Some Topics of Special Interest in Graph Theory

a thesis submitted to

THE SAURASHTRA UNIVERSITY
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for the award of the degree of

DOCTOR OF PHILOSOPHY

in

MATHEMATICS

by

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under the supervision of

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Certificate

This is to certify that the thesis entitled Some Topics of Special Interest in Graph Theory submitted by **Vinodray J. Kaneria** to the **Saurashtra University, RAJKOT** for the award of the degree of Doctor of Philosophy in Mathematics is bonafide record of research work carried out by him under my supervision. The contents embodied in the thesis have not been submitted in part or full to any other Institution or University for the award of any degree or diploma.

Place : RAJKOT.

Date : 24^{th} March 2008.

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Declaration

I hereby declare that the contents embodied in this thesis is the bonafide record of investigations carried out by me under the supervision of **Dr. S. K. Vaidya** in the department of Mathematics, **Saurashtra University, RAJKOT**. The investigations reported here have not been submitted in part or full for the award of any degree or diploma to any other Institution or University.

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Acknowledgement

It is a moment of immense pleasure for me to present this thesis entitled SOME TOPICS OF SPECIAL INTEREST IN GRAPH THEORY to the **Saurashtra University, RAJKOT** for the award of the degree of DOCTOR OF PHILOSOPHY in the subject of **Mathematics.**

I am highly indebted to my research supervisor **Dr. S. K. Vaidya** for his valuable guidance and meticulous care throughout my career. Without his constant encouragement and inspiration this would not have seen the light of the day.

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Vinodray J. Kaneria

Contents

Chapter – 1

Introduction

The attempt to solve famous Königsberg bridge problem was the starting point of graph theory. The great Swiss mathematician Leonhard Euler(1707−1783) had discussed this problem using graphs in 1736 and introduced that how graph theory solves complicated problems which concern to practical situations. After this nothing more was cultivated in this field for around hundred years. In 1848, A. Cayley used this theory for the study of isomers of saturated hydrocarbons. Presentation of four color problem by A. F. Möbious and A. De Morgan during 1840 to 1850 provide the reason to boost up research in the theory of graphs. A game *Around The World* by William Hamilton in 1859 drew attention of several scholars. The fertile period was followed by half a century of relative inactivity. In 1936 D. König published the first book on graph theory which organized the work of several mathematicians and his own. Past 50 years has been a period of intense activities in the field of graph theory. At present thousands of papers have been published and many titles available by eminent authors like Claud Berge, Paul Erdős, Frank Harary, Douglas West, Jonathan Gross and Jay Yellen.

The graph theory and its applications have grown exponentially in the last century. Development of computer science and optimization techniques are responsible for this unprecedented growth. The graph theory has surprising number of applications. It is applied to almost all the fields and in variety of subjects like computer science, physical science, biological science and social science. This theory also helps to understand molecular structure of atoms in chemistry. Easy and optimized electrical network as well as communication network is possible using graphs. Thus it seems that the graph theory enjoys the status of a beautiful *QUEEN* in the field of science and technology.

The graph serves as a mathematical model for any system involving a discrete arrangement of objects. Graph becomes aesthetic appealing due to its diagrammatic representation.

Any field of investigation becomes more interesting when there arise a number of problems that pose challenge to our mind for their eventual solutions, more so when the field itself is just emerging and a whole galore of seemingly related or even unrelated open problems provide motivation for research. The problems arising from the study of different labeling techniques is one of such field. In the present work we have studied and investigated some results which concern to graph labeling techniques particularly graceful labeling, cordial labeling and $3-$equitable labeling.

The present work is motivated through a group discussion sponsored by Department of Science and Technology (DST) at Mary Matha Arts and Science College (Kerala) during $19 - 28$ April 2006.

Moreover we have enumerated minimum number of regular induced subgraphs of K_n. In addition to this we have carried out some investigations which concern to maximal non$-$Hamiltonian graphs.

The work reported in this thesis is divided into nine chapters. This first chapter is of introductory nature. The immediate *Chapter*-2 is intended to provide preliminaries and basic terminology. The next *Chapter*-3 is aimed to give brief idea and survey of various graph labeling techniques. Existing results and latest updates are reported which will serve as a reference material for any researcher.

Graceful labeling of graph is discussed in *Chapter*-4. Here we have discussed the gracefulness of grid graph with some other families of graphs. The results reported here are published in *Proceedings of the International Conference on Emerging Technologies and Applications in Engineering Technology and Sciences* 2008. The detailed discussion about cordial labeling of graphs is carried out in *Chapter*-5. We have contributed nine new families to the theory of cordial graphs. The results reported in this chapter are accepted for publication in reputed journals like *The Mathematics Student*, *Indian Journal of Mathematics and Mathematical Sciences* and *International Journal of Scientific Computing*.

The penultimate *Chapter*-6 is devoted to the discussion of $3-$equitable labeling of graphs. The definitions and survey of existing results is carried out. The results reported in this chapter are our original and already published in *Proceedings of the International Conference on Emerging Technologies and Applications in Engineering Technology and Sciences* 2008.

The labeled graphs are becoming increasingly useful mathematical models for broad range of applications. They are useful for the solutions of problems in additive number theory and coding theory. In *Chapter*-7 we have recorded some of the applications like determination of ambiguities in $X-$ray crystallography, design of good radar type code and laying of optimized communication network addressing system.

Minimum number of regular induced subgraphs of K_n is enumerated in *Chapter*-8. The results reported in this chapter are published in the journal *Mathematics Today*.

The discussion about maximal non$-$Hamiltonian graphs is carried out in last *Chapter*-9. Three powerful conjectures are posed and an algorithm is developed for the construction of maximal non$-$Hamiltonian graphs. These investigations are original and accepted for publication in the *Volume*-24 December 2007 of *Mathematics Today*. List of symbols and references are listed alphabetically at the end of the thesis. The entire thesis is prepared in LaTeX to meet the global standard.

The whole work will give rise to a new trend of research in graph theory in this region. We hope that a group of active researchers will come up in near future.

Chapter – 2

Preliminaries and Basic Terminology

2.1 INTRODUCTION **:** This chapter is devoted to provide all the fundamentals and notations which are useful for the present work. Basic definitions are given and explained with sufficient illustrations. This work becomes more effective due to neat and clean figures.

2.2 BASIC DEFINITIONS :

Definition–2.2.1 **:** A *graph* $G = (V, E)$ consists of two sets, $V = \{v_1, v_2, \ldots\}$ called *vertex set* of G and $E = \{e_1, e_2, \ldots\}$ called *edge set* of G. Sometimes we denote vertex set of G as $V(G)$ and edge set of G as $E(G)$. Elements of V are called *vertices* and elements of E are called *edges*.

Definition–2.2.2 **:** A graph consisting of one vertex and no edge is called a *trivial graph*. A graph which is not trivial is called a *non–trivial graph*.

Definition–2.2.3 **:** The number of edges in a given graph is called *size of the graph*.

Definition–2.2.4 **:** The number of vertices in a given graph is called *order of the graph*.

A graph with order p and size q is denoted as (p, q) graph.

Definition–2.2.5 **:** An edge of a graph that joins a vertex to itself is called a *loop*. A loop is an edge $e = v_i v_i$.

Definition$-$2.2.6 **:** If two vertices of a graph are joined by more than one edge then these edges are called *multiple edges* or *parallel edges*.

Definition$-$2.2.7 **:** If two vertices of a graph are joined by an edge then these vertices are called *adjacent vertices*.

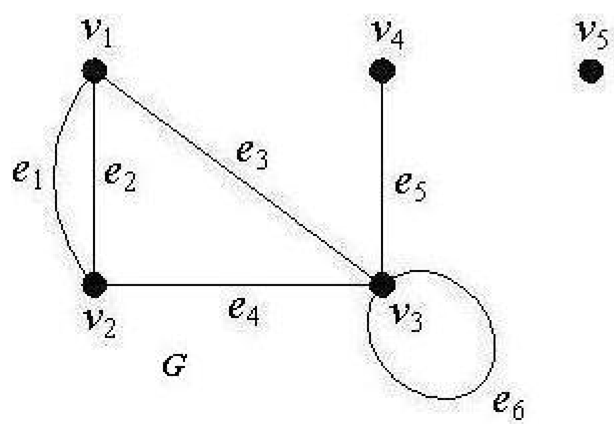
Definition$-$2.2.8 **:** If two or more edges of a graph have a common vertex then these edges are called *incident edges*.

Definition$-$2.2.9 **:** *Degree* of a vertex v of any graph G is defined as the number of edges incident on v, counting twice the number of loops. It is denoted by $d_G(v)$ or $d(v)$.

Definition$-$2.2.10 **:** A vertex of degree one is called a *pendant vertex*.

Definition$-$2.2.11 **:** A vertex of degree zero is called an *isolated vertex*.

Illustration$-$2.2.12 **:** Let us consider the following graph G.



Figure$-$2.1

In above graph G shown in *Figure*$-$2.1

◇ Order of graph G is 5.

◇ Size of graph G is 6.

◇ e_6 is loop.

◇ e_1 and e_2 are multiple edges.

◇ v_2 and v_3 are adjacent vertices.

◇ e_3 and e_5 are incident edges.

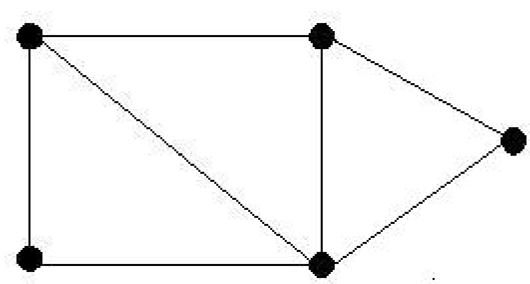
◇ $d(v_3) = 5$, $d(v_2) = 3$.

◇ v_4 is pendant vertex.

◇ v_5 is isolated vertex.

Definition−2.2.13 **:** A graph which has neither loops nor parallel edges is called a *simple graph*.

In the following *Figure*−2.2 a simple graph is shown.

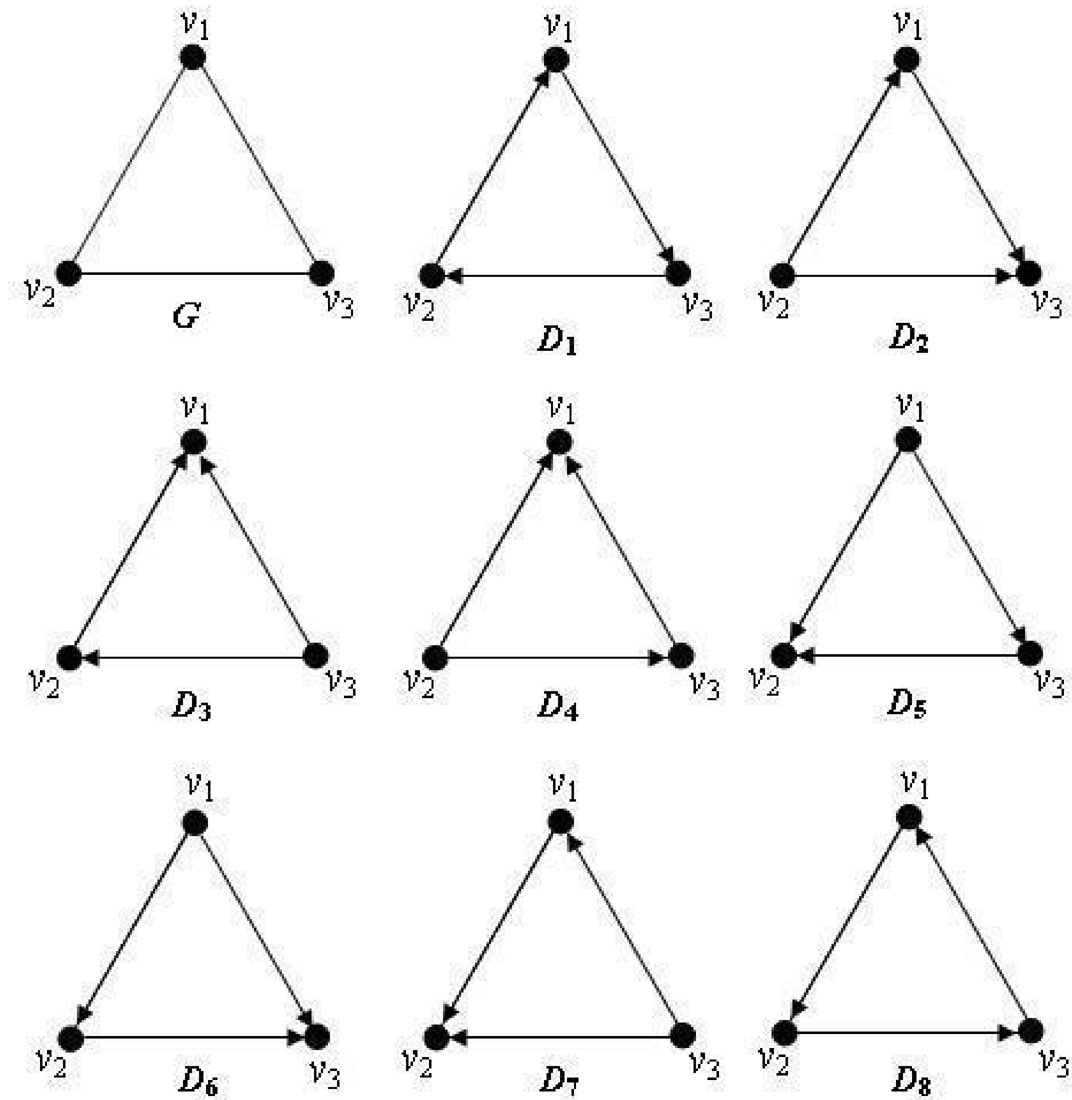


Figure−2.2

Definition−2.2.14 **:** A *directed edge* (or *arc*) is an edge, one of whose end vertices is designated as tail and other end vertex is designated as head. An arc is said to be *directed from* its tail to its head.

Definition−2.2.15 **:** Given a graph G we can obtain a digraph from G by specifying direction to each edge of G. Such a digraph D is called an *orientation*.

In the following *Figure*−2.3 eight different orientations of a graph G are shown.



Figure−2.3

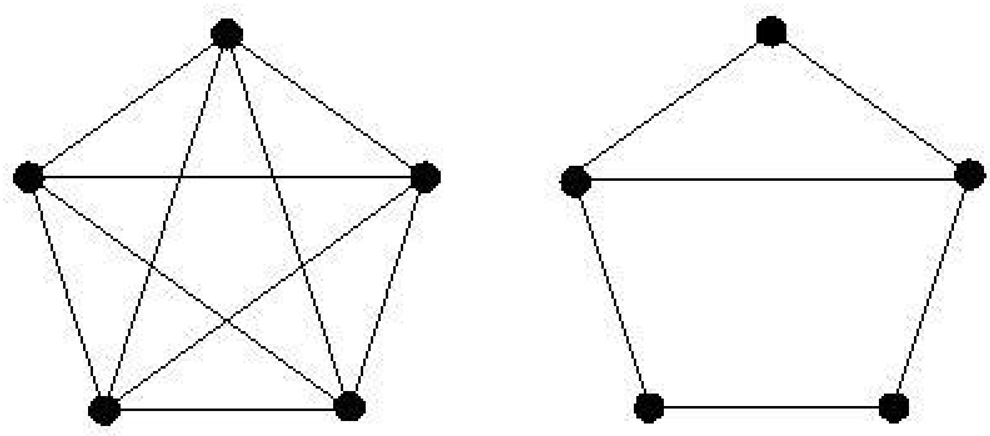
Definition−2.2.16 **:** A *directed graph*(or *digraph*) is a graph each of whose edges is directed.

Definition−2.2.17 **:** A graph in which no edge is directed is called an *undirected graph*.

Definition$-$2.2.18 **:** A graph $G = (V, E)$ is said to be *finite* if V and E both are finite sets.

Definition$-$2.2.19 **:** Let G and H be two graphs. Then H is said to be a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Here G is called *super-graph* of H.

In the following *Figure*$-$2.4 H is a subgraph of G.



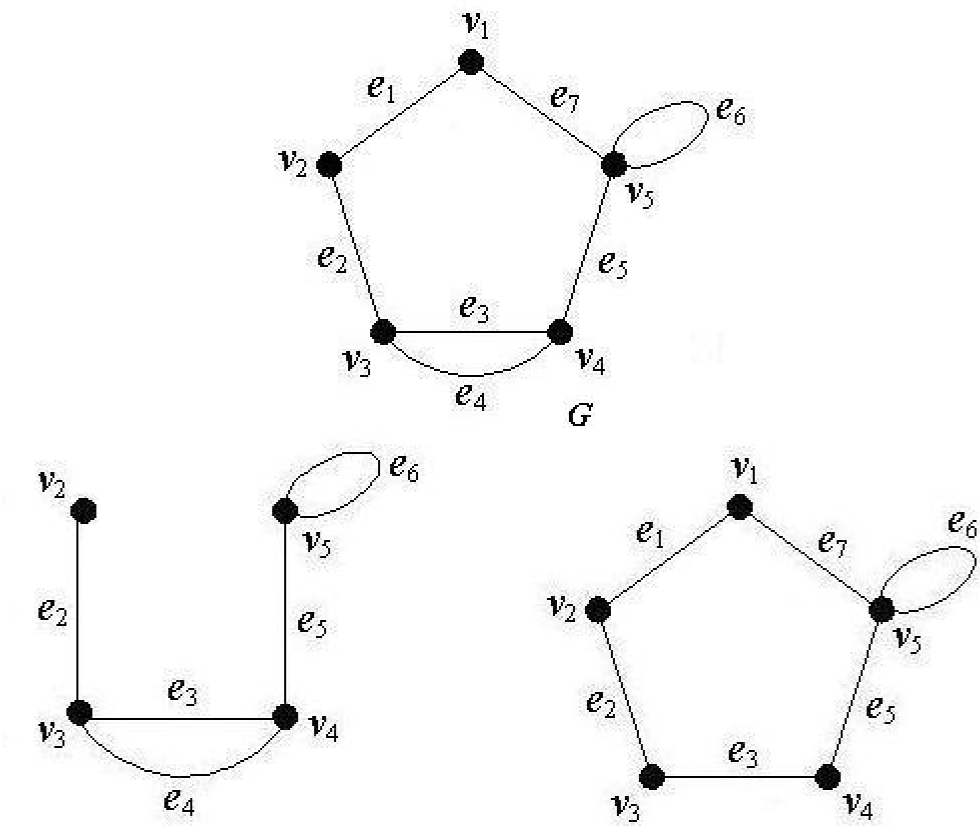
G H

Figure$-$2.4

Definition$-$2.2.20 **:** Deletion of an edge from given graph G forms a subgraph of G which is called *edge deleted subgraph* of G.

Definition$-$2.2.21 **:** The graph obtained by deletion of a vertex from given graph G is called *vertex deleted subgraph* of G.

In the following *Figure*$-$2.5 vertex deleted subgraph and edge deleted subgraph of given graph G are shown.



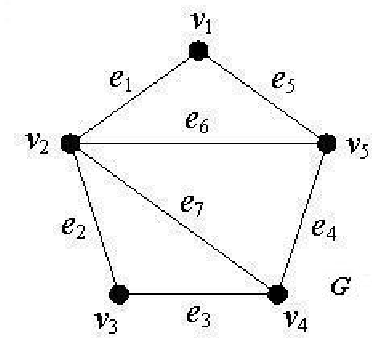
$G - \{v_1\}$ $\qquad\qquad$ $G - \{e_4\}$

Figure−2.5

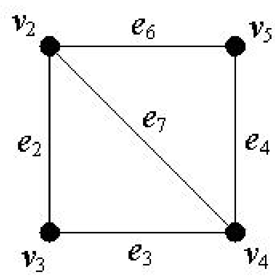
Definition−2.2.22 **:** Let $G = (V, E)$ be a graph. If U is a nonempty subset of the vertex set V of graph G then the *subgraph* $G[U]$ *of* G *induced by* U is defined to be the graph having vertex set U and edge set consisting of those edges of G that have both end vertices in U.

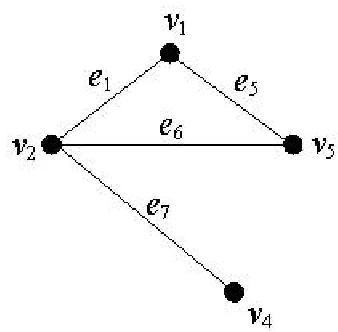
Similarly if F is a nonempty subset of the edge set E of G then the *subgraph* $G[F]$ *of* G *induced by* F is the graph whose vertex set is the set of vertices which are end vertices of edges of F and whose edge set is F.

In the following *Figure*-2.6, $G[U]$ and $G[F]$ are vertex induced subgraph and edge induced subgraph of graph G respectively.



Let $U = \{v_2, v_3, v_4, v_5\}$ and $F = \{e_1, e_5, e_6, e_7\}$





$G[U]$ $G[F]$

Figure-2.6

Definition$-2.2.23$ **:** A subgraph H of a graph G is called *spanning subgraph* of G if $V(H) = V(G)$.

Definition$-2.2.24$ **:** A *walk* is defined as a finite alternating sequence of vertices and edges of the form $v_i e_j v_{i+1} e_{j+1} \ldots e_k v_m$ which begins and ends with vertices such that each edge in the sequence is incident on the vertex preceding and succeeding it in the sequence. A walk from v_0 to v_n is denoted as $v_0 - v_n$ walk. A walk $v_0 - v_0$ is called a *closed walk*.

Definition$-2.2.25$ **:** The number of edges in any walk is called *length of the walk*. A walk is *odd* (or *even*) if its length is odd (or even).

Definition$-2.2.26$ **:** A walk is called a *trail* if no edge is repeated.

Definition$-2.2.27$ **:** A walk in which no vertex is repeated is called a *path*. A path with n vertices is denoted as P_n. A path from v_0 to v_n is denoted as $v_0 - v_n$ path.

Definition$-2.2.28$ **:** A closed path is called a *cycle*. A cycle with n vertices is denoted as C_n.

Illustration$-2.2.29$ **:** Consider the following graph G as shown in *Figure*-2.7.

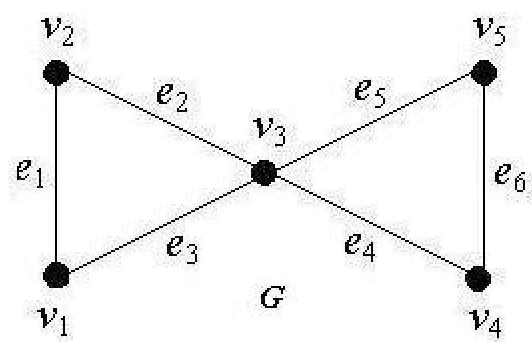


Figure-2.7

Graph G shown in above *Figure*-2.7 is known as *bowtie graph*. For this graph we note the followings :

◇ G is a simple, finite and undirected graph.

◇ $W = v_2e_2v_3e_4v_4e_6v_5e_5v_3e_3v_1$ is a walk.

◇ $P_4 = v_1e_1v_2e_2v_3e_5v_5$ is a path.

◇ $C_3 = v_1e_1v_2e_2v_3e_3v_1$ is a cycle.

Definition$-$2.2.30 **:** A graph which includes exactly one cycle is called a *unicyclic graph.*

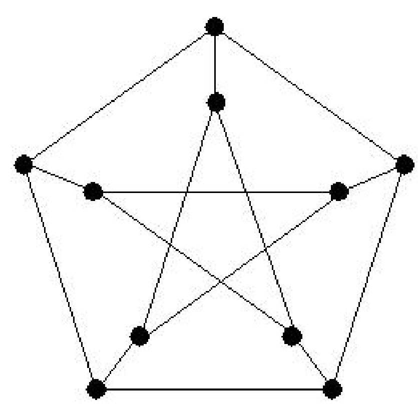
Definition$-$2.2.31 **:** A graph $G = (V, E)$ is said to be *connected* if there is a path between every pair of vertices in G. A graph which is not connected is called a *disconnected graph.*

The graph shown in *Figure*$-$2.2 is connected while the graph shown in *Figure*$-$2.1 is disconnected.

Definition$-$2.2.32 **:** Each maximal connected subgraph of a disconnected graph is called *component of the graph.* Every connected graph has exactly one component.

Definition$-$2.2.33 **:** A graph in which all the vertices having equal degree is called a *regular graph.* If for every vertex v of graph G, $d(v) = k$ for some $k \in N$, then G is $k-$*regular graph.*

In the following *Figure*$-$2.8 a $3-$regular graph on 10 vertices is shown.



Figure$-$2.8

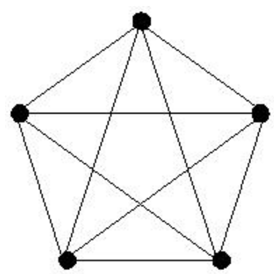
The above graph is known as *Petersen graph* which is a $3-$regular graph with 10 vertices and 15 edges.

Definition−2.2.34 **:** A graph in which the vertices having only two types of degree is called a *bidegreed graph*.

The graph shown in *Figure*−2.7 is a bidegreed graph.

Definition−2.2.35 **:** A simple, connected graph is said to be *complete* if every pair of vertices of G is adjacent. A complete graph on n vertices is denoted by K_n. Note that K_n is $(n-1)$−regular.

In following *Figure*−2.9 K_5 is shown.



K_5

Figure−2.9

Definition−2.2.36 **:** Two vertices of a graph which are adjacent are said to be *neighbours*. The set of all neighbours of a fixed vertex v of G is called the *neighbourhood set* of v. It is denoted by $N(v)$.

In *Figure*−2.7, $N(v_3) = \{v_1, v_2, v_4, v_5\}$.

Definition−2.2.37 **:** A closed trail which covers all the edges of given graph is called an *Eulerian line* or *Eulerian trail*. A graph which has an Eulerian line is called an *Eulerian graph*. The graphs shown in *Figure*−2.7 and *Figure*−2.9 are Eulerian graphs.

Definition−2.2.38 **:** A graph $G = (V, E)$ is said to be *bipartite* if the vertex set can be partitioned into two disjoint subsets V_1 and V_2 such that for every edge $e_i = v_i v_j \in E$, $v_i \in V_1$ and $v_j \in V_2$.

In the following *Figure*-2.10 a bipartite graph is shown.

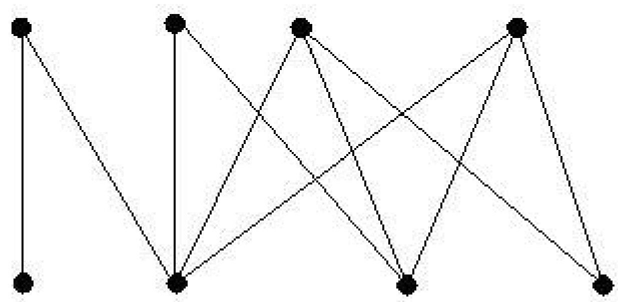


Figure-2.10

Definition$-2.2.39$ **:** A graph $G = (V, E)$ is called $n-$*partite graph* if the vertex set V can be partitioned into n nonempty mutually disjoint sets $V_1, V_2, \ldots, V_n$ such that every edge of G joins the vertices from different subsets. It is often called a *multipartite graph*.

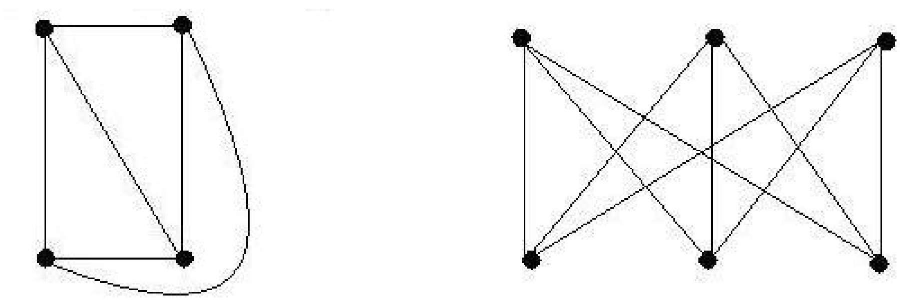
Definition$-2.2.40$ **:** A *complete bipartite graph* is a simple bipartite graph such that two vertices are adjacent if and only if they are in different partite sets. If partite sets are having m and n vertices then the related complete bipartite graph is denoted by $K_{m,n}$.

Definition$-2.2.41$ **:** The $n-$partite graph G is called *complete* $n-$*partite* if for each $i \neq j$, each vertex of the subset V_i is adjacent to every vertex of the subset V_j. The complete $n-$partite graph with $n-$partitions of vertex set is denoted by $K_{m_1, m_2, \ldots, m_n}$.

Definition$-2.2.42$ **:** A graph is said to be *planar* if there exists some geometric representation of G which can be drawn on a plane such that no any two of its edges intersect.

