Some Investigations in the Theory of Graphs

a thesis submitted to

THE
SAURASHTRA UNIVERSITY
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for the award of the degree of

DOCTOR OF PHILOSOPHY

in

MATHEMATICS

by

Gaurang V. Ghodasara

under the supervision of

Dr. S. K. Vaidya
Department of Mathematics
Saurashtra University, RAJKOT − 360 005
INDIA.

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Certificate

This is to certify that the thesis entitled SOME INVESTIGATIONS IN THE THEORY OF GRAPHS submitted by **Gaurang V. Ghodasara** to the **Saurashtra University, RAJKOT (GUJ.)** for the award of the degree of DOCTOR OF PHILOSOPHY in Mathematics is bonafide record of research work carried out by him under my supervision. The contents embodied in the thesis have not been submitted in part or full to any other Institution or University for the award of any degree or diploma.

Place: RAJKOT

Date: 24/03/2008

Dr S. K. Vaidya

Associate Professor

Department of Mathematics

Saurashtra University

RAJKOT – 360005 (Gujarat)

INDIA.

Declaration

I hereby declare that the contents embodied in this thesis is the bonafide record of investigations carried out by me under the supervision of **Dr. S. K. Vaidya** in the Department of Mathematics, **Saurashtra University, RAJKOT**. The investigations reported here have not been submitted in part or full for the award of any degree or diploma to any other Institution or University.

Place: RAJKOT

Date: 24/03/2008

G. V. Ghodasara

Lecturer in Mathematics

Atmiya Institute of Technology and Science

RAJKOT – 360005 (Gujarat)

INDIA.

Acknowledgement

It is a moment of immense pleasure for me to present this thesis entitled Some Investigations in the Theory of Graphs to the **Saurashtra University, RAJKOT (Guj.)** for the award of the degree of Doctor of Philosophy in the subject of **Mathematics**.

I am thankful to the almighty to give me enough strength to reach at this stage of my career. I am highly indebted to my research guide **Dr. S. K. Vaidya** for his constant encouragement, valuable guidance and painstaking care throughout my career. Without his constant motivation and inspiration this would not have seen the light of the day.

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Gaurang V. Ghodasara

CONTENTS

Chapter 1
Introduction

The theory of graphs mainly evolved in last century with the rise of computer age. This theory has variety of applications in diversified fields like computer technology, communication networks, electrical networks etc. Graph Theory is a powerful mathematical tool to explain ecosystems, sociological relationship and structure of molecules. It has solution for traffic problem in cities. Flow of control and databases can be discussed using graph structures.

The great Swiss mathematician Leonhard Euler(1707-1783) used graph to study Königsberg bridge problem during 1736 and introduced that how the graphs are helpful to discuss complicated network problems. For the next 100 years nothing more was explored in the field. In 1847 G. R. Kirchhoff(1824-1887) developed the theory of trees for their applications in electrical networks. Ten years later, A. Cayley(1821-1895) discovered trees while he was trying to enumerate the isomers of hydrocarbons.

It is believed that A. F. Möbious(1790-1868) presented the famous four colour problem in one of his lecture in 1840. About ten years later, A. De Morgan(1806-1871) discussed this problem with his fellow mathematicians in London. De Morgan's letter is regarded as the first systematic reference to the four colour problem. The problem became well known after Cayley published it in 1879 in the first volume of the *Proceedings of The Royal Geographic Society*. To this day, the four colour problem has stimulated an enormous amount of research in the field of graph theory.

The other milestone is due to Sir W. R. Hamilton(1805-1865). In the year 1859 he invented a puzzle and sold it for 25 guineas to a game manufacturer in Dublin. The puzzle was made up of wood whose shape was regular dodecahedron(a polyhedron with 12 faces and 20 corners, each face being a regular pentagon and three edges meeting at each corner). The corners were marked with the names of 20 important cities. The object in the puzzle was to find a route along the edges of the dodecahedron passing through each of the 20 cities exactly once. Although the solution of this problem is easy to obtain but no one has found a necessary and sufficient condition for the existence of such route in an arbitrary graph.

The fertile period was started during 1920 when D. König wrote the first book on the subject. This book was published in 1936. The later part of last century was the period of intense activity in graph theory. Development of computer science and optimization techniques are root cause for this unprecedented growth. Vast amount of papers and more than a dozen books have been published.

In India E. Sampathkumar(University of Mysore) is the first university professor to teach graph theory as a special paper in the final year M.Sc. at Karnataka university, Dharwad, during 1970-1971. In recent time B. D. Acharya, S. A. Choudum, R. Balakrishnan, S. Arumugum, S. B. Rao, V. Swaminathan and many other research groups contributed lot in the topics like domination, graph colouring, signed digraphs, graph labeling etc. Many conferences and seminars are held on graph theory, combinatorics and their applications. These activities are supported by Government of India through its Department of Science and Technology(DST). National Center for Advance Research in Discrete Mathematics(n-CARDMATH) has been established at Kalasalingam University - Krishnankoil(Tamil Nadu). This center provides guidance and motivation to any researcher. The present work is motivated through a group discussion sponsored by DST which was organized during April 19-26, 2006 at Mary Matha Arts and Science College-Mananthavady(Kerala). The focus of the present work is some labeling techniques in graph theory particularly graceful labeling, cordial labeling and 3-equitable labeling. Investigation in this field becomes more interesting because the field itself is rapidly emerging and a whole galore of seemingly related or even unrelated open problems provide motivation for advance research.

The content of this work is divided into nine chapters. This first chapter is of introductory nature.

The immediate *Chapter 2* is intended to provide basic terminology and preliminaries.

The next *Chapter 3* is aimed to provide survey of various techniques of graph labeling. Existing results and latest updates are reported which will serve as a reference material for any scholar.

The next *Chapter 4* is focussed on reconstruction of graph. We have investigated some new results and a conjecture is posed which is stronger than the existing result.

The penultimate *Chapter 5* is aimed to discuss graceful labeling in detail. The discussion is held about gracefulness of grid graph with some other families of graphs. The results reported here are published in the *Proceedings of The International Conference on Emerging Technologies and Applications in Engineering Technology and Sciences 2008*.

The detailed discussion about cordial labeling of graphs is carried out in *Chapter 6*. We have investigated eleven new families of cordial graphs.

The results reported here are accepted for publication in reputed journals like *The Mathematics Student*, *Indian Journal of Mathematics and Mathematical Sciences*, *Proceedings of the International Conference on Emerging Technologies and Applications in Engineering Technology and Sciences 2008* and *International Journal of Scientific Computing*.

The immediate *Chapter 7* relates some graph operations and cordial labeling. Some new results are obtained.

The next *Chapter 8* is intended to discuss 3-equitable labeling of graphs. The results reported in this chapter are novel and published in *Proceedings of the International Conference on Emerging Technologies and Applications in Engineering Technology and Sciences 2008*.

Labeled graphs are becoming increasingly useful mathematical models for broad range of applications. They are useful for the solutions of problems in additive number theory and coding theory. Some applications like determination of ambiguities in X-ray crystallography, design of good radar type codes and laying of optimized communication network are reported in *Chapter 9*. Further scope of research is also given to provide enough motivation. List of symbols is given and references are listed alphabetically at the end of the thesis. The entire thesis is prepared in LaTeX to meet the global standard.

The whole work will establish a new trend of research in the field of graph theory in Gujarat region. We hope that very active research group will come up in near future.

Chapter 2

Basic Terminology and Preliminaries

2.1 Introduction

This chapter is devoted to provide all the fundamentals and notations which are useful for the present work. Basic definitions are given and explained with sufficient illustrations. Figures make this work more effective.

2.2 Basic Definitions

Definition 2.2.1 A *graph* $G = (V, E)$ consists of two sets, $V = \{v_1, v_2, \ldots\}$ called *vertex set* of G and $E = \{e_1, e_2, \ldots\}$ called *edge set* of G. Sometimes we denote vertex set of G as $V(G)$ and edge set of G as $E(G)$. Elements of V are called *vertices* and elements of E are called *edges*.

Definition 2.2.2 A graph consisting of one vertex and no edge is called a *trivial graph*. A graph which is not trivial is called a *non-trivial graph*.

Definition 2.2.3 The number of edges in a given graph is called *size of the graph*.

Definition 2.2.4 The number of vertices in a given graph is called *order of the graph*.

A graph with order p and size q is sometimes denoted as (p, q) graph.

Definition 2.2.5 An edge of a graph that joins a vertex to itself is called a *loop*. A loop is an edge $e = v_iv_i$.

Definition 2.2.6 If two vertices of a graph are joined by more than one edge then these edges are called *multiple edges*.

Definition 2.2.7 If two vertices of a graph are joined by an edge then these vertices are called *adjacent vertices*.

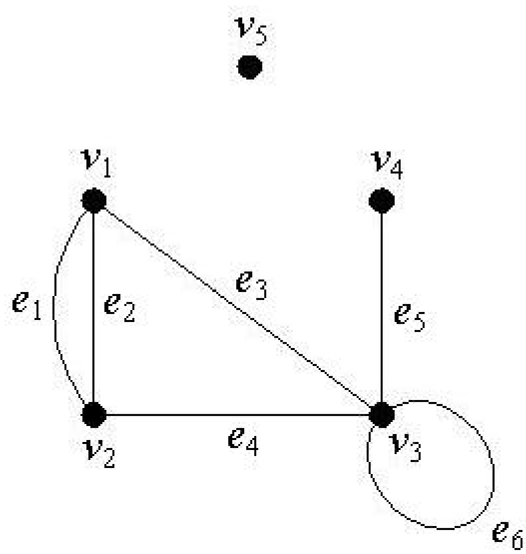
Definition 2.2.8 If two or more edges of a graph have a common vertex then these edges are called *incident edges*.

Definition 2.2.9 *Degree* of a vertex v of any graph G is defined as the number of edges incident on v, counting twice the number of loops. It is denoted by $d(v)$ or $d_G(v)$.

Definition 2.2.10 A vertex of degree one is called a *pendant vertex*.

Definition 2.2.11 A vertex of degree zero is called an *isolated vertex*.

Illustration 2.2.12 Let us consider the following graph G.



G

Figure 2.1

In above graph G shown in *Figure 2.1*

◇ Order of graph G is 5.

◇ Size of graph G is 6.

◇ e_6 is loop.

◇ e_1 and e_2 are multiple edges.

◇ v_2 and v_3 are adjacent vertices.

◇ e_3 and e_5 are incident edges.

◇ $d(v_3) = 5$, $d(v_2) = 3$.

◇ v_4 is pendant vertex.

◇ v_5 is isolated vertex.

Definition 2.2.13 A graph which has neither loops nor parallel edges is called a *simple graph*.

In the following *Figure 2.2* a simple graph is shown.

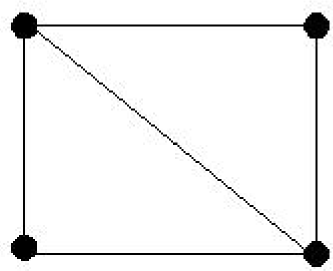


Figure 2.2

Definition 2.2.14 A *directed edge* (or *arc*) is an edge, one of whose end vertices is designated as tail and other end vertex is designated as head. An arc is said to be *directed from* its tail to its head.

Definition 2.2.15 Given a graph G we can obtain a digraph from G by specifying direction to each edge of G. Such a digraph D is called an *orientation*.

In the following *Figure 2.3* eight different orientations of a graph G are shown.

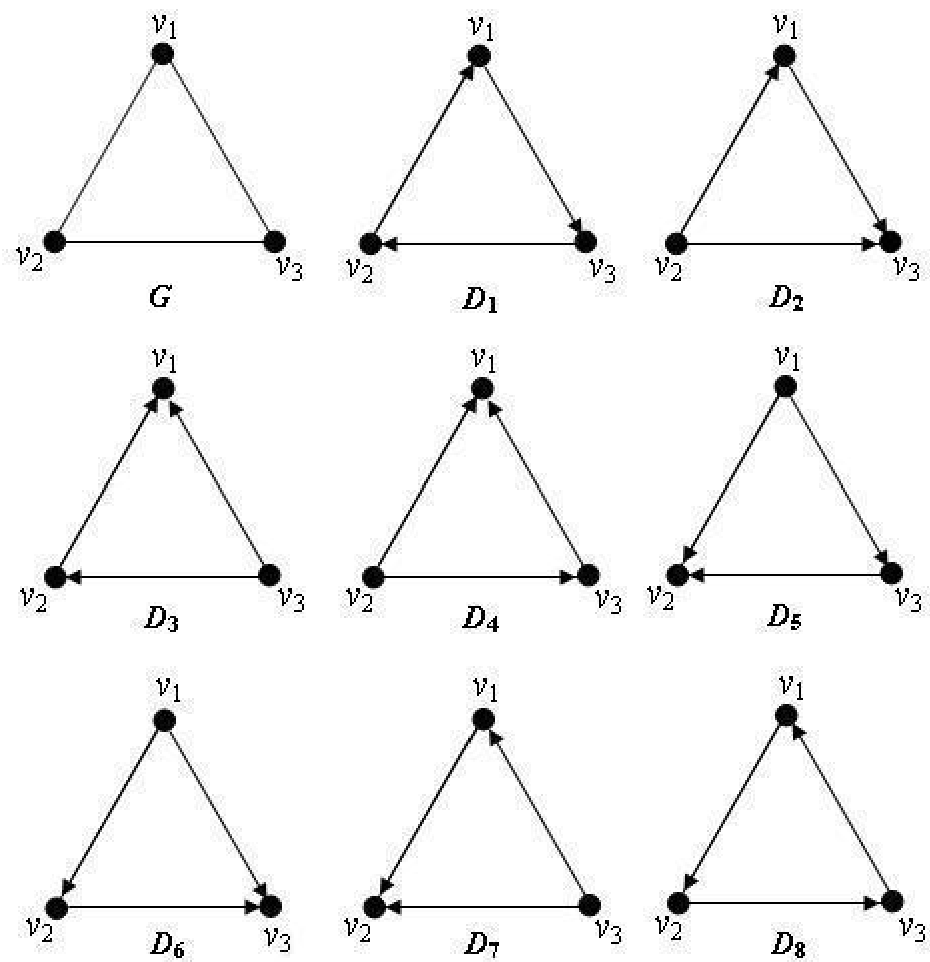


Figure 2.3

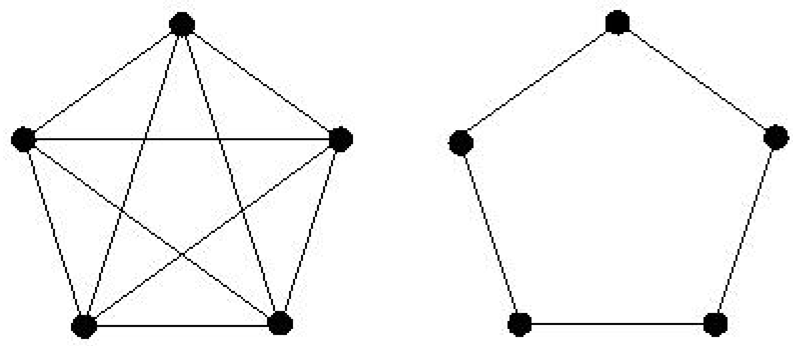
Definition 2.2.16 A *directed graph*(or *digraph*) is a graph each of whose edges is directed.

Definition 2.2.17 A graph in which no edge is directed is called an *undirected graph*.

Definition 2.2.18 A graph $G = (V, E)$ is said to be *finite* if V and E both are finite sets.

Definition 2.2.19 Let G and H be two graphs. Then H is said to be a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Here G is called *supergraph* of H.

In the following *Figure 2.4* H is a subgraph of G.



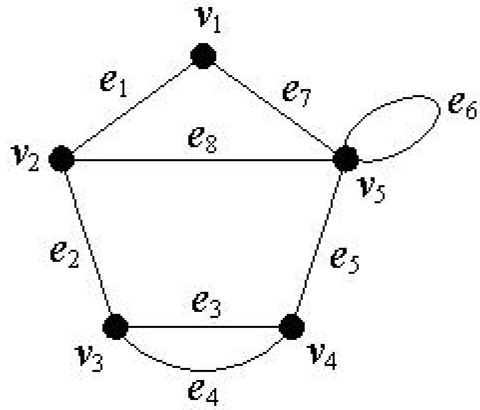
G H

Figure 2.4

Definition 2.2.20 Deletion of an edge from given graph G forms a subgraph of G which is called *edge deleted subgraph* of G.

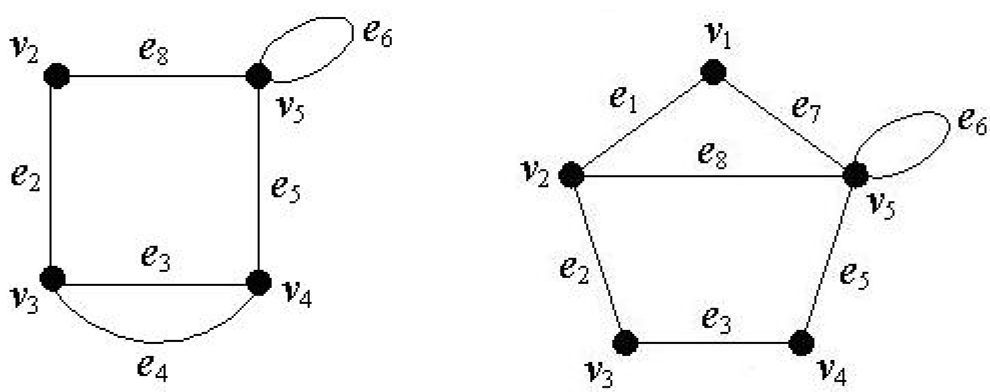
Definition 2.2.21 The graph obtained by deletion of a vertex from given graph G is called *vertex deleted subgraph* of G.

In the following *Figure 2.6* vertex deleted subgraph and edge deleted subgraph of given graph G are shown.



G

Figure 2.5



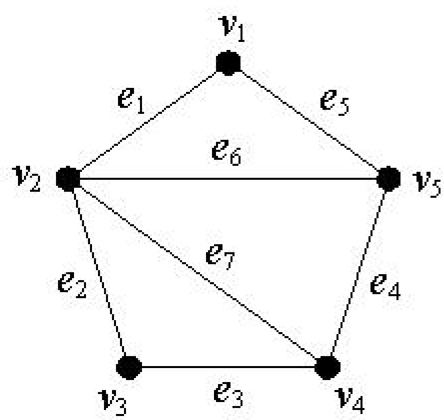
$G - \{v_1\}$ $\qquad$ $G - \{e_4\}$

Figure 2.6

Definition 2.2.22 Let $G = (V, E)$ be a graph. If U is a non-empty subset of the vertex set V of graph G then the *subgraph* $G[U]$ *of* G *induced by* U is defined to be the graph having vertex set U and edge set consisting of those edges of G that have both end vertices in U.

Similarly if F is a non-empty subset of the edge set E of G then the *subgraph* $G[F]$ *of* G *induced by* F is the graph whose vertex set is the set of vertices which are end vertices of edges of F and whose edge set is F.

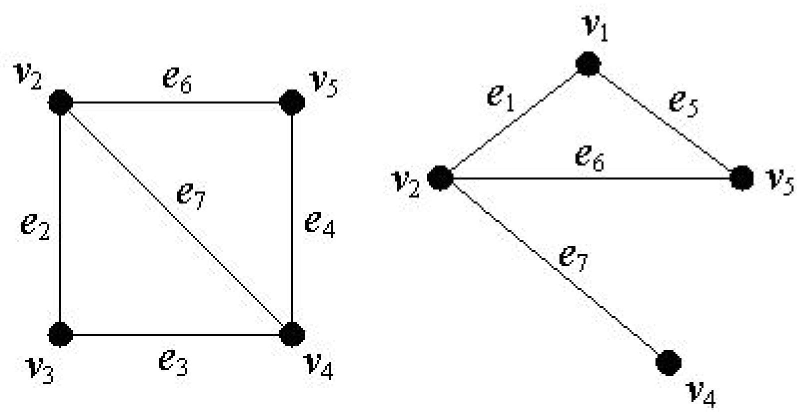
In the following *Figure 2.8* $G[U]$ and $G[F]$ are vertex induced subgraph and edge induced subgraph of graph G respectively.



G

Figure 2.7

Let $U = \{v_2, v_3, v_4, v_5\}$ $\quad\quad$ $F = \{e_1, e_5, e_6, e_7\}$



$G[U]$ $\quad\quad$ $G[F]$

Figure 2.8

Definition 2.2.23 A subgraph H of a graph G is called *spanning subgraph* of G if $V(H) = V(G)$.

Definition 2.2.24 A *walk* is defined as a finite alternating sequence of vertices and edges of the form $v_i e_j v_{i+1} e_{j+1} \ldots e_k v_m$ which begins and ends with vertices such that each edge in the sequence is incident on the vertex preceding and succeeding it in the sequence. A walk from v_0 to v_n is denoted as $v_0 - v_n$ walk. A walk $v_0 - v_0$ is called a *closed walk*.

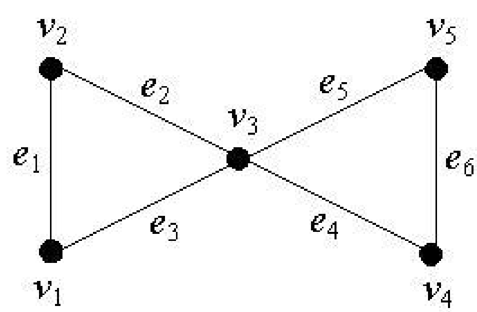
Definition 2.2.25 The number of edges in any walk is called *length of the walk*. A walk is *odd* (or *even*) if its length is odd (or even).

Definition 2.2.26 A walk is called a *trail* if no edge is repeated.

Definition 2.2.27 A walk in which no vertex is repeated is called a *path*. A path with n vertices is denoted as P_n. A path from v_0 to v_n is denoted as $v_0 - v_n$ path.

Definition 2.2.28 A closed path is called a *cycle*. A cycle with n vertices is denoted as C_n.

Illustration 2.2.29 Consider the following graph G as shown in *Figure 2.9*.



G

Figure 2.9

Above graph G shown in *Figure 2.9* is known as *bowtie graph.* For this graph we have the following.

◇ G is a simple, finite and undirected graph.

◇ $W = v_2e_2v_3e_4v_4e_6v_5e_5v_3e_3v_1$ is a walk.

◇ $P_4 = v_1e_1v_2e_2v_3e_5v_5$ is a path.

◇ $C_3 = v_1e_1v_2e_2v_3e_3v_1$ is a cycle.

Definition 2.2.30 A graph $G = (V, E)$ is said to be *connected* if there is a path between every pair of vertices of G. A graph which is not connected is called a *disconnected graph.*

The graph shown in *Figure 2.2* is connected while the graph shown in *Figure 2.1* is disconnected.

Definition 2.2.31 Each maximal connected subgraph of a disconnected graph is called *component of the graph.* Every connected graph has exactly one component.

Definition 2.2.32 A graph which includes exactly one cycle is called a *unicyclic graph.*

Definition 2.2.33 A graph in which all the vertices having equal degree is called a *regular graph.* If for every vertex v of graph G, $d(v) = k$ for some $k \in N$, then G is *k-regular graph.*

In the following *Figure 2.10* a 3-regular graph on 10 vertices is shown.

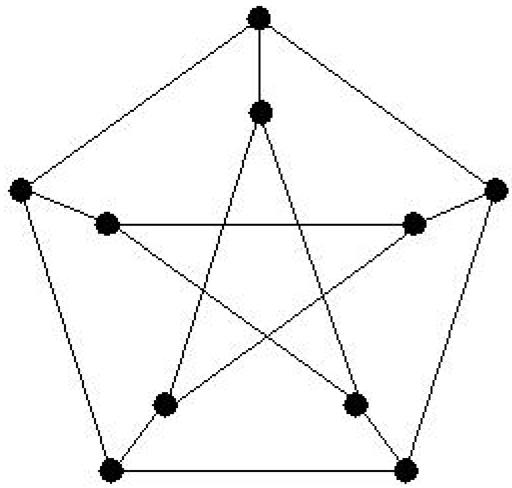


Figure 2.10

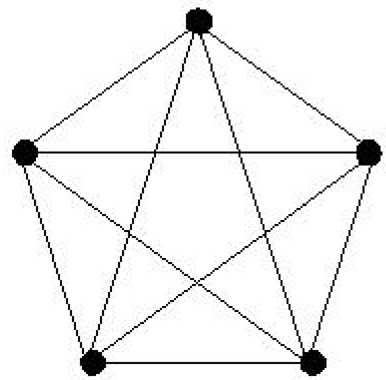
The above graph is known as *Petersen graph* which is a 3-regular graph with 10 vertices and 15 edges.

Definition 2.2.34 A graph in which the vertices having only two types of degree is called a *bidegreed graph*. The graph shown in *Figure 2.9* is a bidegreed graph.

Definition 2.2.35 A simple, connected graph is said to be *complete* if every pair of vertices of G is connected by an edge. A complete graph on n vertices is denoted by K_n.

Note that K_n is $(n-1)$-regular.

In the following *Figure 2.11* K_5 is shown.



K_5

Figure 2.11

Definition 2.2.36 Two vertices of a graph which are adjacent are said to be *neighbours*. The set of all neighbours of a fixed vertex v of G is called the *neighbourhood set* of v. It is denoted by $N(v)$. In *Figure 2.9*, $N(v_3) = \{v_1, v_2, v_4, v_5\}$.

Definition 2.2.37 A closed trail which covers all the edges of given graph is called an *Eulerian line* or *Eulerian trail*. A graph which has an Eulerian line is called an *Eulerian graph*. The graphs shown in *Figure 2.9* and *Figure 2.11* are Eulerian graphs.

Definition 2.2.38 A graph $G = (V, E)$ is said to be *bipartite* if the vertex set can be partitioned into two subsets V_1 and V_2 such that for every edge $e_i = v_iv_j \in E$, $v_i \in V_1$ and $v_j \in V_2$.

In the following *Figure 2.12* a bipartite graph is shown.

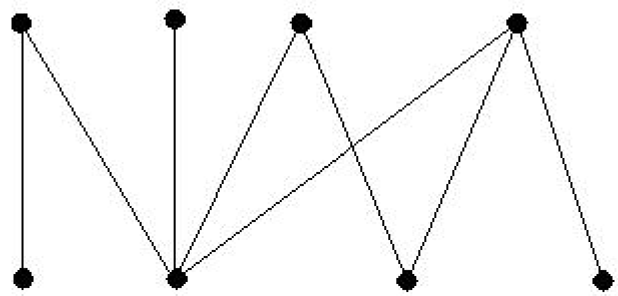


Figure 2.12

Definition 2.2.39 A graph $G = (V, E)$ is called *n-partite graph* if the vertex set V can be partitioned into n nonempty sets $V_1, V_2, \ldots, V_n$ such that every edge of G joins the vertices from different subsets. It is often called a *multipartite graph.*

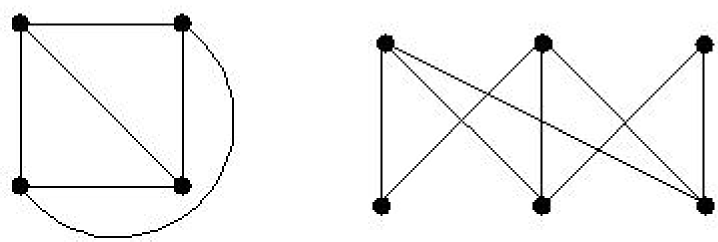
Definition 2.2.40 A *complete bipartite graph* is a simple bipartite graph such that two vertices are adjacent if and only if they are in different partite sets. If partite sets are having m and n vertices then the related complete bipartite graph is denoted by $K_{m,n}$.

Definition 2.2.41 The n-partite graph G is called *complete n-partite* if for each $i \neq j$, each vertex of the subset V_i is adjacent to every vertex of the subset V_j. The complete n-partite graph with n partitions of vertex set is denoted by $K_{m_1, m_2, \ldots, m_n}$.

Definition 2.2.42 A graph is said to be *planar* if there exists some geometric representation of G which can be drawn on a plane such that no any two of its edges intersect.

Definition 2.2.43 A graph that can not be drawn on a plane without a crossover between its edges is called *non planar* graph.

In the following *Figure 2.13* planar and non planar graph are shown.



K_4 Planar graph

$K_{3,3}$ Non planar graph

Figure 2.13

Definition 2.2.44 A simple planar graph is called *maximal planar* if no edge can be added without destroying its planarity.

Definition 2.2.45 A planar graph is *outerplanar* if it can be embedded in the plane so that all its vertices lie on the same region.

Definition 2.2.46 An outerplanar graph is *maximal outerplanar* if no edge can be added without losing outerplanarity.

Definition 2.2.47 A graph which does not contain any cycle is known as *acyclic graph.*

Definition 2.2.48 An acyclic graph is known as *forest.*

Definition 2.2.49 A connected acyclic graph is called a *tree.* Thus every component of a forest is a tree.

In the following *Figure 2.14* a tree T on seven vertices is shown.

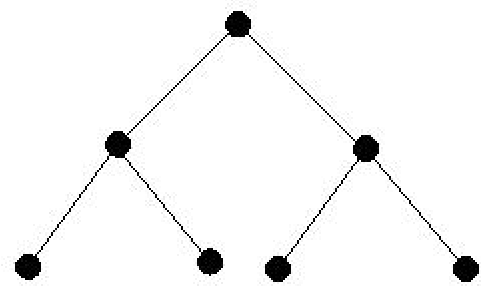


Figure 2.14

Definition 2.2.50 A *spanning tree* of a graph G is a spanning subgraph of G which is a tree. The number of spanning trees of a graph G is denoted by $\tau(G)$.

Definition 2.2.51 A *star graph* with n vertices is a tree with one vertex having degree $n-1$ and other $n-1$ vertices having degree 1. A star graph with $n+1$ vertices is denoted by $K_{1,n}$.

In the following *Figure 2.15* $K_{1,4}$ is shown.

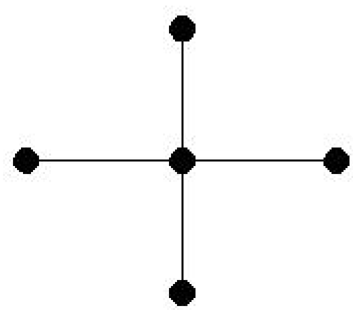
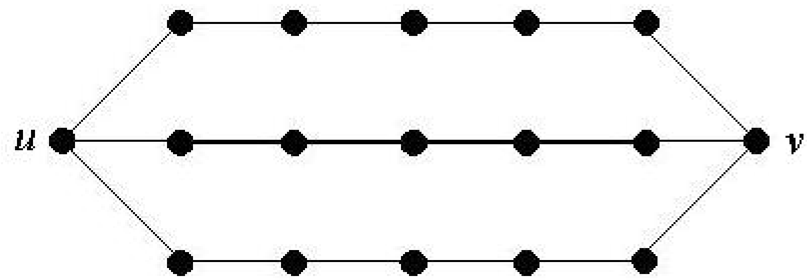


Figure 2.15

Definition 2.2.52 A *banana tree* is a tree which is obtained from a family of stars by joining one end vertex of each star to a new vertex.

Definition 2.2.53 A $t-ply$ $P_t(u,v)$ is a graph with t paths, each of length atleast two and such that no two paths have a vertex in common except the end vertices u and v.

In the following *Figure 2.16* $P_3(u, v)$ is shown.



$P_3(u, v)$

Figure 2.16

Definition 2.2.54 A *caterpillar* is a tree in which a single path (the spine) is incident to (or contains) every edge.

In the following *Figure 2.17* a caterpillar on 10 vertices is shown.

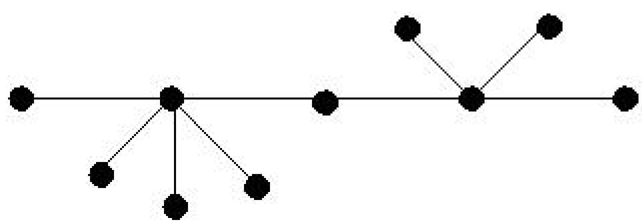


Figure 2.17

Definition 2.2.55 A *lobster* is a tree with the property that the removal of the end vertices leaves a caterpillar.

Definition 2.2.56 A vertex v of a graph G is called a *cut vertex* of G if $G - v$ is disconnected.

Definition 2.2.57 The *vertex connectivity* of a connected graph G is defined as the minimum number of vertices whose removal from G results remaining graph disconnected or K_1. It is denoted by $k(G)$.

A simple graph G is called *n-connected*(where $n \geq 1$) if $k(G) \geq n$.

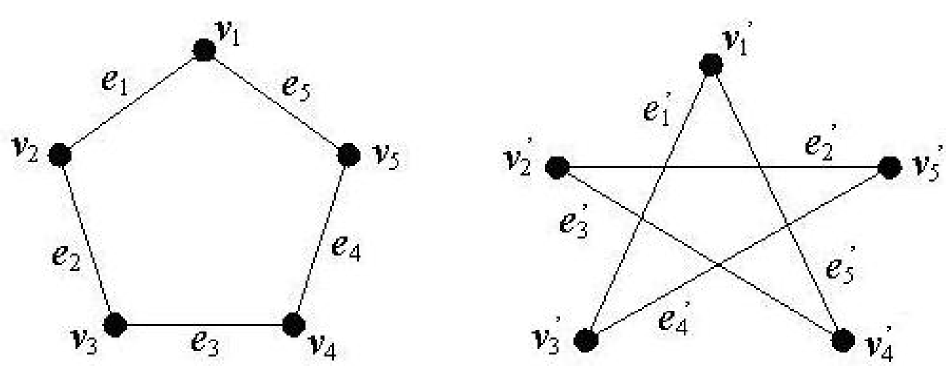
Definition 2.2.58 A connected graph is said to be *separable* if its vertex connectivity is one.

Definition 2.2.59 A *block* of a loopless graph is a maximal connected subgraph H such that no vertex of H is a cut vertex of H.

Definition 2.2.60 A graph $G_1 = (V_1, E_1)$ is said to be *isomorphic* to the graph $G_2 = (V_2, E_2)$ if there exists a bijection between the vertex sets V_1 and V_2 and a bijection between the edge sets E_1 and E_2 such that if e is an edge with end vertices u and v in G_1 then the corresponding edge $e^{'}$ in G_2 has its end vertices $u^{'}$ and $v^{'}$ in G_2 which correspond to u and v respectively.

If such pair of bijections exist then it is called a *graph isomorphism* and it is denoted by $G_1 \cong G_2$.

In the following *Figure 2.18* two isomorphic graphs are shown.



G_1 G_2

Figure 2.18

For the graphs in *Figure 2.18* the vertices v_1, v_2, v_3, v_4, v_5 correspond to vertices $v_1', v_3', v_5', v_2', v_4'$ respectively while edges e_1, e_2, e_3, e_4, e_5 correspond to $e_1', e_4', e_2', e_3', e_5'$ respectively..

¶ **Remark:**

If two graphs are isomorphic then they have

- Same number of vertices
- Same number of edges
- Number of vertices having same degree is equal.

The above facts are not sufficient for the isomorphism of graphs. Consider the graphs shown in *Figure 2.19*.

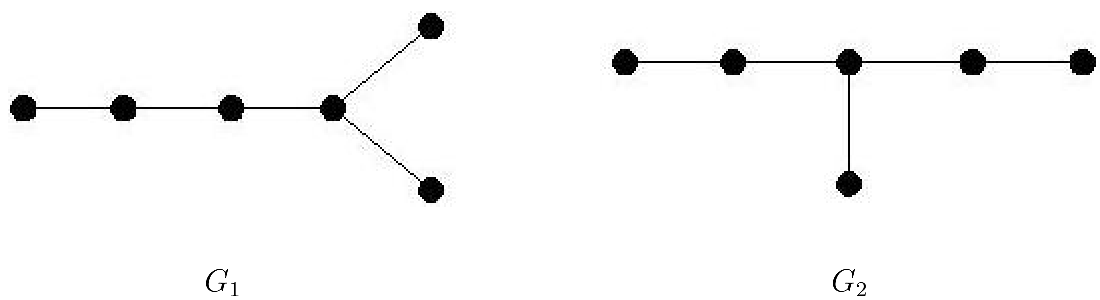
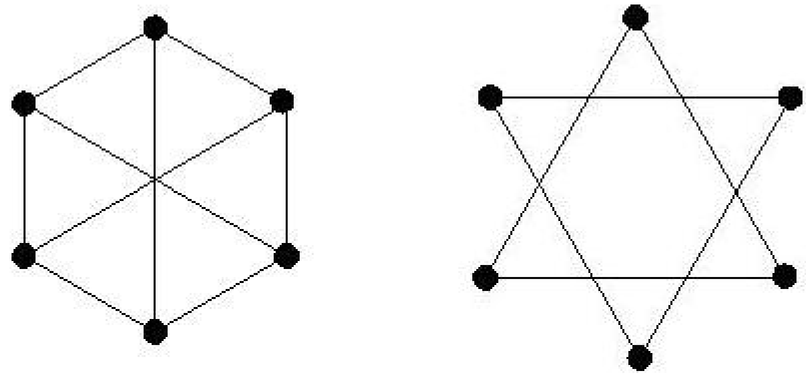


Figure 2.19

Here G_1 and G_2 satisfy above three conditions even though they are not isomorphic. Here bijection does not preserve adjacency as well as incidency.

Definition 2.2.61 The *complement* $\overline{G}$ of a graph $G = (V, E)$ is a graph with vertex set V in which two vertices are adjacent if and only if they are not adjacent in G.

In the following *Figure 2.20* a graph G and its complement is shown.

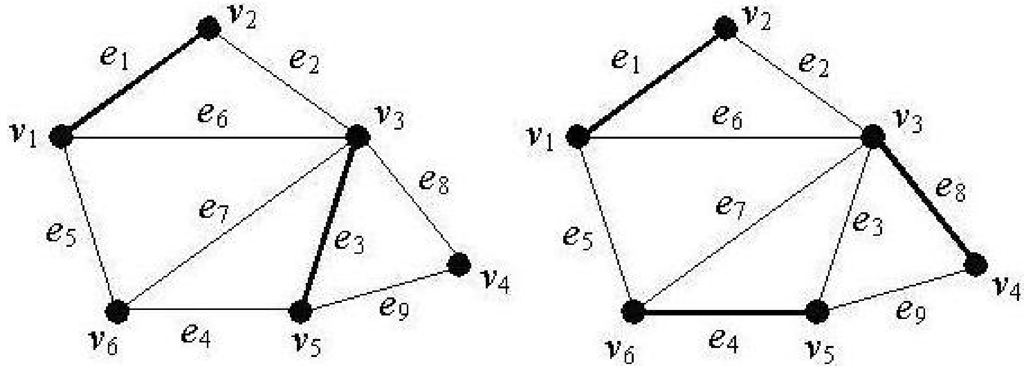


G $\qquad\qquad$ $\overline{G}$

Figure 2.20

Definition 2.2.62 Let $G = (V, E)$ be a graph. A subset M of E is called a *matching* in G if no two of the edges in M are adjacent. In other words, if for any two edges e and f in M the two end vertices of e are both different from the two end vertices of f.

In the following *Figure 2.21* a graph G and its two different matchings are shown.



A graph G with two different matchings

Figure 2.21

In above *Figure 2.21* the sets $M_1 = \{e_1, e_3\}$ and $M_2 = \{e_1, e_4, e_8\}$ are two matchings of graph G.

Definition 2.2.63 If the vertex v of the graph G is the end vertex of some edge in the matching M then v is said to be *M-saturated*.

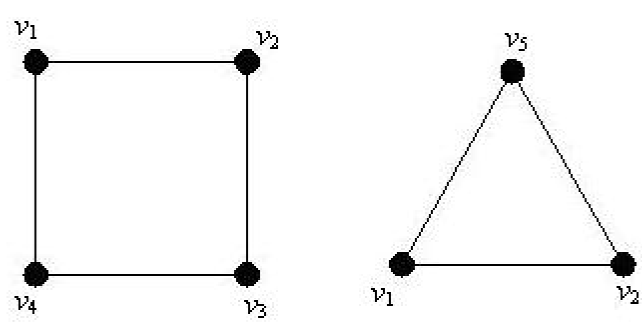
In *Figure 2.21* v_1, v_2, v_3, v_5 are M_1-saturated while every vertex of G is M_2-saturated.

Definition 2.2.64 If M is a matching in graph $G = (V, E)$ such that every vertex is M-saturated then M is called a *perfect matching*.

In *Figure 2.21* the matching $M_2 = \{e_1, e_4, e_8\}$ is a perfect matching.

