
$P_n$	Path graph on $n$ vertices.
$r_n(G)$	Radio number of $G$ .

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$S_n$	Shell on $n$ vertices.
$spl(G)$	Split graph.
$T$	Tree.
$T_n$	Triangular number.
$T(n, l)$	Tadpole graph.
$T(G)$	Total graph of $G$ .
$V(G)$ or $V$	Vertex set of graphs $G$ .
$v_f(n)$	Number of vertices with vertex label $n$ .
$W_n$	Wheel on $n$ vertices.
$\lfloor n \rfloor$	Greatest integer not greater than real number $n$ (Floor of $n$ ).

## **Annexure**

## Some Important Results on Triangular Sum Graphs

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### Abstract

Let  $G = (V, E)$  be a graph with  $p$  vertices and  $q$  edges. A graph  $G$  is said to admit a *triangular sum labeling* if its vertices can be labeled by non-negative integers such that induced edge labels obtained by the sum of the labels of end vertices are the first  $q$  triangular numbers. A graph  $G$  which admits a triangular sum labeling is called a triangular sum graph. In the present work we investigate some classes of graphs which does not admit a triangular sum labeling. Also we show that some classes of graphs can be embedded as an induced subgraph of a triangular sum graph. This work is a nice composition of graph theory and combinatorial number theory.

**Mathematics Subject Classification:** 05C78

**Keywords:** Triangular number, Triangular sum labeling

### 1. Introduction and Definitions

We begin with simple, finite, connected, undirected and non-trivial graph  $G = (V, E)$ , where  $V$  is called the set of vertices and  $E$  is called the set of edges. For various graph theoretic notations and terminology we follow Gross and Yellen [3] and for number theory we follow Burton [1]. We will give brief summary of definitions which are useful for the present investigations.

**Definition 1.1** If the vertices of the graph are assigned values, subject to certain conditions is known as graph labeling.

For detail survey on graph labeling one can refer Gallian [2]. Vast amount of literature is available on different types of graph labeling and more than 1000 research papers have been published so far in last four decades. Most interesting labeling problems have three important ingredients.

- a set of numbers from which vertex labels are chosen.
- a rule that assigns a value to each edge.
- a condition that these values must satisfy.

The present work is aimed to discuss one such labeling known as triangular sum labeling.

**Definition 1.2** A *triangular number* is a number obtained by adding all positive integers less than or equal to a given positive integer  $n$ . If  $n^{\text{th}}$  triangular number is denoted by  $T_n$  then  $T_n = \frac{1}{2}n(n+1)$ . It is easy to observe that there does not exist consecutive integers which are triangular numbers.

**Definition 1.3** A *triangular sum labeling* of a graph  $G$  is a one-to-one function  $f : V \rightarrow N$  ( where  $N$  is the set of all non-negative integers) that induces a bijection  $f^+ : E(G) \rightarrow \{T_1, T_2, \dots, T_n\}$  of the edges of  $G$  defined by  $f^+(uv) = f(u) + f(v)$ ,  $\forall e = uv \in E(G)$ . The graph which admits such labeling is called a triangular sum graph. This concept was introduced by Hegde and Shankaran [4]. In the same paper they obtained a necessary condition for an Eulerian graph to admit a triangular sum labeling. Moreover they investigated some classes of graphs which can be embedded as an induced subgraph of a triangular sum graph. In the present work we investigate some classes of graphs which does not admit a triangular sum labeling.

**Definition 1.4** The *helm* graph  $H_n$  is the graph obtained from a wheel  $W_n = C_n + K_1$  by attaching a pendant edge at each vertex of  $C_n$ .

**Definition 1.5** The graph  $G = \langle W_n : W_m \rangle$  is the graph obtained by joining apex vertices of wheels  $W_n$  and  $W_m$  to a new vertex  $x$ . ( A vertex corresponding to  $K_1$  in  $W_n = C_n + K_1$  is called an apex vertex.)

**Definition 1.6** A chord of a cycle  $C_n$  is an edge joining two non-adjacent vertices of cycle  $C_n$ .

**Definition 1.7** Two chords of a cycle are said to be twin chords if they

form a triangle with an edge of the cycle  $C_n$ .

## 2. Main Results

**Lemma 2.1** In every triangular sum graph  $G$  the vertices with label 0 and 1 are always adjacent.

**Proof:** The edge label  $T_1 = 1$  is possible only when the vertices with label 0 and 1 are adjacent.

**Lemma 2.2** In any triangular sum graph  $G$ , 0 and 1 cannot be the label of vertices of the same triangle contained in  $G$ .

**Proof:** Let  $a_0, a_1$ , and  $a_2$  be the vertices of a triangle. If  $a_0$  and  $a_1$  are labeled with 0 and 1 respectively and  $a_2$  is labeled with some  $x \in N$ , where  $x \neq 0, x \neq 1$ . Such vertex labeling will give rise to edge labels with 1,  $x$ , and  $x + 1$ . In order to admit a triangular sum labeling,  $x$  and  $x + 1$  must be triangular numbers. But it is not possible as we have mentioned in Definition 1.2

**Lemma 2.3** In any triangular sum graph  $G$ , 1 and 2 cannot be the labels of vertices of the same triangle contained in  $G$ .

**Proof:** Let  $a_0, a_1, a_2$  be the vertices of a triangle. Let  $a_0$  and  $a_1$  are labeled with 1 and 2 respectively and  $a_2$  is labeled with some  $x \in N$ , where  $x \neq 1, x \neq 2$ . Such vertex labeling will give rise to edge labels 3,  $x + 1$ , and  $x + 2$ . In order to admit a triangular sum labeling,  $x + 1$  and  $x + 2$  must be triangular numbers, which is not possible due to the fact mentioned in Definition 1.2.

**Theorem 2.4** The Helm graph  $H_n$  is not a triangular sum graph.

**Proof:** Let us denote the apex vertex as  $c_1$ , the consecutive vertices adjacent to  $c_1$  as  $v_1, v_2, \dots, v_n$ , and the pendant vertices adjacent to  $v_1, v_2, \dots, v_n$  as  $u_1, u_2, \dots, u_n$  respectively. If possible  $H_n$  admits a triangular sum labeling  $f : V \rightarrow N$ , then we consider following cases:

**Case 1:**  $f(c_1) = 0$ .

Then according to Lemma 2.1, we have to assign label 1 to exactly one of the vertices from  $v_1, v_2, \dots, v_n$ . Then there is a triangle having the vertices with labels 0 and 1 as adjacent vertices, which contradicts the Lemma 2.2.

**Case 2:** Any one of the vertices from  $v_1, v_2, \dots, v_n$  is labeled with 0. Without loss of generality let us assume that  $f(v_1) = 0$ . Then one of the vertices from  $c_1, v_2, v_n, u_1$  must be labeled with 1. Note that each of the vertices from  $c_1, v_2, v_n, u_1$  is adjacent to  $v_1$ .

**Subcase 1:** If one of the vertices from  $c_1, v_2, v_n$  is labeled with 1. In

each possibility there is a triangle having two of the vertices with labels 0 and 1, which contradicts the Lemma 2.2.

**Subcase 2:** If  $f(u_1) = 1$  then the edge label  $T_2 = 3$  can be obtained by vertex labels 0, 3 or 1, 2. The vertex with label 1 and the vertex with label 2 cannot be adjacent as  $u_1$  is a pendant vertex having label 1 and it is adjacent to the vertex with label 0. Therefore one of the vertices from  $v_2, v_n, c_1$  must receive the label 3. Thus there is a triangle whose two of the vertices are labeled with 0 and 3. Let the third vertex be labeled with  $x$ , with  $x \neq 0$  and  $x \neq 3$ . To admit a triangular sum labeling  $3, x, x + 3$  must be distinct triangular numbers. i.e.  $x$  and  $x + 3$  are two distinct triangular numbers other than 3 having difference 3, which is not possible.

**Case 3:** Any one of the vertices from  $u_1, u_2, \dots, u_n$  is labeled with 0. Without loss of generality we may assume that  $f(u_1) = 0$ . Then according to Lemma 2.1,  $f(v_1) = 1$ . The edge labels  $T_2 = 3$  can be obtained by vertex labels 0, 3 or 1, 2. The vertex with label 0 and the vertex with label 3 cannot be the adjacent vertices as  $u_1$  is a pendant vertex having label 0 and it is adjacent to the vertex with label 1. Therefore one of the vertices from  $v_2, v_n, c_1$  must be labeled with 2. Thus we have a triangle having vertices with labels 1 and 2 which contradicts the Lemma 2.3.

Thus in each of the possibilities discussed above,  $H_n$  does not admits a triangular sum labeling.

**Theorem 2.5** If every edge of a graph  $G$  is an edge of a triangle then  $G$  is not a triangular sum graph.

**Proof:** If  $G$  admits a triangular sum labeling then according to Lemma 2.1 there exists two adjacent vertices having labels 0 and 1 respectively. So there is a triangle having two of the vertices labeled with 0 and 1, which contradicts the Lemma 2.2. Thus  $G$  does not admit a triangular sum labeling.

Following are the immediate corollaries of the previous result.

**Corollary 2.6** The wheel graph  $W_n$  is not a triangular sum graph.

**Corollary 2.7** The fan graph  $f_n = P_{n-1} + K_1$  is not a triangular sum graph.

**Corollary 2.8** The friendship graph  $F_n = nK_3$  is not a triangular sum graph.

**Corollary 2.9** The graph  $g_n$  (the graph obtained by joining all the vertices of  $P_n$  to two additional vertices) is not a triangular sum graph.

**Corollary 2.10** The flower graph (the graph obtained by joining all the pendant vertices of helm graph  $H_n$  with the apex vertex) is not a triangular sum graph.

**Corollary 2.11** The graph obtained by joining apex vertices of two wheel graphs and two apex vertices with a new vertex is not a triangular sum graph.

**Theorem 2.12** The graph  $\langle W_n : W_m \rangle$  is not a triangular sum graph.

**Proof:** Let  $G = \langle W_n : W_m \rangle$ . Let us denote the apex vertex of  $W_n$  by  $u_0$  and the vertices adjacent to  $u_0$  of the wheel  $W_n$  by  $u_1, u_2, \dots, u_n$ . Similarly denote the apex vertex of other wheel  $W_m$  by  $v_0$  and the vertices adjacent to  $v_0$  of the wheel  $W_m$  by  $v_1, v_2, \dots, v_m$ . Let  $w$  be the new vertex adjacent to apex vertices of both the wheels. If possible let  $f : V \rightarrow N$  be one of the possible triangular sum labeling. According to the Lemma 2.1, 0 and 1 are the labels of any two adjacent vertices of the graph  $G$ , we have the following cases:

**Case 1:** If 0 and 1 be the labels of adjacent vertices in  $W_n$  or  $W_m$ , then there is a triangle having two of the vertices labeled with 0 and 1. Which contradicts the Lemma 2.2.

**Case 2:** If  $f(w) = 0$  then according to Lemma 2.1 one of the vertex from  $u_0$  and  $v_0$  is labeled with 1. Without loss of generality we may assume that  $f(u_0) = 1$ . To have an edge label  $T_2 = 3$  we have the following possibilities:

**Subcase 2.1:** One of the vertices from  $u_1, u_2, \dots, u_n$  is labeled with 2. Without loss of generality assume that  $f(u_i) = 2$ , for some  $i \in \{1, 2, 3, \dots, n\}$ . In this situation we will get a triangle having two of its vertices are labeled with 1 and 2, which contradicts the Lemma 2.3.

**Subcase 2.2:** Assume that  $f(v_0) = 3$ . Now to get the edge label  $T_3 = 6$  we have the following subcases:

**Subcase 2.2.1:** Assume that  $f(u_i) = 5$ , for some  $i \in \{1, 2, 3, \dots, n\}$ . In this situation we will get a triangle with distinct vertex labels 1, 5 and  $x$ . Then  $x + 5$  and  $x + 1$  will be the edge labels of two edges with difference 4. It is possible only if  $x = 5$ , but  $x \neq 5$  as we have  $f(u_i) = 5$ .

**Subcase 2.2.2:** Assume that 2 and 4 are the labels of two adjacent vertices from one of the two wheels. So there exists a triangle whose vertex labels are either 1, 2, and 4 or 3, 2, and 4. In either of the situation will give rise to an edge label 5 which is not a triangular number.

**Case 3:** If  $f(w) = 1$  then one of the vertex from  $u_0$  and  $v_0$  is labeled with 0. Without loss of generality assume that  $f(u_0) = 0$ . To have an edge label 3 we have the following possibilities:

**Subcase 3.1:** If  $f(u_i) = 3$  for some  $i \in \{1, 2, 3, \dots, n\}$ . Then there is a triangle having vertex labels as 0, 3,  $x$ , with  $x \neq 3$ . Thus we

have two edge labels  $x + 3$  and  $x$  which are two distinct triangular numbers having difference 3. So  $x = 3$ , which is not possible as  $x \neq 3$ .

**Subcase 3.2:** Assume that  $f(v_0) = 2$ . Now to obtain the edge label  $T_3 = 6$  we have to consider the following possibilities:

- (i)  $6=6+0$ ;                      (ii)  $6=5+1$ ;                      (iii)  $6=4+2$ .
- (i) If  $6 = 6 + 0$  then one of the vertices from  $u_1, u_2, \dots, u_n$  must be labeled with 6. Without loss of generality we may assume that  $f(u_i) = 6$  for some  $i \in \{1, 2, 3, \dots, n\}$ . In this situation there are two distinct triangles having vertex labels  $0, 6, x$  and  $0, 6, y$ , for two distinct triangular numbers  $x$  and  $y$  each of which are different from 0 and 6. Then  $x + 6$  and  $x$  are two distinct triangular numbers having difference 6. This is possible only for  $x = 15$ . On the other hand  $y + 6$  and  $y$  are two distinct triangular numbers having difference 6. Then  $y = 15$ . ( The  $x = y = 15$  which is not possible as  $f$  is one-one)
- (ii) If  $6 = 5 + 1$  and  $f(w) = 1$ , then in this situation label of one of the vertex adjacent to  $w$  must be 5. This is not possible as  $w$  is adjacent to the vertices whose labels are 0 and 2.
- (iii) If  $6 = 2 + 4$ . In this case one of the vertices from  $v_1, v_2, \dots, v_m$  is labeled with 4. Assume that  $f(v_i) = 4$ , for some  $i \in \{1, 2, 3, \dots, m\}$ . In this situation there is a triangle having vertex labels 2, 4 and  $x$  (where  $x$  is a positive integer with  $x \neq 2, x \neq 4$ ). Then  $4+x$  and  $2+x$  will be the edge labels of two edges i.e.  $4+x$  and  $2+x$  are two distinct triangular numbers with difference 2 which is not possible.

Thus we conclude that in each of the possibilities discussed above the graph  $G$  under consideration does not admit a triangular sum labeling.

#### 4. Embedding of some Triangular sum graphs

**Theorem 4.1** Every cycle can be embedded as an induced subgraph of a triangular sum graph.

**Proof:** Let  $G = C_n$  be a cycle with  $n$  vertices. We define labeling  $f : V(G) \rightarrow N$  as follows such that the induced function  $f^+ : E(G) \rightarrow \{T_1, T_2, \dots, T_q\}$  is bijective.

$$f(v_1) = 0$$

$$f(v_2) = 6$$

$$f(v_i) = T_{i+2} - f(v_{i-1}); 3 \leq i \leq n-1$$

$$f(v_n) = T_{f(v_{n-1})-1}$$

Now let  $A = \{T_1, T_2, \dots, T_r\}$  be the set of missing edge labels. i.e. Elements of set  $A$  are the missing triangular numbers between 1 and  $T_{f(v_{n-1})-1}$ . Now add  $r$  pendent vertices which are adjacent to the vertex with label 0 and label these new vertices with labels  $T_1, T_2, \dots, T_r$ . This construction will give rise to edges with labels  $T_1, T_2, \dots, T_r$  such that the resultant supergraph  $H$  admits triangular sum labeling. Thus we proved that every cycle can be embedded as an induced subgraph of a triangular sum graph.

**Example 4.2** In the following *Figure 4.1* embedding of  $C_5$  as an induced subgraph of a triangular sum graph is shown.

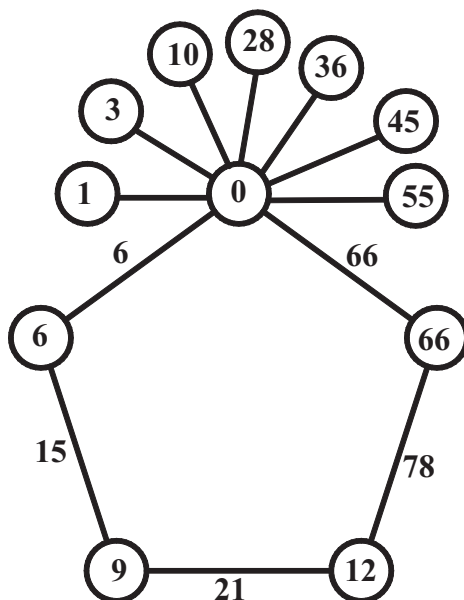


Figure 4.1

**Theorem 4.3** Every cycle with one chord can be embedded as an induced subgraph of a triangular sum graph.

**Proof:** Let  $G = C_n$  be the cycle with one chord. Let  $e = v_1v_k$  be the chord of cycle  $C_n$ . We define labeling as

We define labeling  $f : V(G) \rightarrow N$  as follows such that the induced function  $f^+ : E(G) \rightarrow \{T_1, T_2, \dots, T_q\}$  is bijective.

$$f(v_1) = 0$$

$$f(v_2) = 6$$

$$f(v_i) = T_{i+2} - f(v_{i-1}); 3 \leq i \leq k-1$$

$$f(v_k) = T_{f(v_{k-1})-1}$$

$$f(v_{k+i-1}) = T_{f(v_{k-1})-1+i} - f(v_{k+i-2}); 2 \leq i \leq n-k$$

$$f(v_n) = T_{f(v_{n-1})-1}$$

Now follow the procedure described in Theorem 4.1 and the resultant supergraph H admits triangular sum labeling. Thus we proved that every cycle with one chord can be embedded as an induced subgraph of a triangular sum graph.

**Example 4.4** In the following *Figure 4.2* embedding of  $C_4$  with one chord as an induced subgraph of a triangular sum graph is shown.

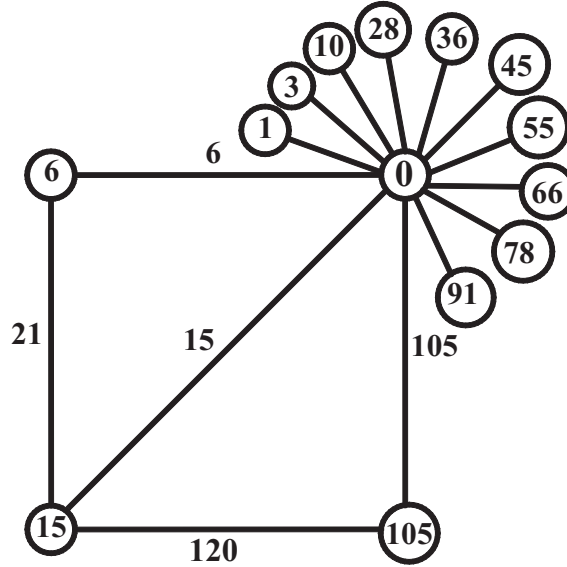


Figure 4.2

**Theorem 4.5** Every cycle with twin chords can be embedded as an induced subgraph of a triangular sum graph.

**Proof:** Let  $G = C_n$  be the cycle with twin chords. Let  $e_1 = v_1v_k$  and  $e_2 = v_1v_{k+1}$  be two chords of cycle  $C_n$ . We define labeling  $f : V(G) \rightarrow N$  as follows such that the induced function  $f^+ : E(G) \rightarrow \{T_1, T_2, \dots, T_q\}$  is bijective.

$$f(v_1) = 0$$

$$f(v_2) = 6$$

$$f(v_i) = T_{i+2} - f(v_{i-1}); 3 \leq i \leq k-1$$

$$f(v_k) = T_{f(v_{k-1})-1}$$

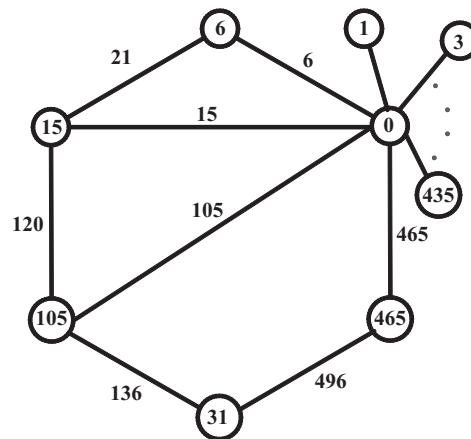
$$f(v_{k+1}) = T_{f(v_k)-1}$$

$$f(v_{k+i}) = T_{f(v_k)-1+i} - f(v_{k+i-1}); 2 \leq i \leq n-k-1$$

$$f(v_n) = T_{f(v_{n-1})-1}$$

Now following the procedure adapted in Theorem 4.1 the resulting supergraph H admits triangular sum labeling. i.e. every cycle with twin chords can be embedded as an induced sub graph of a triangular sum graph.

**Example 4.6** In the following *Figure 4.3* embedding of  $C_6$  with twin chord as an induced subgraph of a triangular sum graph is shown.



*Figure 4.3*

## 5. Concluding Remarks

As every graph does not admit a triangular sum labeling, it is very interesting to investigate classes of graphs which are not triangular sum graphs and to embed classes of graphs as an induced subgraph of a triangular sum graph. We investigate several classes of graphs which does not admit triangular sum labeling. Moreover we show that cycle, cycle with one chord and cycle with twin chords can be embedded as an induced subgraph of a triangular sum graph. This work contribute several new result to the theory of graph labeling.

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## TOTAL PRODUCT CORDIAL GRAPHS INDUCED BY SOME GRAPH OPERATIONS ON CYCLE RELATED GRAPHS

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### Abstract

The aim of this work is to discuss total product cordial graphs in the context of various graph operations. We prove that duplicating the vertices of cycle  $C_n$  altogether produces a total product cordial graph. We also prove that the total graph of  $C_n$ , one point union of  $k$ -copies of  $C_n$  and the middle graph of  $C_n$  admit total product cordial labeling. In addition to this, we show that the graph known as star of cycle and the graph obtained by switching of a vertex in cycle  $C_n$  are total product cordial graphs.

### 1. Introduction

Throughout this work, graph  $G = (V(G), E(G))$ , we mean a finite, connected and undirected graph without loops or multiple edges. For terminology and standard notations, we follow Harary [3]. We will give a brief summary of definitions which are prerequisites for the present investigations.

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**Definition 1.1.** *Duplication* of a vertex  $v_k$  of graph  $G$  produces a new graph  $G'$  by adding a vertex  $v'_k$  with  $N(v_k) = N(v'_k)$ .

In other words, a vertex  $v'_k$  is said to be *duplication* of  $v_k$  if all the vertices which are adjacent to  $v_k$  are now adjacent to  $v'_k$  also.

**Definition 1.2.** The *middle graph*  $M(G)$  of a graph  $G$  is the graph whose vertex set is  $V(G) \cup E(G)$  and in which two vertices are adjacent if and only if either they are adjacent edges of  $G$  or one is a vertex of  $G$  and the other is an edge incident with it.

**Definition 1.3.** The *total graph*  $T(G)$  of a graph  $G$  is the graph whose vertex set is  $V(G) \cup E(G)$  and two vertices are adjacent whenever they are either adjacent or incident in  $G$ .

**Definition 1.4** (Vaidya et al. [6]). A graph obtained by replacing each vertex of a star  $K_{1,n}$  by a graph  $G$  is called *star* of  $G$  denoted as  $G'$ . By the central graph in  $G'$ , we mean the graph which replaces the apex vertex of  $K_{1,n}$ .

**Definition 1.5.** A *one point union*  $C_n^t$  of  $t$  copies of cycles is the graph obtained by taking  $v$  as a common vertex such that any two cycles  $C_n^i$  and  $C_n^j$  ( $i \neq j$ ) are edge disjoint and do not have any vertex in common except  $v$ .

**Definition 1.6.** A *friendship graph*  $F_n$  is a one point union of  $n$  copies of cycle  $C_3$ .

**Definition 1.7.** A *vertex switching*  $G_v$  of a graph  $G$  is the graph obtained by taking a vertex  $v$  of  $G$ , removing all the edges incident to  $v$  and adding edges joining  $v$  to every other vertex which are not adjacent to  $v$  in  $G$ .

**Definition 1.8.** If the vertices of the graph are assigned values subject to certain conditions, then it is known as *graph labeling*.

**Definition 1.9.** Let  $G = (V(G), E(G))$  be a graph. A mapping  $f : V(G) \rightarrow \{0, 1\}$  is called *binary vertex labeling* of  $G$  and  $f(v)$  is called the *label* of vertex  $v$  of  $G$  under  $f$ .

For an edge  $e = uv$ , the induced edge labeling  $f^* : E(G) \rightarrow \{0, 1\}$  is given by  $f^*(e) = f(u)f(v)$ . Let  $v_f(0)$ ,  $v_f(1)$  be the number of vertices of  $G$  having labels 0 and 1, respectively, under  $f$  and let  $e_f(0)$ ,  $e_f(1)$  be the number of edges of  $G$  having labels 0 and 1, respectively, under  $f^*$ .