Definition$-$2.2.43 **:** A graph that can not be drawn on a plane without a crossover between its edges is called *non planar* graph.

In the following *Figure*$-$2.11 planar and non planar graph are shown.



K_4 Planar graph $\qquad$ $K_{3,3}$ Non planar graph

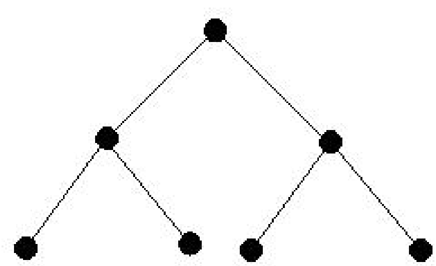
Figure$-$2.11

Definition$-$2.2.44 **:** A graph which does not contain any cycle is known as *acyclic graph*.

Definition$-$2.2.45 **:** An acyclic graph is known as *forest*.

Definition$-$2.2.46 **:** A connected acyclic graph is called a *tree*. Thus every component of a forest is a tree.

In the following *Figure*$-$2.12 a tree T on seven vertices is shown.

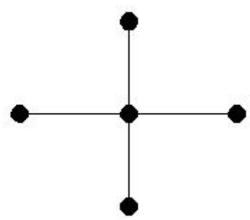


Figure$-$2.12

Definition$-2.2.47$ **:** A *spanning tree* of a graph G is a spanning subgraph of G which is a tree. The number of spanning trees of a graph G is denoted by $\tau(G)$.

Definition$-2.2.48$ **:** A *star graph* with n vertices is a tree with one vertex having degree $n-1$ and other $n-1$ vertices having degree 1. A star graph with $n+1$ vertices is $K_{1,n}$.

In the following $Figure-2.13$ $K_{1,4}$ is shown.

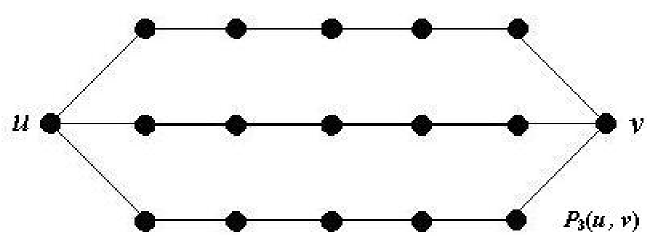


$Figure-2.13$

Definition$-2.2.49$ **:** A *banana tree* is a tree which is obtained from a family of stars by joining one end vertex of each star to a new vertex.

Definition$-2.2.50$ **:** A $t-ply$ $P_t(u,v)$ is a graph with t paths, each of length atleast two and such that no two paths have a vertex in common except the end vertices u and v.

In the following $Figure-2.14$ $P_3(u,v)$ is shown.



$Figure-2.14$

Definition$-2.2.51$ **:** A *caterpillar* is a tree in which a single path (the spine) is incident to (or contains) every edge.

In following *Figure*-2.15 a caterpillar on 10 vertices is shown.

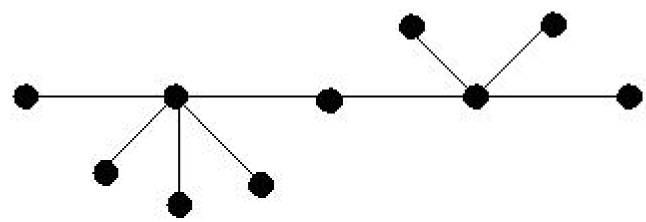


Figure-2.15

Definition$-2.2.52$ **:** A *lobster* is a tree with the property that the removal of the end vertices leaves a caterpillar.

Definition$-2.2.53$ **:** A vertex v of a graph G is called a *cut vertex* of G if $G - v$ is disconnected.

Definition$-2.2.54$ **:** The *vertex connectivity* of a connected graph G is defined as the minimum number of vertices whose removal from G makes remaining graph disconnected.

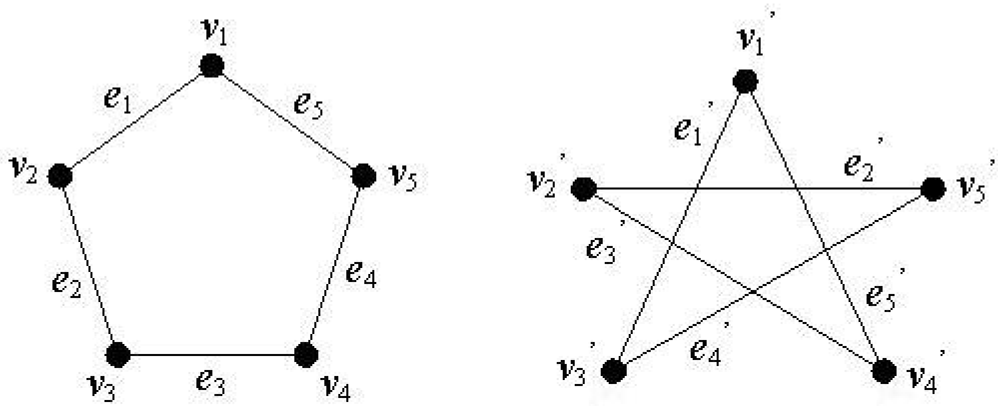
Definition$-2.2.55$ **:** A connected graph is said to be *separable* if its vertex connectivity is one.

Definition$-2.2.56$ **:** A *block* of a loopless graph is a maximal connected subgraph H such that no vertex of H is a cut vertex of H.

Definition$-2.2.57$ **:** A graph $G_1 = (V_1, E_1)$ is said to be *isomorphic* to the graph $G_2 = (V_2, E_2)$ if there exists a bijection between the vertex sets V_1 and V_2 and a bijection between the edge sets E_1 and E_2 such that if e is an edge with end vertices u and v in G_1 then the corresponding edge e' in G_2 has its end vertices u' and v' in G_2 which correspond to u and v respectively.

If such pair of bijections exist then it is called a *graph isomorphism* and it is denoted by $G_1 \cong G_2$.

In the following *Figure*$-$2.16 two isomorphic graphs are shown.



G_1 G_2

Figure$-$2.16

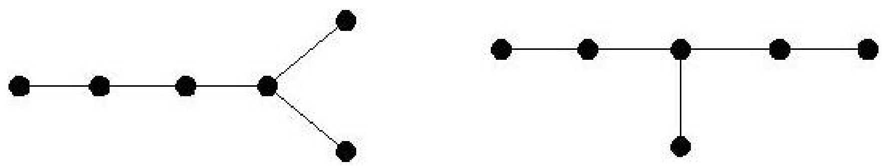
For the graphs in *Figure*$-$2.16 the vertices v_1, v_2, v_3, v_4, v_5 correspond to vertices $v_1^{'}, v_3^{'}, v_5^{'}, v_2^{'}, v_4^{'}$ respectively while edges e_1, e_2, e_3, e_4, e_5 correspond to $e_1^{'}, e_4^{'}, e_2^{'}, e_3^{'}, e_5^{'}$ respectively..

Remark$-$2.2.58 **:**

If two graphs are isomorphic then they have

- Same number of vertices
- Same number of edges
- Number of vertices having same degree is equal.

The above facts are not sufficient for the isomorphism of graphs. Consider the graphs shown in *Figure*$-$2.17.



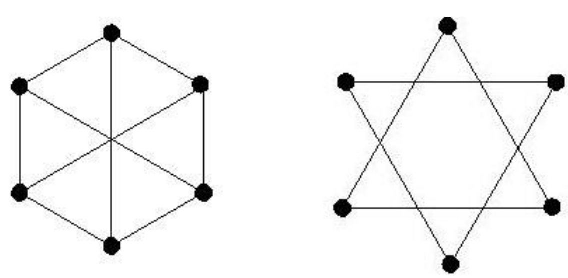
G_1 G_2

Figure-2.17

Here G_1 and G_2 satisfy above three conditions even though they are not isomorphic. Here bijection does not preserve adjacency as well as incidency.

Definition$-2.2.59$ **:** The *complement* $\overline{G}$ of a graph $G = (V, E)$ is a graph with vertex set V in which two vertices are adjacent if and only if they are not adjacent in G.

In following *Figure*-2.18 a graph G and its complement is shown.

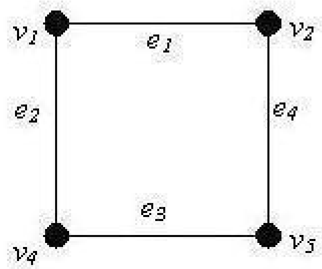


G $\overline{G}$

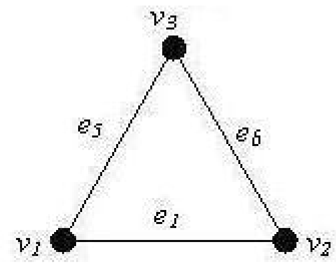
Figure-2.18

Definition$-2.2.60$ **:** If G_1 and G_2 are subgraphs of a graph G then *union of* G_1 *and* G_2 is denoted by $G_1 \cup G_2$ which is the graph consisting of all those vertices which are either in G_1 or in G_2 (or in both) and with edge set consisting of all those edges which are either in G_1 or in G_2 (or in both).

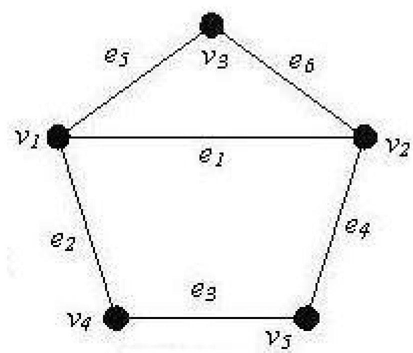
In the following *Figure*-2.19 union of two graphs G_1 and G_2 is shown.



G_1



G_2

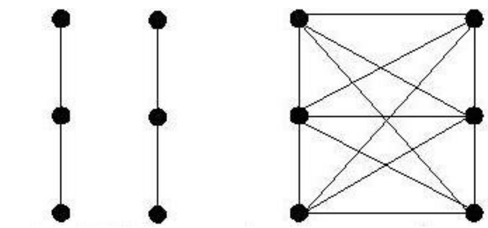


$G_1 \bigcup G_2$

Figure-2.19

Definition$-2.2.61$ **:** Let G and H be two graphs such that $V(G) \cap V(H) = \emptyset$. Then *join of G and H* is denoted by $G + H$. It is the graph with $V(G + H) = V(G) \cup V(H)$, $E(G + H) = E(G) \cup E(H) \cup J$, where $J = \{uv/u \in V(G), v \in V(H)\}$.

In the following *Figure*−2.20 join $G+H$ of two graphs G and H is shown.



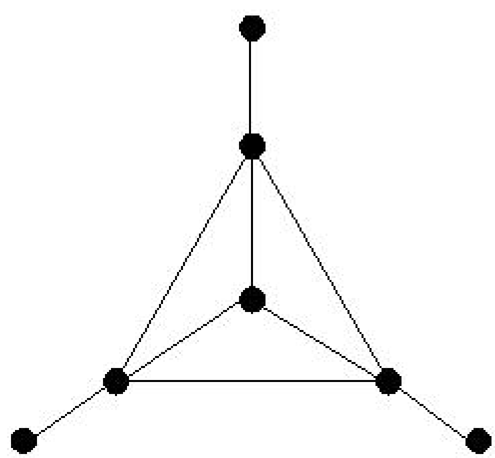
G $\quad$ H $\quad\quad$ $G+H$

Figure−2.20

Definition−2.2.62 **:** The *wheel graph* W_n is join of the graphs C_n and K_1. i.e. $W_n = C_n + K_1$. Here vertices corresponding to C_n are called *rim vertices* and C_n is called *rim* of W_n while the vertex corresponds to K_1 is called *apex vertex*.

Definition−2.2.63 **:** A *helm* $H_n, n \geq 3$ is the graph obtained from the wheel W_n by adding a pendant edge at each vertex on the wheel's rim.

In the following *Figure*−2.21 H_3 is shown.

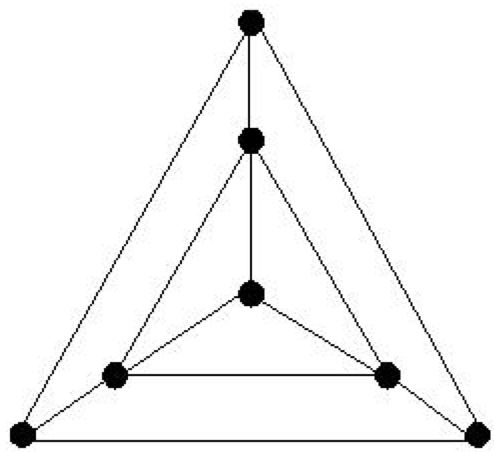


H_3

Figure−2.21

Definition$-2.2.64$ **:** A *closed helm* CH_n is the graph obtained by taking a helm H_n and by adding edges between the pendant vertices.

In the following $Figure{-}2.22$ CH_3 is shown.



CH_3

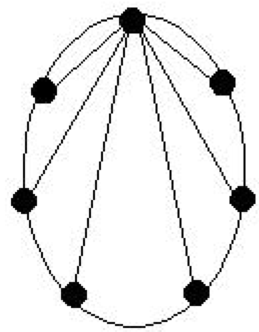
$Figure{-}2.22$

Definition$-2.2.65$ **:** A *web graph* is the graph obtained by joining the pendant vertices of a helm to form a cycle and then adding a single pendant edge to each vertex of this outer cycle.

Definition$-2.2.66$ **:** A *generalized helm* is the graph obtained by taking a web and attaching pendant vertices to all the vertices of the outermost cycle.

Definition$-2.2.67$ **:** A *shell* S_n is the graph obtained by taking $n-3$ concurrent chords in a cycle C_n. The vertex at which all the chords are concurrent is called the *apex*. The shell S_n is also called fan F_{n-1}. i.e. $S_n = F_{n-1} = P_{n-1} + K_1$.

In the following *Figure*-2.23 S_7 (or F_6) is shown.



S_7 or F_6

Figure-2.23

Definition$-2.2.68$ **:** A *multiple shell* $MS\{n_1^{t_1}, n_2^{t_2}, \ldots, n_r^{t_r}\}$ is a graph formed by t_i shells each of order $n_i, 1 \leq i \leq r$ which have a common apex.

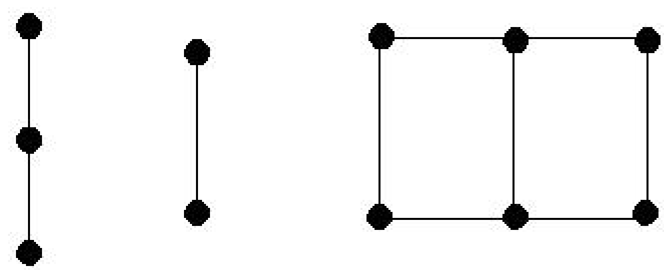
Definition$-2.2.69$ **:** A *triangular cactus* is a connected graph all of whose blocks are triangles.

Definition$-2.2.70$ **:** A $k-$*angular cactus* is a connected graph all of whose blocks are cycles with k vertices.

Definition$-2.2.71$ **:** A *triangular snake* is the graph obtained from a path P_n with vertices $v_1, v_2, \ldots v_n$ by joining v_i and v_{i+1} to a new vertex w_i for $i = 1, 2, \ldots, n-1$.

Definition$-2.2.72$ **:** Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. Then *cartesian product* of G_1 and G_2 which is denoted by $G_1 \times G_2$ is the graph with vertex set $V = V_1 \times V_2$ consisting of vertices $u = (u_1, u_2)$, $v = (v_1, v_2)$ such that u and v are adjacent in $G_1 \times G_2$ whenever ($u_1 = v_1$ and u_2 adjacent to v_2) or ($u_2 = v_2$ and u_1 adjacent to v_1).

In the following *Figure*-2.24 cartesian product of two paths is shown.



G_1 G_2 $G_1 \times G_2$

Figure-2.24

Definition$-2.2.73$ **:** The cartesian product of two paths is known as *grid graph* which is denoted by $P_m \times P_n$. In particular the graph $P_n \times P_2$ is known as *ladder graph*.

Definition$-2.2.74$ **:** The cartesian product of two cycles is known as *torus grid* which is denoted by $C_m \times C_n$.

Definition$-2.2.75$ **:** The graph $K_2 \times K_2 \times \ldots \times K_2$($n$ times) is known as *n-cube*.

Definition$-2.2.76$ **:** Let $G = (V, E)$ be a graph. Let $e = uv$ be an edge of G and w is not a vertex of G. The edge e is *subdivided* when it is replaced by edges $e' = uw$ and $e'' = wv$.

In following *Figure*-2.25 subdivision of an edge is shown.

Figure-2.25 Subdividing an edge

Definition$-2.2.77$ **:** Let $G = (V, E)$ be a graph. If every edge of graph G is subdivided then the resulting graph is called *barycentric subdivision* of G. In other words barycentric subdivision is the graph obtained by inserting a vertex of degree 2 into every edge of original graph. The barycentric subdivision of any graph G is denoted by $S(G)$. It is easy to observe that $|V(S(G))| = |V(G)| + |E(G)|$ and $|E(S(G))| = 2|E(G)|$.

In following *Figure*-2.26 barycentric subdivision of a graph is shown.

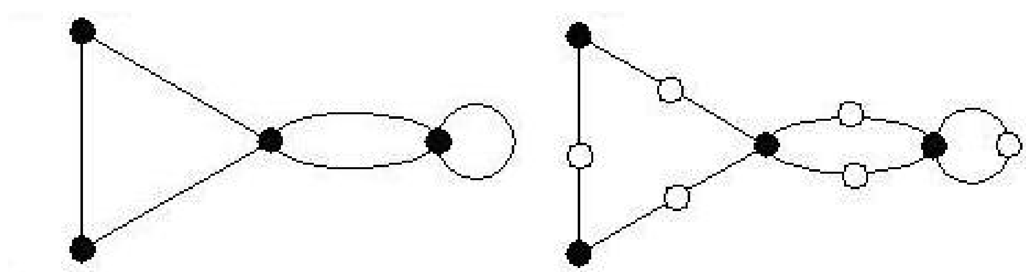


Figure-2.26 A graph and its barycentric subdivision

Definition 2.2.78 Let $e = uv$ be an edge of simple, finite, undirected, connected graph G and $d(u) = k$, $d(v) = l$. Let $N(u) = \{v, u_1, \ldots, u_{n-1}\}$ and $N(v) = \{u, v_1, \ldots, v_{l-1}\}$. A *contraction* on the edge e changes G to a new graph $G * e$, where $V(G * e) = (V(G) - \{u, v\}) \cup \{w\}$, $E(G * e) = E(G - \{u, v\}) \cup \{wu_1, wu_2, \ldots, wu_{k-1}, wv_1, \ldots, wv_{l-1}\}$ and w is new vertex not belonging to G.

2.3 Concluding Remarks :

This chapter was intended to provide all the fundamentals and prerequisites which concern to the present work. Basic definitions like graph, vertex, edge, subgraph etc. are given and explained with the help of illustrations. Common families of graphs like cycle, path, wheel, tree etc. are introduced, notations and terminology are given. We have tried our best to prepare platform for advancement of the subject. Illustrations and figures help for better understanding.

The next chapter is aimed to discuss different graph labeling techniques.

Chapter – 3

Different Graph Labeling Techniques

3.1 Introduction :

Graph labeling was first introduced in $1960's$. At present various graph labeling techniques are available and more than 800 research papers have been published so far. The interest in the field of graph labeling is constantly increasing and it has motivated many researchers. Many graph labeling techniques have applications to practical problems. According to Beineke and Hegde [19] graph labeling serves as a frontier between number theory and structure of graphs. Labeling of graphs have various applications in coding theory, particularly for missile guidance codes, design of good radar type codes, convolution codes with optimal autocorrelation properties. Graph labeling plays vital role in the study of X−ray crystallography, communication network and solution of problems in additive number theory. A detailed study of variety of applications of graph labeling is given by Bloom and Golomb [25]. A systematic survey on graph labeling is updated every two year since last one decade by Gallian [49]. The reference cited here is of latest version of *A Dynamic survey of Graph Labeling*, published by *The Electronics Journal of Combinatorics.*

This chapter is targeted to discuss various graph labeling techniques for graph $G = (V, E)$ with p vertices and q edges.

Throughout the discussion on graph labeling we consider simple, finite and undirected graphs unless or otherwise stated. In the remaining part of this chapter we will concentrate on some important definitions for various labeling techniques and existing results.

3.2 Some Graph Labeling Techniques :

If the vertices of the graph are assigned values subject to certain conditions is known as *graph labeling*.

Most interesting graph labeling problems have three important ingredients :

(1) A set of numbers from which vertex labels are chosen.

(2) A rule that assigns a value to each edge.

(3) A condition that these values must satisfy.

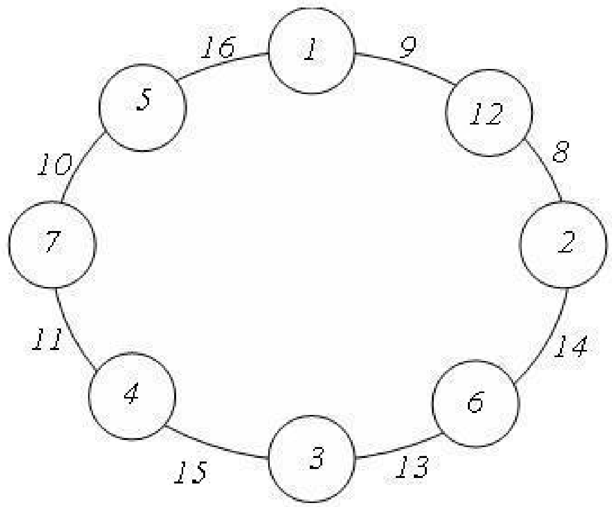
Now discussion about various graph labeling techniques will be carried out in chronological order as they were introduced.

3.2.1 Magic labeling : Magic labeling was introduced by Sedláček [87] in 1963 motivated through the notion of magic squares in number theory.

A function f is called *magic labeling* of a graph G if $f : V\bigcup E \rightarrow \{1, 2, \ldots, p+q\}$ is bijective and for any edge $e = uv, f(u) + f(v) + f(e)$ is constant.

A graph which admits magic labeling is called *magic graph*.

In following $Figure{-}3.1$ magic labeling for C_8 is demonstrated in which for any edge $e = uv, f(u) + f(v) + f(e) = 22$.



Figure−3.1

Some known results about magic labeling are listed below.

Stewart [98] proved that

- K_n is magic for $n = 2$ and all $n \geq 5$.
- $K_{n,n}$ is magic for all $n \geq 3$.
- Fans F_n are magic if and only if $n \geq 3$ and n is odd.
- Wheels W_n are magic for all $n \geq 4$.

For any magic labeling f of graph G, there is a constant $c(f)$ such that for all edges $e = uv \in G$, $f(u) + f(v) + f(e) = c(f)$. *The magic strength* $m(G)$ is defined as the minimum of $c(f)$ where the minimum is taken over all magic labeling of G.

The above definition and some facts listed below were given by S. Avadyappan et al.[13].

- $m(P_{2n}) = 5n + 1$, $m(P_{2n+1}) = 5n + 3$,
- $m(C_{2n}) = 5n + 4$, $m(C_{2n+1}) = 5n + 2$,
- $m(K_{1,n}) = 2n + 4$.

Hegde and Shetty [60] defined $M(G)$ analogous to $m(G)$ as follows. $M(G) = \max\{c(f)\}$, where maximum is taken over all magic labeling f of G.

For any graph G with p vertices and q edges following inequality holds.

$$p + q + 3 \leq m(G) \leq c(f) \leq M(G) \leq 2(p + q).$$

3.2.2 Graceful labeling : Graceful labeling was introduced by Rosa [86] in 1967.

A function f is called *graceful labeling* of a graph G if $f : V \rightarrow \{0, 1, 2, \ldots, q\}$ is injective and the induced function $f^* : E \rightarrow \{1, 2, \ldots, q\}$ defined as $f^*(e = uv) = |f(u) - f(v)|$ is bijective.

A graph which admits graceful labeling is called *graceful graph*.

Initially Rosa named above defined labeling as $\beta-$valuation. Golomb [55] renamed $\beta-$valuation as graceful labeling. We will discuss graceful labeling in detail in *Chapter*-4.

3.2.3 Graceful$-$like labeling : In 1967, Rosa [86] gave another analogue of graceful labeling.

A function f is called *graceful$-$like labeling* of a graph G if $f : V \rightarrow \{0, 1, 2, \ldots, q+1\}$ is injective and the induced function $f^* : E \rightarrow \{1, 2, \ldots, q\}$ or $f^* : E \rightarrow \{1, 2, \ldots, q-1, q+1\}$ defined as $f^*(e = uv) = |f(u) - f(v)|$ is bijective.

Frucht [48] termed such labeling as n*early graceful labeling*. Some known results about graceful$-$like labeling are listed below.

- Frucht [48] has shown that $P_m \bigcup P_n$ admits graceful$-$like labeling with edge labels $\{1, 2, \ldots, q-1, q+1\}$. $G \bigcup K_2$ (where G is graceful graph) admits graceful−like labeling.

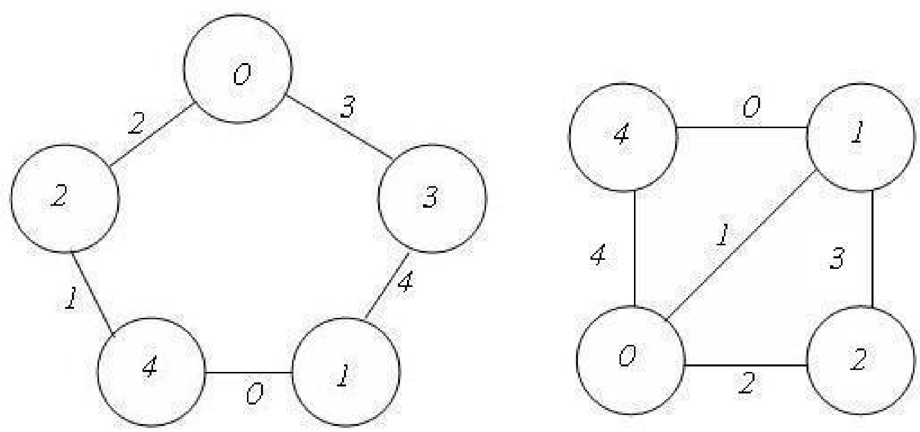
• Seoud and Elsahawi [91] have shown that all cycles admit graceful−like labeling.

• Barrientos [18] proved that cycle C_n is having graceful−like labeling with edge labels $\{1, 2, \ldots, q-1, q+1\}$ if and only if $n \equiv 1$ or 2 (mod 4).

3.2.4 Harmonious labeling : Graham and Sloane [56] introduced harmonious labeling in 1980 during their study of modular versions of additive bases problems stemming from error correcting codes.

A function f is called *harmonious labeling* of a graph G if $f : V \rightarrow \{0, 1, 2, \ldots, q-1\}$ is injective and the induced function $f^* : E \rightarrow \{0, 1, 2, \ldots, q-1\}$ defined as $f^*(e = uv) = (f(u) + f(v))$ mod q is bijective.

A graph which admits harmonious labeling is called *harmonious graph.* We will demonstrate harmonious labeling by means of following examples in *Figure*−3.2.



Figure−3.2

Graham and Sloane observed that if graph G is a tree then exactly two vertices are assigned same vertex label in harmonious labeling. Some known results about harmonious graph are listed below.

• Liu and Zhang [80] proved that every graph is a subgraph of a harmonious graph.

• Graham and Sloane [56] posed a conjecture *Every tree is harmonious.* In connection of above conjecture, Alderd and Mckay [6] proved that trees with 26 or less vertices are harmonious. They also proved that

◇ Caterpillars are harmonious.

◇ Cycles C_n are harmonious if and only if $n \equiv 1, 3 \pmod 4$.

◇ Wheels W_n are harmonious for all n.

◇ $C_m \times P_n$ is harmonious if n is odd.

◇ K_n is harmonious if and only if $n \leq 4$.

◇ $K_{m,n}$ is harmonious if and only if m or $n = 1$.

◇ Fans F_n are harmonious for all n.

• Liu [79] proved that all helms are harmonious.

• Jungreis and Reid [67] proved that grids $P_m \times P_n$ are harmonious if and only if $(m, n) \neq (2, 2)$. In the same paper they proved that $C_m \times P_n$ is harmonious if $m = 4$ and $n \geq 3$.

• Gallian et al.[50] proved that $C_m \times P_n$ is harmonious if $n = 2$ and $m \neq 4$.

3.2.5 Elegant labeling : Elegant labeling was introduced by Chang et al.[36] in 1981.

A function f is called *elegant labeling* of a graph G if $f : V \rightarrow \{0, 1, 2, \ldots, q\}$ is injective and the induced function $f^* : E \rightarrow \{1, 2, \ldots, q\}$ defined as

$f^*(e = uv) = (f(u) + f(v)) \bmod (q + 1)$ is bijective.