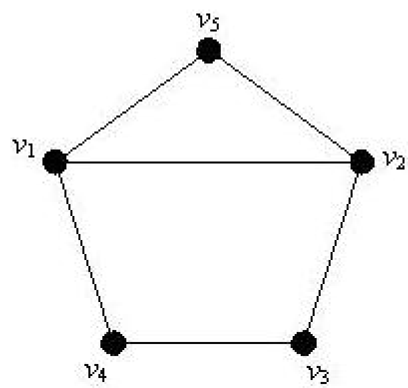
Definition 2.2.65 If G_1 and G_2 are subgraphs of a graph G then *union of* G_1 *and* G_2 is denoted by $G_1 \cup G_2$ which is the graph consisting of all those vertices which are either in G_1 or in G_2 (or in both) and with edge set consisting of all those edges which are either in G_1 or in G_2 (or in both).

In the following *Figure 2.23* union of two graphs G_1 and G_2 is shown.



G_1 G_2

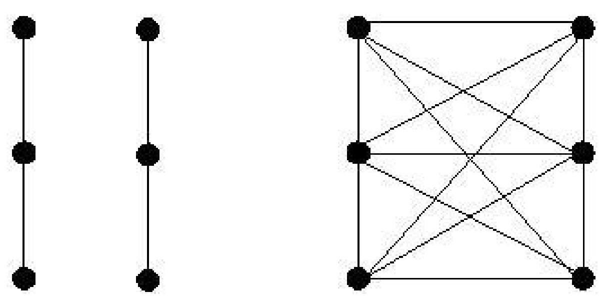
Figure 2.22



$G_1 \cup G_2$

Figure 2.23

Definition 2.2.66 Let G and H be two graphs such that $V(G) \cap V(H) = \emptyset$. Then *join of* G *and* H is denoted by $G+H$. It is the graph with $V(G+H) = V(G) \cup V(H)$, $E(G+H) = E(G) \cup E(H) \cup J$, where $J = \{uv/u \in V(G), v \in V(H)\}$. In the following *Figure 2.24* join $G+H$ of two graphs G and H is shown.



G H $G+H$

Figure 2.24

Definition 2.2.67 The *wheel graph* W_n is join of the graphs C_n and K_1. i.e. $W_n = C_n + K_1$. Here vertices corresponding to C_n are called *rim vertices* and C_n is called *rim* of W_n while the vertex corresponds to K_1 is called *apex vertex*.

Definition 2.2.68 A *helm* $H_n, n \geq 3$ is the graph obtained from the wheel W_n by adding a pendant edge at each vertex on the wheel's rim.

In the following *Figure 2.25* H_3 is shown.

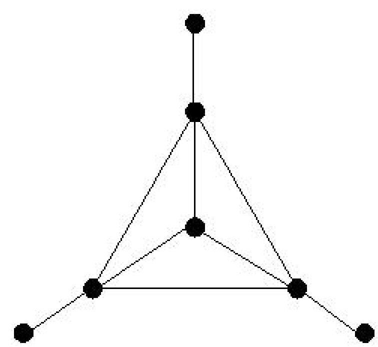


Figure 2.25

Definition 2.2.69 A *closed helm* CH_n is the graph obtained by taking a helm H_n and by adding edges between the pendant vertices.

In the following *Figure 2.26* CH_3 is shown.

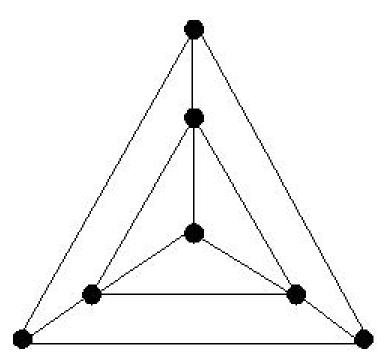


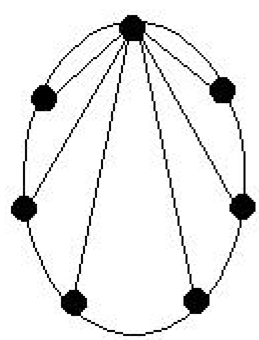
Figure 2.26

Definition 2.2.70 A *web graph* is the graph obtained by joining the pendant vertices of a helm to form a cycle and then adding a single pendant edge to each vertex of this outer cycle.

Definition 2.2.71 A *generalized helm* is the graph obtained by taking a web and attaching pendant vertices to all the vertices of the outermost cycle.

Definition 2.2.72 A *shell* S_n is the graph obtained by taking $n-3$ concurrent chords in a cycle C_n. The vertex at which all the chords are concurrent is called the *apex*. The shell S_n is also called *fan* F_{n-1}. i.e. $S_n = F_{n-1} = P_{n-1} + K_1$.

In the following *Figure 2.27* S_7 (or F_6) is shown.



S_7 or F_6

Figure 2.27

Definition 2.2.73 A *multiple shell* $MS\{n_1^{t_1}, n_2^{t_2}, \ldots, n_r^{t_r}\}$ is a graph formed by t_i shells each of order $n_i, 1 \leq i \leq r$ which have a common apex.

Definition 2.2.74 A *triangular cactus* is a connected graph all of whose blocks are triangles.

Definition 2.2.75 A *k-angular cactus* is a connected graph all of whose blocks are cycles with k vertices.

Definition 2.2.76 A *triangular snake* is the graph obtained from a path $v_1, v_2, \ldots v_n$ by joining v_i and v_{i+1} to a new vertex w_i for $i = 1, 2, \ldots, n-1$.

Definition 2.2.77 Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. Then *cartesian product* of G_1 and G_2 which is denoted by $G_1 \times G_2$ is the graph with vertex set $V = V_1 \times V_2$ consisting of vertices $u = (u_1, u_2)$, $v = (v_1, v_2)$ such that u and v are adjacent in $G_1 \times G_2$ whenever ($u_1 = v_1$ and u_2 adjacent to v_2) or ($u_2 = v_2$ and u_1 adjacent to v_1).

In the following *Figure 2.28* cartesian product of two paths is shown.

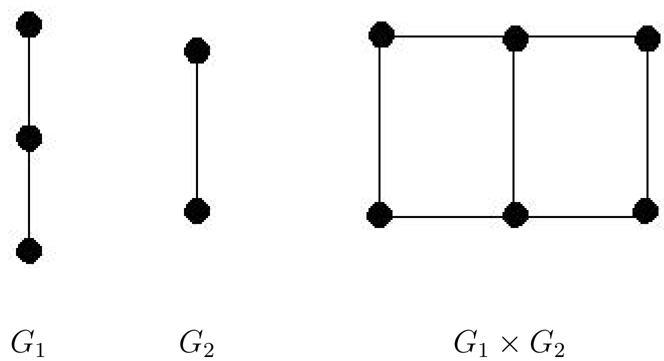


Figure 2.28

Definition 2.2.78 The cartesian product of two paths is known as *grid graph* which is denoted by $P_m \times P_n$. In particular the graph $P_n \times P_2$ is known as *ladder graph*.

Definition 2.2.79 The cartesian product of two cycles is known as *torus grid* which is denoted by $C_m \times C_n$.

Definition 2.2.80 The graph $K_2 \times K_2 \times \ldots \times K_2$($n$ times) is known as *n-cube*.

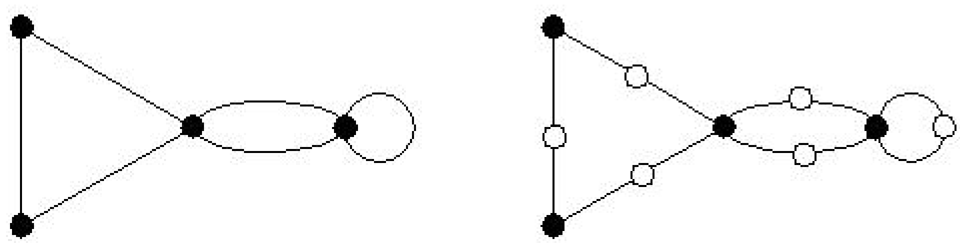
Definition 2.2.81 Let $G = (V, E)$ be a graph. Let $e = uv$ be an edge of G and w is not a vertex of G. The edge e is *subdivided* when it is replaced by edges $e' = uw$ and $e'' = wv$.

In the following *Figure 2.29* subdivision of an edge is shown.

Figure 2.29 Subdividing an edge

Definition 2.2.82 Let $G = (V, E)$ be a graph. If every edge of graph G is subdivided then the resulting graph is called *barycentric subdivision* of G. In other words barycentric subdivision is the graph obtained by inserting a vertex of degree 2 into every edge of original graph. The barycentric subdivision of any graph G is denoted by $S(G)$. It is easy to observe that $|V(S(G))| = |V(G)| + |E(G)|$ and $|E(S(G))| = 2|E(G)|$.

In the following *Figure 2.30* barycentric subdivision of a graph is shown.



A graph and its barycentric subdivision

Figure 2.30

Definition 2.2.83 Let $e = uv$ be an edge of simple, finite, undirected, connected graph G and $d(u) = k$, $d(v) = l$. Let $N(u) = \{v, u_1, \ldots, u_{n-1}\}$ and $N(v) = \{u, v_1, \ldots, v_{l-1}\}$. A *contraction* on the edge e changes G to a new graph $G * e$, where $V(G * e) = (V(G) - \{u, v\}) \cup \{w\}$, $E(G * e) = E(G - \{u, v\}) \cup \{wu_1, wu_2, \ldots, wu_{k-1}, wv_1, \ldots, wv_{l-1}\}$ and w is new vertex not belonging to G.

Definition 2.2.84 The *line graph* (or *edge graph*) of a graph G is the graph whose vertices are the edges of graph G, with $ef \in E(L(G))$ when $e = uv$ and $f = vw$ in G (where $u, v \in V(G)$). The line graph(edge graph) of a graph G is denoted by $L(G)$.

2.3 Concluding Remarks

This chapter was intended to provide all the fundamentals and pre-requisites which concern to the present work. Basic definitions like graph, vertex, edge, subgraph etc. are given and explained with the help of illustrations. Common families of graphs like cycle, path, wheel, tree etc. are introduced, notations and terminology are given. We have tried our best to prepare platform for advancement of the subject. Illustrations and figures help for better understanding.

The next chapter is aimed to discuss different graph labeling techniques.

Chapter 3

Various Techniques of Graph Labeling

3.1 Introduction

Graph labeling were first introduced in $1960's$. At present various graph labeling techniques are available and more than 800 research papers have been published so far. The interest in the field of graph labeling is constantly increasing and it has motivated many researchers. Many graph labeling techniques have applications to practical problems. According to Beineke and Hegde[19] graph labeling serves as a frontier between number theory and structure of graphs. Labeling of graphs have various applications in coding theory, particularly for missile guidance codes, design of good radar type codes, convolution codes with optimal autocorrelation properties. Graph labeling plays vital role in the study of X-ray crystallography, communication network and solution of problems in additive number theory. A detailed study of variety of applications of graph labeling is given by Bloom and Golomb[24]. A systematic survey on graph labeling is updated every two year since last one decade by Gallian[51]. The reference cited here is of latest version of *A Dynamic survey of Graph Labeling*, published by *The Electronics Journal of Combinatorics.*

This chapter is targeted to discuss various graph labeling techniques for graph $G = (V, E)$ with p vertices and q edges. Throughout the discussion on graph labeling we consider simple, finite and undirected graphs unless or otherwise stated. In the remaining part of this chapter we will concentrate upon some important graph labeling techniques and existing results.

3.2 Some Graph Labeling Techniques

If the vertices of the graph are assigned values subject to certain conditions is known as *graph labeling*.

Most interesting graph labeling problems have three important ingredients:

(1) A set of numbers from which vertex labels are chosen.

(2) A rule that assigns a value to each edge.

(3) A condition that these values must satisfy.

Now discussion about various graph labeling techniques will be carried out in chronological order as they were introduced.

3.2.1 Magic Labeling

Magic labeling was introduced by Sedláček[104] in 1963 motivated through the notion of magic squares in number theory.

A function f is called *magic labeling* of a graph G if $f : V \cup E \rightarrow \{1, 2, \ldots, p+q\}$ is bijective and for any edge $e = uv, f(u) + f(v) + f(e)$ is constant.

A graph which admits magic labeling is called *magic graph*.

In the following *Figure 3.1* magic labeling is demonstrated.

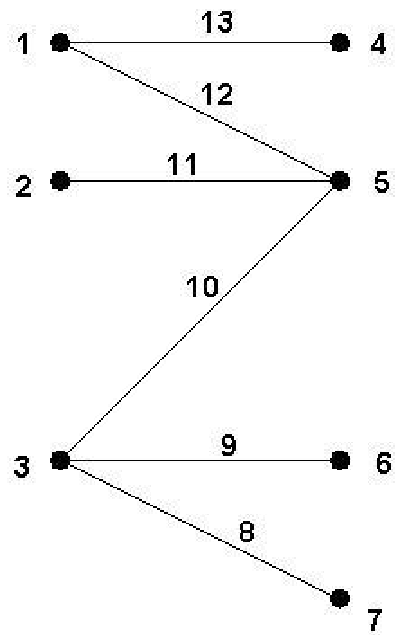


Figure 3.1

Some known results about magic labeling are listed below.

• Stewart[116] proved that

⋄ K_n is magic for $n = 2$ and all $n \geq 5$.

⋄ $K_{n,n}$ is magic for all $n \geq 3$.

⋄ Fans F_n are magic if and only if $n \geq 3$ and n is odd.

⋄ Wheels W_n are magic for all $n \geq 4$.

For any magic labeling f of graph G, there is a constant $c(f)$ such that for all edges $e = uv \in G$, $f(u) + f(v) + f(e) = c(f)$. *The magic strength* $m(G)$ is defined as the minimum of $c(f)$, where the minimum is taken over all magic labeling of G.

The above definition and some facts listed below were given by S. Avadyappan et al.[13].

$\diamond$ $m(P_{2n}) = 5n + 1$, $m(P_{2n+1}) = 5n + 3$,

$\diamond$ $m(C_{2n}) = 5n + 4$, $m(C_{2n+1}) = 5n + 2$,

$\diamond$ $m(K_{1,n}) = 2n + 4$.

• Hegde and Shetty[67] defined $M(G)$ analogous to $m(G)$ as follows: $M(G) = \max\{c(f)\}$, where maximum is taken over all magic labeling f of G.

For any graph G with p vertices and q edges following inequality holds:

$$p + q + 3 \leq m(G) \leq c(f) \leq M(G) \leq 2(p + q).$$

3.2.2 Graceful labeling

Graceful labeling was introduced by Rosa[103] in 1967.

A function f is called *graceful labeling* of a graph G if $f : V \rightarrow \{0, 1, 2, \ldots, q\}$ is injective and the induced function $f^* : E \rightarrow \{1, 2, \ldots, q\}$ defined as $f^*(e = uv) = |f(u) - f(v)|$ is bijective.

A graph which admits graceful labeling is called *graceful graph*.

Initially Rosa named above defined labeling as *$\beta-$valuation*. Golomb[57] renamed $\beta-$valuation as graceful labeling. We will discuss graceful labeling in detail in *Chapter 5*.

3.2.3 Graceful-like labeling

In 1967 Rosa[103] gave another analogue of graceful labeling.

A function f is called *graceful-like labeling* of a graph G if $f : V \rightarrow \{0, 1, 2, \ldots, q+1\}$ is injective and the induced function $f^* : E \rightarrow \{1, 2, \ldots, q\}$ or $f^* : E \rightarrow \{1, 2, \ldots, q-1, q+1\}$ defined as $f^*(e = uv) = |f(u) - f(v)|$ is bijective.

Frucht[50] termed such labeling as *nearly graceful labeling*. Some known results about graceful-like labeling are listed below.

• Frucht[50] has shown that

⋄ $P_m \cup P_n$ admits graceful-like labeling with edge labels $\{1, 2, \ldots, q-1, q+1\}$.

⋄ $G \cup K_2$ (where G is graceful graph) admits graceful-like labeling.

• Seoud and Elsahawi[108] have shown that all cycles admit graceful-like labeling.

• Barrientos[18] proved that cycle C_n is having graceful-like labeling with edge labels $\{1, 2, \ldots, q-1, q+1\}$ if and only if $n \equiv 1$ or $2 (mod 4)$.

3.2.4 Harmonious labeling

Graham and Sloane[58] introduced harmonious labeling in 1980 during their study of modular versions of additive bases problems stemming from error correcting codes.

A function f is called *harmonious labeling* of a graph G if $f : V \rightarrow \{0, 1, 2, \ldots, q-1\}$ is injective and the induced function $f^* : E \rightarrow \{0, 1, 2, \ldots, q-1\}$ defined as $f^*(e = uv) = (f(u) + f(v)) mod q$ is bijective.

A graph which admits harmonious labeling is called *harmonious graph*. We will demonstrate harmonious labeling by means of following examples in *Figure 3.2*.

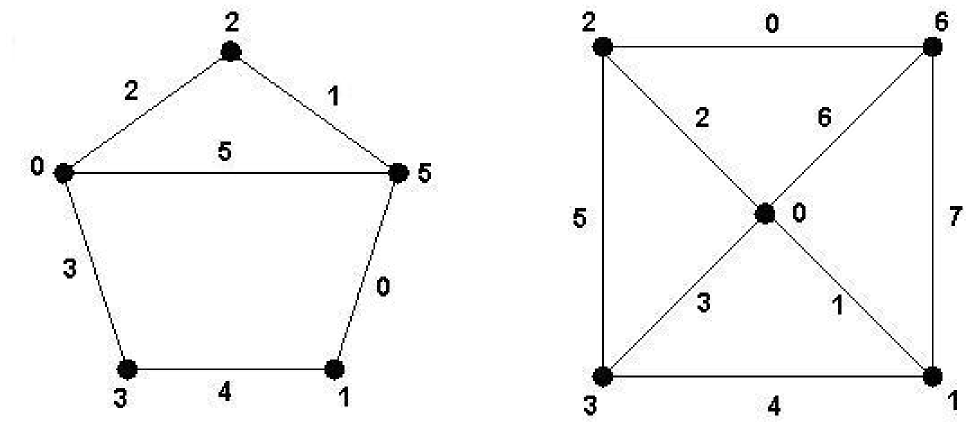


Figure 3.2

Graham and Sloane observed that if graph G is a tree then exactly two vertices are assigned same vertex label. Some known results about harmonious graph are listed below.

• Liu and Zhang[93] proved that every graph is a subgraph of a harmonious graph.

• Graham and Sloane[58] posed a conjecture *Every tree is harmonious.* In connection of above conjecture, Alderd and Mckay[6] proved that trees with 26 or less vertices are harmonious. They also proved that

⋄ Caterpillars are harmonious.

⋄ Cycles C_n are harmonious if and only if $n \equiv 1, 3(mod4)$.

⋄ Wheels W_n are harmonious for all n.

⋄ $C_m \times P_n$ is harmonious if n is odd.

⋄ K_n is harmonious if and only if $n \leq 4$.

⋄ $K_{m,n}$ is harmonious if and only if m or $n = 1$.

⋄ Fans F_n are harmonious for all n.

• Liu[92] proved that all helms are harmonious.

• Jungreis and Reid[76] proved that grids $P_m \times P_n$ are harmonious if and only if $(m, n) \neq (2, 2)$. In the same paper they proved that $C_m \times P_n$ is harmonious if $m = 4$ and $n \geq 3$.

• Gallian et al.[52] proved that $C_m \times P_n$ is harmonious if $n = 2$ and $m \neq 4$.

3.2.5 Elegant labeling

Elegant labeling was introduced by Chang et al.[35] in 1981.

A function f is called *elegant labeling* of a graph G if $f : V \to \{0, 1, 2, \ldots, q\}$ is injective and the induced function $f^* : E \to \{1, 2, \ldots, q\}$ defined as $f^*(e = uv) = (f(u) + f(v)) mod(q + 1)$ is bijective.

A graph which admits elegant labeling is known as *elegant graph*. We will note that as in harmonious labeling it is not necessary to make an exception for trees. Some known results for elegant labeling are listed below.

• Chang et al.[35] proved that

◇ C_n is elegant when $n \equiv 0, 3(mod4)$ and not elegant when $n \equiv 1(mod4)$.

◇ Path P_n is elegant for $n \equiv 1, 2, 3(mod4)$.

• Cahit[30] proved that P_4 is the only path which is not elegant.

• Balakrishnan et al.[15] proved that every simple graph is a subgraph of an elegant graph.

• Deb and Limaye[39] defined *near-elegant labeling* by replacing codomain of edge function f^* by $\{0, 1, \ldots, q - 1\}$ and they proved that triangular snakes where the number of triangles is congruent to $3(mod4)$ are near-elegant.

3.2.6 Prime and Vertex Prime labeling

The concept of prime labeling was originated by Entringer and it was introduced in a paper by Tout et al.[118].

A graph G with p vertices and q edges is said to have a *prime labeling* if $f : V \rightarrow \{1, 2, \ldots, p\}$ is bijective function and for every edge $e = uv$ of G, $(f(u), f(v)) = 1$.

• Around 1980 Entringer conjectured that *All trees have a prime labeling.* So far there has been little progress towards the proof of this conjecture.

• Some known classes of trees having prime labeling are paths, stars, caterpillars, etc.

• Deretsky et al.[42] proved that

⋄ All cycles have prime labeling.

⋄ Disjoint union of C_{2k} and C_n has prime labeling.

⋄ The complete graph K_n does not have a prime labeling for $n \geq 4$.

• Lee et al.[90] proved that W_n has a prime labeling if and only if n is even.

• Seoud et al.[107] proved that all helms, fans, $K_{2,n}$, $K_{3,n}$ (where $n \neq 3, 7$), $P_n + \bar{K}_2$ (where $n = 2$ or n is odd) are having prime labeling. He also proved that $P_n + \bar{K}_m$ does not have prime labeling if $m \geq 3$.

• Seoud and Youssef[109] have shown that $P_n + \bar{K}_2$ has a prime labeling if and only if $n = 2$ or n is odd.

In 1991 Deretsky et al.[42] introduced the notion of dual of prime labeling which is known as vertex prime labeling. According to them a graph with q edges has vertex prime labeling if its edges can be labeled with distinct integers $\{1, 2, \ldots, q\}$ such that for each vertex of degree at least two the greatest common divisor of the labels on its incident edges is 1. Some known results for vertex prime labeling are listed below.

• Deretsky et al.[42] proved that

◇ Forests, all connected graphs are having vertex prime labeling.

◇ $C_{2k} \cup C_n$, $C_{2n} \cup C_{2n} \cup C_{2k+1}$, $C_{2n} \cup C_{2n} \cup C_{2t} \cup C_k$ and $5C_{2m}$ are having vertex prime labeling.

◇ A graph with exactly two components one of them is not an odd cycle has a vertex prime labeling.

◇ 2-regular graph with at least two odd cycles does not have a vertex prime labeling.

◇ He also conjectured that *Any* 2*-regular graph has a vertex prime labeling if and only if it does not have two odd cycles.*

3.2.7 k-graceful labeling

A natural generalization of graceful labeling is the notion of k-graceful labeling which was independently introduced by Slater[113] and by Maheo and Thuillier[97] in 1982.

A function f is called *k-graceful labeling* of a graph G if $f : V \rightarrow \{0, 1, 2, \ldots, k + q - 1\}$ is injective and the induced function $f^* : E \rightarrow \{k, k + 1, k + 2, \ldots, k + q - 1\}$ defined as $f^*(e = uv) = |f(u) - f(v)|$ is bijective. A graph which admits k-graceful labeling is known as *k-graceful graph*. Obviously 1-graceful graphs are the graceful graphs. Some known results for k-graceful graph are listed below.

• Slater[113], Maheo and Thuillier[97] proved that C_n is k-graceful graph if and only if either $n \equiv 0, 1(mod 4)$ with k even and $k \leq \frac{n-1}{2}$ or $n \equiv 3(mod 4)$ with k odd and $k \leq \frac{n^2-1}{2}$.

• Liang and Liu[91] proved that $K_{m,n}$ is k-graceful, for all $m, n \in N$ and for all k.

• Bu et al.[28] proved that $P_n \times P_2$ and $(P_n \times P_2) \cup (P_n \times P_2)$ are k-graceful for all k.

• Acharya[1] proved that a k-graceful Eulerian graph with q edges must satisfies one of following:

(1) $q \equiv 0(mod4)$, $q \equiv 1(mod4)$ if k is even, (2) $q \equiv 3(mod4)$ if k is odd.

3.2.8 Cordial labeling

Cahit[31] introduced the concept of cordial labeling in 1987 as a weaker version of graceful and harmonious labeling.

A function $f : V \rightarrow \{0, 1\}$ is called *binary vertex labeling* of a graph G and $f(v)$ is called *label of the vertex* v of G under f. For an edge $e = uv$, the induced function $f^* : E \rightarrow \{0, 1\}$ is given as $f^*(e = uv) = |f(u) - f(v)|$. Let $v_f(0)$, $v_f(1)$ be number of vertices of G having labels 0 and 1 respectively under f and let $e_f(0)$, $e_f(1)$ be number of edges of G having labels 0 and 1 respectively under f^*. A binary vertex labeling f of a graph G is called *cordial labeling* if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph which admits cordial labeling is called *cordial graph*.

Detail discussion of above defined labeling will be carried out in *Chapter 6*.

3.2.9 Additively graceful labeling

In 1989 Hegde[63] introduced the concept of additively graceful labeling.

A function f is called *additively graceful labeling* of a graph G if $f : V \rightarrow \{0, 1, \ldots, \lceil \frac{q+1}{2} \rceil\}$ is injective and the induced function $f^* : E \rightarrow \{1, 2, \ldots, q\}$

defined as $f^*(e = uv) = f(u) + f(v)$ is bijective. A graph which admits additively graceful labeling is called *additively graceful graph.* Some known results for additively graceful graphs are listed below.

• Hegde[63] proved the following results.

⋄ If G is an additively graceful graph with p vertices and q edges then $q \geq 2p - 4$ and the bounds are best possible.

⋄ The graph for which $q = 2p - 4$ are essentially strongly indexable which will be discussed in 3.2.13.

⋄ The complete graph K_n is additively graceful if and only if $2 \leq n \leq 4$.

⋄ An additively graceful graph is either K_2 or $K_{1,2}$ or has a triangle.

⋄ If G is an additively graceful graph with a triangle then any additively graceful labeling f of G must assign zero to a vertex of triangle in G.

⋄ If an Eulerian graph G with p vertices and q edges is additively graceful then $q \equiv 0, 3(mod 4)$.

⋄ A unicyclic graph G is additively graceful if and only if G is isomorphic to either C_3 or the graph obtained by joining a unique vertex to any one vertex of C_3.

⋄ The graph obtained by joining t new vertices to any two fixed vertices of K_n $(2 \leq n \leq 4)$ is additively graceful.

⋄ He also posed a conjecture *For any additively graceful graph G with p vertices and q edges*, $q \leq \frac{1}{2}(p^2 - 5p + 18)$.

• Jinnah and Singh[75] proved that $P_n \times P_n$ is additively graceful graph.

3.2.10 (k, d)-graceful labeling

Acharya and Hegde[4] generalized the notion of k-graceful labeling to (k, d)-graceful labeling in 1990.

A function f is called (k, d)-*graceful labeling* of a graph G if $f : V \rightarrow \{0, 1, 2, \ldots, k + (q-1)d\}$ is injective and the induced function $f^* : E \rightarrow \{k, k+d, k+2d, \ldots, k+(q-1)d\}$ defined as $f^*(e = uv) = |f(u) - f(v)|$ is bijective. A graph which admits (k, d)-graceful labeling is known as (k, d)-*graceful graph*. Obviously $(1, 1)$-graceful labeling is graceful labeling and $(k, 1)$-graceful labeling is k-graceful labeling. Some known results for (k, d)-graceful labeling are listed below.

• Bu and Zhang[29] proved that $K_{m,n}$ is (k, d)-graceful for all k and d.

• Hegde and Shetty[68] defined a class of trees known as T_p-trees as follows and proved that T_p-trees are (k, d)-graceful for all k and d.

Let T be a tree with adjacent vertices u_0, v_0 and pendant vertices u, v such that the length of the path $u_0 - u$ is same as the length of the path $v_0 - v$. Now delete the edge u_0v_0 and join vertices u and v by an edge uv. Then such a transformation of T is called an *elementary parallel transformation (ept)* and the edge u_0v_0 is called a *transformable edge*. If by a sequence of epts T can be reduce to a path then T is called T_p-*tree*. They also proved that every graph obtained by barycentric subdivision of a T_p-tree is (k, d)-graceful for all k and d.

• Hegde[64] proved that if a graph is (k, d)-graceful for odd k and even d then the graph is bipartite. He also proved that K_n is (k, d)-graceful if and only if $n \leq 4$.

3.2.11 k-equitable labeling

In 1990 Cahit[33] proposed the idea of distributing the vertex and the edge labels among $\{0, 1, 2, \ldots, k-1\}$ as evenly as possible to obtain a generalization of graceful labeling.

A *vertex labeling* of a graph $G = (V, E)$ is a function $f : V \rightarrow \{0, 1, 2, \ldots, k-1\}$ and the value $f(u)$ is called *label of vertex u*. For the vertex labeling function $f : V \rightarrow \{0, 1, 2, \ldots, k-1\}$ the induced function $f^* : E \rightarrow \{0, 1, 2, \ldots, k-1\}$ defined as $f^*(e = uv) = |f(u) - f(v)|$ satisfies the conditions:

(1) $|v_f(i) - v_f(j)| \leq 1$ and

(2) $|e_f(i) - e_f(j)| \leq 1$, $0 \leq i, j \leq k-1$,

where $v_f(i)$ and $e_f(i)$ denote number of vertices and number of edges having label i under f and f^* respectively, $0 \leq i \leq k-1$. Such labeling f is called *k-equitable labeling* for the graph G. A graph which admits k-equitable labeling is called *k-equitable graph*. Obviously 2-equitable labeling is the cordial labeling defined earlier in 3.2.8. When $k = 3$ the labeling is called 3-*equitable labeling* which we will discuss in detail in *Chapter 8*. Some known results for k-equitable graphs are listed below.

• Cahit[33],[34] proved that a graph is graceful if and only if it is $(|E| + 1)$-equitable and he conjectured that *All tree are k-equitable, for all k.*

• Speyer and Szaniszlo[114] proved Cahit's conjecture for $k = 3$.

• Szaniszlo[117] proved that

◇ P_n is k-equitable for all k.

◇ K_n is not k-equitable for $3 \leq k < n$.

$\diamond$ $K_{2,n}$ is k-equitable if and only if $n \equiv (k-1)(mod k)$ or $n \equiv 0, 1, 2, \ldots, (\lfloor \frac{k}{2} \rfloor - 1)(mod k)$ or $n = \lfloor \frac{k}{2} \rfloor$ and k is odd.

$\diamond$ C_n is k-equitable if and only if k meets all of the following conditions:

(1) $n \neq k$,

(2) If $k \equiv 2, 3(mod 4)$ then $n \neq k - 1$ and n is not congruent to $k(mod 2k)$.

• Vickrey[127] discussed the k-equitability of complete multipartite graphs. He proved that for $m \geq 3$ and $k \geq 3$, $K_{m,n}$ is k-equitable if and only if $K_{m,n}$ is one of following graphs:

(1) $K_{4,4}$ for $k = 3$,

(2) $K_{3,k-1}$ for all k and

(3) $K_{m,n}$ for $k > mn$.

3.2.12 Skolem graceful labeling

Lee and Shee[88] introduced the concept of skolem graceful labeling in 1991.

A function f is called *skolem graceful labeling* of a graph G if $f : V \rightarrow \{1, 2, \ldots, p\}$ is bijective and the induced function $f^* : E \rightarrow \{1, 2, \ldots, q\}$ defined as $f^*(e = uv) = |f(u) - f(v)|$ is bijective. A graph which admits skolem graceful labeling is called *skolem graceful graph.* A necessary condition for a graph to be skolem graceful is $p \geq q + 1$. Some known results for skolem graceful graphs are listed below.

• Lee and Wui[89] proved that a connected graph is skolem graceful if and only if it is a graceful tree.

• Yao et al.[129] have shown that a tree with p vertices and with maximum degree at least $\frac{p}{2}$ is skolem graceful.

• Although the disjoint union of trees can not be graceful, they can be skolem graceful.

• Lee and Wui[89] proved that the disjoint union of two or three stars is skolem graceful if and only if at least one star has even size.

• Choudum and Kishore[37] proved that disjoint union of k copies of the star $K_{1,2p}$ is skolem graceful if $k \leq 4p+1$ and the disjoint union of any number of copies of $K_{1,2}$ is skolem-graceful. He also proved that all five stars are skolem graceful.

• Frucht[50] proved that $P_m \cup P_n$ is skolem graceful when $m + n \geq 5$.

• Bhat-Nayak and Deshmukh[23] proved that $P_{n_1} \cup P_{n_2} \cup P_{n_3}$ is skolem graceful when $n_1 < n_2 \leq n_3$, $n_2 = t(n_1+2)+1$, n_1 is even and when $n_1 < n_2 \leq n_3$, $n_2 = t(n_1 + 3) + 1$, n_1 is odd. They also proved that $P_{n_1} \cup P_{n_2} \cup \ldots \cup P_{n_i}$, for $i \geq 4$ is skolem graceful under certain conditions.

3.2.13 Indexable labeling

Acharya and Hegde[5] introduced the concept of indexable labeling in 1991.

A function f is called *indexable labeling* of a graph G if $f : V \rightarrow \{0, 1, 2, \ldots, p-1\}$ is bijective and the induced function $f^* : E \rightarrow N$ defined as $f^*(e = uv) = f(u) + f(v)$ is injective. Here f is called *indexer* of G. A graph which admits indexable labeling is called *indexable graph.*

A graph is said to be *strongly indexable* if $f^*(E) = \{1, 2, \ldots, q\}$. Here f is called *strong indexer* of graph G.

A function f is called (k, d)-*indexable labeling* if $f : V \rightarrow \{0, 1, \ldots, p-1\}$ is bijective and the induced function $f^* : E \rightarrow \{k, k + d, \ldots, k + (q-1)d\}$

defined as $f^*(e = uv) = f(u) + f(v)$ is injective. A (k, d)-*indexable graph* is the graph which admits (k, d)-indexable labeling. A graph is said to be *strongly* (k, d)-*indexable* if $f^*(E) = \{k, k+d, \ldots, k+(q-1)d\}$. Some known results for indexable and (k, d)-indexable graphs are listed below.

• Acharya and Hegde[5] have conjectured that *All unicyclic graphs are indexable.* This conjecture was proved by Arumugam and Germina[12] using *Breadth First Search* (BFS) algorithm[38]. They also proved that all trees are indexable.

• Acharya and Hegde[5] proved that K_2, K_3 and $K_2 \times K_3$ are the only non-trivial regular graphs which are strongly indexable.

• Hegde[65] proved that

⋄ Every graph can be embedded as an induced subgraph of an indexable graph.

⋄ If a connected graph with p vertices and q edges ($q \geq 2$) is (k, d)-indexable then $d \leq 2$.

⋄ $P_m \times P_n$ is indexable for all m and n.

⋄ If G is connected $(1, 2)$-indexable graph then G must be a tree.

⋄ K_n, $n \geq 4$ and wheels W_n are not (k, d)-indexable.

3.2.14 Felicitous labeling

Lee et al.[87] introduced the concept of felicitous labeling in 1991.

A function f is called *felicitous labeling* of a graph G if $f : V \rightarrow \{0, 1, 2, \ldots, q\}$ is injective and the induced function $f^* : E \rightarrow \{0, 1, 2, \ldots, q-1\}$ defined as $f^*(e = uv) = (f(u) + f(v)) mod q$ is bijective. A graph which admits felicitous labeling is called *felicitous graph*. Some known results for felicitous graphs are listed below.

• Balakrishnan and Kumar[16] proved that every graph is a subgraph of felicitous graph.

• Lee et. al.[87] proved that

⋄ Cycles C_n are felicitous except $n \equiv 2 (\text{mod } 4)$.

⋄ $K_{m,n}$ is felicitous when $m, n > 1$.

⋄ $P_2 \cup C_{2n+1}$ is felicitous for all n.

⋄ They also conjectured that *n-cube is felicitous* which was proved by Figueroa-Centeno et. al.[46] in 2001.

• Shee[111] conjectured that $P_m \cup C_n$ *is felicitous when* $n > 2$ *and* $m > 3$, which is still open.

3.3 Concluding Remarks

In this chapter we have discussed various graph labeling techniques in detail. The discussion includes definitions and known results for each labeling technique. This chapter will give broad idea about various labeling techniques and will provide ready reference for any researcher. The penultimate chapter is devoted to the discussion on reconstruction of graphs.

Chapter 4
Graph Reconstruction

4.1 Introduction

There are many unsolved problems in graph theory. Graph reconstruction is one such unsolved problem which was initially posed by Ulam in 1941 and the problem was systematically investigated by Kelly[80] in his Ph.D. dissertation in 1942. Kelly wrote the first paper on reconstruction of graph in 1957. More than 300 papers have been published on this topic even though problem of graph reconstruction is long standing unsolved problem but work on it has been slowed down, may be due to the feeling that existing techniques are not enough to lead to a complete solution.

In this chapter we will give the results and development which have been appeared recently in some selected variations like edge reconstruction, degree associated reconstruction, vertex switching reconstruction etc.

Ramachandran[102] has discussed graph reconstruction briefly. Now we will develop all the terminology and definitions which concern to this chapter.

4.2 Vertex Reconstruction

Definition 4.2.1 A graph H is called *reconstruction* of a graph G if the vertices of G and H can be labeled $v_1, v_2, \ldots, v_n$ and $u_1, u_2, \ldots, u_n$ respectively such that $G - v_i \cong H - u_i$, $\forall i$.

Definition 4.2.2 A vertex deleted subgraph of a graph G in unlabeled form is called *card* of G.

Definition 4.2.3 The collection of cards of G is called *deck* which is denoted as $\mathcal{G}$. Hence $\mathcal{G}$=$\{G_i/G_i = G - v_i, v_i \in G, i = 1, 2, \ldots, n\}$.

Illustrations 4.2.4 For better understanding of above terminology we will consider some illustrations.

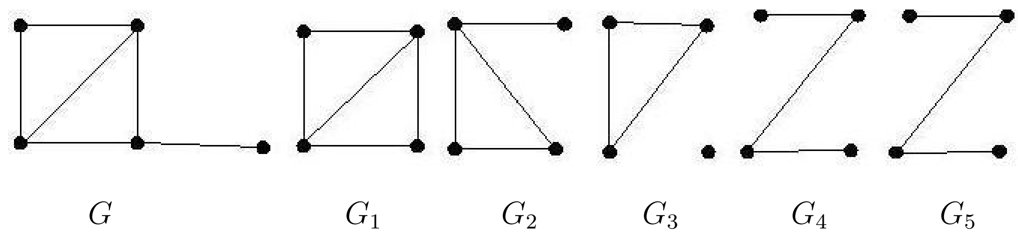


Figure 4.1

In the above *Figure 4.1* for the given graph G, G_1, G_2, G_3, G_4, G_5 are the vertex deleted subgraphs i.e. cards and $\mathcal{G} = \{G_i, i = 1, 2, 3, 4, 5\}$ is a deck of G.

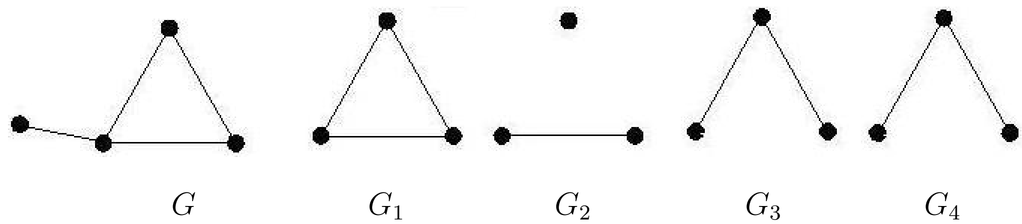


Figure 4.2

In the above *Figure 4.2* G_1, G_2, G_3, G_4 are the vertex deleted subgraphs i.e. cards and $\mathcal{G} = \{G_i, i = 1, 2, 3, 4\}$ is a deck of G.

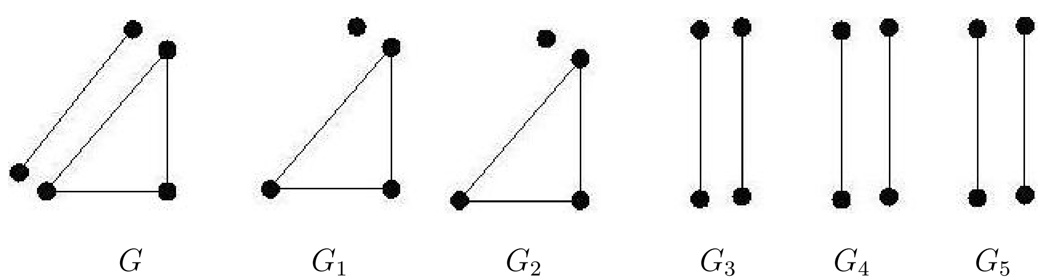


Figure 4.3

In the above *Figure 4.3* G_1, G_2, G_3, G_4, G_5 are the vertex deleted subgraphs i.e. cards and $\mathcal{G} = \{G_i, i = 1, 2, 3, 4, 5\}$ is a deck of G.