Sundaram et al. [4] defined product cordial labeling, in which the edge receives the label as the product of the labels of two end vertices unlike the difference in cordial labeling introduced by Cahit [1]. Sundaram et al. [5] introduced the concept of total product cordial labeling of graph as follows and investigated several results.

**Definition 1.10.** A *total product cordial labeling* of a graph  $G$  is a function  $f : (V(G) \cup E(G)) \rightarrow \{0, 1\}$  such that  $f(xy) = f(x)f(y)$ , where  $x, y \in V(G)$ ,  $xy \in E(G)$  and the total number of 0 and 1 are balanced. That is, if  $v_f(i)$  and  $e_f(i)$  denote the set of vertices and edges, which are labeled as  $i$  for  $i = 0, 1$ , respectively, then  $|(v_f(0) + e_f(0)) - (v_f(1) + e_f(1))| \leq 1$ . If there exists a total product cordial labeling of a graph  $G$ , then we call  $G$  is *total product cordial graph*. For the literature on total product cordial labeling, we refer to 'A dynamic survey of graph labeling' by Gallian [2] and relevant references given in it.

There are three types of problems that can be considered in this area:

1. How total product cordiality is affected under various graph operations;
2. Construct new families of total product cordial graphs by investigating suitable labeling;
3. Given a graph theoretic property  $P$ , characterize the class of graphs with property  $P$  that is total product cordial.

The present work is intended to discuss the problems of the types (i) and (ii).

## 2. Main Results

**Theorem 2.1.** *The graph obtained by duplicating vertices of cycle  $C_n$  altogether is total product cordial.*

**Proof.** Let  $v'_1, v'_2, \dots, v'_n$  be the duplicated vertices corresponding to  $v_1, v_2, \dots, v_n$  of cycle  $C_n$  and  $C'_n$  be the graph resulted due to duplication of vertices.

To define  $f : (V(C'_n) \cup E(C'_n)) \rightarrow \{0, 1\}$ , we consider following two cases:

**Case 1.**  $n$  is even.

We label the vertices as follows:

$$f(v_{2i-1}) = 1, \quad 1 \leq i \leq \frac{n}{2},$$

$$f(v_{2i}) = 0, \quad 1 \leq i \leq \frac{n}{2},$$

$$f(v'_i) = 1, \quad 1 \leq i \leq n.$$

Using above pattern, we have

$$v_f(0) + e_f(0) = v_f(1) + e_f(1) = \frac{5n}{2}.$$

**Case 2.**  $n$  is odd.

We label the vertices as follows:

$$f(v_{2i-1}) = 0, \quad 1 \leq i \leq \frac{n-1}{2},$$

$$f(v_{2i}) = 1, \quad 1 \leq i \leq \frac{n-1}{2},$$

$$f(v_n) = 1,$$

$$f(v'_i) = 1, \quad 1 \leq i \leq n-1,$$

$$f(v'_n) = 0.$$

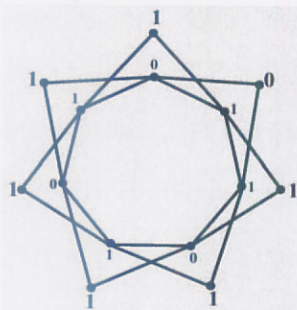
Using above pattern, we have

$$v_f(0) + e_f(0) + 1 = v_f(1) + e_f(1) = \frac{5n-1}{2}.$$

Thus, the graph  $C'_n$  satisfies the condition  $|(v_f(0) + e_f(0)) - (v_f(1) + e_f(1))| \leq 1$ .

That is,  $C'_n$  is total product cordial graph.

**Illustration 2.2.** In the following Figure 2.1, total product cordial labeling of  $C'_7$  is shown:



**Figure 2.1.**  $C_7$  and its total product cordial labeling.

**Theorem 2.3.**  $T(C_n)$  is total product cordial graph.

**Proof.** Let  $v_1, v_2, \dots, v_n$  be the vertices of cycle  $C_n$  and  $e'_1, e'_2, \dots, e'_n$  be the vertices in  $T(C_n)$  corresponding to the edges  $e_1, e_2, \dots, e_n$  in  $C_n$ .

To define  $f : V(T(C_n)) \cup E(T(C_n)) \rightarrow \{0, 1\}$ , we consider following two cases:

**Case 1.**  $n$  is even.

We label the vertices as follows:

$$f(e'_i) = 0, \quad 1 \leq i \leq \frac{n}{2},$$

$$f(e'_i) = 1, \quad \frac{n}{2} + 1 \leq i \leq n,$$

$$f(v_i) = 0, \quad 1 \leq i \leq \frac{n}{2} - 1,$$

$$f(v_i) = 1, \quad \frac{n}{2} \leq i \leq n.$$

**Case 2.**  $n$  is odd.

We label the vertices as follows:

$$f(e'_i) = 1, \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor + 1,$$

$$f(e'_i) = 0, \quad \left\lfloor \frac{n}{2} \right\rfloor + 2 \leq i \leq n,$$

$$f(v_i) = 1, \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor + 1,$$

$$f(v_i) = 0, \quad \left\lfloor \frac{n}{2} \right\rfloor + 2 \leq i \leq n.$$

In view of the labeling pattern defined above, we have

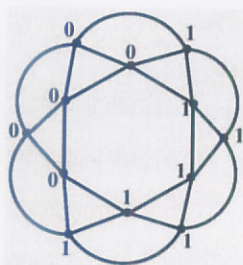
$$v_f(0) + e_f(0) = v_f(1) + e_f(1) = 3n.$$

Thus, we conclude that the graph  $T(C_n)$  satisfies the condition

$$|(v_f(0) + e_f(0)) - (v_f(1) + e_f(1))| \leq 1.$$

Hence,  $f$  is a total product cordial labeling of  $T(C_n)$ .

**Illustration 2.4.** The total product cordial labeling of graph  $T(C_6)$  is shown in Figure 2.2:



**Figure 2.2.**  $T(C_6)$  and its total product cordial labeling.

**Theorem 2.5.** The star of cycle  $C_n$  admits total product cordial labeling.

**Proof.** Let  $v_1, v_2, \dots, v_n$  be the vertices of central cycle  $C_n$  and  $v_{ij}$  be the vertices of cycle  $C_n^j$ , where  $1 \leq j \leq n$ , which are adjacent to the  $i$ th vertex of central cycle  $C_n$ .

To define  $f : (V(C'_n) \cup E(C'_n)) \rightarrow \{0, 1\}$ , we consider following two cases:

**Case 1.**  $n$  is even.

We label the vertices as follows:

$$f(v_i) = 1, \quad 1 \leq i \leq n,$$

$$f(v_{1j}) = 1, \quad 1 \leq j \leq \frac{n}{2},$$

$$f(v_{1j}) = 0, \quad \frac{n}{2} + 1 \leq j \leq n,$$

$$f(v_{ij}) = 0, \quad 1 \leq j \leq \frac{n}{2}, \quad 2 \leq i \leq n,$$

$$f(v_{ij}) = 1, \quad \frac{n}{2} + 1 \leq j \leq n, \quad 2 \leq i \leq n.$$

Using above pattern, we have

$$v_f(0) + e_f(0) = v_f(1) + e_f(1) = n^2 + \frac{3n}{2}.$$

**Case 2.**  $n$  is odd.

We label the vertices as follows:

$$f(v_i) = 1, \quad 1 \leq i \leq n,$$

$$f(v_{ij}) = 0, \quad 1 \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor + 1, \quad 1 \leq i \leq n,$$

$$f(v_{ij}) = 1, \quad \left\lfloor \frac{n}{2} \right\rfloor + 2 \leq j \leq n, \quad 1 \leq i \leq n.$$

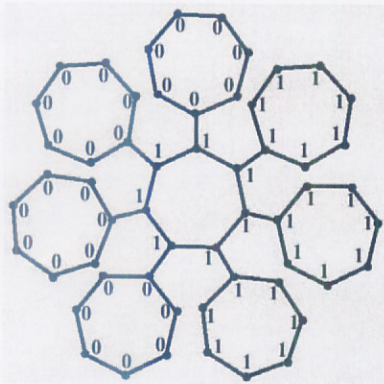
Using above pattern, we have

$$v_f(0) + e_f(0) = v_f(1) + e_f(1) + 1 = 2n \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor + 2n + 1.$$

Thus  $|(v_f(0) + e_f(0)) - (v_f(1) + e_f(1))| \leq 1$ .

Hence,  $f$  is a total product cordial labeling for the graph star of cycle.

**Illustration 2.6.** The total product cordial labeling for the star of cycle  $C_7$  is shown in Figure 2.3:



**Figure 2.3.** Total product cordial labeling for star of cycle  $C_7$ .

**Theorem 2.7.** *The one point union of  $k$  copies of cycle  $C_n$  admits total product cordial labeling.*

**Proof.** Let  $v_{ij}$  be the  $i$ th vertex of  $j$ th copy of cycle  $C_j$ . Let  $v_1$  be the common vertex of all the cycles. Then without loss of generality, we start the label assignment from  $v_1$ .

To define  $f : (V(C_n^k) \cup E(C_n^k)) \rightarrow \{0, 1\}$ , we consider following four cases:

**Case 1.**  $n \in N (n \geq 3)$  and  $k$  is even.

$$f(v_1) = 1,$$

$$f(v_{ij}) = 1, \quad 1 \leq j \leq \frac{k}{2}, \quad 2 \leq i \leq n,$$

$$f(v_{ij}) = 0, \quad \frac{k}{2} + 1 \leq j \leq k, \quad 2 \leq i \leq n.$$

In view of the labeling pattern defined above, we have

$$v_f(0) + e_f(0) + 1 = v_f(1) + e_f(1) = 2n.$$

**Case 2.**  $n > 3$  is even,  $k$  is odd.

$$f(v_1) = 1,$$

$$f(v_{ij}) = 1, \quad 1 \leq j \leq \left\lfloor \frac{k}{2} \right\rfloor, \quad 2 \leq i \leq n,$$

$$f(v_{ij}) = 0, \quad \left\lfloor \frac{k}{2} \right\rfloor + 1 \leq j \leq k - 2, \quad 2 \leq i \leq n,$$

$$f(v_{ij}) = 1, \quad j = k - 1, \quad 2 \leq i \leq \frac{n}{2},$$

$$f(v_{ij}) = 0, \quad j = k - 1, \quad \frac{n}{2} + 1 \leq i \leq n,$$

$$f(v_{3j}) = 1, \quad j = k,$$

$$f(v_{ij}) = 0, \quad j = k, \quad 2 \leq i \leq n, \quad i \neq 3.$$

In view of the labeling pattern defined above, we have

$$v_f(0) + e_f(0) + 1 = v_f(1) + e_f(1) = 2n \left\lfloor \frac{k}{2} \right\rfloor - \left\lfloor \frac{k}{2} \right\rfloor + n.$$

**Case 3.**  $n \geq 3$  is odd,  $k$  is odd.

$$f(v_1) = 1,$$

$$f(v_{ij}) = 1, \quad 1 \leq j \leq \left\lfloor \frac{k}{2} \right\rfloor, \quad 2 \leq i \leq n,$$

$$f(v_{ij}) = 0, \quad \left\lfloor \frac{k}{2} \right\rfloor + 1 \leq j \leq k-1, \quad 2 \leq i \leq n,$$

$$f(v_{ij}) = 1, \quad j = k, \quad 2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor + 1,$$

$$f(v_{ij}) = 0, \quad j = k, \quad \left\lfloor \frac{n}{2} \right\rfloor + 2 \leq i \leq n.$$

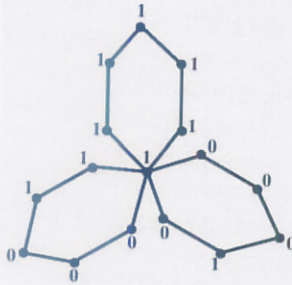
In view of the labeling pattern defined above, we have

$$v_f(0) + e_f(0) + 1 = v_f(1) + e_f(1) = 2n \left\lfloor \frac{k}{2} \right\rfloor - \left\lfloor \frac{k}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor + \frac{n+1}{2}.$$

Thus in all the four cases,  $|(v_f(0) + e_f(0)) - (v_f(1) + e_f(1))| \leq 1$ .

Hence, the graph obtained by one point union of  $k$  copies of cycle admits total product cordial labeling.

**Illustration 2.8.** In the following Figure 2.4, the one point union of three copies of cycle  $C_6$  and its total product cordial labeling are as shown in Figure 2.4:



**Figure 2.4.** Total product cordial labeling for the one point union of three copies of cycle  $C_6$ .

**Corollary 2.9.** *Friendship graph  $F_n$  admits total product cordial labeling.*

The proof is obvious from the Cases 2 and 4 of the above Theorem 2.7.

**Theorem 2.10.**  *$M(C_n)$  is total product cordial graph.*

**Proof.** Let  $v_1, v_2, \dots, v_n$  be the vertices of cycle  $C_n$  and  $e'_1, e'_2, \dots, e'_n$  be the vertices in  $M(C_n)$  corresponding to the edges  $e_1, e_2, \dots, e_n$  in  $C_n$ .

To define  $f : V(M(C_n)) \cup E(M(C_n)) \rightarrow \{0, 1\}$ , we consider following two cases:

**Case 1.**  $n$  is even.

$$f(v_i) = 1, \quad 1 \leq i \leq n,$$

$$f(e'_{2i-1}) = 0, \quad 1 \leq i \leq \frac{n}{2},$$

$$f(e'_{2i}) = 1, \quad 1 \leq i \leq \frac{n}{2}.$$

Using above pattern, we have

$$v_f(0) + e_f(0) = v_f(1) + e_f(1) = \frac{5n}{2}.$$

**Case 2.**  $n$  is odd.

$$f(v_i) = 1, \quad 1 \leq i \leq n-1,$$

$$f(v_i) = 0, \quad i = n,$$

$$f(e'_{2i-1}) = 1, \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor + 1,$$

$$f(e'_{2i}) = 0, \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor.$$

Using above pattern, we have

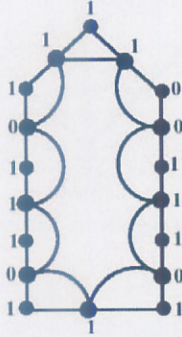
$$v_f(0) + e_f(0) + 1 = v_f(1) + e_f(1) + 1 = 2n + \left\lfloor \frac{n}{2} \right\rfloor + 1.$$

In view of the above defined pattern,  $M(C_n)$  satisfies the condition

$$|(v_f(0) + e_f(0)) - (v_f(1) + e_f(1))| \leq 1.$$

Hence,  $M(C_n)$  admits total product cordial labeling.

**Illustration 2.11.** In the following Figure 2.5, total product cordial labeling for  $M(C_9)$  is shown:



**Figure 2.5.** Total product cordial labeling for  $M(C_9)$ .

**Theorem 2.12.** The graph obtained by switching of an arbitrary vertex in cycle  $C_n$  admits total product cordial labeling.

**Proof.** Let  $v_1, v_2, \dots, v_n$  be the successive vertices of  $C_n$ . Then  $G_v$  denotes the vertex switching of  $G$  with respect to the vertex  $v$  of  $G$ . Without loss of generality, let the switched vertex be  $v_1$  and we initiate the labeling from this switched vertex  $v_1$ .

To define  $f : (V(G_{v_1}) \cup E(G_{v_1})) \rightarrow \{0, 1\}$ , we consider following four cases:

**Case 1.**  $n$  is even,  $n = 2k$ ,  $k = 3, 5, 7, 9, \dots$

$$f(v_1) = 0,$$

$$f(v_i) = 1, \quad 2 \leq i \leq \frac{n}{2} + 1,$$

$$f\left(v_{\frac{n}{2}+1+i}\right) = 0, \quad 1 \leq i \leq \left\lfloor \frac{n}{4} \right\rfloor,$$

$$f\left(v_{\frac{n}{2}+\left\lfloor \frac{n}{4} \right\rfloor+1+i}\right) = 1, \quad 1 \leq i \leq \left\lfloor \frac{n}{4} \right\rfloor.$$

Using above pattern, we have

$$v_f(0) + e_f(0) = v_f(1) + e_f(1) + 1 = n + 2 \left\lfloor \frac{n}{4} \right\rfloor - 1.$$

**Case 2.**  $n$  is even,  $n = 2k$ ,  $k = 4, 6, 8, \dots$

$$f(v_1) = 0,$$

$$f(v_i) = 1, \quad 2 \leq i \leq \frac{n}{2} + 1,$$

$$f\left(v_{\frac{n}{2}+1+i}\right) = 0, \quad 1 \leq i \leq \frac{n}{4} - 1,$$

$$f\left(v_{\frac{n}{2}+\frac{n}{4}-1+i}\right) = 1, \quad 1 \leq i \leq \frac{n}{4}.$$

Using above pattern, we have

$$v_f(0) + e_f(0) + 1 = v_f(1) + e_f(1) = \frac{3n}{2} - 2.$$

**Case 3.**  $n$  is odd.

$$f(v_1) = 1,$$

$$f(v_i) = 0, \quad 2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor + 1,$$

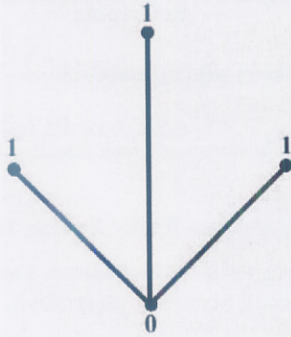
$$f\left(v_{\left\lfloor \frac{n}{2} \right\rfloor + 1 + i}\right) = 1, \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor.$$

Using above pattern, we have

$$v_f(0) + e_f(0) = v_f(1) + e_f(1) = 3 \left\lfloor \frac{n}{2} \right\rfloor - 1.$$

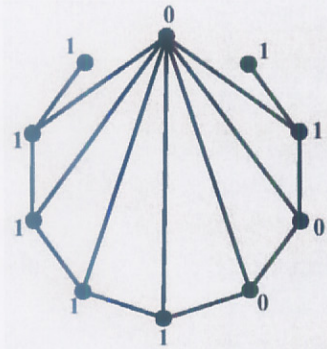
Thus,  $G_{v_1}$  satisfies the condition  $|(v_f(0) + e_f(0)) - (v_f(1) + e_f(1))| \leq 1$ . Hence,  $G_{v_1}$  admits total product cordial labeling.

**Case 4.** The vertex switching of cycle  $C_4$  is an acyclic graph and its total product cordial labeling is given in following Figure 2.6:



**Figure 2.6.** Total product cordial labeling for a vertex switching of  $C_4$ .

**Illustration 2.13.** In the following Figure 2.7, the graph obtained by switching of a vertex in cycle  $C_{10}$  and its total product cordial labeling are shown:



**Figure 2.7.** Vertex switching in  $C_{10}$  and its total product cordial labeling.

### 3. Concluding Remarks

Labeling of discrete structure is the potential area of research due to its diversified applications. We discuss here total product cordial labeling in the context of some graph operations. We contribute six new results to the theory of total product cordial labeling. This work is an effort to provide total product cordial labeling for the graphs resulted from the graph operations on given graphs. Analogous results can be investigated for various families of graph and for different graph labeling problems.

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# Prime Cordial Labeling for Some Graphs

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## Abstract

We present here prime cordial labeling for the graphs obtained by some graph operations on given graphs.

**Keywords:** Prime cordial labeling, Total graph, Vertex switching

## 1. Introduction

We begin with simple, finite, connected and undirected graph  $G = (V(G), E(G))$ . For all standard terminology and notations we follow (Harary F., 1972). We will give brief summary of definitions which are useful for the present investigations.

*Definition 1.1* If the vertices of the graph are assigned values subject to certain conditions then it is known as *graph labeling*.

For a dynamic survey on graph labeling we refer to (Gallian J., 2009). A detailed study on variety of applications of graph labeling is reported in (Bloom G. S., 1977, p. 562-570).

*Definition 1.2* Let  $G$  be a graph. A mapping  $f: V(G) \rightarrow \{0, 1\}$  is called *binary vertex labeling* of  $G$  and  $f(v)$  is called the label of the vertex  $v$  of  $G$  under  $f$ .

For an edge  $e = uv$ , the induced edge labeling  $f^*: E(G) \rightarrow \{0, 1\}$  is given by  $f^*(e) = |f(u) - f(v)|$ . Let  $v_f(0)$ ,  $v_f(1)$  be the number of vertices of  $G$  having labels 0 and 1 respectively under  $f$  while  $e_f(0)$ ,  $e_f(1)$  be the number of edges having labels 0 and 1 respectively under  $f^*$ .

*Definition 1.3* A binary vertex labeling of a graph  $G$  is called a *cordial labeling* if  $|v_f(0) - v_f(1)| \leq 1$  and  $|e_f(0) - e_f(1)| \leq 1$ . A graph  $G$  is *cordial* if it admits cordial labeling.

The concept of cordial labeling was introduced by (Cahit I., 1987, p.201-207). After this many researchers have investigated graph families or graphs which admit cordial labeling. Some labeling schemes are also introduced with minor variations in cordial theme. Some of them are product cordial labeling, total product cordial labeling and prime cordial labeling. The present work is focused on prime cordial labeling.

*Definition 1.4* A *prime cordial labeling* of a graph  $G$  with vertex set  $V(G)$  is a bijection  $f: V(G) \rightarrow \{1, 2, 3, \dots, |V(G)|\}$  defined by  $f(e = uv) = 1$  ; if  $\gcd(f(u), f(v)) = 1$   
 $= 0$  ; otherwise

and  $|e_f(0) - e_f(1)| \leq 1$ . A graph which admits prime cordial labeling is called a prime cordial graph.

*Definition 1.5* Let  $G$  be a graph with two or more vertices then the *total graph*  $T(G)$  of a graph  $G$  is the graph whose vertex set is  $V(G) \cup E(G)$  and two vertices are adjacent whenever they are either adjacent or incident in  $G$ .

*Definition 1.6* The *composition* of two graphs  $G_1$  and  $G_2$  denoted by  $G_1[G_2]$  has vertex set  $V(G_1[G_2]) = V(G_1) \times V(G_2)$  and edge set  $E(G_1[G_2]) = \{(u_1, v_1)(u_2, v_2) / u_1u_2 \in E(G_1) \text{ or } [u_1 = u_2 \text{ and } v_1v_2 \in E(G_2)]\}$

*Definition 1.7* A *vertex switching*  $G_v$  of a graph  $G$  is the graph obtain by taking a vertex  $v$  of  $G$ , removing all the edges incident to  $v$  and adding edges joining  $v$  to every other vertex which are not adjacent to  $v$  in  $G$ .

## 2. Main Results

*Theorem 2.1*  $T(P_n)$  is prime cordial graph,  $\forall n \geq 5$ .

*Proof:* If  $v_1, v_2, v_3, \dots, v_n$  and  $e_1, e_2, e_3, \dots, e_n$  be the vertices and edges of  $P_n$  then  $v_1, v_2, v_3, \dots, v_n, e_1, e_2, e_3, \dots, e_n$  are vertices of  $T(P_n)$ .

We define vertex labeling  $f: V(T(P_n)) \rightarrow \{1, 2, 3, \dots, |V(G)|\}$  as follows. We consider following four cases.

Case 1:  $n = 3, 5$

For the graph  $T(P_3)$  the possible pairs of labels of adjacent vertices are (1,2), (1,3), (1,4), (1,5), (2,3), (2,4), (2,5), (3,4), (3,5), (4,5). Then obviously  $e_f(0) = 1, e_f(1) = 6$ . That is,  $|e_f(0) - e_f(1)| = 5$  and in all other possible arrangement of vertex labels  $|e_f(0) - e_f(1)| > 5$ . Therefore  $T(P_3)$  is not a prime cordial graph.

The case when  $n=5$  is to be dealt separately. The graph  $T(P_5)$  and its prime cordial labeling is shown in Fig 1.

Case 2:  $n$  odd,  $n \geq 7$

$$f(v_1) = 2, f(v_2) = 4,$$

$$f(v_{i+2}) = 2(i+3), \quad 1 \leq i \leq \lfloor n/2 \rfloor - 2$$

$$f(v_{\lfloor n/2 \rfloor + 1}) = 3, f(v_{\lfloor n/2 \rfloor + 2}) = 1, f(v_{\lfloor n/2 \rfloor + 3}) = 7,$$

$$f(v_{\lfloor n/2 \rfloor + 2 + i}) = 4i + 9, \quad 1 \leq i \leq \lfloor n/2 \rfloor - 2$$

$$f(e_i) = f(v_{\lfloor n/2 \rfloor}) + 2i, \quad 1 \leq i \leq \lfloor n/2 \rfloor - 1,$$

$$f(e_{\lfloor n/2 \rfloor}) = 6, f(e_{\lfloor n/2 \rfloor + 1}) = 9, f(e_{\lfloor n/2 \rfloor + 2}) = 5,$$

$$f(e_{\lfloor n/2 \rfloor + i + 1}) = 4i + 7, \quad 1 \leq i \leq \lfloor n/2 \rfloor - 2$$

In this case we have  $e_f(0) = e_f(1) + 1 = 2(n-1)$

Case 3:  $n = 2, 4, 6$

For the graph  $T(P_2)$  the possible pairs of labels of adjacent vertices are (1,2), (1,3), (2,3). Then obviously  $e_f(0) = 0, e_f(1) = 3$ . Therefore  $T(P_2)$  is not a prime cordial graph.