A graph which admits elegant labeling is known as *elegant graph.* We will note that as in harmonious labeling it is not necessary to make an exception for trees. Some known results for elegant labeling are listed below.

• Chang et al.[36] proved that C_n is elegant when $n \equiv 0, 3 \pmod 4$ and not elegant when $n \equiv 1 \pmod 4$ and path P_n is elegant when $n \equiv 1, 2, 3 \pmod 4$.

• Cahit [30] proved that P_4 is the only path which is not elegant.

• Balakrishnan et al.[16] proved that every simple graph is a subgraph of an elegant graph.

• Deb and Limaye [40] defined *near−elegant labeling* by replacing codomain of edge function f^* by $\{0, 1, \dots, q-1\}$ and they proved that triangular snakes where the number of triangles is congruent to 3 (mod 4) are near−elegant.

3.2.6 Prime and vertex prime labeling : The concept of prime labeling was originated by Entringer and it was introduced in a paper by Tout et al.[100].

A graph G with p vertices and q edges is said to have a *prime labeling* if $f : V \rightarrow \{1, 2, \dots, p\}$ is bijective function and for every edge $e = uv$ of G, $(f(u), f(v)) = 1$.

• Around 1980 Entringer conjectured that *All tree have a prime labeling.* So far there has been little progress towards the proof of this conjecture.

• Some known classes of trees having prime labeling are paths, stars, caterpillars, etc.

• Deretsky et al.[42] proved that

◇ All cycles have prime labeling.

◇ Disjoint union of C_{2k} and C_n have prime labeling.

◇ The complete graph K_n does not have a prime labeling for $n \geq 4$.

• Lee et al.[77] proved that W_n have prime labeling if and only if n is even.

• Seoud et al.[90] proved that all helms, fans, $K_{2,n}$, $K_{3,n}$ (where $n \neq 3, 7$), $P_n + \bar{K}_2$ (where $n = 2$ or n is odd) are having prime labeling. He also proved that $P_n + \bar{K}_m$ does not have prime labeling if $m \geq 3$.

• Seoud and Youssef [92] shown that $P_n + \bar{K}_2$ is having prime labeling if and only if $n = 2$ or n is odd.

In 1991 Deretsky et al.[42] introduced the notion of dual of prime labeling which is known as *vertex prime labeling*. According to them a graph with q edges has vertex prime labeling if its edges can be labeled with distinct integers $\{1, 2, \ldots, q\}$ such that for each vertex of degree at least two the greatest common divisor of the labels on its incident edges is 1. Some known results for vertex prime labeling are listed below.

• Deretsky et al.[42] proved that

◇ Forests, all connected graphs are having vertex prime labeling.

◇ $C_{2k} \bigcup C_n$, $C_{2n} \bigcup C_{2n} \bigcup C_{2k+1}$, $C_{2n} \bigcup C_{2n} \bigcup C_{2t} \bigcup C_k$ and $5C_{2m}$ are having vertex prime labeling.

◇ A graph with exactly two component one of them is not an odd cycle has a vertex prime labeling.

◇ $2-$regular graph with at least two odd cycles does not have a vertex prime labeling.

◇ He also conjectured that $2-$*regular graph has a vertex prime labeling if and only if it does not have two odd cycles.*

3.2.7 $k-$**Graceful labeling :** A natural generalization of graceful labeling is the notion of $k-$graceful labeling which was independently introduced by Slater [96] and by Maheo and Thuillier [83] in 1982.

A function f is called $k-$*graceful labeling* of a graph G if $f : V \rightarrow \{0, 1, 2, \ldots, k + q - 1\}$ is injective and the induced function $f^* : E \rightarrow \{k, k + 1, k + 2, \ldots, k + q - 1\}$ defined as $f^*(e = uv) = |f(u) - f(v)|$ is bijective. A graph which admits $k-$graceful labeling is known as $k-$*graceful graph*. Obviously $1-$graceful graphs are the graceful graphs. Some known results for $k-$graceful graph are listed below.

• Slater [96], Maheo and Thuillier [83] proved that C_n is $k-$graceful graph if and only if either $n \equiv 0, 1 \pmod 4$ with $k-$even and $k \leq \frac{n-1}{2}$ or $n \equiv 3 \pmod 4$ with $k-$odd and $k \leq \frac{n^2-1}{2}$.

• Liang and Liu [78] proved that $K_{m,n}$ is $k-$graceful, for all $m, n \in N$ and for all k.

• Bu et al.[28] proved that $P_n \times P_2$ and $(P_n \times P_2) \bigcup (P_n \times P_2)$ are $k-$graceful for all k.

• Acharya [1] proved that a $k-$graceful Eulerian graph with q edges must satisfies one of the following:

(1) $q \equiv 0 \pmod 4$, $q \equiv 1 \pmod 4$ if k is even, (2) $q \equiv 3 \pmod 4$ if k is odd.

3.2.8 Cordial labeling : Cahit [31] introduced the concept of cordial labeling in 1987 as a weaker version of graceful and harmonious labeling.

A function $f : V \rightarrow \{0, 1\}$ is called *binary vertex labeling* of a graph G and $f(v)$ is called *label of the vertex* v of G under f. For an edge $e = uv$, the induced function $f^* : E \rightarrow \{0, 1\}$ is given as $f^*(e = uv) = |f(u) - f(v)|$. Let $v_f(0)$, $v_f(1)$ be number of vertices of G having labels 0 and 1 respectively under f and let $e_f(0)$, $e_f(1)$ be number of edges of G having labels 0 and 1 respectively under f^*. A binary vertex labeling f of a graph G is called *cordial labeling* if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph which admits cordial labeling is called *cordial graph*.

Detailed discussion of above defined labeling will be carried out in *Chapter*-5.

3.2.9 Additively graceful labeling : In 1989 Hegde [57] introduced the concept of additively graceful labeling.

A function f is called *additively graceful labeling* of a graph G if $f : V \rightarrow \{0, 1, \ldots, \lceil \frac{q+1}{2} \rceil\}$ is injective and the induced function $f^* : E \rightarrow \{1, 2, \ldots, q\}$ defined as $f^*(e = uv) = f(u) + f(v)$ is bijective. A graph which admits additively graceful labeling is called *additively graceful graph.* Some known results on additively graceful graph are listed below.

• Hegde [57] proved the following results.

$\diamond$ If G is an additively graceful graph with p vertices and q edges then $q \geq 2p - 4$ and the bounds are best possible.

$\diamond$ The graph for which $q = 2p - 4$ are essentially strongly indexable which is discussed in 3.2.13.

$\diamond$ The complete graph K_n is additively graceful if and only if $2 \leq n \leq 4$.

$\diamond$ An additively graceful graph is either K_2 or $K_{1,2}$ or has a triangle.

$\diamond$ If G is an additively graceful graph with a triangle then any additively graceful labeling f of G must assign zero to a vertex of triangle in G.

$\diamond$ If an Eulerian graph G with p vertices and q edges is additively graceful then $q \equiv 0, 3 \pmod 4$.

$\diamond$ A unicyclic graph G is additively graceful if and only if G is isomorphic to either C_3 or the graph obtained by joining a unique vertex to any one vertex of C_3.

$\diamond$ The graph obtained by joining t new vertices to any two fixed vertices of K_n $(2 \leq n \leq 4)$ is additively graceful.

$\diamond$ He also posed a conjecture *For any additively graceful graph G with p vertices and q edges $q \leq \frac{1}{2}(p^2 - 5p + 18)$.*

• Jinnah and Singh [66] proved that $P_n \times P_n$ is additively graceful graph.

3.2.10 $(k, d)-$**Graceful labeling :** Acharya and Hegde [4] generalized the notion of $k-$graceful labeling to $(k, d)-$graceful labeling in 1990.

A function f is called $(k, d)-$*graceful labeling* of a graph G if $f : V \rightarrow \{0, 1, 2, \ldots, k + (q - 1)d\}$ is injective and the induced function $f^* : E \rightarrow \{k, k + d, k + 2d, \ldots, k + (q - 1)d\}$ defined as $f^*(e = uv) = |f(u) - f(v)|$ is bijective. A graph which admits $(k, d)-$graceful labeling is known as $(k, d)-$*graceful graph*. Obviously $(1, 1)-$graceful labeling is graceful labeling and $(k, 1)-$graceful labeling is $k-$graceful labeling. Some known results for $(k, d)-$graceful labeling are listed below.

• Bu and Zhang [29] proved that $K_{m,n}$ is $(k, d)-$graceful for all k and d.

• Hegde and Shetty [61] defined a class of trees known as T_p-trees as follows and proved that T_p-trees are $(k, d)-$graceful for all k and d.

Let T be a tree with adjacent vertices u_0, v_0 and pendant vertices u, v such that the length of the path $u_0 - u$ is same as the length of the path

$v_0 - v$. Now delete the edge u_0v_0 and join vertices u and v by an edge uv. Then such a transformation of T is called an *elementary parallel transformation (ept)* and the edge u_0v_0 is called a *transformable edge*. If by a sequence of ept$'$s T can be reduced to a path then T is called T_p-*tree*. They also proved that every graph obtained by barycentric subdivision of a T_p-tree is $(k, d)-$graceful for all k and d.

• Hegde [59] proved that if a graph is $(k, d)-$graceful for odd k and even d then the graph is bipartite. He also proved that K_n is $(k, d)-$graceful if and only if $n \leq 4$.

3.2.11 $k-$equitable labeling : In 1990 Cahit [33] proposed the idea of distributing the vertex and the edge labels among $\{0, 1, 2, \ldots, k-1\}$ as evenly as possible to obtain a generalization of graceful labeling.

A vertex labeling of a graph $G = (V, E)$ is a function $f : V \to \{0, 1, 2, \ldots, k-1\}$ and the value $f(u)$ is called *label of vertex u*. For the vertex labeling function $f : V \to \{0, 1, 2, \ldots, k-1\}$, induced function $f^* : E \to \{0, 1, 2, \ldots, k-1\}$ defined as $f^*(e = uv) = |f(u) - f(v)|$ which satisfies the conditions:

(1) $|v_f(i) - v_f(j)| \leq 1$ and

(2) $|e_f(i) - e_f(j)| \leq 1$, $0 \leq i, j \leq k-1$,

where $v_f(i)$ and $e_f(i)$ denotes number of vertices and number of edges having label i under f and f^* respectively, $0 \leq i \leq k-1$. Such labeling f is called $k-$*equitable labeling* for the graph G. A graph which admits $k-$equitable labeling is called $k-$*equitable graph*. Obviously $2-$equitable labeling is the cordial labeling defined earlier in 3.2.8. When $k = 3$ the labeling is called $3-$*equitable labeling* which we will discuss in detail in *Chapter*-6. Some known results for $k-$equitable graph are listed below.

• Cahit [33],[34] proved that a graph is graceful if and only if it is $(|E|+1)-$equitable and he conjectured that all tree are $k-$equitable, for all k.

• Speyer and Szaniszlo [97] proved Cahit$'$s conjecture for $k=3$.

• Szaniszlo [99] proved that

$\diamond$ P_n is $k-$equitable for all k.

$\diamond$ K_n is not $k-$equitable for $3 \leq k < n$.

$\diamond$ $K_{2,n}$ is $k-$equitable if and only if $n \equiv k-1 \pmod{k}$ or $n \equiv 0, 1, 2, \ldots, (\lfloor \frac{k}{2} \rfloor - 1) \pmod{k}$ or $n = \lfloor \frac{k}{2} \rfloor$ and k is odd.

$\diamond$ C_n is $k-$equitable if and only if k meets all of the following conditions:

(1) $n \neq k$,

(2) if $k \equiv 2, 3 \pmod{4}$ then $n \neq k-1$ and n is not congruent to $k \pmod{2k}$.

• Vickrey [109] discussed the $k-$equitability of complete multipartite graphs. He proved that for $m \geq 3$ and $k \geq 3$, $K_{m,n}$ is $k-$equitable if and only if $K_{m,n}$ is one of following graphs:

(1) $K_{4,4}$ for $k=3$,

(2) $K_{3,k-1}$ for all k and

(3) $K_{m,n}$ for $k > mn$.

3.2.12 Skolem graceful labeling : Lee and Shee [75] introduced the concept of skolem graceful labeling in 1991.

A function f is called *skolem graceful labeling* of a graph G if $f : V \rightarrow \{1, 2, \ldots, p\}$ is bijective and the induced function $f^* : E \rightarrow \{1, 2, \ldots, q\}$ defined as $f^*(e = uv) = |f(u) - f(v)|$ is bijective. A graph which admits skolem graceful labeling is called *skolem graceful graph*. A necessary condition for a graph to be skolem graceful is $p \geq q+1$. Some known results for skolem graceful graphs are listed below.

• Lee and Wui [76] proved that a connected graph is skolem graceful if and only if it is a graceful tree.

• Yao et al.[111] have shown that a tree with p vertices and with maximum degree at least $\frac{p}{2}$ is skolem graceful.

• Although the disjoint union of trees can not be graceful, they can be skolem graceful.

• Lee and Wui [76] proved that the disjoint union of two or three stars is skolem graceful if and only if at least one star has even size.

• Choudum and Kishore [38] proved that disjoint union of k copies of the star $K_{1,2p}$ is skolem graceful if $k \leq 4p+1$ and the disjoint union of any number of copies of $K_{1,2}$ is skolem graceful. He also proved that all five stars are skolem graceful.

• Frucht [48] proved that $P_m \bigcup P_n$ is skolem graceful when $m+n \geq 5$.

• Bhat$-$Nayak and Deshmukh [23] proved that $P_{n_1} \bigcup P_{n_2} \bigcup P_{n_3}$ is skolem graceful when $n_1 < n_2 \leq n_3$, $n_2 = t(n_1+2)+1$ (where n_1 is even) and when $n_1 < n_2 \leq n_3$, $n_2 = t(n_1+3)+1$ (where n_1 is odd). They also proved that $P_{n_1} \bigcup P_{n_2} \bigcup \ldots \bigcup P_{n_i}$, for $i \geq 4$ is skolem graceful under certain conditions.

3.2.13 Indexable labeling : Acharya and Hegde [5] introduced the concept of indexable labeling in 1991.

A function f is called *indexable labeling* of a graph G if $f : V \rightarrow \{0, 1, 2, \ldots\ldots, p-1\}$ is bijective and the induced function $f^* : E \rightarrow N$

defined as $f^*(e = uv) = f(u) + f(v)$ is injective. Here f is called *indexer* of G. A graph which admits indexable labeling is called *indexable graph.* A graph is said to be *strongly indexable* if $f^*(E) = \{1, 2, \ldots, q\}$. Here f is called *strong indexer* of graph G. A function f is called $(k, d)-$*indexable labeling* if $f : V \to \{0, 1, 2, \ldots, p-1\}$ is bijective and the induced function $f^* : E \to \{k, k+d, \ldots, k+(q-1)d\}$ defined as $f^*(e = uv) = f(u)+f(v)$ is injective. A $(k, d)-$*indexable graph* is the graph which admits $(k, d)-$indexable labeling. A graph is said to be *strongly* $(k, d)-$*indexable* if $f^*(E) = \{k, k + d, \ldots, k + (q-1)d\}$.

Some known results on indexable and $(k, d)-$indexable graphs are listed below.

• Acharya and Hegde [5] have conjectured that *All unicyclic graphs are indexable.* This conjecture was proved by Arumugam and Germina [12] using breadth first search (BFS) algorithm [39]. They also proved that all trees are indexable.

• Acharya and Hegde [5] proved that K_2, K_3 and $K_2 \times K_3$ are the only non-trivial regular graphs which are strongly indexable.

• Hegde [58] proved that

⋄ Every graph can be embedded as an induced subgraph of indexable graph.

⋄ If a connected graph with p vertices and q edges ($q \geq 2$) is $(k, d)-$indexable then $d \leq 2$.

⋄ $P_m \times P_n$ is indexable for all m and n.

⋄ If G is connected $(1, 2)-$indexable graph then G must be a tree.

⋄ K_n, $n \geq 4$ and wheels W_n are not $(k, d)-$indexable.

3.2.14 Felicitous labeling : Lee et al.[74] introduced the concept of felicitous labeling in 1991.

A function f is called *felicitous labeling* of a graph G if $f : V \rightarrow \{0, 1, 2, \ldots \ldots, q\}$ is injective and the induced function $f^* : E \rightarrow \{0, 1, 2, \ldots \ldots, q - 1\}$ defined as $f^*(e = uv) = (f(u) + f(v)) \bmod q$ is bijective. A graph which admits felicitous labeling is called *felicitous graph*. Some known results on felicitous graphs are listed below.

• Balakrishnan and Kumar [15] proved that every graph is subgraph of a felicitous graph.

• Lee et al.[74] proved that

◇ Cycles C_n are felicitous except $n \equiv 2 \pmod 4$

◇ $K_{m,n}$ is felicitous when $m, n > 1$

◇ $P_2 \bigcup C_{2n+1}$ is felicitous for all n.

◇ They also conjectured that *$n-$cube is felicitous* which was proved by Figueroa$-$Centeno et al.[45] in 2001.

• Shee [94] conjectured that $P_m \bigcup C_n$ *is felicitous when* $n > 2$ *and* $m > 3$ which is still open.

3.3 Concluding Remarks :

In this chapter we have discussed various graph labeling techniques in detail. The discussion includes definitions and known results for each labeling. This chapter will give broad idea about various labeling techniques and will provide ready reference for any researcher. The penultimate chapter is devoted to the discussion on graceful labeling.

Chapter – 4

Graceful Labeling of Graphs

4.1 Introduction :

In *Chapter*−3 we have discussed various types of graph labeling while this chapter is aimed to discuss graceful labeling in detail. Some new classes of graceful graphs are investigated and some open problems are given at the end. As we mentioned earlier the graceful labeling was introduced by Rosa during 1967.

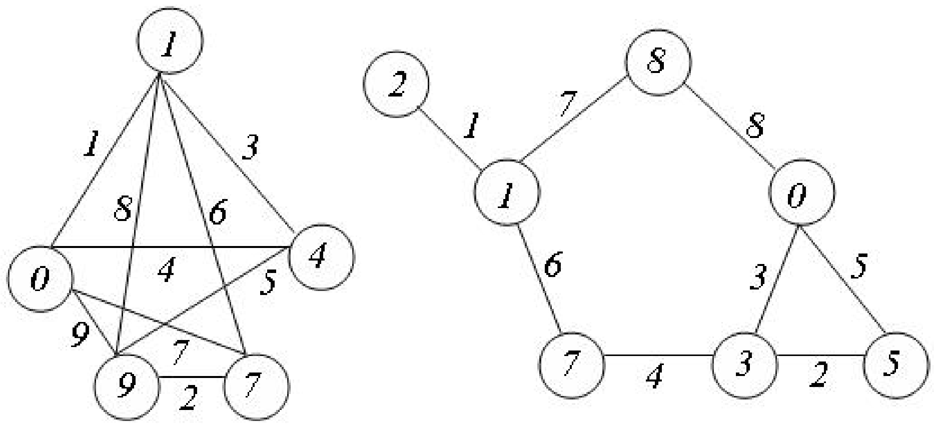
In the immediate section we will recall the definition of graceful labeling for ready reference.

4.2 Some Basic Definitions and Important Results :

Definition−4.2.1 **:** If the vertices of the graph are assigned values subject to certain conditions is known as *graph labeling*.

Definition−4.2.2 **:** A function f is called *graceful labeling* of a graph G if $f : V \rightarrow \{0, 1, 2, \ldots, q\}$ is injective and the induced function $f^* : E \rightarrow \{1, 2, \ldots, q\}$ defined as $f^*(e = uv) = |f(u) - f(v)|$ is bijective. A graph which admits graceful labeling is called *graceful graph*.

In the following *Figure*−4.1 some graceful graphs and their graceful labeling are shown.

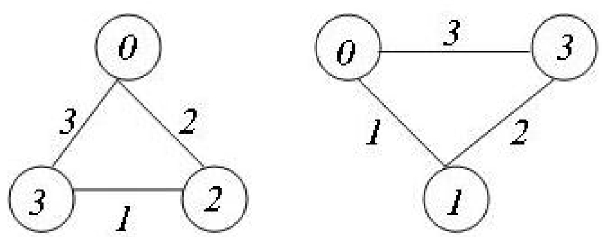


Figure−4.1

Some obvious facts and known results are listed below.

• Any graceful graph will always have vertices with labels 0 and q and these vertices are adjacent. One can visualize this from *Figure*−4.1.

• Graceful labeling is not unique. This fact is demonstrated in the following *Figure*−4.2.



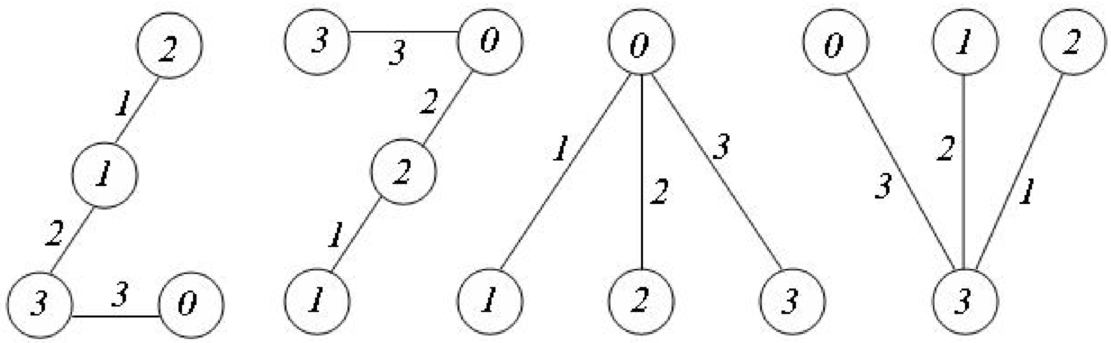
Figure−4.2

• Supergraph of a graceful graph need not be graceful. e.g. K_4 if graceful but K_5 is not.

• Subgraph of a graceful graph need not be a graceful graph. e.g. $W_5 = C_5 + K_1$ is graceful while C_5 is not.

• If $\{a_1, a_2, \ldots, a_p\} \subseteq \{0, 1, \ldots, q\}$ is a graceful labeling of any graph G, then $\{q - a_i / i = 1, 2, \ldots, p\}$ is also graceful labeling for the graph G.

• There are $q!$ connected graceful graphs with q edges. For example there are $3! = 6$ graceful graphs with 3 edges as shown in the above $Figure-4.2$ and following $Figure-4.3$.



$Figure-4.3$

• All the graphs with $p \leq 5$ are graceful except C_5, K_5 and Bowtie graph.

• Rosa [86] proved that the cycle C_n is graceful if and only if $n \equiv 0, 3 \pmod 4$.

• Frucht [47], Hoede and Kuiper [63] proved that all wheels $W_n = C_n + K_1$ are graceful graphs.

• Golomb [55] proved that the complete graph K_n is graceful if and only if $n \leq 4$.

• Rosa [86] and Golomb [55] proved that the complete bipartite graphs are graceful.

• Aravamudhan and Murugan [11] have shown that the complete tripartite graph $K_{1,m,n}$ is graceful.

• Beutner and Harborth [21] showed that $K_n - e$ (K_n with one edge deleted) is graceful only if $n \leq 5$, $K_n - 2e$ and $K_n - 3e$ are graceful only if $n \leq 6$.

• Ringel−Kotzig conjecture about gracefulness of trees is still an open problem and it has motivated good number of research papers. The conjecture is *All trees are graceful*. In [64] Kotzig called the effort to prove this conjecture as a *disease*. The trees known to be graceful are Caterpillars, Paths and Star Graphs.

• Ayel and Favaron [14] proved that all Helms are graceful.

• Kang et al.[69] proved that Webs are graceful.

• Bermond [20] conjectured that *Lobsters are graceful*.

• Morgan [84] proved that all lobsters with perfect matchings are graceful.

• Chen et al.[37] and Bhat−Nayak and Deshmukh [22] proved that banana trees are graceful.

• Aldred and Mckay [6] used a computer program to show that trees with at most 27 vertices are graceful.

Despite of many efforts the graceful tree conjecture remains an open problem and this problem has motivated some new graph labeling techniques.

• Truszczynski [101] studied unicyclic graphs and conjectured that *All unicyclic graphs except C_n, where $n \equiv 1$ or 2 (mod 4) are graceful*.

Because of the immense diversity of unicyclic graphs, a proof of above conjecture seems out of reach in the near future.

• Delorme et al.[41] and Ma and Feng [81] proved that cycle with a chord is graceful.

• Gracefulness of cycle with k consecutive chords is also investigated by Koh et al.[71],[72] and Goh and Lim [54].

• Koh and Rogers [72] conjectured that *Cycle with triangle [denoted as $C_n(p, q, r)$] is graceful if and only if* $n \equiv 0, 1$ *(mod 4).*

Next section is aimed to discuss gracefulness of some product related graphs. This section also includes investigations carried out by us.

4.3 Gracefulness of Some Product Related Graphs:

We have defined the cartesian product of two graphs in *Chapter*−2. This definition has attracted many researchers. Some results of product related graphs are listed below.

• Acharya and Gill [3] proved that grid graph $P_m \times P_n$ is graceful.

• Maheo [82] gave the graceful labeling for $P_m \times P_2$ which can be readily be extended to all grids.

• Kathiresan [70] proved that the graph obtained from subdividing each step of ladders $P_n \times P_2$ exactly once is graceful.

• Acharya [1] proved that certain subgraph of grid graphs are graceful.

• Huang and Skiena [65] proved that $C_m \times P_n$ is graceful for all n, when m is even and for all n with $3 \leq n \leq 12$ when m is odd.

• Jungreis and Reid [67] proved that torus grid $C_m \times C_n$ is graceful when $m \equiv 0$ (mod 4) and n is even.

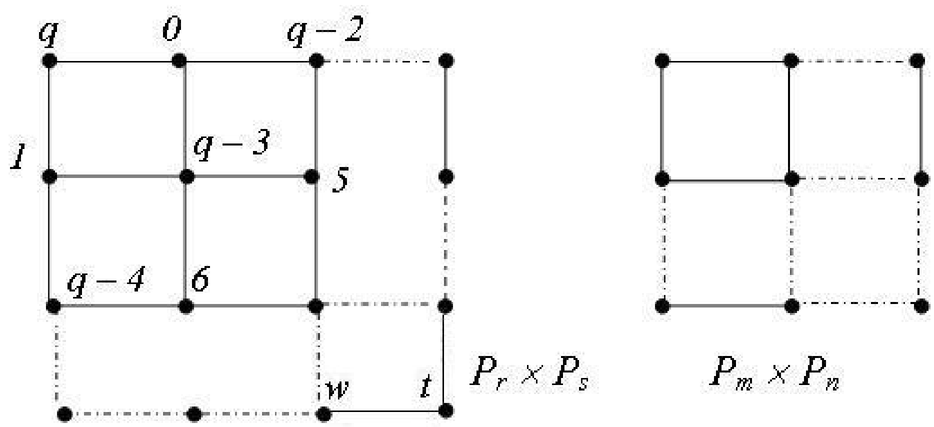
A complete solution for the problem of graceful torus grid will most likely involve a large number of cases.

We have also investigated some new families of product related graphs. We will provide detail proof of these results.

Theorem$-$4.3.1 **:** The graph $G = (P_m \times P_n) \bigcup (P_r \times P_s)$, where $m, n, r, s \in N \setminus \{1\}$ is graceful.

Proof : It is obvious that the graph G has number of vertices $p = rs + mn$ and number of edges $q = 2(rs + mn) - (m + n + r + s)$. According to *Definition*$-$4.2.2 the available vertex labels are $0, 1, \ldots, q$.

Now label the vertices of $(P_r \times P_s)$ by the labels $q, 0, 1, q-2, q-3, q-4, 4, 5, \ldots$ etc. This labeling sequence is having two sequential patterns, one is increasing and other is decreasing. Such labeling will give rise to edge labeling as decreasing sequence of labels $q, q-1, \ldots, q+r+s+1-2rs$. Such vertex labeling pattern is shown in *Figure*$-$4.4.



Figure$-$4.4

Now our task is to label the vertices of $(P_m \times P_n)$. It will depend on the vertex labels of the last grid of $(P_r \times P_s)$. Let w and t be vertex labels of last grid of $(P_r \times P_s)$. These labels produce edge label $q + r + s + 1 - 2rs = 2mn + 1 - (m + n)$. At this stage we have to consider following two possibilities.

Case - I: $w < t$. Then w must be a label from increasing sequence of labels and $t - w = 2mn + 1 - (m + n)$. Now available vertex labels are $t + 1, t - 1, t - 2, \ldots, w + 2, w + 1$, which are in number $2mn + 1 - (m + n)$.