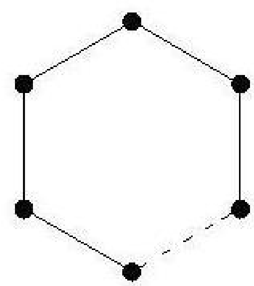


Figure 4.4

In above *Figure 4.4* each card of $\mathcal{G}$ is a path of $(n-1)$ vertices i.e. each card is P_{n-1}.

At this stage we will also note that number of elements in any deck $\mathcal{G}$ is same as number of vertices of G.

Definition 4.2.5 Let $\mathcal{G}$ and $\mathcal{H}$ be decks of graphs G and H respectively then we say $\mathcal{G} = \mathcal{H}$ provided they have the same number of elements and each $\mathcal{G}_i = \mathcal{H}_i$, $i = 1, 2, \ldots, n$.

Definition 4.2.6 Let G and H be two simple graphs. If $\mathcal{G} = \mathcal{H}$ then H is said to be *reconstruction* of G and graph G is known as *reconstructible* graph.

Reconstruction Conjecture (RC):

All graphs with at least three vertices are reconstructible.

In 1964 Harary[62] reformulated RC as follows.

Reconstruction Conjecture (RC): *Any graph G with at least three vertices is uniquely determined upto isomorphism by its deck.*

Here one can obviously ask question that why must any graph G has at least three vertices?

For this simple reason is that there are non-isomorphic graphs G and H

on two vertices for which $\mathcal{G} = \mathcal{H}$. In the following *Figure 4.5* two such graphs are shown for which G is not isomorphic to H but $\mathcal{G} = \mathcal{H}$.

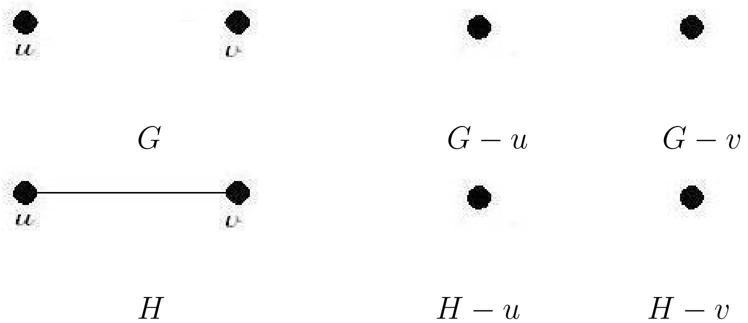


Figure 4.5

We can look at the RC in another way as follows.

Given a collection of subgraphs of the form $\mathcal{G}$, then exactly one graph G can be uniquely recaptured from $\mathcal{G}$. In this case we say that G is reconstructible from $\mathcal{G}$. Two things are needed for such reconstruction one is $|V(G)|$ and the other is $|E(G)|$. $|V(G)|$ is straight forward as we noted earlier it is simply number of graphs in $\mathcal{G}$ and $|E(G)| = \frac{\sum_{v \in V(G)} |E(G-v)|}{|V(G)|-2}$, where v is any arbitrary vertex of graph G. It is also very interesting to know how often a particular graph H occurs as a non-spanning subgraph of G. Let us denote the number of occurrence of H as non-spanning subgraph of G by $S(H, G)$. The following known result is very useful to know $S(H, G)$.

Theorem 4.2.7 Let $S(H, G)$ and $S(H, G - v)$ be the number of subgraphs of G and $G - v$ respectively which are isomorphic to H, where $|V(H)| < |V(G)|$ then $S(H, G) = \frac{\sum_{v \in V(G)} S(H,G)}{|V(G)|-|V(H)|}$.

Above result is very useful and throw some light about reconstructible graphs. The following fundamental *Lemma 4.2.8* is very important in this regard which was given by Kelly.

Lemma 4.2.8 For any two graphs F and G such that $|V(F)| < |V(G)|$, the number $S(F, G)$ of subgraphs of G isomorphic to F is reconstructible. (Two subgraphs isomorphic to F are counted as different if they have different vertex set or edge set).

As a consequence of above result we have following corollary.

Corollary 4.2.9 For any two graphs F and G such that $|V(F)| < |V(G)|$, the number of subgraphs of G which are isomorphic to F and include a given vertex v is reconstructible from the deck of G (this number is infact $S(F, G) - S(F, G - v)$).

Using above results following are the classes of reconstructible graph as mentioned by Ramachandran[102].

- All regular graphs are reconstructible.
- All disconnected graphs are reconstructible.
- Trees are reconstructible.
- Unicyclic graphs are reconstructible.
- Cactus are reconstructible.
- Maximal planar graphs are reconstructible.
- Outer planar graphs are reconstructible.
- Separable graphs without end vertices are reconstructible.

We have also investigated following results in connection of reconstruction of graphs.

Theorem 4.2.10 Forests are reconstructible.

Proof: Consider a forest G with n vertices. Then it is obvious that its deck $\mathcal{G}$ contains n copies of forests say $f_1, f_2, \ldots, f_n$. Now let H be another graph obtained from deck $\mathcal{G}$. Then clearly $|V(H)| = n$ and

$$|E(H)| = \frac{\sum_{v\in V(H)}|E(H-v)|}{|V(H)|-2}$$
$$< \frac{n(n-2)}{n-2} = n.$$

i.e. $|E(H)| \leq n-1$.

Therefore to prove that H is a forest it remains to show that H is acyclic. If possible assume that H is not acyclic. Then H contains atleast one cycle, say C. As H is disconnected cycle C will be contained in any one component of H. It is clear that H itself can't be cycle as we have $|E(H)| < n$. Therefore $\exists$ a vertex $v \in V(H)$ where v not belongs to C. Therefore $H-v$ still contains a cycle which is not possible as each graph in $\mathcal{G}$ is a forest.

Thus H must be acyclic graph. i.e. H is forest. Thus we can obtain a forest H from the deck of forest G. Therefore forests are reconstructible.

Lemma 4.2.11 For any tree T only pendant vertices are not cut vertices of T.

Proof: Let v be any vertex of T with $d(v) > 1$. Therefore there are at least two vertices, say u and w which are adjacent to v. As T is a tree there is exactly one path between u and w which passes through v. Note that u and w cannot be adjacent otherwise u, v and w will form a triangle which is not possible as T is a tree. Hence no path exists between u and w in $T-v$ which implies that $T-v$ is disconnected. Hence v is a cut vertex of T. As v is an arbitrary vertex we have proved that every vertex v with $d(v) > 1$ is a cut vertex of T.

Moreover removal of any pendant vertex will not effect connectedness of T. It follows that any pendant vertices are not cut vertices of T. Hence only pendant vertices of tree T are not cut vertices of T.

Lemma 4.2.12 Let T be a tree with $|V(T)| > 2$. If there are exactly two vertices of T which are not cut vertices of T then T must be a path graph.

Proof: Let u and v be two vertices of T which are not cut vertices of T. Then u and v are the only pendant vertices of T according to previous *Lemma 4.2.11*. To prove that T is a path graph it suffices to prove that $d(w) = 2$ for any vertex $w \in V(T)$ different than u and v. If possible let $d(w) > 2$ and also assume that w is on the path between u and v. Then there are at least three vertices in T which are adjacent to w. Let v_1, v_2, v_3 be adjacent with w otherwise they will form a triangle which is not possible as T is a tree.

Further assume that v_1 be a vertex on the path from u to w and v_2 be a vertex on the path from w to v. As v_1 and v_3 are adjacent to w and there is exactly one path between u and w which contains v_1, say $P = u, u_1, \ldots, u_n, v_1, w$. Then v_3 cannot be on this path. Similarly v_2 and v_3 are adjacent to w and there is exactly one path between w and v, say $P' = w, v_2, u_1', \ldots, u_n', v$. Then v_3 cannot be on this path. Hence there is another subtree say T' which doesn't contain paths P and P' but contains path between w and v_3. As T' is a tree it must involve atleast one pendant vertex. Thus we have one more pendant vertex which is distinct from u and v.

Thus there are three pendant vertices in T. Hence by previous *Lemma 4.2.11* there are three different vertices which are not cut vertices of T. This contradicts the fact that T has exactly two vertices which are not cut vertices of T. Therefore our assumption $d(w) > 2$ is wrong.

$\Rightarrow d(w) = 2, \forall\ w \in V(T)$.

$\Rightarrow T$ must be a path graph.

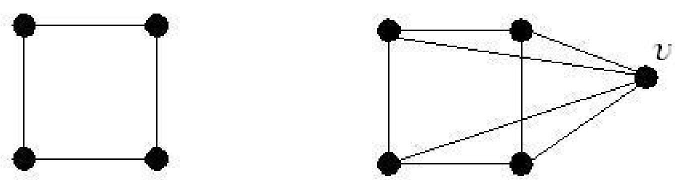
Theorem 4.2.13 Paths are reconstructible.

Proof: Consider deck $\mathcal{G}$ of acyclic graphs where exactly two graphs are connected i.e. exactly two cards in $\mathcal{G}$ are connected which implies that there are exactly two vertices of G which are not cut vertices of G. Then by previous *Lemma 4.2.12* G must be a path graph. Thus we can recover G from $\mathcal{G}$ and this recovered graph is a path graph. Therefore paths are reconstructible.

4.3 Reconstruction and Suspension of Graph

Definition 4.3.1 Let G be a graph. If a new vertex v is joined to each of the pre-existing vertices of G then the resulting graph is called the *suspension* of G from v (or *join of* G *and* v) which is denoted as $G + v$.

In the following *Figure 4.6* graph G and its suspension $G + v$ are shown.



G $\qquad$ $G + v$

Figure 4.6

Following theorem relates above concept with reconstruction of graph.

Theorem 4.3.2 The suspension $G + v$ of any graph G is reconstructible.

Proof: If G is reconstructible then it can be uniquely determined from the collection $\mathcal{G} = \{G - v'/v' \in V(G)\}$. Now consider the collection $\mathcal{G}^+$ consisting of one copy of G and n copies of graphs $(G - v') + v$. Since there is only one way of joining vertex v to any graph and G is uniquely determined from $\mathcal{G}$, it follows that $G + v$ can be uniquely determined from the collection $\mathcal{G}^+$.

4.4 A Step Forward in the Direction of RC

In 1988 Yang Yongzhi[130] has proved that RC is true if all 2-connected graphs are reconstructible.

We have investigated a powerful result which support above existing observation.

Theorem 4.4.1 Let G be a graph with $|V(G)| > 2$. Then G is 2-connected if and only if each graph in $\mathcal{G}$ is connected.

Proof: First assume that G is 2-connected. i.e. $k(G) \geq 2$. Then any vertex of G is not a cut vertex of G which implies that $G - v$ is connected $\forall$ $v \in V(G)$. Therefore each graph in $\mathcal{G}$ is connected. Conversely suppose that each graph in $\mathcal{G}$ is connected i.e. $\forall v \in V(G)$, $(G - v)$ is connected i.e. v is not a cut vertex of G.

i.e. Any vertex $v \in V(G)$ is not cut vertex of G.

i.e. $k(G) \geq 2$.

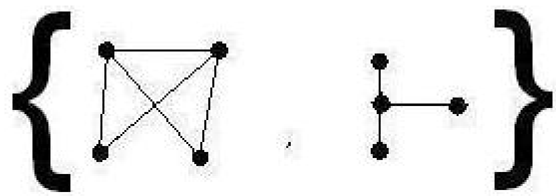
i.e. G is 2-connected.

Above result can be combined with Yang Yongzhi's existing result as follows:

Our Conjecture: *All those graphs G for which all cards in $\mathcal{G}$ are connected and reconstructible then RC is true.*

Consider following examples in connection with above observation.

Example 1



$\mathcal{G}$

Figure 4.7

Here each card is connected. The graph obtained from it is shown in following *Figure 4.8*.

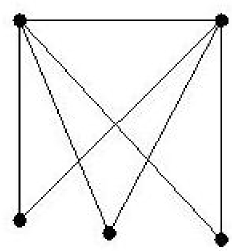
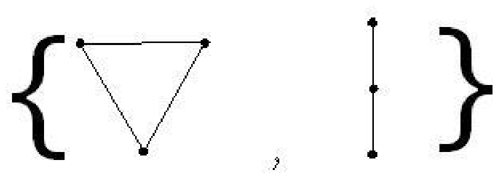


Figure 4.8

Example 2



$\mathcal{G}$

Figure 4.9

Here each card is connected. The graph obtained from it is shown in following *Figure 4.10*.

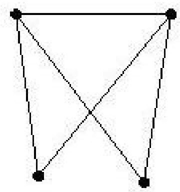


Figure 4.10

Thus in above examples graph G is reconstructed from $\mathcal{G}$ which is a deck of connected cards.

In 1976 Krishnamoorthy and Parthasarathy[83] have proved that critical blocks are reconstructible. (A block is *critical* if removal of any vertex v from this block will be a separable graph.) Hence RC is true if all blocks G having a vertex v such that $G - v$ is also a block are reconstructible.

Our observation is stronger than above existing result because Krishnamoorthy and Parathasarathy have considered $\mathcal{G}$ to be collection of blocks which are 2-connected while we are considering $\mathcal{G}$ to be collection of connected graphs not necessarily be 2-connected that means our collection $\mathcal{G}$ contains cut vertices while their does not.

4.5 Edge Reconstruction

In previous section we had discussed reconstruction of graph from vertex deleted subgraphs. In this section we will discuss reconstruction of graph using edge deleted subgraphs.

Definition 4.5.1 Let G be a simple graph with atleast four edges and let $\mathcal{G}_e$ denotes the collection of all its edge deleted subgraphs of the form $G - e$. i.e. $\mathcal{G}_e = \{G - e/e \in E(G)\}$. Now take another simple graph H with atleast four edges and H_e denotes the collection of all edge deleted subgraphs of the form $H - f$ i.e. $\mathcal{H}_e = \{H - f / f \in E(H)\}$.

If $\mathcal{G}_e = \mathcal{H}_e$ then H is said to be *reconstruction* of G and graph G is known as *edge reconstructible graph.*

Edge Reconstruction Conjecture:

All graphs with atleast four edges are reconstructible from the collection of edge deleted subgraphs. (Collection of edge deleted subgraphs is also known as edge deck). Here one can obviously ask question that why must any graph G have atleast four edges?

For this simple reason is that there are non-isomorphic graphs G and H on two and three edges for which $\mathcal{G}_e = \mathcal{H}_e$. In the following *Figure 4.11 and Figure 4.12* two such graphs are shown for which G is not isomorphic to H but $\mathcal{G}_e = \mathcal{H}_e$.

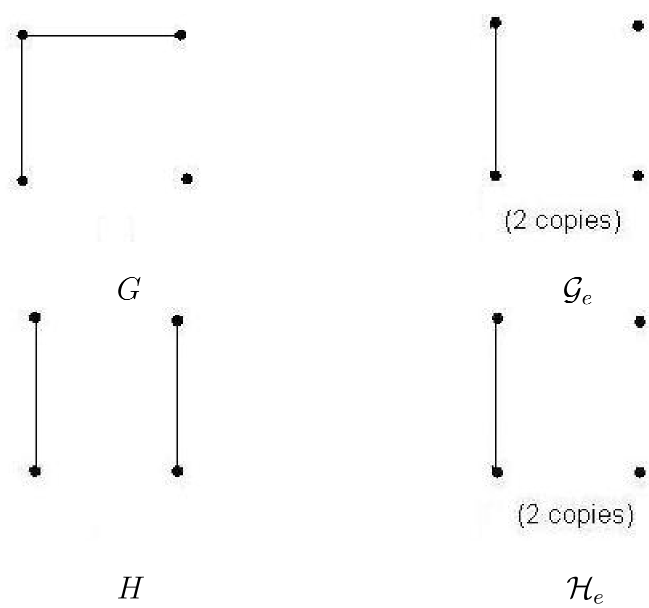


Figure 4.11

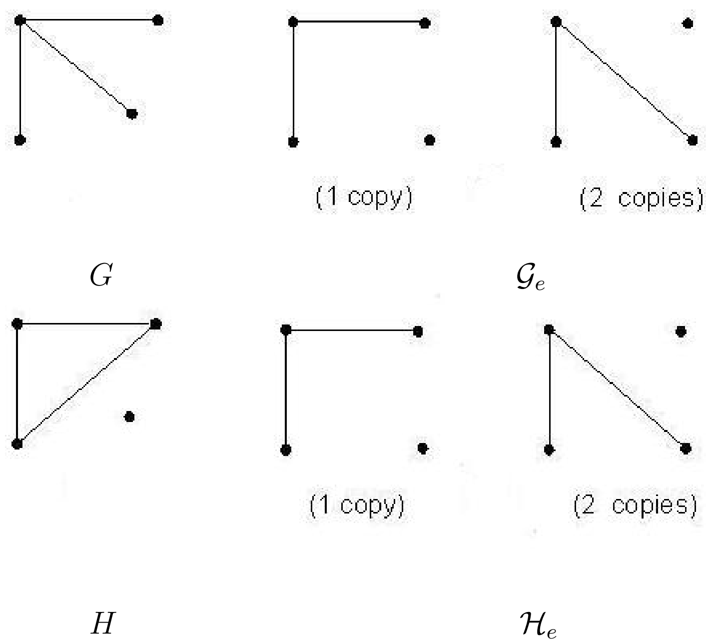


Figure 4.12

For better understanding of edge reconstruction consider following examples in which graph G is reconstructed from edge deck $\mathcal{G}_e$.

Example 1

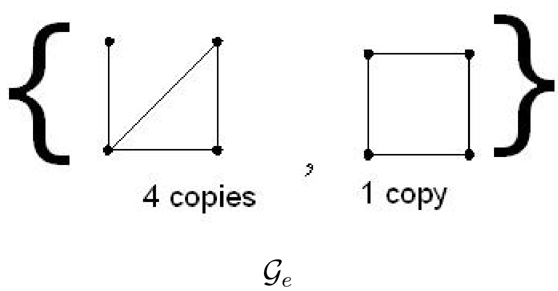


Figure 4.13

The graph obtained from above deck is shown in following *Figure 4.14*.

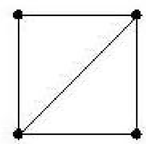
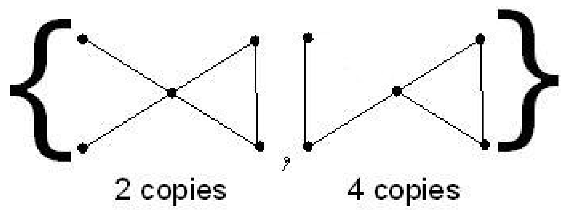


Figure 4.14

Example 2



$\mathcal{G}_e$

Figure 4.15

The graph obtained from above deck is shown in following *Figure 4.16*.

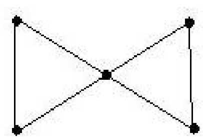


Figure 4.16

Following are some known results about edge reconstruction.

Theorem 4.5.2 (Hemminger[26])

A graph is edge reconstructible if and only if its edge graph is reconstructible.

Theorem 4.5.3 (Greenwell[59])

If G is reconstructible and has no isolated vertices then G is edge reconstructible. Using this result with results for vertex reconstruction, several classes of graphs and other parameters are edge reconstructible.

Following are some known results.

• Lovasz[94] proved that a graph G with p vertices and q edges is edge reconstructible if $q > \frac{1}{4}p(p-1)$.

• Müller[99] improved this result as follows.

◇ G is edge reconstructible if $2^{q-1} > p!$.

◇ A graph G is edge reconstructible if $q > p \times \log_2 p$.

We have tried to relate regularity of graph with edge reconstruction as follows.

Theorem 4.5.4 If G is a regular graph then the vertices in graphs of $\mathcal{G}_e$ have only two different kinds of degree.

Proof: Let G be a k-regular graph. i.e. $d(v) = k$, $\forall v \in V(G)$. Then deletion of edge e will be responsible to decrease the degree of vertices u and v by one and remaining vertices will have degree k. Thus the vertices of $G - e$ will have only two different kinds of degree namely k and $k-1$. As e was an arbitrary edge of the graph G we can say that $\mathcal{G}_e$ has graphs having only two different kinds of degree.

4.6 Edge Reconstruction of Wheel Graph

We are familiar with wheel graph W_n which is defined in *Chapter 2*. In this chapter we will take up it in connection of edge reconstruction. Consider W_4, W_5, W_6 as shown in *Figure 4.17, Figure 4.19, Figure 4.21* respectively and their deck are shown in *Figure 4.18, Figure 4.20, Figure 4.22*.

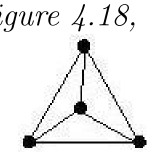
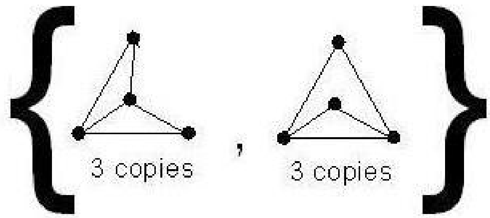


Figure 4.17



$\mathcal{G}_e$

Figure 4.18

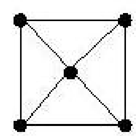
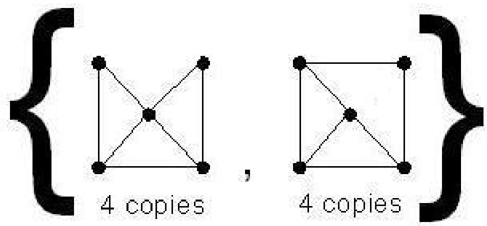


Figure 4.19



$\mathcal{G}_e$

Figure 4.20

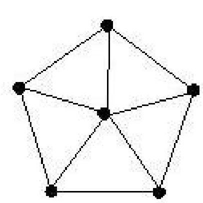
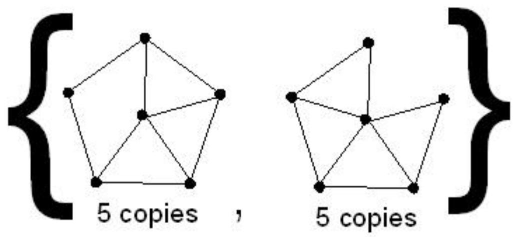


Figure 4.21



$\mathcal{G}_e$

Figure 4.22

Observing above figures carefully we can make following statement: The edge deck of wheel graph W_n contains $(n-1)$ copies of two types of graphs each of which having $(2n-3)$ edges. Thus wheel graphs are edge reconstructible according to Lovasz[94] and Müller[99].

4.7 Some More Classes of Edge Reconstructible Graphs

• Lauri[84] proved the edge reconstructibility of planar graphs with minimum degree 5.

• Fiorini and Lauri[47] proved the edge reconstructibility of 4-connected planar graphs of minimum degree 4. They have also proved that 3-connected graphs which triangulate a surface are edge reconstructible.

• Myrvold, Ellingham and Hoffman[100] proved that bidegreed graphs are edge reconstructible, moreover they have also shown that all graphs which do not have three consecutive integers in their degree sequence are edge reconstructible.

Thus in this section we have studied all the latest updates about edge reconstructibility of graph. In the next section we will study vertex switching reconstruction in detail.

4.8 Vertex Switching Reconstruction

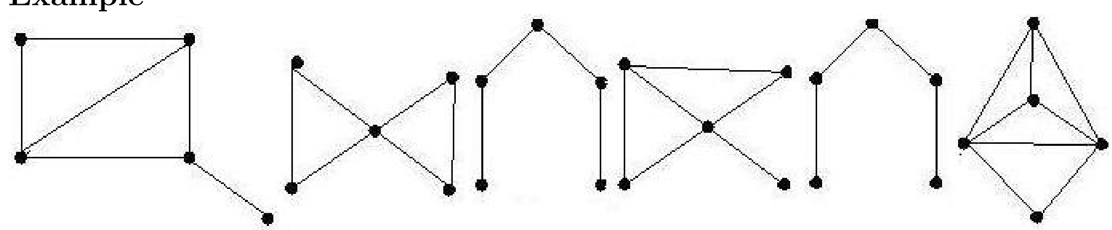
Vertex switching reconstruction was first considered by Stanley[115].

Definition 4.8.1 A *vertex switching* G_v of a graph G is obtained by taking a vertex v of G, removing all edges incident with v and adding edges joining v to every vertex not adjacent to v in G.

Definition 4.8.2 The collection $\{G_v : v \in V(G)\}$ of unlabeled graph G is called the *vertex switching deck* of G.

For better understanding of above terminology we consider following example.

Example



G G_1 G_2 G_3 G_4 G_5

Figure 4.23

Here for a given graph G, $\{G_1, G_2, G_3, G_4, G_5\}$ is vertex switching deck.

Definition 4.8.3 A graph G is called *vertex switching reconstructible* if any graph with the same vertex switching deck as G is isomorphic to G. The following are some known results:

- If G has n vertices and $n \equiv 1, 2, 3 (mod 4)$, then G is vertex switching reconstructible.
- When $n \neq 4$, number of edges and degree sequence are vertex switching reconstructible.

- Disconnected graphs of order $n \neq 4$ are vertex switching reconstructible.
- Triangle free graphs are vertex switching reconstructible.
- Regular graphs of order $n \neq 4$ are vertex switching reconstructible.

4.9 Concluding Remarks

In this chapter reconstruction of graph is discussed in detail. We derived that suspension of graph, path graphs and forests are reconstructible. Edge reconstruction and vertex switching reconstruction are studied in detail. A conjecture is posed in the support of long standing problem of graph reconstruction is the salient feature of this chapter. The next chapter is intended to discuss graceful labeling of graphs.

Chapter 5
Some Graceful Graphs

5.1 Introduction

In *Chapter 3* we have discussed various types of graph labeling while this chapter is aimed to discuss graceful labeling in detail. Some new classes of graceful graphs are investigated and some open problems are given at the end. As we mentioned earlier the graceful labeling was introduced by Rosa[103] during 1967.

In the immediate section we will recall the definition of graceful labeling for ready reference.

5.2 Some Basic Definitions and Known Results

Definition 5.2.1 If the vertices of the graph are assigned values subject to certain conditions is known as *graph labeling*.

Definition 5.2.2 A function f is called *graceful labeling* of a graph G if $f : V \rightarrow \{0, 1, 2, \ldots, q\}$ is injective and the induced function $f^* : E \rightarrow \{1, 2, \ldots, q\}$ defined as $f^*(e = uv) = |f(u) - f(v)|$ is bijective. A graph which admits graceful labeling is called *graceful graph*.

In the following *Figure 5.1* two graceful graphs and their graceful labeling are shown.

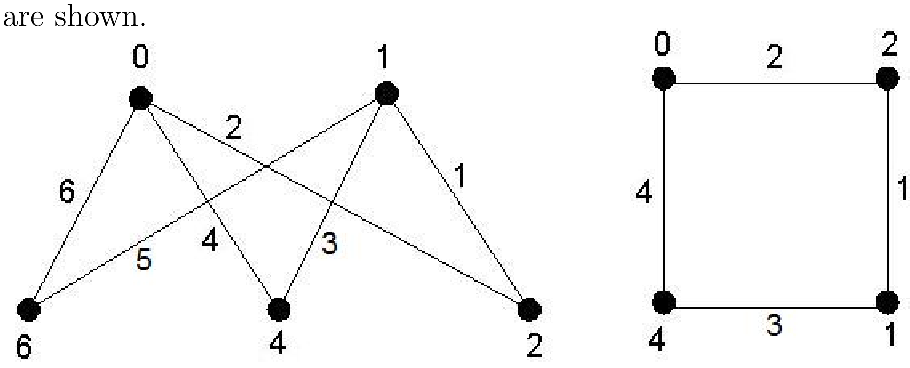


Figure 5.1

Some obvious facts and known results are listed below.

• Any graceful graph will always have vertices with labels 0 and q. These vertices are always adjacent. One can visualize this from *Figure 5.1*.

• Graceful labeling is not unique. This fact is demonstrated in the following *Figure 5.2*.

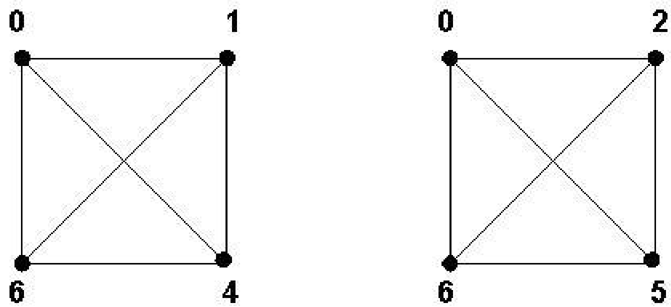


Figure 5.2

• Supergraph of a graceful graph need not be graceful. e.g. K_4 if graceful but K_5 is not.

• Subgraph of a graceful graph need not be a graceful graph. e.g. $W_5 = C_5 + K_1$ is graceful while C_5 is not.

• If $\{a_1, a_2, \ldots, a_p\} \subseteq \{0, 1, \ldots, q\}$ is a graceful labeling of any graph G then $\{q - a_i / i = 1, 2, \ldots, p\}$ is also graceful labeling for the graph G.

• There are $q!$ connected graceful graphs with q edges. For example there are $3! = 6$ graceful graphs with 3 edges as shown in following *Figure 5.3*.

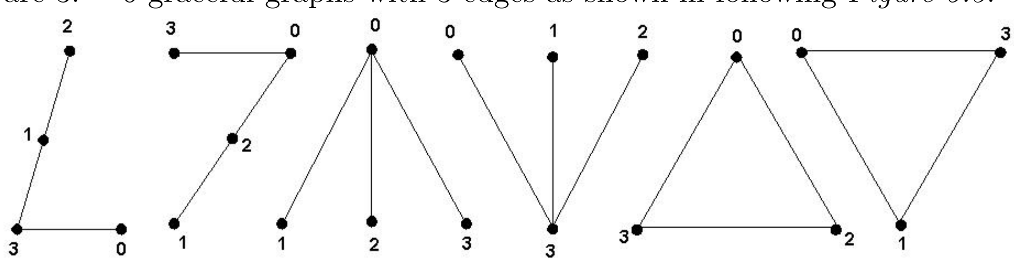


Figure 5.3

• All the graphs with $p \leq 5$ (where p denotes number of vertices) are graceful except C_5, K_5 and Bowtie graph.

• Rosa[103] proved that cycle C_n is graceful if and only if $n \equiv 0, 3(mod4)$.

• Frucht[49], Hoede and Kuiper[72] proved that all wheels $W_n = C_n + K_1$ are graceful graphs.

• Golomb[57] proved that the complete graph K_n is graceful if and only if $n \leq 4$.

• Rosa[103] and Golomb[57] proved that the complete bipartite graphs $K_{m,n}$ are graceful for all m and n.

• Aravamudhan and Murugan[11] have shown that the complete tripartite graph $K_{1,m,n}$ is graceful for all m and n.

• Beutner and Harborth[21] showed that $K_n - e$ is graceful only if $n \leq 5$, $K_n - 2e$ and $K_n - 3e$ are graceful only if $n \leq 6$.

• The Ringel-Kotzig conjecture about gracefulness of trees is still an open problem and it has motivated good number of research papers. The conjecture is *All trees are graceful.* In [73] Kotzig has called the effort to prove this conjecture as a *disease.* The trees known to be graceful are caterpillars, paths, star graphs etc.

• Ayel and Favaron[14] proved that all helms are graceful.

• Kang et al.[78] proved that webs are graceful.

• Bermond[20] conjectured that *Lobsters are graceful.*

• Morgan[98] proved that all lobsters with perfect matching are graceful.

• Chen et al.[36], Bhat-Nayak and Deshmukh[22] proved that banana trees are graceful.

• Aldred and Mckay[6] used a computer program to show that trees with 27 or less vertices are graceful.

Despite of many efforts the graceful tree conjecture remains an open problem but this problem has motivated some new graph labeling techniques.

• Truszczynski[119] studied unicyclic graphs and conjectured that *All unicyclic graphs except* C_n*, where* $n \equiv 1, 2(mod4)$ *are graceful.*

Because of the immense diversity of unicyclic graphs, proof of above conjecture seems out of reach in the near future.

• Delorme et al.[40], Ma and Feng[95] proved that cycle with a chord is graceful.

• Gracefulness of cycle with k consecutive chords is also investigated by Koh et al.[81],[82], Goh and Lim[56].

• Koh and Rogers[82] conjectured that *Cycle with triangle denoted as* $C_n(p, q, r)$ *is graceful if and only if* $n \equiv 0, 1(mod4)$.

Next section is aimed to discuss gracefulness of some product related graphs which also includes investigations carried out by us.

5.3 Gracefulness of Some Product Related Graphs

We have defined the cartesian product of two graphs in *Chapter 2*. This definition has attracted many researchers. Some known results regarding product related graphs are listed below.

• Acharya and Gill[3] proved that grid graph $P_m \times P_n$ is graceful.

• Maheo[96] gave the graceful labeling for $P_m \times P_2$ which can be readily be extended to all grids.

• Kathiresan[79] proved that the graph obtained from subdividing each step of ladder $P_n \times P_2$ exactly once are graceful.

• Acharya[1] proved that certain subgraphs of grid graphs are graceful.

• Huang and Skiena[74] proved that $C_m \times P_n$ is graceful for all n, when m is even and for all n with $3 \leq n \leq 12$ when m is odd.

• Jungreis and Reid[76] proved that torus grid $C_m \times C_n$ is graceful when $m \equiv 0(mod4)$ and n is even.

A complete solution for the problem of graceful torus grid will most likely involve a large number of cases.

We have also investigated some new families of product related graphs. We will provide detail proof of these results.

Theorem 5.3.1 The graph $G = (P_m \times P_n) \cup (P_r \times P_s)$, where $m, n, r, s \in N \setminus \{1\}$ is graceful.

Proof: It is obvious that the graph G has number of vertices $p = rs + mn$ and number of edges $q = 2(rs+mn)-(m+n+r+s)$. According to *Definition 5.2.2* the available vertex labels are $0, 1, \ldots, q$.

Now label the vertices of $(P_r \times P_s)$ by the labels $q, 0, 1, q-2, q-3, q-4, 4, 5, \ldots$ etc. This labeling sequence is having two sequential patterns, one is increasing and other is decreasing. Such labeling will give rise to edge labeling as decreasing sequence of labels $q, q-1, \ldots, q+r+s+1-2rs$. Such vertex labeling pattern is shown in *Figure 5.4*.

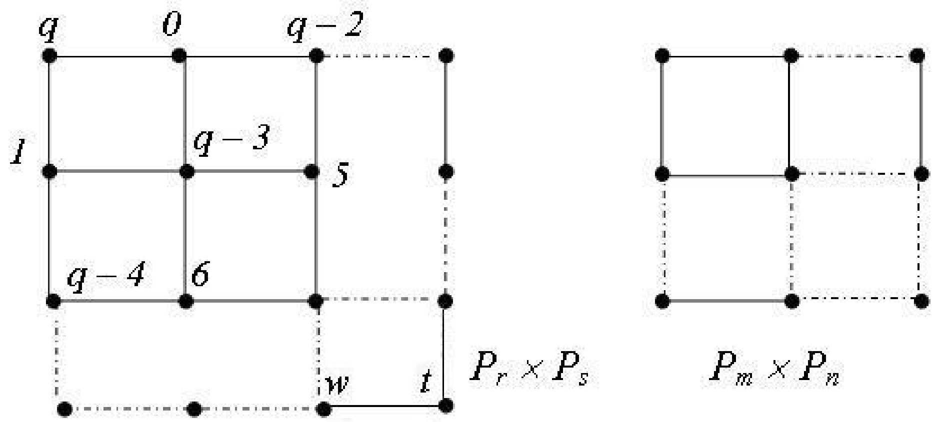


Figure 5.4

Now our task is to label the vertices of $(P_m \times P_n)$. It will depend on the vertex labels of the last grid of $(P_r \times P_s)$. Let w and t be vertex labels of last grid of $(P_r \times P_s)$. These labels produce edge label $q + r + s + 1 - 2rs = 2mn + 1 - (m + n)$. At this stage we have to consider following two possibilities.

<u>Case-1:</u> $w < t$. Then w must be a label from increasing sequence of labels and $t - w = 2mn + 1 - (m + n)$. Now available vertex labels are $t + 1, t - 1, t - 2, \ldots, w + 2, w + 1$, which are in number $2mn + 1 - (m + n)$.

We will use these labels for labeling of vertices of $(P_m \times P_n)$. This vertex labeling sequence is $t + 1 = 2mn - (m + n) + w + 2, w + 2, w + 3, 2mn - (m + n) + w, 2mn - (m + n) + w - 1, 2mn - (m + n) + w - 2, w + 7, w + 8, \ldots$ etc. This labeling sequence is having two sequential patterns, one is increasing and other is decreasing. Such labeling will give rise to edge labeling as decreasing sequence of labels $2mn - (m + n), \ldots, 2, 1$. Thus we have labeled all the $rs + mn$ vertices of G gracefully.

Case-2: $w > t$. Then w must be a label from decreasing sequence of labels and $w - t = 2mn + 1 - (m + n)$. Now available vertex labels are $w-1, w-2, \ldots, t+2, t+1, t-1$, which are in number $2mn+1-(m+n)$.

We will use these labels for labeling of vertices of $(P_m \times P_n)$. This vertex labeling sequence is $t-1, w-2 = 2mn-(m+n)+t-1, w-3, t+1, t+2, t+3, w-7, \ldots$ etc. This labeling sequence is having two sequential patterns, one is increasing and other is decreasing. Such labeling will give rise to edge labeling as decreasing sequence of labels $2mn - (m + n), \ldots, 2, 1$. Thus we have labeled all the $rs + mn$ vertices of G gracefully.

Therefore $G = (P_r \times P_s) \cup (P_m \times P_n)$ is graceful graph.

Illustration 5.3.2 For better understanding of labeling pattern discussed in above *Theorem 5.3.1* let us consider $G = (P_3 \times P_3) \cup (P_3 \times P_2)$. For the graph G, $p = 15$ and $q = 19$. Therefore for graceful labeling of G available vertex labels are $0, 1, \ldots, 19$. As per procedure employed in *Theorem 5.3.1* we first label vertices of $P_3 \times P_3$ by $19, 0, 17, 1, 16, 4, 15, 5, 13$ and $P_3 \times P_2$ by $14, 7, 8, 12, 11, 10$. This will produce edge labels $1, 2, \ldots, 19$ as shown in *Figure 5.5*. Thus G is a graceful graph.

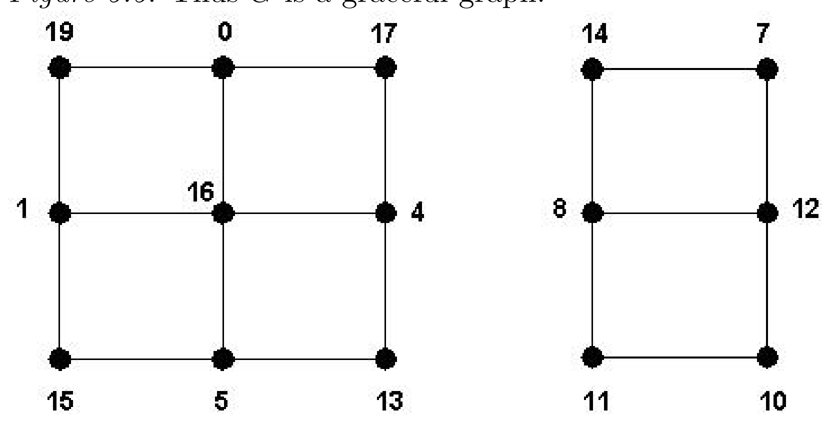


Figure 5.5

Theorem 5.3.3 The graph $G = C_{2f+3} \cup (P_m \times P_n) \cup (P_r \times P_s)$ (where $m, n, r, s \in N \setminus \{1\}$ and $f = 2(mn + rs) - (m + n + r + s)$) is graceful.