For the graph  $T(P_4)$  the possible pairs of labels of adjacent vertices are (1,2), (1,3), (1,4), (1,5), (1,6), (1,7), (2,3), (2,4), (2,5), (2,6), (2,7), (3,4), (3,5), (3,6), (3,7), (4,5), (4,6), (4,7), (5,6), (5,7), (6,7). Then obviously  $e_f(0) = 4, e_f(1) = 7$ . That is,  $|e_f(0) - e_f(1)| = 3$  and in all other possible arrangement of vertex labels  $|e_f(0) - e_f(1)| > 3$ . Thus  $T(P_4)$  is not a prime cordial graph.

The case when  $n=6$  is to be dealt separately. The graph  $T(P_6)$  and its prime cordial labeling is shown in Fig 2.

Case 4:  $n$  even,  $n \geq 8$

$$f(v_1) = 2, f(v_2) = 4,$$

$$f(v_{i+2}) = 2(i+3), \quad 1 \leq i \leq n/2 - 3$$

$$f(v_{n/2}) = 6, f(v_{n/2+1}) = 9, f(v_{n/2+2}) = 5,$$

$$f(v_{n/2+2+i}) = 4i + 7, \quad 1 \leq i \leq n/2 - 2$$

$$f(e_i) = f(v_{n/2-1}) + 2i, \quad 1 \leq i \leq n/2 - 1,$$

$$f(e_{n/2}) = 3, f(e_{n/2+1}) = 1, f(e_{n/2+2}) = 7,$$

$$f(e_{n/2+2+i}) = 4i + 9, \quad 1 \leq i \leq n/2 - 3$$

In this case we have  $e_f(0) = e_f(1) + 1 = 2(n-1)$

That is,  $T(P_n)$  is a prime cordial graph,  $\forall n \geq 5$ .

*Illustration 2.2* Consider the graph  $T(P_7)$ . The labeling is as shown in Fig 3.

*Theorem 2.3*  $T(C_n)$  is prime cordial graph,  $\forall n \geq 5$ .

*Proof:* If  $v_1, v_2, v_3, \dots, v_n$  and  $e_1, e_2, e_3, \dots, e_n$  be the vertices and edges of  $C_n$  then  $v_1, v_2, v_3, \dots, v_n, e_1, e_2, e_3, \dots, e_n$  are vertices of  $T(C_n)$ .

We define vertex labeling  $f: V(T(C_n)) \rightarrow \{1, 2, 3, \dots, |V(G)|\}$  as follows. We consider following four cases.

Case 1:  $n = 4$

For the graph  $T(C_4)$  the possible pair of labels of adjacent vertices are (1,2), (1,3), (1,4), (1,5), (1,6), (1,7), (1,8), (2,3), (2,4), (2,5), (2,6), (2,7), (2,8), (3,4), (3,5), (3,6), (3,7), (3,8), (4,5), (4,6), (4,7), (4,8), (5,6), (5,7), (5,8), (6,7), (6,8), (7,8). Then obviously  $e_f(0) = 6, e_f(1) = 10$ . That is,  $|e_f(0) - e_f(1)| = 4$  and all other possible arrangement of vertex labels will yield  $|e_f(0) - e_f(1)| > 4$ . Thus  $T(C_4)$  is not a prime cordial graph.

Case 2:  $n$  even,  $n \geq 6$ 

$$\begin{aligned}
f(v_1) &= 2, f(v_2) = 8, \\
f(v_{i+2}) &= 4i + 10, \quad 1 \leq i \leq n/2 - 3 \\
f(v_{n/2}) &= 12, f(v_{n/2+1}) = 3, f(v_{n/2+2}) = 9, f(v_{n/2+3}) = 7, \\
f(v_{n/2+2+i}) &= 4i + 9, \quad 1 \leq i \leq n/2 - 3 \\
f(e_1) &= 4, f(e_2) = 10, \\
f(e_{i+2}) &= 4(i + 3), \quad 1 \leq i \leq n/2 - 3, \\
f(e_{n/2}) &= 6, f(e_{n/2+1}) = 1, f(e_{n/2+2}) = 5, \\
f(e_{n/2+1+i}) &= 4i + 7, \quad 1 \leq i \leq n/2 - 2
\end{aligned}$$

In view of the labeling pattern defined above we have

$$e_f(0) = e_f(1) = 2n.$$

Case 3:  $n = 3$ 

For the graph  $T(C_3)$  the possible pairs of labels of adjacent vertices are (1,2), (1,3), (1,4), (1,5), (1,6), (2,3), (2,4), (2,5), (2,6), (3,4), (3,5), (3,6), (4,5), (4,6), (5,6). Then obviously  $e_f(0) = 4$ ,  $e_f(1) = 8$ . That is,  $|e_f(0) - e_f(1)| = 4$  and all other possible arrangement of vertex labels will yield  $|e_f(0) - e_f(1)| > 4$ . Thus  $T(C_3)$  is not a prime cordial graph.

Case 4:  $n$  odd,  $n \geq 5$ 

$$\begin{aligned}
f(v_1) &= 2, \\
f(v_{i+1}) &= 4(i + 1), \quad 1 \leq i \leq \lfloor n/2 \rfloor - 1 \\
f(v_{\lfloor n/2 \rfloor + 1}) &= 6, f(v_{\lfloor n/2 \rfloor + 2}) = 9, f(v_{\lfloor n/2 \rfloor + 3}) = 5, \\
f(v_{\lfloor n/2 \rfloor + 3 + i}) &= 4i + 7, \quad 1 \leq i \leq n - \lfloor n/2 \rfloor - 3 \\
f(e_1) &= 4, \\
f(e_{i+1}) &= 4i + 6, \quad 1 \leq i \leq \lfloor n/2 \rfloor - 1, \\
f(e_{\lfloor n/2 \rfloor + 1}) &= 3, f(e_{\lfloor n/2 \rfloor + 2}) = 1, f(e_{\lfloor n/2 \rfloor + 3}) = 7, \\
f(e_{\lfloor n/2 \rfloor + 3 + i}) &= 4i + 9, \quad 1 \leq i \leq n - \lfloor n/2 \rfloor - 3
\end{aligned}$$

In view of the labeling pattern defined above we have

$$e_f(0) = e_f(1) = 2n.$$

Thus  $f$  is a prime cordial labeling of  $T(C_n)$ .

*Illustration 2.4* Consider the graph  $T(C_6)$ . The labeling is as shown in Fig 4.

**Theorem 2.5**  $P_2[P_m]$  is prime cordial graph  $\forall m \geq 5$ .

*Proof:* Let  $u_1, u_2, u_3, \dots, u_m$  be the vertices of  $P_m$  and  $v_1, v_2$  be the vertices of  $P_2$ . We define vertex labeling  $f: V(P_2[P_m]) \rightarrow \{1, 2, 3, \dots, |V(G)|\}$  as follows. We consider following four cases.

Case 1:  $m = 2, 4$ 

For the graph  $P_2[P_2]$  the possible pairs of labels of adjacent vertices are (1,2), (1,3), (1,4), (2,3), (2,4), (3,4). Then obviously  $e_f(0) = 1$ ,  $e_f(1) = 5$ . That is,  $|e_f(0) - e_f(1)| = 4$  and in all other possible arrangement of vertex labels we have  $|e_f(0) - e_f(1)| > 4$ . Therefore  $P_2[P_2]$  is not a prime cordial graph.

For the graph  $P_2[P_4]$  the possible pairs of labels of adjacent vertices are (1,2), (1,3), (1,4), (1,5), (1,6), (1,7), (1,8), (2,3), (2,4), (2,5), (2,6), (2,7), (2,8), (3,4), (3,5), (3,6), (3,7), (3,8), (4,5), (4,6), (4,7), (4,8), (5,6), (5,7), (5,8), (6,7), (6,8), (7,8). Then obviously  $e_f(0) = 7$ ,  $e_f(1) = 9$ . i.e.  $|e_f(0) - e_f(1)| = 2$  and in all other possible arrangement of vertex labels we have  $|e_f(0) - e_f(1)| > 2$ . Thus  $P_2[P_4]$  is not a prime cordial graph.

Case 2:  $m$  even,  $m \geq 6$ 

$$\begin{aligned}
f(u_1, v_1) &= 2, f(u_2, v_1) = 8, \\
f(u_{2+i}, v_1) &= 4i + 10, \quad 1 \leq i \leq m/2 - 3 \\
f(u_{m/2}, v_1) &= 12, \\
f(u_{m/2+i}, v_1) &= 4i - 3, \quad 1 \leq i \leq m/2
\end{aligned}$$

$$f(u_1, v_2) = 4, f(u_2, v_2) = 10,$$

$$f(u_{2+i}, v_2) = 4i + 12, \quad 1 \leq i \leq m/2 - 3$$

$$f(u_{m/2}, v_2) = 6, f(u_{m/2+1}, v_2) = 3,$$

$$f(u_{m/2+1+i}, v_2) = 4i + 3, \quad 1 \leq i \leq m/2 - 1$$

Using above pattern we have

$$e_f(0) = e_f(1) = \frac{5n-4}{2}$$

Case 3:  $m = 3$

For the graph  $P_2[P_3]$  the possible pairs of labels of adjacent vertices are (1,2), (1,3), (1,4), (1,5), (1,6), (2,3), (2,4), (2,5), (2,6), (3,4), (3,5), (3,6), (4,5), (4,6), (5,6). Then obviously  $e_f(0) = 4$ ,  $e_f(1) = 7$ . That is,  $|e_f(0) - e_f(1)| = 3$  and in all other possible arrangement of vertex labels we have  $|e_f(0) - e_f(1)| > 3$ . Thus  $P_2[P_3]$  is not a prime cordial graph.

Case 4:  $m$  odd,  $m \geq 5$

$$f(u_i, v_1) = 4(1+i), \quad 1 \leq i \leq \lfloor n/2 \rfloor - 1$$

$$f(u_{\lfloor n/2 \rfloor}, v_1) = 2,$$

$$f(u_{\lfloor n/2 \rfloor+1}, v_1) = 6, \quad f(u_{\lfloor n/2 \rfloor+2}, v_1) = 9, \quad f(u_{\lfloor n/2 \rfloor+3}, v_1) = 5,$$

$$f(u_{\lfloor n/2 \rfloor+2+i}, v_1) = 4i + 7, \quad 1 \leq i \leq \lfloor n/2 \rfloor - 2$$

$$f(u_1, v_2) = 4,$$

$$f(u_{1+i}, v_2) = 4i + 6, \quad 1 \leq i \leq \lfloor n/2 \rfloor - 1$$

$$f(u_{\lfloor n/2 \rfloor+1}, v_2) = 3, \quad f(u_{\lfloor n/2 \rfloor+2}, v_2) = 1, \quad f(u_{\lfloor n/2 \rfloor+3}, v_2) = 7,$$

$$f(u_{\lfloor n/2 \rfloor+2+i}, v_2) = 4i + 9, \quad 1 \leq i \leq \lfloor n/2 \rfloor - 2$$

Using above pattern we have

$$e_f(0) = e_f(1) + 1 = 2n + \lfloor n/2 \rfloor - 1.$$

Thus in case 2 and case 4 the graph  $P_2[P_m]$  satisfies the condition  $|e_f(0) - e_f(1)| \leq 1$ .

That is,  $P_2[P_m]$  is a prime cordial graph  $\forall m \geq 5$ .

*Illustration 2.6* Consider the graph  $P_2[P_5]$ . The prime cordial labeling is as shown in Fig 5.

*Theorem 2.7* Two cycles joined by a path  $P_m$  is a prime cordial graph.

*Proof:* Let  $G$  be the graph obtained by joining two cycles  $C_n$  and  $C'_n$  by a path  $P_m$ . Let  $v_1, v_2, v_3, \dots, v_n, v'_1, v'_2, v'_3, \dots, v'_n$  be the vertices of  $C_n, C'_n$  respectively. Here  $u_1, u_2, u_3, \dots$  are the vertices of  $P_m$ . We define vertex labeling  $f: V(G) \rightarrow \{1, 2, 3, \dots, |V(G)|\}$  as follows. We consider following four cases.

Case 1:  $m$  odd,  $m \geq 5$

$$f(u_1) = f(v_1) = 2, f(v_2) = 4,$$

$$f(v_{i+2}) = 2(i+3), \quad 1 \leq i \leq n-2$$

$$f(u_{i+1}) = f(v_n) + 2i, \quad 1 \leq i \leq \lfloor m/2 \rfloor - 2$$

$$f(u_{\lfloor m/2 \rfloor}) = 6, \quad f(u_{\lfloor m/2 \rfloor+1}) = 3, \quad f(u_{\lfloor m/2 \rfloor+2}) = 1$$

$$f(u_{\lfloor m/2 \rfloor+2+i}) = 2i + 3, \quad 1 \leq i \leq \lfloor m/2 \rfloor - 1$$

$$f(v'_1) = f(u_m), \quad f(v'_{i+1}) = f(v'_1) + 2i, \quad 1 \leq i \leq n-1$$

In view of the above defined labeling pattern we have

$$e_f(0) = e_f(1) = n + \lfloor m/2 \rfloor.$$

Case 2:  $m = 3$

$$f(u_1) = f(v_1) = 6, f(v_2) = 2, f(v_3) = 4,$$

$$f(v_{i+3}) = 2(i+3), \quad 1 \leq i \leq n-3$$

$$f(u_2) = 3, f(v'_1) = f(u_3) = 1, \quad f(v'_2) = 5$$

$$f(v'_{2+i}) = 2i + 5, \quad 1 \leq i \leq n-2$$

In view of the above defined labeling pattern we have

$$e_f(0) = e_f(1) = n + 1$$

Case 3:  $m$  even,  $m \geq 4$

$$f(u_1) = f(v_1) = 2, f(v_2) = 4,$$

$$f(v_{i+2}) = 2(i+3), \quad 1 \leq i \leq n-2$$

$$f(u_{i+1}) = f(v_n) + 2i, \quad 1 \leq i \leq m/2 - 2$$

$$f(u_{m/2}) = 6, \quad f(u_{m/2+1}) = 3, \quad f(u_{m/2+2}) = 1$$

$$f(u_{m/2+2+i}) = 2i+3, \quad 1 \leq i \leq m/2 - 2$$

$$f(v'_1) = f(u_m), \quad f(v'_{i+1}) = f(v'_1) + 2i, \quad 1 \leq i \leq n-1$$

In view of the above defined labeling pattern we have

$$e_f(0) = e_f(1) + 1 = n + m/2$$

Case 4:  $m = 2$

$$f(u_1) = f(v_1) = 2$$

$$f(v_{i+1}) = 2(i+1), \quad 1 \leq i \leq n-1$$

$$f(v'_1) = f(u_2) = 1$$

$$f(v'_{i+1}) = 2i+1, \quad 1 \leq i \leq n-1$$

In view of the above defined labeling pattern we have

$$e_f(0) + 1 = e_f(1) = n + 1.$$

Thus in all cases graph  $G$  satisfies the condition  $|e_f(0) - e_f(1)| \leq 1$ .

That is  $G$  is a prime cordial graph.

*Illustration 2.8* Consider the graph joining to copies of  $C_5$  by the path  $P_7$ . The prime cordial labeling is as shown in Fig 6.

*Theorem 2.9* The graph obtained by switching of an arbitrary vertex in cycle  $C_n$  admits prime cordial labeling except  $n = 5$ .

*Proof*: Let  $v_1, v_2, \dots, v_n$  be the successive vertices of  $C_n$  and  $G_v$  denotes the graph obtained by switching of a vertex  $v$ . Without loss of generality let the switched vertex be  $v_1$  and we initiate the labeling from the switched vertex  $v_1$ .

To define  $f: V(G_{v_1}) \rightarrow \{1, 2, 3, \dots, |V(G)|\}$  we consider following four cases.

Case 1:  $n = 4$

The case when  $n=4$  is to be dealt separately. The graph  $G_{v_1}$  and its prime cordial labeling is shown in Fig 7.

Case 2:  $n$  even,  $n \geq 6$

$$f(v_1) = 2, f(v_2) = 1, f(v_3) = 4$$

$$f(v_{3+i}) = 2(i+3), \quad 1 \leq i \leq n/2 - 3$$

$$f(v_{n/2+1}) = 6, f(v_{n/2+2}) = 3$$

$$f(v_{n/2+2+i}) = 2i+3, \quad 1 \leq i \leq n/2 - 2$$

Using above pattern we have

$$e_f(0) = e_f(1) + 1 = n - 2$$

Case 3:  $n = 5$

For the graph  $G_{v_1}$  the possible pairs of labels of adjacent vertices are (1,2), (1,3), (1,4), (1,5), (2,3), (2,4), (2,5), (3,4), (3,5), (4,5). Then obviously  $e_f(0) = 1$ ,  $e_f(1) = 4$ . That is,  $|e_f(0) - e_f(1)| = 3$  and in all other possible arrangement of vertex labels we have  $|e_f(0) - e_f(1)| > 3$ . Thus,  $G_{v_1}$  is not a prime cordial graph.

Case 4:  $n$  odd,  $n \geq 7$

$$f(v_1) = 2, f(v_2) = 1, f(v_3) = 4$$

$$f(v_{3+i}) = 2(i+3), \quad 1 \leq i \leq \lfloor n/2 \rfloor - 3$$

$$f(v_{\lfloor n/2 \rfloor + 1}) = 6, f(v_{\lfloor n/2 \rfloor + 2}) = 3$$

$$f(v_{\lfloor n/2 \rfloor + 2 + i}) = 2i + 3, 1 \leq i \leq \lfloor n/2 \rfloor - 1$$

Using above pattern we have

$$e_f(0) + 1 = e_f(1) = n - 2$$

Thus in cases 1, 2 and 4  $f$  satisfies the condition for prime cordial labeling. That is,  $Gv_1$  is a prime cordial graph.

*Illustration 2.10* Consider the graph obtained by switching the vertex in  $C_7$ . The prime cordial labeling is as shown in Fig 8.

### 3. Concluding Remarks

It is always interesting to investigate whether any graph or graph families admit a particular type of graph labeling? Here we investigate five results corresponding to prime cordial labeling. Analogous work can be carried out for other graph families and in the context of different graph labeling problems.

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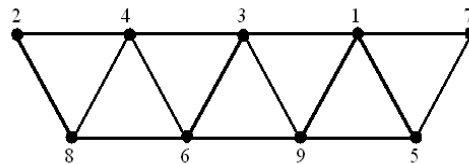


Figure 1.  $T(P_5)$  and its prime cordial labeling

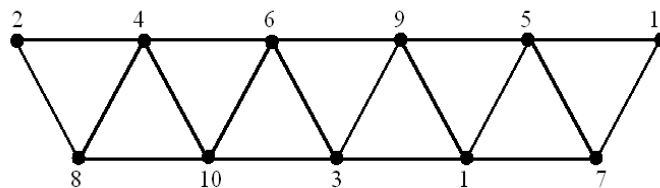


Figure 2.  $T(P_6)$  and its prime cordial labeling

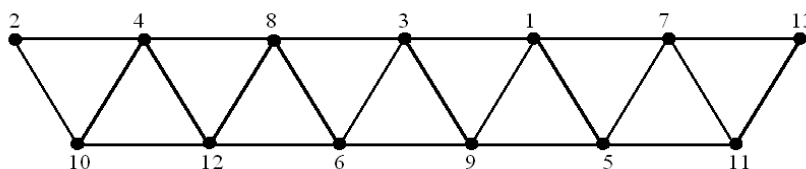
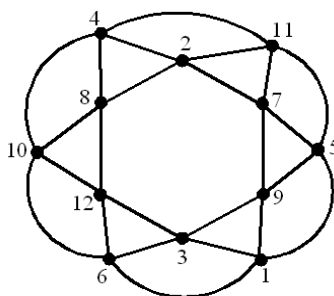
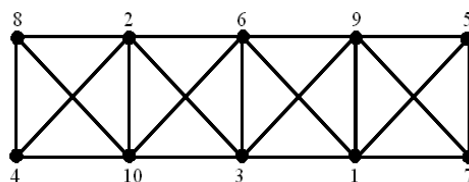
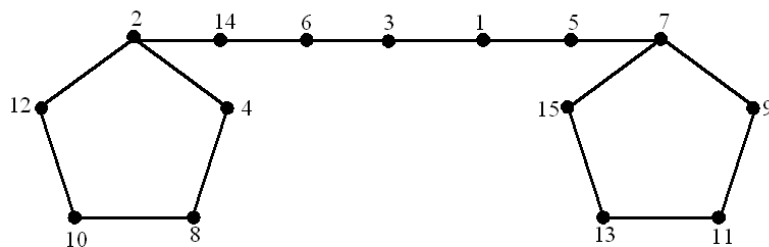
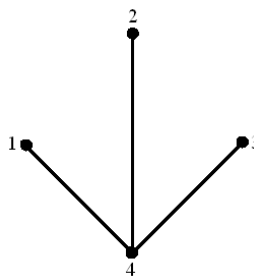


Figure 3.  $T(P_7)$  and its prime cordial labeling

Figure 4.  $T(C_6)$  and its prime cordial labelingFigure 5.  $P_2[P_5]$  and its prime cordial labelingFigure 6. Two cycles  $C_5$  join by  $P_7$  and its prime cordial labelingFigure 7. Vertex switching in  $C_4$  and its prime cordial labeling

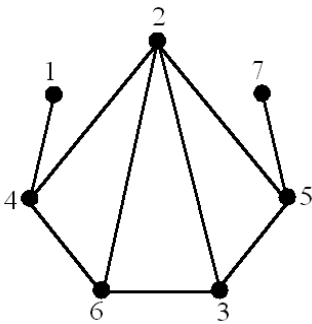


Figure 8. Vertex switching in  $C_7$  and its prime cordial labeling

# L(2,1)-Labeling in the Context of Some Graph Operations

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## Abstract

Let  $G = (V(G), E(G))$  be a connected graph. For integers  $j \geq k$ ,  $L(j, k)$ -labeling of a graph  $G$  is an integer labeling of the vertices in  $V$  such that adjacent vertices receive integers which differ by at least  $j$  and vertices which are at distance two apart receive labels which differ by at least  $k$ . In this paper we discuss  $L(2, 1)$ -labeling (or distance two labeling) in the context of some graph operations.

**Keywords:** Graph Labeling,  $\lambda$ - Number,  $\lambda'$ - Number

## 1. Introduction

We begin with finite, connected, undirected graph  $G = (V(G), E(G))$  without loops and multiple edges. For standard terminology and notations we refer to (West, D., 2001). We will give brief summary of definitions and information which are prerequisites for the present work.

**Definition 1.1** Duplication of a vertex  $v_k$  of graph  $G$  produces a new graph  $G'$  by adding a vertex  $v'_k$  with  $N(v'_k) = N(v_k)$ . In other words a vertex  $v'_k$  is said to be duplication of  $v_k$  if all the vertices which are adjacent to  $v_k$  are now adjacent to  $v'_k$  also.

**Definition 1.2** Let  $G$  be a graph. A graph  $H$  is called a *supersubdivision* of  $G$  if  $H$  is obtained from  $G$  by replacing every edge  $e_i$  of  $G$  by a complete bipartite graph  $K_{2,m_i}$  (for some  $m_i$  and  $1 \leq i \leq q$ ) in such a way that the ends of each  $e_i$  are merged with the two vertices of 2-vertices part of  $K_{2,m_i}$  after removing the edge  $e_i$  from graph  $G$ .

A new family of graph introduced in (Vaidya, S., 2008, p.54-64) defined as follows.

**Definition 1.3** A graph obtained by replacing each vertex of a star  $K_{1,n}$  by a graph  $G$  is called *star of  $G$*  denoted as  $G'$ . The central graph in  $G'$  is the graph which replaces apex vertex of  $K_{1,n}$ .

**Definition 1.4** If the vertices of the graph are assigned values subject to certain conditions then it is known as *graph labeling*.

The unprecedented growth of wireless communication is recorded but the available radio frequencies allocated to these communication networks are not enough. Proper allocation of frequencies is demand of the time. The interference by unconstrained transmitters will interrupt the communications. This problem was taken up in (Hale, W., 1980, p.1497-1514) in terms of graph labeling. In a private communication with Griggs during 1988 Roberts proposed a variation in channel assignment problem. According to him any two close transmitters must receive different channels in order to avoid interference. Motivated by this problem the concept of  $L(2,1)$ -labeling was introduced by (Yeh, R., 1990) and

(Griggs, J., and Yeh, R., 1992, p.586-595) which is defined as follows.

**Definition 1.5** For a graph  $G$ ,  $L(2, 1)$ -labeling (or distance two labeling) with span  $k$  is a function  $f : V(G) \rightarrow \{0, 1, \dots, k\}$  such that the following conditions are satisfied:

$$(1) |f(x) - f(y)| \geq 2 \text{ if } d(x, y) = 1$$

$$(2) |f(x) - f(y)| \geq 1 \text{ if } d(x, y) = 2$$

In other words the  $L(2, 1)$ -labeling of a graph is an abstraction of assigning integer frequencies to radio transmitters such that (1) Transmitters that are one unit of distance apart receive frequencies that differ by at least two and (2) Transmitters that are two units of distance apart receive frequencies that differ by at least one. The *span* of  $f$  is the largest number in  $f(V)$ . The minimum span taken over all  $L(2, 1)$ -labeling of  $G$ , denoted as  $\lambda(G)$  is called the  $\lambda$ -number of  $G$ . The minimum label in  $L(2, 1)$ -labeling of  $G$  is assumed to be 0.