We will use these labels for labeling of vertices $(P_m \times P_n)$. This vertex labeling sequence is $t + 1 = 2mn - (m + n) + w + 2, w + 2, w + 3, 2mn - (m+n)+w, 2mn-(m+n)+w-1, 2mn-(m+n)+w-2, w+7, w+8, \ldots$ etc. This labeling sequence is having two sequential patterns, one is increasing and other is decreasing. Such labeling will give rise to edge labeling as decreasing sequence of labels $2mn - (m + n), \ldots, 2, 1$. Thus we have labeled all the $rs + mn$ vertices of G gracefully.

Case - II: $w > t$. Then w must be a label from decreasing sequence of labels and $w - t = 2mn + 1 - (m + n)$. Now available vertex labels are $w - 1, w - 2, \ldots, t + 2, t + 1, t - 1$, which are in number $2mn + 1 - (m + n)$.

We will use these labels for labeling of vertices of $(P_m \times P_n)$. This vertex labeling sequence is $t-1, w-2 = 2mn-(m+n)+t-1, w-3, t+1, t+2, t+3, w-7, \ldots$ etc. This labeling sequence is having two sequential patterns, one is increasing and other is decreasing. Such labeling will give rise to edge labeling as decreasing sequence of labels $2mn - (m + n), \ldots, 2, 1$. Thus we have labeled all the $rs + mn$ vertices of G gracefully.

Therefore $G = (P_r \times P_s) \bigcup (P_m \times P_n)$ is graceful graph.

Illustration$-$4.3.2: For better understanding of labeling pattern discussed in above *Theorem* $-$4.3.1 let us consider $G = (P_3 \times P_4) \bigcup (P_4 \times P_2)$. For the graph G $p = 20$ and $q = 27$. Therefore for graceful labeling of G available vertex labels are $0, 1, \ldots, 27$. As per procedure employed in *Theorem*$-$4.3.1 we first label vertices of $P_3 \times P_4$ by $27, 0, 25, 4, 1, 24, 5, 20, 23, 6, 19, 8$ and $P_4 \times P_2$

by $7, 17, 16, 9, 10, 14, 13, 12$. this will produce edge labels $1, 2, \ldots, 27$ as shown in *Figure*-4.5. Thus G is a graceful graph.

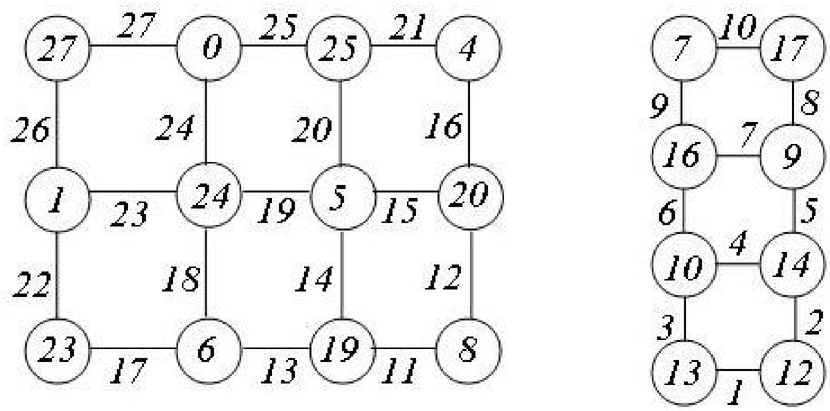


Figure-4.5

Theorem$-4.3.3$ **:** The graph $G = C_{2f+3} \bigcup (P_m \times P_n) \bigcup (P_r \times P_s)$, where $m, n, r, s \in N \setminus \{1\}$ and $f = 2(mn + rs) - (m + n + r + s)$, is graceful.

Proof : It is obvious that G will have number of vertices $p = 2f+3+mn+rs$ and number of edges $q = 3f + 3$. Let $u_1, u_2, \ldots, u_{2f+3}$ be successive vertices of C_{2f+3}. Now label $f + 2$ vertices $u_1, u_3, \ldots, u_{2f+3}$ of C_{2f+3} by the labels $0, 1, 2, \ldots, f + 1$ respectively and label the remaining $f + 1$ vertices $u_2, u_4, \ldots, u_{2f+2}$ of C_{2f+3} by the labels $3f + 3, 3f + 2, \ldots, 2f + 3$ respectively. Thus all the vertices of C_{2f+3} are labeled. This vertex labeling will give rise to edge label according to *Definition*$-4.2.2$ as $3f + 3, 3f + 2, \ldots, f + 2, f + 1$. Now our task is to label the vertices of $(P_m \times P_n) \bigcup (P_r \times P_s)$ for which the available vertex labels are $2f + 2, 2f + 1, \ldots, f + 2$ and required edge labels for $(P_m \times P_n) \bigcup (P_r \times P_s)$ are $f, f - 1, \ldots\ldots, 2, 1$. Since available vertex labels are $f + 1$ and required edge labels are f, we first label the vertices of $(P_m \times P_n)\bigcup(P_r \times P_s)$ by $0, 1, \ldots, f$, as in *Theorem*$-4.3.1$. Then we add $f + 2$ to all the vertex labels of $(P_m \times P_n)\bigcup(P_r \times P_s)$ will produce edge labels $1, 2, \ldots, f$ for $(P_m \times P_n)\bigcup(P_r \times P_s)$. Thus we have labeled $G = C_{2f+3}\bigcup(P_m \times P_n)\bigcup(P_r \times P_s)$ gracefully. Therefore G is graceful graph.

4.4 Gracefulness of Union of Grid Graph with Complete Bipartite Graph and Path Graph :

Bu and Cao [27] have discussed gracefulness of $K_{m,n}$ and its union with path graph. Seoud and Youssef [93] have shown that $K_{m,n}\bigcup K_{p,q}$ $(m, n, p, q \geq 2)$, $K_{m,n}\bigcup K_{p,q}\bigcup K_{r,s}$ $(m, n, p, q, r, s \geq 2$ and $(p, q) \neq (2, 2))$ are graceful graphs. In this section we will discuss gracefulness of union of grid graph with complete bipartite graph and path graph.

Theorem$-4.4.1$ **:** $G = K_{m,n}\bigcup(P_r \times P_s)$, $r, s \geq 2$ is graceful graph.

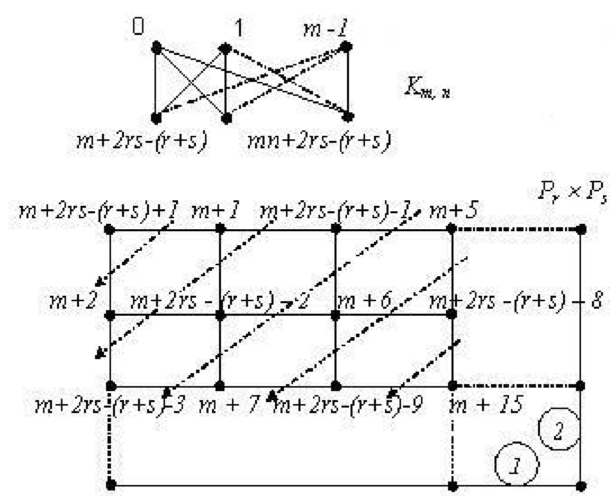
Proof : Here total number of vertices $p = m + n + rs$ and number of edges $q = mn + 2rs - (r + s)$.

Now label the vertices of $K_{m,n}$ by the labels $0, 1, \ldots, m - 1, m + 2rs - (r + s), 2m + 2rs - (r + s), \ldots, q = mn + 2rs - (r + s)$, which give rise to edge labels as $q, q - 1, \ldots, 2rs - (r + s) + 1$ to edges of $K_{m,n}$. Now our task is to label the vertices of $(P_r \times P_s)$, for which the available vertex labels are $m + 1, m + 2, \ldots, m + 2rs - (r + s) - 1$ and $m + 2rs - (r + s) + 1$.

Let us denote the vertices of the grid graph $P_r \times P_s$ by $v_{11}, v_{12}, \ldots, v_{1n}, v_{21}, \ldots, v_{mn}$. Now label the vertex v_{11} by $m + 2rs - (r + s) + 1$, v_{12} by $m + 1$, v_{21} by $m + 2$, v_{13} by $m + 2rs - (r + s) - 1$, v_{22} by $m + 2rs - (r + s) - 2$, v_{31} by $m + 2rs - (r + s) - 3$, v_{14} by $m + 5$, v_{23} by $m + 6$, v_{32} by $m + 7$, v_{41} by $m + 8$,

v_{15} by $m + 2rs - (r + s) - 7$, v_{24} by $m + 2rs - (r + s) - 8$ etc. This will give rise edge labeling as $2rs - (r + s), 2rs - (r + s) - 1, 2rs - (r + s) - 2, \ldots, 2, 1$. For the vertex labeling and edge labeling following pattern has been observed.

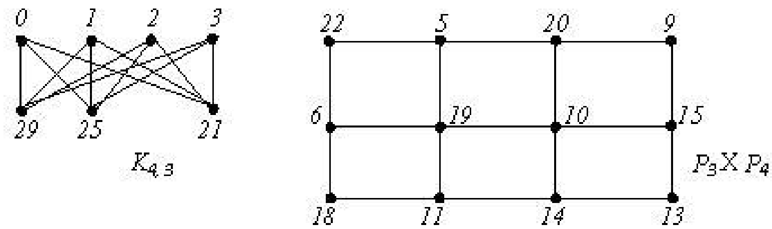
(1) In each square of grid the difference between two labels of main diagonal is always one.
(2) In the labeling of vertices two sequential patterns has been found. One is increasing and another is decreasing. This will give rise to edge labeling into decreasing sequence of labels $2rs - (r + s), 2rs - (r + s) - 1, 2rs - (r + s) - 2, \ldots, 2, 1$, such labeling pattern for vertices and edges is shown by down arrows in the *Figure*−4.6.



Figure−4.6

Thus we have labeled all the vertices of graph $K_{m,n} \bigcup (P_r \times P_s)$ gracefully, where $m, n, r, s \in N \setminus \{1\}$ and the graph becomes a graceful graph.

Illustration$-$4.4.2**:** For better understanding of above discussed labeling pattern, let us see graceful labeling pattern for the graph $G = K_{4,3} \bigcup (P_3 \times P_4)$. For the graph G, $p = 19$ and $q = 29$. Therefore for graceful labeling available vertex labels are $0, 1, 2, \ldots, 29$. As per the procedure employed in *Theorem*$-$4.4.1 we first label the vertices of $K_{4,3}$ by $0, 1, 2, 3, 29, 25, 21$ and $(P_3 \times P_4)$ by $22, 5, 20, 9, 6, 19, 10, 15, 18, 11, 14, 13$. This will produce edge labels $1, 2, \ldots, 29$ as shown in *Figure*$-$4.7. Thus G is a graceful graph.

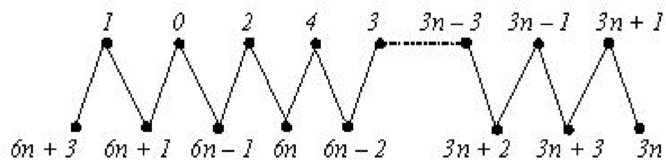


Figure$-$4.7

Lemma$-$4.4.3 **:** Using $0, 1, \ldots, t-2$ and t vertex labels one can produce $1, 2, \ldots, t-1$ edge labels for path graph P_t, $t \geq 3$.

Proof : There are six cases to be considered as follows:

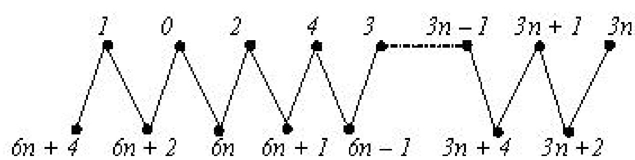
<u>Case - I:</u> $t \equiv 3 \pmod 6$. In this case $t = 6n + 3$ for some non$-$negative integer n. Now $P_t = P_{6n+3}$ is a path of length $6n + 2$, it has $6n + 3$ vertices and for this available vertex labels are $0, 1, 2, \ldots, 6n + 1$ and $6n + 3$. Let us denote these vertices by $u_1, u_2, \ldots, u_{6n+3}$. We shall label the vertices $u_2, u_4, \ldots, u_{6n+2}$ according to the sequence $1, 0, 2, 4, 3, 5, 7, 6, \ldots, 3n-3, 3n-1, 3n+1$. Now label the remaining vertices $u_1, u_3, \ldots, u_{6n+3}$ according to the sequence $6n + 3, 6n + 1, 6n - 1, 6n, \ldots, 3n + 2, 3n + 3, 3n$, as shown in *Figure*$-$4.8.



Figure−4.8

Such vertex labeling will give rise to edge labeling for P_t as $6n+2, 6n, 6n+1, 6n-1, 6n-3, \ldots, 3, 4, 2, 1$.

<u>Case - II:</u> $t \equiv 4 \pmod 6$. Then $t = 6n + 4$, for some $n \in N\bigcup\{0\}$. Here available vertex labels are $0, 1, 2, \ldots, 6n+2$ and $6n+4$. We shall label the vertices $u_2, u_4, \ldots, u_{6n+2}$ according to the sequence $1, 0, 2, 4, 3, \ldots, 3n-3, 3n-1, 3n+1, 3n$ and label the remaining vertices $u_1, u_3, \ldots, u_{6n+3}$ according to the sequence $6n+4, 6n+2, 6n, 6n+1, 6n-1, \ldots, 3n+3, 3n+4, 3n+2$ as shown in *Figure*−4.9. Such vertex labeling will give rise to edge labeling for P_t as $6n+3, 6n+1, 6n+2, 6n, 6n-2, \ldots, 4, 5, 3, 1, 2$.



Figure−4.9

<u>Case - III:</u> $t \equiv 5 \pmod 6$. Then $t = 6n + 5$, for some $n \in N\bigcup\{0\}$. Here available vertex labels are $0, 1, 2, \ldots, 6n+3$ and $6n+5$. We shall label the vertices at even places according to the sequence $1, 0, 2, 4, 3, \ldots, 3n-3, 3n-1, 3n+1, 3n$ and label the remaining vertices according to the sequence $6n+5, 6n+3, 6n+1, 6n+2, 6n, \ldots, 3n+5, 3n+2, 3n+3$. Such vertex labeling will give rise to edge labeling for P_t as $6n+4, 6n+2, 6n+3, 6n+1, \ldots, 5, 6, 4, 1, 2, 3$.

<u>**Case - IV:**</u> $t \equiv 0 \pmod 6$. Then $t = 6n$, for some $n \in N$. Here available vertex labels are $0, 1, 2, \ldots, 6n-2$ and $6n$. We shall label the vertices at even places according to the sequence $1, 0, 2, 4, 3, \ldots, 3n-4, 3n-2, 3n-3, 3n$ and label the remaining vertices according to the sequence $6n, 6n-2, 6n-4, \ldots, 3n+3, 3n+1, 3n-1$. Such vertex labeling will give rise to edge labeling for P_{6n} as $6n-1, 6n-3, 6n-2, \ldots, 7, 5, 3, 4, 2, 1$.

<u>**Case - V:**</u> $t \equiv 1 \pmod 6$. Then $t = 6n+1$, for some $n \in N$. We shall label the vertices at even places according to the sequence $1, 0, 2, 4, 3, \ldots, 3n-3, 3n-2, 3n-3, 3n-1$ and label the remaining vertices according to the sequence $6n+1, 6n-1, 6n-3, \ldots, 3n+2, 3n, 3n+1$. Such vertex labeling will give rise to edge labeling for P_t as $6n, 6n-2, 6n-1, 6n-3, \ldots, 5, 3, 1, 2$.

<u>**Case - VI:**</u> $t \equiv 2 \pmod 6$. Then $t = 6n+2$, for some $n \in N$. We shall label the vertices at even places according to the sequence $1, 0, 2, 4, 3, \ldots, 3n-2, 3n-3, 3n, 3n-1$ and label the remaining vertices according to the sequence $6n+2, 6n, 6n-2, 6n-1, \ldots, 3n+5, 3n+2, 3n+3, 3n+1$, will give rise to edge labeling for P_t as $6n+1, 6n-1, 6n, \ldots, 4, 5, 6, 3, 1, 2$.

Thus in any case one can produce $1, 2, \ldots, t-1$ edge labels for P_t, $t \geq 3$, using $0, 1, 2, \ldots, t-2$ and t vertex labels.

Remark$-4.4.4$ **:** From the above *Lemma*$-4.4.3$ following observations are obvious:

(1) By adding n in each term of the sequence $1, 2, \ldots, t-2, t$ (which are vertex labels for P_t $t \geq 3$) one can produce edge labels $1, 2, \ldots, t-1$ for P_t, $t \geq 3$.

(2) By subtracting each term of the sequence $1, 2, \ldots, t-2, t$ (which are vertex labels for P_t) from $n+t$ one can produce edge labels $1, 2, \ldots, t-1$ for P_t, $t \geq 3$.

Theorem$-4.4.5$ **:** The graph $G = (P_r \times P_s) \bigcup P_t$ is graceful, where $t \in N \setminus \{2\}$ and $r, s \in N \setminus \{1\}$.

Proof : Here for the graph G under consideration number of vertices $p = rs + t$ and number of edges $q = 2rs - (r+s) + t - 1$. According to *Definition*$-4.2.2$ the available vertex labels are $0, 1, \ldots, q$.

Now label the vertices of $P_r \times P_s$ by the labels $q, 0, 1, q-2, q-3, q-4, 6, 7, \ldots$etc. As we discussed in *Theorem*$-4.4.1$ two labeling pattern has been observed. Such vertex labeling will give rise to edge labeling as decreasing sequence of labels $q, q-1, \ldots, q - 2rs + r + s + 1$ which is shown in *Figure*-4.10.

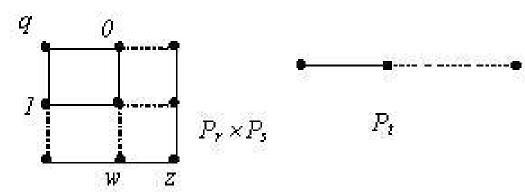


Figure-4.10

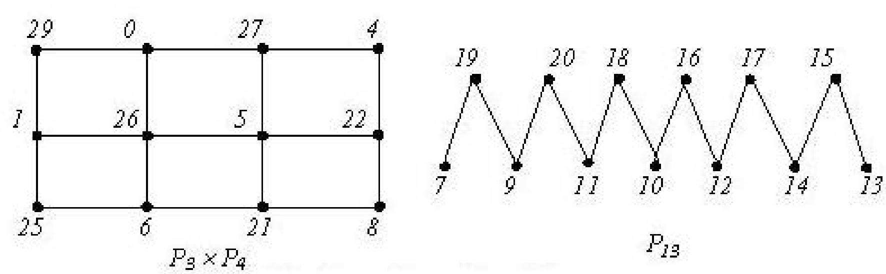
Now our task is to label the vertices of P_t. It will depend on the vertex labels of the last grid of $P_r \times P_s$. Let w and z be vertex labels of last grid of $P_r \times P_s$. These labels produce edge label $q - 2rs + r + s + 1 = t$. At this stage following two cases are to be considered.

<u>Case - I:</u> $w < z$. Then w must be a label from increasing sequence of labels and $z - w = t$. Now available vertex labels are $z + 1 = t + w + 1, z - 1 = t + w - 1, z - 2 = t + w - 2, \ldots, w + 2, w + 1$, which are in number t. Using these labels we can label P_t according to *Remark*$-$4.4.4 and produce edge labels $1, 2, \ldots, t-1$.

<u>Case - II:</u> $w > z$. Then w must be a label from decreasing sequence of labels and $w - z = t$. Now available vertex labels are $w - 1 = t + z - 1, w - 2 = t + z - 2, \ldots, z + 2, z + 1, z - 1$, which are in number t. Using these labels one can label the vertices of P_t according to *Remark*$-$4.4.4 and produce edge labels $1, 2, \ldots, t-1$.

Therefore $G = (P_r \times P_s) \bigcup P_t$ is graceful, where $r, s \in N \setminus \{1\}$ and $t \in N \setminus \{2\}$.

Illustration$-$4.4.6**:** For better understanding of the above discussed labeling pattern, let us see graceful labeling pattern for the graph $G = (P_3 \times P_4) \bigcup P_{13}$. For this graph G, $p = 25$ and $q = 29$. So for graceful labeling of G, available vertex labels are $0, 1, 2, \ldots, 29$. According to *Theorem*$-$4.4.5 one can label $(P_3 \times P_4)$ by $29, 0, 27, 4, 1, 26, 5, 22, 25, 6, 21, 8$ and P_{13} by $7, 19, 9, 20, 11, 18, 10, 16, 12, 17, 14, 15, 13$. This will give rise to edge labels $29, 28, \ldots, 13$ for grid graph $(P_3 \times P_4)$ and $12, 10, 11, 9, \ldots$
$\ldots, 5, 3, 1, 2$ for P_{13} according to *Case* - V of *Lemma*$-$4.4.3 such labeling pattern is shown in *Figure*$-$4.11. Hence G is a graceful graph.



Figure$-$4.11

Theorem$-4.4.7$ **:** The graph $G = K_{m,n} \bigcup (P_r \times P_s) \bigcup P_t$ is graceful, where $t \in N \setminus \{2\}$ and $m, n, r, s \in N \setminus \{1\}$.

Proof : The graph G has number of vertices $p = m+n+rs+t$ and number of edges $e = mn+2rs-(r+s)+t-1 = mn+q$ say, where $q = 2rs-(r+s)+t-1$ is the number of edges in $(P_r \times P_s) \bigcup P_t$.

Now label the vertices of $K_{m,n}$ by labels $0, 1, \ldots, m-1, m+q, 2m+q, \ldots, e = mn+q$, which will give rise to edge labels as $e, e-1, \ldots, q+1$ for the edges of $K_{m,n}$. Now our task is to label the vertices of $(P_r \times P_s) \bigcup P_t$, for which the available vertex labels are in number $q+1$. These are $m, m+1, m+2, \ldots, m+q-1$ and $m+q+1$. Now by adding $m+1$ in all the vertex labels of $(P_r \times P_s) \bigcup P_t$ reported in *Theorem - 4.4.5* one can produce edge labels $1, 2, \ldots, q$. Thus we have labeled $G = K_{m,n} \bigcup (P_r \times P_s) \bigcup P_t$ gracefully and hence G is a graceful graph.

4.5 Gracefulness of Union of Two Path Graphs with Grid Graph and Complete Bipartite Graph :

It is obvious that union of two path graphs can not be graceful as number of vertices of $P_n \bigcup P_t$ is more than the number of labels available for its gracefulness. In connection of *Lemma*$-4.4.3$, we have following remarks.

Remark$-4.5.1$ **:** Using $n, n+1, \ldots, n+t-2, n+t$, for n$\in$ N one can produce $1, 2, \ldots, t-1$ edge labels for path graph P_t(where $t \geq 3$). In order to produce $s, s+1, \ldots, t-1$ edge labels for path graph P_{t-s} using above vertex labels one can proceed as either of the following two ways.

(i) Using $n+s, n+s+1, \ldots, n+t-2, n+t$, (where $n, s \in N$) one can produce $1, 2, \ldots, t-s-1$ edge labels for path graph P_{t-s}. Now choose half of the total number of vertex labels from the above mentioned sequence of vertex labels into their numerically increasing order (one less than half of the total number when n is odd) and subtract s from each selected vertex labels. This will produce edge labels $s, s+1, \ldots, t-1$ for P_{t-s}.

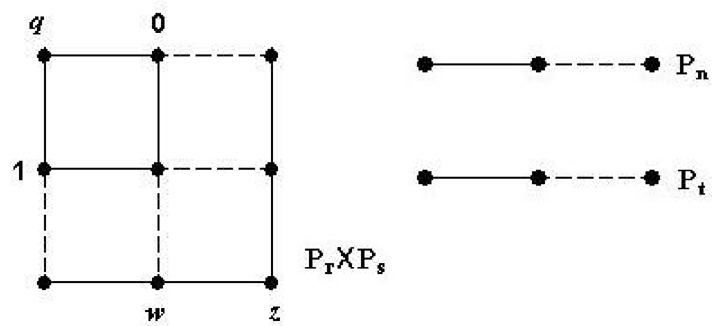
(ii) Using $n+s, n+s+2, \ldots, n+t-1, n+t$, (where $n \in N$) will produce edge labels as $1, 2, \ldots, t-s-1$ for P_{t-s}. Now choose half of the total number of vertex labels from the above mentioned sequence of vertex labels according to their numerically increasing order (one less than half of the total number when n is odd) and subtract s from each selected vertex labels. This will produce edge labels $s, s+1, \ldots, t-1$ for P_{t-s}.

Remark$-4.5.2$ **:** If we label the grid graph $(P_r \times P_s)$ by using increasing and decreasing sequence of vertex labels in diagonal pattern then there are $min\{r, s\} - 1$ vertex labels are not used after graceful labeling of $(P_r \times P_s)$. Moreover if $K_{r,s}$ is labeled by t vertex labels (where $t \leq max\{r, s\}$) $0, 1, \ldots, t-1$ and remaining by $t, 2t, \ldots, rs$ then there are t vertex labels namely $1+t, 2+t, \ldots\ldots, 2t-1, 2t+1$ which are not used in the graceful labeling of $K_{r,s}$.

Theorem$-4.5.3$ **:** The graph $G = P_n \cup P_t \cup (P_r \times P_s)$,where $t < min\{r, s\}, r, s \geq 3$ is graceful.

Proof : Here total number of vertices $p = n + t + rs$ and total number of edges $q = n + t + 2rs - (r + s - 2)$.

Now label the vertices of $(P_r \times P_s)$ by labels $q, 0, 1, q-2, q-3, q-4 \ldots$ etc. This labeling sequence is having two sequential patterns, one is increasing and other is decreasing. Such labeling pattern will give rise to edge labeling as decreasing sequence of labels $q, q-1, \ldots, q+r+s+1-2rs$, which is shown in *Figure*−4.12.



Figure−4.12

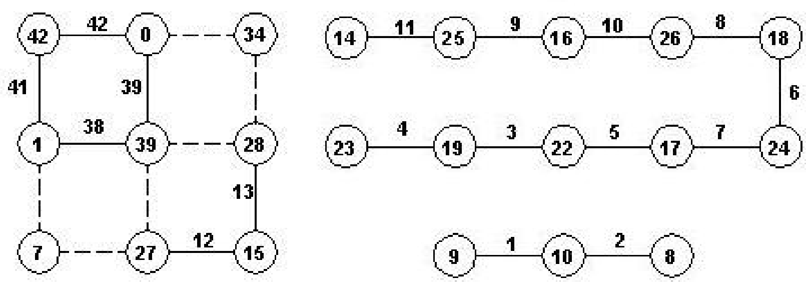
Now our task is to label the vertices of P_n. It will depend on the vertex labels of the grid graph $(P_r \times P_s)$. Let w and z be vertex labels of last grid of $(P_r \times P_s)$.

<u>Case - I:</u> $w < z$. Then w must be a label from increasing sequence of labels and $z - w = q + r + s + 1 - 2rs = n + t - 1$. Now available vertex labels are $z + 1, z - 1, \ldots, w + 2, w + 1$ which are total $n + t - 1$.

<u>Case - II:</u> $w > z$. Then w must be a label from the decreasing sequence of labels and $n + t - 1 = w - z$. Now available vertex labels are $w - 1, w - 2, \ldots, z + 2, z + 1, z - 1$, which are in number $n + t - 1$.

Using these labels one can label the vertices of P_n according to *Remark*−4.5.1 which will give rise to edge labels as $n + t - 2, n + t - 3, \ldots, t$. Now to label P_t one can use vertex labels which are not used in graceful labeling of grid graph. This labels will give rise to edge labels $1, 2, \ldots, t - 1$ for P_t. Thus graph G under consideration admits graceful labeling.

Illustration−4.5.4**:** For better understanding of above defined labeling pattern consider the graph $G = P_{10} \cup P_3 \cup (P_5 \times P_4)$ with $q = 42$. The graceful labeling of G is shown in *Figure*−4.13.

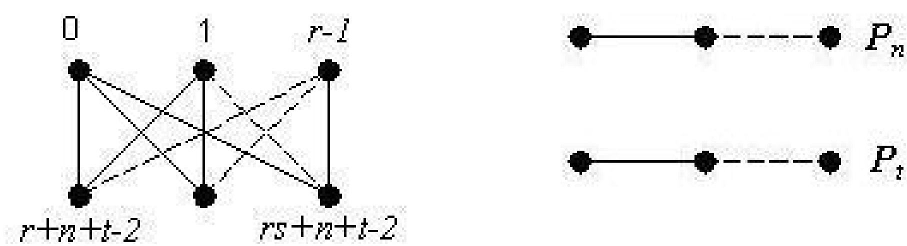


Figure−4.13

Theorem−4.5.5 **:** The graph $G = P_n \cup P_t \cup K_{r,s}$,where $t \leq max\{r, s\}, r, s \geq 3$ is graceful.