Proof: It is obvious that G will have number of vertices $p = 2f+3+mn+rs$ and number of edges $q = 3f + 3$. Let $u_1, u_2, \ldots, u_{2f+3}$ be successive vertices of C_{2f+3}. Now label $f + 2$ vertices $u_1, u_3, \ldots, u_{2f+3}$ of C_{2f+3} by the labels $0, 1, 2, \ldots, f + 1$ respectively and label the remaining $f + 1$ vertices $u_2, u_4, \ldots, u_{2f+2}$ of C_{2f+3} by the labels $3f+3, 3f+2, \ldots, 2f+3$ respectively. Thus all the vertices of C_{2f+3} are labeled. This vertex labeling will give rise to edge labels according to *Definition 5.2.2* as $3f+3, 3f+2, \ldots, f+2, f+1$. Now our task is to label the vertices of $(P_m \times P_n) \cup (P_r \times P_s)$ for which the available vertex labels are $2f + 2, 2f + 1, \ldots, f + 2$ and required edge labels for $(P_m \times P_n) \cup (P_r \times P_s)$ are $f, f - 1, \ldots, 2, 1$. Since available vertex labels are $f + 1$ and required edge labels are f, we first label the vertices of $(P_m \times P_n) \cup (P_r \times P_s)$ by $0, 1, \ldots, f$ as in *Theorem 5.3.1*. Then we add $f + 2$ to all the vertex labels of $(P_m \times P_n) \cup (P_r \times P_s)$ will produce edge labels $1, 2, \ldots, f$ for $(P_m \times P_n) \cup (P_r \times P_s)$. Thus we have labeled $G = C_{2f+3} \cup (P_m \times P_n) \cup (P_r \times P_s)$ gracefully. Therefore G is graceful graph.

5.4 Gracefulness of Union of Grid Graph with Complete Bipartite Graph and Path Graph

Bu and Cao[27] have discussed gracefulness of $K_{m,n}$ and its union with path graph. Seoud and Youssef[110] have shown that $K_{m,n} \cup K_{p,q}$ $(m, n, p, q \geq 2)$, $K_{m,n} \cup K_{p,q} \cup K_{r,s}$, $(m, n, p, q, r, s \geq 2$ and $(p, q) \neq (2, 2))$ are graceful graphs. In this section we will discuss gracefulness of union of grid graph with complete bipartite graph and path graph.

Theorem 5.4.1 $G = K_{m,n} \cup (P_r \times P_s)$, $r, s \geq 2$ is graceful graph.

Proof: Here total number of vertices $p = m + n + rs$ and total number of edges $q = mn + 2rs - (r + s)$.

Now label the vertices of $K_{m,n}$ by the labels $0, 1, \ldots, m - 1, m + 2rs - (r + s), 2m + 2rs - (r + s), \ldots, q = mn + 2rs - (r + s)$, which give rise to edge labels as $q, q - 1, \ldots, 2rs - (r + s) + 1$ to edges of $K_{m,n}$. Now our task is to label the vertices of $(P_r \times P_s)$ for which the available vertex labels are $m + 1, m + 2, \ldots, m + 2rs - (r + s) - 1$ and $m + 2rs - (r + s) + 1$.

Let us denote the vertices of the grid graph $P_r \times P_s$ by $v_{11}, v_{12}, \ldots, v_{1n}, v_{21}, \ldots, v_{mn}$. Now label the vertex v_{11} by $m + 2rs - (r + s) + 1$, v_{12} by $m + 1$, v_{21} by $m + 2$, v_{13} by $m + 2rs - (r + s) - 1$, v_{22} by $m + 2rs - (r + s) - 2$, v_{31} by $m + 2rs - (r + s) - 3$, v_{14} by $m + 5$, v_{23} by $m + 6$, v_{32} by $m + 7$, v_{41} by $m + 8$, v_{15} by $m + 2rs - (r + s) - 7$, v_{24} by $m + 2rs - (r + s) - 8$ etc. This will give rise to edge labels as $2rs - (r + s), 2rs - (r + s) - 1, 2rs - (r + s) - 2, \ldots, 2, 1$. For the vertex labeling and edge labeling following pattern has been observed.

(1) In each square of grid the difference between two labels of main diagonal is always one.

(2) In the labeling of vertices two sequential patterns have been found, one is increasing and another is decreasing. This will give rise to edge labeling into decreasing sequence of labels $2rs - (r + s), 2rs - (r + s) - 1, 2rs - (r + s) - 2, \ldots, 2, 1$. Such labeling pattern for vertices and edges is shown by down arrows in following *Figure 5.6*.

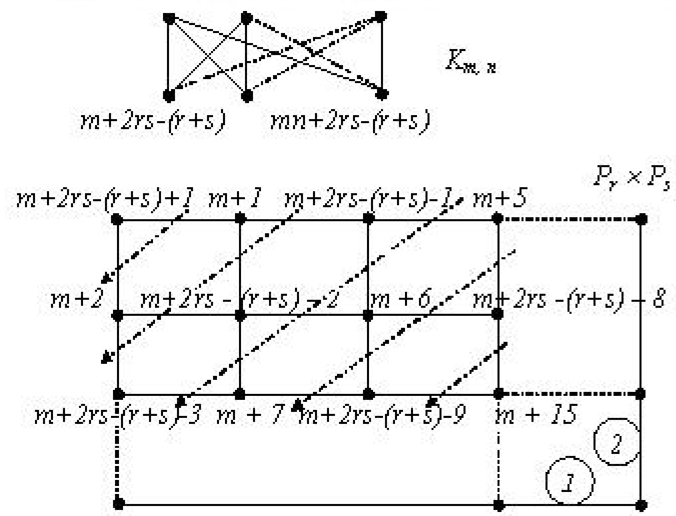


Figure 5.6

Thus we have labeled all the vertices of graph $K_{m,n} \cup (P_r \times P_s)$ gracefully, where $m, n, r, s \in N \backslash \{1\}$ and hence the graph is graceful graph.

Illustration 5.4.2 For better understanding of above discussed labeling pattern let us consider the graph $G = K_{4,3} \cup (P_3 \times P_4)$. For this graph G, $p = 19$ and $q = 29$. Therefore for graceful labeling available vertex labels are $0, 1, 2, \ldots, 29$. As per the procedure employed in *Theorem 5.4.1* we first label the vertices of $K_{4,3}$ by $0, 1, 2, 3, 29, 25, 21$ and the vertices of $(P_3 \times P_4)$ by $22, 5, 20, 9, 6, 19, 10, 15, 18, 11, 14, 13$. This will produce edge labels $1, 2, \ldots, 29$ as shown in following *Figure 5.7*. Thus G is a graceful graph.

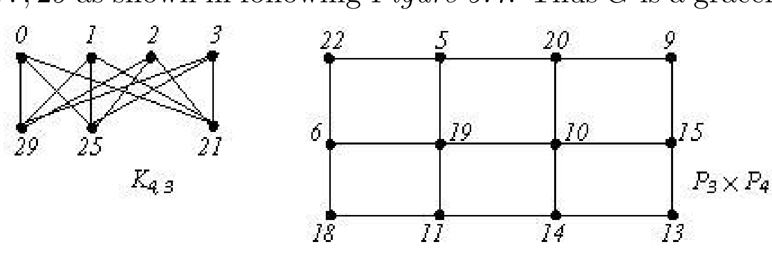


Figure 5.7

Lemma 5.4.3 Using $0, 1, \ldots, t-2$ and t vertex labels one can produce $1, 2, \ldots, t-1$ edge labels for path graph P_t, $t \geq 3$.

Proof: There are six cases to be considered as follows:

<u>Case-1:</u> $t \equiv 3(mod6)$.

In this case $t = 6n+3$ for some non-negative integer n. Then for P_t available vertex labels are $0, 1, 2, \ldots, 6n+1$ and $6n+3$. Let us denote these vertices by $u_1, u_2, \ldots, u_{6n+3}$. We shall label the vertices $u_2, u_4, \ldots, u_{6n+2}$ according to the sequence $1, 0, 2, 4, 3, 5, 7, 6, \ldots, 3n-3, 3n-1, 3n+1$. Now label the remaining vertices $u_1, u_3, \ldots, u_{6n+3}$ according to the sequence $6n+3, 6n+1, 6n-1, 6n, \ldots, 3n+2, 3n+3, 3n$, as shown in *Figure 5.8*.

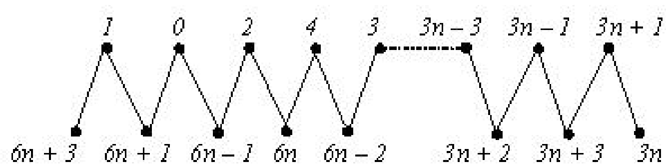


Figure 5.8

Such vertex labeling will give rise to edge labeling for P_t as $6n+2, 6n, 6n+1, 6n-1, 6n-3, \ldots, 3, 4, 2, 1$.

<u>Case-2:</u> $t \equiv 4(mod6)$.

Then $t = 6n+4$ for some $n \in N \cup \{0\}$. Here available vertex labels are $0, 1, 2, \ldots, 6n+2$ and $6n+4$. We shall label the vertices $u_2, u_4, \ldots, u_{6n+2}$ according to the sequence $1, 0, 2, 4, 3, \ldots, 3n-3, 3n-1, 3n+1, 3n$ and label the remaining vertices $u_1, u_3, \ldots, u_{6n+3}$ according to the sequence $6n+4, 6n+2, 6n, 6n+1, 6n-1, \ldots, 3n+3, 3n+4, 3n+2$ as shown in *Figure 5.9*. Such vertex labeling will give rise to edge labels $6n+3, 6n+1, 6n+2, 6n, 6n-2, \ldots, 4, 5, 3, 1, 2$.

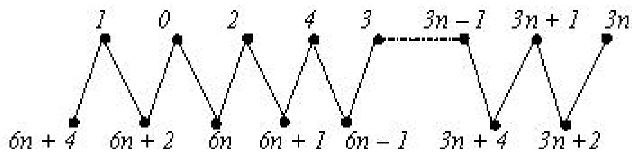


Figure 5.9

<u>Case-3:</u> $t \equiv 5(mod 6)$

Then $t = 6n + 5$ for some $n \in N \cup \{0\}$. Here available vertex labels are $0, 1, 2, \ldots, 6n + 3$ and $6n + 5$. We shall label the vertices at even places according to the sequence $1, 0, 2, 4, 3, \ldots, 3n - 3, 3n - 1, 3n + 1, 3n$ and label the remaining vertices according to the sequence $6n + 5, 6n + 3, 6n + 1, 6n + 2, 6n, \ldots, 3n + 5, 3n + 2, 3n + 3$. Such vertex labeling will give rise to edge labels $6n + 4, 6n + 2, 6n + 3, 6n + 1, \ldots, 5, 6, 4, 1, 2, 3$.

<u>Case-4:</u> $t \equiv 0(mod 6)$

Then $t = 6n$ for some $n \in N$. Here available vertex labels are $0, 1, 2, \ldots, 6n - 2$ and $6n$. We shall label the vertices at even places according to the sequence $1, 0, 2, 4, 3, \ldots, 3n - 4, 3n - 2, 3n - 3, 3n$ and label the remaining vertices according to the sequence $6n, 6n - 2, 6n - 4, \ldots, 3n + 3, 3n + 1, 3n - 1$. Such vertex labeling will give rise to edge labels $6n - 1, 6n - 3, 6n - 2, \ldots, 7, 5, 3, 4, 2, 1$.

<u>Case-5:</u> $t \equiv 1(mod 6)$

Then $t = 6n + 1$ for some $n \in N$. We shall label the vertices at even places according to the sequence $1, 0, 2, 4, 3, \ldots, 3n - 3, 3n - 2, 3n - 3, 3n - 1$ and label the remaining vertices according to the sequence $6n + 1, 6n - 1, 6n - 3, \ldots, 3n + 2, 3n, 3n + 1$. Such vertex labeling will give rise to edge labels $6n, 6n - 2, 6n - 1, 6n - 3, \ldots, 5, 3, 1, 2$.

Case-6: $t \equiv 2(mod 6)$

Then $t = 6n + 2$ for some $n \in N$. We shall label the vertices at even places according to the sequence $1, 0, 2, 4, 3, \ldots, 3n-2, 3n-3, 3n, 3n-1$ and label the remaining vertices according to the sequence $6n+2, 6n, 6n-2, 6n-1, \ldots, 3n+5, 3n+2, 3n+3, 3n+1$. Such vertex labeling will give rise to edge labels $6n+1, 6n-1, 6n, \ldots, 4, 5, 6, 3, 1, 2$.

Thus in any case one can produce $1, 2, \ldots, t-1$ edge labels for P_t, $t \geq 3$, using $0, 1, 2, \ldots, t-2$ and t vertex labels.

Remark 5.4.4 From the above *Lemma 5.4.3* following observations are obvious:

(1) By adding n in each term of the sequence $1, 2, \ldots, t-2, t$ (which are vertex labels for P_t) one can produce edge labels $1, 2, \ldots, t-1$ for P_t, $t \geq 3$.

(2) By subtracting each term of the sequence $1, 2, \ldots, t-2, t$ (which are vertex labels for P_t) from $n+t$ one can produce edge labels $1, 2, \ldots, t-1$ for P_t, $t \geq 3$.

Theorem 5.4.5 The graph $G = (P_r \times P_s) \cup P_t$ is graceful, where $t \in N \setminus \{2\}$ and $r, s \in N \setminus \{1\}$.

Proof: Here for the graph G under consideration number of vertices $p = rs + t$ and number of edges $q = 2rs - (r+s) + t - 1$. According to *Definition 5.2.2* the available vertex labels are $0, 1, \ldots, q$.

Now label the vertices of $P_r \times P_s$ by the labels $q, 0, 1, q-2, q-3, q-4, 6, 7, \ldots$etc. As we discussed in *Theorem 5.4.1* two labeling patterns have been observed. Such vertex labeling will give rise to edge labeling as decreasing sequence of labels $q, q-1, \ldots, q-2rs+r+s+1$ which is shown in *Figure 5.10*.

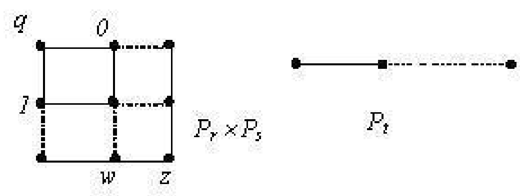


Figure 5.10

Now our task is to label the vertices of P_t. It will depend on the vertex labels of the last grid of $P_r \times P_s$. Let w and z be vertex labels of last grid of $P_r \times P_s$. These labels produce edge label $q - 2rs + r + s + 1 = t$. At this stage following two cases are to be considered.

Case-1: $w < z$. Then w must be a label from increasing sequence of labels and $z - w = t$. Now available vertex labels are $z + 1 = t + w + 1, z - 1 = t + w - 1, z - 2 = t + w - 2, \ldots, w + 2, w + 1$, which are in number t. Using these labels we can label P_t according to *Remark 5.4.4* and produce edge labels $1, 2, \ldots, t - 1$.

Case-2: $w > z$. Then w must be a label from decreasing sequence of labels and $w - z = t$. Now available vertex labels are $w - 1 = t + z - 1, w - 2 = t + z - 2, \ldots, z + 2, z + 1, z - 1$, which are in number t. Using these labels one can label the vertices of P_t according to *Remark 5.4.4* and produce edge labels $1, 2, \ldots, t - 1$.

Therefore $G = (P_r \times P_s) \cup P_t$ is graceful, where $r, s \in N \setminus \{1\}$ and $t \in N \setminus \{2\}$.

Illustration 5.4.6 For better understanding of the above discussed labeling pattern, let us consider the graph $G = (P_3 \times P_4) \cup P_{13}$. For this graph G, $p = 25$ and $q = 29$. So for graceful labeling of G, available vertex labels are $0, 1, 2, \ldots, 29$. According to *Theorem 5.4.5* one can label $(P_3 \times P_4)$ by 29, 0, 27, 4, 1, 26, 5, 22, 25, 6, 21, 8 and P_{13} by 7, 19, 9, 20, 11, 18, 10, 16, 12, 17, 14, 15, 13. This will give rise to edge labels $29, 28, \ldots, 13$ for grid graph $(P_3 \times P_4)$ and 12, 10, 11, 9,..., 5, 3, 1, 2 for P_{13} according to Case-5 of *Lemma 5.4.3*. Such labeling pattern is shown in *Figure 5.11*. Hence G is a graceful graph.

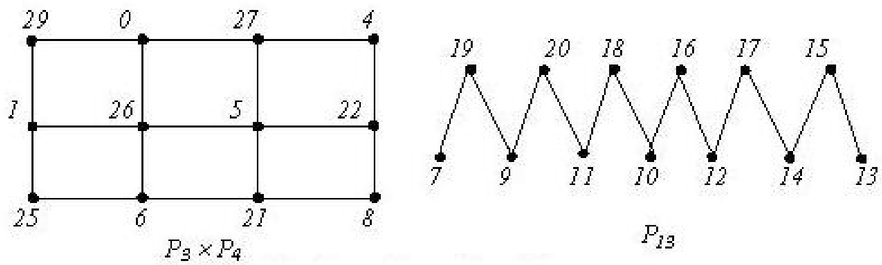


Figure 5.11

Theorem 5.4.7 The graph $G = K_{m,n} \cup (P_r \times P_s) \cup P_t$ is graceful where $t \in N \setminus \{2\}$ and $m, n, r, s \in N \setminus \{1\}$.

Proof: The graph G has number of vertices $p = m + n + rs + t$ and number of edges $e = mn + 2rs - (r + s) + t - 1 = mn + q$ where $q = 2rs - (r + s) + t - 1$ is the number of edges in $(P_r \times P_s) \cup P_t$.

Now label the vertices of $K_{m,n}$ by labels $0, 1, \ldots, m-1, m+q, 2m+q, \ldots, e = mn + q$, which will give rise to edge labels as $e, e-1, \ldots, q+1$ for the edges of $K_{m,n}$. Now our task is to label the vertices of $(P_r \times P_s) \cup P_t$, for which the available vertex labels are in number $q + 1$. These are $m, m+1, m+2, \ldots, m+q-1$ and $m+q+1$. Now by adding $m+1$ in all the vertex labels of $(P_r \times P_s) \cup P_t$ reported in *Theorem 5.4.5* one can produce edge labels $1, 2, \ldots, q$. Thus we have labeled $G = K_{m,n} \cup (P_r \times P_s) \cup P_t$ gracefully and hence G is a graceful graph.

5.5 Gracefulness of Union of Two Path Graphs with Grid Graph and Complete Bipartite Graph

It is obvious that union of two path graphs can not be graceful as number of vertices of $P_n \cup P_t$ is more than the number of labels available for its gracefulness. In connection of *Lemma 5.4.3* we have following remarks.

Remark 5.5.1 Using $n, n+1, \ldots, n+t-2, n+t$, for n∈ N one can produce $1, 2, \ldots, t-1$ edge labels for path graph P_t(where $t \geq 3$). In order to produce $s, s+1, \ldots, t-1$ edge labels for path graph P_{t-s} using above vertex labels one can proceed as either of the following two ways.

(i) Using $n+s, n+s+1, \ldots, n+t-2, n+t$, (where $n, s \in N$) one can produce $1, 2, \ldots, t-s-1$ edge labels for path graph P_{t-s}. Now choose half of the total number of vertex labels from the above mentioned sequence of vertex labels into their numerically increasing order (one less than half of the total number when n is odd) and subtract s from each selected vertex labels. This will produce edge labels $s, s+1, \ldots, t-1$ for P_{t-s}.

(ii) Using $n+s, n+s+2, \ldots, n+t-1, n+t$, (where $n \in N$) one can produce edge labels as $1, 2, \ldots, t-s-1$ for P_{t-s}. Now choose half of the total number of vertex labels from the above mentioned sequence of vertex labels according to their numerically increasing order (one less than half of the total number when n is odd) and subtract s from each selected vertex labels. This will produce edge labels $s, s+1, \ldots, t-1$ for P_{t-s}.

¶ **Remark 5.5.2** If we label the grid graph ($P_r \times P_s$) by using increasing and decreasing sequence of vertex labels in diagonal pattern then there are $min\{r, s\} - 1$ vertex labels which are not used after graceful labeling of $(P_r \times P_s)$. Moreover if $K_{r,s}$ is labeled by t vertex labels (where $t \leq max\{r, s\}$)

$0, 1, \ldots, t-1$ and remaining by $t, 2t, \ldots, rs$ then there are t vertex labels namely $1+t, 2+t, \ldots, 2t-1, 2t+1$ which are not used in the graceful labeling of $K_{r,s}$.

Theorem 5.5.3 The graph $G = P_n \cup P_t \cup (P_r \times P_s)$, where $t < min\{r, s\}, r, s \geq 3$ is graceful.

Proof: Here total number of vertices $p = n + t + rs$ and total number of edges $q = n + t + 2rs - (r + s - 2)$.

Now label the vertices of $(P_r \times P_s)$ by labels $q, 0, 1, q-2, q-3, q-4 \ldots$ etc. This labeling sequence is having two sequential patterns, one is increasing and other is decreasing. Such labeling pattern will give rise to edge labeling as decreasing sequence of labels $q, q-1, \ldots, q+r+s+1-2rs$, which is shown in *Figure 5.12*.

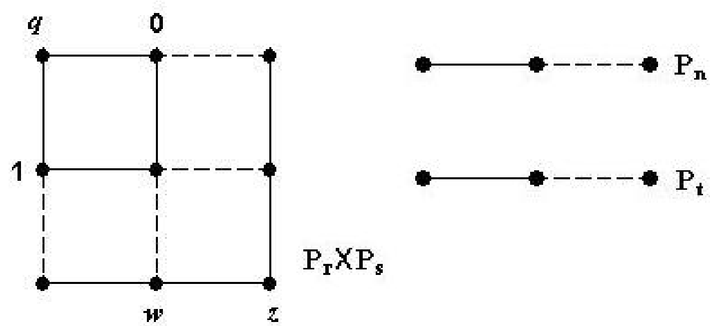


Figure 5.12

Now our task is to label the vertices of P_n. It will depend on the vertex labels of the grid graph $(P_r \times P_s)$. Let w and z be vertex labels of last grid of $(P_r \times P_s)$.

Case-1: $w < z$. Then w must be a label from increasing sequence of labels and $z - w = q + r + s + 1 - 2rs = n + t - 1$. Now available vertex labels are $z+1, z-1, \ldots, w+2, w+1$ which are total $n + t - 1$.

Case-2: $w > z$. Then w must be a label from the decreasing sequence of labels and $n + t - 1 = w - z$. Now available vertex labels are $w - 1, w - 2, \ldots, z + 2, z + 1, z - 1$, which are in number $n + t - 1$.

Using these labels one can label the vertices of P_n according to *Remark 5.5.1* which will give rise to edge labels as $n + t - 2, n + t - 3, \ldots, t$. Now to label P_t one can use vertex labels which are not used in graceful labeling of grid graph. This labels will give rise to edge labels $1, 2, \ldots, t - 1$ for P_t. Thus graph G under consideration admits graceful labeling.

Illustration 5.5.4 For better understanding of above defined labeling pattern consider the graph $G = P_{10} \cup P_3 \cup (P_5 \times P_4)$. Here $q = 42$. The graceful labeling of G is as shown in following *Figure 5.13*.

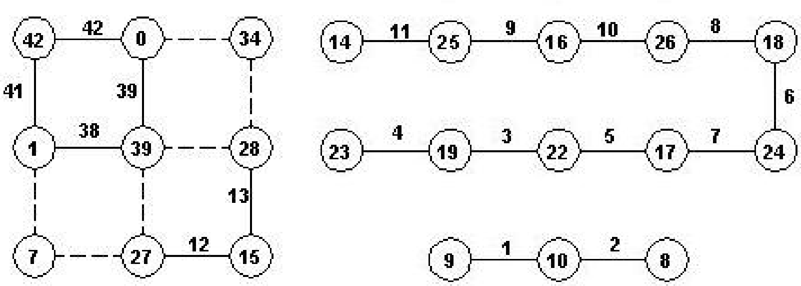


Figure 5.13

Theorem 5.5.5 The graph $G = P_n \cup P_t \cup K_{r,s}$, where $t \leq max\{r, s\}, (r, s \geq 3)$ is graceful.

Proof: Here total number of vertices $p = n + t + r + s$ and total number of edges $q = rs + n + t - 2$.

Now label the vertices of $K_{r,s}$ by labels $0, 1, \ldots, r-1$, $r+n+t-2, \ldots, rs+n + t - 2 = q$ (assuming $r \geq s$) as shown in *Figure 5.14*. This will give rise to edge labels as $q, q - 1, \ldots, n + t - 1$ of $K_{r,s}$.

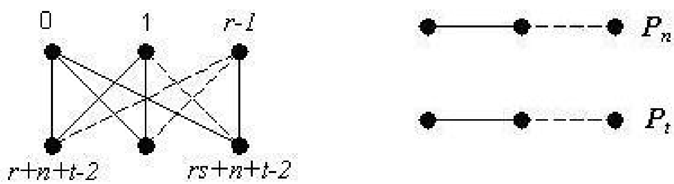


Figure 5.14

Now our task is to label the vertices of P_n and then P_t for which the available vertex labels are $r, r+1, r+2, \ldots, r+n+t-3, r+n+t-1$. These are in number $n+t-1$ and $2r+n+t-3, 2r+n+t-1, 2r+n+t, \ldots, 3r+n+t-3$, which are in number r. Using these labels according to *Remark 5.5.2* one can label P_n and P_t which give rise to edge labels as $n+t-2, n+t-3, \ldots, t$ and $t-1, t-2, \ldots, 2, 1$ respectively. Thus we have labeled all the vertices of graph G under consideration gracefully.

Illustration 5.5.6 For better understanding of above defined labeling pattern consider the graph $G = P_{10} \cup P_5 \cup (K_{4,5})$. Here $q = 33$. The graceful labeling of G is as shown in following *Figure 5.15.*

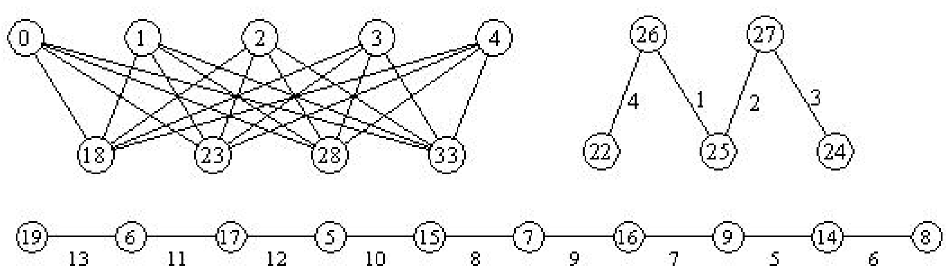


Figure 5.15

5.6 Some Open Problems

¶ One can discuss gracefulness of union of grid graph with wheels, cycles, Petersen graph etc.

¶ One can derive parallel results for other type of labeling like harmonious, (k, d)-graceful, skolem graceful, k-equitable etc.

¶ One can discuss gracefulness in the context of various graph operations like contraction, barycentric subdivision etc.

¶ One can investigate graceful labeling for the star of cycle, which is defined in *Chapter 6*.

5.7 Concluding Remarks

The graceful labeling of graph is stronger in its class. Grid graph is very interesting family of graphs. Here we have discussed the gracefulness of grid graph with some other families. The results obtained here are new and of very general nature. This work throws light on the gracefulness of disconnected graphs which is very less cultivated field. Illustrations provide better understanding of the derived results. This work contributes eight new results to the theory of graceful graphs. The next *Chapter 6* is aimed to discuss cordial labeling of graphs.

Chapter 6
Some Cordial Graphs

6.1 Introduction

In *Chapter 3* we have discussed various types of graph labeling while this chapter is aimed to discuss the cordial labeling of graphs in detail. Some new families of cordial graphs are investigated and some open problems are also posed.

Many researchers have studied cordiality of graphs. As we mentioned in *Chapter 3*, Cahit[31] introduced cordial graphs in 1987 as a weaker version of graceful and harmonious graphs. In the immediate section we will recall the definition of cordial graph and will provide detail survey on cordial graphs.

6.2 Some Basic Definitions and Important Results

Definition 6.2.1 If the vertices of the graph are assigned values subject to certain conditions is known as *graph labeling*.

For detail survey on graph labeling one can refer Gallian[51].

Definition 6.2.2 Let $G = (V, E)$ be a graph. A function $f : V(G) \rightarrow \{0, 1\}$ is called *binary vertex labeling* of G and $f(v)$ is called *label of the vertex v* under f.

For an edge $e = uv$ the induced function $f^* : E(G) \rightarrow \{0, 1\}$ is given by $f^*(e) = |f(u) - f(v)|$. Let $v_f(0)$, $v_f(1)$ be the number of vertices of G having labels 0 and 1 respectively under f and let $e_f(0)$, $e_f(1)$ be the number of edges having labels 0 and 1 respectively under f^*.

Definition 6.2.3 A binary vertex labeling of a graph G is called a *cordial labeling* if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$.

A graph which admits cordial labeling is called a *cordial graph*.

Vast amount of literature is available in printed and in electronic form about cordial labeling. Some known families of cordial graphs are listed below.

• As investigated by Cahit[32]

◇ Every tree is cordial.

◇ Complete bipartite graphs $K_{m,n}$ are cordial.

◇ Complete graphs K_n are cordial if and only if $n \leq 3$.

◇ Maximal outer planar graphs are cordial.

◇ Eulerian graph is not cordial if its number of edges congruent to $2(mod4)$.

◇ All fans $F_n = P_n + K_1$ are cordial.

◇ Wheels $W_n = C_n + K_1$ are cordial if and only if n is not congruent to $3(mod4)$.

◇ k-angular cactus with t cycles is cordial if and only if kt is not congruent to $2(mod4)$.

• Ho et al.[70] proved that

◇ Unicyclic graph is cordial except C_{4k+2}.

◇ Generalized Petersen graph $P(n, k)$ is cordial if and only if n is not congruent to $2(mod4)$.

• Lee and Liu[86], Du[44] proved that complete n-partite graph is cordial if and only if at most three of its partite sets have odd cardinality.

• Seoud and Maqsoud[106] proved that if G is a graph with n vertices and m edges and every vertex has odd degree then G is not cordial when $m + n \equiv 2(mod4)$.

• Andar et al. in [7],[8],[9] and [10] proved that

◇ Multiple shells are cordial.

◇ t-ply graph $P_t(u, v)$ is cordial except when it is Eulerian and the number of edges is congruent to $2(mod 4)$.

◇ Helms, closed helms and generalized helms are cordial.

• In [10], Andar et al. showed that a cordial labeling g of a graph G can be extended to a cordial labeling of the graph obtained from G by attaching $2m$ pendant edges at each vertex of G. They also proved that a cordial labeling g of a graph G with p vertices can be extended to a cordial labeling of the graph obtained from G by attaching $2m + 1$ pendant edges at each vertex of G if and only if G does not satisfy either of the following conditions:

(1) G has an even number of edges and $p \equiv 2(mod 4)$.

(2) G has an odd number of edges and either $p \equiv 1(mod 4)$ with $e_g(1) = e_g(0) + i(G)$ or $p \equiv 3(mod 4)$ with $e_g(0) = e_g(1) + i(G)$, where $i(G) = min\{|e_g(0) - e_g(1)|\}$.

6.3 Cordial Labeling For Some Cycle Related Graphs

We have investigated some new families of cordial graphs. In this section we will give cordial labeling for cycle with one chord, cycle with twin chords and cycle with triangle. Before proving these results let us provide some important definitions.

Definition 6.3.1 A *chord* of a cycle C_n is an edge joining two non-adjacent vertices of cycle C_n.

Definition 6.3.2 Two chords of a cycle are said to be *twin chords* if they form a triangle with an edge of the cycle C_n.

For positive integers n and p with $3 \leq p \leq n - 2$, $C_{n,p}$ is the graph consisting of a cycle C_n with a pair of twin chords with which the edges of C_n form cycles C_p, C_3 and C_{n+1-p} without chords.

Definition 6.3.3 A *cycle with triangle* is a cycle with three chords which by themselves form a triangle.

For positive integers p, q, r and $n \geq 6$ with $p+q+r+3=n$, $C_n(p,q,r)$ denotes a cycle with triangle whose edges form the edges of cycles C_{p+2}, C_{q+2} and C_{r+2} without chords.

Theorem 6.3.4 Cycles with one chord are cordial.

Proof: Let $u_1, u_2, \ldots, u_n$ be consecutive vertices of cycle C_n and $e = u_1u_3$ be a chord of cycle C_n. The vertices u_1, u_2, u_3 forms a triangle with chord e. To define labeling function $f : V(G) \to \{0,1\}$ we consider following cases.

Case 1: $n \equiv 0, 1(mod 4)$

In this case we define labeling f as:

$f(u_i) = 0$; if $i \equiv 1, 2(mod 4)$

$= 1$; if $i \equiv 0, 3(mod 4)$, $1 \leq i \leq n$.

Case 2: $n \equiv 2(mod 4)$

In this case we define labeling f as:

$f(u_n) = 0$, $f(u_{n-1}) = 1$ and

$f(u_i) = 0$; if $i \equiv 1, 2(mod 4)$

$= 1$; if $i \equiv 0, 3(mod 4)$, $1 \leq i \leq n-2$.

Case 3: $n \equiv 3(mod 4)$

In this case we define labeling f as:

$f(u_1) = 1$ and

$f(u_i) = 0$; if $i \equiv 1, 2(mod 4)$

$= 1$; if $i \equiv 0, 3(mod 4)$, $2 \leq i \leq n$.

The labeling pattern defined above covers all possible arrangement of vertices. In each case, the graph G under consideration satisfies the conditions $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ as shown in following *Table 6.1.* i.e. G admits cordial labeling.

Let $n = 4a + b$, where $a \in N$.

b	vertex conditions	edge conditions
0	$v_f(0)=v_f(1)$	$e_f(0)+1=e_f(1)$
1	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$
2	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$
3	$v_f(0)+1=v_f(1)$	$e_f(0)=e_f(1)$

Table 6.1

Illustration 6.3.5 For better understanding of above defined labeling pattern let us consider cycle C_5 with one chord (it is related to Case-1). The labeling is shown in following *Figure 6.1.*

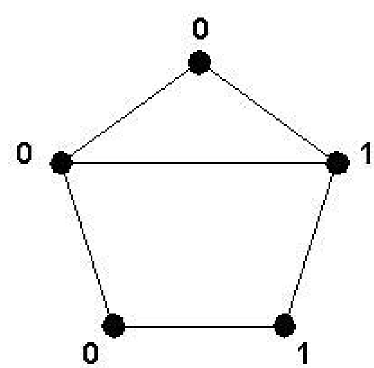


Figure 6.1

Theorem 6.3.6 Cycles with twin chords are cordial, where chords form two triangles and one cycle C_{n-2}.

Proof: Let G be the cycle with twin chords, where chords form two triangles and one cycle C_{n-2}. Here number of vertices $p = n$ and number of edges $q = n + 2$. Let $u_1, u_2, \ldots, u_n$ be successive vertices of G. Let $e_1 = u_n u_2$ and $e_2 = u_n u_3$ be the chords of cycle C_n. To define labeling function $f : V(G) \rightarrow \{0, 1\}$ we consider following cases.

Case 1: $n \equiv 0(mod4)$

In this case we define labeling f as:

$f(u_i) = 0$; if $i \equiv 1, 2(mod4)$

$\quad = 1$; if $i \equiv 0, 3(mod4)$, $1 \leq i \leq n$.

Case 2: $n \equiv 1, 2, 3(mod4)$

In this case we define labeling f as:

$f(u_i) = 0$; if $i \equiv 0, 1(mod4)$

$\quad = 1$; if $i \equiv 2, 3(mod4)$, $1 \leq i \leq n$.

The labeling pattern defined above covers all possible arrangement of vertices. In each case, the graph G under consideration satisfies the conditions $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ as shown in following *Table 6.2.* i.e. G admits cordial labeling.

Let $n = 4a + b$, where $n \in N$, $n \geq 5$.

b	vertex conditions	edge conditions
0,2	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)$
1	$v_f(0)=v_f(1)+1$	$e_f(0)+1=e_f(1)$
3	$v_f(0)+1=v_f(1)$	$e_f(0)+1=e_f(1)$

Table 6.2

Illustration 6.3.7 For better understanding of above defined labeling pattern let us consider cycle C_7 with twin chords (it is related to Case-2). The labeling is shown in following *Figure 6.2.*

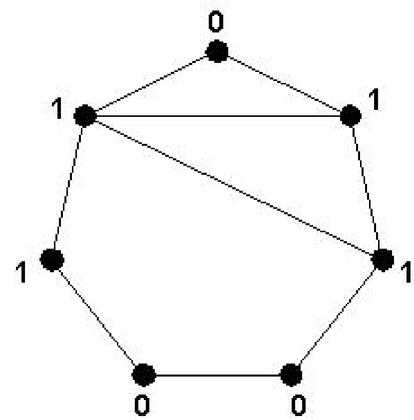


Figure 6.2

Theorem 6.3.8 Cycles with triangle $C_n(1, 1, n-5)$ is cordial except $n \equiv 3(mod4)$.

Proof: Let G be cycle with triangle $C_n(1, 1, n-5)$. Let $u_1, u_2, \ldots, u_n$ be successive vertices of G. Let u_1, u_3 and u_5 be the vertices of triangle formed by edges $e_1 = u_1u_3$, $e_2 = u_3u_5$ and $e_3 = u_1u_5$.

Note that for the case $n \equiv 3(mod4)$, graph G is an Eulerian graph with number of edges congruent to $2(mod4)$. Then in this case G is not cordial as proved by Cahit[32]. So it remains to consider following cases to define labeling function $f : V(G) \to \{0, 1\}$.

Case 1: $n \equiv 0, 1(mod4)$

In this case we define labeling f as:

$f(u_i) = 0$; if $i \equiv 1, 2(mod4)$

$\quad = 1$; if $i \equiv 0, 3(mod4)$, $1 \leq i \leq n$.

Case 2: $n \equiv 2(mod4)$

In this case we define labeling f as:

$f(u_n) = 0$, $f(u_{n-1}) = 1$ and

$f(u_i) = 0$; if $i \equiv 1, 2(mod4)$

$= 1$; if $i \equiv 0, 3(mod4)$, $1 \leq i \leq n-2$.

The labeling pattern defined above covers all possible arrangement of vertices. In each case, the graph G under consideration satisfies the conditions $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ as shown in following *Table 6.3.* i.e. G admits cordial labeling.

Let $n = 4a + b$, where $n \in N$, $n \geq 6$.

b	vertex conditions	edge conditions
0	$v_f(0)=v_f(1)$	$e_f(0)+1=e_f(1)$
1	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$
2	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$

Table 6.3

Illustration 6.3.9 For better understanding of above defined labeling pattern let us consider cycle C_6 with triangle (it is related to Case-2). The labeling is shown in following *Figure 6.3.*

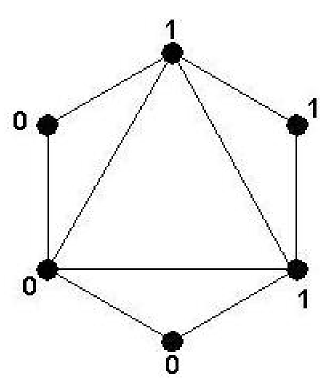


Figure 6.3

In the immediate section we will prove some more results.

6.4 Path Union of Graphs and Cordial Labeling

Definition 6.4.1 Let G be a graph and $G_1, G_2, \ldots, G_n$, $n \geq 2$ be n copies of graph G. Then the graph obtained by adding an edge from G_i to G_{i+1} (for $i = 1, 2, \ldots, n-1$) is called *path union* of G.

Shee and Ho[112] introduced above concept. They also proved that path union of Petersen graph, trees, wheels and unicyclic graphs are cordial.

We have investigated cordial labeling for path union of finite number of copies of cycle with chord, cycle with twin chords and cycle with triangle.

Theorem 6.4.2 The path union of finite number of copies of cycle C_n with one chord is cordial, where chord forms a triangle with edges of the cycle.

Proof: Let G be the path union of cycle C_n with one chord and $G_1, G_2, \ldots, G_k$ be k copies of cycle C_n with one chord, where $|G_i| = n$, for each i. Let us denote the consecutive vertices of graph G_i by $\{u_{i1}, u_{i2}, \ldots, u_{in}\}$, for $i = 1, 2, \ldots, k$. Let u_{i1}, u_{i2}, u_{i3} forms a triangle with chord e. Let $e_i = u_{i3}u_{(i+1)1}$ be the edge joining G_i and G_{i+1}, for $i = 1, 2, \ldots, k-1$. To define labeling function $f : V(G) \rightarrow \{0, 1\}$ we consider following cases.

<u>Case 1</u>: $n \equiv 0(mod4)$

In this case we define labeling as:

$f(u_{ij}) = 0$; if $j \equiv 0, 3(mod4)$

$\quad = 1$; if $j \equiv 1, 2(mod4)$, when i is even, $1 \leq i \leq k$, $1 \leq j \leq n$.

$f(u_{ij}) = 0$; if $j \equiv 1, 2(mod4)$

$\quad = 1$; if $j \equiv 0, 3(mod4)$, when i is odd, $1 \leq i \leq k$, $1 \leq j \leq n$.