**Definition 1.6** An injective  $L(2, 1)$ -labeling is called an  $L'(2, 1)$ -labeling and the minimum span taken over all such  $L'(2, 1)$ -labeling is called  $\lambda'$ -number of the graph.

The  $L(2, 1)$ -labeling has been extensively studied in the recent past by many researchers (Georges, J., 1995, p.141-159), (Georges, J.P., 2001, p.28-35), (Georges, J., 1996, p.47-57), (Georges, J., 1994, p.103-111), (Liu, D., 1997, p.13-22), (Shao, Z., 2005, p.668-671). Practically it is observed that the interference might go beyond two levels. This observation motivated (Chartrand, G., 2001, p.77-85) to introduce the concept of radio labeling which is the extension of  $L(2, 1)$ -labeling when the interference is beyond two levels to the largest possible - the diameter of  $G$ . We investigate three results corresponding to  $L(2, 1)$ -labeling and  $L'(2, 1)$ -labeling each.

## 2. Main Results

**Theorem 2.1** Let  $C'_n$  be the graph obtained by duplicating all the vertices of the cycle  $C_n$  at a time then  $\lambda(C'_n) = 7$ . (where  $n > 3$ )

**Proof:** Let  $v'_1, v'_2, \dots, v'_n$  be the duplicated vertices corresponding to  $v_1, v_2, \dots, v_n$  of cycle  $C_n$ .

To define  $f : V(C'_n) \rightarrow N \cup \{0\}$ , we consider following four cases.

**Case 1:**  $n \equiv 0 \pmod{3}$  (where  $n > 5$ )

We label the vertices as follows.

$$\begin{aligned} f(v_i) &= 0, i = 3j - 2, 1 \leq j \leq \frac{n}{3} \\ f(v_i) &= 2, i = 3j - 1, 1 \leq j \leq \frac{n}{3} \\ f(v_i) &= 4, i = 3j, 1 \leq j \leq \frac{n}{3} \\ f(v'_i) &= 7, i = 3j - 2, 1 \leq j \leq \frac{n}{3} \\ f(v'_i) &= 6, i = 3j - 1, 1 \leq j \leq \frac{n}{3} \\ f(v'_i) &= 5, i = 3j, 1 \leq j \leq \frac{n}{3} \end{aligned}$$

**Case 2:**  $n \equiv 1 \pmod{3}$  (where  $n > 5$ )

We label the vertices as follows.

$$\begin{aligned} f(v_i) &= 0, i = 3j - 2, 1 \leq j \leq \lfloor \frac{n}{3} \rfloor \\ f(v_i) &= 2, i = 3j - 1, 1 \leq j \leq \lfloor \frac{n}{3} \rfloor \\ f(v_i) &= 4, i = 3j, 1 \leq j \leq \lfloor \frac{n}{3} \rfloor \\ f(v_{n-3}) &= 0, f(v_{n-2}) = 3, f(v_{n-1}) = 1, f(v_n) = 4 \\ f(v'_i) &= 7, i = 3j - 2, 1 \leq j \leq \lfloor \frac{n}{3} \rfloor \\ f(v'_i) &= 6, i = 3j - 1, 1 \leq j \leq \lfloor \frac{n}{3} \rfloor \\ f(v'_i) &= 5, i = 3j, 1 \leq j \leq \lfloor \frac{n}{3} \rfloor \\ f(v'_{n-3}) &= 7, f(v'_{n-2}) = 7, f(v'_{n-1}) = 6, f(v'_n) = 5 \end{aligned}$$

**Case 3:**  $n \equiv 2 \pmod{3}$  (where  $n > 5$ )

We label the vertices as follows.

$$\begin{aligned} f(v_i) &= 0, i = 3j - 2, 1 \leq j \leq \lfloor \frac{n}{3} \rfloor \\ f(v_i) &= 2, i = 3j - 1, 1 \leq j \leq \lfloor \frac{n}{3} \rfloor \\ f(v_i) &= 4, i = 3j, 1 \leq j \leq \lfloor \frac{n}{3} \rfloor \\ f(v_{n-1}) &= 1, f(v_n) = 3 \\ f(v'_1) &= 6, f(v'_n) = 7 \end{aligned}$$

$$\begin{aligned}f(v'_i) &= 6, i = 3j - 1, 1 \leq j \leq \lfloor \frac{n}{3} \rfloor \\f(v'_i) &= 5, i = 3j, 1 \leq j \leq \lfloor \frac{n}{3} \rfloor \\f(v'_i) &= 7, i = 3j + 1, 1 \leq j \leq \lfloor \frac{n}{3} \rfloor\end{aligned}$$

**Case 4:**  $n = 4, 5$

These cases are to be dealt separately. The  $L(2, 1)$ -labeling for the graphs obtained by duplicating all the vertices at a time in the cycle  $C_n$  when  $n = 4, 5$  are as shown in Fig 1

Thus in all the possibilities  $R_f = \{0, 1, 2, \dots, 7\} \subset N \cup \{0\}$ .

i.e.  $\lambda(C'_n) = 7$ .

**Remark** The  $L(2, 1)$ -labeling for the graph obtained by duplicating all the vertices of the cycle  $C_3$  is shown in Fig 2  
Thus  $R_f = \{0, 1, 2, \dots, 6\} \subset N \cup \{0\}$ .

i.e.  $\lambda(C'_3) = 6$ .

**Illustration 2.2** Consider the graph  $C_6$  and duplicate all the vertices at a time. The  $L(2, 1)$ -labeling is as shown in Fig 3.

**Theorem 2.3** Let  $C'_n$  be the graph obtained by duplicating all the vertices at a time of the cycle  $C_n$  then  $\lambda'(C'_n) = p - 1$ , where  $p$  is the total number vertices in  $C'_n$  (where  $n > 3$ ).

**Proof:** Let  $v'_1, v'_2, \dots, v'_n$  be the duplicated vertices corresponding to  $v_1, v_2, \dots, v_n$  of cycle  $C_n$ .

To define  $f : V(C'_n) \rightarrow N \cup \{0\}$ , we consider following two cases.

**Case 1:**  $n > 5$

$$\begin{aligned}f(v_i) &= 2i - 7, 4 \leq i \leq n \\f(v_i) &= f(v_n) + 2, 1 \leq i \leq 3 \\f(v'_i) &= 2i - 2, 1 \leq i \leq n\end{aligned}$$

Now label the vertices of  $C'_n$  using the above defined pattern we have  $R_f = \{0, 1, 2, \dots, p - 1\} \subset N \cup \{0\}$

This implies that  $\lambda'(C'_n) = p - 1$ .

**Case 2:**  $n = 4, 5$

These cases to be dealt separately. The  $L'(2, 1)$ -labeling for the graphs obtained by duplicating all the vertices at a time in the cycle  $C_n$  when  $n = 4, 5$  are as shown in the following Fig 4.

**Remark** The  $L'(2, 1)$ -labeling for the graphs obtained by duplicating all the vertices at a time in the cycle  $C_3$  is shown in the following Fig 5.

Thus  $R_f = \{0, 1, 2, \dots, 6\} \subset N \cup \{0\}$ .

i.e.  $\lambda'(C'_3) = 6$ .

**Illustration 2.4** Consider the graph  $C_6$  and duplicating all the vertices at a time. The  $L'(2, 1)$ -labeling is as shown in Fig 6.

**Theorem 2.5** Let  $C'_n$  be the graph obtained by taking arbitrary supersubdivision of each edge of cycle  $C_n$  then

**1** For  $n$  even

$$\lambda(C'_n) = \Delta + 2$$

**2** For  $n$  odd

$$\lambda(C'_n) = \begin{cases} \Delta + 2; & \text{if } s + t + r < \Delta, \\ \Delta + 3; & \text{if } s + t + r = \Delta, \\ s + t + r + 2; & \text{if } s + t + r > \Delta \end{cases}$$

where  $v_k$  is a vertex with label 2,

$s$  is number of subdivision between  $v_{k-2}$  and  $v_{k-1}$ ,

$t$  is number of subdivision between  $v_{k-1}$  and  $v_k$ ,

$r$  is number of subdivision between  $v_k$  and  $v_{k+1}$ ,

$\Delta$  is the maximum degree of  $C'_n$ .

**Proof:** Let  $v_1, v_2, \dots, v_n$  be the vertices of cycle  $C_n$ . Let  $C'_n$  be the graph obtained by arbitrary super subdivision of cycle  $C_n$ .

It is obvious that for any two vertices  $v_i$  and  $v_{i+2}$ ,  $N(v_i) \cap N(v_{i+2}) = \emptyset$   
 To define  $f : V(C'_n) \rightarrow N \cup \{0\}$ , we consider following two cases.

**Case 1:**  $C_n$  is even cycle

$$\begin{aligned} f(v_{2i-1}) &= 0, 1 \leq i \leq \frac{n}{2} \\ f(v_{2i}) &= 1, 1 \leq i \leq \frac{n}{2} \end{aligned}$$

If  $P_{ij}$  is the number of supersubdivisions between  $v_i$  and  $v_j$  then for the vertex  $v_1$ ,  $|N(v_1)| = P_{12} + P_{n1}$ . Without loss of generality we assume that  $v_1$  is the vertex with maximum degree i.e.  $d(v_1) = \Delta$ . suppose  $u_1, u_2, \dots, u_\Delta$  be the members of  $N(v_1)$ . We label the vertices of  $N(v_1)$  as follows.

$$f(u_i) = 2 + i, 1 \leq i \leq \Delta$$

As  $N(v_1) \cap N(v_3) = \emptyset$  then it is possible to label the vertices of  $N(v_3)$  using the vertex labels of the members of  $N(v_1)$  in accordance with the requirement for  $L(2, 1)$ -labeling. Extending this argument recursively upto  $N(v_{n-1})$  it is possible to label all the vertices of  $C'_n$  using the distinct numbers between 0 and  $\Delta + 2$ .

$$\text{i.e. } R_f = \{0, 1, 2, \dots, \Delta + 2\} \subset N \cup \{0\}$$

$$\text{Consequently } \lambda(C'_n) = \Delta + 2.$$

**Case 2:**  $C_n$  is odd cycle

Let  $v_1, v_2, \dots, v_n$  be the vertices of cycle  $C_n$ .

Without loss of generality we assume that  $v_1$  is a vertex with maximum degree and  $v_k$  be the vertex with minimum degree.

Define  $f(v_k) = 2$  and label the remaining vertices alternatively with labels 0 and 1 such that  $f(v_1) = 0$ . Then either  $f(v_{k-1}) = 1$ ;  $f(v_{k+1}) = 0$  OR  $f(v_{k-1}) = 0$ ;  $f(v_{k+1}) = 1$ . We assign labeling in such a way that  $f(v_{k-1}) = 1$ ;  $f(v_{k+1}) = 0$ .

Now following the procedure adapted in case (1) it is possible to label all the vertices except the vertices between  $v_{k-1}$  and  $v_k$ . Label the vertices between  $v_{k-1}$  and  $v_k$  using the vertex labels of  $N(v_1)$  except the labels which are used earlier to label the vertices between  $v_{k-2}$ ,  $v_{k-1}$  and between  $v_k$ ,  $v_{k+1}$ .

If there are  $p$  vertices  $u_1, u_2, \dots, u_p$  are left unlabeled between  $v_{k-1}$  and  $v_k$  then label them as follows,

$$f(u_i) = \max\{\text{labels of the vertices between } v_{k-2} \text{ and } v_{k-1}, \text{ labels of the vertices between } v_k \text{ and } v_{k+1}\} + i, 1 \leq i \leq p$$

Now if  $s$  is the number of subdivisions between  $v_{k-2}$  and  $v_{k-1}$

$t$  is the number of subdivisions between  $v_{k-1}$  and  $v_k$

$r$  is the number of subdivisions between  $v_k$  and  $v_{k+1}$

then (1)  $R_f = \{0, 1, 2, \dots, \Delta + 2\} \subset N \cup \{0\}$ , when  $s + t + r < \Delta$

$$\text{i.e. } \lambda(C'_n) = \Delta + 2$$

(2)  $R_f = \{0, 1, 2, \dots, \Delta + 3\} \subset N \cup \{0\}$ , when  $s + t + r = \Delta$

$$\text{i.e. } \lambda(C'_n) = \Delta + 3$$

(3)  $R_f = \{0, 1, 2, \dots, s + t + r + 2\} \subset N \cup \{0\}$ , when  $s + t + r > \Delta$

$$\text{i.e. } \lambda(C'_n) = s + t + r + 2$$

**Illustration 2.6** Consider the graph  $C_8$ . The  $L(2, 1)$ -labeling of  $C'_8$  is shown in Fig 7.

**Theorem 2.7** Let  $G'$  be the graph obtained by taking arbitrary supersubdivision of each edge of graph  $G$  with number of vertices  $n \geq 3$  then  $\lambda'(G') = p - 1$ , where  $p$  is the total number of vertices in  $G'$ .

**Proof:** Let  $v_1, v_2, \dots, v_n$  be the vertices of any connected graph  $G$  and let  $G'$  be the graph obtained by taking arbitrary supersubdivision of  $G$ . Let  $u_k$  be the vertices which are used for arbitrary supersubdivision of the edge  $v_i v_j$  where  $1 \leq i \leq n$ ,  $1 \leq j \leq n$  and  $i < j$

Here  $k$  is a total number of vertices used for arbitrary supersubdivision.

We define  $f : V(G') \rightarrow N \cup \{0\}$  as

$$f(v_i) = i - 1, \text{ where } 1 \leq i \leq n$$

Now we label the vertices  $u_i$  in the following order.

First we label the vertices between  $v_1$  and  $v_{1+j}$ ,  $1 \leq j \leq n$  then following the same procedure for  $v_2, v_3, \dots, v_n$

$$f(u_i) = f(v_n) + i, 1 \leq i \leq k$$

Now label the vertices of  $G'$  using the above defined pattern we have  $R_f = \{0, 1, 2, \dots, p - 1\} \subset N \cup \{0\}$

This implies that  $\lambda'(G') = p - 1$ .

**Illustration 2.8** Consider the graph  $P_4$  and its supersubdivision. The  $L'(2, 1)$ -labeling is as shown in Fig 8.

**Theorem 2.9** Let  $C'_n$  be the graph obtained by taking star of a cycle  $C_n$  then  $\lambda(C'_n) = 5$ .

**Proof:** Let  $v_1, v_2, \dots, v_n$  be the vertices of cycle  $C_n$  and  $v_{ij}$  be the vertices of cycle  $C_n$  which are adjacent to the  $i^{th}$  vertex of cycle  $C_n$ .

To define  $f : V(C'_n) \rightarrow N \cup \{0\}$ , we consider following four cases.

**Case 1:**  $n \equiv 0 \pmod{3}$

$$\begin{aligned} f(v_i) &= 0, i = 3j - 2, 1 \leq j \leq \frac{n}{3} \\ f(v_i) &= 2, i = 3j - 1, 1 \leq j \leq \frac{n}{3} \\ f(v_i) &= 4, i = 3j, 1 \leq j \leq \frac{n}{3} \end{aligned}$$

Now we label the vertices  $v_{ij}$  of star of a cycle according to the label of  $f(v_i)$ .

(1) when  $f(v_i) = 0, i = 3j - 2, 1 \leq j \leq \frac{n}{3}$

$$\begin{aligned} f(v_{ik}) &= 3, k = 3p - 2, 1 \leq p \leq \frac{n}{3} \\ f(v_{ik}) &= 5, k = 3p - 1, 1 \leq p \leq \frac{n}{3} \\ f(v_{ik}) &= 1, k = 3p, 1 \leq p \leq \frac{n}{3} \end{aligned}$$

(2) when  $f(v_i) = 2, i = 3j - 1, 1 \leq j \leq \frac{n}{3}$

$$\begin{aligned} f(v_{ik}) &= 5, k = 3p - 2, 1 \leq p \leq \frac{n}{3} \\ f(v_{ik}) &= 3, k = 3p - 1, 1 \leq p \leq \frac{n}{3} \\ f(v_{ik}) &= 1, k = 3p, 1 \leq p \leq \frac{n}{3} \end{aligned}$$

(3) when  $f(v_i) = 4, i = 3j, 1 \leq j \leq \frac{n}{3}$

$$\begin{aligned} f(v_{ik}) &= 1, k = 3p - 2, 1 \leq p \leq \frac{n}{3} \\ f(v_{ik}) &= 3, k = 3p - 1, 1 \leq p \leq \frac{n}{3} \\ f(v_{ik}) &= 5, k = 3p, 1 \leq p \leq \frac{n}{3} \end{aligned}$$

**Case 2:**  $n \equiv 1 \pmod{3}$

$$\begin{aligned} f(v_i) &= 0, i = 3j - 2, 1 \leq j \leq \lfloor \frac{n}{3} \rfloor \\ f(v_i) &= 2, i = 3j - 1, 1 \leq j \leq \lfloor \frac{n}{3} \rfloor \\ f(v_i) &= 5, i = 3j, 1 \leq j \leq \lfloor \frac{n}{3} \rfloor \\ f(v_n) &= 3 \end{aligned}$$

Now we label the vertices of star of a cycle  $v_{ij}$  according to label of  $f(v_i)$ .

(1) when  $f(v_i) = 0, i = 3j - 2, 1 \leq j \leq \lfloor \frac{n}{3} \rfloor$

$$\begin{aligned} f(v_{ik}) &= 4, k = 3p - 2, 1 \leq p \leq \lfloor \frac{n}{3} \rfloor \\ f(v_{ik}) &= 2, k = 3p - 1, 1 \leq p \leq \lfloor \frac{n}{3} \rfloor \\ f(v_{ik}) &= 0, k = 3p, 1 \leq p \leq \lfloor \frac{n}{3} \rfloor - 1 \\ f(v_{i(n-1)}) &= 5, \\ f(v_{in}) &= 1 \end{aligned}$$

(2) when  $f(v_i) = 2, i = 3j - 1, 1 \leq j \leq \lfloor \frac{n}{3} \rfloor$

$$\begin{aligned} f(v_{ik}) &= 4, k = 3p - 2, 1 \leq p \leq \lfloor \frac{n}{3} \rfloor \\ f(v_{ik}) &= 0, k = 3p - 1, 1 \leq p \leq \lfloor \frac{n}{3} \rfloor \\ f(v_{ik}) &= 2, k = 3p, 1 \leq p \leq \lfloor \frac{n}{3} \rfloor - 1 \\ f(v_{i(n-1)}) &= 3, \\ f(v_{in}) &= 1 \end{aligned}$$

(3) when  $f(v_i) = 5, i = 3j, 1 \leq j \leq \lfloor \frac{n}{3} \rfloor$

$$\begin{aligned}
 f(v_{ik}) &= 1, k = 3p - 2, 1 \leq p \leq \lfloor \frac{n}{3} \rfloor \\
 f(v_{ik}) &= 3, k = 3p - 1, 1 \leq p \leq \lfloor \frac{n}{3} \rfloor \\
 f(v_{ik}) &= 5, k = 3p, 1 \leq p \leq \lfloor \frac{n}{3} \rfloor - 1 \\
 f(v_{i(n-1)}) &= 0, \\
 f(v_{in}) &= 4
 \end{aligned}$$

(4) when  $f(v_i) = 3, i = n$

$$\begin{aligned}
 f(v_{ik}) &= 1, k = 3p - 2, 1 \leq p \leq \lfloor \frac{n}{3} \rfloor \\
 f(v_{ik}) &= 5, k = 3p - 1, 1 \leq p \leq \lfloor \frac{n}{3} \rfloor \\
 f(v_{ik}) &= 3, k = 3p, 1 \leq p \leq \lfloor \frac{n}{3} \rfloor - 1 \\
 f(v_{i(n-1)}) &= 0 \\
 f(v_{in}) &= 4
 \end{aligned}$$

Case 3:  $n \equiv 2 \pmod{3}, n \neq 5$

$$\begin{aligned}
 f(v_i) &= 1, i = 3j - 2, 1 \leq j \leq \lfloor \frac{n}{3} \rfloor - 1 \\
 f(v_i) &= 3, i = 3j - 1, 1 \leq j \leq \lfloor \frac{n}{3} \rfloor - 1 \\
 f(v_i) &= 5, i = 3j, 1 \leq j \leq \lfloor \frac{n}{3} \rfloor - 1 \\
 f(v_{n-4}) &= 0, f(v_{n-3}) = 2, f(v_{n-2}) = 5, f(v_{n-1}) = 0, f(v_n) = 4
 \end{aligned}$$

Now we label the vertices  $v_{ij}$  of star of a cycle according to the label of  $f(v_i)$ .

(1) when  $f(v_i) = 1, i = 3j - 2, 1 \leq j \leq \lfloor \frac{n}{3} \rfloor - 1$

$$\begin{aligned}
 f(v_{ik}) &= 5, k = 3p - 2, 1 \leq p \leq \lfloor \frac{n}{3} \rfloor \\
 f(v_{ik}) &= 3, k = 3p - 1, 1 \leq p \leq \lfloor \frac{n}{3} \rfloor \\
 f(v_{ik}) &= 1, k = 3p, 1 \leq p \leq \lfloor \frac{n}{3} \rfloor \\
 f(v_{i(n-1)}) &= 4, \\
 f(v_{in}) &= 0
 \end{aligned}$$

(2) when  $f(v_i) = 3, i = 3j - 1, 1 \leq j \leq \lfloor \frac{n}{3} \rfloor - 1$

$$\begin{aligned}
 f(v_{ik}) &= 0, k = 3p - 2, 1 \leq p \leq \lfloor \frac{n}{3} \rfloor \\
 f(v_{ik}) &= 2, k = 3p - 1, 1 \leq p \leq \lfloor \frac{n}{3} \rfloor \\
 f(v_{ik}) &= 4, k = 3p, 1 \leq p \leq \lfloor \frac{n}{3} \rfloor \\
 f(v_{i(n-1)}) &= 1, \\
 f(v_{in}) &= 5
 \end{aligned}$$

(3) when  $f(v_i) = 5, i = 3j, 1 \leq j \leq \lfloor \frac{n}{3} \rfloor - 1$  and  $i = n - 2$

$$\begin{aligned}
 f(v_{ik}) &= 1, k = 3p - 2, 1 \leq p \leq \lfloor \frac{n}{3} \rfloor \\
 f(v_{ik}) &= 3, k = 3p - 1, 1 \leq p \leq \lfloor \frac{n}{3} \rfloor \\
 f(v_{ik}) &= 5, k = 3p, 1 \leq p \leq \lfloor \frac{n}{3} \rfloor \\
 f(v_{i(n-1)}) &= 2, \\
 f(v_{in}) &= 4
 \end{aligned}$$

(4) when  $f(v_i) = 0, i = n - 4, n - 1$

$$\begin{aligned}
 f(v_{ik}) &= 3, k = 3p - 2, 1 \leq p \leq \lfloor \frac{n}{3} \rfloor \\
 f(v_{ik}) &= 5, k = 3p - 1, 1 \leq p \leq \lfloor \frac{n}{3} \rfloor \\
 f(v_{ik}) &= 1, k = 3p, 1 \leq p \leq \lfloor \frac{n}{3} \rfloor - 1 \\
 f(v_{i(n-2)}) &= 2, \\
 f(v_{i(n-1)}) &= 4, \\
 f(v_{in}) &= 1
 \end{aligned}$$

(5) when  $f(v_i) = 2, i = n - 3$

$$\begin{aligned}
 f(v_{ik}) &= 4, k = 3p - 2, 1 \leq p \leq \lfloor \frac{n}{3} \rfloor \\
 f(v_{ik}) &= 0, k = 3p - 1, 1 \leq p \leq \lfloor \frac{n}{3} \rfloor \\
 f(v_{ik}) &= 2, k = 3p, 1 \leq p \leq \lfloor \frac{n}{3} \rfloor \\
 f(v_{i(n-1)}) &= 5, \\
 f(v_{in}) &= 1
 \end{aligned}$$

**(6)** when  $f(v_i) = 4, i = n$

$$\begin{aligned}
 f(v_{ik}) &= 2, k = 3p - 2, 1 \leq p \leq \lfloor \frac{n}{3} \rfloor \\
 f(v_{ik}) &= 0, k = 3p - 1, 1 \leq p \leq \lfloor \frac{n}{3} \rfloor \\
 f(v_{ik}) &= 4, k = 3p, 1 \leq p \leq \lfloor \frac{n}{3} \rfloor \\
 f(v_{i(n-1)}) &= 1, \\
 f(v_{in}) &= 5
 \end{aligned}$$

**Case 4:**  $n = 5$

This case is to be dealt separately. The  $L(2, 1)$ -labeling for the graph obtained by taking star of the cycle  $C_5$  is shown in Fig 9. Thus in all the possibilities we have  $\lambda(C'_n) = 5$

**Illustration 2.10** Consider the graph  $C_7$ , the  $L(2, 1)$ -labeling is as shown in Fig 10.

**Theorem 2.11** Let  $G'$  be the graph obtained by taking star of a graph  $G$  then  $\lambda'(G') = p - 1$ , where  $p$  be the total number of vertices of  $G'$ .