Proof : Here total number of vertices $p = n + t + r + s$ and total number of edges $q = rs + n + t - 2$.

Now label the vertices of $K_{r,s}$ by labels $0, 1, \ldots, r-1$, $r+n+t-2, \ldots, rs+n+t-2 = q$(assuming $r \geq s$) as shown in *Figure*−4.14. This will give rise to edge labels as $q, q-1, \ldots, n+t-1$ of $K_{r,s}$.



Figure−4.14

Now our task is to label the vertices of P_n and then P_t for which the available vertex labels are $r, r+1, r+2, \ldots, r+n+t-3, r+n+t-1$. These are in number $n+t-1$ and $2r+n+t-3, 2r+n+t-1, 2r+n+t, \ldots, 3r+n+t-3$, which are in number r. Using these labels according to *Remark*$-4.5.2$ one can label P_n and P_t which give rise to edge labels as $n+t-2, n+t-3, \ldots, t$ and $t-1, t-2, \ldots, 2, 1$ respectively. Thus we have labeled all the vertices of graph G under consideration gracefully.

Illustration$-4.5.6$**:** For better understanding of above defined labeling pattern consider the graph $G = P_{10} \cup P_5 \cup (K_{4,5})$ with $q = 33$. The graceful labeling of G is shown in *Figure*-4.15.

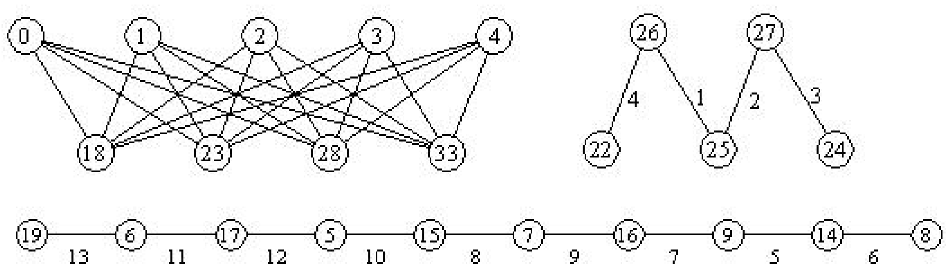


Figure-4.15

4.6 Some Open Problems :

• One can discuss gracefulness of union of grid graph with wheels, cycles, Petersen graph etc.

• One can derive parallel results for other type of labeling like harmonious, $(k,d)-$graceful, skolem graceful, $k-$equitable etc.

• One can discuss gracefulness in the context of various graph operations like contraction and barycentric subdivision.

• One can investigate graceful labeling for the star of cycle, which is defined in *Chapter*-5.

4.7 Concluding Remarks :

The graceful labeling of graph is stronger in its class. Grid graph is very interesting family of graphs. Here we have discussed the gracefulness of union of grid graph with some other families. The results obtained here are new and of very general nature. This work throws light on the gracefulness of disconnected graphs which is very less cultivated field. Illustrations provide better understanding of the derived results. This work contributes eight new results to the theory of graceful graphs. The next *Chapter*-5 is aimed to discuss cordial labeling of graphs.

Chapter – 5

Cordial Labeling of Graphs

5.1 Introduction :

In *Chapter*−3 we have discussed various types of graph labeling, while this chapter is aimed to discuss cordial labeling of graphs in detail. Some new families of cordial graphs are investigated and some open problems are also posed.

Many researchers have studied cordiality of graphs. As we mentioned in *Chapter*−3, Cahit [31] introduced cordial graph in 1987 as a weaker version of graceful and harmonious graphs. In the immediate section we will recall the definition of cordial graphs and will provide detail survey on cordial graphs.

5.2 Some Basic Definitions and Important Results :

Definition−5.2.1 **:** If the vertices of the graph are assigned values subject to certain conditions is known as *graph labeling*.

Definition−5.2.2 **:** A function $f : V \to \{0, 1\}$ is called *binary vertex labeling* of a graph G and $f(v)$ is called *label of the vertex* v of G under f.

For an edge $e = uv$, the induced function $f^* : E \to \{0, 1\}$ is given as $f^*(e = uv) = |f(u) - f(v)|$. Let $v_f(0)$, $v_f(1)$ be number of vertices of G having labels 0 and 1 respectively under f and let $e_f(0)$, $e_f(1)$ be number of edges of G having labels 0 and 1 respectively under f^*.

Definition$-$5.2.3 **:** A binary vertex labeling f of a graph G is called *cordial labeling* if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph which admits cordial labeling is called *cordial graph*.

Vast amount of literature is available in printed or in electronic form. Some known families of cordial graphs are listed below.

• As investigated by Cahit [32]

◇ Every tree is cordial.

◇ Complete bipartite graphs $K_{m,n}$ are cordial.

◇ Complete graphs K_n are cordial iff $n \leq 3$.

◇ Maximal outer planar graphs are cordial.

◇ Eulerian graph is not cordial if its size congruent to 2 (mod 4).

◇ All fans $F_n = P_n + K_1$ are cordial.

◇ Wheels $W_n = C_n + K_1$ are cordial iff n is not congruent to 3 (mod 4).

◇ $k-$angular cactus with t cycles is cordial iff kt is not congruent to 2 (mod 4).

• Ho et al.[62], proved that

◇ Unicyclic graph is cordial except C_{4k+2}.

◇ Generalized Petersen graph $P(n, k)$ is cordial iff n is not congruent to 2 (mod 4).

• Lee and Liu [73], Du[43] proved that complete $n-$partite graph is cordial if and only if at most three of its partite sets have odd cardinality.

• Seoud and Maqsoud [88] proved that if G is a graph with n vertices and m edges and every vertex has odd degree then G is not cordial when $m+n \equiv 2$ (mod 4).

• Andar et al. in [7],[8],[9] and [10] proved that

◇ Multiple shells are cordial.

◇ $t-$ply graph $P_t(u, v)$ is cordial except when it is Eulerian and the number of edges is congruent to 2 (mod 4).

◇ Helms, closed helms and generalized helms are cordial.

◇ Flowers are cordial.

• In [10], Andar et al. showed that a cordial labeling of a graph G can be extended to a cordial labeling of the graph obtained from G by attaching $2m$ pendant edges at each vertex of G. They also proved that a cordial labeling of a graph G with p vertices can be extended to a cordial labeling of the graph obtained from G by attaching $2m + 1$ pendant edges at each vertex of G if and only if G does not satisfy either of the following conditions: (1) G has an even number of edges and $p \equiv 2 \pmod 4$; (2) G has an odd number of edges and either $p \equiv 1 \pmod 4$ with $e_g(1) = e_g(0) + i(G)$ or $p \equiv 3$ (mod 4) with $e_g(0) = e_g(1) + i(G)$, where $i(G) = min\{|e_g(0) - e_g(1)|\}$.

5.3 Cordial Labeling for Some Cycle Related Graphs:

We have investigated some new families of cordial graphs. In this section we will give cordial labeling for cycle with one chord, cycle with twin chords and cycle with triangle. Let us provide some important definitions.

Definition$-$5.3.1 **:** *Chord* of a cycle C_n is an edge joining two non$-$adjacent vertices of cycle C_n.

Definition$-$5.3.2 **:** Two chords of a cycle are said to be *twin chords* if they form a triangle with an edge of the cycle C_n.

For positive integers n and p with $3 \leq p \leq n - 2$, $C_{n,p}$ is the graph consisting of a cycle C_n with a pair of twin chords with which the edges of C_n form cycles C_p, C_3 and C_{n+1-p} without chords.

Definition$-$5.3.3 **:** A *cycle with triangle* is a cycle with three chords which by themselves form a triangle.

For positive integers p, q, r and $n \geq 6$ with $p+q+r+3 = n$, $C_n(p,q,r)$ denotes a cycle with triangle whose edges form the edges of C_n, cycles C_{p+2}, C_{q+2} and C_{r+2} without chords.

Theorem$-$5.3.4 **:** Cycles with one chord are cordial.

Proof : Let $u_1, u_2, \ldots, u_n$ be consecutive vertices of cycle C_n and $e = u_1u_3$ be a chord of cycle C_n. The vertices u_1, u_2, u_3 forms a triangle with chord e. To define labeling function $f : V(G) \rightarrow \{0, 1\}$ we consider following cases.

Case$-$1 : $n \equiv 0, 1 \pmod 4$

$f(u_i) = 0$; if $i \equiv 1, 2 \pmod 4$

$= 1$; if $i \equiv 0, 3 (\bmod\ 4)$, $1 \leq i \leq n$.

Case$-$2 : $n \equiv 2 \pmod 4$

$f(u_i) = 0$; if $i \equiv 1, 2 \pmod 4$

$= 1$; if $i \equiv 0, 3 \pmod 4$, $1 \leq i \leq n-2$ and $f(u_n) = 0$, $f(u_{n-1}) = 1$.

Case$-$3 : $n \equiv 3 \pmod 4$

$f(u_1) = 1$ and $f(u_i) = 0$; if $i \equiv 1, 2 \pmod 4$

$= 1$; if $i \equiv 0, 3 \pmod 4$, $2 \leq i \leq n$.

The labeling pattern defined above covers all possible arrangement of vertices. In each case the graph G under consideration satisfies the conditions $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ as shown in following *Table*$-$5.1. i.e. G admits cordial labeling.

Let $n = 4a + b$, $a \in N$.

b	vertex conditions	edge conditions
0	$v_f(0)=v_f(1)$	$e_f(0)+1=e_f(1)$
1	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$
2	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$
3	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$

Table−5.1

Theorem−5.3.5 **:** Cycles with twin chords are cordial, where chords form two triangles and one cycle C_{n-2}.

Proof : Let G be the cycle with twin chords, where chords form two triangles and one cycle C_{n-2}. Here number of vertices $p = n$ and number of edges $q = n + 2$. Let $u_1, u_2, \ldots, u_n$ be successive vertices of G. Let $e_1 = u_n u_2$ and $e_2 = u_n u_3$ be the chords of cycle C_n. To define labeling function $f : V(G) \to \{0, 1\}$ we consider following cases.

Case−1 : $n \equiv 0 \pmod 4$

$f(u_i) = 0$; if $i \equiv 1, 2 \pmod 4$

$\quad = 1$; if $i \equiv 0, 3 \pmod 4$, $1 \le i \le n$.

Case−2 : $n \equiv 1, 2, 3 \pmod 4$

$f(u_i) = 0$; if $i \equiv 0, 1 \pmod 4$

$\quad = 1$; if $i \equiv 2, 3 \pmod 4$, $1 \le i \le n$.

The labeling pattern defined above covers all possible arrangement of vertices. In each case, the graph G under consideration satisfies the conditions $|v_f(0) - v_f(1)| \le 1$ and $|e_f(0) - e_f(1)| \le 1$ as shown in following *Table*−5.2. i.e. G admits cordial labeling.

Let $n = 4a + b$, $a \in N$.

b	vertex conditions	edge conditions
0,2	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)$
1,3	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)+1$

Table-5.2

Theorem$-5.3.6$ **:** Cycles with triangle $C_n(1,1,n-5)$ is cordial except $n \equiv 3 \pmod 4$.

Proof : Let G be cycle with triangle $C_n(1,1,n-5)$. Let $u_1, u_2, \ldots, u_n$ be successive vertices of G. Let u_1, u_3 and u_5 be the vertices of triangle formed by edges $e_1 = u_1u_3$, $e_2 = u_3u_5$ and $e_3 = u_1u_5$.

Note that for the case $n \equiv 3 \pmod 4$, graph G is an Eulerian graph with number of edges congruent to 2 (mod 4). Then in this case G is not cordial as proved by Cahit [32]. So it remains to consider following cases to define labeling function $f : V(G) \to \{0,1\}$.

Case-1 : $n \equiv 0, 1 \pmod 4$

$f(u_i) = 0, 1$; if $i \equiv 1, 2 \pmod 4$

$\quad\quad = 1$; if $i \equiv 0, 3 \pmod 4$, $1 \leq i \leq n$.

Case-2 : $n \equiv 2 \pmod 4$

$f(u_n) = 0$, $f(u_{n-1}) = 1$ and

$f(u_i) = 0$; if $i \equiv 1, 2 (\text{mod } 4)$

$\quad\quad = 1$; if $i \equiv 0, 3 \pmod 4$, $1 \leq i \leq n-2$.

The labeling pattern defined above covers all possible arrangement of vertices. In each case the graph G under consideration satisfies the conditions $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ as shown in following *Table*-5.3. i.e. G admits cordial labeling.

Let $n = 4a + b$, $n \in N$, $n \geq 6$.

b	vertex conditions	edge conditions
0	$v_f(0)=v_f(1)$	$e_f(0)+1=e_f(1)$
1	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$
2	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$

Table−5.3

5.4 Path Union of Graphs and Cordial Labeling :

Definition−5.4.1 **:** Let G be a graph and $G_1, G_2, \ldots, G_n$, $n \geq 2$ be n copies of graph G. Then the graph obtained by adding an edge from G_i to G_{i+1} (for $i = 1, 2, \ldots, n-1$) is called *path union* of G.

Shee and Ho [95] introduced the concept of path union. They also proved that path union of Petersen graph, trees, wheels and unicyclic graphs are cordial. We have investigated cordial labeling for path union of finite number of copies of cycle with one chord.

Theorem−5.4.2 **:** The path union of finite number of copies of cycle C_n with one chord is cordial, where chord forms a triangle with edges of the cycle.

Proof : Let G be the path union of cycle C_n with one chord and $G_1, G_2, \ldots, G_k$ be k copies of cycle C_n with one chord, where $|G_i| = n$, for each i. Let us denote the consecutive vertices of graph G_i by $\{u_{i1}, u_{i2}, \ldots, u_{in}\}$, for $i = 1, 2, \ldots, k$. Let u_{i1}, u_{i2}, u_{i3} forms a triangle with chord e. Let $e_i = u_{i3}u_{(i+1)1}$ be the edge joining G_i and G_{i+1}, for $i = 1, 2, \ldots, k-1$. To define labeling function $f : V(G) \rightarrow \{0, 1\}$ we consider following cases.

Case−1 : $n \equiv 0 \pmod 4$

$f(u_{ij}) = 0$; if $j \equiv 0, 3 \pmod 4$

$\quad = 1$; if $j \equiv 1, 2 \pmod 4$, when i is even, $1 \leq i \leq k$, $1 \leq j \leq n$.

$f(u_{ij}) = 0$; if $j \equiv 1, 2 \pmod 4$

$\quad\quad = 1$; if $j \equiv 0, 3 \pmod 4$, when i is odd, $1 \leq i \leq k$, $1 \leq j \leq n$.

Case-2 : $n \equiv 1 \pmod 4$

When $i \equiv 0, 1 \pmod 4$

$f(u_{ij}) = 0$; if $j \equiv 0, 3 \pmod 4$

$\quad\quad = 1$; if $j \equiv 1, 2 \pmod 4$, $1 \leq i \leq k$, $1 \leq j \leq n$.

When $i \equiv 2, 3 \pmod 4$

$f(u_{ij}) = 0$; if $j \equiv 1, 2 \pmod 4$

$\quad\quad = 1$; if $j \equiv 0, 3 \pmod 4$, $1 \leq i \leq k$, $1 \leq j \leq n$.

Case-3 : $n \equiv 2 \pmod 4$

$f(u_{in-1}) = 1$, $f(u_{in}) = 0$ and

$f(u_{ij}) = 0$; if $j \equiv 1, 2 \pmod 4$

$\quad\quad = 1$; if $j \equiv 0, 3 \pmod 4$, $1 \leq i \leq k$, $1 \leq j \leq n-2$.

Case-4 : $n \equiv 3 \pmod 4$

When $i \equiv 0, 1 \pmod 4$

$f(u_{i1}) = 0$ and

$f(u_{ij}) = 0$; if $j \equiv 2, 3 \pmod 4$

$\quad\quad = 1$; if $j \equiv 0, 1 \pmod 4$, $1 \leq i \leq k$, $2 \leq j \leq n$.

When $i \equiv 2, 3 \pmod 4$

$f(u_{i1}) = 1$ and

$f(u_{ij}) = 0$; if $j \equiv 1, 2 \pmod 4$

$\quad\quad = 1$; if $j \equiv 0, 3 \pmod 4$, $1 \leq i \leq k$, $2 \leq j \leq n$.

The labeling pattern defined above covers all possible arrangement of vertices. In each case the graph G under consideration satisfies the conditions $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ as shown in following *Table*-5.4.

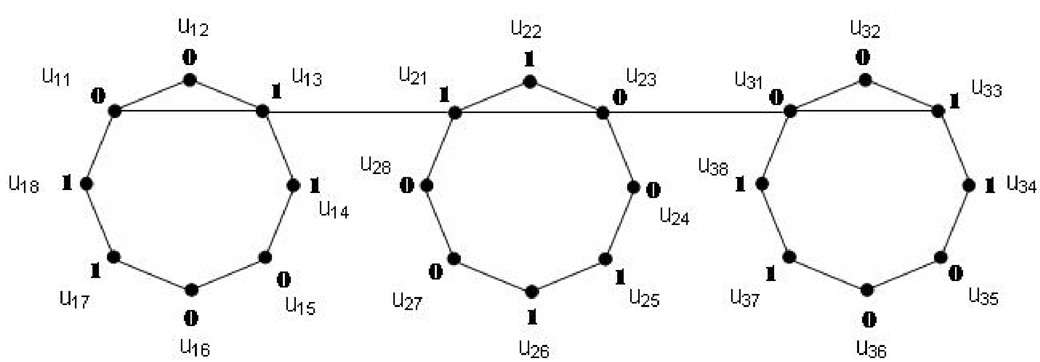
i.e. G admits cordial labeling.

Let $n = 4a + b$, $n \in N$, $n \geq 4$.

b	d	vertex conditions	edge conditions
0	0,1,2,3	$v_f(0)=v_f(1)$	$e_f(0)+1=e_f(1)$
1	0,2	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$
	1	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$
	3	$v_f(0)+1=v_f(1)$	$e_f(0)=e_f(1)$
2	0,1,2,3	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$
3	0,2	$v_f(0)=v_f(1)$	$e_f(0)+1=e_f(1)$
	1	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$
	3	$v_f(0)+1=v_f(1)$	$e_f(0)=e_f(1)$

Table$-$5.4

Illustration$-$5.4.3 **:** For better understanding of above defined labeling pattern let us consider graph G which is path union of three copies of cycle C_8 with one chord (It is the case related with $n \equiv 0 \pmod 4$, $k = 3$). The labeling pattern is shown in *Figure*$-$5.5.



G

Figure$-$5.5

5.5 Some More Cordial Graphs :

We introduce here a graph called star of a graph.

Definition$-$5.5.1 **:** A graph obtained by replacing each vertex of star graph $K_{1,n}$ by a graph G is called *star of* G. We denote it as G^*. We name central graph in G^* is the graph which replaces central vertex of graph $K_{1,n}$.

We have investigated cordial labeling for star of cycle.

Theorem$-$5.5.2 **:** Star of cycles C_n^* is cordial for all n.

Proof : Let $v_1, v_2, \ldots, v_n$ be successive vertices of central cycle of C_n^* and $u_{i1}, u_{i2}, \ldots, u_{in}$ be successive vertices of other cycles $C_n^{(i)}$ (except central cycle), $i = 1, 2, \ldots, n$. Let e_i be the edge such that $e_i = u_{i1}v_i$. Moreover, let us denote the vertex of cycle $C_n^{(i)}$ which is adjacent to a vertex v_i labeled by 0 as $u_{ij}^{(0)}$ and similarly denote the vertex of cycle $C_n^{(i)}$ which is adjacent to a vertex v_i labeled by 1 as $u_{ij}^{(1)}$. To define required labeling $f : V(C_n^*) \rightarrow \{0, 1\}$ we consider following cases.

<u>Case$-$1 :</u> $n \equiv 0 \pmod 4$

$f(v_i) = 0$; if $i \equiv 0, 1 \pmod 4$

$\qquad = 1$; if $i \equiv 2, 3 \pmod 4$, $1 \leq i \leq n$.

$f(u_{ij}^{(0)}) = 0$; if $j \equiv 0, 3 \pmod 4$

$\qquad = 1$; if $j \equiv 1, 2 \pmod 4$, $1 \leq j \leq n$, $1 \leq i \leq n$.

$f(u_{ij}^{(1)}) = 0$; if $j \equiv 2, 3 \pmod 4$

$\qquad = 1$; if $j \equiv 0, 1 \pmod 4$, $1 \leq j \leq n$, $1 \leq i \leq n$.

<u>Case$-$2 :</u> $n \equiv 1 \pmod 4$

$f(v_i) = 0$; if $i \equiv 0, 1 \pmod 4$

$\qquad = 1$; if $i \equiv 2, 3 \pmod 4$, $1 \leq i \leq n$.

$f(u_{ij}^{(0)}) = 0$; if $j \equiv 0, 3 \pmod 4$

$\quad\quad = 1$; if $j \equiv 1, 2 \pmod 4$, $1 \leq j \leq n$, $1 \leq i \leq n$.

$f(u_{ij}^{(1)}) = 0$; if $j \equiv 1, 2 \pmod 4$

$\quad\quad = 1$; if $j \equiv 0, 3 \pmod 4$, $1 \leq j \leq n$, $1 \leq i \leq n$.

<u>Case−3 :</u> $n \equiv 2 \pmod 4$

$f(v_i) = 0$; if $i \equiv 0, 2 \pmod 4$

$\quad\quad = 1$; if $i \equiv 1, 3 \pmod 4$, $1 \leq i \leq n$.

$f(u_{ij}^{(0)}) = 0$; if $j \equiv 0, 1 \pmod 4$

$\quad\quad = 1$; if $j \equiv 2, 3 \pmod 4$, $1 \leq j \leq n$, $1 \leq i \leq n$.

$f(u_{in}^{(1)}) = 1$, $f(u_{in-1}^{(1)}) = 0$ and

$f(u_{ij}^{(1)}) = 0$; if $j \equiv 0, 3 \pmod 4$

$\quad\quad = 1$; if $j \equiv 1, 2 \pmod 4$, $1 \leq j \leq n - 2$, $1 \leq i \leq n$.

<u>Case−4 :</u> $n \equiv 3 \pmod 4$

$f(v_i) = 0$; if $i \equiv 0, 1 \pmod 4$

$\quad\quad = 1$; if $i \equiv 2, 3 \pmod 4$, $1 \leq i \leq n$.

$f(u_{ij}^{(0)}) = 0$; if $j \equiv 0, 1 \pmod 4$

$\quad\quad = 1$; if $j \equiv 2, 3 \pmod 4$, $1 \leq j \leq n$, $1 \leq i \leq n$.

$f(u_{ij}^{(1)}) = 0$; if $j \equiv 2, 3 \pmod 4$

$\quad\quad = 1$; if $j \equiv 0, 1 \pmod 4$, $1 \leq j \leq n$, $1 \leq i \leq n$.

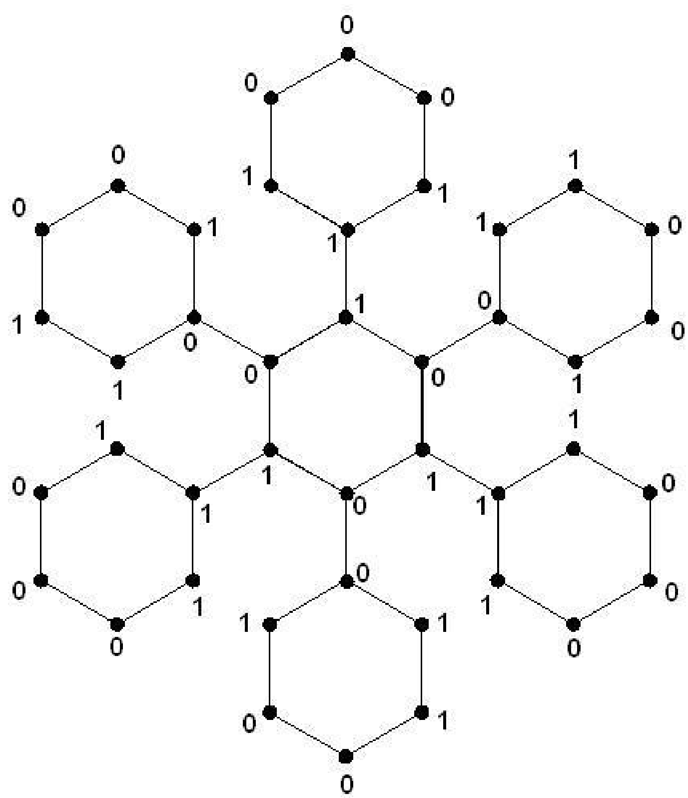
The graph under consideration satisfies the condition $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ in each case as shown in following *Table*−5.6.

Let $n = 4a + b$, $n \in N$, $n \geq 3$

b	vertex conditions	edge conditions
0	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)$
1	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$
2	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)$
3	$v_f(0)=v_f(1)$	$e_f(0)+1=e_f(1)$

Table−5.6

Illustration−5.5.3 **:** For better understanding of above defined labeling pattern let us consider star of cycle C_6 (It is related to *Case*−3). The cordial labeling of star of C_6 is as shown in *Figure*−5.7



Figure−5.7

Theorem$-$5.5.4 **:** The graph obtained by joining two copies of cycle C_n by a path of arbitrary length is cordial.

Proof : Let $\{u_1, u_2, \ldots, u_n\}$ be the vertices of first copy of cycle C_n. Let $\{v_1, \ldots, v_k\}$ be the vertices of path P_k with $u_1 = v_1$ and $\{w_1, \ldots, w_n\}$ be the vertices of second copy of cycle C_n with $v_k = w_1$. To define labeling function $f : V(G) \rightarrow \{0, 1\}$ we consider following cases.

Case 1: $n \equiv 0(\text{mod } 4)$, $k \equiv 0(\text{mod } 4)$

$f(u_i) = 0$; if $i \equiv 0, 3(\text{mod } 4)$

$\quad = 1$; if $i \equiv 1, 2(\text{mod } 4)$, $1 \leq i \leq n$.

$f(v_j) = 0$; if $j \equiv 0, 3(\text{mod } 4)$

$\quad = 1$; if $j \equiv 1, 2(\text{mod } 4)$, $1 \leq j \leq k$.

$f(w_i) = 0$; if $i \equiv 1, 2(\text{mod } 4)$

$\quad = 1$; if $i \equiv 0, 3(mod 4)$, $1 \leq i \leq n$.

Case 2: $n \equiv 0(\text{mod } 4)$, $k \equiv 1, 2(\text{mod } 4)$

$f(u_i) = 0$; if $i \equiv 1, 2(\text{mod } 4)$

$\quad = 1$; if $i \equiv 0, 3(\text{mod } 4)$, $1 \leq i \leq n$.

$f(v_j) = 0$; if $j \equiv 1, 2(\text{mod } 4)$

$\quad = 1$; if $j \equiv 0, 3(\text{mod } 4)$, $1 \leq j \leq k$.

$f(w_i) = 0$; if $i \equiv 1, 2(\text{mod } 4)$

$\quad = 1$; if $i \equiv 0, 3(\text{mod } 4)$, $1 \leq i \leq n$.

Case 3: $n \equiv 0(\text{mod } 4)$, $k \equiv 3(\text{mod } 4)$ and $n \equiv 3(\text{mod } 4)$, $k \equiv 0(\text{mod } 4)$

$f(u_i) = 0$; if $i \equiv 1, 2(\text{mod } 4)$

$\quad = 1$; if $i \equiv 0, 3(\text{mod } 4)$, $1 \leq i \leq n$.

$f(v_j) = 0$; if $j \equiv 1, 2(\text{mod } 4)$

$\quad = 1$; if $j \equiv 0, 3(\text{mod } 4)$, $1 \leq j \leq k$.

$f(w_i) = 0;$ if $i \equiv 0, 3 (\mathrm{mod}\ 4)$

$= 1;$ if $i \equiv 1, 2 (\mathrm{mod}\ 4)$, $1 \leq i \leq n$.

<u>Case 4:</u> $n \equiv 1 (\mathrm{mod}\ 4)$, $k \equiv 0 (\mathrm{mod}\ 4)$

$f(u_i) = 0;$ if $i \equiv 1, 2 (\mathrm{mod}\ 4)$

$= 1;$ if $i \equiv 0, 3 (\mathrm{mod}\ 4)$, $1 \leq i \leq n$.

$f(v_j) = 0;$ if $j \equiv 0, 1 (\mathrm{mod}\ 4)$

$= 1;$ if $j \equiv 2, 3 (\mathrm{mod}\ 4)$, $1 \leq j \leq k$.

$f(w_i) = 0;$ if $i \equiv 1, 2 (\mathrm{mod}\ 4)$

$= 1;$ if $i \equiv 0, 3 (\mathrm{mod}\ 4)$, $1 \leq i \leq n$.