Case 2: $n \equiv 1(mod4)$

In this case we define labeling as:

When $i \equiv 0, 1(mod4)$

$f(u_{ij}) = 0$; if $j \equiv 1, 2(mod4)$

$= 1$; if $j \equiv 0, 3(mod4)$, $1 \leq i \leq k$, $1 \leq j \leq n$.

When $i \equiv 2, 3(mod4)$

$f(u_{ij}) = 0$; if $j \equiv 0, 3(mod4)$

$= 1$; if $j \equiv 1, 2(mod4)$, $1 \leq i \leq k$, $1 \leq j \leq n$.

Case 3: $n \equiv 2(mod4)$

In this case we define labeling as:

$f(u_{in-1}) = 1$, $f(u_{in}) = 0$ and

$f(u_{ij}) = 0$; if $j \equiv 1, 2(mod4)$

$= 1$; if $j \equiv 0, 3(mod4)$, $1 \leq i \leq k$, $1 \leq j \leq n - 2$.

Case 4: $n \equiv 3(mod4)$

In this case we define labeling as:

When $i \equiv 0, 1(mod4)$

$f(u_{i1}) = 0$ and

$f(u_{ij}) = 0$; if $j \equiv 0, 3(mod4)$

$= 1$; if $j \equiv 1, 2(mod4)$, $1 \leq i \leq k$, $2 \leq j \leq n$.

When $i \equiv 2, 3(mod4)$

$f(u_{i1}) = 1$ and

$f(u_{ij}) = 0$; if $j \equiv 1, 2(mod4)$

$= 1$; if $j \equiv 0, 3(mod4)$, $1 \leq i \leq k$, $2 \leq j \leq n$.

The labeling pattern defined above covers all possible arrangement of vertices. In each case, the graph G under consideration satisfies the conditions

$|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ as shown in following *Table 6.4*. i.e. G admits cordial labeling.

Let $n = 4a + b$, $k = 4c + d$ where $n, k \in N$, $n \geq 4$.

b	d	vertex conditions	edge conditions
0	0,1,2,3	$v_f(0)=v_f(1)$	$e_f(0)+1=e_f(1)$
1	0,2	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$
	1	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$
	3	$v_f(0)+1=v_f(1)$	$e_f(0)=e_f(1)$
2	0,1,2,3	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$
3	0,2	$v_f(0)=v_f(1)$	$e_f(0)+1=e_f(1)$
	1	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$
	3	$v_f(0)+1=v_f(1)$	$e_f(0)=e_f(1)$

Table 6.4

Illustrations 6.4.3 For better understanding of above defined labeling pattern let us consider few examples.

Example 1 Consider a path union of three copies of cycle C_8 with one chord (it is the case related to $n \equiv 0(mod4)$, $k = 3$). The labeling pattern is shown in *Figure 6.4*.

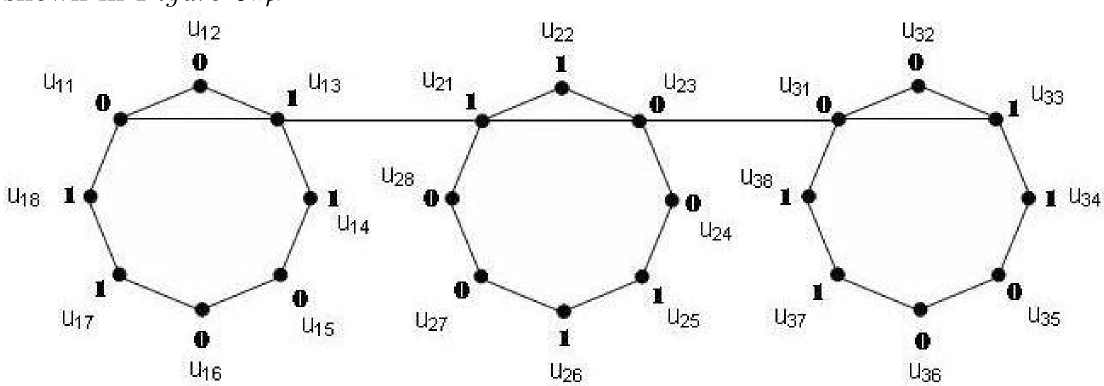


Figure 6.4

Example 2 Consider a path union of four copies of cycle C_5 with one chord (it is the case related to $n \equiv 1(mod4)$, $k = 4$). The labeling pattern is shown in *Figure 6.5*.

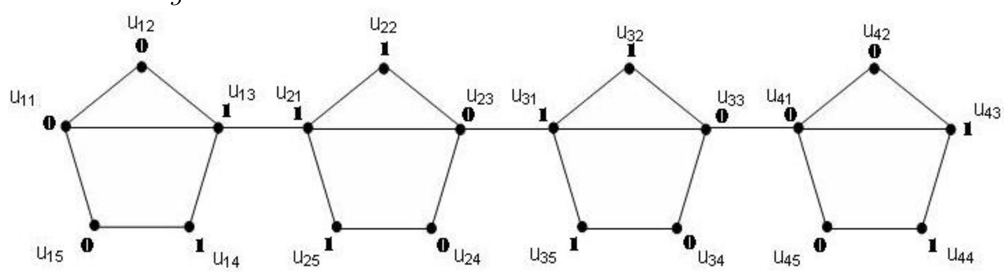


Figure 6.5

Example 3 Consider a path union of four copies of cycle C_6 with one chord (it is the case related to $n \equiv 2(mod4)$, $k = 4$). The labeling pattern is shown in *Figure 6.6*.

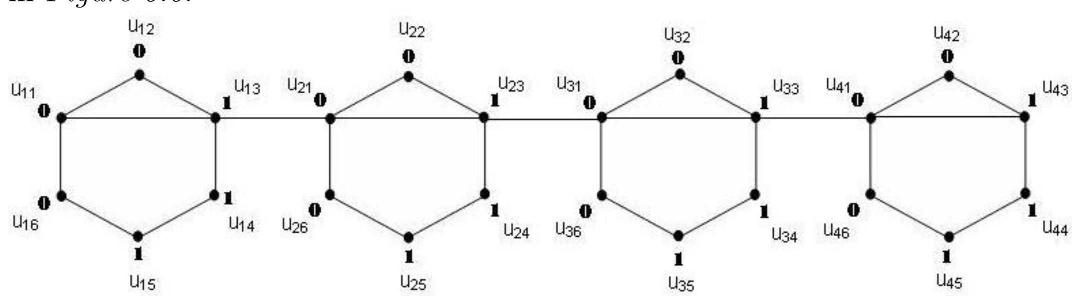


Figure 6.6

Theorem 6.4.4 The path union of finite number of copies of cycle C_n with twin chords is cordial.

Proof: Let G be the path union of cycle C_n with twin chords and $G_1, G_2, \ldots,$ G_k be k copies of cycle C_n with twin chords, where $|G_i| = n$, for each i. Let us denote the successive vertices of graph G_i by $\{u_{i1}, u_{i2}, \ldots u_{in}\}$, for $i = 1, 2, \ldots k$. Let $e_i = u_{i3}u_{(i+1)1}$ be the edge joining G_i and G_{i+1}, for

$i = 1, 2, \ldots k-1$. To define labeling function $f : V(G) \rightarrow \{0, 1\}$, we consider following cases.

Case 1: $n \equiv 0(mod4)$

In this case we define labeling as:

When $i \equiv 1, 2(mod4)$

$f(u_{ij}) = 0$; if $j \equiv 0, 1(mod4)$

$= 1$; if $j \equiv 2, 3(mod4)$,$1 \leq i \leq k$, $1 \leq j \leq n$.

When $i \equiv 0, 3(mod4)$

$f(u_{ij}) = 0$; if $j \equiv 2, 3(mod4)$

$= 1$; if $j \equiv 0, 1(mod4)$,$1 \leq i \leq k$, $1 \leq j \leq n$.

Case 2: $n \equiv 1(mod4)$

In this case we define labeling as:

When $i \equiv 0, 1(mod4)$

$f(u_{ij}) = 0$; if $j \equiv 0, 1(mod4)$

$= 1$; if $j \equiv 2, 3(mod4)$, $1 \leq i \leq k$, $1 \leq j \leq n$.

When $i \equiv 2, 3(mod4)$

$f(u_{ij}) = 0$; if $j \equiv 0, 3(mod4)$

$= 1$; if $j \equiv 1, 2(mod4)$, $1 \leq i \leq k$, $1 \leq j \leq n$.

Case 3: $n \equiv 2(mod4)$

In this case we define labeling as:

When $i \equiv 0, 1(mod4)$

$f(u_{i1}) = 0$, $f(u_{i2}) = 1$ and

$f(u_{ij}) = 0$; if $j \equiv 1, 2(mod4)$

$= 1$; if $j \equiv 0, 3(mod4)$, $1 \leq i \leq k$, $3 \leq j \leq n$.

When $i \equiv 2, 3(mod4)$

$f(u_{i1}) = 1$, $f(u_{i2}) = 0$ and

$f(u_{ij}) = 0$; if $j \equiv 0, 3(mod4)$

$= 1$; if $j \equiv 1, 2(mod4)$, $1 \leq i \leq k$, $3 \leq j \leq n$.

Case 4: $n \equiv 3(mod4)$

In this case we define labeling as:

$f(u_{ij}) = 0$; if $j \equiv 0, 1(mod4)$

$= 1$; if $j \equiv 2, 3(mod4)$, when i is odd, $1 \leq i \leq k$, $1 \leq j \leq n$.

$f(u_{ij}) = 0$; if $j \equiv 2, 3(mod4)$

$= 1$; if $j \equiv 0, 1(mod4)$, when i is even, $1 \leq i \leq k$, $1 \leq j \leq n$.

The labeling pattern defined above covers all possible arrangement of vertices. In each case, the graph G under consideration satisfies the conditions $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ as shown in following *Table 6.5* i.e. G admits cordial labeling.

Let $n = 4a + b$, $k = 4c + d$, where $n, k \in N$, $n \geq 5$.

b	d	vertex conditions	edge conditions
0	0,2	$v_f(0)=v_f(1)$	$e_f(0)+1=e_f(1)$
	1,3	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)$
1	0,2	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$
	1	$v_f(0)+1=v_f(1)$	$e_f(0)=e_f(1)+1$
	3	$v_f(0)+1=v_f(1)$	$e_f(0)+1=e_f(1)$
2	0,2	$v_f(0)=v_f(1)$	$e_f(0)+1=e_f(1)$
	1,3	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)$
3	0,2	$v_f(0)=v_f(1)$	$e_f(0)+1=e_f(1)$
	1,3	$v_f(0)+1=v_f(1)$	$e_f(0)+1=e_f(1)$

Table 6.5

Illustrations 6.4.5 For better understanding of above defined labeling pattern let us consider few examples.

Example 1 Consider a path union of three copies of cycle C_5 with twin chords(it is the case related to $n \equiv 1(mod4)$, $k = 3$). The labeling pattern is shown in *Figure 6.7*.

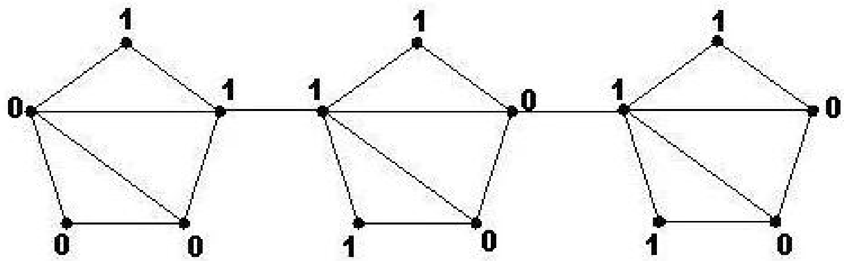


Figure 6.7

Example 2 Consider a path union of four copies of cycle C_6 with twin chords(it is the case related to $n \equiv 2(mod4)$, $k = 4$). The labeling pattern is shown in *Figure 6.8*.

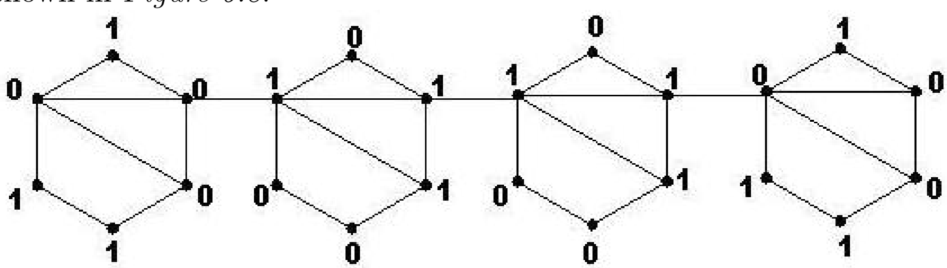


Figure 6.8

Theorem 6.4.6 The path union of finite number of copies of cycle with triangle $C_n(1, 1, n-5)$ is cordial.

Proof: Let G be the path union of cycle C_n with triangle and $G_1, G_2, \ldots G_k$ be k copies of cycle C_n with triangle, where $|G_i| = n$, for each i. Let us denote the successive vertices of graph G_i by $\{u_{i1}, u_{i2}, \ldots u_{in}\}$, for $i = 1, 2, \ldots k$.

Let $e_i = u_{i4}u_{(i+1)2}$ be the edge joining G_i and G_{i+1}, for $i = 1, 2, \ldots k-1$. To define labeling function $f : V(G) \rightarrow \{0, 1\}$ we consider following cases.

Case 1: $n \equiv 0(mod4)$

In this case we define labeling as:

$f(u_{ij}) = 0$; if $j \equiv 0, 3(mod4)$

$= 1$; if $j \equiv 1, 2(mod4)$, when i is even, $1 \leq i \leq k$, $1 \leq j \leq n$.

$f(u_{ij}) = 0$; if $j \equiv 1, 2(mod4)$

$= 1$; if $j \equiv 0, 3(mod4)$, when i is odd, $1 \leq i \leq k$, $1 \leq j \leq n$.

Case 2: $n \equiv 1(mod4)$

In this case we define labeling as:

When $i \equiv 0, 1(mod4)$

$f(u_{ij}) = 0$; if $j \equiv 1, 2(mod4)$

$= 1$; if $j \equiv 0, 3(mod4)$, $1 \leq i \leq k$, $1 \leq j \leq n$.

When $i \equiv 2, 3(mod4)$

$f(u_{ij}) = 0$; if $j \equiv 0, 3(mod4)$

$= 1$; if $j \equiv 1, 2(mod4)$, $1 \leq i \leq k$, $1 \leq j \leq n$.

Case 3: $n \equiv 2(mod4)$

In this case we define labeling as:

$f(u_{in-1}) = 1$, $f(u_{in}) = 0$ and

$f(u_{ij}) = 0$; if $j \equiv 0, 3(mod4)$

$= 1$; if $j \equiv 1, 2(mod4)$, $1 \leq i \leq k$, $1 \leq j \leq n-2$.

Case 4: $n \equiv 3(mod4)$

In this case we define labeling as:

When G has even number of copies,

For $i \equiv 0(mod4)$

$f(u_{i1}) = 1, f(u_{i2}) = 0, f(u_{i3}) = 1$ and

$f(u_{ij}) = 0$; if $j \equiv 2, 3(mod 4)$

$= 1$; if $j \equiv 0, 1(mod 4)$, $1 \leq i \leq k$, $4 \leq j \leq n$.

For $i \equiv 1(mod 4)$

$f(u_{ij}) = 0$; if $j \equiv 0, 1(mod 4)$

$= 1$; if $j \equiv 2, 3(mod 4)$, $1 \leq i \leq k$, $1 \leq j \leq n$.

For $i \equiv 2(mod 4)$

$f(u_{i1}) = 0, f(u_{i2}) = 1, f(u_{i3}) = 0$ and

$f(u_{ij}) = 0$; if $j \equiv 0, 1(mod 4)$

$= 1$; if $j \equiv 2, 3(mod 4)$, $1 \leq i \leq k$, $4 \leq j \leq n$.

For $i \equiv 3(mod 4)$

$f(u_{ij}) = 0$; if $j \equiv 2, 3(mod 4)$

$= 1$; if $j \equiv 0, 1(mod 4)$, $1 \leq i \leq k$, $1 \leq j \leq n$.

When G has odd number of copies,

For $G = G_1$,

$f(u_{ij}) = 0$; if $j \equiv 0, 1(mod 4)$

$= 1$; if $j \equiv 2, 3(mod 4)$, $1 \leq i \leq k$, $1 \leq j \leq n$.

For $G = G_2 = G_3$,

$f(u_{i1}) = 0, f(u_{i2}) = 1, f(u_{i3}) = 0$ and

$f(u_{ij}) = 0$; if $j \equiv 0, 1(mod 4)$

$= 1$; if $j \equiv 2, 3(mod 4)$, $1 \leq i \leq k$, $4 \leq j \leq n$.

We define labeling pattern for remaining copies as:

For $i \equiv 0(mod 4)$

$f(u_{ij}) = 0$; if $j \equiv 2, 3(mod 4)$

$= 1$; if $j \equiv 0, 1(mod 4)$, $4 \leq i \leq k$, $1 \leq j \leq n$.

For $i \equiv 1(mod4)$

$f(u_{i1}) = 1, f(u_{i2}) = 0,\ f(u_{i3}) = 1$

$f(u_{ij}) = 0$; if $j \equiv 2, 3(mod4)$

$= 1$; if $j \equiv 0, 1(mod4)$, $4 \leq i \leq k$, $4 \leq j \leq n$.

For $i \equiv 2(mod4)$

$f(u_{ij}) = 0$; if $j \equiv 0, 1(mod4)$

$= 1$; if $j \equiv 2, 3(mod4)$, $4 \leq i \leq k$, $1 \leq j \leq n$.

For $i \equiv 3(mod4)$

$f(u_{i1}) = 0, f(u_{i2}) = 1,\ f(u_{i3}) = 0$

$f(u_{ij}) = 0$; if $j \equiv 1, 0(mod4)$

$= 1$; if $j \equiv 2, 3(mod4)$, $4 \leq i \leq k$, $4 \leq j \leq n$.

The labeling pattern defined above covers all possible arrangement of vertices. In each case, the graph G under consideration satisfies the conditions $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ as shown in following *Table 6.6*. i.e. G admits cordial labeling.

Let $n = 4a + b$, $k = 4c + d$, where $n, k \in N$, $n \geq 6$.

b	d	vertex conditions	edge conditions
0	0,1,2,3	$v_f(0)=v_f(1)$	$e_f(0)+1=e_f(1)$
1	0,2	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$
	1	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$
	3	$v_f(0)+1=v_f(1)$	$e_f(0)=e_f(1)$
2	0,1,2,3	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$
3	0,2	$v_f(0)+1=v_f(1)$	$e_f(0)=e_f(1)$
	1,3	$v_f(0)+1=v_f(1)$	$e_f(0)+1=e_f(1)$

Table 6.6

Illustration 6.4.7 For better understanding of above defined labeling pattern let us consider path union of four copies of cycle C_6 with triangle (it is the case related to $n \equiv 2(mod4)$, $k = 4$). The labeling pattern is shown in *Figure 6.9*.

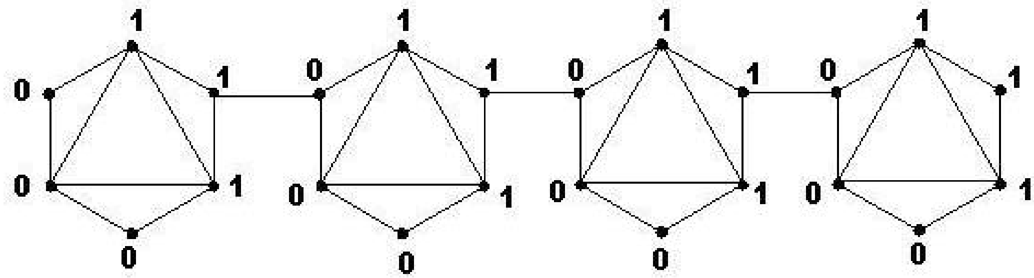


Figure 6.9

¶ **Remark 6.4.8**

In *Theorems 6.4.2* to *6.4.6* we consider edges between end vertices of chord but it is also possible to discuss cordiality when edges are attached to other vertices.

In the next section some more cordial graphs are investigated.

6.5 Some More Cordial Graphs

We introduce a graph called *star of a graph* as follows.

Definition 6.5.1 A graph obtained by replacing each vertex of star graph $K_{1,n}$ by a graph G is called *star of* G. We denote it as G^*. We name *central graph* in G^* is the graph which replaces central vertex of graph $K_{1,n}$.

We have investigated cordial labeling for star of cycle, star of wheel and star of Petersen graph.

Theorem 6.5.2 Star of cycle C_n^* is cordial for all n.

Proof: Let $v_1, v_2, \dots v_n$ be successive vertices of central cycle of C_n^* and $u_{i1}, u_{i2}, \dots u_{in}$ be successive vertices of other cycles $C_n^{(i)}$(except central cycle), $i = 1, 2 \dots n$. Let e_i be the edge such that $e_i = u_{i1}v_i$. Moreover, let us denote the vertex of cycle $C_n^{(i)}$ which is adjacent to a vertex v_i labeled by 0 as $u_{ij}^{(0)}$ and similarly denote the vertex of cycle $C_n^{(i)}$ which is adjacent to a vertex v_i labeled by 1 as $u_{ij}^{(1)}$. To define required labeling $f : V(C_n^*) \to \{0, 1\}$ we consider following cases.

<u>Case 1</u>: $n \equiv 0(mod4)$

In this case define labeling f as

$f(v_i) = 0$; if $i \equiv 0, 1(mod4)$

$\quad = 1$; if $i \equiv 2, 3(mod4)$, $1 \leq i \leq n$.

$f(u_{ij}^{(0)}) = 0$; if $j \equiv 0, 3(mod4)$

$\quad = 1$; if $j \equiv 1, 2(mod4)$, $1 \leq j \leq n$, $1 \leq i \leq n$.

$f(u_{ij}^{(1)}) = 0$; if $j \equiv 2, 3(mod4)$

$\quad = 1$; if $j \equiv 0, 1(mod4)$, $1 \leq j \leq n$, $1 \leq i \leq n$.

<u>Case 2</u>: $n \equiv 1(mod4)$

In this case define labeling f as

$f(v_i) = 0$; if $i \equiv 0, 1(mod4)$

$\quad = 1$; if $i \equiv 2, 3(mod4)$, $1 \leq i \leq n$.

$f(u_{ij}^{(0)}) = 0$; if $j \equiv 0, 3(mod4)$

$\quad = 1$; if $j \equiv 1, 2(mod4)$, $1 \leq j \leq n$, $1 \leq i \leq n$.

$f(u_{ij}^{(1)}) = 0$; if $j \equiv 1, 2(mod4)$

$\quad = 1$; if $j \equiv 0, 3(mod4)$, $1 \leq j \leq n$, $1 \leq i \leq n$.

Case 3: $n \equiv 2(mod4)$

In this case define labeling f as

$f(v_i) = 0$; if $i \equiv 0, 2(mod4)$

$\quad = 1$; if $i \equiv 1, 3(mod4)$, $1 \le i \le n$.

$f(u_{ij}^{(0)}) = 0$; if $j \equiv 0, 1(mod4)$

$\quad = 1$; if $j \equiv 2, 3(mod4)$, $1 \le j \le n$, $1 \le i \le n$.

$f(u_{in}^{(1)}) = 1$, $f(u_{in-1}^{(1)}) = 0$ and

$f(u_{ij}^{(1)}) = 0$; if $j \equiv 0, 3(mod4)$

$\quad = 1$; if $j \equiv 1, 2(mod4)$, $1 \le j \le n-2$, $1 \le i \le n$.

Case 4: $n \equiv 3(mod4)$

In this case define labeling f as

$f(v_i) = 0$; if $i \equiv 0, 1(mod4)$

$\quad = 1$; if $i \equiv 2, 3(mod4)$, $1 \le i \le n$.

$f(u_{ij}^{(0)}) = 0$; if $j \equiv 0, 1(mod4)$

$\quad = 1$; if $j \equiv 2, 3(mod4)$, $1 \le j \le n$, $1 \le i \le n$.

$f(u_{ij}^{(1)}) = 0$; if $j \equiv 2, 3(mod4)$

$\quad = 1$; if $j \equiv 0, 1(mod4)$, $1 \le j \le n$, $1 \le i \le n$.

The graph under consideration satisfies the condition $|v_f(0) - v_f(1)| \le 1$ and $|e_f(0) - e_f(1)| \le 1$ in each case as shown in following *Table 6.7.* i.e. C_n^* admits cordial labeling.

Let $n = 4a + b$, where $n \in N$, $n \geq 3$

b	vertex conditions	edge conditions
0	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)$
1	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$
2	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)$
3	$v_f(0)=v_f(1)$	$e_f(0)+1=e_f(1)$

Table 6.7

Illustration 6.5.3 For better understanding of above defined labeling pattern let us consider C_6^* (it is related to Case-3). The cordial labeling of C_6^* is as shown in *Figure 6.10*.

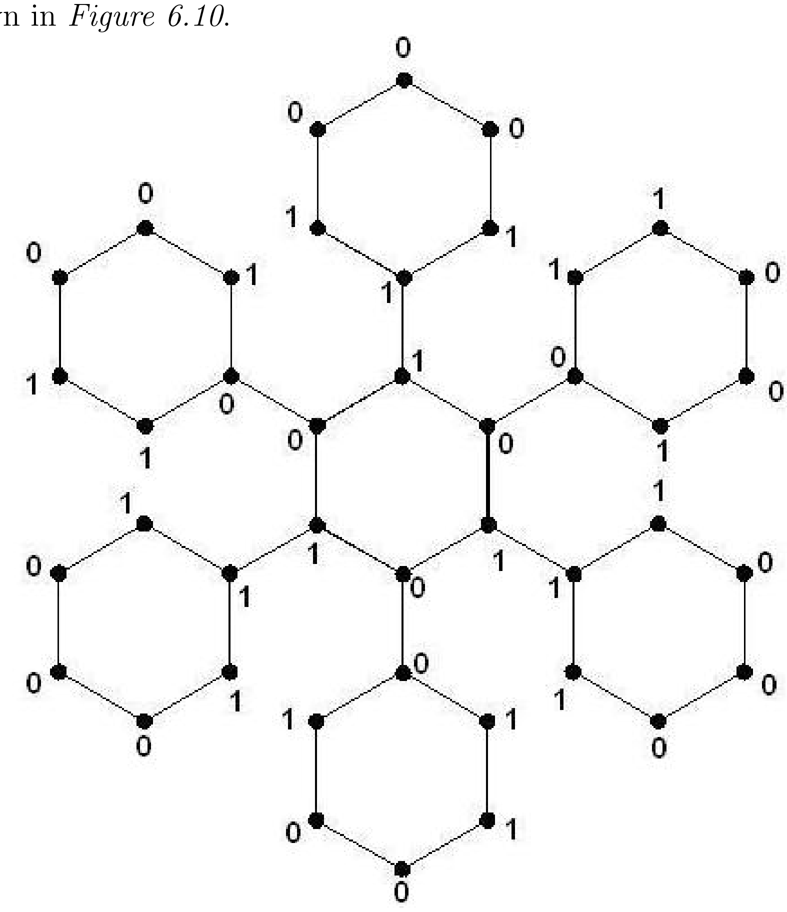


Figure 6.10

Theorem 6.5.4 Star of wheel W_n^* is cordial for all n.

Proof: Let $v_1, v_2, \ldots, v_n$ be successive vertices of central wheel and let v_0 be the apex vertex of central wheel of W_n^*. Further let $u_{i1}, u_{i2}, \ldots, u_{in}$ be successive vertices of wheel $W_n^{(i)}$ (except central wheel), $i = 1, 2, \ldots, n$ and let u_{i0} be apex vertex of wheel $W_n^{(i)}$ (except central wheel), $i = 1, 2, \ldots, n$. Let e_i be the edge such that $e_i = u_{i1}v_i$, $i = 1, 2, \ldots, n$.

Moreover, let us denote the vertex of wheel $W_n^{(i)}$ which is adjacent to a vertex v_i labeled by 0 as $u_{ij}^{(0)}$ and similarly denote the vertex of wheel $W_n^{(i)}$ which is adjacent to a vertex v_i labeled by 1 as $u_{ij}^{(1)}$.

To define required labeling $f : V(W_n^*) \rightarrow \{0, 1\}$ we consider following cases.

Case-1: $n \equiv 0(mod 4)$

In this case define labeling f as

$f(v_0) = 0$ and

$f(v_i) = 0$; if $i \equiv 1, 2(mod 4)$

$= 1$; if $i \equiv 0, 3(mod 4)$, $1 \leq i \leq n$

$f(u_{i0}^{(0)}) = 1$ and

$f(u_{ij}^{(0)}) = 0$; if $j \equiv 0, 3(mod 4)$

$= 1$; if $j \equiv 1, 2(mod 4)$, $1 \leq j \leq n$ and for fixed i, $1 \leq i \leq n$

$f(u_{i0}^{(1)}) = 0$ and

$f(u_{ij}^{(1)}) = 0$; if $j \equiv 0, 3(mod 4)$

$= 1$; if $j \equiv 1, 2(mod 4)$, $1 \leq j \leq n$ and for fixed i, $1 \leq i \leq n$

Case-2: $n \equiv 1(mod 4)$

In this case define labeling f as

$f(v_0) = 1$ and

$f(v_i) = 0$; if $i \equiv 1, 2(mod4)$

$= 1$; if $i \equiv 0, 3(mod4)$, $1 \leq i \leq n$

$f(u_{i0}^{(0)}) = 0$ and

$f(u_{ij}^{(0)}) = 0$; if $j \equiv 0, 3(mod4)$

$= 1$; if $j \equiv 1, 2(mod4)$, $1 \leq j \leq n$ and for fixed i, $1 \leq i \leq n$

$f(u_{i0}^{(1)}) = 0$ and

$f(u_{ij}^{(1)}) = 0$; if $j \equiv 0, 3(mod4)$

$= 1$; if $j \equiv 1, 2(mod4)$, $1 \leq j \leq n$ and for fixed i, $1 \leq i \leq n$

Case-3: $n \equiv 2(mod4)$

In this case define labeling f as

$f(v_0) = 0$ and

$f(v_i) = 0$; if i is odd,

$= 1$; if i is even, $1 \leq i \leq n$

$f(u_{i0}^{(0)}) = 0$, $f(u_{in}^{(0)}) = 0$, $f(u_{in-1}^{(0)}) = 1$ and

$f(u_{ij}^{(0)}) = 0$; if $j \equiv 1, 2(mod4)$

$= 1$; if $j \equiv 0, 3(mod4)$, $1 \leq j \leq n - 2$ and for fixed i, $1 \leq i \leq n$

$f(u_{i0}^{(1)}) = 1$, $f(u_{in}^{(1)}) = 1$, $f(u_{in-1}^{(1)}) = 0$ and

$f(u_{ij}^{(1)}) = 0$; if $j \equiv 0, 3(mod4)$

$= 1$; if $j \equiv 1, 2(mod4)$, $1 \leq j \leq n - 2$ and for fixed i, $1 \leq i \leq n$

Case-4: $n \equiv 3(mod4)$

In this case define labeling f as

$f(v_0) = 1$ and

$f(v_i) = 0$; if $i \equiv 0, 3(mod4)$

$= 1$; if $i \equiv 1, 2(mod4)$, $1 \leq i \leq n$

$f(u_{i0}^{(0)}) = 1$ and

$f(u_{ij}^{(0)}) = 0$; if $j \equiv 0, 3(mod4)$

$= 1$; if $j \equiv 1, 2(mod4)$, $1 \leq j \leq n$ and for fixed i, $1 \leq i \leq n$

$f(u_{i0}^{(1)}) = 0$ and

$f(u_{ij}^{(1)}) = 0$; if $j \equiv 2, 3(mod4)$

$= 1$; if $j \equiv 0, 1(mod4)$, $1 \leq j \leq n$ and for fixed i, $1 \leq i \leq n$

The graph under consideration satisfies the condition $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ in each case as shown in following *Table 6.8*. i.e. Graph W_n^* admits cordial labeling.

Let $n = 4a + b$, where $n \in N, n \geq 3$

b	vertex conditions	edge conditions
0	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$
1	$v_f(0)=v_f(1)$	$e_f(0)+1=e_f(1)$
2	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$
3	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$

Table 6.8

Illustration 6.5.5 Let us demonstrate the above defined labeling pattern by means of following example.

Consider W_5^* (it is the case related to $n \equiv 1(mod4)$). The cordial labeling of W_5^* is as shown in *Figure 6.11*.

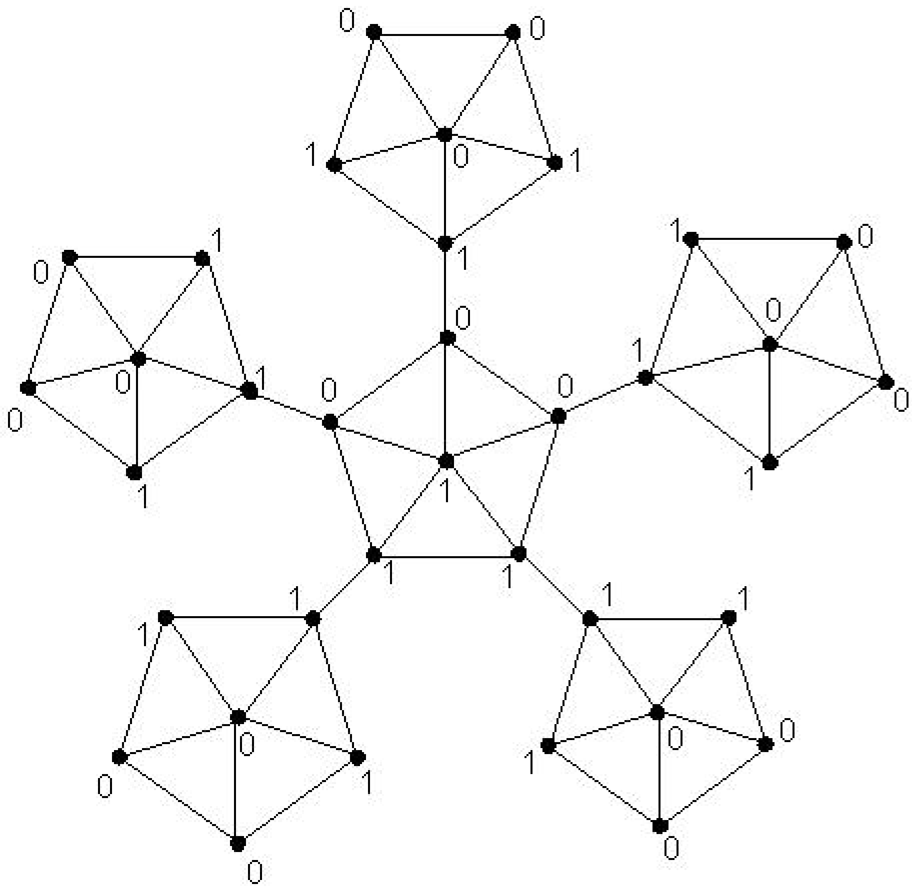


Figure 6.11

Theorem 6.5.6 Star of Petersen graph is cordial.

Proof: Let v_1, v_2, v_3, v_4, v_5 be external vertices and $v_6, v_7, v_8, v_9, v_{10}$ be internal vertices of central Petersen graph such that v_i is adjacent to v_{i+5}, $i = 1, 2, 3, 4, 5$. Let $u_{i1}, u_{i2}, u_{i3}, u_{i4}, u_{i5}$ be external vertices and $u_{i6}, u_{i7}, u_{i8}, u_{i9}, u_{i10}$ be internal vertices of i^{th} copy of Petersen graph (except central Petersen graph). Let e_i be the edge such that $e_i = u_{i1}v_i$, for $i = 1, 2, 3, 4, 5$.

Now define labeling $f : V(G) \to \{0, 1\}$ as follows.

$f(v_i) = 0$; if $i = 3, 4, 8, 9, 10$

$= 1$; if $i = 1, 2, 5, 6, 7$.

$f(u_{ij}) = 0$; if $j = 1, 4, 5, 9, 10$

$= 1$; if $j = 2, 3, 6, 7, 8$, for $i = 1, 2, 3, 4, 5$.

Above defined labeling pattern is shown in following *Figure 6.12*.

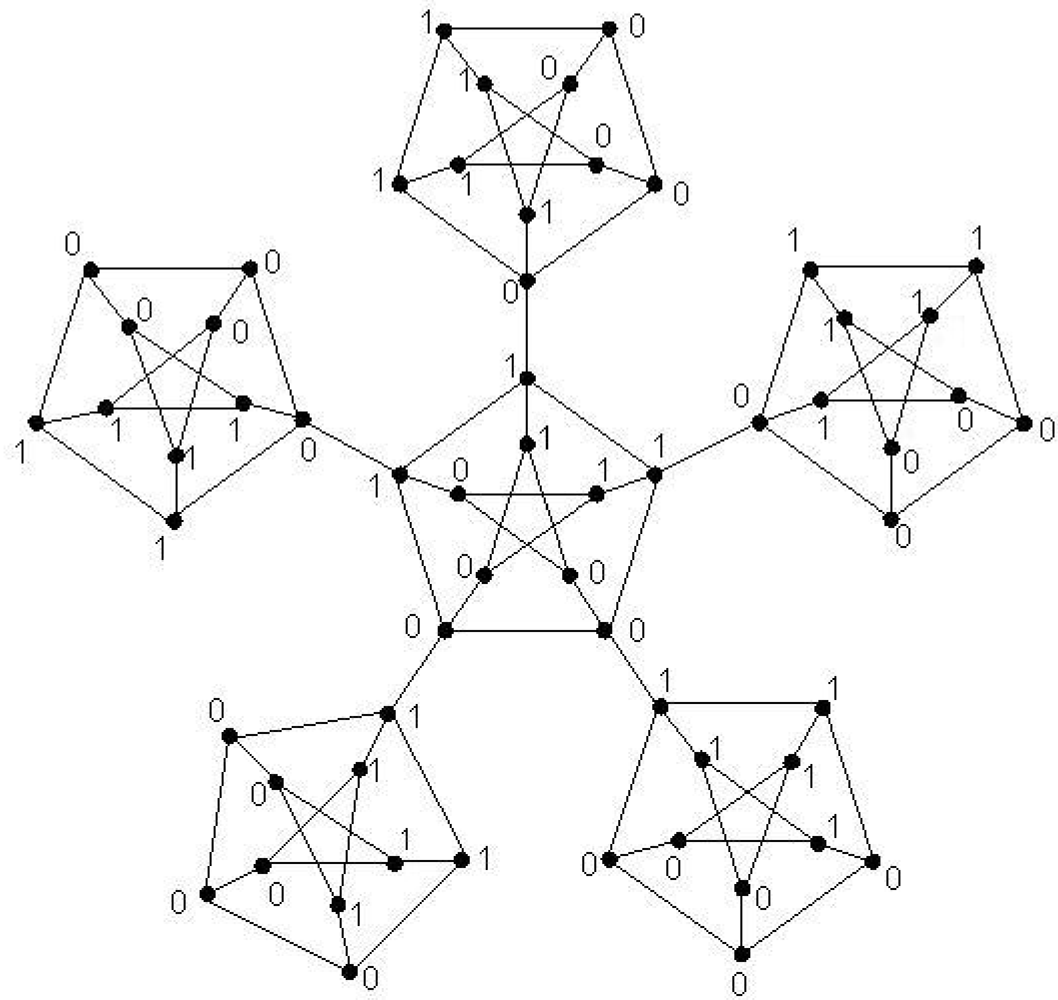


Figure 6.12

From above *Figure 6.12* one can easily see that $v_f(0) = v_f(1)$ and $e_f(0) = e_f(1)+1$. Thus star of Petersen graph is cordial as it admits cordial labeling.

In the next two results we will discuss cordiality of the graphs obtained by attachment of a path of arbitrary length between two copies of wheels and Petersen graph respectively.

Theorem 6.5.7 The graph obtained by joining two copies of wheel by a path of arbitrary length is cordial.

Proof: Let G be the graph obtained by joining two copies of wheel by a path of arbitrary length. Let $u_1, u_2, \ldots, u_n$ be successive vertices of first copy of wheel W_n and $w_1, w_2, \ldots, w_n$ be successive vertices of second copy of wheel W_n. Let u_0 and w_0 be apex vertices of first and second copy of wheel respectively. Let $v_1, v_2, \ldots, v_k$ be vertices of path P_k with $v_1 = u_1$ and $v_k = w_1$. To define required labeling $f : V(G) \rightarrow \{0, 1\}$ we consider following cases.