**Proof:** Let  $v_1, v_2, \dots, v_n$  be the vertices of any connected graph  $G$ . Let  $v_{ij}$  be the vertices of a graph which is adjacent to the  $i^{th}$  vertex of graph  $G$ . By the definition of a star of a graph the total number of vertices in a graph  $G'$  are  $n(n + 1)$ .

To define  $f : V(G') \rightarrow N \cup \{0\}$

$$f(v_{i1}) = i - 1, \quad 1 \leq i \leq n$$

for  $1 \leq i \leq n$  do the labeling as follows:

$$\begin{aligned}
 f(v_i) &= f(v_{ni}) + 1 \\
 f(v_{1(i+1)}) &= f(v_i) + 1 \\
 f(v_{(j+1)(i+1)}) &= f(v_{j(i+1)}) + 1, 1 \leq j \leq n - 1
 \end{aligned}$$

Thus  $\lambda'(G') = p - 1 = n^2 + n - 1$

**Illustration 2.12** Consider the star of a graph  $K_4$ , the  $L'(2, 1)$ -labeling is shown in Fig 11.

### 3 Concluding Remarks

Here we investigate some new results corresponding to  $L(2, 1)$ -labeling and  $L'(2, 1)$ -labeling. The  $\lambda$ -number is completely determined for the graphs obtained by duplicating the vertices altogether in a cycle, arbitrary supersubdivision of a cycle and star of a cycle. We also determine  $\lambda'$ -number for some graph families. This work is an effort to relate some graph operations and  $L(2, 1)$ -labeling. All the results reported here are of very general nature and  $\lambda$ -number as well as  $\lambda'$ -number are completely determined for the larger graphs resulted from the graph operations on standard graphs which is the salient features of this work. It is also possible to investigate some more results corresponding to other graph families.

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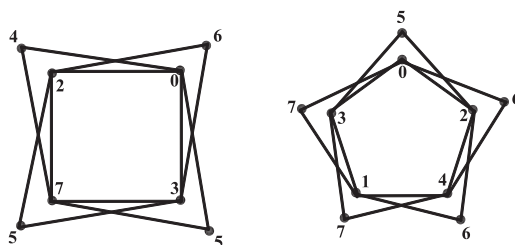


Figure 1. vertex duplication in  $C_4, C_5$  and  $L(2, 1)$ -labeling

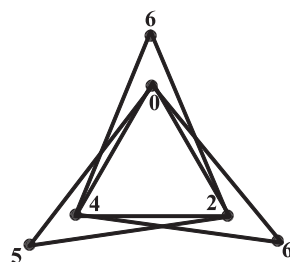


Figure 2. vertex duplication in  $C_3$  and  $L(2, 1)$ -labeling

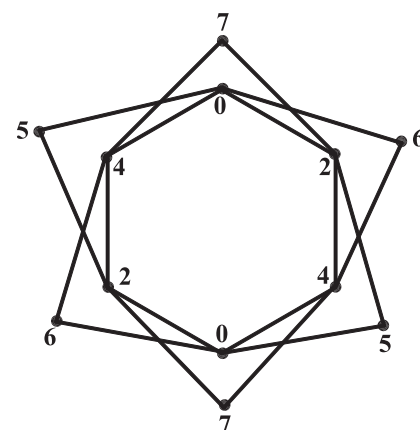


Figure 3. vertex duplication in  $C_6$  and  $L(2, 1)$ -labeling

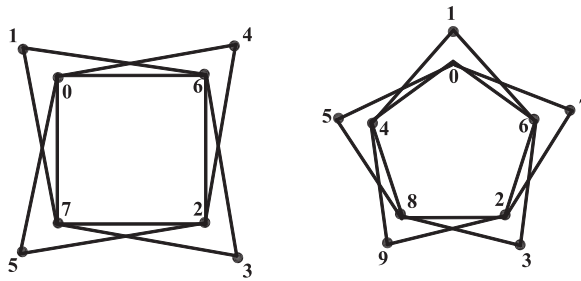


Figure 4. vertex duplication in  $C_4, C_5$  and  $L'(2, 1)$ -labeling

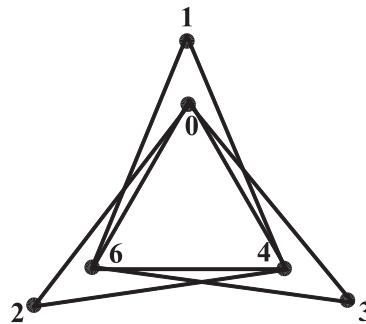


Figure 5. vertex duplication in  $C_3$  and  $L'(2, 1)$ -labeling

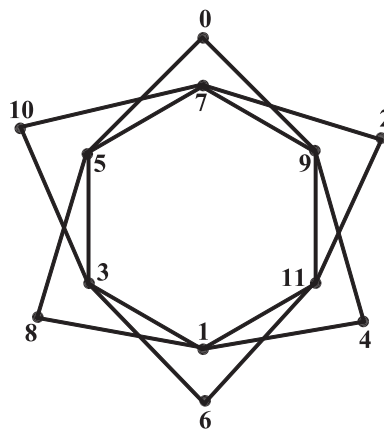
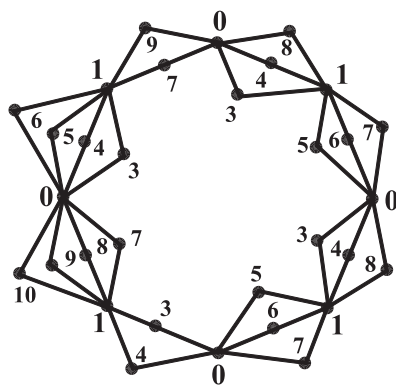
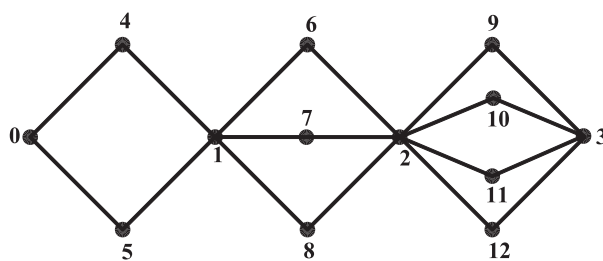
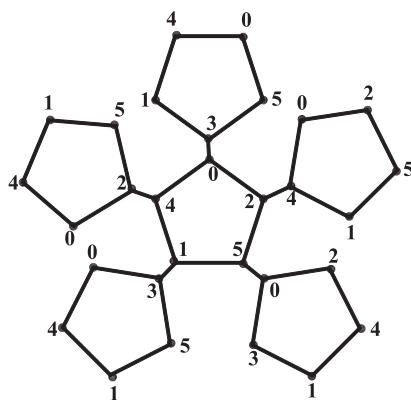
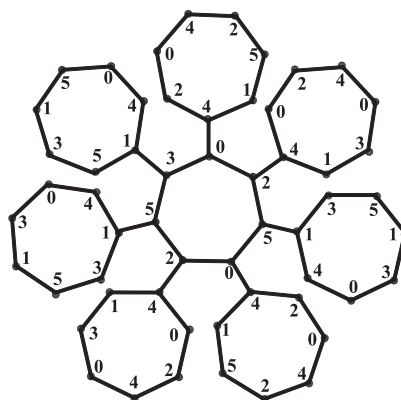
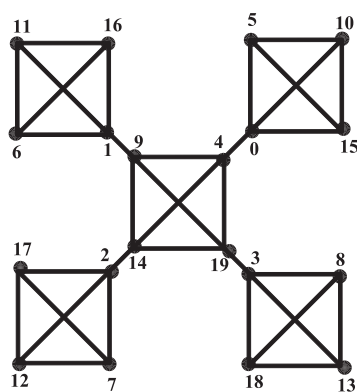


Figure 6. vertex duplication in  $C_6$  and  $L'(2, 1)$ -labeling

Figure 7.  $L(2, 1)$ -labeling of  $C'_8$ Figure 8.  $L'(2, 1)$ -labeling of  $P'_4$ Figure 9.  $L(2, 1)$ -labeling for star of cycle  $C_5$

Figure 10.  $L(2, 1)$ -labeling for star of cycle  $C_7$ Figure 11.  $L'(2, 1)$ -labeling for star of a complete graph  $K_4$

## Fibonacci and Super Fibonacci Graceful Labeling of Some Graphs\*

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P.L.Vihol<sup>2</sup>

**Abstract:** In the present work we discuss the existence and non-existence of Fibonacci and super Fibonacci graceful labeling for certain graphs. We also show that the graph obtained by switching a vertex in cycle  $C_n$ , (where  $n \geq 6$ ) is not super Fibonacci graceful but it can be embedded as an induced subgraph of a super Fibonacci graceful graph.

**Key words:** Graceful Labeling; Fibonacci Graceful Labeling; Super Fibonacci Graceful Labeling

### 1. INTRODUCTION

Graph labeling where the vertices are assigned values subject to certain conditions. The problems arising from the effort to study various labeling schemes of the elements of a graph is a potential area of challenge. Most of the labeling techniques found their origin with 'graceful labeling' introduced by Rosa (1967). The famous graceful tree conjecture and many illustrious works on graceful graphs brought a tide of different graph labeling techniques. Some of them are Harmonious labeling, Elegant labeling, Edge graceful labeling, Odd graceful labeling etc. A comprehensive survey on graph labeling is given in Gallian (2010). The present work is aimed to provide Fibonacci graceful labeling of some graphs.

Throughout this work graph  $G = (V(G), E(G))$  we mean a simple, finite, connected and undirected graph with  $p$  vertices and  $q$  edges. For standard terminology and notations in graph theory we follow Gross and Yellen (1998) while for number theory we follow Niven and Zuckerman (1972). We will give brief summary of definitions and other information which are useful for the present investigations.

**Definition 1.1** A *vertex switching*  $G_v$  of a graph  $G$  is obtained by taking a vertex  $v$  of  $G$ , removing all edges incidence to  $v$  and adding edges joining  $v$  to every vertex which are not adjacent to  $v$  in  $G$ .

**Definition 1.2** Consider two copies of fan ( $F_n = P_n + K_1$ ) and define a new graph known as *joint sum* of  $F_n$  is the graph obtained by connecting a vertex of first copy with a vertex of second copy.

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**Definition 1.3** A function  $f$  is called *graceful* labeling of graph if  $f : V(G) \rightarrow \{0, 1, 2, \dots, q\}$  is injective and the induced function  $f^* : E(G) \rightarrow \{1, 2, \dots, q\}$  defined as  $f^*(e = uv) = |f(u) - f(v)|$  is bijective. A graph  $G$  is called graceful if it admits *graceful labeling*.

**Definition 1.4** The *Fibonacci numbers*  $F_0, F_1, F_2, \dots$  are defined by  $F_0, F_1, F_2, \dots$  and  $F_{n+1} = F_n + F_{n-1}$ .

**Definition 1.5** The function  $f : V(G) \rightarrow \{0, 1, 2, \dots, F_q\}$  (where  $F_q$  is the  $q^{th}$  Fibonacci number) is said to be *Fibonacci graceful* if  $f^* : E(G) \rightarrow \{F_1, F_2, \dots, F_q\}$  defined by  $f^*(uv) = |f(u) - f(v)|$  is bijective.

**Definition 1.6** The function  $f : V(G) \rightarrow \{0, F_1, F_2, \dots, F_q\}$  (where  $F_q$  is the  $q^{th}$  Fibonacci number) is said to be *Super Fibonacci graceful* if the induced edge labeling  $f^* : E(G) \rightarrow \{F_1, F_2, \dots, F_q\}$  defined by  $f^*(uv) = |f(u) - f(v)|$  is bijective.

Above two concepts were introduced by Kathiresan and Amutha [5]. Deviating from the definition 1.1 they assume that  $F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, \dots$  and proved that

- $K_n$  is Fibonacci graceful if and only if  $n \leq 3$ .
- If  $G$  is Eulerian and Fibonacci graceful then  $q \equiv 0(mod 3)$ .
- Every path  $P_n$  of length  $n$  is Fibonacci graceful.
- $P_n^2$  is a Fibonacci graceful graph.
- Caterpillars are Fibonacci graceful.
- The bistar  $B_{m,n}$  is Fibonacci graceful but not Super Fibonacci graceful for  $n \geq 5$ .
- $C_n$  is Super Fibonacci graceful if and only if  $n \equiv 0(mod 3)$ .
- Every fan  $F_n$  is Super Fibonacci graceful.
- If  $G$  is Fibonacci or Super Fibonacci graceful then its pendant edge extension  $G'$  is Fibonacci graceful.
- If  $G_1$  and  $G_2$  are Super Fibonacci graceful in which no two adjacent vertices have the labeling 1 and 2, then their union  $G_1 \cup G_2$  is Fibonacci graceful.
- If  $G_1, G_2, \dots, G_n$  are super Fibonacci graceful graphs in which no two adjacent vertices are labeled with 1 and 2 then amalgamation of  $G_1, G_2, \dots, G_n$  obtained by identifying the vertices having labels 0 is also a super Fibonacci graceful.

In the present work we prove that

- Trees are Fibonacci graceful.
- Wheels are not Fibonacci graceful.
- Helms are not Fibonacci graceful.

The graph obtained by

- Switching of a vertex in a cycle  $C_n$  is Fibonacci graceful.
- Joint Sum of two copies of fan is Fibonacci graceful.
- Switching of a vertex in a cycle  $C_n$  is super Fibonacci graceful except  $n \geq 6$ .
- Switching a vertex of cycle  $C_n$  for  $n \geq 6$  can be embedded as an induced subgraph of a super Fibonacci graceful graph.

**Observation 1.7** If in a triangle edges receives Fibonacci numbers from vertex labels than they are always consecutive.

## 2. MAIN RESULTS

**Theorem 2.1** Trees are Fibonacci graceful.

**Proof:** Consider a vertex with minimum eccentricity as the root of tree T. Let this vertex be  $v$ . Without loss of generality at each level of tree T we initiate the labeling from left to right. Let  $P^1, P^2, P^3, \dots, P^n$  be the children of  $v$ .

Define  $f: V(T) \rightarrow \{0, 1, 2, \dots, F_q\}$  in the following manner.

$$f(v) = 0, f(P^1) = F_1$$

Now if  $P_{1i}^1 (1 \leq i \leq t)$  are children of  $P^1$  then

$$f(P_{1i}^1) = f(P^1) + F_{i+1}, 1 \leq i \leq t$$

If there are  $r$  vertices at level two of  $P^1$  and out of these  $r$  vertices,  $r_1$  be the children of  $P_{11}^1$  then label them as follows,

$$f(P_{11i}^1) = f(P_{11}^1) + F_{i+1}, 1 \leq i \leq r_1$$

Let there are  $r_2$  vertices, which are children of  $P_{12}^1$  then label them as follows,

$$f(P_{12i}^1) = f(P_{12}^1) + F_{i+1}, 1 \leq i \leq r_2$$

Following the same procedure to label all the vertices of a subtree with root as  $P^1$ .

we can assign label to each vertex of the subtree with roots as  $P^2, P^3, \dots, P^{n-1}$  and define  $f(P^{i+1}) = F_{f_i+1}$ , where  $F_{f_i}$  is the  $f_i^{th}$  Fibonacci number assign to the last edge of the tree rooted at  $P^i$ .

Now for the vertex  $P^n$ . Define  $f(P^n) = F_q$

Let us denote  $P_{ij}^n$ , where  $i$  is the level of vertex and  $j$  is number of vertices at  $i^{th}$  level.

At this stage one has to be cautious to avoid the repetition of vertex labels in right most branch. For that we first assign vertex label to that vertex which is adjacent to  $F_q$  and is a internal vertex of the path whose length is largest among all the paths whose origin is  $F_q$  (That is,  $F_q$  is a root). Without loss of generality we consider this path to be a left most path to  $F_q$  and continue label assignment from left to right as stated earlier.

If  $P_{li}^n$  ( $1 \leq i \leq s$ ) be the children of  $P^n$  then define

$$f(P_{li}^n) = f(P^n) - F_{q-i}, 1 \leq i \leq s$$

If there are  $P_{2i}^n$  ( $1 \leq i \leq b$ ) vertices at level two of  $P^n$  and out of these  $b$  vertices,  $b_1$  be the children of  $P_{11}^n$ . Then label them as follows.

$$f(P_{2i}^n) = f(P_{11}^n) - F_{q-s-i}, 1 \leq i \leq b_1$$

If there are  $b_2$  vertices, which are children of  $P_{12}^n$  then label them as follows,

$$f(P_{2(b_1+i)}^1) = f(P_{12}^n) - F_{q-s-b_1-i}, 1 \leq i \leq b_2$$

We will also consider the situation when all the vertices of subtree rooted at  $F_q$  is having all the vertices of degree two after  $i^{th}$  level then we define labeling as follows.

$$f(P_{i1}^n) = f(P_{(i-1)1}^n) + (-1)^{i-1} F_{q-(\text{labeled vertices in the branch})}$$

Continuing in this fashion unless all the vertices of a subtree with root as  $P^n$  are labeled.

Thus we have labeled all the vertices of each level. That is, T admits Fibonacci Graceful Labeling.

That is, trees are Fibonacci Graceful.

The following Figure 1 will provide better under standing of the above defined labeling pattern.

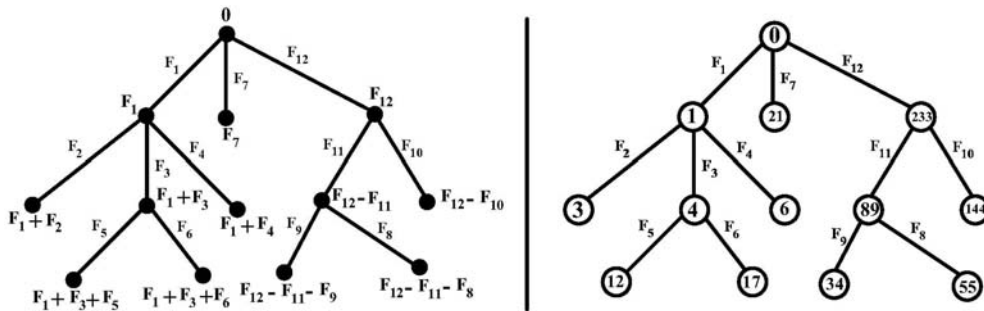


Figure 1: A Tree And its Fibonacci Graceful Labeling

**Theorem 2.2** Wheels are not Fibonacci graceful.

**Proof:** Let  $v$  be the apex vertex of the wheel  $W_n$  and  $v_1, v_2, \dots, v_n$  be the rim vertices.

Define  $f : V(W_n) \rightarrow \{0, 1, 2, \dots, F_q\}$

We consider following cases.

**Case 1:** Let  $f(v) = 0$

so, the vertices  $v_1, v_2, \dots, v_n$  must be label with Fibonacci numbers.

Let  $f(v_1) = F_q$  then  $f(v_2) = F_{q-1}$  or  $f(v_2) = F_{q-2}$ .

If  $f(v_2) = F_{q-2}$  then  $f(v_n) = F_{q-1}$  is not possible as  $f(v_1 v_n) = f(v v_2) = F_{q-2}$ .

If  $f(v_2) = F_{q-1}$  then  $f(v_n) \neq F_{q-2}$  otherwise  $f(v_1 v_n) = f(v v_2) = F_{q-1}$ .

If  $f(v_n) = F_p$  be the Fibonacci number other then  $F_{q-1}$  and  $F_{q-2}$  then  $|f(v_n) - f(v_1)| = |F_p - F_q|$  can not be Fibonacci number for  $|p - q| > 2$

**Case 2:** If  $v_1$  is a rim vertex then define  $f(v_1) = 0$

If  $f(v_2) = F_q$  then the apex vertex must be labeled with  $F_{q-1}$  or  $F_{q-2}$ .

**Sub Case 1:** Let  $f(v) = F_{q-1}$

Now  $f(v_n)$  must be labeled with either by  $F_{q-2}$  or by  $F_{q-3}$ .

If  $f(v_n) = F_{q-2}$  then  $f(v_1 v_n) = f(v v_2) = F_{q-2}$

and if  $f(v_n) = F_{q-3}$  then  $f(v v_n) = f(v v_2) = F_{q-2}$

**Sub Case 2:** Let  $f(v) = F_{q-2}$

Now  $f(v_n)$  must be label with either by  $F_{q-1}$  or by  $F_{q-3}$  or by  $F_{q-4}$ .

if  $f(v_n) = F_{q-1}$  then  $f(v_1 v_n) = f(v v_2) = F_{q-1}$

if  $f(v_n) = F_{q-3}$  then

$$f(v_1 v_2) = F_q$$

$$f(v v_1) = F_{q-2}$$

$$f(v v_2) = F_{q-1}$$

$$f(v_n v_1) = F_{q-3}$$

$$f(v v_n) = F_{q-4}$$

For  $W_3$ ,  $f(v_2 v_3)$  can not be Fibonacci number. Now for  $n > 3$  let us assume that  $f(v_3) = k$

which is not Fibonacci number because for  $f(v_3) = F_{q-1}$ , we have  $f(v v_1) = f(v_2 v_3) = F_{q-2}$ .

now we have following cases. (1)  $F_{q-2} < k < F_q$ , (2)  $k < F_{q-2} < F_q$

In (1) we have.....

$$F_q - k = F_s$$

$$k - F_{q-2} = F_{s'}$$

$$F_q - F_{q-2} = F_s + F_{s'} \Rightarrow F_{q-1} = F_s + F_{s'} \text{ is possible only when } s = q-2 \text{ and } s' = q-3,$$

$$\text{then } f(v_2 v_3) = f(v v_1) \text{ and } f(v v_3) = f(v_1 v_n)$$

In (2) we have.....

$$F_q - k = F_s$$

$$F_{q-2} - k = F_{s'}$$

$$F_q - F_{q-2} = F_s + F_{s'} \Rightarrow F_{q-1} = F_s + F_{s'} \text{ is possible only when } s = q-2 \text{ and } s' = q-3,$$

$$\text{then } f(v_2 v_3) = f(v v_1) \text{ and } f(v v_3) = f(v_1 v_n)$$

Thus, we can not find a number  $f(v_3) = k$  for which  $f(v_2 v_3)$  and  $f(v v_3)$  are the distinct Fibonacci numbers.

For  $f(v_n) = F_{q-4}$  we can argue as above.

**Sub Case 3:** If  $f(v) = F_q$

Then we do not have two Fibonacci numbers corresponding to  $f(v_1)$  and  $f(v_n)$  such that the edges will receive distinct Fibonacci numbers.

Thus we conclude that wheels are not Fibonacci graceful.

**Theorem 2.3** Helms are not Fibonacci graceful.

**Proof:** Let  $H_n$  be the helm and  $v'_1, v'_2, v'_3, \dots, v'_n$  be the pendant vertices corresponding to it. If 0 is the label of any of the rim vertices of wheel corresponding to  $H_n$  then all the possibilities to admit Fibonacci graceful labeling is ruled out as we argued in above Theorem 2.2. Thus possibilities of 0 being the label of any of the pendant vertices is remained at our disposal.

Define  $f: V(H_n) \rightarrow \{0, 1, 2, \dots, F_q\}$

Without loss of generality we assume  $f(v'_1) = 0$  then  $f(v_1) = F_q$

Let  $f(v_2) = p$  and  $f(v) = r$

In the following Figures 2(1) to 2(3) the possible labeling is demonstrated. In first two arrangements the possibility of  $H_3$  being Fibonacci graceful is washed out by the similar arrangements for wheels are not Fibonacci graceful held in Theorem 2.2. For the remaining arrangement as shown in Figure 2(3) we have to consider following two possibilities.

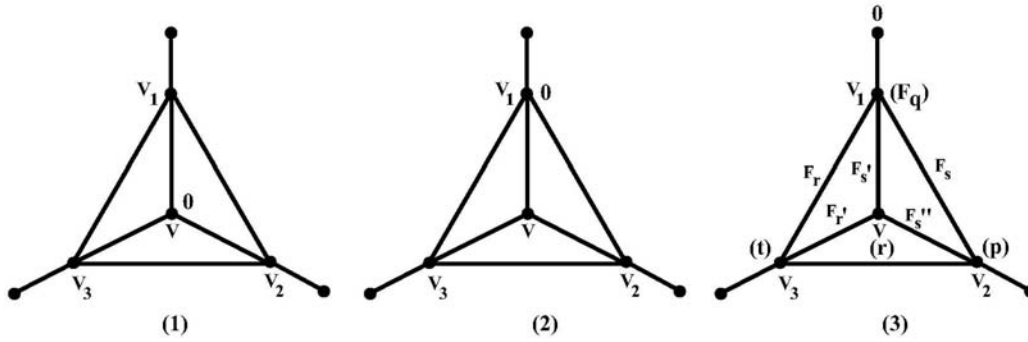


Figure 2: Ordinary Labeling in H3

**Case 1:**  $p < r < F_q$

$$F_q - p = F_s$$

$$F_q - r = F_{s'}$$

$$r - p = F_{s''} \text{ then}$$

$$F_{s'} + F_{s''} - F_s = 0 \Rightarrow F_s = F_{s'} + F_{s''}$$

**Case 2:**  $r < p < F_q$

$$F_q - p = F_s$$

$$F_q - r = F_{s'}$$

$$p - r = F_{s''} \text{ then}$$

$$F_s + F_{s''} - F_{s'} = 0 \Rightarrow F_{s'} = F_s + F_{s''}$$

Now let  $f(v_3) = t$  then consider the case  $p < r < t < F_q$ ,

$$F_s = F_{s'} + F_{s''}$$

$$F_{s'} = F_r + F_{r'}$$

From these two equations we have...

$$F_{s'} = F_r + F_{r'} = F_s - F_{s''}$$

so we have  $F_r < F_{r'} < F_{s'} < F_{s''} < F_s$  and they are consecutive Fibonacci numbers according to Observation 1.7 .