<u>Case 5:</u> $n \equiv 1 (\mathrm{mod}\ 4)$, $k \equiv 1 (\mathrm{mod}\ 4)$

$f(u_i) = 0;$ if $i \equiv 1, 2 (\mathrm{mod}\ 4)$

$= 1;$ if $i \equiv 0, 3 (\mathrm{mod}\ 4)$, $1 \leq i \leq n$.

$f(v_k) = 1$ and

$f(v_j) = 0;$ if $j \equiv 0, 1 (\mathrm{mod}\ 4)$

$= 1;$ if $j \equiv 2, 3 (\mathrm{mod}\ 4)$, $1 \leq j \leq k - 1$.

$f(w_i) = 0;$ if $i \equiv 2, 3 (\mathrm{mod}\ 4)$

$= 1;$ if $i \equiv 0, 1 (\mathrm{mod}\ 4)$, $1 \leq i \leq n$.

<u>Case 6:</u> $n \equiv 1 (\mathrm{mod}\ 4)$, $k \equiv 2 (\mathrm{mod}\ 4)$ and

$n \equiv 2 (\mathrm{mod}\ 4)$, $k \equiv 3 (\mathrm{mod}\ 4)$

$f(u_i) = 0;$ if $i \equiv 1, 2 (\mathrm{mod}\ 4)$

$= 1;$ if $i \equiv 0, 3 (\mathrm{mod}\ 4)$, $1 \leq i \leq n$.

$f(v_j) = 0;$ if $j \equiv 0, 1 (\mathrm{mod}\ 4)$

$= 1;$ if $j \equiv 2, 3 (\mathrm{mod}\ 4)$, $1 \leq j \leq k$.

$f(w_i) = 0;$ if $i \equiv 2, 3 (\mathrm{mod}\ 4)$

$= 1;$ if $i \equiv 0, 1 (\mathrm{mod}\ 4)$, $1 \leq i \leq n$.

Case 7: $n \equiv 1 (\text{mod } 4)$, $k \equiv 3 (\text{mod } 4)$

$f(u_i) = 0$; if $i \equiv 0, 1 (\text{mod } 4)$

$= 1$; if $i \equiv 2, 3 (\text{mod } 4)$, $1 \leq i \leq n$.

$f(v_k) = 0$ and

$f(v_j) = 0$; if $j \equiv 0, 1 (\text{mod } 4)$

$= 1$; if $j \equiv 2, 3 (\text{mod } 4)$, $1 \leq j \leq k - 1$.

$f(w_i) = 0$; if $i \equiv 0, 1 (\text{mod } 4)$

$= 1$; if $i \equiv 2, 3 (\text{mod } 4)$, $1 \leq i \leq n$.

Case 8: $n \equiv 2 (\text{mod } 4)$, $k \equiv 0 (\text{mod } 4)$

$f(u_n) = 0$, $f(u_{n-1}) = 1$ and

$f(u_i) = 0$; if $i \equiv 1, 2 (\text{mod } 4)$

$= 1$; if $i \equiv 0, 3 (\text{mod } 4)$, $1 \leq i \leq n - 2$.

$f(v_j) = 0$; if $j \equiv 1, 2 (\text{mod } 4)$

$= 1$; if $j \equiv 0, 3 (\text{mod } 4)$, $1 \leq j \leq k$.

$f(w_i) = 0$; if $i \equiv 2, 3 (\text{mod } 4)$

$= 1$; if $i \equiv 0, 1 (\text{mod } 4)$, $1 \leq i \leq n$.

Case 9: $n \equiv 2 (\text{mod } 4)$, $k \equiv 1 (\text{mod } 4)$

$f(u_i) = 0$; if $i \equiv 0, 1 (\text{mod } 4)$

$= 1$; if $i \equiv 2, 3 (\text{mod } 4)$, $1 \leq i \leq n$.

$f(v_j) = 0$; if $j \equiv 1, 2 (\text{mod } 4)$

$= 1$; if $j \equiv 0, 3 (\text{mod } 4)$, $1 \leq j \leq k$.

$f(w_i) = 0$; if $i \equiv 2, 3 (\text{mod } 4)$

$= 1$; if $i \equiv 0, 1 (\text{mod } 4)$, $1 \leq i \leq n$.

Case 10: $n \equiv 2(\text{mod } 4)$, $k \equiv 2(\text{mod } 4)$

$f(u_n) = 0$, $f(u_{n-1}) = 1$ and

$f(u_i) = 0$; if $i \equiv 1, 2(\text{mod } 4)$

$= 1$; if $i \equiv 0, 3(\text{mod } 4)$, $1 \leq i \leq n - 2$.

$f(v_j) = 0$; if $j \equiv 0, 1(\text{mod } 4)$

$= 1$; if $j \equiv 2, 3(\text{mod } 4)$, $1 \leq j \leq k$.

$f(w_i) = 0$; if $i \equiv 2, 3(\text{mod } 4)$

$= 1$; if $i \equiv 0, 1(\text{mod } 4)$, $1 \leq i \leq n$.

Case 11: $n \equiv 3(\text{mod } 4)$, $k \equiv 1(\text{mod } 4)$

$f(u_i) = 0$; if $i \equiv 0, 3(\text{mod } 4)$

$= 1$; if $i \equiv 1, 2(\text{mod } 4)$, $1 \leq i \leq n$.

$f(v_k) = 0$ and

$f(v_j) = 0$; if $j \equiv 0, 3(\text{mod } 4)$

$= 1$; if $j \equiv 1, 2(\text{mod } 4)$, $1 \leq j \leq k - 1$.

$f(w_i) = 0$; if $i \equiv 0, 1(\text{mod } 4)$

$= 1$; if $i \equiv 2, 3(\text{mod } 4)$, $1 \leq i \leq n$.

Case 12: $n \equiv 3(\text{mod } 4)$, $k \equiv 2(\text{mod } 4)$

$f(u_i) = 0$; if $i \equiv 1, 2(\text{mod } 4)$

$= 1$; if $i \equiv 0, 3(\text{mod } 4)$, $1 \leq i \leq n$.

$f(v_j) = 0$; if $j \equiv 1, 2(\text{mod } 4)$

$= 1$; if $j \equiv 0, 3(\text{mod } 4)$, $1 \leq j \leq k$.

$f(w_i) = 0$; if $i \equiv 0, 1(\text{mod } 4)$

$= 1$; if $i \equiv 2, 3(\text{mod } 4)$, $1 \leq i \leq n$.

Case 13: $n \equiv 3 (\text{mod } 4)$, $k \equiv 3 (\text{mod } 4)$

$f(u_i) = 0$; if $i \equiv 1, 2 (\text{mod } 4)$

$\qquad = 1$; if $i \equiv 0, 3 (\text{mod } 4)$, $1 \leq i \leq n$.

$f(v_k) = 0$ and $f(v_j) = 0$; if $j \equiv 1, 2 (\text{mod } 4)$

$\qquad = 1$; if $j \equiv 0, 3 (\text{mod } 4)$, $1 \leq j \leq k - 1$.

$f(w_i) = 0$; if $i \equiv 0, 1 (\text{mod } 4)$

$\qquad = 1$; if $i \equiv 2, 3 (\text{mod } 4)$, $1 \leq i \leq n$.

Let $n = 4a + b, k = 4c + d, i = 4s + r, j = 4x + y$, where $n, k, i, j \in N$

b	d	r	y	$f(u_i)$	$f(v_j)$	$f(w_i)$	vertex labeling to be dealt seperately	vertex conditions	edge conditions
0	0	0	0	0	0	1	–	$v_f(0)=v_f(1)$	$e_f(0)+1=e_f(1)$
		1	1	1	1	0			
		2	2	1	1	0			
		3	3	0	0	1			
	1	0	0	1	1	1	–	$v_f(0)+1=v_f(1)$	$e_f(0)=e_f(1)$
		1	1	0	0	0			
		2	2	0	0	0			
		3	3	1	1	1			
	2	0	0	1	1	1	–	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$
		1	1	0	0	0			
		2	2	0	0	0			
		3	3	1	1	1			
	3	0	0	1	1	0	–	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$
		1	1	0	0	1			
		2	2	0	0	1			
		3	3	1	1	0			
1	0	0	0	1	0	1	–	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$
		1	1	0	0	0			
		2	2	0	1	0			
		3	3	1	1	1			
	1	0	0	1	0	1	$f(v_k)=1$	$v_f(0)+1=v_f(1)$	$e_f(0)=e_f(1)$
		1	1	0	0	1			
		2	2	0	1	0			
		3	3	1	1	0			

1	2	0	0	1	0	1	—	$\nu_f(0)=\nu_f(1)$	$e_f(0)=e_f(1)+1$
		1	1	0	0	1			
		2	2	0	1	0			
		3	3	1	1	0			
	3	0	0	0	0	0	$f(v_k)=0$	$\nu_f(0)=\nu_f(1)+1$	$e_f(0)=e_f(1)$
		1	1	0	0	0			
		2	2	1	1	1			
		3	3	1	1	1			
2	0	0	0	1	1	1	$f(u_n)=0, f(u_{n-1})=1$	$\nu_f(0)=\nu_f(1)$	$e_f(0)=e_f(1)+1$
		1	1	0	0	1			
		2	2	0	0	0			
		3	3	1	1	0			
	1	0	0	0	1	1	$f(w_n)=0, f(w_{n-1})=1$	$\nu_f(0)=\nu_f(1)+1$	$e_f(0)=e_f(1)$
		1	1	0	0	0			
		2	2	1	0	0			
		3	3	1	1	1			
	2	0	0	1	0	1	$f(u_n)=0, f(u_{n-1})=1$	$\nu_f(0)=\nu_f(1)$	$e_f(0)+1=e_f(1)$
		1	1	0	0	1			
		2	2	0	1	0			
		3	3	1	1	0			
	3	0	0	1	0	1	—	$\nu_f(0)=\nu_f(1)+1$	$e_f(0)=e_f(1)$
		1	1	0	0	1			
		2	2	0	1	0			
		3	3	1	1	0			
3	0	0	0	1	1	0	—	$\nu_f(0)=\nu_f(1)$	$e_f(0)+1=e_f(1)$
		1	1	0	0	1			
		2	2	0	0	1			
		3	3	1	1	0			
	1	0	0	0	0	0	$f(v_k)=0$	$\nu_f(0)+1=\nu_f(1)$	$e_f(0)=e_f(1)$
		1	1	1	1	0			
		2	2	1	1	1			
		3	3	0	0	1			
	2	0	0	1	1	0	—	$\nu_f(0)=\nu_f(1)$	$e_f(0)+1=e_f(1)$
		1	1	0	0	0			
		2	2	0	0	1			
		3	3	1	1	1			
	3	0	0	1	1	0	$f(v_k)=0$	$\nu_f(0)=\nu_f(1)+1$	$e_f(0)=e_f(1)$
		1	1	0	0	0			
		2	2	0	0	1			
		3	3	1	1	1			

Table−5.8

The labeling pattern defined above covers all possible arrangement of vertices. In each case the graph G under consideration satisfies the conditions $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ as shown in above *Table*-5.8. i.e. G admits cordial labeling.

For better understanding of above defined labeling pattern let us consider an example.

Example$-5.5.5$ **:** Consider a graph obtained by joining two copies of cycle C_5 by a path P_5 (It is the case related with $n \equiv 1(\text{mod } 4)$, $k \equiv 1(\text{mod } 4)$). The labeling pattern is shown in *Figure*-5.9.

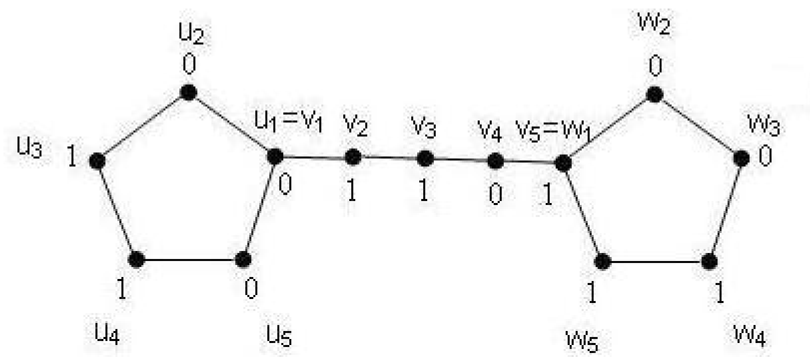


Figure-5.9

Theorem$-5.5.6$ **:** The graph G obtained by joining two copies of cycle C_n with one chord by a path of arbitrary length is cordial.

Proof : Let $u_1, \ldots, u_n$ be consecutive vertices of first copy of cycle C_n with one chord, $v_1, \ldots, v_k$ be consecutive vertices of path P_k with $u_1 = v_1$ and $w_1, \ldots, w_n$ be consecutive vertices of second copy of cycle C_n with one chord, where $v_k = w_1$. To define labeling function $f : V(G) \rightarrow \{0, 1\}$ there are sixteen cases. We shall define it according to following *Table*-5.10.

Let $n = 4a + b, k = 4c + d, i = 4s + r, j = 4x + y$, where $n, k, i, j \in N$ and $n \geq 4$

b	d	r	y	$f(u_i)$	$f(v_j)$	$f(w_i)$	vertex labeling to be dealt seperately	vertex conditions	edge conditions
0	0	0	0	1	1	0	—	$v_f(0)=v_f(1)$	$e_f(0)+1=e_f(1)$
		1	1	0	0	1			
		2	2	0	0	1			
		3	3	1	1	0			
	1	0	0	1	1	0	$f(v_k)=1$	$v_f(0)+1=v_f(1)$	$e_f(0)=e_f(1)$
		1	1	0	0	1			
		2	2	0	0	1			
		3	3	1	1	0			
	2	0	0	1	1	1	—	$v_f(0)=v_f(1)$	$e_f(0)+1=e_f(1)$
		1	1	0	0	0			
		2	2	0	0	0			
		3	3	1	1	1			
	3	0	0	1	1	1	$f(v_k)=0$	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$
		1	1	0	0	0			
		2	2	0	0	0			
		3	3	1	1	1			
1	0	0	0	1	1	1	—	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$
		1	1	0	0	1			
		2	2	0	0	0			
		3	3	1	1	0			
	1	0	0	1	1	0	—	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$
		1	1	0	0	0			
		2	2	0	0	1			
		3	3	1	1	1			
	2	0	0	1	0	0	—	$v_f(0)=v_f(1)$	$e_f(0)+1=e_f(1)$
		1	1	0	0	1			
		2	2	0	1	1			
		3	3	1	1	0			
	3	0	0	1	0	0	—	$v_f(0)+1=v_f(1)$	$e_f(0)=e_f(1)$
		1	1	0	0	1			
		2	2	0	1	1			
		3	3	1	1	0			
2	0	0	0	1	0	1	$f(u_n)=0, f(u_{n-1})=1$	$v_f(0)=v_f(1)$	$e_f(0)+1=e_f(1)$
		1	1	0	0	0			
		2	2	0	1	0			
		3	3	1	1	1			
	1	0	0	1	0	0	$f(u_n)=0, f(u_{n-1})=1,$ $f(v_k)=1,$ $f(w_n)=1, f(w_{n-1})=0$	$v_f(0)+1=v_f(1)$	$e_f(0)=e_f(1)$
		1	1	0	0	1			
		2	2	0	1	1			
		3	3	1	1	0			

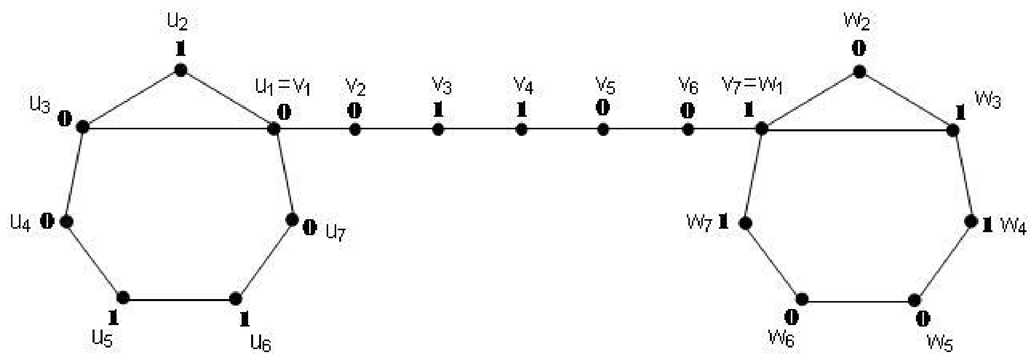
2	2	0	0	1	0	0	$f(u_n)=0, f(u_{n-1})=1,$ $f(w_n)=1, f(w_{n-1})=0$	$v_f(0)=v_f(1)$	$e_f(0)+1=e_f(1)$
		1	1	0	0	1			
		2	2	0	1	1			
		3	3	1	1	0			
	3	0	0	1	0	1	$f(u_n)=0, f(u_{n-1})=1,$ $f(v_k)=0,$ $f(w_n)=0, f(w_{n-1})=1$	$v_f(0)+1=v_f(1)$	$e_f(0)=e_f(1)$
		1	1	0	0	0			
		2	2	0	1	0			
		3	3	1	1	1			
3	0	0	0	0	1	1	$f(u_1)=0, f(w_1)=1$	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$
		1	1	1	0	0			
		2	2	1	0	0			
		3	3	0	1	1			
	1	0	0	1	1	1	$f(u_1)=1, f(v_{k-1})=0,$ $f(v_k)=1, f(w_1)=1$	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$
		1	1	0	1	0			
		2	2	0	0	0			
		3	3	1	0	1			
	2	0	0	1	0	0	$f(u_1)=0, f(u_2)=1,$ $f(w_1)=1, f(w_2)=0$	$v_f(0)=v_f(1)$	$e_f(0)+1=e_f(1)$
		1	1	0	0	1			
		2	2	0	1	1			
		3	3	1	1	0			
	3	0	0	0	1	1	$f(u_1)=0, f(w_1)=1$	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$
		1	1	1	0	0			
		2	2	1	0	0			
		3	3	0	1	1			

$Table\!-\!5.10$

The labeling pattern defined above covers all possible arrangement of vertices. In each case the graph G under consideration satisfies the conditions $|v_f(0)-v_f(1)| \leq 1$ and $|e_f(0)-e_f(1)| \leq 1$ as shown in above $Table\!-\!5.10$. i.e. G admits cordial labeling.

For better understanding of above defined labeling pattern let us consider an example.

Example$-5.5.7$ **:** Consider a graph obtained by joining two copies of cycles C_7 with one chord by a path P_7 (It is the case related with $n \equiv 3(\text{mod } 4)$, $k \equiv 3(\text{mod } 4)$). The labeling pattern is shown in $Figure\!-\!5.11$.



Figure−5.11

Theorem−5.5.8 **:** The graph G obtained by joining two cycles with twin chords by a path of arbitrary length is cordial where chords form two triangles and one cycle C_{n-2}.

Proof : Let $u_1, \ldots, u_n$ be successive vertices of first copy of cycle C_n such that u_1, u_2, u_3 form a triangle with one of the chord and $d(u_1) = 4, d(u_3) = d(u_4) = 3$ while $d(u_2) = 2$ and $d(u_i) = 2$, for $5 \leq i \leq n$. Let $w_1, \ldots, w_n$ be the successive vertices of second copy of cycle C_n such that w_1, w_2, w_3 form a triangle with one of the twin chords and $d(w_1) = 4, d(w_3) = d(w_4) = 3$ while $d(w_2) = 2$ and $d(w_i) = 2$, for $5 \leq i \leq n$. Let $v_1, \ldots, v_k$ be the successive vertices of path P_k with $v_1 = u_i$, for $i = 3$ or $i = 1$ or $i = 4$ and $v_k = w_1$. To define labeling function $f : V(G) \rightarrow \{0, 1\}$ there are following cases.

Case-A $v_1 = u_3$

Case-B $v_1 = u_1$

Case-C $v_1 = u_4$

We shall define the labeling function according to *Table*−5.12 to *Table*−5.14.

Let $n = 4a + b, k = 4c + d, i = 4s + r, j = 4x + y$, where $n, k, i, j \in N$ and $n \geq 5$

b	d	r	y	$f(u_i)$	$f(v_j)$	$f(w_i)$	vertex labeling to be dealt seperately	vertex conditions	edge conditions
0	0	0	0	1	1	1	–	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$
		1	1	1	0	1			
		2	2	0	0	0			
		3	3	0	1	0			
	1	0	0	1	1	0	–	$v_f(0)+1=v_f(1)$	$e_f(0)=e_f(1)$
		1	1	1	0	0			
		2	2	0	0	1			
		3	3	0	1	1			
	2	0	0	1	0	1	–	$v_f(0)=v_f(1)$	$e_f(0)+1=e_f(1)$
		1	1	1	0	1			
		2	2	0	1	0			
		3	3	0	1	0			
	3	0	0	1	1	1	–	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$
		1	1	1	0	1			
		2	2	0	0	0			
		3	3	0	1	0			
1	0	0	0	1	1	0	$f(w_n)=0$	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$
		1	1	1	0	1			
		2	2	0	0	1			
		3	3	0	1	0			
	1	0	0	1	0	1	–	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$
		1	1	0	1	1			
		2	2	0	1	0			
		3	3	1	0	0			
	2	0	0	1	0	1	–	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$
		1	1	0	1	1			
		2	2	0	1	0			
		3	3	1	0	0			
	3	0	0	0	0	1	–	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$
		1	1	0	1	0			
		2	2	1	1	0			
		3	3	1	0	1			
2	0	0	0	0	0	0	$f(w_1)=0$	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$
		1	1	0	1	1			
		2	2	1	1	1			
		3	3	1	0	0			
	1	0	0	1	0	0	–	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$
		1	1	0	1	1			
		2	2	0	1	1			
		3	3	1	0	0			
	2	0	0	1	0	0	–	$v_f(0)=v_f(1)$	$e_f(0)+1=e_f(1)$
		1	1	0	1	1			
		2	2	0	1	1			
		3	3	1	0	0			
	3	0	0	1	0	0	$f(w_1)=0$	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$
		1	1	0	1	1			
		2	2	0	1	1			
		3	3	1	0	0			

3	0	0	0	0	0	0	–	$v_f(0)=v_f(1)$	$e_f(0)+1=e_f(1)$
		1	1	0	1	1			
		2	2	1	1	1			
		3	3	1	0	0			
	1	0	0	0	0	1	$f(w_1)=1$	$v_f(0)+1=v_f(1)$	$e_f(0)=e_f(1)$
		1	1	0	1	0			
		2	2	1	1	0			
		3	3	1	0	1			
	2	0	0	0	0	1	–	$v_f(0)=v_f(1)$	$e_f(0)+1=e_f(1)$
		1	1	0	1	1			
		2	2	1	1	0			
		3	3	1	0	0			
	3	0	0	0	0	1	$f(v_k)=1$	$v_f(0)+1=v_f(1)$	$e_f(0)=e_f(1)$
		1	1	0	1	1			
		2	2	1	1	0			
		3	3	1	0	0			

$Table-5.12$

Let $n = 4a + b, k = 4c + d, i = 4s + r, j = 4x + y$, where $n, k, i, j \in N$ and $n \geq 5$

b	d	r	y	$f(u_i)$	$f(v_j)$	$f(w_i)$	vertex labeling to be dealt seperately	vertex conditions	edge conditions
0	0	0	0	1	0	0	–	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$
		1	1	1	1	0			
		2	2	0	1	1			
		3	3	0	0	1			
	1	0	0	1	0	1	–	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$
		1	1	1	1	1			
		2	2	0	1	0			
		3	3	0	0	0			
	2	0	0	1	0	1	–	$v_f(0)=v_f(1)$	$e_f(0)+1=e_f(1)$
		1	1	1	1	1			
		2	2	0	1	0			
		3	3	0	0	0			
	3	0	0	1	0	0	–	$v_f(0)+1=v_f(1)$	$e_f(0)=e_f(1)$
		1	1	1	1	0			
		2	2	0	1	1			
		3	3	0	0	1			
1	0	0	0	1	1	1	–	$v_f(0)=v_f(1)$	$e_f(0)+1=e_f(1)$
		1	1	1	1	1			
		2	2	0	0	0			
		3	3	0	0	0			
	1	0	0	0	0	1	–	$v_f(0)+1=v_f(1)$	$e_f(0)=e_f(1)$
		1	1	1	1	1			
		2	2	1	1	0			
		3	3	0	0	0			
	2	0	0	0	1	0	–	$v_f(0)=v_f(1)$	$e_f(0)+1=e_f(1)$
		1	1	1	1	0			
		2	2	1	0	1			
		3	3	0	0	1			
	3	0	0	0	0	0	–	$v_f(0)+1=v_f(1)$	$e_f(0)=e_f(1)$
		1	1	1	1	0			
		2	2	1	1	1			
		3	3	0	0	1			

3	0	0	0	1	0	0	–	$v_f(0)=v_f(1)$	$e_f(0)+1=e_f(1)$
		1	1	1	1	0			
		2	2	0	1	1			
		3	3	0	0	1			
	1	0	0	1	0	0	$f(v_k)=0$	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$
		1	1	1	1	0			
		2	2	0	1	1			
		3	3	0	0	1			
	2	0	0	1	0	1	$f(w_1)=1$	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$
		1	1	1	1	0			
		2	2	0	1	0			
		3	3	0	0	1			
	3	0	0	1	0	1	$f(v_k)=1$	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$
		1	1	1	1	1			
		2	2	0	1	0			
		3	3	0	0	0			
2	0	0	0	0	0	1	–	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$
		1	1	1	1	0			
		2	2	1	1	0			
		3	3	0	0	1			
	1	0	0	0	0	0	$f(v_k)=0$	$v_f(0)+1=v_f(1)$	$e_f(0)=e_f(1)$
		1	1	1	1	0			
		2	2	1	1	1			
		3	3	0	0	1			
	2	0	0	0	1	1	–	$v_f(0)=v_f(1)$	$e_f(0)+1=e_f(1)$
		1	1	1	1	0			
		2	2	1	0	0			
		3	3	0	0	1			
	3	0	0	0	0	1	–	$v_f(0)+1=v_f(1)$	$e_f(0)=e_f(1)$
		1	1	1	1	0			
		2	2	1	1	0			
		3	3	0	0	1			

Table−5.13

Let $n = 4a+b, k = 4c+d, i = 4s+r, j = 4x+y$, where $n, k, i, j \in N$ and $n \geq 5$

b	d	r	y	$f(u_i)$	$f(v_j)$	$f(w_i)$	vertex labeling to be dealt seperately	vertex conditions	edge conditions
0	0	0	0	1	0	0	–	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$
		1	1	1	1	0			
		2	2	0	1	1			
		3	3	0	0	1			
	1	0	0	1	0	1	–	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$
		1	1	1	1	1			
		2	2	0	1	0			
		3	3	0	0	0			
	2	0	0	1	0	1	–	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$
		1	1	1	1	1			
		2	2	0	1	0			
		3	3	0	0	0			
	3	0	0	1	0	0	–	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$
		1	1	1	1	0			
		2	2	0	1	1			
		3	3	0	0	1			

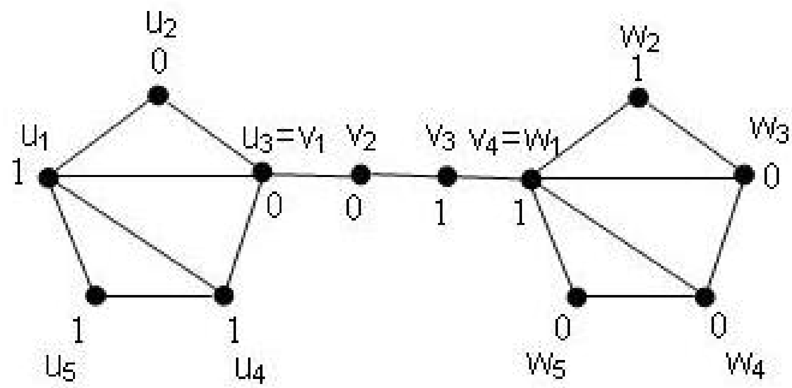
1	0	0	0	1	0	1	–	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$
		1	1	1	1	0			
		2	2	0	1	0			
		3	3	0	0	1			
	1	0	0	1	1	0	–	$v_f(0)+1=v_f(1)$	$e_f(0)=e_f(1)$
		1	1	1	1	1			
		2	2	0	0	1			
		3	3	0	0	0			
	2	0	0	0	0	1	–	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$
		1	1	0	0	1			
		2	2	1	1	0			
		3	3	1	1	0			
	3	0	0	1	0	1	–	$v_f(0)+1=v_f(1)$	$e_f(0)=e_f(1)$
		1	1	1	1	0			
		2	2	0	1	0			
		3	3	0	0	1			
2	0	0	0	1	0	0	$f(w_1)=0$	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$
		1	1	1	1	1			
		2	2	0	1	1			
		3	3	0	0	0			
	1	0	0	0	0	1	–	$v_f(0)+1=v_f(1)$	$e_f(0)=e_f(1)$
		1	1	1	0	0			
		2	2	1	1	0			
		3	3	0	1	1			
	2	0	0	0	1	1	–	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$
		1	1	1	0	0			
		2	2	1	0	0			
		3	3	0	1	1			
	3	0	0	0	1	0	$f(v_k)=0$	$v_f(0)+1=v_f(1)$	$e_f(0)=e_f(1)$
		1	1	1	0	0			
		2	2	1	0	1			
		3	3	0	1	1			
3	0	0	0	1	0	0	–	$v_f(0)=v_f(1)$	$e_f(0)+1=e_f(1)$
		1	1	1	1	0			
		2	2	0	1	1			
		3	3	0	0	1			
	1	0	0	1	0	1	$f(w_1)=1$	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$
		1	1	1	1	0			
		2	2	0	1	0			
		3	3	0	0	1			
	2	0	0	1	0	1	$f(w_1)=1$	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$
		1	1	1	1	0			
		2	2	0	1	0			
		3	3	0	0	1			
	3	0	0	1	0	0	$f(w_1)=0$	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$
		1	1	1	1	1			
		2	2	0	1	1			
		3	3	0	0	0			

Table-5.14

The labeling pattern defined above covers all possible arrangement of vertices. In each case the graph G under consideration satisfies the conditions $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ as shown in above *Table*-5.12 to 5.14. i.e. G admits cordial labeling.