Case 1: $n \equiv 0(mod4)$, $k \equiv 0(mod4)$; $n \equiv 1(mod4)$, $k \equiv 0(mod4)$;

$n \equiv 1(mod4)$, $k \equiv 3(mod4)$; $n \equiv 2(mod4)$, $k \equiv 0(mod4)$;

$n \equiv 2(mod4)$, $k \equiv 3(mod4)$

In this case define labeling f as

$f(u_0) = 0$ and

$f(u_i) = 0$; if $i \equiv 0, 3(mod4)$

$= 1$; if $i \equiv 1, 2(mod4)$; $1 \leq i \leq n$

$f(v_j) = 0$; if $j \equiv 0, 3(mod4)$

$= 1$; if $j \equiv 1, 2(mod4)$; $1 \leq j \leq k$

$f(w_0) = 1$ and

$f(w_i) = 0$; if $i \equiv 1, 2(mod4)$

$= 1$; if $i \equiv 0, 3(mod4)$; $1 \leq i \leq n$

Case 2: $n \equiv 0(mod4)$, $k \equiv 1,2(mod4)$

In this case define labeling f as

$f(u_0) = 0$ and

$f(u_i) = 0$; if $i \equiv 0,3(mod4)$

$= 1$; if $i \equiv 1,2(mod4)$; $1 \leq i \leq n$

$f(v_j) = 0$; if $j \equiv 0,3(mod4)$

$= 1$; if $j \equiv 1,2(mod4)$; $1 \leq j \leq k$

$f(w_0) = 1$ and

$f(w_i) = 0$; if $i \equiv 0,3(mod4)$

$= 1$; if $i \equiv 1,2(mod4)$; $1 \leq i \leq n$

Case 3: $n \equiv 0(mod4)$, $k \equiv 3(mod4)$, $n \equiv 1(mod4)$, $k \equiv 2(mod4)$

In this case define labeling f as

$f(u_0) = 0$ and

$f(u_i) = 0$; if $i \equiv 0,3(mod4)$

$= 1$; if $i \equiv 1,2(mod4)$; $1 \leq i \leq n$

$f(v_j) = 0$; if $j \equiv 2,3(mod4)$

$= 1$; if $j \equiv 0,1(mod4)$; $1 \leq j \leq k$

$f(w_0) = 1$ and

$f(w_i) = 0$; if $i \equiv 1,2(mod4)$

$= 1$; if $i \equiv 0,3(mod4)$; $1 \leq i \leq n$

Case 4: $n \equiv 1(mod4)$, $k \equiv 1(mod4)$

In this case define labeling f as

$f(u_0) = 0$ and

$f(u_i) = 0$; if $i \equiv 0,3(mod4)$

$= 1$; if $i \equiv 1,2(mod4)$; $1 \leq i \leq n$

$f(v_j) = 0$; if $j \equiv 0, 3(mod4)$

$= 1$; if $j \equiv 1, 2(mod4)$; $1 \leq j \leq k$

$f(w_0) = 0$ and

$f(w_i) = 0$; if $i \equiv 0, 3(mod4)$

$= 1$; if $i \equiv 1, 2(mod4)$; $1 \leq i \leq n$

<u>Case 5</u>: $n \equiv 2(mod4)$, $k \equiv 1(mod4)$

In this case define labeling f as

$f(u_0) = 0$ and

$f(u_i) = 0$; if $i \equiv 0, 3(mod4)$

$= 1$; if $i \equiv 1, 2(mod4)$; $1 \leq i \leq n$

$f(v_k) = 0$ and

$f(v_j) = 0$; if $j \equiv 0, 3(mod4)$

$= 1$; if $j \equiv 1, 2(mod4)$; $1 \leq j \leq k-1$

$f(w_0) = 0$ and

$f(w_i) = 0$; if $i \equiv 0, 1(mod4)$

$= 1$; if $i \equiv 2, 3(mod4)$; $1 \leq i \leq n$

<u>Case 6</u>: $n \equiv 2(mod4)$, $k \equiv 2(mod4)$

In this case define labeling f as

$f(u_0) = 0$ and

$f(u_i) = 0$; if $i \equiv 0, 3(mod4)$

$= 1$; if $i \equiv 1, 2(mod4)$; $1 \leq i \leq n$

$f(v_j) = 0$; if $j \equiv 2, 3(mod4)$

$= 1$; if $j \equiv 0, 1(mod4)$; $1 \leq j \leq k$

$f(w_0) = 1$ and

$f(w_i) = 0$; if $i \equiv 1, 2(mod4)$

$= 1$; if $i \equiv 0, 3(mod4)$; $1 \leq i \leq n$

Case 7: $n \equiv 3(mod4)$, $k \equiv 0(mod4)$, $n \equiv 3(mod4)$, $k \equiv 3(mod4)$

In this case define labeling f as

$f(u_0) = 1$ and

$f(u_i) = 0$; if $i \equiv 0, 3(mod4)$

$= 1$; if $i \equiv 1, 2(mod4)$; $1 \leq i \leq n$

$f(v_j) = 0$; if $j \equiv 0, 3(mod4)$

$= 1$; if $j \equiv 1, 2(mod4)$; $1 \leq j \leq k$

$f(w_0) = 0$ and

$f(w_i) = 0$; if $i \equiv 1, 2(mod4)$

$= 1$; if $i \equiv 0, 3(mod4)$; $1 \leq i \leq n$

Case 8: $n \equiv 3(mod4)$, $k \equiv 1(mod4)$, $n \equiv 3(mod4)$, $k \equiv 2(mod4)$

In this case define labeling f as

$f(u_0) = 1$ and

$f(u_i) = 0$; if $i \equiv 0, 3(mod4)$

$= 1$; if $i \equiv 1, 2(mod4)$; $1 \leq i \leq n$

$f(v_j) = 0$; if $j \equiv 0, 3(mod4)$

$= 1$; if $j \equiv 1, 2(mod4)$; $1 \leq j \leq k$

$f(w_0) = 0$ and

$f(w_i) = 0$; if $i \equiv 2, 3(mod4)$

$= 1$; if $i \equiv 0, 1(mod4)$; $1 \leq i \leq n$

The graph G under consideration satisfies the condition $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ in each case as shown in following *Table 6.9.* i.e. G admits cordial labeling.

Let $n = 4a + b$, $k = 4c + d$, where $n, k \in N$, $n \geq 3$.

b	d	vertex conditions	edge conditions
0	0	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$
	1	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$
	2	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$
	3	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$
1	0	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$
	1	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$
	2	$v_f(0)=v_f(1)$	$e_f(0)+1=e_f(1)$
	3	$v_f(0)+1=v_f(1)$	$e_f(0)=e_f(1)$
2	0	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$
	1	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$
	2	$v_f(0)=v_f(1)$	$e_f(0)+1=e_f(1)$
	3	$v_f(0)+1=v_f(1)$	$e_f(0)=e_f(1)$
3	0	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$
	1	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$
	2	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$
	3	$v_f(0)+1=v_f(1)$	$e_f(0)=e_f(1)$

Table 6.9

Illustration 6.5.8 For better understanding of above defined labeling pattern consider the graph G obtained by joining two copies of wheel W_6 by path P_7 (it is the case related to Case-1). The cordial labeling pattern is shown in following *Figure 6.13*.

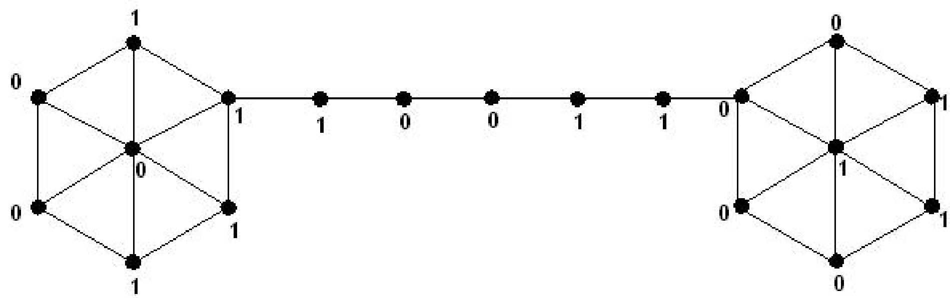


Figure 6.13

Theorem 6.5.9 The graph obtained by joining two copies of Petersen graph by a path of arbitrary length is cordial.

Proof: Let G be the graph obtained by joining two copies of Petersen graph by a path of arbitrary length. Let $u_1, u_2, \ldots, u_5$ and $u_6, u_7, \ldots, u_{10}$ respectively be external and internal vertices of first copy of Petersen graph. Here each u_i is adjacent to u_{i+5}, $i = 1, 2, 3, 4, 5$. Similarly let $w_1, w_2, \ldots, w_5$ and $w_6, w_7, \ldots, w_{10}$ respectively be external and internal vertices of second copy of Petersen graph. Here each w_i is adjacent to w_{i+5}, $i = 1, 2, 3, 4, 5$. Let $v_1, v_2 \ldots v_k$ be successive vertices of path P_k with $v_1 = u_1$ and $v_k = w_1$.

To define required labeling $f : V(G) \rightarrow \{0, 1\}$ we consider following cases.

Case 1: $k \equiv 0(mod4)$

In this case, define labeling f as

$f(u_i) = 0$; if $i = 3, 4, 7, 10$

$\quad = 1$; if $i = 1, 2, 5, 6, 8, 9$

$f(v_j) = 0$; if $j \equiv 0, 3(mod4)$

$\quad = 1$; if $j \equiv 1, 2(mod4)$; $1 \leq j \leq k$

$f(w_i) = 0$; if $i = 1, 2, 5, 6, 8, 9$

$\quad = 1$; if $i = 3, 4, 7, 10$

Case 2: $k \equiv 1(mod4)$

In this case, define labeling f as

$f(u_i) = 0$; if $i = 3, 4, 7, 8, 9$

$= 1$; if $i = 1, 2, 5, 6, 10$

$f(v_j) = 0$; if $j \equiv 2, 3(mod4)$

$= 1$; if $j \equiv 0, 1(mod4)$; $1 \leq j \leq k$

$f(w_i) = 0$; if $i = 3, 4, 7, 10$

$= 1$; if $i = 1, 2, 5, 6, 8, 9$

Case 3: $k \equiv 2(mod4)$

In this case, define labeling f as

$f(u_i) = 0$; if $i = 3, 4, 8, 9, 10$

$= 1$; if $i = 1, 2, 5, 6, 7$

$f(v_j) = 0$; if $j \equiv 2, 3(mod4)$

$= 1$; if $j \equiv 0, 1(mod4)$; $1 \leq j \leq k$

$f(w_i) = 0$; if $i = 1, 2, 5, 6, 7$

$= 1$; if $i = 3, 4, 8, 9, 10$

Case 4: $k \equiv 3(mod4)$

In this case, define labeling f as

$f(u_i) = 0$; if $i = 3, 4, 8, 9, 10$

$= 1$; if $i = 1, 2, 5, 6, 7$

$f(v_j) = 0$; if $j \equiv 0, 3(mod4)$

$= 1$; if $j \equiv 1, 2(mod4)$; $1 \leq j \leq k$

$f(w_i) = 0$; if $i = 1, 2, 5, 6, 8, 9$

$= 1$; if $i = 3, 4, 7, 10$

The graph G under consideration satisfies the condition $|v_f(0)-v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ in each case as shown in following *Table 6.10.* i.e. G admits cordial labeling.

Let $k = 4c + d$, where $k \in N$

d	vertex conditions	edge conditions
0	$v_f(0)=v_f(1)$	$e_f(0)+1=e_f(1)$
1	$v_f(0)+1=v_f(1)$	$e_f(0)=e_f(1)$
2	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$
3	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$

Table 6.10

Illustration 6.5.10

For better understanding of above defined labeling pattern let us consider the graph G obtained by joining two copies of Petersen graph by path P_5 (it is the case related to Case-2). The cordial labeling pattern is as shown in *Figure 6.14.*

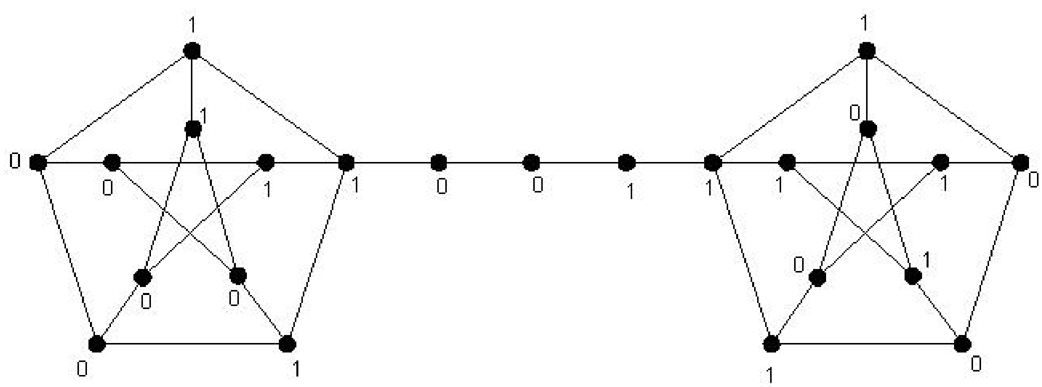


Figure 6.14

6.6 Some Open Problems

¶ In connection of cordial labeling of path union, instead of taking one edge between two graphs one can think path of arbitrary length between any two graphs. Then *Theorem 6.5.7* and *Theorem 6.5.9* reported in previous section will be special cases.

¶ One can derive results similar to the previous section for multiple shells, helms, closed helms etc.

¶ One can investigate cordial labeling for star of some other graphs.

6.7 Concluding Remarks

In this chapter cordial labeling is discussed in detail and survey of some existing results is carried out. Eleven new results are obtained. Result of *Theorem 6.5.2* is accepted for publication in *Proceedings of The International Conference on Emerging Technology and Applications in Engineering, Technology and Sciences* (2008). Results of *Theorem 6.5.4*, *Theorem 6.5.6*, *Theorem 6.5.7* and *Theorem 6.5.9* are accepted for publication in *International Journal of Scientific Computing* (December 2007) while results of *Theorem 6.3.4* and *Theorem 6.4.2* are accepted for publication in *IJMMS*(June 2008(1)). All these research papers are collaborative work of Vaidya et al.[121],[122],[124]. Hint for further research is given in the form of some open problems. Investigations carried out here are novel and original. Labeling pattern is given in vary elegant way and it is demonstrated by means of several examples. Neat figures add beauty to the work. In the penultimate chapter cordial labeling is discussed in the context of some graph operations.

Chapter 7

Some Graph Operations and Cordial Labeling

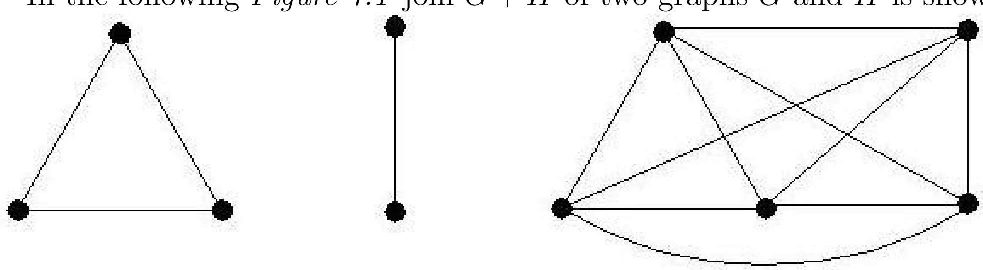
7.1 Introduction

The previous chapter provides brief account of cordial labeling while this chapter is targeted to discuss cordial labeling in the context of different graph operations.

7.2 Join of Two Graphs and Cordial Labeling

Definition 7.2.1 Let G and H be two graphs such that $V(G) \cap V(H) = \emptyset$. Then *join of* G *and* H is denoted by $G+H$. It is the graph with $V(G+H) = V(G) \cup V(H)$, $E(G+H) = E(G) \cup E(H) \cup J$, where $J = \{uv/u \in V(G), v \in V(H)\}$.

In the following *Figure 7.1* join $G+H$ of two graphs G and H is shown.



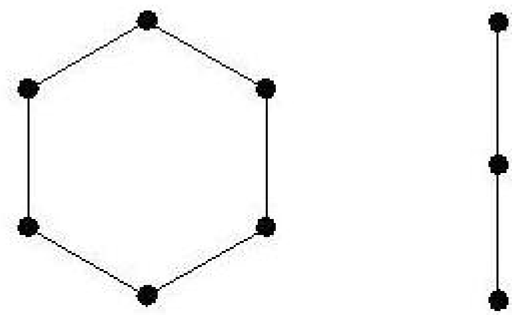
$G = C_3$ $\qquad$ $H = P_2$ $\qquad$ $G + H = C_3 + P_2$

Figure 7.1

Cordiality of join of two graphs can be intimately discussed in reference of size of the graph in following way.

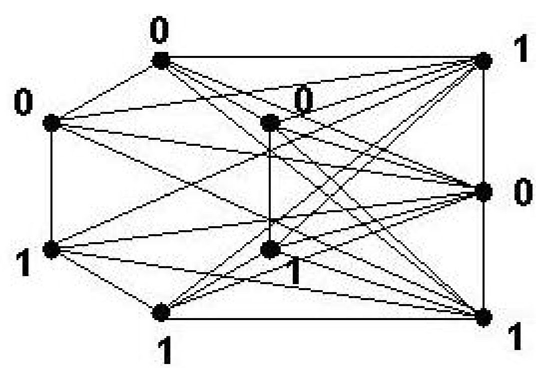
• Youssef[131] has proved that if G and H are cordial and both have even size then $G+H$ is cordial. In this context we have the following aspects.

∘ Let $G = C_6$ and $H = P_3$. Then G is of even size and not cordial(see [70]), H is of even size and cordial(see [32]) while $G+H$ is cordial as shown in *Figure 7.3*.



$G = C_6$ $H = P_3$

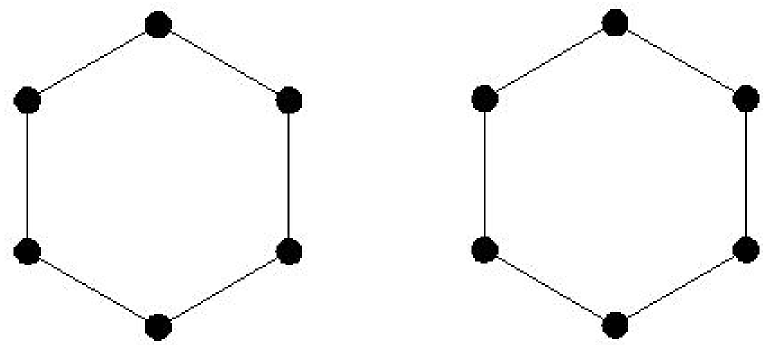
Figure 7.2



$G + H = C_6 + P_3$

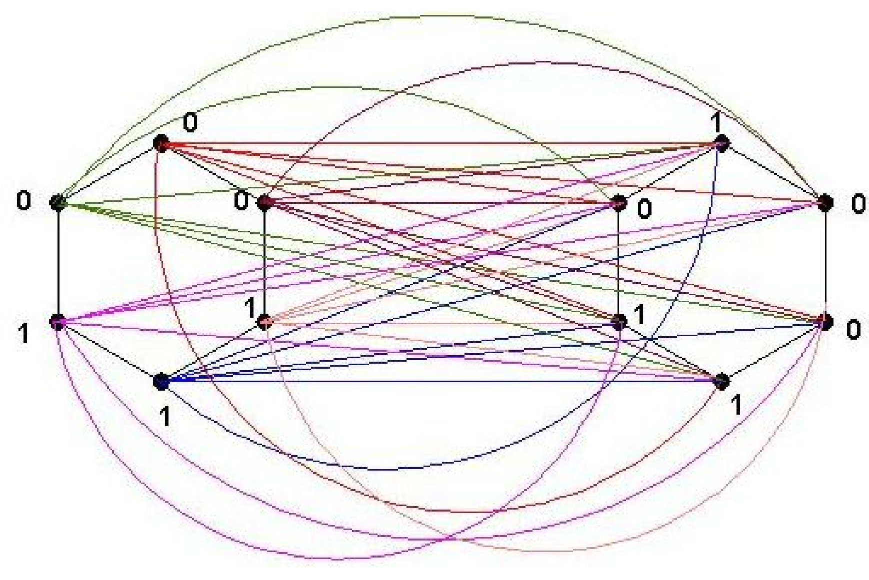
Figure 7.3

∘ Let $G = H = C_6$ then G and H both of even size and not cordial as proved by Ho et al.[70] while $G + H$ is cordial as shown in *Figure 7.5*.



$G = C_6$ $H = C_6$

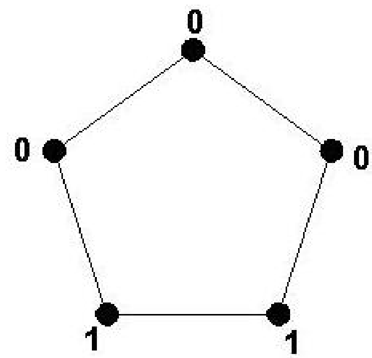
Figure 7.4

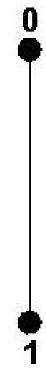


$G + H = C_6 + C_6$

Figure 7.5

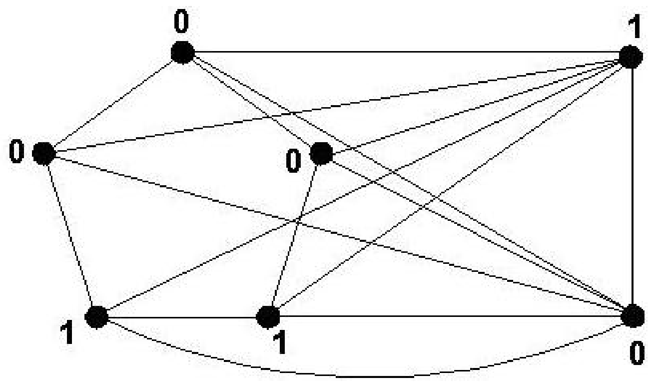
∘ Let $G = C_5$ and $H = K_2$. Then G and H both are of odd size and cordial and $G + H$ is also cordial as shown in following *Figure 7.7*.





$G = C_5$ $H = K_2$

Figure 7.6



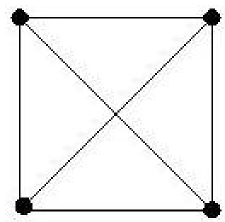
$G + H = C_5 + K_2$

Figure 7.7

∘ Let $G = H = K_2$. Then G and H both are of odd size and cordial(see [32]) while $G + H$ is not cordial as proved by Cahit[32].

$G = K_2$ $H = K_2$

Figure 7.8



$G + H = K_4$

Figure 7.9

Some other known results are listed below.

• Seoud, Diab and Elsahawi[107] have proved the following.

⋄ $P_m + P_n$ is cordial for all m and n except $(m, n) = (2, 2)$.

⋄ $C_m + C_n$ is cordial if m is not congruent to $0 (mod 4)$ and n is not congruent to $2 (mod 4)$.

⋄ $C_n + K_{1,m}$ is cordial for n is not congruent to $3 (mod 4)$ and odd m except $(n, m) = (3, 1)$.

• Diab[43] proved that $C_m + P_n$ is cordial if and only if $(m, n) \neq (3, 3), (3, 2)$ or $(3, 1)$. In the same paper he showed that $P_m + K_{1,n}$ is cordial if and only if $(m, n) \neq (1, 2)$.

7.3 Union of Two Graphs and Cordial Labeling

Definition 7.3.1 If G_1 and G_2 are subgraphs of a graph G then union of G_1 and G_2 is denoted by $G_1 \cup G_2$ which is the graph consisting of all those vertices which are either in G_1 or in G_2 (or in both) and with edge set consisting of all those edges which are either in G_1 or in G_2 (or in both).

Youssef[132] has proved that if G and H are cordial and one has even size then $G \cup H$ is cordial. In connection to this result we have the following aspects.

○ Let us consider cycle C_5 with one chord and $G_1 = C_4$ and $G_2 = C_3$. Then G_1 is of even size and cordial, G_2 is of odd size and cordial. Also $G_1 \cup G_2$ is cordial as proved in *Theorem 6.3.4*. See the following *Figure 7.12*.

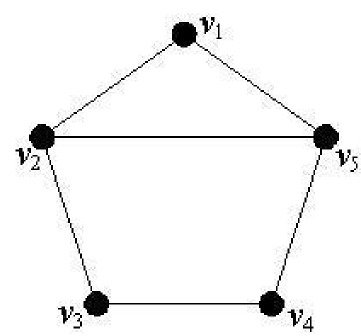
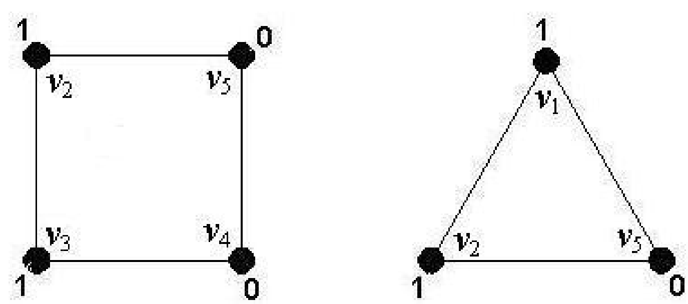
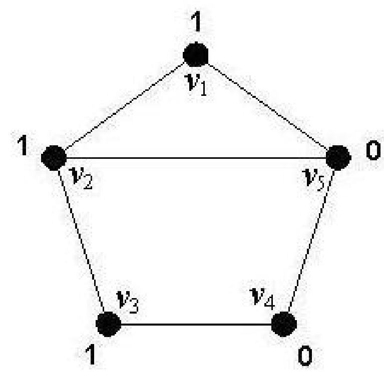


Figure 7.10



$G_1 = C_4$ $G_2 = C_3$

Figure 7.11



$G_1 \cup G_2$

Figure 7.12

○ Let us consider the complete graph K_4.

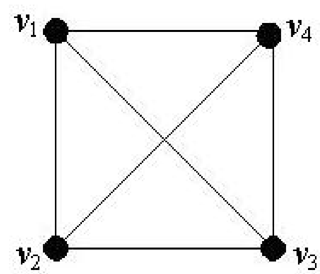
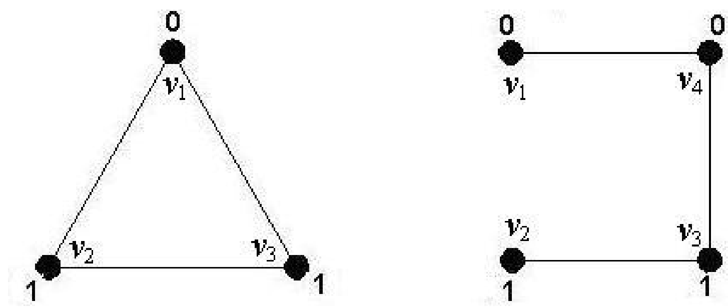


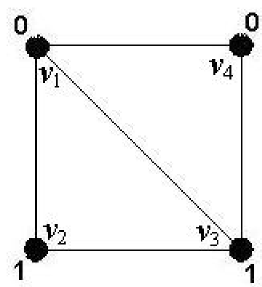
Figure 7.13

Let $G_1 = C_3$ and $G_2 = P_4$. Then G_1 and G_2 both are of odd size and cordial. Also $G_1 \cup G_2$ is cordial as proved in *Theorem 6.3.4*. See the following *Figure 7.15*.



$G_1 = C_3$ $G_2 = P_4$

Figure 7.14



$G_1 \cup G_2$

Figure 7.15

∘ Let us consider compete graph K_7.

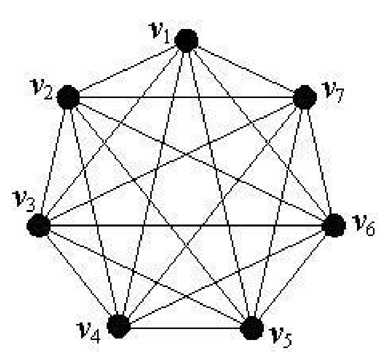
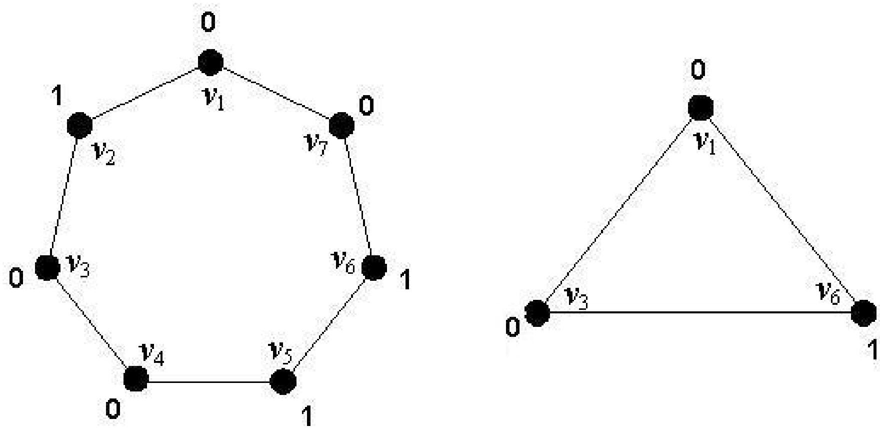


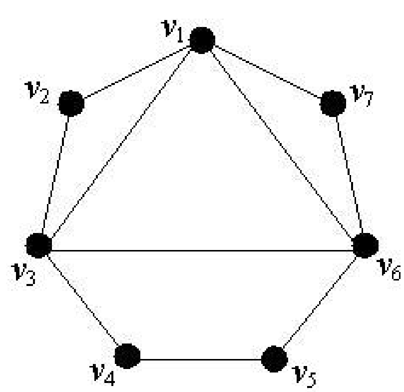
Figure 7.16

Let $G_1 = C_7$ and $G_2 = C_3$. Then G_1 and G_2 both are of odd size and cordial while $G_1 \cup G_2$ is not cordial as it is an Eulerian graph with number of edges congruent to $2(mod4)$(see [32]).



$G_1 = C_7$ $G_2 = C_3$

Figure 7.17



$G_1 \cup G_2$

Figure 7.18

Some other known results are listed below.

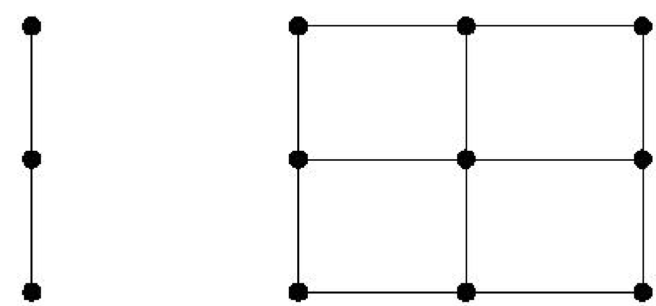
• Diab[43] proved that $P_m \cup K_{1,n}$ is cordial if and only if $(m,n) \neq (1,2)$. In the same paper he proved that $C_m \cup K_{1,n}$ is cordial for all m and n.

• Youssef[132] proved that $C_m \cup C_n$ is cordial if and only if $m+n$ is not congruent to $2(mod4)$.

7.4 Cartesian Product of Two Graphs and Cordial Labeling

Definition 7.4.1 Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. Then *cartesian product of* G_1 *and* G_2 which is denoted by $G_1 \times G_2$ is the graph with vertex set $V = V_1 \times V_2$ consisting of vertices $u = (u_1, u_2)$, $v = (v_1, v_2)$ such that u and v are adjacent in $G_1 \times G_2$ whenever ($u_1 = v_1$ and u_2 adjacent to v_2) or ($u_2 = v_2$ and u_1 adjacent to v_1).

In the following *Figure 7.19* cartesian product $P_3 \times P_3$ is shown.



P_3 $\qquad\qquad$ $P_3 \times P_3$

Figure 7.19

Some known results are listed below.

• Ho et al.[71] proved that the cartesian product of two cordial graphs of even size is cordial.

To see this consider $G = H = P_3$. Here G and H both are cordial and of even size. Then $G \times H = P_3 \times P_3$ is also cordial according to the following existing results.

• Lee et al.[85] proved that

⋄ The cartesian product of an arbitrary number of paths is cordial.

⋄ The cartesian product of two cycles is cordial if and only if atleast one of them is even.

⋄ The cartesian product of an arbitrary number of cycles is cordial if atleast one of them has length a multiple of 4 or atleast two of them are even.

• Seoud and Abdel Maqsoud[106] proved that $C_n \times P_m$ is cordial except for the case $C_{4k+2} \times P_2$.

7.5 Reconstruction of Graphs and Cordial Labeling

Reconstruction of graph is discussed in detail in *Chapter 4*. In this section we will discuss cordial labeling in the context of reconstruction of graph. We observe that cordiality of a graph may or may not be reconstructible. In support of this observation we have the following:

• There are some graphs which are not cordial although their deck contains cordial cards.

Consider cycle C_n where $n \equiv 2(mod4)$. Such cycle C_n is not cordial as proved by Ho et al.[70]. Here deck $\mathcal{G}$ contains n copies of path P_n which are cordial as proved by Cahit[32].

• There are some graphs which are cordial but their deck contains some graphs which are not cordial.

Tadpole $T(l,1)$ where $l \equiv 2(mod 4)$ is cordial as proved by Ho et al.[70]. Here deck $\mathcal{G}$ contains one copy of cycle C_l and l copies of trees. Here cycle C_l where $l \equiv 2(mod 4)$ is not cordial as proved by Ho et. al.[70] while l copies of trees are cordial as proved by Cahit[32].

• The wheel graph $W_n = C_n + K_1$ is reconstructible from the deck of one copy of cycle C_n and n copies of fans $F_{n-1} = P_{n-1} + K_1$. Here in the deck :

1. When $n \equiv 0, 1(mod 4)$ cycle C_n as well as fans F_{n-1} are cordial and the graph reconstructed from it is W_n which is also cordial.

2. When $n \equiv 2(mod 4)$ cycle C_n is not cordial and fans F_{n-1} are cordial. The graph reconstructed from these deck is W_n where $n \equiv 2(mod 4)$ which is cordial.

3. When $n \equiv 3(mod 4)$ cycle C_n as well as fans F_{n-1} are cordial but graph W_n reconstructed from this deck is not cordial.

¶ **Remark**

For the cordiality of wheel, cycle and fan one can refer the related references mentioned in *Chapter 6*.

7.6 Contraction of Graphs and Cordial Labeling

Definition 7.6.1 Let $e = uv$ be an edge of simple, finite, undirected, connected graph G and $d(u) = k$, $d(v) = l$. Let $N(u) = \{v, u_1, \dots, u_{n-1}\}$ and $N(v) = \{u, v_1, \dots, v_{l-1}\}$. A contraction on the edge e changes G to a new graph $G * e$ where $V(G * e) = (V(G) - \{u, v\}) \cup \{w\}$, $E(G * e) = E(G - \{u, v\}) \cup \{wu_1, wu_2, \dots, wu_{k-1}, wv_1, \dots, wv_{l-1}\}$ and w is new vertex not belonging to G.

Note that

- Contraction of cycle C_n is cycle C_{n-1}.
- Contraction of wheel $W_n = C_n + K_1$ is either fan F_{n-1} or wheel W_{n-1}.
- Contraction of K_n is K_{n-1}.
- Contraction of P_n is P_{n-1}.

In the context of above definition we have following observations.

Observation 1: Contraction of cycle C_n is cordial except $n \equiv 3(mod4)$ because as proved by Ho et al.[70] unicyclic graphs are cordial except C_{4k+2}.

Observation 2: Contraction of complete graph K_n is cordial if and only if $n \leq 4$ because Cahit[32] proved that K_n is cordial if and only if $n \leq 3$.

Definition 7.6.2 A collection of edge contracted subgraph of a graph G is called *contraction deck* of G which is denoted as $\mathcal{G}^*$ and it is defined as $\mathcal{G}^* = \{G * e/e \in G\}$. Each element of $\mathcal{G}^*$ is called *card*. We will have two more observations in connection of above definition.

Observation 3: Contraction deck of wheel graph $W_n = C_n + K_1$ (where $n \equiv 0(mod4)$) contains some cordial as well as some non-cordial cards because contraction deck of the wheel graph contains fans F_{n-1} and wheel W_{n-1}. As proved by Cahit[32] all fans are cordial but wheels W_n are cordial except $n \equiv 3(mod4)$.

Observation 4: Contraction deck of Tadpole $T(l, r)$ contains all cordial cards except $l \equiv 2(mod4)$ and $r = 1$ because contraction deck of tadpole $T(l, 1)$, where $l \equiv 2(mod4)$ contains fans and a cycle C_l where $l \equiv 2(mod4)$. As proved by Ho et al.[70] unicyclic graphs are cordial except C_{4k+2}.

7.7 Vertex Switching and Cordial Labeling

Definition 7.7.1 A *vertex switching* G_v of a graph G is obtained by taking a vertex v of G, removing all edges incident to v and adding edges joining v to every vertex not adjacent to v in G.

We will discuss cordiality in the context of above definition. We will discuss cordiality of vertex switching in some cycle related graphs.

Theorem 7.7.2 Vertex switching of cycle C_n is cordial.

Proof: Let $G = C_n$ and $v_1, v_2, \ldots, v_n$ be successive vertices of C_n. G_v denotes the vertex switching of G with respect to the vertex v of G. Here note that in each of the following cases the labeling pattern starts from the switched vertex which is considered as v_1. To define binary vertex labeling $f : V(G_v) \rightarrow \{0, 1\}$ following cases to be considered.

Case 1: $n \equiv 0, 1, 2(mod4)$

In this case we define labeling f as:

$f(v_i) = 0$; if $i \equiv 0, 1(mod4)$

$\quad = 1$; if $i \equiv 2, 3(mod4)$, $1 \leq i \leq n$.

Case 2: $n \equiv 3(mod4)$

In this case we define labeling f as:

$f(v_i) = 0$; if $i \equiv 2, 3(mod4)$

$\quad = 1$; if $i \equiv 0, 1(mod4)$, $1 \leq i \leq n$.

The labeling pattern defined above covers all the possibility of vertex switching. In each case the graph under consideration satisfies the conditions for cordiality as shown in following *Table 7.1.* i.e. G_v admits cordial labeling.

Let $n = 4a + b$, $n \in N, n \geq 3$.

b	vertex conditions	edge conditions
0	$v_f(0)=v_f(1)$	$e_f(0)+1=e_f(1)$
1,3	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)+1$
2	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$

Table 7.1

Illustrations 7.7.3 For better understanding of above defined labeling pattern let us consider few examples.

Example 1 Consider cycle C_7 (it is the case related to $n \equiv 3(mod4)$). The labeling pattern is shown in *Figure 7.20*.

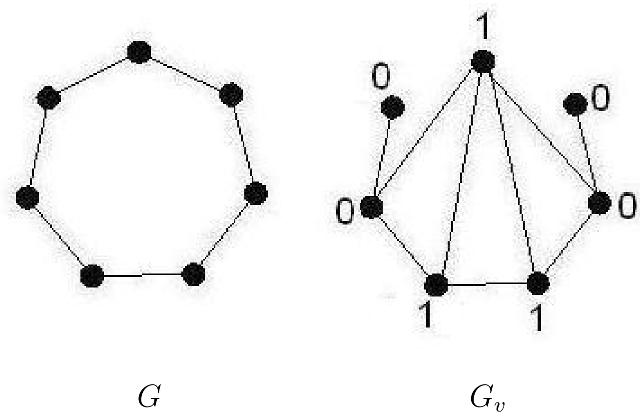
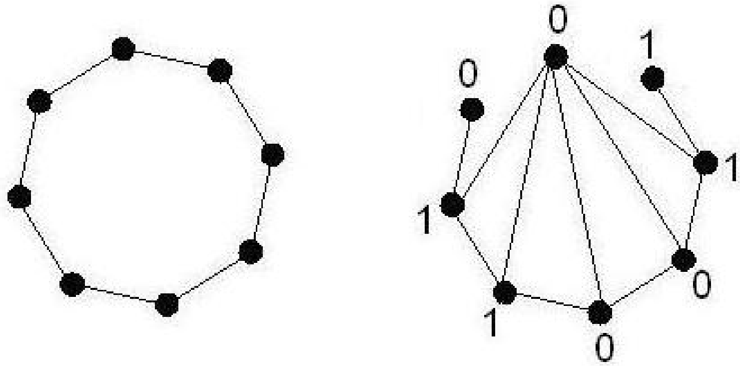


Figure 7.20

Example 2 Consider cycle C_8 (it is the case related to $n \equiv 0(mod4)$). The labeling pattern is shown in *Figure 7.21*.



G $\qquad\qquad$ G_v

Figure 7.21

Theorem 7.7.4 Vertex switching of cycle C_n with one chord is cordial.