For  $r \geq p, t$  we have  $F_s = F_{s'} + F_{s''}$  and  $F_r = F_{s'} + F_{r'}$  so we have

$$F_{s'} = F_s - F_{s''} \text{ and } F_{s'} = F_r - F_{r'} \text{ which is not possible.}$$

similar argument can be made for  $r \leq p, t$  .

i.e. we have either  $p < r < t$  or  $t < r < p$ .

As  $F_{s'} < F_{s''} < F_s$ , so we can say that with  $f(vv_2) = F_{s''}$  the edges of the triangle with vertices  $f(v)$ ,  $f(v_2)$  and  $f(v_3)$  will not have Fibonacci numbers such that  $F_{s''} = \text{sum of two Fibonacci numbers}$ .

Similar arguments can also be made for  $t < r < p < F_q$ .

Hence Helms are not Fibonacci graceful graphs.

**Theorem 2.4** The graph obtained by switching of a vertex in cycle  $C_n$  admits Fibonacci graceful labeling.

**Proof:** Let  $v_1, v_2, v_3, \dots, v_n$  be the vertices of cycle  $C_n$  and  $C'_n$  be the graph resulted from switching of the vertex  $v_1$ .

Define  $f: V(C'_n) \rightarrow \{0, 1, 2, \dots, F_q\}$  as follows.

$$f(v_1) = 0$$

$$f(v_2) = F_q - 1$$

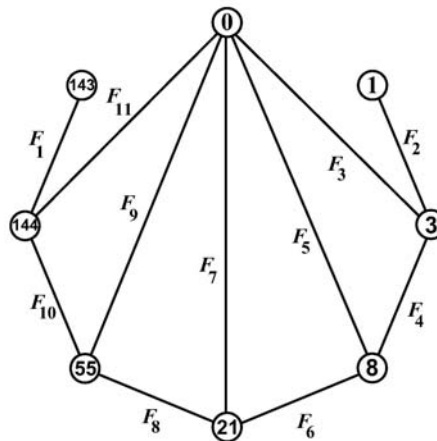
$$f(v_3) = F_q$$

$$f(v_{i+3}) = F_{q-2i}, 1 \leq i \leq n-3$$

Above defined function  $f$  admits Fibonacci graceful labeling.

Hence we have the result.

**Illustration 2.5** Consider the graph  $C'_8$ . The Fibonacci graceful labeling is as shown in Figure 3.



**Figure 3: Fibonacci Graceful Labeling of  $C'_8$**

**Theorem 2.6** The graph obtained by joint sum of two copies of fans ( $F_n = P_n + K_1$ ) is Fibonacci graceful.

**Proof:** Let  $v_1, v_2, \dots, v_n$  and  $v'_1, v'_2, v'_3, \dots, v'_m$  be the vertices of  $F_n^1$  and  $F_m^2$  respectively. Let  $v$  be the apex vertex of  $F_n^1$  and  $v'$  be the apex vertex of  $F_m^2$  and let  $G$  be the joint sum of two fans.

Define  $f: V(G) \rightarrow \{0, 1, 2, \dots, F_q\}$  as follows.

$$f(v) = 0$$

$$f(v') = F_q$$

$$f(v_i) = F_{2i-1}, 1 \leq i \leq n$$

$$f(v'_1) = F_q - F_{2n+1}$$

$$f(v'_2) = F_q - F_{2n+2}$$

$$f(v'_{2+i}) = F_q - F_{2n+2+2i}, 1 \leq i \leq m-2$$

In view of the above defined pattern the graph  $G$  admits Fibonacci graceful labeling.

**Illustration 2.7** Consider the Joint Sum of two copies of  $F_4$ . The Fibonacci graceful labeling is as shown in Figure 4.

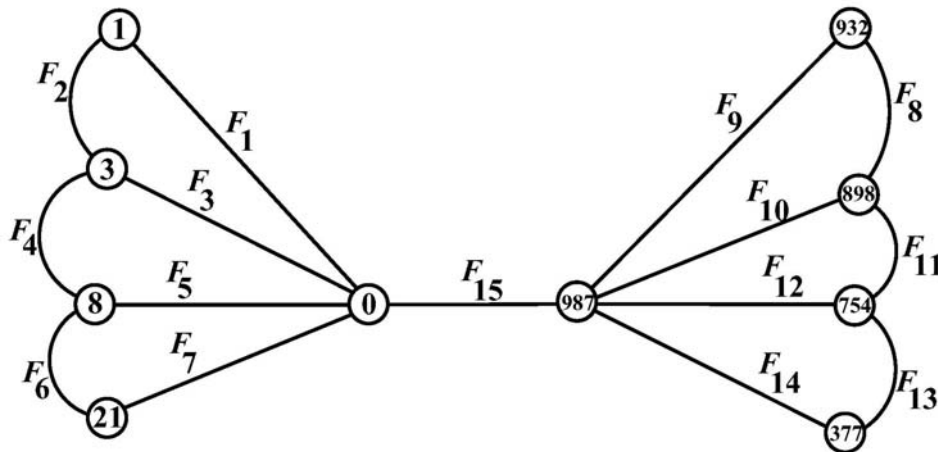


Figure 4: Fibonacci Graceful Labeling of Joint Sum of  $F_4$

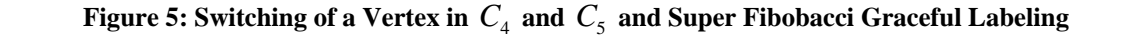
**Theorem 2.8** The graph obtained by Switching of a vertex in a cycle  $C_n$  is super Fibonacci graceful except  $n \geq 6$ .

**Proof:** We consider here two cases.

**case 1:**  $n = 3, 4, 5$

For  $n = 3$  the graph obtained by switching of a vertex is a disconnected graph which is not desirable for the Fibonacci graceful labeling.

Super Fibonacci graceful labeling of switching of a vertex in  $C_n$  for  $n = 4, 5$  is as shown in Figure 5.



**case 2:**  $n \geq 6$  The graph shown in Figure 6 will be the subgraph of all the graphs obtained by switching of a vertex in  $C_n (n \geq 6)$ .

In Figure 7 all the possible assignment of vertex labels is shown which demonstrates the repetition of edge labels.

(1) In *Fig8(a)* edge label  $F_{q-1}$  is repeated as

$$|F_q - F_{q-2}| = F_{q-1} \text{ \& } |F_{q-1} - 0| = F_{q-1}$$

(2) In Fig8(b) edge label  $F_{q-1}$  is repeated as

$$|F_q - F_{q-2}| = F_{q-1} \text{ \& } |F_{q-1} - 0| = F_{q-1}$$

(3) In Fig8(c) edge label  $F_p$  is repeated as  $|F_{p+1} - F_{p-1}| = F_p \text{ \& } |F_p - 0| = F_p$ ,

where  $F_p$  is any Fibonacci number.

(4) In Fig8(d) edge label  $F_p$  is repeated as  $|F_{p+2} - F_{p+1}| = F_p \text{ \& } |F_p - 0| = F_p$ ,

where  $F_p$  is any Fibonacci number.

(5) In Fig8(e) edge label  $F_{p-1}$  is repeated as  $|F_p - F_{p-2}| = F_{p-1} \text{ \& } |F_{p-1} - 0| =$

$F_{p-1}$ , where  $F_p$  is any Fibonacci number.

(6) In Fig8(f) edge label  $F_{q-2}$  is repeated as

$$|F_{q-1} - F_{q-3}| = F_{q-2} \text{ \& } |F_q - F_{q-1}| = F_{q-2}$$

(7) In Fig8(g) edge label  $F_{q-1}$  is repeated as

$$|F_q - F_{q-2}| = F_{q-1} \text{ \& } |F_{q-1} - 0| = F_{q-1}$$

(8) In Fig8(h) edge label  $F_{q-1}$  is repeated as

$$|F_q - F_{q-2}| = F_{q-1} \text{ \& } |F_{q-1} - 0| = F_{q-1}$$

**Theorem 2.9** The graph obtained by Switching of a vertex in cycle  $C_n$  for  $n \geq 6$  can be embedded as an induced subgraph of a super Fibonacci graceful graph.

**Proof:** Let  $v_1, v_2, v_3, \dots, v_n$  be the vertices of  $C_n$  and  $v_1$  be the switched vertex.

Define  $f: V(G) \rightarrow \{0, F_1, F_2, \dots, F_{q+3}\}$

$$f(v_1) = 0$$

$$f(v_{i+1}) = F_{2i-1}, 1 \leq i \leq n-1$$

Now it remains to assign Fibonacci numbers  $F_1, F_{q+2}$  and  $F_{q+3}$ . Put 3 vertices in the graph. Join first vertex  $v'$  labeled with  $F_2$  to the vertex  $v_3$ . Now join second vertex  $v''$  labeled with  $F_{q+3}$  to the vertex  $v_1$  and vertex  $v'''$  labeled with  $F_{q+2}$  to the vertex  $v''$ .

Thus the resultant graph is a super Fibonacci graceful graph.

**Illustration 2.10** In the following Figure 8 the graph obtained by switching of a vertex in cycle  $C_6$  and its super Fibonacci graceful labeling of its embedding is shown.

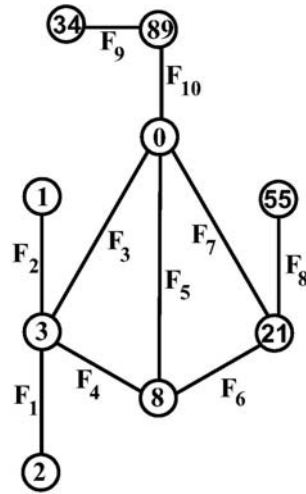


Figure 8: A Super Fibonacci Graceful Embedding

### 3. CONCLUDING REMARKS

Here we have contributed seven new results to the theory of Fibonacci graceful graphs. It has been proved that trees, vertex switching of cycle  $C_n$ , joint sum of two fans are Fibonacci graceful while wheels and helms are not Fibonacci graceful. We have also discussed super Fibonacci graceful labeling and show that the graph obtained by switching of a vertex in cycle  $C_n$  ( $n \geq 6$ ) does not admit super Fibonacci graceful labeling but it can be embedded as an induced subgraph of a super Fibonacci graceful graph.

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# Cordial labeling for middle graph of some graphs

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### ABSTRACT

This paper is aimed to discuss cordial graphs in the context of middle graph of a graph. We present here cordial labeling for the middle graphs of path, crown, star and tadpole.

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### Introduction

We begin with simple, finite and undirected graph  $G = (V(G), E(G))$ . In the present work  $|V(G)|$  and  $|E(G)|$  denote the number of vertices and edges in the graph  $G$  respectively. For all other terminology and notations we follow Harary[1]. We will give brief summary of definitions which are useful for the present investigations.

**Definition -1.1 :** If the vertices of the graph are assigned values subject to certain conditions then is known as graph labeling. An extensive survey on graph labeling we refer to Gallian[2]. According to Beineke and Hegde[3] graph labeling serves as a frontier between number theory and structure of graphs. A detailed study of variety of applications of graph labeling is reported in Bloom and Golomb[4].

**Definition -1.2 :** Let  $G$  be a graph. A mapping  $f: V(G) \rightarrow \{0,1\}$  is called binary vertex labeling of  $G$  and  $f(v)$  is called the *label* of the vertex  $v$  of  $G$  under  $f$ .

For an edge  $e = uv$ , the induced edge labeling  $f^*: E(G) \rightarrow \{0,1\}$  is given by  $f^*(e) = |f(u) - f(v)|$ .

Let  $v_f(0)$ ,  $v_f(1)$  be the number of vertices of  $G$  having labels 0 and 1 respectively under  $f$  and let  $e_f(0)$ ,  $e_f(1)$  be the number of edges having labels 0 and 1 respectively under  $f^*$ .

**Definition -1.3 :** A binary vertex labeling of a graph  $G$  is called a cordial labeling if  $|v_f(0) - v_f(1)| \leq 1$  and

$|e_f(0) - e_f(1)| \leq 1$ . A graph  $G$  is cordial if it admits cordial labeling.

The concept of cordial labeling was introduced by Cahit[5] and he proved that every tree is cordial. In the same paper he proved that  $K_n$  is cordial if and only if  $n \leq 3$ . Ho et al.[6]

proved that unicyclic graph is cordial unless it is  $C_{4k+2}$ . Andar et al.[7] has discussed cordiality of multiple shells. Vaidya et al.[8],[9],[10],[11] have also discussed the cordiality of various graphs.

**Definition -1.4 :** The middle graph  $M(G)$  of a graph  $G$  is the graph whose vertex set is  $V(G) \cup E(G)$  and in which two vertices are adjacent if and only if either they are adjacent edges of  $G$  or one is a vertex of  $G$  and the other is an edge incident with it.

In the present investigations we prove that the middle graphs of path, crown (The Crown  $(C_n \odot K_1)$  is obtained by joining a single pendant edge to each vertex of  $C_n$ ), star and tadpole (Tadpole  $T(n,l)$  is a graph in which path  $P_l$  is attached to any one vertex of cycle  $C_n$ ) admit cordial labeling.

### Main Results

**Theorem - 2.1:** The middle graph  $M(G)$  of an Eulerian graph

$$|E(M(G))| = \frac{\sum_{i=1}^n d(v_i)^2 + 2e}{2}$$

$G$  is Eulerian and

**Proof:** Let  $G$  be an Eulerian graph. If  $v_1, v_2, v_3, \dots, v_n$  are vertices of  $G$  and  $e_1, e_2, e_3, \dots, e_q$  are edges of  $G$  then  $v_1,$

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$v_2, v_3, \dots, v_n, e_1, e_2, \dots, e_q$  are the vertices of  $M(G)$ . Then it is obvious that if  $d(v_i)$  is even in  $G$  then it remains even in  $M(G)$ .

Now it remains to show that  $d(e_i)$  is even in  $M(G)$ . For that if  $v'$  and  $v''$  are the vertices adjacent to any vertex  $e_i$  then  $d(e_i) = (d(v') - 1) + (d(v'') - 1)$

$= d(v') + d(v'') - 2$  which is even as both  $d(v')$  and  $d(v'')$  are even for  $1 \leq i \leq q$ .

Therefore  $M(G)$  is an Eulerian graph. It is also obvious that the  $d(v_i)$  number of edges are incident with each vertex  $v_i$  of  $G$  which forms a complete graph  $K_{d(v_i)}$  in  $M(G)$ .

Now if the total number of edges in  $M(G)$  be denoted as  $|E(M(G))|$  then

$$\begin{aligned} |E(M(G))| &= d(v_1) + d(v_2) + d(v_3) + \dots + d(v_n) + |E(K_{d(v_1)})| + |E(K_{d(v_2)})| + |E(K_{d(v_3)})| + \dots + |E(K_{d(v_n)})| \\ &= d(v_1) + d(v_2) + d(v_3) + \dots + d(v_n) + \frac{d(v_1)(d(v_1)-1)}{2} + \frac{d(v_2)(d(v_2)-1)}{2} + \dots + \frac{d(v_n)(d(v_n)-1)}{2} \\ &= \frac{d(v_1)^2}{2} + \frac{d(v_2)^2}{2} + \dots + \frac{d(v_n)^2}{2} + \frac{d(v_1)}{2} + \frac{d(v_2)}{2} + \dots + \frac{d(v_n)}{2} \\ &= \frac{\sum_{i=1}^n d(v_i)^2 + \sum_{i=1}^n d(v_i)}{2} \end{aligned}$$

But  $\sum_{i=1}^n d(v_i) = 2e$ , Hence  $|E(M(G))| = \frac{\sum_{i=1}^n d(v_i)^2 + 2e}{2}$  proved.

**Corollary - 2.2 :** The middle graph  $M(G)$  of any graph  $G$  is not cordial when  $|E(M(G))| = \frac{\sum_{i=1}^n d(v_i)^2 + 2e}{2} \equiv 2 \pmod{4}$ .

**Proof :** By Theorem 2.1, for  $M(G)$  of any graph  $G$ ,

$$|E(M(G))| = \frac{\sum_{i=1}^n d(v_i)^2 + 2e}{2}.$$

Then as proved by Cahit[5] an Eulerian graph with  $e \equiv 2 \pmod{4}$  is not cordial.

**Theorem - 2.3 :**  $M(P_n)$  is a cordial graph.

**Proof:** If  $v_1, v_2, \dots, v_n$  and  $e_1, e_2, \dots, e_n$  are respectively the vertices and edges of  $P_n$  then  $v_1, v_2, \dots, v_n, e_1, e_2, \dots, e_n$  are the vertices of  $M(P_n)$ .

To define  $f: V(M(P_n)) \rightarrow \{0, 1\}$ , we consider following four cases.

**Case 1:**  $n$  is odd,  $n = 2k + 1, k = 1, 3, 5, 7, \dots$

In this case  $|V(M(P_n))| = 2n - 1, |E(M(P_n))| = 2n + 2k - 3$

We label the vertices as follows.

$$\begin{aligned} f(v_{2i-1}) &= 0 \text{ for } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor + 1 \\ f(v_{2i}) &= 1 \text{ for } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \end{aligned}$$

$$\begin{aligned} f(e_{4i-3}) &= 1 \\ f(e_{4i-2}) &= 1 \end{aligned} \left\{ \begin{aligned} 1 \leq i \leq \left\lfloor \frac{n}{4} \right\rfloor + 1 \\ f(e_{4i-1}) = 0 \\ f(e_{4i}) = 0 \end{aligned} \right\} 1 \leq i \leq \left\lfloor \frac{n}{4} \right\rfloor$$

In view of the above defined labeling pattern we have  $v_f(0) + 1 = v_f(1) = n, e_f(0) = e_f(1) + 1 = n + k - 1$

**Case 2:**  $n$  odd,  $n = 2k + 1, k = 2, 4, 6, \dots$

In this case  $|V(M(P_n))| = 2n - 1, |E(M(P_n))| = 2n + 2k - 3$

We label the vertices as follows.

$$\begin{aligned} f(v_{2i-1}) &= 0 \text{ for } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor + 1 \\ f(v_{2i}) &= 1 \text{ for } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \end{aligned}$$

$$\begin{aligned} f(e_{4i-3}) &= 0 \\ f(e_{4i-2}) &= 0 \\ f(e_{4i-1}) &= 1 \\ f(e_{4i}) &= 1 \end{aligned} \left\{ 1 \leq i \leq \left\lfloor \frac{n}{4} \right\rfloor \right.$$

In view of the above defined labeling pattern we have  $v_f(0) = v_f(1) + 1 = n, e_f(0) = e_f(1) + 1 = n + k - 1$

**Case 3:**  $n$  even,  $n = 2k, k = 1, 3, 5, 7, \dots$

In this case  $|V(M(P_n))| = 2n - 1, |E(M(P_n))| = 2n + 2k - 4$

We label the vertices as follows.

$$\begin{aligned} f(v_{2i-1}) &= 0 \\ f(v_{2i}) &= 1 \end{aligned} \left\{ 1 \leq i \leq \frac{n}{2} \right.$$

$$f(e_{4i-3}) = 0 \text{ for } 1 \leq i \leq \left\lfloor \frac{n}{4} \right\rfloor + 1$$

$$\begin{aligned} f(e_{4i-2}) &= 0 \\ f(e_{4i-1}) &= 1 \\ f(e_{4i}) &= 1 \end{aligned} \left\{ 1 \leq i \leq \left\lfloor \frac{n}{4} \right\rfloor \right.$$

**Case 4:**  $n$  even,  $n = 2k, k = 2, 4, 6, \dots$

In this case  $|V(M(P_n))| = 2n - 1$ ,  
 $|E(M(P_n))| = 2n + 2k - 4$   
 We label the vertices as follows.

$$\left. \begin{array}{l} f(v_{2i-1}) = 0 \\ f(v_{2i}) = 1 \end{array} \right\} 1 \leq i \leq \frac{n}{2}$$

$$\left. \begin{array}{l} f(e_{4i-3}) = 0 \\ f(e_{4i-2}) = 0 \end{array} \right\} 1 \leq i \leq \frac{n}{4}$$

$$f(e_{4i-1}) = 1 \text{ for } 1 \leq i \leq \frac{n}{4}$$

$$f(e_{4i}) = 1 \text{ for } 1 \leq i \leq \frac{n}{4} - 1$$

In above two cases we have  $v_f(0) = v_f(1) + 1 = n$ ,  
 $e_f(0) = e_f(1) = n + k - 2$

Thus in all the four cases  $f$  satisfies the condition for cordial labeling. That is,  $M(P_n)$  is a cordial graph.

**Illustration - 2.4 :** In the following Figure 2.1  $M(P_7)$  and its cordial labeling is shown.

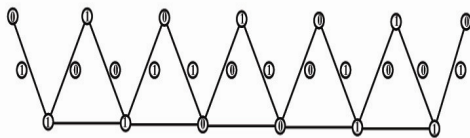


Figure 2.1  $M(P_7)$  and its cordial labeling

**Theorem - 2.5 :** The middle graph of crown is a cordial graph.

**Proof:** Consider the crown  $C_n \odot K_1$  in which  $v_1, v_2, \dots, v_n$  be the vertices of cycle  $C_n$  and  $v'_1, v'_2, \dots, v'_n$  be the pendant vertices attached at each vertex of  $C_n$ . Let  $e_1, e_2, \dots, e_n$  and  $e'_1, e'_2, \dots, e'_n$  are vertices corresponding to edges of  $C_n \odot K_1$  in  $M(C_n \odot K_1)$ .

To define  $f : V(M(C_n \odot K_1)) \rightarrow \{0, 1\}$  we consider following three cases.

**Case 1:**  $n$  is odd,  $n = 2k + 1, k = 2, 4, 6, \dots$

In this case  $|V(M(C_n \odot K_1))| = 4n$  and  
 $|E(M(C_n \odot K_1))| = 6n + 2 \left\lfloor \frac{n}{2} \right\rfloor + 1$

We label the vertices as follows.

$$\left. \begin{array}{l} f(v_{2i-1}) = 0 \\ f(v_{2i}) = 1 \end{array} \right\} 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$$

$$f(v_n) = 1$$

$$f(v'_{2i-1}) = 1 \text{ for } 1 \leq i \leq \left\lfloor \frac{n}{4} \right\rfloor + 1$$

$$f(v'_{2i}) = 0 \text{ for } 1 \leq i \leq \left\lfloor \frac{n}{4} \right\rfloor$$

$$\left. \begin{array}{l} f(v'_{2\lfloor \frac{n}{4} \rfloor + 2i}) = 1 \\ f(v'_{2\lfloor \frac{n}{4} \rfloor + 2i+1}) = 0 \end{array} \right\} 1 \leq i \leq \left\lfloor \frac{n}{4} \right\rfloor$$

$$\left. \begin{array}{l} f(e_{2i-1}) = 1 \\ f(e_{2i}) = 0 \end{array} \right\} 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$$

$$f(e_n) = 0$$

$$f(e'_{2i-1}) = 0 \text{ for } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor + 1$$

$$f(e'_{2i}) = 1 \text{ for } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$$

In view of the above defined pattern

$$v_f(0) = v_f(1) = 2n, \quad e_f(0) + 1 = e_f(1) = 3n + \left\lfloor \frac{n}{2} \right\rfloor + 1$$

**Case 2:**  $n$  is odd,  $n = 2k + 1, k = 1, 3, 5, 7, \dots$

In this case  $|V(M(C_n \odot K_1))| = 4n$  and  
 $|E(M(C_n \odot K_1))| = 6n + 2 \left\lfloor \frac{n}{2} \right\rfloor + 1$

We label the vertices as follows.

$$\left. \begin{array}{l} f(v'_{2\lfloor \frac{n}{4} \rfloor + 2i}) = 1 \\ f(v'_{2\lfloor \frac{n}{4} \rfloor + 2i+1}) = 0 \end{array} \right\} 1 \leq i \leq \left\lfloor \frac{n}{4} \right\rfloor + 1$$

Now label the remaining vertices as in case 1.

In view of the above defined pattern we have

$$v_f(0) = v_f(1) = 2n, \quad e_f(0) = e_f(1) + 1 = 3n + \left\lfloor \frac{n}{2} \right\rfloor + 1$$

**Case 3:**  $n$  is even,  $n = 2k, k = 2, 3, \dots$

In this case  $|V(M(C_n \odot K_1))| = 3n$  and  
 $|E(M(C_n \odot K_1))| = 7n$

We label the vertices as follows.

$$\left. \begin{aligned} f(v_{2i-1}) &= 0 \\ f(v_{2i}) &= 1 \end{aligned} \right\} 1 \leq i \leq \frac{n}{2}$$

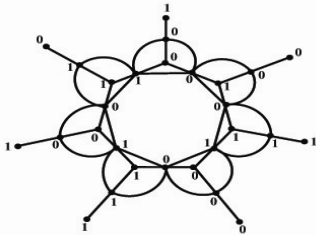
$$\begin{aligned} f(v'_i) &= 1 \text{ for } 1 \leq i \leq n \\ f(e_i) &= 0 \text{ for } 1 \leq i \leq n \\ \left. \begin{aligned} f(e'_{2i-1}) &= 1 \\ f(e'_{2i}) &= 0 \end{aligned} \right\} 1 \leq i \leq \frac{n}{2} \end{aligned}$$

In view of the above defined pattern we have

$$v_f(0) = v_f(1) = \frac{3n}{2}, \quad e_f(0) = e_f(1) = 3n + \frac{n}{2}$$

Thus in all the cases described above  $f$  admits cordial labeling for the graph under consideration. That is, middle graph of the crown is a cordial graph.