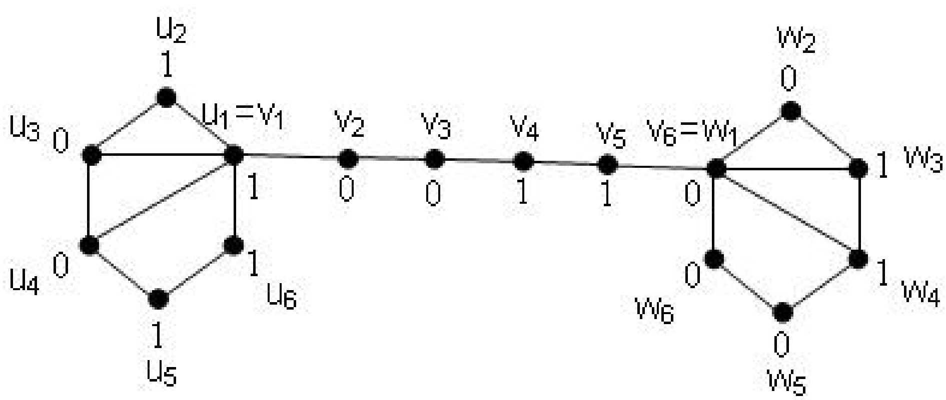
Let us demonstrate above labeling pattern by means of following examples.

Example$-$5.5.9 **:** Consider a graph obtained by joining two copies of cycles C_5 with twin chords by a path P_4 (It is the case related with *Case*$-A$, $n \equiv 1(\text{mod } 4)$, $k \equiv 0(\text{mod } 4)$). The labeling pattern is shown in *Figure*$-$5.15.



Figure$-$5.15

Example$-$5.5.10 **:** Consider a graph obtained by joining two copies of cycles C_6 with twin chords by a path P_6 (It is the case related with *Case*$-B$, $n \equiv 2(\text{mod } 4)$, $k \equiv 2(\text{mod } 4)$). The labeling pattern is shown in *Figure*$-$5.16.



Figure$-$5.16

Example$-5.5.11$ **:** Consider a graph obtained by joining two copies of cycles C_8 with twin chords by a path P_7 (It is the case related with *Case*$-C$, $n \equiv 0(\text{mod } 4)$, $k \equiv 3\ (\text{mod } 4)$). The labeling pattern is shown in *Figure*-5.17.

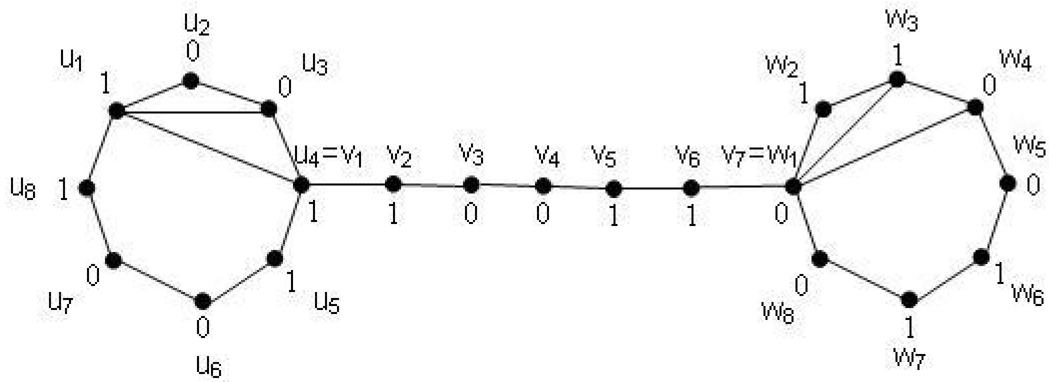


Figure-5.17

Theorem$-5.5.12$ **:** $K_n \bigcup K_n$, where $n = t^2$, for some $t \in N$ is cordial graph.

Proof : It is obvious that t^2-t is an even integer. Let $u_1, u_2, \ldots, u_n$ be successive vertices of first copy of K_n and $v_1, \ldots, v_n$ be successive vertices of second copy of K_n. We shall define the labeling function $f : V(K_n \bigcup K_n) \rightarrow \{0, 1\}$ by

$f(u_i) = 0$ if $i \in \{1, 2, \ldots, \frac{t^2-t}{2}\}$

$\quad = 1$ if $i \in \{\frac{t^2-t}{2} + 1, \ldots, n\}$ and

$f(v_j) = 0$ if $j \in \{\frac{t^2-t}{2} + 1, \ldots, n\}$

$\quad = 1$ if $j \in \{1, 2, \ldots, \frac{t^2-t}{2}\}$.

The labeling pattern of this graph is shown in following *Figure*-5.18.

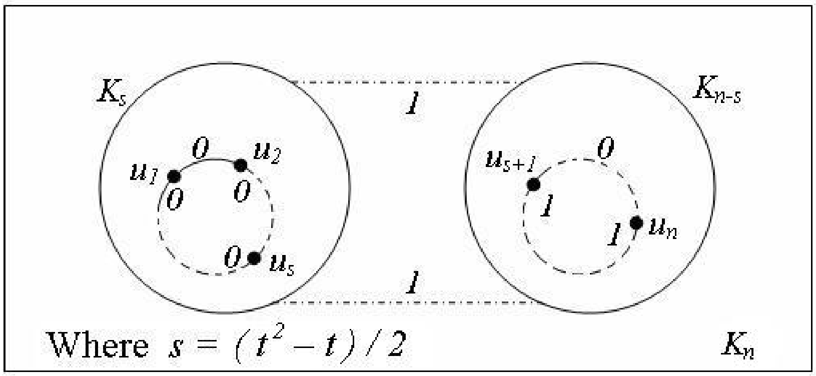


Figure-5.18

For the graph under consideration $v_f(0) = v_f(1)$ and

$$\begin{aligned} e_f(0) &= 2[(\tfrac{t^2-t}{2}) \times (\tfrac{t^2-t}{2} - 1) + \{t^2 - (\tfrac{t^2-t}{2})\} \times \{t^2 - (\tfrac{t^2-t}{2}) - 1\}] \\ &= \tfrac{1}{2}[(t^2 - t) \times (t^2 - t - 2) + (t^2 + t) \times (t^2 + t - 2)] \\ &= t^4 - t^2 \\ &= t^2(t^2 - 1) \\ &= \tfrac{1}{2} \text{ number of edges in } K_n \bigcup K_n. \end{aligned}$$

Therefore $e_f(0) = e_f(1)$.

Thus the graph under consideration satisfies the conditions $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. Therefore the graph $K_n \bigcup K_n$, where $n = t^2$, for some $t \in N$ is cordial graph.

5.6 Some Open Problems :

• In connection of cordial labeling of path union, instead of taking one edge between two graphs one can think path of arbitrary length between any two graphs. Then *Theorem*−5.4.2 reported in previous section will be a spacial case.

• One can derive results similar to the previous section for multiple shells, helms etc.

• One can discuss cordiality in the context of various graph operations like barycentric subdivision and contraction.

• One can investigate cordial labeling for star of some other graphs.

5.7 Concluding Remarks :

In this chapter cordial labeling is discussed in detail and survey of some existing results is carried out. Nine new results are obtained. Hint for further results is given in the form of open problems. Investigations carried out here are novel and important. Labeling pattern is given in vary elegant way and it is demonstrated by means of sufficient examples.

The penultimate chapter is aimed to discuss 3−equitable labeling of graphs.

Chapter – 6

3−Equitable Labeling Of Graphs

6.1 Introduction :

In the previous *Chapter*−5 we have discussed cordiality of various graphs while this chapter is aimed to discuss 3−equitable labeling of graphs in detail. Four new 3−equitable graphs are investigated.

6.2 Some Definitions and Existing Results :

As we mentioned in *Chapter*−3 Cahit [33] has defined $k-$equitable labeling in 1990. Here we will discuss 3−equitable labeling which is particular type of $k-$equitable labeling defined as follows.

Definition−6.2.1 : Let $G = (V, E)$ be a graph. A mapping $f : V(G) \rightarrow\{0,1,2\}$ is called *ternary vertex* labeling of G and $f(v)$ is called *label* of the vertex v of G under f.

For an edge $e = uv$ the induced edge labeling $f^* : E(G) \rightarrow\{0,1,2\}$ is given by $f^*(e)=|f(u) - f(v)|$. Let $v_f(0)$,$v_f(1)$,$v_f(2)$ be the number of vertices of G having labels 0,1 and 2 respectively under f and let $e_f(0)$,$e_f(1)$,$e_f(2)$ be the number of edges having labels 0,1 and 2 respectively under f^*.

Definition$-$6.2.2 A ternary vertex labeling of a graph G is called $3-$*equitable labeling* if $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$, $0 \leq i, j \leq 2$. A graph G is called $3-$*equitable graph* if it admits $3-$equitable labeling.

Some known families of $3-$equitable graphs are listed below.

• Cahit [32],[33] proved that

⋄ C_n is $3-$equitable if and only if n is not congruent to 3(mod 6).

⋄ An Eulerian graph with $q \equiv 3$(mod 6) is not $3-$equitable where q is the number of edges.

⋄ All caterpillars are $3-$equitable.

⋄ He conjectured that *A triangular cactus with* n *blocks is* $3-$*equitable if and only if* n *is even.*

⋄ Every tree with fewer than five end vertices has a $3-$equitable labeling.

• Seoud and Abdel Maqsoud [89] proved that

⋄ A graph with p vertices and q edges in which every vertex has odd degree is not $3-$equitable if $p \equiv 0(\text{mod } 3)$ and $q \equiv 3(mod\ 6)$.

⋄ All fans except $P_2 + K_1$ are $3-$equitable.

⋄ P_n^2 is $3-$equitable for all n except 3.

⋄ $K_{m,n}$, $3 \leq m \leq n$ is $3-$equitable if and only if $(m, n) = (4, 4)$.

• Bapat and Limaye [17] proved that

⋄ Helms H_n, $n \geq 4$ are $3-$equitable.

⋄ Flowers are $3-$equitable.

• Youssef [112] proved that $W_n = C_n + K_1$ is $3-$equitable for all $n \geq 4$.

In the next section we will give brief account of some new results investigated by us.

6.3 Some Cycle Related 3−equitable Graphs :

We have investigated some new families of cycle related 3−equitable graphs. In this section we will give 3−equitable labeling for cycle with one chord, cycle with twin chords and cycle with triangle.

Theorem−6.3.1 **:** Cycle with one chord is 3−equitable.

Proof : Let G be the cycle with one chord. Let $v_1, v_2, \dots, v_n$ be successive vertices of cycle C_n. Let $e_1 = v_2v_n$ be chord of a cycle C_n. To define ternary vertex labeling $f : V(G) \rightarrow$ {0,1,2} we consider the following cases and we shall define labeling for them as follows.

Case−1 : $n \equiv 0, 4, 5 (\text{mod } 6)$

$$
\begin{aligned}
f(v_i) &= 0 \text{ ; if } i \equiv 2, 5 (\text{mod } 6) \\
&= 1 \text{ ; if } i \equiv 0, 1 (\text{mod } 6) \\
&= 2 \text{ ; if } i \equiv 3, 4 (\text{mod } 6),\ 1 \leq i \leq n.
\end{aligned}
$$

Case−2 : $n \equiv 1 (\text{mod } 6)$

$$
\begin{aligned}
f(v_i) &= 0 \text{ ; if } i \equiv 3, 4 (\text{mod } 6) \\
&= 1 \text{ ; if } i \equiv 0, 1 (\text{mod } 6) \\
&= 2 \text{ ; if } i \equiv 2, 5 (\text{mod } 6),\ 1 \leq i \leq n.
\end{aligned}
$$

Case−3 : $n \equiv 2 (\text{mod } 6)$

$$
\begin{aligned}
f(v_{n-1}) = 0,\ f(v_n) = 2 \text{ and } f(v_i) &= 0 \text{ ; if } i \equiv 0, 3 (\text{mod } 6) \\
&= 1 \text{ ; if } i \equiv 1, 2 (\text{mod } 6) \\
&= 2 \text{ ; if } i \equiv 4, 5 (\text{mod } 6),\ 1 \leq i \leq n-2.
\end{aligned}
$$

Case−4 : $n \equiv 3 (\text{mod } 6)$

$f(v_{n-1}) = 0,\ f(v_n) = 2$ and label remaining vertices as in Case−2.

The labeling pattern defined above covers all possible arrangement of vertices. In each case the graph G under consideration satisfies the conditions

$|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ $(0 \leq i, j \leq 2)$ as shown in following *Table*-6.1. i.e. G admits $3-$equitable labeling.

Let $n = 4a + b$, $a \epsilon N$.

b	vertex conditions	edge conditions
0	$v_f(0)=v_f(1)=v_f(2)$	$e_f(0)+1=e_f(1)=e_f(2)+1$
1	$v_f(0)+1=v_f(1)=v_f(2)+1$	$e_f(0)=e_f(1)=e_f(2)+1$
2	$v_f(0)=v_f(1)+1=v_f(2)$	$e_f(0)=e_f(1)=e_f(2)$
3	$v_f(0)=v_f(1)=v_f(2)$	$e_f(0)+1=e_f(1)=e_f(2)+1$
4	$v_f(0)+1=v_f(1)+1=v_f(2)$	$e_f(0)+1=e_f(1)=e_f(2)$
5	$v_f(0)=v_f(1)+1=v_f(2)$	$e_f(0)=e_f(1)=e_f(2)$

Table-6.1

For better understanding of above defined labeling pattern let us consider an example.

Example$-6.3.2$ **:** Consider cycle C_7 with one chord. The labeling pattern is shown in *Figure*-6.2. (It is the case related to $n \equiv 1 (\text{mod } 6)$)

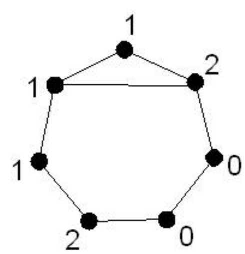


Figure-6.2

Theorem$-6.3.3$ **:** Cycle with twin chords where chords form two triangles and one cycle C_{n-2} is $3-$equitable.

Proof : Let G be the cycle with twin chords where chords form two triangle and one cycle C_{n-2}. Let $v_1, v_2, \ldots, v_n$ be successive vertices of cycle C_n and $e_1 = v_2v_n$ and $e_2 = v_3v_n$ be two chords of a cycle C_n. To define ternary vertex labeling $f : V(G) \rightarrow$ {0,1,2} we consider the following cases and we shall define labeling for them as follows.

Case-1 : $n \equiv 0 (\text{mod } 6)$

$f(v_i) = 0$; if $i \equiv 1, 2(\text{mod } 6)$

$= 1$; if $i \equiv 4, 5(\text{mod } 6)$

$= 2$; if $i \equiv 0, 3(\text{mod } 6)$, $1 \leq i \leq n$.

Case-2 : $n \equiv 1(\text{mod } 6)$

$f(v_i) = 0$; if $i \equiv 2, 5(\text{mod } 6)$

$= 1$; if $i \equiv 3, 4(\text{mod } 6)$

$= 2$; if $i \equiv 0, 1(\text{mod } 6)$, $1 \leq i \leq n$.

Case-3 : $n \equiv 2, 3, 4, 5(\text{mod } 6)$

$f(v_i) = 0$; if $i \equiv 2, 5(\text{mod } 6)$

$= 1$; if $i \equiv 0, 1(\text{mod } 6)$

$= 2$; if $i \equiv 3, 4(\text{mod } 6)$, $1 \leq i \leq n$.

The labeling pattern defined above covers all possible arrangement of vertices. In each case the graph G under consideration satisfies the conditions $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ $(0 \leq i, j \leq 2)$ as shown in following *Table*-6.3. i.e. G admits 3$-$equitable labeling.

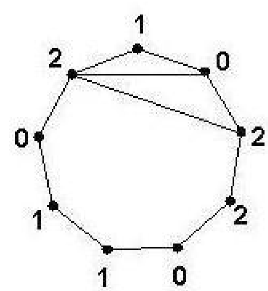
Let $n = 4a + b$, $n \epsilon N$, $n \geq 5$.

b	Vertex Conditions	Edge Conditions
0	$v_f(0)=v_f(1)=v_f(2)$	$e_f(0)=e_f(1)+1=e_f(2)$
1,4	$v_f(0)+1=v_f(1)+1=v_f(2)$	$e_f(0)=e_f(1)=e_f(2)$
2	$v_f(0)=v_f(1)=v_f(2)+1$	$e_f(0)+1=e_f(1)=e_f(2)+1$
3	$v_f(0)=v_f(1)=v_f(2)$	$e_f(0)+1=e_f(1)=e_f(2)$
5	$v_f(0)=v_f(1)+1=v_f(2)$	$e_f(0)+1=e_f(1)+1=e_f(2)$

Table-6.3

For better understanding of above defined labeling pattern let us consider an example.

Example$-$6.3.4 **:** Consider cycle C_9 with twin chords. The labeling pattern is as shown in *Figure*$-$6.4. (It is the case related to $n \equiv 3(\text{mod } 6)$)



Figure$-$6.4

Theorem$-$6.3.5**:** Cycle with triangle $C_n(1, 1, n-5)$ is 3$-$equitable except $n \equiv 0(\text{mod } 6)$.

Proof : Let G be cycle with triangle $C_n(1, 1, n-5)$. Let $v_1, v_2, \ldots, v_n$ be successive vertices of G. Let v_1, v_3 and v_5 be the vertices of triangle formed by edges $e_1 = v_1v_3$, $e_2 = v_3v_5$ and $e_3 = v_1v_5$.To define ternary vertex labeling $f : V(G) \rightarrow$\{0,1,2\} we consider the following cases and we shall define labeling for them as follows.

Case$-$1 : $n \equiv 0(\text{mod } 6)$

Here graph G is an Eulerian graph with number of edges congruent to 3(mod 6). Then in this case G is not 3$-$equitable as proved by Cahit [32].

Case$-$2 : $n \equiv 1(\text{mod } 6)$

$$f(v_1) = 2,\ f(v_2) = 1 \text{ and } f(v_i) = 0 \text{ ; if } i \equiv 1, 4(\text{mod } 6)$$
$$= 1 \text{ ; if } i \equiv 2, 3(\text{mod } 6)$$
$$= 2 \text{ ; if } i \equiv 0, 5(\text{mod } 6),\ 3 \leq i \leq n.$$

Case-3 : $n \equiv 2 (\text{mod } 6)$

$f(v_1) = 2,\ f(v_2) = 0$ and

$f(v_i) = 0$; if $i \equiv 1, 4 (\text{mod } 6)$

$= 1$; if $i \equiv 0, 5 (\text{mod } 6)$

$= 2$; if $i \equiv 3, 2 (\text{mod } 6)$, $3 \leq i \leq n$.

Case-4 : $n \equiv 3 (\text{mod } 6)$

$f(v_{n-2}) = 0,\ f(v_{n-1}) = 2,\ f(v_n) = 1$ and

$f(v_i) = 0$; if $i \equiv 0, 3 (\text{mod } 6)$

$= 1$; if $i \equiv 4, 5 (\text{mod } 6)$

$= 2$; if $i \equiv 1, 2 (\text{mod } 6)$, $1 \leq i \leq n - 3$.

Case-5 : $n \equiv 4 (\text{mod } 6)$

$f(v_{n-3}) = 1,\ f(v_{n-2}) = 0,\ f(v_{n-1}) = 2, f(v_n) = 0$ and

$f(v_i) = 0$; if $i \equiv 1, 4 (\text{mod } 6)$

$= 1$; if $i \equiv 0, 5 (\text{mod } 6)$

$= 2$; if $i \equiv 2, 3 (\text{mod } 6)$, $1 \leq i \leq n - 4$.

Case-6 : $n \equiv 5 (\text{mod } 6)$

$f(v_{n-2}) = 1,\ f(v_{n-1}) = 0,\ f(v_n) = 0$ and

$f(v_i) = 0$; if $i \equiv 0, 3 (\text{mod } 6)$

$= 1$; if $i \equiv 4, 5 (\text{mod } 6)$

$= 2$; if $i \equiv 1, 2 (\text{mod } 6)$, $1 \leq i \leq n - 3$.

The labeling pattern defined above covers all possible arrangement of vertices. In each case the graph G under consideration satisfies the conditions $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ $(0 \leq i, j \leq 2)$ as shown in following *Table*-6.5. i.e. G admits $3-$equitable labeling.

Let $n = 4a + b$, $n \epsilon N$, $n \geq 6$.

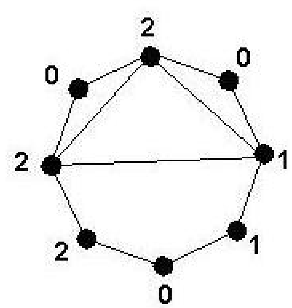
b	vertex conditions	edge conditions
1	$v_f(0)+1=v_f(1)+1=v_f(2)$	$e_f(0)+1=e_f(1)=e_f(2)+1$
2	$v_f(0)=v_f(1)+1=v_f(2)$	$e_f(0)+1=e_f(1)=e_f(2)$
3	$v_f(0)=v_f(1)=v_f(2)$	$e_f(0)=e_f(1)=e_f(2)$
4	$v_f(0)+1=v_f(1)+1=v_f(2)$	$e_f(0)=e_f(1)=e_f(2)$
5	$v_f(0)=v_f(1)+1=v_f(2)+1$	$e_f(0)+1=e_f(1)+1=e_f(2)$

Table−6.5

Remark−6.3.6 **:** In the above *Theorem*−6.3.5 we have discussed the 3−equitable labeling of $C_n(1, 1, n - 5)$ but it is also possible to develop 3−equitable labeling when three chords are making possible triangle with respect to given cycle. For the sake of brevity that discussion is not included here.

For better understanding of above defined labeling pattern let us consider an example.

Example−6.3.7 **:** Consider cycle C_8 with triangle. The labeling pattern is as shown in *Figure*−6.6. (It is the case related to $n \equiv 2(\text{mod } 6)$)



Figure−6.6

We have defined star of a graph in *Chapter*−5 as *Definition*−5.5.1. In that connection we will prove following result.

Theorem$-$6.3.8 **:** Star of cycles C_n^* is 3$-$equitable for all n.

Proof : Let $v_1, v_2, \ldots, v_n$ be successive vertices of central cycle of C_n^* and $u_{i1}, \ldots, u_{in}$ be successive vertices of other cycles $C_n^{(i)}$ (except central cycle), $i = 1, 2, \ldots, n$. Let e_i be the edge such that $e_i = u_{i1}v_i$. Moreover, let us denote the vertex of cycle $C_n^{(i)}$ which is adjacent to a vertex v_i labeled by 0 as $u_{ij}^{(0)}$, the vertex of cycle $C_n^{(i)}$ which is adjacent to a vertex v_i labeled by 1 as $u_{ij}^{(1)}$ and the vertex of cycle $C_n^{(i)}$ which is adjacent to a vertex v_i labeled by 2 as $u_{ij}^{(2)}$. To define required labeling $f : V(C_n^*) \to \{0, 1, 2\}$ we consider following cases and we shall define labeling function for them as follows.

<u>Case$-$1</u>: $n \equiv 0 (\text{mod } 6)$

$f(v_i) = 0$; if $i \equiv 0, 3 (\text{mod } 4)$

$= 1$; if $i \equiv 1, 2 (\text{mod } 6)$

$= 2$; if $i \equiv 4, 5 (\text{mod } 6)$, $1 \leq i \leq n$

$f(u_{ij}^{(0)}) = 0$; if $j \equiv 0, 3 (\text{mod } 6)$

$= 1$; if $j \equiv 1, 2 (\text{mod } 6)$

$= 2$; if $j \equiv 4, 5 (\text{mod } 6)$, $1 \leq j \leq n$,$1 \leq i \leq n$

$f(u_{ij}^{(1)}) = 0$; if $j \equiv 0, 3 (\text{mod } 6)$

$= 1$; if $j \equiv 1, 2 (\text{mod } 6)$

$= 2$; if $j \equiv 4, 5 (\text{mod } 6)$, $1 \leq j \leq n$, $1 \leq i \leq n$

$f(u_{ij}^{(2)}) = 0$; if $j \equiv 1, 4 (\text{mod } 6)$

$= 1$; if $j \equiv 0, 5 (\text{mod } 6)$

$= 2$; if $j \equiv 2, 3 (\text{mod } 6)$, $1 \leq j \leq n$, $1 \leq i \leq n$

<u>Case$-$2</u>: $n \equiv 1 (\text{mod } 6)$

$f(v_i) = 0$; if $i \equiv 0, 3 (\text{mod } 6)$

$= 1$; if $i \equiv 1, 2 (\text{mod } 6)$

$= 2$; if $i \equiv 4, 5 (\text{mod } 6)$, $1 \leq i \leq n$

$f(u_{ij}^{(0)}) = 0$; if $j \equiv 2, 5 (\text{mod } 6)$

$= 1$; if $j \equiv 0, 1 (\text{mod } 6)$

$= 2$; if $j \equiv 3, 4 (\text{mod } 6)$, $1 \leq j \leq n$, $1 \leq i \leq n$

$f(u_{i1}^{(1)}) = 2$ and

$f(u_{ij}^{(1)}) = 0$; if $j \equiv 1, 4 (\text{mod } 6)$

$= 1$; if $j \equiv 2, 3 (\text{mod } 6)$

$= 2$; if $j \equiv 0, 5 (\text{mod } 6)$, $2 \leq j \leq n$, $1 \leq i \leq n$

$f(u_{ij}^{(2)}) = 0$; if $j \equiv 1, 4 (\text{mod } 6)$

$= 1$; if $j \equiv 2, 3 (\text{mod } 6)$

$= 2$; if $j \equiv 0, 5 (\text{mod } 6)$, $1 \leq j \leq n$, $1 \leq i \leq n$

Case−3: $n \equiv 2 (\text{mod } 6)$

$f(v_i) = 0$; if $i \equiv 2, 5 (\text{mod } 6)$

$= 1$; if $i \equiv 0, 1 (\text{mod } 6)$

$= 2$; if $i \equiv 3, 4 (\text{mod } 6)$, $1 \leq i \leq n$

$f(u_{ij}^{(0)}) = 0$; if $j \equiv 1, 4 (\text{mod } 6)$

$= 1$; if $j \equiv 0, 5 (\text{mod } 6)$

$= 2$; if $j \equiv 2, 3 (\text{mod } 6)$, $1 \leq j \leq n$, $1 \leq i \leq n$

$f(u_{in}^{(1)}) = 1$, $f(u_{in-1}^{(1)}) = 2$ and

$f(u_{ij}^{(1)}) = 0$; if $j \equiv 0, 3 (\text{mod } 6)$

$= 1$; if $j \equiv 1, 2 (\text{mod } 6)$

$= 2$; if $j \equiv 4, 5 (\text{mod } 6)$, $1 \leq j \leq n - 2$, $1 \leq i \leq n$

$f(u_{ij}^{(2)}) = 0$; if $j \equiv 2, 5 (\text{mod } 6)$

$= 1$; if $j \equiv 0, 1 (\text{mod } 6)$

$= 2$; if $j \equiv 3, 4 (\text{mod } 6)$, $1 \leq j \leq n$, $1 \leq i \leq n$

Case$-$4: $n \equiv 3 (\mathrm{mod}\ 6)$

$f(v_i) = 0$; if $i \equiv 1, 4 (\mathrm{mod}\ 6)$

$= 1$; if $i \equiv 0, 5 (\mathrm{mod}\ 6)$

$= 2$; if $i \equiv 2, 3 (\mathrm{mod}\ 6)$, $1 \leq i \leq n$

$f(u_{ij}^{(0)}) = 0$; if $j \equiv 1, 4 (\mathrm{mod}\ 6)$

$= 1$; if $j \equiv 0, 5 (\mathrm{mod}\ 6)$

$= 2$; if $j \equiv 2, 3 (\mathrm{mod}\ 6)$, $1 \leq j \leq n$, $1 \leq i \leq n$

Let n_1 denotes the number of cycles whose one end vertex u_{ij} (for some j) is adjacent to vertex v_i which is labeled by 1. Here note that number of vertices in central cycle which are labeled by 1 is even.