Proof : Let G be the cycle C_n with one chord and $v_1, v_2, \ldots, v_n$ be successive vertices of C_n. G_v denotes the vertex switching of G with respect to the vertex v of G. Here note that in each of the following cases the labeling pattern starts from the switched vertex which is considered as v_1. To define binary vertex labeling $f : V(G_v) \to \{0, 1\}$ following cases to be considered.

<u>Case A</u>: Vertex switching of a vertex having $d(v) = 2$ and it is adjacent with both end vertices of chord.

<u>Subcase 1</u>: $n \equiv 0(mod4)$

In this case we define labeling f as:

$f(v_1) = 0$, $f(v_n) = 1$ and

$f(v_i) = 0$; if $i \equiv 0, 3(mod4)$

$\quad = 1$; if $i \equiv 1, 2(mod4)$, $2 \leq i \leq n-1$.

<u>Subcase 2</u>: $n \equiv 1(mod4)$

In this case we define labeling f as:

$f(v_i) = 0$; if $i \equiv 2, 3(mod4)$

$= 1$; if $i \equiv 0, 1(mod4)$, $1 \leq i \leq n$.

<u>Subcase 3</u>: $n \equiv 2(mod4)$

In this case we define labeling f as:

$f(v_1) = 0$, $f(v_n) = 1$ and

$f(v_i) = 0$; if $i \equiv 2, 3(mod4)$

$= 1$; if $i \equiv 0, 1(mod4)$, $2 \leq i \leq n-1$.

<u>Subcase 4</u>: $n \equiv 3(mod4)$

In this case we define labeling f as:

$f(v_1) = 0$, $f(v_{n-1}) = 1$, $f(v_n) = 0$ and

$f(v_i) = 0$; if $i \equiv 2, 3(mod4)$

$= 1$; if $i \equiv 0, 1(mod4)$, $2 \leq i \leq n-2$.

<u>Case B</u>: Vertex switching of a vertex having $d(v) = 3$.

<u>Subcase 1</u>: $n \equiv 0(mod4)$

In this case we define labeling f as:

$f(v_i) = 0$; if $i \equiv 1, 2(mod4)$

$= 1$; if $i \equiv 0, 3(mod4)$, $1 \leq i \leq n$.

<u>Subcase 2</u>: $n \equiv 1(mod4)$

In this case we define labeling f as:

$f(v_n) = 0$ and

$f(v_i) = 0$; if $i \equiv 2, 3(mod4)$

$= 1$; if $i \equiv 0, 1(mod4)$, $1 \leq i \leq n-1$.

<u>Subcase 3</u>: $n \equiv 2(mod4)$

In this case we define labeling f as:

$f(v_1) = 1$, $f(v_n) = 0$ and

$f(v_i) = 0$; if $i \equiv 0, 1(mod4)$

$= 1$; if $i \equiv 2, 3(mod4)$, $2 \leq i \leq n - 1$.

Subcase 4: $n \equiv 3(mod4)$

In this case we define labeling f as:

$f(v_i) = 0$; if $i \equiv 1, 2(mod4)$

$= 1$; if $i \equiv 0, 3(mod4)$, $1 \leq i \leq n$.

Case C: Vertex switching of a remaining vertex which are having $d(v) = 2$.

Subcase 1: $n \equiv 0(mod4)$

In this case first label both the end vertices of chord by label 0. For remaining vertices we define labeling f as:

$f(v_1) = 1$,

$f(v_i) = 0$; if $i \equiv 0, 1(mod4)$

$= 1$; if $i \equiv 2, 3(mod4)$, $1 \leq i \leq n - 2$.

Subcase 2: $n \equiv 1, 2, 3(mod4)$

In this case we define labeling f as:

$f(v_i) = 0$; if $i \equiv 0, 1(mod4)$

$= 1$; if $i \equiv 2, 3(mod4)$, $1 \leq i \leq n$.

The labeling pattern defined above covers all the possibility of vertex switching. In each cases A, B and C the graph under consideration satisfies the conditions for cordiality as shown in following *Table 7.2*, *Table 7.3* and *Table 7.4* respectively. i.e. In each case G_v admits cordial labeling.

Let $n = 4a + b$, $a \in N$.

b	vertex conditions	edge conditions
0,2	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)$
1	$v_f(0)+1=v_f(1)$	$e_f(0)=e_f(1)$
3	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$

Table 7.2

b	vertex conditions	edge conditions
0,2	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)$
1,3	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$

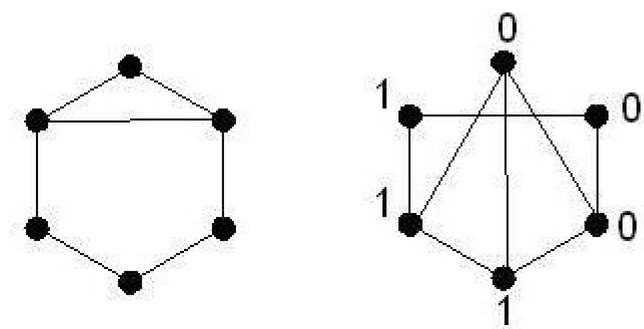
Table 7.3

b	vertex conditions	edge conditions
0,2	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)$
1	$v_f(0)+1=v_f(1)$	$e_f(0)=e_f(1)$
3	$v_f(0)+1=v_f(1)$	$e_f(0)=e_f(1)$

Table 7.4

Illustrations 7.7.5 For better understanding of above defined labeling pattern let us consider few examples.

Example 1 Consider cycle C_6 with one chord (it is the case related to <u>**Case A**</u>, $n \equiv 2(mod4)$). The labeling pattern is as shown in *Figure 7.22*.



G G_v

Figure 7.22

Example 2 Consider cycle C_7 with one chord (it is the case related to <u>**Case B**</u>, $n \equiv 3(mod4)$). The labeling pattern is as shown in *Figure 7.23*.

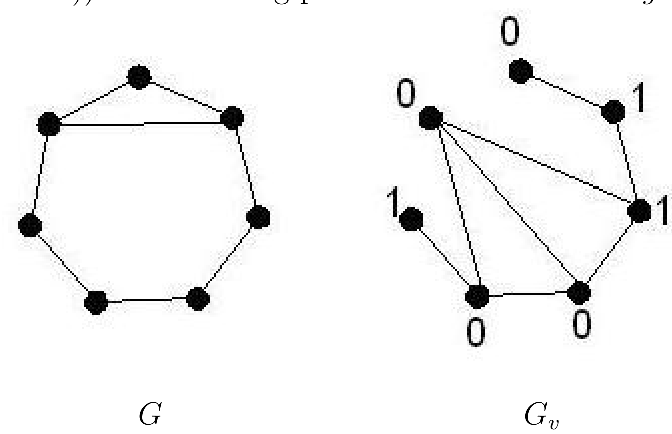


Figure 7.23

Example 3 Consider cycle C_8 with one chord (it is the case related to <u>**Case C**</u>, $n \equiv 0(mod4)$). The labeling pattern is as shown in *Figure 7.24*.

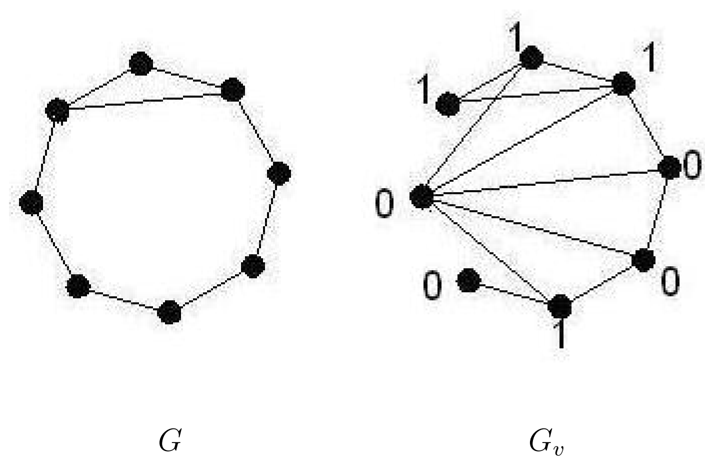


Figure 7.24

7.8 Concluding Remarks

This chapter was aimed to discuss cordial labeling in the context of various graph operations like reconstruction, contraction, join etc. Some new results are obtained. Proofs are given in very elegant way and illustrations provide better understanding of the derived results. Looking to the current trend of research in the field of graph labeling this approach will provide new direction to any researcher.

The penultimate *Chapter 8* is devoted to the discussion of 3-equitable labeling of graphs.

Chapter 8
Some 3-equitable Graphs

8.1 Introduction

In *Chapter 3* we have discussed various types of graph labeling while this chapter is targeted to discuss a particular type of k-equitable labeling known as *3-equitable labeling*. Some new families of 3-equitable graphs are investigated and some open problems are also given.

In 1990 Cahit[33] introduced the notion of 3-equitable labeling. In the succeeding section we will give some definitions and important results.

8.2 Some Definitions and Important Results

Definition 8.2.1 Let $G = (V, E)$ be a graph. A mapping $f : V(G) \rightarrow \{0, 1, 2\}$ is called *ternary vertex labeling* of G and $f(v)$ is called *label of the vertex* v of G under f.

For an edge $e = uv$, the induced edge labeling $f^* : E(G) \rightarrow \{0, 1, 2\}$ is given by $f^*(e) = |f(u) - f(v)|$. Let $v_f(0)$, $v_f(1)$, $v_f(2)$ be the number of vertices of G having labels 0, 1 and 2 respectively under f and let $e_f(0)$, $e_f(1)$, $e_f(2)$ be the number of edges having labels 0, 1 and 2 respectively under f^*.

Definition 8.2.2 A ternary vertex labeling of a graph G is called *3-equitable labeling* if $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$, $0 \leq i, j \leq 2$.

A graph which admits 3-equitable labeling is called *3-equitable graph*.

Some known families of 3-equitable graphs are listed below.

• Cahit[32],[33] proved that

⋄ C_n is 3-equitable if and only if n is not congruent to $3(mod 6)$.

⋄ An Eulerian graph with $q \equiv 3(mod 6)$ is not 3-equitable where q is the number of edges.

⋄ All caterpillars are 3-equitable.

$\diamond$ He conjectured that *A triangular cactus with* n *blocks is 3-equitable if and only if* n *is even.*

$\diamond$ Every tree with fewer than five end vertices has a 3-equitable labeling.

• Seoud and Abdel Maqsoud[105] proved that

$\diamond$ A graph with p vertices and q edges in which every vertex has odd degree is not 3-equitable if $p \equiv 0(mod3)$ and $q \equiv 3(mod6)$.

$\diamond$ All fans except $P_2 + K_1$ are 3-equitable.

$\diamond$ P_n^2 is 3-equitable for all n except 3.

$\diamond$ $K_{m,n}$, $3 \leq m \leq n$ is 3-equitable if and only if $(m, n) = (4, 4)$.

• Bapat and Limaye[17] proved that Helms H_n, $n \geq 4$ are 3-equitable.

• Youssef[131] proved that $W_n = C_n + K_1$ is 3-equitable for all $n \geq 4$.

In the immediate section we will provide brief account of results investigated by us about 3-equitable labeling of some cycle related graphs.

8.3 3-equitable Labeling For Some Cycle Related Graphs

We have defined star of a graph in *Chapter 6* as *Definition 6.5.1.* In that connection we will prove following result.

Theorem 8.3.1 Star of cycle C_n^* is 3-equitable for all n.

Proof: Let $v_1, v_2, \ldots, v_n$ be successive vertices of central cycle of C_n^* and $u_{i1}, u_{i2}, \ldots, u_{in}$ be successive vertices of other cycles $C_n^{(i)}$ (except central cycle), $i = 1, 2, \ldots, n$. Let e_i be the edge such that $e_i = u_{i1}v_i$. Moreover, let us denote the vertex of cycle $C_n^{(i)}$ which is adjacent to a vertex v_i labeled by 0 as $u_{ij}^{(0)}$, the vertex of cycle $C_n^{(i)}$ which is adjacent to a vertex v_i labeled by

1 as $u_{ij}^{(1)}$ and the vertex of cycle $C_n^{(i)}$ which is adjacent to a vertex v_i labeled by 2 as $u_{ij}^{(2)}$. To define required labeling $f : V(C_n^*) \to \{0, 1, 2\}$ we consider following cases.

<u>Case 1</u>: $n \equiv 0(mod6)$

In this case define labeling f as

$f(v_i) = 0$; if $i \equiv 0, 3(mod4)$

$= 1$; if $i \equiv 1, 2(mod6)$

$= 2$; if $i \equiv 4, 5(mod6)$, $1 \le i \le n$.

$f(u_{ij}^{(0)}) = 0$; if $j \equiv 0, 3(mod6)$

$= 1$; if $j \equiv 1, 2(mod6)$

$= 2$; if $j \equiv 4, 5(mod6)$, $1 \le j \le n$, $1 \le i \le n$.

$f(u_{ij}^{(1)}) = 0$; if $j \equiv 0, 3(mod6)$

$= 1$; if $j \equiv 1, 2(mod6)$

$= 2$; if $j \equiv 4, 5(mod6)$, $1 \le j \le n$, $1 \le i \le n$.

$f(u_{ij}^{(2)}) = 0$; if $j \equiv 1, 4(mod6)$

$= 1$; if $j \equiv 0, 5(mod6)$

$= 2$; if $j \equiv 2, 3(mod6)$, $1 \le j \le n$, $1 \le i \le n$.

<u>Case 2</u>: $n \equiv 1(mod6)$

In this case define labeling f as

$f(v_i) = 0$; if $i \equiv 0, 3(mod6)$

$= 1$; if $i \equiv 1, 2(mod6)$

$= 2$; if $i \equiv 4, 5(mod6)$, $1 \le i \le n$.

$f(u_{ij}^{(0)}) = 0$; if $j \equiv 2, 5(mod6)$

$= 1$; if $j \equiv 0, 1(mod6)$

$= 2$; if $j \equiv 3, 4(mod6)$, $1 \le j \le n$, $1 \le i \le n$.

$f(u_{i1}^{(1)}) = 2$ and

$f(u_{ij}^{(1)}) = 0$; if $j \equiv 1, 4(mod6)$

$= 1$; if $j \equiv 2, 3(mod6)$

$= 2$; if $j \equiv 0, 5(mod6)$, $2 \leq j \leq n$, $1 \leq i \leq n$.

$f(u_{ij}^{(2)}) = 0$; if $j \equiv 1, 4(mod6)$

$= 1$; if $j \equiv 2, 3(mod6)$

$= 2$; if $j \equiv 0, 5(mod6)$, $1 \leq j \leq n$, $1 \leq i \leq n$.

Case 3: $n \equiv 2(mod6)$

In this case define labeling f as

$f(v_i) = 0$; if $i \equiv 2, 5(mod6)$

$= 1$; if $i \equiv 0, 1(mod6)$

$= 2$; if $i \equiv 3, 4(mod6)$, $1 \leq i \leq n$.

$f(u_{ij}^{(0)}) = 0$; if $j \equiv 1, 4(mod6)$

$= 1$; if $j \equiv 0, 5(mod6)$

$= 2$; if $j \equiv 2, 3(mod6)$, $1 \leq j \leq n$, $1 \leq i \leq n$.

$f(u_{in}^{(1)}) = 1$, $f(u_{in-1}^{(1)}) = 2$ and

$f(u_{ij}^{(1)}) = 0$; if $j \equiv 0, 3(mod6)$

$= 1$; if $j \equiv 1, 2(mod6)$

$= 2$; if $j \equiv 4, 5(mod6)$, $1 \leq j \leq n-2$, $1 \leq i \leq n$.

$f(u_{ij}^{(2)}) = 0$; if $j \equiv 2, 5(mod6)$

$= 1$; if $j \equiv 0, 1(mod6)$

$= 2$; if $j \equiv 3, 4(mod6)$, $1 \leq j \leq n$, $1 \leq i \leq n$.

Case 4: $n \equiv 3(mod6)$

In this case define labeling f as

$f(v_i) = 0$; if $i \equiv 1, 4(mod6)$

$= 1$; if $i \equiv 0, 5(mod6)$

$= 2$; if $i \equiv 2, 3(mod6)$, $1 \leq i \leq n$.

$f(u_{ij}^{(0)}) = 0$; if $j \equiv 1, 4(mod6)$

$= 1$; if $j \equiv 0, 5(mod6)$

$= 2$; if $j \equiv 2, 3(mod6)$, $1 \leq j \leq n$, $1 \leq i \leq n$.

Let n_1 denotes the number of cycles whose one end vertex u_{ij} (for some j) is adjacent to vertex v_i which is labeled by 1. Here note that number of vertices in central cycle which are labeled by 1 is even.

To label $\frac{n_1}{2}$ such cycles we define labeling f as

$f(u_{in}^{(1)}) = 1$, $f(u_{in-1}^{(1)}) = 2$, $f(u_{in-2}^{(1)}) = 0$ and

$f(u_{ij}^{(1)}) = 0$; if $j \equiv 0, 3(mod6)$

$= 1$; if $i \equiv 4, 5(mod6)$

$= 2$; if $i \equiv 1, 2(mod6)$, $1 \leq j \leq n-3$, $1 \leq i \leq n$.

To label remaining $\frac{n_1}{2}$ such cycles we define labeling f as

$f(u_{ij}^{(1)}) = 0$; if $j \equiv 2, 5(mod6)$

$= 1$; if $j \equiv 0, 1(mod6)$

$= 2$; if $j \equiv 3, 4(mod6)$, $1 \leq j \leq n$, $1 \leq i \leq n$.

$f(u_{ij}^{(2)}) = 0$; if $j \equiv 1, 4(mod6)$

$= 1$; if $j \equiv 2, 3(mod6)$

$= 2$; if $j \equiv 0, 5(mod6)$, $1 \leq j \leq n$, $1 \leq i \leq n$.

Case 5: $n \equiv 4(mod6)$

In this case define labeling f as

$f(v_i) = 0$; if $i \equiv 0, 3(mod4)$

$= 1$; if $i \equiv 1, 2(mod6)$

$= 2$; if $i \equiv 4, 5(mod6)$, $1 \leq i \leq n$.

$f(u_{ij}^{(0)}) = 0$; if $j \equiv 0, 3(mod6)$

$= 1$; if $j \equiv 4, 5(mod6)$

$= 2$; if $j \equiv 1, 2(mod6)$, $1 \leq j \leq n$, $1 \leq i \leq n$.

$f(u_{ij}^{(1)}) = 0$; if $j \equiv 0, 3(mod6)$

$= 1$; if $j \equiv 1, 2(mod6)$

$= 2$; if $j \equiv 4, 5(mod6)$, $1 \leq j \leq n$, $1 \leq i \leq n$.

$f(u_{ij}^{(2)}) = 0$; if $j \equiv 1, 4(mod6)$

$= 1$; if $j \equiv 0, 5(mod6)$

$= 2$; if $j \equiv 2, 3(mod6)$, $1 \leq j \leq n$, $1 \leq i \leq n$.

Case 6: $n \equiv 5(mod6)$

In this case define labeling f as

$f(v_i) = 0$; if $i \equiv 2, 5(mod4)$

$= 1$; if $i \equiv 0, 1(mod6)$

$= 2$; if $i \equiv 3, 4(mod6)$, $1 \leq i \leq n$.

$f(u_{ij}^{(0)}) = 0$; if $j \equiv 0, 3(mod6)$

$= 1$; if $j \equiv 4, 5(mod6)$

$= 2$; if $j \equiv 1, 2(mod6)$, $1 \leq j \leq n$, $1 \leq i \leq n$.

$f(u_{ij}^{(1)}) = 0$; if $j \equiv 2, 5(mod6)$

$= 1$; if $j \equiv 0, 1(mod6)$

$= 2$; if $j \equiv 3, 4(mod6)$, $1 \leq j \leq n$, $1 \leq i \leq n$.

$f(u_{ij}^{(2)}) = 0$; if $j \equiv 2, 5(mod6)$

$= 1$; if $j \equiv 3, 4(mod6)$

$= 2$; if $j \equiv 0, 1(mod6)$, $1 \leq j \leq n$, $1 \leq i \leq n$.

The above defined labeling pattern covers all possible arrangement of vertices. In each case the graph under consideration satisfies the condition $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$, $0 \leq i, j \leq 2$ as shown in *Table 8.1*. i.e. Graph C_n^* admits 3-equitable labeling.

Let $n = 4a + b$, $n \in N$, $n \geq 3$.

b	vertex conditions	edge conditions
0,3	$v_f(0)=v_f(1)=v_f(2)$	$e_f(0)=e_f(1)=e_f(2)$
1	$v_f(0)+1=v_f(1)=v_f(2)$	$e_f(0)=e_f(1)=e_f(2)$
2	$v_f(0)=v_f(1)=v_f(2)$	$e_f(0)=e_f(1)+1=e_f(2)$
4	$v_f(0)+1=v_f(1)=v_f(2)$	$e_f(0)=e_f(1)=e_f(2)$
5	$v_f(0)=v_f(1)=v_f(2)$	$e_f(0)+1=e_f(1)=e_f(2)$

Table 8.1

Illustration 8.3.2 For better understanding of above defined labeling pattern let us consider C_5^* (it related to Case-6). The 3-equitable labeling of C_5^* is as shown in *Figure 8.1*.

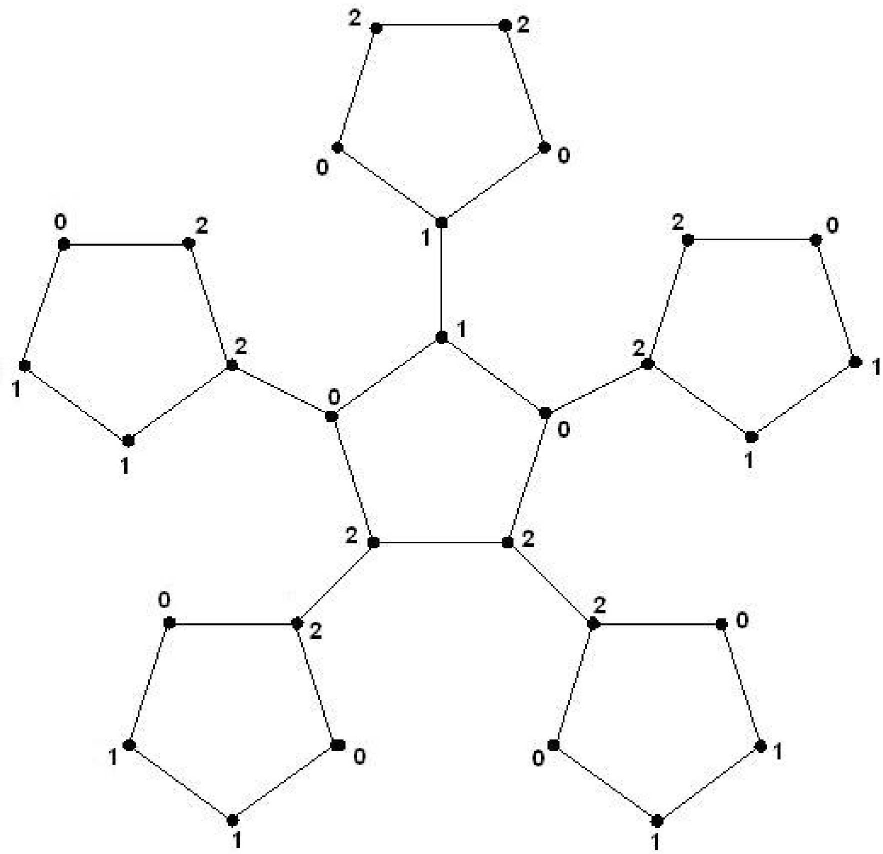


Figure 8.1

Before proving next theorem we will provide one useful definition.

Definition 8.3.3 $T(l, r)$ is the graph called *tadpole* in which path of length r is attached to any one vertex of cycle C_l. $T(l, r)$ has $l + r$ vertices.

Theorem 8.3.4 Tadpoles $T(l, r)$ are 3-equitable, for all l and r.

Proof: Let $G = T(l, r)$. The number of vertices in G is $p = l + r$ and number of edges in G is $q = l + r$. Let $\{v_1, v_2, ..., v_l\}$ be the vertices of cycle C_l and $\{v_{l+1}, v_{l+2}, ..., v_{l+r}\}$ be the vertices of path of length r attached to any one vertex of cycle C_l. Let v_l be the vertex adjacent to first vertex v_{l+1}

of path P_r. To define labeling function $f : V(G) \rightarrow \{0, 1, 2\}$ we consider following cases.

Case 1:

(a) $l \equiv 0(mod6)$, $r \equiv 0, 1, 4, 5(mod6)$

In this case define labeling f as

$f(v_i) = 0$; if $i \equiv 0, 3(mod6)$

$= 1$; if $i \equiv 1, 2(mod6)$

$= 2$; if $i \equiv 4, 5(mod6)$, $1 \leq i \leq l + r$.

(b) $l \equiv 0(mod6)$, $r \equiv 2, 3(mod6)$

In this case define labeling f as

$f(v_i) = 0$; if $i \equiv 2, 5(mod6)$

$= 1$; if $i \equiv 0, 1(mod6)$

$= 2$; if $i \equiv 3, 4(mod6)$, $1 \leq i \leq l + r$.

Case 2: $l \equiv 1(mod6)$, $r \equiv 0, 1, 2, 3, 4, 5(mod6)$

In this case define labeling f as

$f(v_i) = 0$; if $i \equiv 2, 5(mod6)$

$= 1$; if $i \equiv 0, 1(mod6)$

$= 2$; if $i \equiv 3, 4(mod6)$, $1 \leq i \leq l + r$.

Case 3:

(a) $l \equiv 2(mod6)$, $r \equiv 2, 3, 4, 5(mod6)$

In this case define labeling f as

$f(v_i) = 0$; if $i \equiv 0, 3(mod6)$

$= 1$; if $i \equiv 1, 2(mod6)$

$= 2$; if $i \equiv 4, 5(mod6)$, $1 \leq i \leq l + r$.

(b) $l \equiv 2(mod6)$, $r \equiv 0,1(mod6)$

In this case define labeling f as

$f(v_1) = 2$, $f(v_l) = 0$,

For $2 \leq i \leq l-1$,

$f(v_i) = 0$; if $i \equiv 1,4(mod6)$

$= 1$; if $i \equiv 2,3(mod6)$

$= 2$; if $i \equiv 0,5(mod6)$, $1 \leq i \leq l+r$.

For $l+1 \leq i \leq l+r$,

$f(v_i) = 0$; if$i \equiv 2,5(mod6)$

$= 1$; if $i \equiv 3,4(mod6)$

$= 2$; if $i \equiv 0,1(mod6)$, $1 \leq i \leq l+r$.

Case 4:

(a) $l \equiv 3(mod6)$, $r \equiv 1,2,3,4(mod6)$

In this case define labeling f as

$f(v_i) = 0$; if $i \equiv 0,3(mod6)$

$= 1$; if $i \equiv 1,2(mod6)$

$= 2$; if $i \equiv 4,5(mod6)$, $1 \leq i \leq l+r$.

(b) $l \equiv 3(mod6)$, $r \equiv 0,5(mod6)$

In this case define labeling f as

$f(v_i) = 0$; if $i \equiv 2,5(mod6)$

$= 1$; if $i \equiv 0,1(mod6)$

$= 2$; if $i \equiv 3,4(mod6)$, $1 \leq i \leq l+r$.

Case 5:

(a) $l \equiv 4(mod6)$, $r \equiv 0,1,2,3(mod6)$

In this case define labeling f as

$f(v_i) = 0$; if $i \equiv 0, 3(mod6)$

$= 1$; if $i \equiv 1, 2(mod6)$

$= 2$; if $i \equiv 4, 5(mod6)$, $1 \leq i \leq l + r$.

(b) $l \equiv 4(mod6)$, $r \equiv 4(mod6)$

In this case, we define labeling f as

$f(v_i) = 0$; if $i \equiv 1, 4(mod6)$

$= 1$; if $i \equiv 0, 5(mod6)$

$= 2$; if $i \equiv 2, 3(mod6)$, $1 \leq i \leq l + r$.

(c) $l \equiv 4(mod6)$, $r \equiv 5(mod6)$

In this case, we define labeling f as

For $1 \leq i \leq l + 3$,

$f(v_i) = 0$; if $i \equiv 0, 3(mod6)$

$= 1$; if $i \equiv 1, 2(mod6)$

$= 2$; if $i \equiv 4, 5(mod6)$, $1 \leq i \leq l + r$.

$f(v_{l+4}) = 2$,

For $l + 5 \leq i \leq l + r$,

$f(v_i) = 0$; if $i \equiv 2, 5(mod6)$

$= 1$; if $i \equiv 3, 4(mod6)$

$= 2$; if $i \equiv 0, 1(mod6)$, $1 \leq i \leq l + r$.

<u>Case 6</u>:

(a) $l \equiv 5(mod6)$, $r \equiv 0, 1, 2, 5(mod6)$

In this case define labeling f as

$f(v_i) = 0$; if $i \equiv 0, 3(mod6)$

$= 1$; if $i \equiv 1, 2(mod6)$

$= 2$; if $i \equiv 4, 5(mod6)$, $1 \leq i \leq l + r$.

(b) $l \equiv 5(mod6)$, $r \equiv 3(mod6)$

In this case define labeling f as

$f(v_i) = 0$; if $i \equiv 1, 4(mod6)$,

$= 1$; if $i \equiv 0, 5(mod6)$,

$= 2$; if $i \equiv 2, 3(mod6)$.

(c) $l \equiv 5(mod6)$, $r \equiv 4(mod6)$

In this case define labeling f as

For $1 \leq i \leq l + 1$,

$f(v_i) = 0$; if $i \equiv 0, 3(mod6)$,

$= 1$; if $i \equiv 1, 2(mod6)$,

$= 2$; if $i \equiv 4, 5(mod6)$.

$f(v_{l+2}) = 0$, $f(v_{l+3}) = 2$,

For $l + 4 \leq i \leq l + r$,

$f(v_i) = 0$; if $i \equiv 2, 5(mod6)$,

$= 1$; if $i \equiv 3, 4(mod6)$,

$= 2$; if $i \equiv 0, 1(mod6)$.

The above defined labeling pattern covers all possible arrangement of vertices. In each case the graph under consideration satisfies the condition $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$, $0 \leq i, j \leq 2$. i.e. $T(l, r)$ admits 3-equitable labeling.

Illustration 8.3.5 For better understanding of above defined labeling pattern let us consider tadpole $T(11, 4)$ (it related to Case-6(c)). The 3-equitable labeling of tadpole $T(11, 4)$ is as shown in *Figure 8.2*.

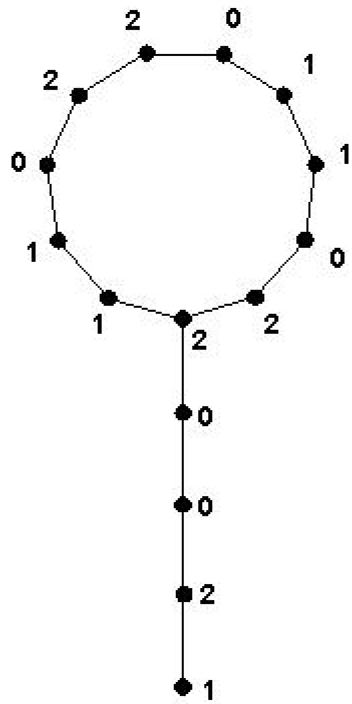


Figure 8.2

Now we will investigate one more family of 3-equitable graph.

Theorem 8.3.6 The graph obtained by joining two copies of cycle by a path of arbitrary length is 3-equitable.

Proof: Let G be the graph obtained by joining two copies of cycle C_n by path P_k. Let $u_1, u_2, \ldots, u_n$ be successive vertices of first copy of cycle C_n and $w_1, w_2, \ldots, w_n$ be successive vertices of second copy of cycle C_n. Let $v_1, v_2, \ldots, v_k$ be successive vertices of path P_k with $u_1 = v_1$ and $v_k = w_1$.

To define labeling function $f : V(G) \rightarrow \{0, 1, 2\}$ we consider following cases.

Case 1: $n \equiv 0(mod6)$, $k \equiv 0, 3(mod6)$

In this case define labeling f as

$f(u_i) = 0$; if $i \equiv 0, 3(mod6)$

$= 1$; if $i \equiv 1, 2(mod6)$

$= 2$; if $i \equiv 4, 5(mod6)$, $1 \leq i \leq n$.

$f(v_j) = 0$; if $j \equiv 0, 3(mod6)$

$= 1$; if $j \equiv 1, 2(mod6)$

$= 2$; if $j \equiv 4, 5(mod6)$, $1 \leq j \leq k$.

$f(w_i) = 0$; if $i \equiv 1, 4(mod6)$

$= 1$; if $i \equiv 2, 3(mod6)$

$= 2$; if $i \equiv 0, 5(mod6)$, $1 \leq i \leq n$.

Case 2: $n \equiv 0(mod6)$, $k \equiv 1, 2(mod6)$

In this case define labeling f as

$f(u_i) = 0$; if $i \equiv 0, 3(mod6)$

$= 1$; if $i \equiv 1, 2(mod6)$

$= 2$; if $i \equiv 4, 5(mod6)$, $1 \leq i \leq n$.

$f(v_j) = 0$; if $j \equiv 0, 3(mod6)$

$= 1$; if $j \equiv 1, 2(mod6)$

$= 2$; if $j \equiv 4, 5(mod6)$, $1 \leq j \leq k$.

$f(w_i) = 0$; if $i \equiv 0, 3(mod6)$

$= 1$; if $i \equiv 1, 2(mod6)$

$= 2$; if $i \equiv 4, 5(mod6)$, $1 \leq i \leq n$.

Case 3: $n \equiv 0(mod6)$, $k \equiv 4, 5(mod6)$

In this case define labeling f as

$f(u_i) = 0$; if $i \equiv 0, 3(mod6)$

$= 1$; if $i \equiv 1, 2(mod6)$

$= 2$; if $i \equiv 4, 5(mod6)$, $1 \leq i \leq n$.

$f(v_j) = 0$; if $j \equiv 0, 3 (mod 6)$

$= 1$; if $j \equiv 1, 2 (mod 6)$

$= 2$; if $j \equiv 4, 5 (mod 6)$, $1 \leq j \leq k$.

$f(w_i) = 0$; if $i \equiv 0, 3 (mod 6)$

$= 1$; if $i \equiv 4, 5 (mod 6)$

$= 2$; if $i \equiv 1, 2 (mod 6)$, $1 \leq i \leq n$.

Case 4: $n \equiv 1 (mod 6)$, $k \equiv 0 (mod 6)$

In this case define labeling f as

$f(u_i) = 0$; if $i \equiv 2, 5 (mod 6)$

$= 1$; if $i \equiv 0, 1 (mod 6)$

$= 2$; if $i \equiv 3, 4 (mod 6)$, $1 \leq i \leq n$.

$f(v_j) = 0$; if $j \equiv 2, 5 (mod 6)$

$= 1$; if $j \equiv 0, 1 (mod 6)$

$= 2$; if $j \equiv 3, 4 (mod 6)$, $1 \leq j \leq k$.

$f(w_i) = 0$; if $i \equiv 0, 3 (mod 6)$

$= 1$; if $i \equiv 1, 2 (mod 6)$

$= 2$; if $i \equiv 4, 5 (mod 6)$, $1 \leq i \leq n$.

Case 5: $n \equiv 1 (mod 6)$, $k \equiv 1 (mod 6)$

In this case define labeling f as

$f(u_i) = 0$; if $i \equiv 2, 5 (mod 6)$

$= 1$; if $i \equiv 0, 1 (mod 6)$

$= 2$; if $i \equiv 3, 4 (mod 6)$, $1 \leq i \leq n$.

$f(v_k) = 2$ and

$f(v_j) = 0$; if $j \equiv 2, 5(mod 6)$

$= 1$; if $j \equiv 0, 1(mod 6)$

$= 2$; if $j \equiv 3, 4(mod 6)$, $1 \leq j \leq k-1$.

$f(w_i) = 0$; if $i \equiv 2, 5(mod 6)$

$= 1$; if $i \equiv 3, 4(mod 6)$

$= 2$; if $i \equiv 0, 1(mod 6)$, $1 \leq i \leq n$.

<u>Case 6</u>: $n \equiv 1(mod 6)$, $k \equiv 2, 3(mod 6)$

In this case define labeling f as

$f(u_i) = 0$; if $i \equiv 1, 4(mod 6)$

$= 1$; if $i \equiv 2, 3(mod 6)$

$= 2$; if $i \equiv 0, 5(mod 6)$, $1 \leq i \leq n$.

$f(v_j) = 0$; if $j \equiv 1, 4(mod 6)$

$= 1$; if $j \equiv 2, 3(mod 6)$

$= 2$; if $j \equiv 0, 5(mod 6)$, $1 \leq j \leq k$.

$f(w_n) = 2$,

$f(w_i) = 0$; if $i \equiv 0, 3(mod 6)$

$= 1$; if $i \equiv 1, 2(mod 6)$

$= 2$; if $i \equiv 4, 5(mod 6)$, $1 \leq i \leq n-1$.

<u>Case 7</u>: $n \equiv 1(mod 6)$, $k \equiv 4(mod 6)$

In this case define labeling f as

$f(u_i) = 0$; if $i \equiv 1, 4(mod 6)$

$= 1$; if $i \equiv 2, 3(mod 6)$

$= 2$; if $i \equiv 0, 5(mod 6)$, $1 \leq i \leq n$.

$f(v_j) = 0$; if $j \equiv 1, 4(mod6)$

$= 1$; if $j \equiv 2, 3(mod6)$

$= 2$; if $j \equiv 0, 5(mod6)$, $1 \leq j \leq k$.

$f(w_n) = 2$,

$f(w_i) = 0$; if $i \equiv 1, 4(mod6)$

$= 1$; if $i \equiv 0, 5(mod6)$

$= 2$; if $i \equiv 2, 3(mod6)$, $1 \leq i \leq n-1$.

Case 8: $n \equiv 1(mod6)$, $k \equiv 5(mod6)$

In this case define labeling f as

$f(u_n) = 2$,

$f(u_i) = 0$; if $i \equiv 1, 4(mod6)$

$= 1$; if $i \equiv 0, 5(mod6)$

$= 2$; if $i \equiv 2, 3(mod6)$, $1 \leq i \leq n-1$.

$f(v_j) = 0$; if $j \equiv 1, 4(mod6)$

$= 1$; if $j \equiv 2, 3(mod6)$

$= 2$; if $j \equiv 0, 5(mod6)$, $1 \leq j \leq k$.

$f(w_i) = 0$; if $i \equiv 2, 5(mod6)$

$= 1$; if $i \equiv 3, 4(mod6)$

$= 2$; if $i \equiv 0, 1(mod6)$, $1 \leq i \leq n$.

Case 9: $n \equiv 2(mod6)$, $k \equiv 0(mod6)$

In this case define labeling f as

$f(u_n) = 2$, $f(u_{n-1}) = 0$,

$f(u_i) = 0$; if $i \equiv 0, 3(mod6)$

$= 1$; if $i \equiv 1, 2(mod6)$

$= 2$; if $i \equiv 4, 5(mod6)$, $1 \leq i \leq n-2$.

$f(v_j) = 0$; if $j \equiv 0, 3(mod6)$

. $= 1$; if $j \equiv 1, 2(mod6)$

$= 2$; if $j \equiv 4, 5(mod6)$, $1 \leq j \leq k$.

$f(w_i) = 0$; if $i \equiv 1, 4(mod6)$

$= 1$; if $i \equiv 2, 3(mod6)$

$= 2$; if $i \equiv 0, 5(mod6)$, $1 \leq i \leq n$.

Case 10: $n \equiv 2(mod6)$, $k \equiv 1(mod6)$

In this case define labeling f as

$f(u_i) = 0$; if $i \equiv 1, 4(mod6)$

$= 1$; if $i \equiv 2, 3(mod6)$

$= 2$; if $i \equiv 0, 5(mod6)$, $1 \leq i \leq n$.

$f(v_k) = 1$,

$f(v_j) = 0$; if $j \equiv 1, 4(mod6)$

$= 1$; if $j \equiv 0, 5(mod6)$

$= 2$; if $j \equiv 2, 3(mod6)$, $1 \leq j \leq k - 1$.

$f(w_n) = 2$, $f(w_{n-1}) = 0$,

$f(w_i) = 0$; if $i \equiv 0, 3(mod6)$

$= 1$; if $i \equiv 1, 2(mod6)$

$= 2$; if $i \equiv 4, 5(mod6)$, $1 \leq i \leq n - 2$.

Case-11: $n \equiv 2(mod6)$, $k \equiv 2(mod6)$

In this case define labeling f as

$f(u_i) = 0$; if $i \equiv 1, 4(mod6)$

$= 1$; if $i \equiv 0, 5(mod6)$

$= 2$; if $i \equiv 2, 3(mod6)$, $1 \leq i \leq n$.