**Illustration - 2.6 :** In the following Figure 2.2 cordial labeling for  $M(C_7 \odot K_1)$  is shown.



**Figure 2.2**  $M(C_7 \odot K_1)$  and its cordial labeling

**Theorem - 2.7 :**  $M(K_{1,n})$  is a cordial graph.

**Proof:** Let  $v, v_1, v_2, \dots, v_n$  be the vertices of star  $K_{1,n}$  with  $v$  as an apex vertex and  $e_1, e_2, \dots, e_n$  be the vertices in  $M(K_{1,n})$  corresponding to the edges  $e_1, e_2, \dots, e_n$  in  $K_{1,n}$ .

To define  $f: V(M(K_{1,n})) \rightarrow \{0, 1\}$ , we consider following two cases.

**Case 1:**  $n = 2k + 1, k = 2, 3, 4, \dots$

In this case  $|V(M(K_{1,n}))| = 2n + 1$ ,

$$|E(M(K_{1,n}))| = 2n\left(\left\lfloor \frac{k}{2} \right\rfloor + 1\right)$$

or

$$|E(M(K_{1,n}))| = 2n\left(\left\lfloor \frac{k}{2} \right\rfloor + 1\right) + 2k + 1$$

depending upon  $k = 2, 4, 6, 8, \dots$  or  $k = 3, 5, 7, 9, \dots$

$$f(e_{2i-1}) = 0, \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor + 1$$

$$f(e_{2i}) = 1, \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$$

$$f(v_{n-i}) = p_i, \text{ where } p_i = 1, \text{ if } i \text{ is even,}$$

$$= 0, \text{ if } i \text{ is odd,} \quad 0 \leq i \leq \left\lfloor \frac{k}{2} \right\rfloor - 1$$

$$f(v_{n-\left\lfloor \frac{k}{2} \right\rfloor - i}) = f(e_{n-\left\lfloor \frac{k}{2} \right\rfloor - i}), \quad 0 \leq i \leq n - \left\lfloor \frac{k}{2} \right\rfloor - 1$$

$$f(v) = 1$$

Using above pattern if  $k = 2, 3, 6, 7, \dots$  then

$$v_f(0) + 1 = v_f(1) = n + 1 \quad \text{and if } k = 4, 5, 8, 9, \dots \text{ then}$$

$$v_f(0) = v_f(1) + 1 = n + 1$$

If  $k = 2, 4, 6, 8, \dots$  then  $e_f(0) = e_f(1) = n\left(\left\lfloor \frac{k}{2} \right\rfloor + 1\right)$  and if

$$k = 3, 5, 7, \dots \text{ then } e_f(0) = e_f(1) + 1 = n\left(\left\lfloor \frac{k}{2} \right\rfloor + 1\right) + k + 1$$

**Case 2:**  $n = 2k, k = 2, 3, 4, \dots$

In this case  $|V(M(K_{1,n}))| = 2n + 1$ ,

$$|E(M(K_{1,n}))| = 2n\left(\frac{k}{2} + 1\right) - k$$

or

$$|E(M(K_{1,n}))| = 2n\left(\left\lfloor \frac{k}{2} \right\rfloor + 1\right) + 2\left\lfloor \frac{k}{2} \right\rfloor - 1$$

depending upon  $k = 2, 4, 6, 8, \dots$  or  $k = 3, 5, 7, 9, \dots$

$$f(e_{2i-1}) = 0, \quad 1 \leq i \leq \frac{n}{2}$$

$$f(e_{2i}) = 1, \quad 1 \leq i \leq \frac{n}{2}$$

$$f(v_{n-i}) = p_i, \text{ where } p_i = 0, \text{ if } i \text{ is even,}$$

$$= 1, \text{ if } i \text{ is odd,} \quad 0 \leq i \leq \left\lfloor \frac{k}{2} \right\rfloor - 1$$

$$f(v_{n-\left\lfloor \frac{k}{2} \right\rfloor - i}) = f(e_{n-\left\lfloor \frac{k}{2} \right\rfloor - i}), \quad 0 \leq i \leq n - \left\lfloor \frac{k}{2} \right\rfloor - 1$$

$$f(v) = 1$$

Using above pattern if  $k = 2, 3, 6, 7, \dots$  then

$$v_f(0) = v_f(1) + 1 = n + 1 \quad \text{and if } k = 4, 5, 8, 9, \dots \text{ then}$$

$$v_f(0) + 1 = v_f(1) = n + 1$$

If  $k = 2, 4, 6, 8, \dots$  then  $e_f(0) = e_f(1) = n\left(\frac{k}{2} + 1\right) - \frac{k}{2}$  and

$$\text{if } k = 3, 5, 7, \dots \text{ then } e_f(0) = e_f(1) + 1 = n\left(\left\lfloor \frac{k}{2} \right\rfloor + 1\right) + \left\lfloor \frac{k}{2} \right\rfloor.$$

Also note that for  $n = 2$  we have  $v_f(0) = v_f(1) + 1 = 3$  and  $e_f(0) + 1 = e_f(1) = 3$ .

Thus in all the cases described above  $f$  admits cordial labeling for  $M(K_{1,n})$ . That is,  $M(K_{1,n})$  admits cordial labeling.

**Illustration - 2.8 :** In the following Figure 2.3 cordial labeling of  $M(K_{1,6})$  is shown.

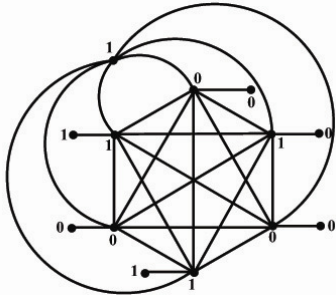


Figure 2.3  $M(K_{1,6})$  and its cordial labeling

**Theorem - 2.9 :**  $M(T(n, l+1))$  is a cordial graph.

**Proof:** Consider the tadpole  $T(n, l+1)$  in which  $v_1, v_2, \dots, v_n$  be the vertices of cycle  $C_n$  and  $v'_1, v'_2, v'_3, \dots, v'_l$  be the vertices of the path attached to the cycle  $C_n$ . Also let  $e_1, e_2, \dots, e_n$  and  $e'_1, e'_2, \dots, e'_l$  be the vertices in  $M(T(n, l+1))$  corresponding to the edges of cycle  $C_n$  and path  $P_n$  respectively in  $T(n, l+1)$ .

To define  $f : V(M(T(n, l+1))) \rightarrow \{0, 1\}$ , we consider the following cases.

**Case 1: n is odd**

**Subcase 1:**  $n = 2k + 1$ ,  $k = 2, 4, 6, \dots$  and  $l = 2j$ ,  $j = 2, 4, 6, \dots$

In this subcase  $|V(M(T(n, l+1)))| = 2n + 2l$ ,

$$|E(M(T(n, l+1)))| = 2 \left\lfloor \frac{n}{2} \right\rfloor + 2n + 2l + 6$$

$$\left. \begin{array}{l} f(v_{2i-1}) = 1 \\ f(v_{2i}) = 0 \end{array} \right\} 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$$

$$f(e_{4i-3}) = 0, 1 \leq i \leq \left\lfloor \frac{n}{4} \right\rfloor + 1$$

$$f(e_{4i-2}) = 0, 1 \leq i \leq \left\lfloor \frac{n}{4} \right\rfloor$$

$$\left. \begin{array}{l} f(e_{4i-1}) = 1 \\ f(e_{4i}) = 1 \end{array} \right\} 1 \leq i \leq \left\lfloor \frac{n}{4} \right\rfloor$$

$$f(v_n) = f(v'_1) = 1 \text{ (when } v'_1 \text{ is attached to } v_n)$$

$$\left. \begin{array}{l} f(v'_{2i}) = 0 \\ f(v'_{2i+1}) = 1 \end{array} \right\} 1 \leq i \leq \left\lfloor \frac{l}{2} \right\rfloor$$

$$\left. \begin{array}{l} f(e'_{4i-3}) = f(e'_{4i-2}) = 0 \\ f(e'_{4i-1}) = f(e'_{4i}) = 1 \end{array} \right\} 1 \leq i \leq \left\lfloor \frac{l}{4} \right\rfloor$$

In view of the above defined labeling pattern

$$v_f(0) = v_f(1) = n + l, e_f(0) = e_f(1) = \left\lfloor \frac{n}{2} \right\rfloor + n + l + 3$$

**Subcase 2:**  $n = 2k + 1$ ,  $k = 2, 4, 6, \dots$  and  $l = 2j$ ,  $j = 3, 5, 7, \dots$

In this subcase  $|V(M(T(n, l+1)))| = 2n + 2l$ ,

$$|E(M(T(n, l+1)))| = 2 \left\lfloor \frac{n}{2} \right\rfloor + 2n + 2l + 8$$

$$f(e'_{n-1}) = 0, f(e'_n) = 1$$

$$f(v_n) = 1, f(v'_1) = 1 \text{ (when } v'_1 \text{ is attached to } v_1)$$

remaining vertices are labeled as in subcase 1.

In view of the above defined labeling pattern

$$v_f(0) = v_f(1) = n + l, e_f(0) = e_f(1) = \left\lfloor \frac{n}{2} \right\rfloor + n + l + 4$$

For  $l = 2$  we have  $e_f(0) = e_f(1) = 11$ .

**Subcase 3:**  $n = 2k + 1$ ,  $k = 2, 4, 6, \dots$  and  $l = 2j + 1$ ,  $j = 1, 3, 5, 7, \dots$

In this subcase  $|V(M(T(n, l+1)))| = 2n + 2l$ ,

$$|E(M(T(n, l+1)))| = 2 \left\lfloor \frac{n}{2} \right\rfloor + 2n + 2l + 5$$

$$f(v_n) = f(v'_1) = 1 \text{ (when } v'_1 \text{ is attached to } v_n)$$

$$f(v'_{2i}) = 0, 1 \leq i \leq \left\lfloor \frac{l}{2} \right\rfloor + 1$$

$$f(v'_{2i+1}) = 1, 1 \leq i \leq \left\lfloor \frac{l}{2} \right\rfloor$$

$$\left. \begin{array}{l} f(e'_{4i-3}) = f(e'_{4i-2}) = 1 \\ f(e'_{4i-1}) = 0 \end{array} \right\} 1 \leq i \leq \left\lfloor \frac{l}{4} \right\rfloor + 1$$

$$f(e'_{4i}) = 0, 1 \leq i \leq \left\lfloor \frac{l}{4} \right\rfloor$$

remaining vertices are labeled as in subcase 1.

Using above pattern we have

$$v_f(0) = v_f(1) = n + l,$$

$$e_f(0) + 1 = e_f(1) = \left\lfloor \frac{n}{2} \right\rfloor + n + l + 3$$

For  $l = 1$  we have  $e_f(0) + 1 = e_f(1) = 10$ .

**Subcase 4:**  $n = 2k + 1$ ,  $k = 2, 4, 6, \dots$  and  $l = 2j + 1$ ,  $j = 2, 4, 6, \dots$

In this subcase  $|V(M(T(n, l + 1)))| = 2n + 2l$ ,

$$|E(M(T(n, l + 1)))| = 2 \left\lfloor \frac{n}{2} \right\rfloor + 2n + 2l + 7$$

$$f(v_n) = f(v'_1) = 1 \text{ (when } v'_1 \text{ is attached to } v_n)$$

$$f(v'_{2i}) = 0, \quad 1 \leq i \leq \left\lfloor \frac{l}{2} \right\rfloor + 1$$

$$f(v'_{2i+1}) = 1, \quad 1 \leq i \leq \left\lfloor \frac{l}{2} \right\rfloor$$

$$f(e'_{4i-3}) = 1, \quad 1 \leq i \leq \left\lfloor \frac{l}{4} \right\rfloor + 1$$

$$\left. \begin{aligned} f(e'_{4i-2}) &= 1 \\ f(e'_{4i-1}) &= f(e'_{4i}) = 0 \end{aligned} \right\} 1 \leq i \leq \left\lfloor \frac{l}{4} \right\rfloor$$

remaining vertices are labeled as in subcase 1.

Using above pattern we have

$$v_f(0) = v_f(1) = n + l,$$

$$e_f(0) + 1 = e_f(1) = \left\lfloor \frac{n}{2} \right\rfloor + n + l + 4$$

**Subcase 5:**  $n = 2k + 1$ ,  $k = 1, 3, 5, 7, \dots$  and  $l = 2j$ ,  $j = 2, 4, 6, \dots$

In this subcase  $|V(M(T(n, l + 1)))| = 2n + 2l$ ,

$$|E(M(T(n, l + 1)))| = 2 \left\lfloor \frac{n}{2} \right\rfloor + 2n + 2l + 6$$

$$\left. \begin{aligned} f(v_{2i-1}) &= 1 \\ f(v_{2i}) &= 0 \end{aligned} \right\} 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$$

$$\left. \begin{aligned} f(e_{4i-3}) &= f(e_{4i-2}) = 0 \\ f(e_{4i-1}) &= 1 \end{aligned} \right\} 1 \leq i \leq \left\lfloor \frac{n}{4} \right\rfloor + 1$$

$$f(e_{4i}) = 1, \quad 1 \leq i \leq \left\lfloor \frac{n}{4} \right\rfloor$$

$$f(v_n) = f(v'_1) = 1 \text{ (when } v'_1 \text{ is attached to } v_n)$$

$$\left. \begin{aligned} f(v'_{2i}) &= 1 \\ f(v'_{2i+1}) &= 0 \end{aligned} \right\} 1 \leq i \leq \left\lfloor \frac{l}{2} \right\rfloor$$

$$\left. \begin{aligned} f(e'_{4i-3}) &= f(e'_{4i-2}) = 0 \\ f(e'_{4i-1}) &= f(e'_{4i}) = 1 \end{aligned} \right\} 1 \leq i \leq \frac{l}{4}$$

Using above pattern we have

$$v_f(0) = v_f(1) = n + l, \quad e_f(0) = e_f(1) = \left\lfloor \frac{n}{2} \right\rfloor + n + l + 3$$

**Subcase 6:**  $n = 2k + 1$ ,  $k = 1, 3, 5, 7, \dots$  and  $l = 2j$ ,  $j = 3, 5, 7, \dots$

In this subcase  $|V(M(T(n, l + 1)))| = 2n + 2l$ ,

$$|E(M(T(n, l + 1)))| = 2 \left\lfloor \frac{n}{2} \right\rfloor + 2n + 2l + 8$$

$$f(v_n) = f(v'_1) = 1 \text{ (when } v'_1 \text{ is attached to } v_n)$$

$$\left. \begin{aligned} f(v'_{2i}) &= 0 \\ f(v'_{2i+1}) &= 1 \end{aligned} \right\} 1 \leq i \leq \frac{l}{2}$$

$$\left. \begin{aligned} f(e'_{4i-3}) &= f(e'_{4i-2}) = 0 \\ f(e'_{4i-1}) &= f(e'_{4i}) = 1 \end{aligned} \right\} 1 \leq i \leq \left\lfloor \frac{l}{4} \right\rfloor$$

$$f(e'_{n-1}) = 0, f(e'_n) = 1$$

remaining vertices are labeled as in subcase 5.

Using above pattern we have

$$v_f(0) = v_f(1) = n + l, \quad e_f(0) = e_f(1) = \left\lfloor \frac{n}{2} \right\rfloor + n + l + 4$$

For  $l = 2$  we have  $e_f(0) = e_f(1) = 8$ .

**Subcase 7:**  $n = 2k + 1$ ,  $k = 1, 3, 5, 7, \dots$  and  $l = 2j + 1$ ,  $j = 1, 3, 5, \dots$

In this subcase  $|V(M(T(n, l + 1)))| = 2n + 2l$ ,

$$|E(M(T(n, l + 1)))| = 2 \left\lfloor \frac{n}{2} \right\rfloor + 2n + 2l + 5$$

$$f(v_n) = f(v'_1) = 1 \text{ (when } v'_1 \text{ is attached to } v_n)$$

$$f(v'_{2i}) = 0, \quad 1 \leq i \leq \left\lfloor \frac{l}{2} \right\rfloor + 1$$

$$f(v'_{2i+1}) = 1, \quad 1 \leq i \leq \left\lfloor \frac{l}{2} \right\rfloor$$

$$\left. \begin{array}{l} f(e'_{4i-3}) = f(v'_{4i-2}) = 1 \\ f(e'_{4i-1}) = 0 \end{array} \right\} 1 \leq i \leq \left\lfloor \frac{l}{4} \right\rfloor + 1$$

$$f(e'_{4i}) = 0, \quad 1 \leq i \leq \left\lfloor \frac{l}{4} \right\rfloor$$

remaining vertices are labeled as in subcase 5.

Using above pattern we have

$$v_f(0) = v_f(1) = n + l,$$

$$e_f(0) = e_f(1) + 1 = \left\lfloor \frac{n}{2} \right\rfloor + n + l + 3$$

For  $l = 1$  we have  $e_f(0) = e_f(1) + 1 = 7$ .

**Subcase 8:**  $n = 2k + 1$ ,  $k = 1, 3, 5, 7, \dots$  and  $l = 2j + 1$ ,  $j = 2, 4, 6, \dots$

In this subcase  $|V(M(T(n, l+1)))| = 2n + 2l$ ,

$$|E(M(T(n, l+1)))| = 2 \left\lfloor \frac{n}{2} \right\rfloor + 2n + 2l + 7$$

$$f(v_n) = f(v'_1) = 1 \text{ (when } v'_1 \text{ is attached to } v_n)$$

$$f(v'_{2i}) = 0, \quad 1 \leq i \leq \left\lfloor \frac{l}{2} \right\rfloor + 1$$

$$f(v'_{2i+1}) = 1, \quad 1 \leq i \leq \left\lfloor \frac{l}{2} \right\rfloor$$

$$f(e'_{4i-3}) = 1, \quad 1 \leq i \leq \left\lfloor \frac{l}{4} \right\rfloor + 1$$

$$\left. \begin{array}{l} f(e'_{4i-2}) = 1 \\ f(e'_{4i-1}) = f(e'_{4i}) = 0 \end{array} \right\} 1 \leq i \leq \left\lfloor \frac{l}{4} \right\rfloor$$

remaining vertices are labeled as in subcase 5.

Using above pattern we have

$$v_f(0) = v_f(1) = n + l,$$

$$e_f(0) = e_f(1) + 1 = \left\lfloor \frac{n}{2} \right\rfloor + n + l + 4$$

**Case 2: n is even**

**Subcase 1:**  $n = 2k$ ,  $k = 2, 4, 6, \dots$  and  $l = 2j$ ,  $j = 2, 4, 6, \dots$

In this subcase  $|V(M(T(n, l+1)))| = 2n + 2l$ ,  $|E(M(T(n, l+1)))| = 3n + 2l + 5$

$$\left. \begin{array}{l} f(v_{2i-1}) = 1 \\ f(v_{2i}) = 0 \end{array} \right\} 1 \leq i \leq \frac{n}{2}$$

$$\left. \begin{array}{l} f(e_{4i-3}) = f(e_{4i-2}) = 0 \\ f(e_{4i-1}) = f(e_{4i}) = 1 \end{array} \right\} 1 \leq i \leq \frac{n}{4}$$

$$f(v'_1) = 1 \text{ (when } v'_1 \text{ is attached to } v_1)$$

$$\left. \begin{array}{l} f(v'_{2i}) = 0 \\ f(v'_{2i+1}) = 1 \end{array} \right\} 1 \leq i \leq \frac{l}{2}$$

$$\left. \begin{array}{l} f(e'_{4i-3}) = f(e'_{4i-2}) = 0 \\ f(e'_{4i-1}) = f(e'_{4i}) = 1 \end{array} \right\} 1 \leq i \leq \frac{l}{4}$$

Using above pattern we have

$$v_f(0) = v_f(1) = n + l, \quad e_f(0) = e_f(1) + 1 = \frac{3n}{2} + l + 3$$

**Subcase 2:**  $n = 2k$ ,  $k = 2, 4, 6, \dots$  and  $l = 2j$ ,  $j = 3, 5, 7, \dots$

In this subcase  $|V(M(T(n, l+1)))| = 2n + 2l$ ,  $|E(M(T(n, l+1)))| = 3n + 2l + 7$

$$f(v'_1) = 1 \text{ (when } v'_1 \text{ is attached to } v_1)$$

$$\left. \begin{array}{l} f(e'_{4i-3}) = f(e'_{4i-2}) = 0 \\ f(e'_{4i-1}) = f(e'_{4i}) = 1 \end{array} \right\} 1 \leq i \leq \left\lfloor \frac{l}{4} \right\rfloor$$

$$f(e'_{n-1}) = 0, \quad f(e'_n) = 1$$

remaining vertices are labeled as in subcase 1 of case (2).

Using above pattern we have

$$v_f(0) = v_f(1) = n + l, \quad e_f(0) + 1 = e_f(1) = \frac{3n}{2} + l + 4$$

For  $l = 2$  we have  $e_f(0) + 1 = e_f(1) = 10$ .

**Subcase 3:**  $n = 2k$ ,  $k = 2, 4, 6, \dots$  and  $l = 2j + 1$ ,  $j = 1, 3, 5, 7, \dots$

In this subcase  $|V(M(T(n, l+1)))| = 2n + 2l$ ,  $|E(M(T(n, l+1)))| = 3n + 2l + 4$

$$f(v'_1) = 1 \text{ (when } v'_1 \text{ is attached to } v_1)$$

$$f(v'_{2i}) = 0, \quad 1 \leq i \leq \left\lfloor \frac{l}{2} \right\rfloor + 1$$

$$f(v'_{2i+1}) = 1, \quad 1 \leq i \leq \left\lfloor \frac{l}{2} \right\rfloor$$

$$\left. \begin{array}{l} f(e'_{4i-3}) = f(e'_{4i-2}) = 1 \\ f(e'_{4i-1}) = 0 \end{array} \right\} 1 \leq i \leq \left\lfloor \frac{l}{4} \right\rfloor + 1$$

$$f(e'_{4i}) = 0, \quad 1 \leq i \leq \left\lfloor \frac{l}{4} \right\rfloor$$

remaining vertices are labeled as in subcase 1 of case (2).  
Using above pattern we have

$$v_f(0) = v_f(1) = n + l, \quad e_f(0) = e_f(1) = \frac{3n}{2} + l + 2$$

For  $l = 1$  we have  $e_f(0) = e_f(1) = 8$ .

**Subcase 4:**  $n = 2k$ ,  $k = 2, 4, 6, \dots$  and  $l = 2j + 1$ ,  
 $j = 2, 4, 6, \dots$

In this subcase  $|V(M(T(n, l+1)))| = 2n + 2l$ ,  
 $|E(M(T(n, l+1)))| = 3n + 2l + 6$

$$f(v'_1) = 1 \text{ (when } v'_1 \text{ is attached to } v_1)$$

$$f(v'_{2i}) = 0, \quad 1 \leq i \leq \left\lfloor \frac{l}{2} \right\rfloor + 1$$

$$f(v'_{2i+1}) = 1, \quad 1 \leq i \leq \left\lfloor \frac{l}{2} \right\rfloor$$

$$f(e'_{4i-3}) = 1, \quad 1 \leq i \leq \left\lfloor \frac{l}{4} \right\rfloor + 1$$

$$\left. \begin{array}{l} f(e'_{4i-2}) = 1 \\ f(e'_{4i-1}) = f(e'_{4i}) = 0 \end{array} \right\} 1 \leq i \leq \left\lfloor \frac{l}{4} \right\rfloor$$

remaining vertices are labeled as in subcase 1 of case (2).  
Using above pattern we have

$$v_f(0) = v_f(1) = n + l, \quad e_f(0) = e_f(1) = \frac{3n}{2} + l + 3.$$

**Subcase 5:**  $n = 2k$ ,  $k = 3, 5, 7, \dots$  and  $l = 2j$ ,  
 $j = 2, 4, 6, \dots$

In this subcase  $|V(M(T(n, l+1)))| = 2n + 2l$ ,  
 $|E(M(T(n, l+1)))| = 3n + 2l + 5$

$$\left. \begin{array}{l} f(v'_{2i-1}) = 1 \\ f(v'_{2i}) = 0 \end{array} \right\} 1 \leq i \leq \frac{n}{2}$$

$$\left. \begin{array}{l} f(e'_{4i-3}) = f(e'_{4i-2}) = 0 \\ f(e'_{4i-1}) = f(e'_{4i}) = 1 \end{array} \right\} 1 \leq i \leq \left\lfloor \frac{n}{4} \right\rfloor$$

$$f(e'_{n-1}) = 0, f(e'_n) = 1$$

$$f(v'_1) = 1 \text{ (when } v'_1 \text{ is attached to } v_1)$$

$$\left. \begin{array}{l} f(v'_{2i}) = 0 \\ f(v'_{2i+1}) = 1 \end{array} \right\} 1 \leq i \leq \frac{l}{2}$$

$$\left. \begin{array}{l} f(e'_{4i-3}) = f(e'_{4i-2}) = 0 \\ f(e'_{4i-1}) = f(e'_{4i}) = 1 \end{array} \right\} 1 \leq i \leq \frac{l}{4}$$

Using above pattern we have

$$v_f(0) = v_f(1) = n + l, \quad e_f(0) + 1 = e_f(1) = \frac{3n}{2} + l + 3$$

**Subcase 6:**  $n = 2k$ ,  $k = 3, 5, 7, \dots$  and  $l = 2j$ ,  
 $j = 3, 5, 7, \dots$

In this subcase  $|V(M(T(n, l+1)))| = 2n + 2l$ ,  
 $|E(M(T(n, l+1)))| = 3n + 2l + 7$

$$f(v'_1) = 1 \text{ (when } v'_1 \text{ is attached to } v_1)$$

$$\left. \begin{array}{l} f(e'_{4i-3}) = f(e'_{4i-2}) = 0 \\ f(e'_{4i-1}) = f(e'_{4i}) = 1 \end{array} \right\} 1 \leq i \leq \left\lfloor \frac{l}{4} \right\rfloor$$

$$f(e'_{n-1}) = 1, f(e'_n) = 0$$

remaining vertices are labeled as in subcase 5 of case (2).  
Using above pattern we have

$$v_f(0) = v_f(1) = n + l, \quad e_f(0) + 1 = e_f(1) = \frac{3n}{2} + l + 4$$

For  $l = 2$  we have  $e_f(0) + 1 = e_f(1) = 13$

**Subcase 7:**  $n = 2k$ ,  $k = 3, 5, 7, \dots$  and  $l = 2j + 1$ ,  
 $j = 1, 3, 5, 7, \dots$

In this subcase  $|V(M(T(n, l+1)))| = 2n + 2l$ ,  
 $|E(M(T(n, l+1)))| = 3n + 2l + 4 + 4 \left\lfloor \frac{j}{2} \right\rfloor$

$$f(v'_1) = 0 \text{ (when } v'_1 \text{ is attached to } v_2)$$

$$f(v'_{2i}) = 1, \quad 1 \leq i \leq \left\lfloor \frac{l}{2} \right\rfloor + 1$$

$$f(v'_{2i+1}) = 0, \quad 1 \leq i \leq \left\lfloor \frac{l}{2} \right\rfloor$$

$$\left. \begin{array}{l} f(e'_{4i-3}) = f(e'_{4i-2}) = 0 \\ f(e'_{4i-1}) = 1 \end{array} \right\} 1 \leq i \leq \left\lfloor \frac{l}{4} \right\rfloor + 1$$

$$f(e'_{4i}) = 1, \quad 1 \leq i \leq \left\lfloor \frac{l}{4} \right\rfloor$$

remaining vertices are labeled as in subcase 5 of case (2).  
Using above pattern we have

$$v_f(0) = v_f(1) = n + l, \quad e_f(0) = e_f(1) = \frac{3n}{2} + l + 2 + 2 \left\lfloor \frac{j}{2} \right\rfloor$$

For  $l = 1$  we have  $e_f(0) = e_f(1) = 11$

**Subcase 8:**  $n = 2k$ ,  $k = 3, 5, 7, \dots$  and  $l = 2j + 1$ ,  
 $j = 2, 4, 6, \dots$

In this subcase  $|V(M(T(n, l+1)))| = 2n + 2l$ ,

$$|E(M(T(n, l+1)))| = 3n + 2l + 2 + 4 \left\lfloor \frac{j}{2} \right\rfloor$$

$$f(v'_1) = 0 \text{ (when } v'_1 \text{ is attached to } v_2)$$

$$f(v'_{2i}) = 1, \quad 1 \leq i \leq \left\lfloor \frac{l}{2} \right\rfloor + 1$$

$$f(v'_{2i+1}) = 0, \quad 1 \leq i \leq \left\lfloor \frac{l}{2} \right\rfloor$$

$$f(e'_{4i-3}) = 0, \quad 1 \leq i \leq \left\lfloor \frac{l}{4} \right\rfloor + 1$$

$$\left. \begin{aligned} f(e'_{4i-2}) &= 0 \\ f(e'_{4i-1}) &= f(e'_{4i}) = 1 \end{aligned} \right\} 1 \leq i \leq \left\lfloor \frac{l}{4} \right\rfloor$$

remaining vertices are labeled as in subcase 5 of case (2).

Using above pattern we have

$$v_f(0) = v_f(1) = n + l,$$

$$e_f(0) = e_f(1) = \frac{3n}{2} + l + 1 + 2 \left\lfloor \frac{j}{2} \right\rfloor$$

Thus in all the cases described above  $f$  admits cordial labeling for  $M(T(n, l+1))$ . That is,  $M(T(n, l+1))$  admits cordial labeling.

**Illustration - 2.10 :** In the following Figure 2.4 cordial labeling of  $M(T(6, 5))$  is shown.

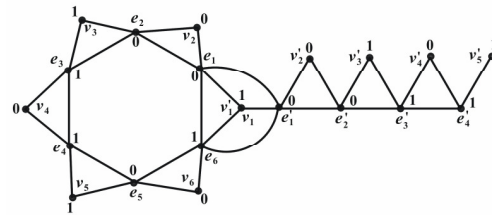


Figure 2.4  $M(T(6, 5))$  and its cordial labeling

### Concluding Remarks

Labeling of discrete structure is a potential area of research due to its diversified applications. We discuss here cordial labeling in the context middle graph of a graph. We contribute six new results to the theory of cordial labeling. It is possible to investigate analogous results for various families of graph and in the context of different graph labeling problems which is the open area of research.

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# Embedding and Np-Complete Problems for 3-Equitable Graphs

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## Abstract

We present here some important results in connection with 3-equitable graphs. We prove that any graph  $G$  can be embedded as an induced subgraph of a 3-equitable graph. We have also discussed some properties which are invariant under embedding. This work rules out any possibility of obtaining any forbidden subgraph characterization for 3-equitable graphs.

**Keywords:** Embedding, NP-Complete, 3-Equitable Graph.

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2000 Mathematics Subject Classification: 05C78

## 1. INTRODUCTION

We begin with simple, finite, connected and undirected graph  $G = (V(G), E(G))$ , where  $V(G)$  is called set of vertices and  $E(G)$  is called set of edges of a graph  $G$ . For all other terminology and notations in graph theory we follow West [1] and for number theory we follow Niven and Zuckerman [2].

**Definition 1.1** The assignment of numbers to the vertices of a graph with certain condition(s) is called graph labeling.

For detailed survey on graph labeling we refer to Gallian [3]. Vast amount of literature is available on different types of graph labeling and more than 1200 papers have been published in past four decades. As stated in Beineke and Hegde [4] graph labeling serves as a frontier between number theory and structure of graphs. Most of the graph labeling techniques trace their origin to that one introduced by Rosa [5].

### Definition 1.2

Let  $G = (V(G), E(G))$  be a graph with  $p$  vertices and  $q$  edges. Let  $f : V \rightarrow \{0, 1, 2, \dots, q\}$  be an injection. For each edge  $uv \in E$ , define  $f^*(uv) = |f(u) - f(v)|$ . If  $f^*(E) = \{1, 2, \dots, q\}$  then  $f$  is called  $\beta$ -valuation. Golomb [6] called such labeling as a graceful labeling and this is now the familiar term.

### Definition 1.3

For a mapping  $f : V(G) \rightarrow \{0, 1, 2, \dots, k-1\}$  and an edge  $e = uv$  of  $G$ , we define  $f(e) = |f(u) - f(v)|$ . The labeling  $f$  is called a  $k$ -equitable labeling if the number of vertices with the label  $i$  and the number of vertices with the label  $j$  differ by at most 1 and the number of edges with the label  $i$  and the number of edges with label  $j$  differ by at most 1. By  $v_f(i)$  we mean the number of vertices with the label  $i$  and by  $e_f(i)$  we mean the number of edges with the label  $i$ .

Thus for  $k$  - equitable labeling we must have  $|v_f(i) - v_f(j)| \leq 1$  and  $|e_f(i) - e_f(j)| \leq 1$ , where  $0 \leq i, j \leq k-1$ .

For  $k = 2$ ,  $f$  is called cordial labeling and for  $k = 3$ ,  $f$  is called 3-equitable labeling. We focus on 3-equitable labeling.

A graph  $G$  is *3-equitable* if it admits a 3-equitable labeling. This concept was introduced by Cahit [7]. There are four types of problems that can be considered in this area.

- (1) How 3-equitability is affected under various graph operations.
- (2) Construct new families of 3-equitable graphs by finding suitable labeling.
- (3) Given a graph theoretic property  $P$  characterize the class of graphs with property  $P$  that are 3-equitable.
- (4) Given a graph  $G$  having the graph theoretic property  $P$ , is it possible to embed  $G$  as an induced subgraph of a 3-equitable graph  $G$ , having the property  $P$ ?

The problems of first three types are largely investigated but the problems of last type are of great importance. Such problems are extensively explored recently by Acharya et al [8] in the context of graceful graphs. We present here an affirmative answer for planar graphs, trianglefree graphs and graphs with given chromatic number in the context of 3-equitable graphs. As a consequence we deduce that deciding whether the chromatic number is less than or equal to  $k$ , where  $k \geq 3$ , is NP-complete even for 3-equitable graphs. We obtain similar result for clique number also.

## 2. Main Results

### Theorem 2.1

Any graph  $G$  can be embedded as an induced subgraph of a 3-equitable graph.

**Proof:** Let  $G$  be the graph with  $n$  vertices. Without loss of generality we assume that it is always possible to label the vertices of any graph  $G$  such that the vertex conditions for 3-equitable graphs are satisfied. i.e.  $|v_f(i) - v_f(j)| \leq 1$ ,  $0 \leq i, j \leq 2$ . Let  $V_0$ ,  $V_1$  and  $V_2$  be the set of vertices with label 0, 1 and 2 respectively. Let  $E_0$ ,  $E_1$  and  $E_2$  be the set of edges with label 0, 1 and 2 respectively. Let  $n(V_0)$ ,  $n(V_1)$  and  $n(V_2)$  be the number of elements in sets  $V_0$ ,  $V_1$  and  $V_2$  respectively. Let  $n(E_0)$ ,  $n(E_1)$  and  $n(E_2)$  be the number of elements in sets  $E_0$ ,  $E_1$  and  $E_2$  respectively.

**Case 1:**  $n \equiv 0 \pmod{3}$

**Subcase 1:**  $n(E_0) \neq n(E_1) \neq n(E_2)$ .

Suppose  $n(E_0) < n(E_1) < n(E_2)$ . Let  $|n(E_2) - n(E_0)| = r > 1$  and  $|n(E_2) - n(E_1)| = s > 1$ . The new graph  $H$  can be obtained by adding  $r + s$  vertices to the graph  $G$ .

Define  $r + s = p$  and consider a partition of  $p$  as  $p = a + b + c$  with  $|a - b| \leq 1$ ,  $|b - c| \leq 1$  and  $|c - a| \leq 1$ .

Now out of new  $p$  vertices label  $a$  vertices with 0,  $b$  vertices with 1 and  $c$  vertices with 2. i.e. label the vertices  $u_1, u_2, \dots, u_a$  with 0,  $v_1, v_2, \dots, v_b$  with 1 and  $w_1, w_2, \dots, w_c$  with 2. Now we adapt the following procedure.

**Step 1:** To obtain required number of edges with label 1.

- Join  $s$  number of elements  $v_i$  to the arbitrary element of  $V_0$ .
- If  $b < s$  then join  $(s - b)$  number of elements  $u_1, u_2, \dots, u_{s-b}$  to the arbitrary element of  $V_1$ .
- If  $a < s - b$  then join  $(s - a - b)$  number of vertices  $w_1, w_2, \dots, w_{s-a-b}$  to the arbitrary element of  $V_1$ .

Above construction will give rise to required number of edges with label 1.

**Step 2:** To obtain required number of edges with label 0.

- Join remaining number of  $u_i$ 's (which are left at the end of step 1) to the arbitrary element of  $V_0$ .

- Join the remaining number of  $v_i$ 's (which are left at the end of step 1) to the arbitrary element of  $V_1$ .
- Join the remaining number of  $w_i$ 's (which are left at the end of step 1) to the arbitrary element of  $V_2$ .

As a result of above procedure we have the following vertex conditions and edge conditions.

$$|v_f(0) - v_f(1)| = |n(V_0) + a - n(V_1) - b| \leq 1,$$

$$|v_f(1) - v_f(2)| = |n(V_1) + b - n(V_2) - c| \leq 1,$$

$$|v_f(2) - v_f(0)| = |n(V_2) + c - n(V_0) - a| \leq 1$$

and

$$|e_f(0) - e_f(1)| = |n(E_0) + n(E_2) - n(E_0) - n(E_1) - n(E_2) + n(E_1)| = 0,$$

$$|e_f(1) - e_f(2)| = |n(E_1) + n(E_2) - n(E_1) - n(E_2)| = 0,$$

$$|e_f(2) - e_f(0)| = |n(E_2) - n(E_0) - n(E_2) + n(E_0)| = 0.$$

Similarly one can handle the following cases.

$$n(E_0) < n(E_2) < n(E_1),$$

$$n(E_2) < n(E_0) < n(E_1),$$

$$n(E_1) < n(E_2) < n(E_0),$$

$$n(E_2) < n(E_1) < n(E_0),$$

$$n(E_1) < n(E_0) < n(E_2).$$

**Subcase 2:**  $n(E_i) = n(E_j) < n(E_k), i \neq j \neq k, 0 \leq i, j, k \leq 2$

Suppose  $n(E_0) = n(E_1) < n(E_2)$

$$|n(E_2) - n(E_0)| = r$$

$$|n(E_2) - n(E_1)| = r$$

The new graph  $H$  can be obtained by adding  $2r$  vertices to the graph  $G$ .

Define  $2r = p$  and consider a partition of  $p$  as  $p = a + b + c$  with  $|a - b| \leq 1, |b - c| \leq 1$  and  $|c - a| \leq 1$ .

Now out of new  $p$  vertices, label  $a$  vertices with 0,  $b$  vertices with 1 and  $c$  vertices with 2. i.e. label the vertices  $u_1, u_2, \dots, u_a$  with 0,  $v_1, v_2, \dots, v_b$  with 1 and  $w_1, w_2, \dots, w_c$  with 2. Now we adapt the following procedure.

### **Step 1:**

To obtain required number of edges with label 0.

- Join  $r$  number of elements  $u_i$ 's to the arbitrary element of  $V_0$ .
- If  $a < r$  then join  $(r - a)$  number of elements  $v_1, v_2, \dots, v_{r-a}$  to the arbitrary element of  $V_1$ .
- If  $b < r - a$  then join  $(r - a - b)$  number of vertices  $w_1, w_2, \dots, w_{r-a-b}$  to the arbitrary element of  $V_2$ .

Above construction will give rise to required number of edges with label 0.

### **Step 2:**

To obtain required number of edges with label 1.

- Join remaining number of  $w_i$ 's (which are not used at the end of step 1) to the arbitrary element of  $V_1$ .
- Join the remaining number of  $v_i$ 's (which are not used at the end of step 1) to the arbitrary element of  $V_0$ .
- Join the remaining number of  $u_i$ 's (which are not used at the end of step 1) to the arbitrary element

of  $V_1$ .

Similarly we can handle the following possibilities.

$$n(E_1) = n(E_2) < n(E_0)$$

$$n(E_0) = n(E_2) < n(E_1)$$

**Subcase 3:**  $n(E_i) < n(E_j) = n(E_k), i \neq j \neq k, 0 \leq i, j, k \leq 2$

Suppose  $n(E_2) < n(E_0) = n(E_1)$

Define  $|n(E_2) - n(E_0)| = r$

The new graph  $H$  can be obtained by adding  $r$  vertices to the graph  $G$  as follows.

Consider a partition of  $r$  as  $r = a + b + c$  with  $|a - b| \leq 1, |b - c| \leq 1$  and  $|c - a| \leq 1$ .

Now out of new  $r$  vertices label  $a$  vertices with 0,  $b$  vertices with 1 and  $c$  vertices with 2 i.e. label the vertices  $u_1, u_2, \dots, u_a$  with 0,  $v_1, v_2, \dots, v_b$  with 1 and  $w_1, w_2, \dots, w_c$  with 2. Now we adapt the following procedure.

### Step 1:

To obtain required number of edges with label 2.

- Join  $r$  number of vertices  $w_i$ 's to the arbitrary element of  $V_0$ .
- If  $c < r$  then join  $r - c$  number of elements  $u_1, u_2, \dots, u_{r-c}$  to the arbitrary element of  $V_2$ .

Above construction will give rise to required number of edges with label 2.

At the end of this step if the required number of 2 as edge labels are generated then we have done. If not then move to step 2. This procedure should be followed in all the situations described earlier when  $n(E_2) < n(E_0)$  or  $n(E_2) < n(E_1)$ .

### Step 2:

To obtain the remaining (at the end of step 1) number of edges with label 2.

- If  $k$  number of edges are required after joining all the vertices with label 0 and 2 then add  $k$  number of vertices labeled with 0,  $k$  number of vertices labeled with 1 and  $k$  number of vertices labeled with 2. Then vertex conditions are satisfied.
- Now we have  $k$  number of new vertices with label 2,  $k$  number of new vertices with label 0 and  $2k$  number of new vertices with label 1.
- Join  $k$  new vertices with label 2 to the arbitrary element of the set  $V_0$ .
- Join  $k$  new vertices with label 0 to the arbitrary element of the set  $V_2$ .
- Join  $k$  new vertices with label 1 to the arbitrary element of set  $V_0$ .
- Join  $k$  new vertices with label 1 to the arbitrary element of the set  $V_1$ .

**Case 2:**  $n \equiv 1 \pmod{3}$ .

**Subcase 1:**  $n(E_i) \neq n(E_j) \neq n(E_k), i \neq j \neq k, 0 \leq i, j, k \leq 2$ .

Suppose  $n(E_0) < n(E_1) < n(E_2)$  Let  $|n(E_2) - n(E_0)| = r > 1$  and  $|n(E_2) - n(E_1)| = s > 1$ .

Define  $r + s = p$  and consider a partition of  $p$  such that  $p = a + b + c$  with

$$|n(V_0) + a - n(V_1) - b| \leq 1$$

$$|n(V_1) + b - n(V_2) - c| \leq 1$$

$$|n(V_0) + a - n(V_2) - c| \leq 1.$$

Now we can follow the procedure which we have discussed in case-1.

**Case 3:**  $n \equiv 2 \pmod{3}$

We can proceed as case-1 and case-2.

Thus in all the possibilities the graph  $H$  resulted due to above construction satisfies the conditions for 3-equitable graph. That is, any graph  $G$  can be embedded as an induced subgraph of a 3-

equitable graph.

For the better understanding of result derived above consider following illustrations.

### Illustration 2.2

For a Graph  $G = C_9$  we have  $n(E_0) = 0$ ,  $n(E_1) = 6$ ,  $n(E_2) = 3$ .

Now  $|n(E_1) - n(E_0)| = 6 = r$ ,  $|n(E_1) - n(E_2)| = 3 = s$ .

This is the case related to subcase (1) of case (1).

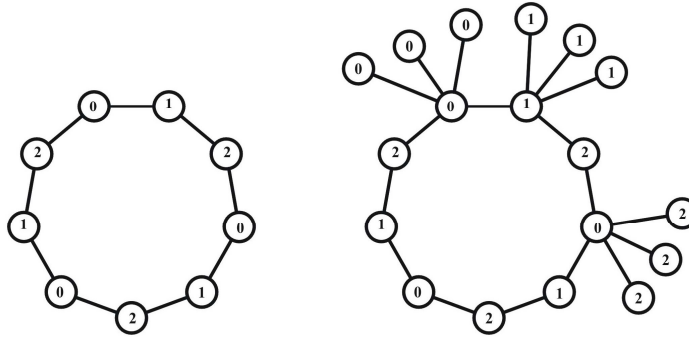


FIGURE 1:  $C_9$  and its 3-equitable embedding

**Procedure to construct  $H$  :**

#### Step 1:

- Add  $p = r + s = 6 + 3 = 9$  vertices in  $G$  and partition  $p$  as  $p = a + b + c = 3 + 3 + 3$ .
- Label 3 vertices with 0 as  $a = 3$ .
- Label 3 vertices with 1 as  $b = 3$ .
- Label 3 vertices with 2 as  $c = 3$ .

#### Step 2:

- Join the vertices with 0 and 1 to the arbitrary element of the set  $V_0$  and  $V_1$  respectively.
- Join the vertices with label 2 to the arbitrary element of set  $V_0$ .

The resultant graph  $H$  is shown in Figure 1 is 3-equitable.

### Illustration 2.3

Consider a Graph  $G = K_4$  as shown in following Figure 2 for which  $n(E_0) = 1$ ,  $n(E_1) = 4$ ,  $n(E_2) = 1$ .

Here  $|n(E_1) - n(E_0)| = 3 = r$ ,  $|n(E_1) - n(E_2)| = 3 = s$  i.e.  $r = s$ .

This is the case related to subcase (2) of case (2).

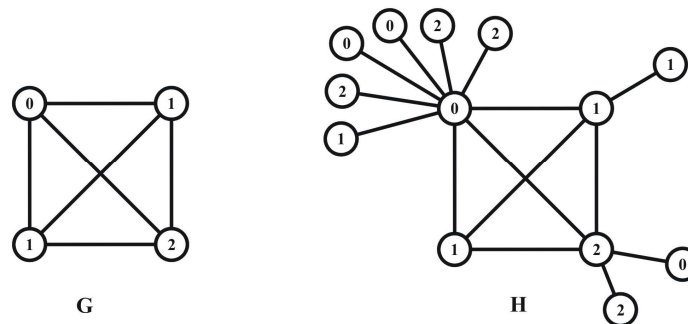


FIGURE 2:  $K_4$  and its 3-equitable embedding

**Procedure to construct  $H$  :**

**Step 1:**

- Add  $p = 2r = 3 + 3 = 6$  vertices in  $G$  and partition  $p$  as  $p = a + b + c = 2 + 1 + 3$ .
- Label 2 vertices with 0 as  $a = 2$ .
- Label 1 vertex with 1 as  $b = 1$ .
- Label 3 vertices with 2 as  $c = 3$ .

**Step 2:**

- Join the vertices with label 0 to the arbitrary element of the set  $V_0$  and join one vertex with label 2 to the arbitrary element of  $V_2$ .
- join the remaining vertices with label 2 with the arbitrary element of set  $V_0$ .

**Step 3:**

- Now add three more vertices and label them as 0, 1 and 2 respectively.
- Now join the vertices with label 0 and 2 with the arbitrary elements of  $V_2$  and  $V_0$  respectively.
- Now out of the remaining two vertices with label 1 join one vertex with arbitrary element of set  $V_0$  and the other with the arbitrary element of set  $V_1$ .

The resultant graph  $H$  shown in Figure 2 is 3-equitable.

**Corollary 2.4** Any planar graph  $G$  can be embedded as an induced subgraph of a planar 3-equitable graph.

**Proof:** If  $G$  is planar graph. Then the graph  $H$  obtained by Theorem 2.1 is a planar graph.

**Corollary 2.5** Any triangle-free graph  $G$  can be embedded as an induced subgraph of a triangle free 3-equitable graph.

**Proof:** If  $G$  is triangle-free graph. Then the graph  $H$  obtained by Theorem 2.1 is a triangle-free graph.

**Corollary 2.6** The problem of deciding whether the chromatic number  $\chi \leq k$ , where  $k \geq 3$  is NP-complete even for 3-equitable graphs.

**Proof:** Let  $G$  be a graph with chromatic number  $\chi(G) \geq 3$ . Let  $H$  be the 3-equitable graph constructed in Theorem 2.1, which contains  $G$  as an induced subgraph. Since  $H$  is constructed by adding only pendant vertices to  $G$ . We have  $\chi(H) = \chi(G)$ . Since the problem of deciding whether the chromatic number  $\chi \leq k$ , where  $k \geq 3$  is NP-complete [9]. It follows that deciding whether the chromatic number  $\chi \leq k$ , where  $k \geq 3$ , is NP-complete even for 3-equitable graphs.

**Corollary 2.7** The problem of deciding whether the clique number  $\omega(G) \geq k$  is NP-complete even when restricted to 3-equitable graphs.

**Proof:** Since the problem of deciding whether the clique number of a graph  $\omega(G) \geq k$  is NP-complete [9] and  $\omega(H) = \omega(G)$  for the 3-equitable graph  $H$  constructed in Theorem 2.1, the above result follows.

### 3. Concluding Remarks

In this paper, we have considered the general problem. Given a graph theoretic property  $P$  and a graph  $G$  having  $P$ , is it possible to embed  $G$  as an induced subgraph of a 3-equitable graph  $H$  having the property  $P$ ? As a consequence we derive that deciding whether the chromatic number  $\chi \leq k$ , where  $k \geq 3$ , is NP-complete even for 3-equitable graphs. We obtain similar result for clique number. Moreover this work rules out any possibility of forbidden subgraph characterization for 3-equitable graph. Analogous work for other graph theoretic parameters like domination number, total domination number, fractional domination number etc. and graphs admitting various other types of labeling can be carried out for further research.

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