$f(v_k) = 0$,

$f(v_j) = 0$; if $j \equiv 1, 4(mod6)$

$= 1$; if $j \equiv 2, 3(mod6)$

$= 2$; if $j \equiv 0, 5(mod6)$, $1 \leq j \leq k - 1$.

$f(w_i) = 0$; if $i \equiv 1, 4(mod6)$

$= 1$; if $i \equiv 2, 3(mod6)$

$= 2$; if $i \equiv 0, 5(mod6)$, $1 \leq i \leq n$.

Case-12: $n \equiv 2(mod6)$, $k \equiv 3(mod6)$

In this case define labeling f as

$f(u_n) = 0$, $f(u_{n-1}) = 2$,

$f(u_i) = 0$; if $i \equiv 1, 4(mod6)$

$= 1$; if $i \equiv 2, 3(mod6)$

$= 2$; if $i \equiv 0, 5(mod6)$, $1 \leq i \leq n - 2$.

$f(v_k) = 2$,

$f(v_j) = 0$; if $j \equiv 1, 4(mod6)$

$= 1$; if $j \equiv 2, 3(mod6)$

$= 2$; if $j \equiv 0, 5(mod6)$, $1 \leq j \leq k - 1$.

$f(w_n) = 0$,

$f(w_i) = 0$; if $i \equiv 0, 3(mod6)$

$= 1$; if $i \equiv 4, 5(mod6)$

$= 2$; if $i \equiv 1, 2(mod6)$, $1 \leq i \leq n - 1$.

Case-13: $n \equiv 2(mod6)$, $k \equiv 4(mod6)$

In this case define labeling f as

$f_(u_n) = 0$, $f(u_{n-1}) = 2$,

$f(u_i) = 0$; if $i \equiv 0, 3(mod6)$

$= 1$; if $i \equiv 1, 2(mod6)$

$= 2$; if $i \equiv 4, 5(mod6)$, $1 \leq i \leq n-2$.

$f(v_j) = 0$; if $j \equiv 0, 3(mod6)$

$= 1$; if $j \equiv 1, 2(mod6)$

$= 2$; if $j \equiv 4, 5(mod6)$, $1 \leq j \leq k$.

$f(w_n) = 2$, $f(w_{n-1}) = 1$,

$f(w_i) = 0$; if $i \equiv 0, 3(mod6)$

$= 1$; if $i \equiv 4, 5(mod6)$

$= 2$; if $i \equiv 1, 2(mod6)$, $1 \leq i \leq n-2$.

Case-14: $n \equiv 2(mod6)$, $k \equiv 5(mod6)$

In this case define labeling f as

$f(u_i) = 0$; if $i \equiv 2, 5(mod6)$

$= 1$; if $i \equiv 0, 1(mod6)$

$= 2$; if $i \equiv 3, 4(mod6)$, $1 \leq i \leq n$.

$f(v_j) = 0$; if $j \equiv 0, 3(mod6)$

$= 1$; if $j \equiv 1, 2(mod6)$

$= 2$; if $j \equiv 4, 5(mod6)$, $1 \leq j \leq k$.

$f(w_i) = 0$; if $i \equiv 2, 5(mod6)$

$= 1$; if $i \equiv 3, 4(mod6)$

$= 2$; if $i \equiv 0, 1(mod6)$, $1 \leq i \leq n$.

Case-15: $n \equiv 3(mod6)$, $k \equiv 0(mod6)$

In this case define labeling f as

$f(u_i) = 0$; if $i \equiv 1, 4(mod6)$

$= 1$; if $i \equiv 0, 5(mod6)$

$= 2$; if $i \equiv 2, 3(mod6)$, $1 \leq i \leq n$.

$f(v_j) = 0$; if $j \equiv 1, 4(mod6)$

$= 1$; if $j \equiv 0, 5(mod6)$

$= 2$; if $j \equiv 2, 3(mod6)$, $1 \leq j \leq k$.

$f(w_i) = 0$; if $i \equiv 0, 3(mod6)$

$= 1$; if $i \equiv 1, 2(mod6)$

$= 2$; if $i \equiv 4, 5(mod6)$, $1 \leq i \leq n$.

Case-16: $n \equiv 3(mod6)$, $k \equiv 1(mod6)$

In this case define labeling f as

$f(u_i) = 0$; if $i \equiv 1, 4(mod6)$

$= 1$; if $i \equiv 0, 5(mod6)$

$= 2$; if $i \equiv 2, 3(mod6)$, $1 \leq i \leq n$.

$f(v_j) = 0$; if $j \equiv 1, 4(mod6)$

$= 1$; if $j \equiv 2, 3(mod6)$

$= 2$; if $j \equiv 0, 5(mod6)$, $1 \leq j \leq k$.

$f(w_i) = 0$; if $i \equiv 1, 4(mod6)$

$= 1$; if $i \equiv 2, 3(mod6)$

$= 2$; if $i \equiv 0, 5(mod6)$, $1 \leq i \leq n$.

Case-17: $n \equiv 3(mod6)$, $k \equiv 2(mod6)$

In this case define labeling f as

$f(u_i) = 0$; if $i \equiv 1, 4(mod6)$

$= 1$; if $i \equiv 2, 3(mod6)$

$= 2$; if $i \equiv 0, 5(mod6)$, $1 \leq i \leq n$.

$f(v_k) = 0,$

$f(v_j) = 0;$ if $j \equiv 1, 4(mod 6)$

$= 1;$ if $j \equiv 2, 3(mod 6)$

$= 2;$ if $j \equiv 0, 5(mod 6)$, $1 \leq j \leq k - 1.$

$f(w_i) = 0;$ if $i \equiv 1, 4(mod 6)$

$= 1;$ if $i \equiv 0, 5(mod 6)$

$= 2;$ if $i \equiv 2, 3(mod 6)$, $1 \leq i \leq n.$

Case-18: $n \equiv 3(mod 6)$, $k \equiv 3(mod 6)$

In this case define labeling f as

$f(u_i) = 0;$ if $i \equiv 1, 4(mod 6)$

$= 1;$ if $i \equiv 0, 5(mod 6)$

$= 2;$ if $i \equiv 2, 3(mod 6)$, $1 \leq i \leq n.$

$f(v_j) = 0;$ if $j \equiv 1, 4(mod 6)$

$= 1;$ if $j \equiv 2, 3(mod 6)$

$= 2;$ if $j \equiv 0, 5(mod 6)$, $1 \leq j \leq k.$

$f(w_i) = 0;$ if $i \equiv 0, 3(mod 6)$

$= 1;$ if $i \equiv 1, 2(mod 6)$

$= 2;$ if $i \equiv 4, 5(mod 6)$, $1 \leq i \leq n.$

Case-19: $n \equiv 3(mod 6)$, $k \equiv 4, 5(mod 6)$

In this case define labeling f as

$f(u_i) = 0;$ if $i \equiv 0, 3(mod 6)$

$= 1;$ if $i \equiv 1, 2(mod 6)$

$= 2;$ if $i \equiv 4, 5(mod 6)$, $1 \leq i \leq n.$

$f(v_j) = 0$; if $j \equiv 0, 3(mod6)$

$= 1$; if $j \equiv 1, 2(mod6)$

$= 2$; if $j \equiv 4, 5(mod6)$, $1 \leq j \leq k$.

$f(w_i) = 0$; if $i \equiv 0, 3(mod6)$

$= 1$; if $i \equiv 4, 5(mod6)$

$= 2$; if $i \equiv 1, 2(mod6)$, $1 \leq i \leq n$.

Case-20: $n \equiv 4(mod6)$, $k \equiv 0(mod6)$

In this case define labeling f as

$f(u_n) = 0$,

$f(u_i) = 0$; if $i \equiv 0, 3(mod6)$

$= 1$; if $i \equiv 4, 5(mod6)$

$= 2$; if $i \equiv 1, 2(mod6)$, $1 \leq i \leq n - 1$.

$f(v_j) = 0$; if $j \equiv 0, 3(mod6)$

$= 1$; if $j \equiv 4, 5(mod6)$

$= 2$; if $j \equiv 1, 2(mod6)$, $1 \leq j \leq k$.

$f(w_n) = 2$,

$f(w_i) = 0$; if $i \equiv 1, 4(mod6)$

$= 1$; if $i \equiv 2, 3(mod6)$

$= 2$; if $i \equiv 0, 5(mod6)$, $1 \leq i \leq n - 1$.

Case-21: $n \equiv 4(mod6)$, $k \equiv 1(mod6)$

In this case define labeling f as

$f(u_n) = 0$,

$f(u_i) = 0$; if $i \equiv 0, 3(mod6)$

$= 1$; if $i \equiv 4, 5(mod6)$

$= 2$; if $i \equiv 1, 2(mod6)$, $1 \leq i \leq n - 1$.

$f(v_k) = 1,$

$f(v_j) = 0;$ if $j \equiv 0, 3(mod6)$

$= 1;$ if $j \equiv 4, 5(mod6)$

$= 2;$ if $j \equiv 1, 2(mod6)$, $1 \leq j \leq k-1$.

$f(w_i) = 0;$ if $i \equiv 0, 3(mod6)$

$= 1;$ if $i \equiv 1, 2(mod6)$

$= 2;$ if $i \equiv 4, 5(mod6)$, $1 \leq i \leq n$

Case-22: $n \equiv 4(mod6)$, $k \equiv 2(mod6)$

In this case define labeling f as

$f(u_i) = 0;$ if $i \equiv 0, 3(mod6)$

$= 1;$ if $i \equiv 1, 2(mod6)$

$= 2;$ if $i \equiv 4, 5(mod6)$, $1 \leq i \leq n$.

$f(v_j) = 0;$ if $j \equiv 2, 5(mod6)$

$= 1;$ if $j \equiv 0, 1(mod6)$

$= 2;$ if $j \equiv 3, 4(mod6)$, $1 \leq j \leq k$.

$f(w_i) = 0;$ if $i \equiv 1, 4(mod6)$

$= 1;$ if $i \equiv 0, 5(mod6)$

$= 2;$ if $i \equiv 2, 3(mod6)$, $1 \leq i \leq n$.

Case-23: $n \equiv 4(mod6)$, $k \equiv 3(mod6)$

In this case define labeling f as

$f(u_i) = 0;$ if $i \equiv 0, 3(mod6)$

$= 1;$ if $i \equiv 1, 2(mod6)$

$= 2;$ if $i \equiv 4, 5(mod6)$, $1 \leq i \leq n$.

$f(v_j) = 0$; if $j \equiv 0, 3(mod6)$

$= 1$; if $j \equiv 1, 2(mod6)$

$= 2$; if $j \equiv 4, 5(mod6)$, $1 \leq j \leq k$.

$f(w_i) = 0$; if $i \equiv 1, 4(mod6)$

$= 1$; if $i \equiv 0, 5(mod6)$

$= 2$; if $i \equiv 2, 3(mod6)$, $1 \leq i \leq n$.

Case-24: $n \equiv 4(mod6)$, $k \equiv 4(mod6)$

In this case define labeling f as

$f(u_i) = 0$; if $i \equiv 0, 3(mod6)$

$= 1$; if $i \equiv 1, 2(mod6)$

$= 2$; if $i \equiv 4, 5(mod6)$, $1 \leq i \leq n$.

$f(v_j) = 0$; if $j \equiv 0, 3(mod6)$

$= 1$; if $j \equiv 1, 2(mod6)$

$= 2$; if $j \equiv 4, 5(mod6)$, $1 \leq j \leq k$.

$f(w_n) = 0$,

$f(w_i) = 0$; if $i \equiv 0, 3(mod6)$

$= 1$; if $i \equiv 4, 5(mod6)$

$= 2$; if $i \equiv 1, 2(mod6)$, $1 \leq i \leq n-1$.

Case-25: $n \equiv 4(mod6)$, $k \equiv 5(mod6)$

In this case define labeling f as

$f(u_i) = 0$; if $i \equiv 2, 5(mod6)$

$= 1$; if $i \equiv 3, 4(mod6)$

$= 2$; if $i \equiv 0, 1(mod6)$, $1 \leq i \leq n$.

$f(v_j) = 0$; if $j \equiv 2, 5(mod6)$

$= 1$; if $j \equiv 3, 4(mod6)$

$= 2$; if $j \equiv 0, 1(mod6)$, $1 \leq j \leq k$.

$f(w_i) = 0$; if $i \equiv 1, 4(mod6)$

$= 1$; if $i \equiv 0, 5(mod6)$

$= 2$; if $i \equiv 2, 3(mod6)$, $1 \leq i \leq n$.

<u>Case-26</u>: $n \equiv 5(mod6)$, $k \equiv 0(mod6)$

In this case define labeling f as

$f(u_i) = 0$; if $i \equiv 0, 3(mod6)$

$= 1$; if $i \equiv 1, 2(mod6)$

$= 2$; if $i \equiv 4, 5(mod6)$, $1 \leq i \leq n$.

$f(v_j) = 0$; if $j \equiv 0, 3(mod6)$

$= 1$; if $j \equiv 1, 2(mod6)$

$= 2$; if $j \equiv 4, 5(mod6)$, $1 \leq j \leq k$.

$f(w_i) = 0$; if $i \equiv 1, 4(mod6)$

$= 1$; if $i \equiv 2, 3(mod6)$

$= 2$; if $i \equiv 0, 5(mod6)$, $1 \leq i \leq n$.

<u>Case-27</u>: $n \equiv 5(mod6)$, $k \equiv 1(mod6)$

In this case define labeling f as

$f(u_i) = 0$; if $i \equiv 1, 4(mod6)$

$= 1$; if $i \equiv 0, 5(mod6)$

$= 2$; if $i \equiv 2, 3(mod6)$, $1 \leq i \leq n$.

$f(v_k) = 2$,

$f(v_j) = 0$; if $j \equiv 1, 4(mod6)$

$= 1$; if $j \equiv 2, 3(mod6)$

$= 2$; if $j \equiv 0, 5(mod6)$, $1 \leq j \leq k - 1$.

$f(w_i) = 0$; if $i \equiv 2, 5(mod6)$

$= 1$; if $i \equiv 3, 4(mod6)$

$= 2$; if $i \equiv 0, 1(mod6)$, $1 \leq i \leq n$.

Case-28: $n \equiv 5(mod6)$, $k \equiv 2(mod6)$

In this case define labeling f as

$f(u_i) = 0$; if $i \equiv 2, 5(mod6)$

$= 1$; if $i \equiv 0, 1(mod6)$

$= 2$; if $i \equiv 3, 4(mod6)$, $1 \leq i \leq n$.

$f(v_j) = 0$; if $j \equiv 0, 3(mod6)$

$= 1$; if $j \equiv 1, 2(mod6)$

$= 2$; if $j \equiv 4, 5(mod6)$, $1 \leq j \leq k$.

$f(w_i) = 0$; if $i \equiv 0, 3(mod6)$

$= 1$; if $i \equiv 1, 2(mod6)$

$= 2$; if $i \equiv 4, 5(mod6)$, $1 \leq i \leq n$.

Case-29: $n \equiv 5(mod6)$, $k \equiv 3(mod6)$

In this case define labeling f as

$f(u_i) = 0$; if $i \equiv 1, 4(mod6)$

$= 1$; if $i \equiv 2, 3(mod6)$

$= 2$; if $i \equiv 0, 5(mod6)$, $1 \leq i \leq n$.

$f(v_j) = 0$; if $j \equiv 1, 4(mod6)$

$= 1$; if $j \equiv 0, 5(mod6)$

$= 2$; if $j \equiv 2, 3(mod6)$, $1 \leq j \leq k$.

$f(w_i) = 0$; if $i \equiv 0, 3(mod6)$

$= 1$; if $i \equiv 4, 5(mod6)$

$= 2$; if $i \equiv 1, 2(mod6)$, $1 \leq i \leq n$.

Case-30: $n \equiv 5(mod6)$, $k \equiv 4, 5(mod6)$

In this case define labeling f as

$f(u_i) = 0$; if $i \equiv 2, 5(mod6)$

$= 1$; if $i \equiv 0, 1(mod6)$

$= 2$; if $i \equiv 3, 4(mod6)$, $1 \leq i \leq n$.

$f(v_j) = 0$; if $j \equiv 0, 3(mod6)$

$= 1$; if $j \equiv 1, 2(mod6)$

$= 2$; if $j \equiv 4, 5(mod6)$, $1 \leq j \leq k$.

$f(w_i) = 0$; if $i \equiv 0, 3(mod6)$

$= 1$; if $i \equiv 4, 5(mod6)$

$= 2$; if $i \equiv 1, 2(mod6)$, $1 \leq i \leq n$.

The above defined labeling pattern covers all possible arrangement of vertices. In each case the graph under consideration satisfies the condition $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$, $0 \leq i, j \leq 2$. i.e. G admits 3-equitable labeling.

Illustration 8.3.7 For better understanding of above defined labeling pattern let us consider the graph obtained by joining two copies of cycle C_{10} by path P_9 (it is related to Case-23). The 3-equitable labeling is as shown in *Figure 8.3*.

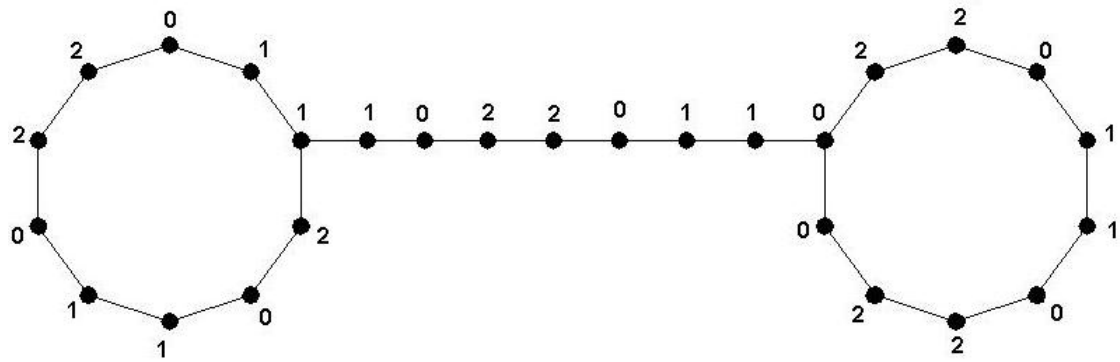


Figure 8.3

8.4 Some Open Problems

¶ In connection of 3-equitable labeling of star of a graph one can investigate 3-equitable labeling for star of wheel, Petersen graph, cycle with chord, cycle with twin chords, cycle with triangle etc.

¶ In *Theorem 8.3.6* we have considered path of arbitrary length between two copies of cycle but one can investigate 3-equitable labeling for the graph obtained by joining a path of arbitrary length between two copies of cycle with chord, cycle with twin chords, cycle with triangle, Petersen graph etc.

¶ One can discuss 3-equitable labeling in the context of various graph operations like contraction, barycentric subdivision etc.

8.5 Concluding Remarks

This chapter was aimed to discuss 3-equitable labeling in detail. Three new results are obtained. The labeling pattern is given and demonstrated by means of enough illustrations. The results obtained here are new and original. Result of *Theorem 8.3.1* is accepted for publication in *Proceedings of The International Conference on Emerging Technology and Applications in Engineering, Technology and Sciences* (2008) which is a collaborative work of Vaidya et al.[121]. This work contributes three new graphs to the theory of 3-equitable graphs. Some open problems are also given which will indicate the scope of further research.

In the next chapter we will discuss applications of graph labeling.

Chapter 9
Applications of Graph Labeling

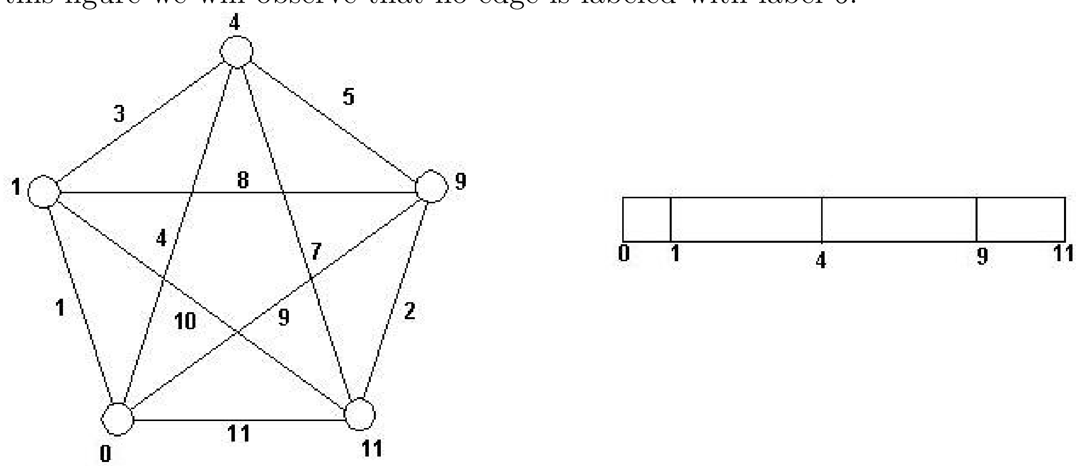
9.1 Introduction

Labeled graphs are becoming more interesting due to their broad range of applications. This family has variety of applications in diversified fields. Labeled graphs have vital applications to coding theory, particularly in the development of missile guidance codes, design of radar type codes and convolution codes with optimal autocorrelation properties. Optimal circuit layouts and solution of problem of number theory can be discussed in the context of graph labeling. Ambiguity in X-ray crystallography can also be explained using graph labeling techniques. A detail survey on such applications is systematically studied by Bloom and Golomb[24]. We will discuss some interesting applications reported in that paper. Some of these applications are also recorded in Germina[55].

9.2 Semigraceful Labeling and Golomb Ruler

We have discussed graceful labeling and graceful graphs in *Chapter 5*. As we noted there K_n is graceful if and only if $n \leq 4$. In other words it is not possible to label vertices with numbers $\{0, 1, 2 \ldots, {}_nC_2\}$ such that each edge can be labeled distinctly using labels $\{1, 2 \ldots, {}_nC_2\}$. This problem has motivated Golomb to define semigraceful labeling. According to him if the constraint *edge labels to be consecutive integers* is relaxed then such labeling is called *semigraceful labeling* and the graph which admits such labeling is called *semigraceful graph*. In other words semigraceful graph on n vertices does not use all the labels from $\{1, 2 \ldots, {}_nC_2\}$ but some edge labels are missing. In general vertex labels in semigraceful labeling may exceed ${}_nC_2$ or repeat or both. Semigraceful labeling is *optimal* if it minimizes the largest edge label which is denoted by $G(K_n)$.

In the following *Figure 9.1(a)* a semigraceful labeling for K_5 is shown. In this figure we will observe that no edge is labeled with label 6.

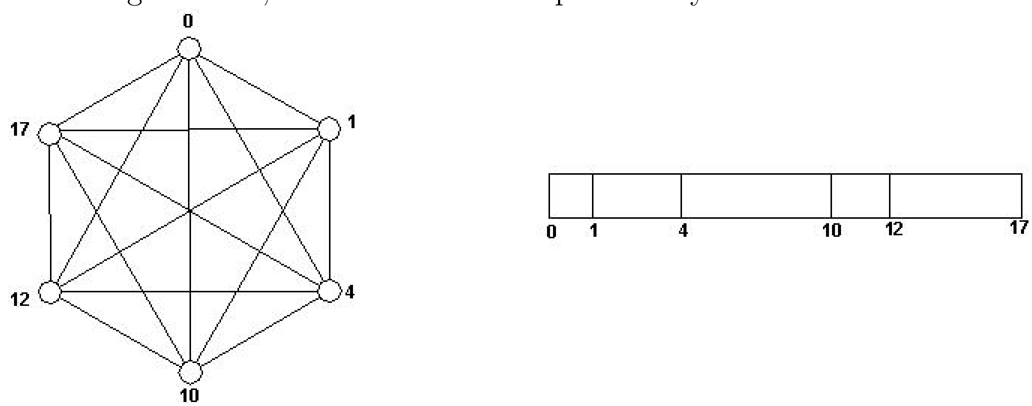


(a) *(b)*

Figure 9.1

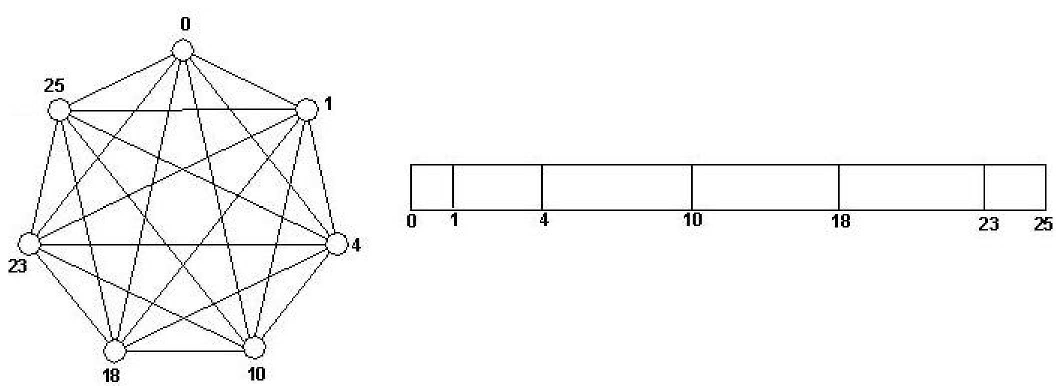
Golomb observed an important equivalence for the coding theory context between a semigraceful labeling which minimizes $G(K_n)$. He developed a special ruler on which n division marks (including the ends) are placed. The positions of the division marks correspond to the number placed on the end vertices of K_n. The edge labels of K_n thus exactly correspond to the set of measurements which can be made on the ruler. Such ruler is named by Gardner[53] as a *Golomb Ruler*. In *Figure 9.1(b)* a ruler corresponding to semigraceful labeling for K_5 is shown. As we mentioned earlier no edge is labeled with 6. Equivalently from *Figure 9.1(b)* we can see that it is not possible to measure length 6 directly by the Golomb Ruler. All optimal rulers have been found for $n \leq 11$ and are summarized in Bloom and Golomb[25]. Such ruler will be able to measure ${}_nC_2$ lengths which are numerically equal to edge labels of K_n and they measure non-redundant minimal length.

In *Figure 9.2* to *9.4* we provide semigraceful labeling and equivalent Golomb rulers for K_6, K_7, K_8 respectively. These rulers will measure maximum lengths of 17, 25 and 34 units in optimal way.



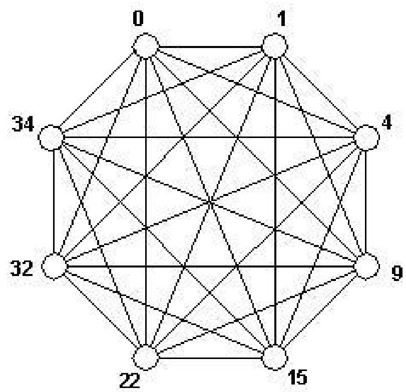
(a) *(b)*

Figure 9.2

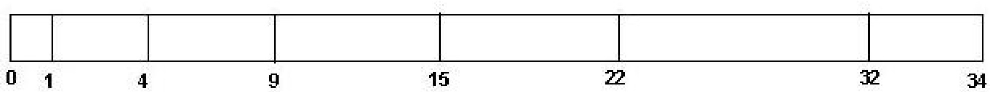


(a) *(b)*

Figure 9.3



(a)



(b)

Figure 9.4

It is also possible to provide other pattern of labeling and corresponding ruler. Such rulers are called *homometric rulers*. For example for K_6 it is possible to provide semigraceful labeling using vertex labels $0, 1, 4, 10, 15, 17$ or $0, 1, 4, 11, 13, 17$ or $0, 1, 8, 12, 14, 17$.

In the following *Table 9.1* we have summarized the particulars regarding possible semigraceful labeling of K_n for $n \leq 11$.

Nodes	Length	Divisions	Marks at
2	1	1	0,1
3	3	1,2	0,1,3
4	6	1,3,2	0,1,4,6
5	11	1,3,5,2	0,1,4,9,11
		2,5,1,3	0,2,7,8,11
6	17	1,3,6,2,5	0,1,4,10,12,17
		1,3,6,5,2	0,1,4,10,15,17
		1,7,3,2,4	0,1,8,11,13,17
		1,7,4,2,3	0,1,8,12,14,17
7	25	1,3,6,8,5,2	0,1,4,10,18,23,25
		1,6,4,9,3,2	0,1,7,11,20,23,25
		1,10,5,3,4,2	0,1,11,16,19,23,25
		2,1,7,6,5,4	0,2,3,10,16,21,25
		2,5,6,8,1,3	0,2,7,13,21,22,25
8	34	1,3,5,6,7,10,2	0,1,4,9,15,22,32,34
9	44	1,4,7,13,2,8,6,3	0,1,5,12,25,27,35,41,44
10	55	1,5,4,13,3,8,7,12,2	0,1,6,10,23,26,34,41,53,55
11	72	1,3,9,15,5,14,7,10,6,2	0,1,4,13,28,33,47,54,64,70,72
		1,8,10,5,7,21,4,2,11,3	0,1,9,19,24,31,52,56,58,69,72

Table 9.1

The discovery of Golomb Rulers with more marks as well as method for generating such class remains an open problem. The Golomb Rulers discussed above have several applications in coding theory, X-ray crystallography etc. In the remaining part of this chapter we will discuss such applications.

9.3 Generation of Radar Type Codes

In the previous section we have discussed Golomb Ruler in detail and also seen the possibility to measure the lengths (distances) with that ruler. In coding context distance interval is replaced by time interval. Let us consider a time mark ruler corresponding to K_5 shown in *Figure 9.1.* One can generate a radar code from this ruler by transmitting a sequence of five pulses at times corresponding to the marks on the ruler. i.e. 0,1,4,9,11. We observe that there is a 1 unit time interval between the onset of the first and second pulses, 3 units time interval between the second and third, 5 units time interval between third and fourth and 2 units between the last two. The time duration between the emission of the signal and its return is determined by correlating all incoming sequences of 11 time units duration with the original sequence. Let each pulse be of one unit duration. Thus, when an incoming string matches the original as shown in following *Figure 9.5(a)* then a signal of strength 5 is generated as shown in following *Figure 9.5(b)*.

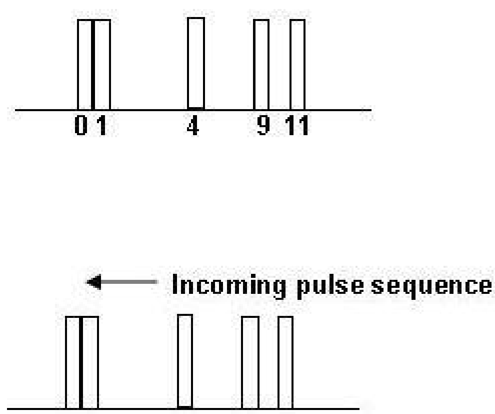


Figure 9.5 (a)

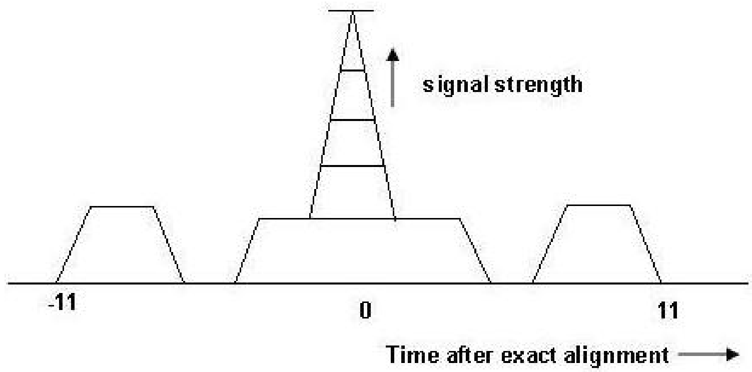
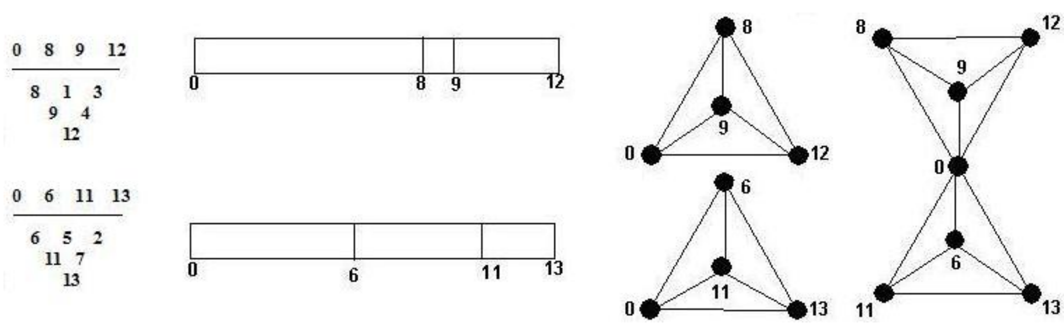


Figure 9.5 (b)

In the same *Figure 9.5(b)* we can see that a dip in the autocorrelation occurs at ±6 time units, since there are no pulses which are aligned with a 6 unit shift of the pulse sequence out of its synch position. Six, of course, is the only distance of 11 or fewer units that the original ruler can not measure. We have also seen that it is the only number which is missing in labeling of K_5.

Eckler[45] investigated the problem related to above application for designing missile guidance codes. In an air borne missile, receiver passes all incoming signal trains down a delay line. If the line is tapped in several places which correspond to the actual time interval between incoming pulses, then the sum of those pulses will exceed a threshold and initiate some control action.

The command code for such a missile contains two or more different commands. Thus, in terms of instrumentation the delay line must be tapped by sets of leads corresponding to the delays between pulses for each command. In order to make code insensitive to random interference pulses (such as electrical storms or jamming effects) all of the delays pulses for one command must totally differ from those for every command. It is also desirable to use the shortest code-word durations possible in order to minimize the delay line and to decrease the time during which interference could occur. Thus Eckler calculated $(d-1)$ intervals for the d pulses associated with of n different commands. In synch these commands give on reception by the missile, an autocorrelation of height d. Out-of-synch, the maximum autocorrelation is 1, and the noiseless cross-correlation between commands also never exceeds 1. This problem is analogous to find a set of n rulers of different lengths with $(d-1)$ marks on it. The marks on these rulers permit measuring each length in only one way. Moreover, the longest of these rulers must be as short as possible. Alternatively the problem corresponds to label as gracefully as possible a disconnected graph with n components. Each component is a complete graph on $(d-1)$ vertices. For this each component of the composite graph has a vertex labeled with 0. In the following *Figure 9.6* 2-message, 4-pulse missile code with minimum duration is shown.



(a) (b) (c) (d)

Figure 9.6

In the above *Figure 9.6*,

(a) Difference triangles

(b) Rulers

(c) Disconnected graph with 2 components

(d) Connected graph

9.4 X-ray Crystallography and Golomb Ruler

Ruler models are very much useful in X-ray crystallography. It sometimes happens that distinct crystal structures will give rise to identical X-ray diffraction patterns. These inherent ambiguities in the X-ray analysis of crystal structures have been studied by Patterson[101], Garrido[54] and Franklin[48].

For any crystal structure positions of atoms are determined by measurements made on X-ray diffraction patterns. These measurements indicate the set of distances between atoms in the crystal lattice, but in general do not necessarily specify the absolute positions of the atoms without any ambiguity. Mathematically, finite sets of integers $R = \{0 = a_1 < a_2 < \ldots < a_n\}$ and $S = \{0 = b_1 < b_2 < \ldots < a_n = b_n\}$ corresponding to two atom positions may have exactly the same set of differences $D(R) = D(S) = \{|a_i - a_j| : i < j\}$. Since the diffraction pattern determines the set of differences $D(R)$, it is impossible to determine which of the homometric sets R or S produced it, and consequently which crystal lattice give rise to the diffraction pattern. This homometric set problem may be viewed as a determination of non-equivalent rulers, which make identical sets of measurements. The sets R and S designate the positions of the marks of two rulers and $D(R)$ and $D(S)$ are their respective sets of ${}_nC_2$ measurements.

Thus the class of diffraction patterns corresponds to a set of differences, which has no repeated elements, i.e., to a non-redundant set. Two equivalent rulers are shown in *Figure 9.7.* Also there are no non-redundant rulers with fewer than 6 points or of length less than 17.

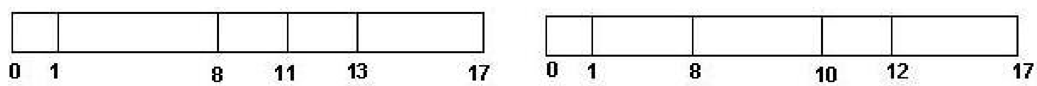


Figure 9.7

Measurements made by the rulers are 1,2,3,4,5,6,7,8,9,10,11,12,13,16,17.

The shortest non-redundant homometric pairs of rulers and the ${}_6C_2 = 15$ intervals which they measure.

9.5 Communication Network Labeling

In a small communication network, it may be desirable to assign each user terminal *node number*(vertex label) subject to the constraint that all the resulting edges (communication links) receive distinct numbers. In this way, the numbers of any two communicative terminals automatically specify (by simple subtraction) the link number of the connecting path; and conversely the path number uniquely corresponds to the pair of user terminals which it interconnects.

Properties of a potential numbering system for such networks have been explored under the guise of gracefully labeled graphs, that is, the properties of graceful graphs provide design parameters for an appropriate communication network.

If a graphical model of any communication network can not be labeled gracefully, there is a possibility of using semigraceful labeling in which the constraint requiring *the edge labels to be consecutive integers* is relaxed.

The most important question for utilizing a *graceful addressing and identification system* involve being better able to determine whether an arbitrary model of a communications network is in a graceful configuration. If it is, how should it be labeled? If it isn't, can it be embedded into a graceful structure easily? or should it be labeled semigracefully? Moreover, determination needs to be made of growth provisions for any addressing scheme, i.e., of algorithms for labeling a graph in which new vertices and edges have been added to a gracefully labeled graph.

9.6 Scope of Further Research

¶ One can explore the related ruler problems which have similar applications to communications network. This includes the problems of finding the shortest rulers with k marks which measure all integer lengths from 1 to n, either (i) allowing the same length to be measured in more than one way, or (ii) not allowing the same length to be measured in more than one way.

¶ One can study the structure of different crystals using the ruler model. This approach will give rise to interdisciplinary research work.

¶ One can develop the graph model for communication network using other labeling techniques like harmonious labeling, k-equitable labeling etc.

9.7 Concluding Remarks

Graph labelings present a common context for many applied and theoretical problems. Some of these are illustrated in the current chapter. Graph labeling and diversified applications are held together by common thread. This chapter creates an impression of graph labeling as a unifying model which has vital potential to provide solutions for practical purposes. Graph labeling techniques may work as a powerful unifying model with biotechnology, information technology and new generation communication network. One can develop new labeling technique and discover its applications to diversified area.

List of Symbols

$\|B\|$	Cardinality of set B.
CH_n	Closed helm on n vertices.
C_n	Cycle with n vertices.
C_n^*	Star of cycle C_n.
$E(G)$ or E	Edge set of graph G.
F_n	Fan on n vertices.
$\overline{G}$	Complement of G.
$G \cup H$	Union of graphs G and H.
$G \cap H$	Intersection of graphs G and H.
$G \times H$	Cartesian product of graphs G and H.
$G + H$	Join of graphs G and H.
$G \cong H$	G is isomorphic to H.
$G = (V, E)$	A graph G with vertex set V and edge se E.
$G + v$	Suspension of graph G and vertex v.
$G * e$	Contraction of edge e in graph G.
$G - e$	Graph G with one edge deleted.
$G - v$	Graph G with one vertex deleted.
H_n	Helm on n vertices.
K_n	Complete graph on n vertices.
$K_{m,n}$	Complete bipartite graph.
$N(v)$	Neighbourhood of vertex v.

P_n	Path graph on n vertices.
S_n	Shell on n vertices.
T	Tree.
$T(G)$	Spanning tree of graph G.
$V(G)$ or V	Vertex set of graphs G.
W_n	Wheel on n vertices.
(a, b)	Greatest Common Divisor of integers a and b.
$d(v)$ or $d_G(v)$	Degree of a vertex v of graph G.
$\triangle(G)$	Maximum degree of a vertex in graph G.
$\delta(G)$	Minimum degree of a vertex in graph G.
$e_f(n)$	Number of edges with edge label n.
${}_nC_r$ or ${}^nC_r$	r Combinations of an n objects.
$\lceil n \rceil$	Least integer not less than real number n (Ceiling of n).
$\lfloor n \rfloor$	Greatest integer not greater than real number n (Floor of n).
(p, q)	A graph with order p and size q.
$v_f(n)$	Number of vertices with vertex label n.

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