To label $\frac{n_1}{2}$ such cycles we define labeling f as

$f(u_{in}^{(1)}) = 1$, $f(u_{in-1}^{(1)}) = 2$, $f(u_{in-2}^{(1)}) = 0$ and

$f(u_{ij}^{(1)}) = 0$; if $j \equiv 0, 3 (\mathrm{mod}\ 6)$

$= 1$; if $i \equiv 4, 5 (\mathrm{mod}\ 6)$

$= 2$; if $i \equiv 1, 2 (\mathrm{mod}\ 6)$, $1 \leq j \leq n - 3$, $1 \leq i \leq n$.

To label remaining $\frac{n_1}{2}$ such cycles we define labeling f as

$f(u_{ij}^{(1)}) = 0$; if $j \equiv 2, 5 (\mathrm{mod}\ 6)$

$= 1$; if $j \equiv 0, 1 (\mathrm{mod}\ 6)$

$= 2$; if $j \equiv 3, 4 (\mathrm{mod}\ 6)$, $1 \leq j \leq n$, $1 \leq i \leq n$

$f(u_{ij}^{(2)}) = 0$; if $j \equiv 1, 4 (\mathrm{mod}\ 6)$

$= 1$; if $j \equiv 2, 3 (\mathrm{mod}\ 6)$

$= 2$; if $j \equiv 0, 5 (\mathrm{mod}\ 6)$, $1 \leq j \leq n$, $1 \leq i \leq n$

Case$-$5: $n \equiv 4 (\mathrm{mod}\ 6)$

$f(v_i) = 0$; if $i \equiv 0, 3 (\mathrm{mod}\ 4)$

$= 1$; if $i \equiv 1, 2 (\mathrm{mod}\ 6)$

$= 2$; if $i \equiv 4, 5 \pmod 6$, $1 \leq i \leq n$

$f(u_{ij}^{(0)}) = 0$; if $j \equiv 0, 3 \pmod 6$

$= 1$; if $j \equiv 4, 5 \pmod 6$

$= 2$; if $j \equiv 1, 2 \pmod 6$, $1 \leq j \leq n$, $1 \leq i \leq n$

$f(u_{ij}^{(1)}) = 0$; if $j \equiv 0, 3 \pmod 6$

$= 1$; if $j \equiv 1, 2 \pmod 6$

$= 2$; if $j \equiv 4, 5 \pmod 6$, $1 \leq j \leq n$, $1 \leq i \leq n$

$f(u_{ij}^{(2)}) = 0$; if $j \equiv 1, 4 \pmod 6$

$= 1$; if $j \equiv 0, 5 \pmod 6$

$= 2$; if $j \equiv 2, 3 \pmod 6$, $1 \leq j \leq n$, $1 \leq i \leq n$

Case$-$6: $n \equiv 5 \pmod 6$

$f(v_i) = 0$; if $i \equiv 2, 5 \pmod 4$

$= 1$; if $i \equiv 0, 1 \pmod 6$

$= 2$; if $i \equiv 3, 4 \pmod 6$, $1 \leq i \leq n$

$f(u_{ij}^{(0)}) = 0$; if $j \equiv 0, 3 \pmod 6$

$= 1$; if $j \equiv 4, 5 \pmod 6$

$= 2$; if $j \equiv 1, 2 \pmod 6$, $1 \leq j \leq n$, $1 \leq i \leq n$

$f(u_{ij}^{(1)}) = 0$; if $j \equiv 2, 5 \pmod 6$

$= 1$; if $j \equiv 0, 1 \pmod 6$

$= 2$; if $j \equiv 3, 4 \pmod 6$, $1 \leq j \leq n$, $1 \leq i \leq n$

$f(u_{ij}^{(2)}) = 0$; if $j \equiv 2, 5 \pmod 6$

$= 1$; if $j \equiv 3, 4 \pmod 6$

$= 2$; if $j \equiv 0, 1 \pmod 6$, $1 \leq j \leq n$, $1 \leq i \leq n$

The above defined labeling pattern covers all possible arrangement of vertices. In each case the graph under consideration satisfies the condition

$|v_f(i)-v_f(j)| \leq 1$ and $|e_f(i)-e_f(j)| \leq 1$, $0 \leq i,j \leq 2$ as shown in *Table*-6.7.
i.e. graph C_n^* admits $3-$equitable labeling.
Let $n = 4a + b$, $n \epsilon N$, $n \geq 3$.

b	vertex conditions	edge conditions
0,3	$v_f(0)=v_f(1)=v_f(2)$	$e_f(0)=e_f(1)=e_f(2)$
1	$v_f(0)+1=v_f(1)=v_f(2)$	$e_f(0)=e_f(1)=e_f(2)$
2	$v_f(0)=v_f(1)=v_f(2)$	$e_f(0)=e_f(1)+1=e_f(2)$
4	$v_f(0)+1=v_f(1)=v_f(2)$	$e_f(0)=e_f(1)=e_f(2)$
5	$v_f(0)=v_f(1)=v_f(2)$	$e_f(0)+1=e_f(1)=e_f(2)$

Table-6.7

Example$-6.3.9$ **:** For better understanding of above defined labeling pattern let us consider star of cycle C_5 (It related to Case-6). The $3-$equitable labeling of star of C_5 is as shown in *Figure*-6.8.

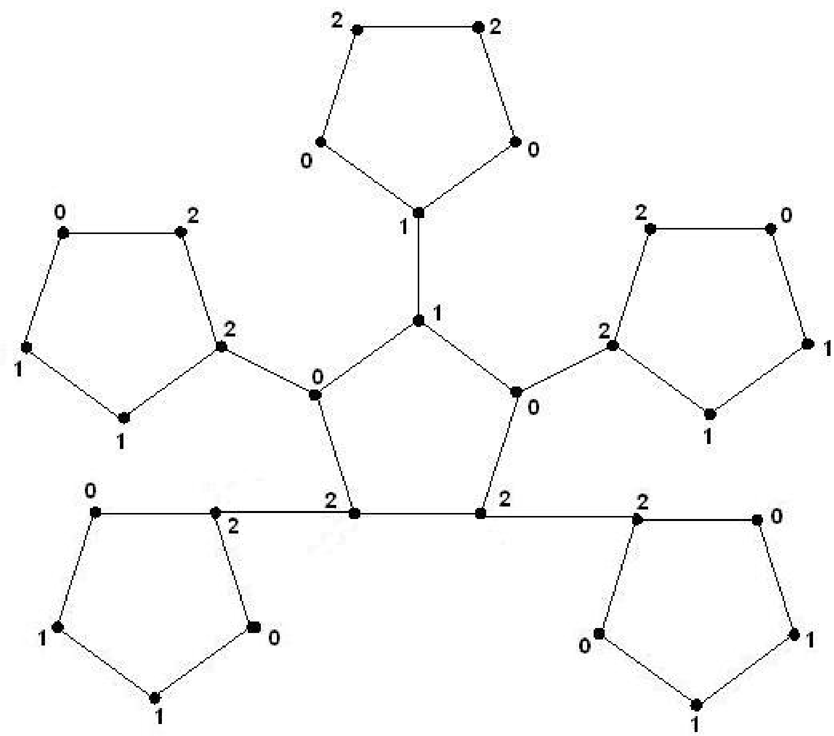


Figure-6.8

6.4 Some Open Problems :

• One can discuss 3−equitable labeling in the context of various graph operations like barycentric subdivision and contraction.

• One can investigate 3−equitable labeling for path union of cycles, cycle with one chord, cycle with twin chords, cycle with triangle etc.

• One can investigate the results for 3−equitable labeling parallel to results investigated as in *Section*−5.5 for cordial labeling.

6.5 Concluding Remarks :

In this chapter 3−equitable labeling is discussed in detail and survey of some existing results is carried out. The results obtained here are novel and labeling pattern is given in very elegant way which is demonstrated by means of examples.

The penultimate chapter is aimed to discuss applications of graph labeling.

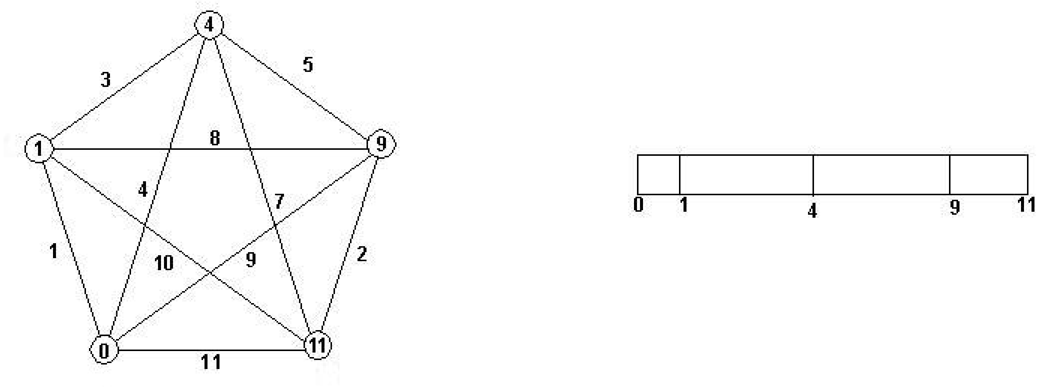
Chapter – 7

Applications of Graph Labeling

7.1 Introduction : Labeled graphs are becoming more interesting due to their broad range of applications. This family has variety of applications in diversified fields. Labeled graphs have vital applications to coding theory, particularly in the development of missile guidance codes, design of radar type codes and convolution codes with optimal autocorrelation properties. Optimal circuit layouts and solution of problem of number theory can be discussed in the context of graph labeling. Ambiguity in X−ray crystallography can also be explained using graph labeling techniques. A detailed survey on such applications is systematically studied by Bloom and Golomb [25]. We will discuss some interesting applications reported in that paper. Some of these applications are also recorded in Germina [53].

7.2 Semigraceful Labeling and Golomb Ruler : We have discussed graceful labeling and graceful graphs in *Chapter*−4. As we noted there K_n is graceful if and only if $n \leq 4$. In other words it is not possible to label vertices with numbers $\{0, 1, 2, \ldots, {}_nC_2\}$ such that each edge can be labeled distinctly using labels $\{1, 2, \ldots\ldots, {}_nC_2\}$. This problem has motivated Golomb to define semigraceful labeling. According to him if the constraint *edge labels to be consecutive integers* is relaxed then such labeling is called *semigraceful labeling* and the graph which admits such labeling is called *semigraceful graph*. In other words semigraceful graph on n vertices does not use all the labels from $\{1, 2, \ldots, {}_nC_2\}$ but some edge labels are missing. In general vertex labels in semigraceful labeling may exceed ${}_nC_2$ or repeat or both. Semigraceful labeling is *optimal* if it minimizes the largest edge label which is denoted by $G(K_n)$.

In the following *Figure*$-$7.1(a) a semigraceful labeling for K_5 is shown. In this figure we will observe that no edge is labeled with label 6.

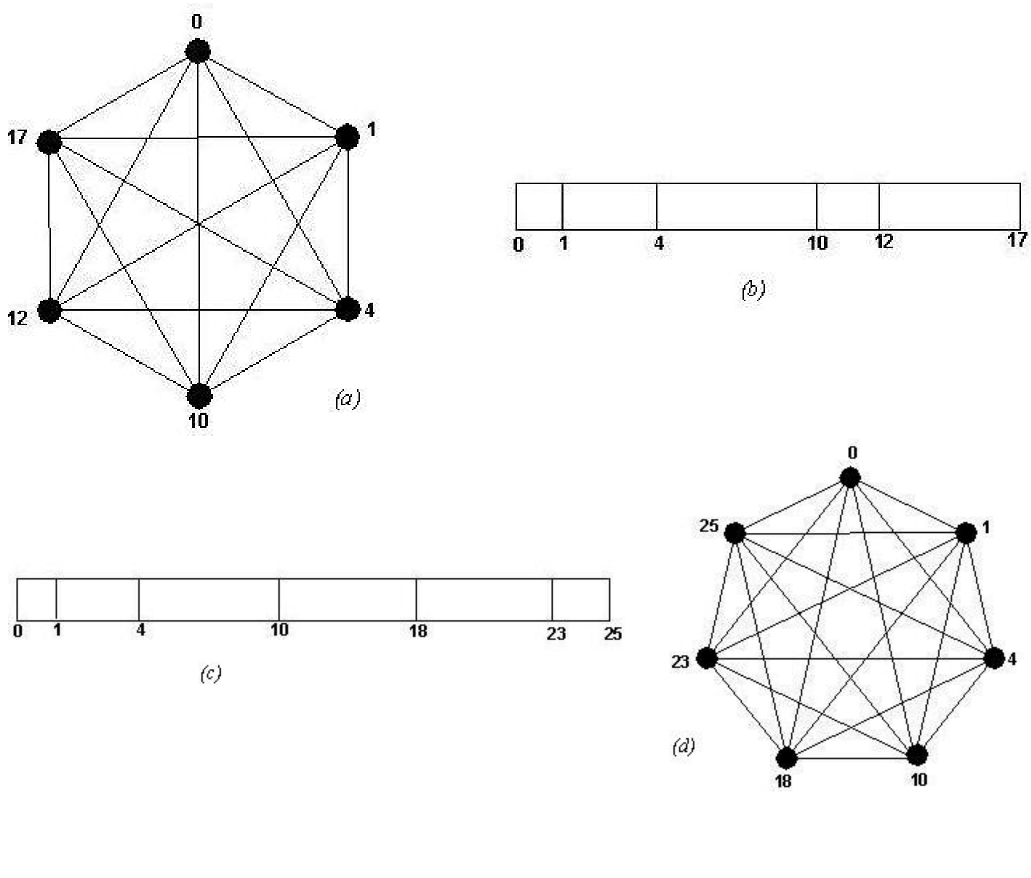


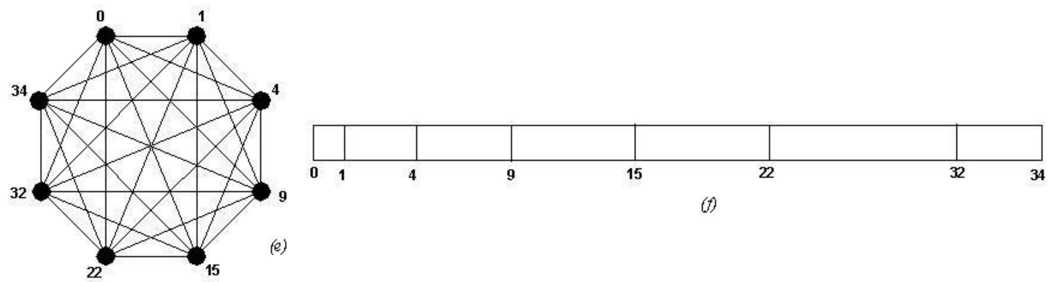
(a) *(b)*

Figure$-$7.1

Golomb observed an important equivalence for the coding theory context between a semigraceful labeling which minimizes $G(K_n)$. He developed a special ruler on which n division marks(including the ends) are placed. The positions of the division marks correspond to the number placed on the end vertices of K_n. The edge labels of K_n thus exactly correspond to the set of measurements which can be made on the ruler. Such ruler is named by Gradner [51] as a *Golomb Ruler*. In *Figure*$-$7.1(b) a ruler corresponding to semigraceful labeling for K_5 is shown. As we mentioned earlier no edge is labeled with 6. Equivalently from *Figure*−7.1(b) we can see that it is not possible to measure length 6 directly by the Golomb Ruler. All optimal rulers have been found for $n \leq 11$ and are summarized in Bloom and Golomb [24]. Such ruler will be able to measure ${}_nC_2$ lengths which are numerically equal to edge labels of K_n and they measure non−redundant minimal length.

In *Figure*−7.2 we provide semigraceful labeling and equivalent Golomb rulers for K_6, K_7, K_8 respectively. These rulers will measure maximum lengths of 17, 25 and 34 units in optimal way.





Figure−7.2

It is also possible to provide other pattern of labeling and corresponding ruler. Such rulers are called *homometric rulers*. For example for K_6 it is possible to provide semigraceful labeling using vertex labels $0, 1, 4, 10, 15, 17$ or $0, 1, 4, 11, 13, 17$ or $0, 1, 8, 12, 14, 17$.

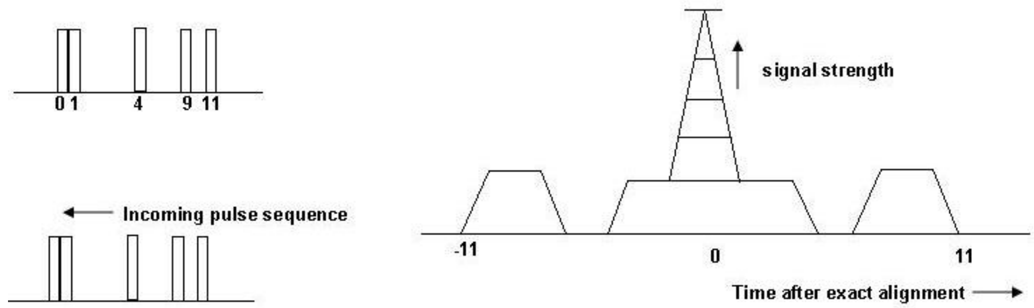
In the following *Table*−7.3 we have summarized the particulars regarding possible semigraceful labeling of K_n for $n \leq 11$.

Nodes	Length	Divisions	Marks at
2	1	1	0,1
3	3	1,2	0,1,3
4	6	1,3,2	0,1,4,6
5	11	1,3,5,2	0,1,4,9,11
		2,5,1,3	0,2,7,8,11
6	17	1,3,6,2,5	0,1,4,10,12,17
		1,3,6,5,2	0,1,4,10,15,17
		1,7,3,2,4	0,1,8,11,13,17
		1,7,4,2,3	0,1,8,12,14,17
7	25	1,3,6,8,5,2	0,1,4,10,18,23,25
		1,6,4,9,3,2	0,1,7,11,20,23,25
		1,10,5,3,4,2	0,1,11,16,19,23,25
		2,1,7,6,5,4	0,2,3,10,16,21,25
		2,5,6,8,1,3	0,2,7,13,21,22,25
8	34	1,3,5,6,7,10,2	0,1,4,9,15,22,32,34
9	44	1,4,7,13,2,8,6,3	0,1,5,12,25,27,35,41,44
10	55	1,5,4,13,3,8,7,12,2	0,1,6,10,23,26,34,41,53,55
11	72	1,3,9,15,5,14,7,10,6,2	0,1,4,13,28,33,47,54,64,70,72
		1,8,10,5,7,21,4,2,11,3	0,1,9,19,24,31,52,56,58,69,72

Table−7.3

The discovery of Golomb Rulers with more marks as well as method for generating such class remains an open problem. The Golomb Rulers discussed above have several applications in coding theory, X−ray crystallography etc. In the remaining part of this chapter we will discuss such applications.

7.3 Generation of Radar Type Codes : In the previous section we have discussed Golomb Ruler in detail and also seen the possibility to measure the lengths(distances) with that ruler. In coding context distance interval is replaced by time interval. Let us consider a time mark ruler corresponding to K_5 shown in *Figure*−7.1. One can generate a radar code from this ruler by transmitting a sequence of five pulses at times corresponding to the marks on the ruler. i.e. 0,1,4,9,11. We observe that there is a 1 unit time interval between the onset of the first and second pulses, 3 units time interval between the second and third, 5 units time interval between third and fourth and 2 units between the last two. The time duration between the emission of the signal and its return is determined by correlating all incoming sequences of 11 time units duration with the original sequence. Let each pulse be of one unit duration. Thus, when an incoming string matches the original as shown in following *Figure*−7.4*(a).* Then a signal of strength 5 is generated as shown in following *Figure*−7.4*(b).*



(a) *(b)*

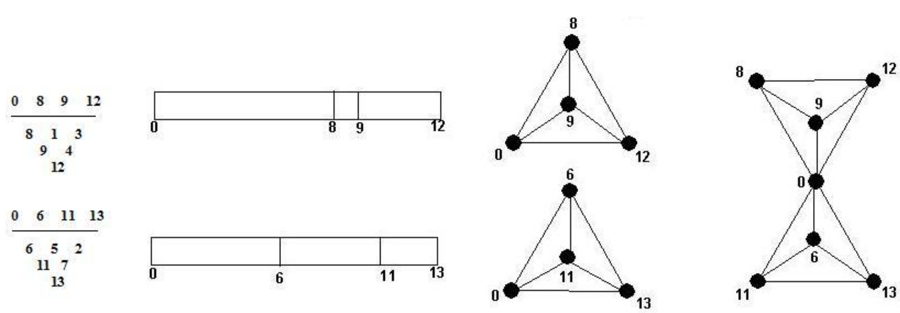
Figure−7.4

In the same *Figure*–7.4*(b)* we can see that a dip in the autocorrelation occurs at ± 6 time units, since there are no pulses which are aligned with a 6 unit shift of the pulse sequence out of its synch position. Six, of course, is the only distance of 11 or fewer units that the original ruler can not measure. We have also seen that it is the only number which is missing in labeling of K_5.

Eckler [44] investigated the problem related to above application for designing missile guidance codes. In an air borne missile, receiver passes all incoming signal trains down a delay line. If the line is tapped in several places which correspond to the actual time interval between incoming pulses, then the sum of those pulses will exceed a threshold and initiate some control action.

The command code for such a missile contains two or more different commands. Thus, in terms of instrumentation the delay line must be tapped by sets of leads corresponding to the delays between pulses for each command. In order to make code insensitive to random interference pulses (such as electrical storms or jamming effects) all of the delays pulses for one command must totally differ from those for every command. It is also desirable to use the shortest code word durations possible in order to minimize the delay line and to decrease the time during which interference could occur. Thus Eckler calculated $(d-1)$ intervals for the d pulses associated with of n different commands. In synch these commands give on reception by the missile, an autocorrelation of height d. Out-of-synch, the maximum autocorrelation is 1, and the noiseless cross correlation between commands also never exceed 1. This problem is analogous to find a set of n rulers of different lengths with $(d-1)$ marks on it. The marks on these rulers permit measuring each length in only one way. Moreover, the longest of these rulers must be as short as possible. Alternatively the problem corresponds to label as gracefully as possible a disconnected graph with n components. Each component is a complete graph on $(d-1)$ vertices. For this labeling each component of the composite graph has a vertex labeled with 0.

In the following $Figure-7.5$ $2-$message, $4-$pulse missile code with minimum duration is shown.



(a) (b) (c) (d)

$Figure-7.5$

In above *Figure*$-$7.5,

(a) Difference triangles

(b) Rulers

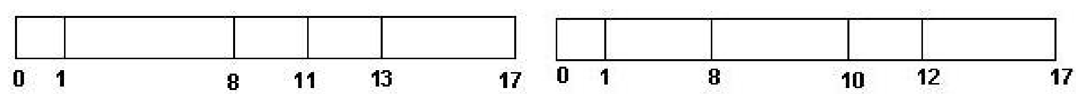
(c) Disconnected graph with 2 components

(d) Connected graph

7.4 X$-$ray Crystallography and Golomb Ruler : Ruler models are very much useful in X$-$ray crystallography. It sometimes happens that distinct crystal structures will give rise to identical X$-$ray diffraction patterns. These inherent ambiguities in the X$-$ray analysis of crystal structures have been studied by Patterson [85], Garrido [52] and Franklin [46].

For any crystal structure positions of atoms are determined by measurements made on X$-$ray diffraction patterns. These measurements indicate the set of distances between atoms in the crystal lattice, but in general do not necessarily specify the absolute positions of the atoms without any ambiguity. Mathematically, finite sets of integers $R = \{0 = a_1 < a_2 < \ldots < a_n\}$ and $S = \{0 = b_1 < b_2 < \ldots < a_n = b_n\}$ corresponding to two atom positions may have exactly the same set of differences $D(R) = D(S) = \{|a_i - a_j| : i < j\}$. Since the diffraction pattern determines the set of differences $D(R)$, it is impossible to determine which of the homometric sets R or S produced it, and consequently which crystal lattice give rise to the diffraction pattern. This homometric set problem may be viewed as a determination of non$-$equivalent rulers, which make identical sets of measurements. The sets R and S designate the positions of the marks of two rulers and $D(R)$ and $D(S)$ are their respective sets of ${}_nC_2$ measurements.

Thus the class of diffraction patterns corresponds to a set of differences, which has no repeated elements, that is, to a non−redundant set. Two equivalent rulers are shown in *Figure*−7.6. Also there are no non−redundant rulers with fewer than 6 points or of length less than 17.



Figure−7.6

Measurements made by the rulers are 1,2,3,4,5,6,7,8,9,10,11,12,13,16,17.

The shortest non−redundant homometric pairs of rulers and the ${}_6C_2 = 15$ intervals which they measure.

7.5 Communication Network Labeling **:** In a small communication network, it may be desirable to assign each user terminal *node number*(vertex label) subject to the constraint that all the resulting edges (communication links) receive distinct numbers. In this way, the numbers of any two communicative terminals automatically specify (by simple subtraction) the link number of the connecting path and conversely the path number uniquely corresponds to the pair of user terminals which it interconnects.

Properties of a potential numbering system for such networks have been explored under the guise of gracefully labeled graphs, that is, the properties of graceful graphs provide design parameters for an appropriate communication network.

If a graphical model of any communication network can not be labeled gracefully, there is a possibility of using semigraceful labeling in which the constraint requiring *the edge labels to be consecutive integers* is relaxed.

The most important question for utilizing a *graceful addressing and identification system* involve being better able to determine whether an arbitrary model of a communications network is in a graceful configuration. If it is, how should it be labeled? If it isn't, can it be embedded into a graceful structure easily? or should it be labeled semigracefully? Moreover, determination needs to be made of growth provisions for any addressing scheme, that is, of algorithms for labeling a graph in which new vertices and edges have been added to a gracefully labeled graph.

7.6 Scope of Further Research :

• One can explore the related ruler problems which have similar applications to communications network. This includes the problems of finding the shortest rulers with k marks which measure all integer lengths from 1 to n, either (i) allowing the same length to be measured in more than one way, or (ii) not allowing the same length to be measured in more than one way.

• One can study the structure of different crystals using the ruler model. This approach will give rise to interdisciplinary research work.

• One can develop the graph model for communication network using other labeling techniques like harmonious labeling, $k-$equitable labeling etc.

7.7 Concluding Remarks : Graph labeling presents a common context for many applied and theoretical problems. Some of these are illustrated in the current chapter. Graph labeling and diversified applications are held together by common thread. This chapter creates an impression of graph labeling as a unifying model which has vital potential to provide solutions for practical purposes. Graph labeling techniques may work as a powerful unifying model with biotechnology, information technology and new generation communication network. One can develop new labeling technique and discover its applications to diversified area.

Chapter – 8

Regular Induced Subgraphs of K_n

8.1 Introduction :

We begin with some preliminaries for present chapter. Let $G = (V, E)$ be a simple graph with vertex set V and edge set E. We shall denote *degree of vertex* v of graph G by $d_G(v)$, is the number of vertices adjacent to v in G. The maximum degree for any vertex v of G is $\triangle(G)$ and the minimum degree for any vertex v of G is $\delta(G)$. If $\triangle(G) = \delta(G)$, then G is called a *regular graph.* If $\triangle(G) = \delta(G) = k$, then G is called a $k-$*regular graph.* G is called *even regular* or *odd regular* as k is even or odd accordingly. We note that if G is an odd regular graph then the number of vertices in G can not be odd, as degree sum of all vertices is even.

We shall denote Z_n as a group of integer modulo n and rZ_n as a subgroup of Z_n generated by the element r for some $r \in \{1, 2, \ldots, n\}$. ${}^nC_r$ has the usual meaning of combination notation. In 1889 Cayley [35] enumerated number of spanning trees for the graph K_n. In present chapter we have enumerated even regular induced subgraphs of K_n.

8.2 Induced Subgraphs of K_n :

We introduce following definition.

Definition$-8.2.1$ **:** Let $G = (V, E)$ be a graph with n vertices $v_1, \ldots, v_n$. Construct a cycle by starting from any vertex $v_s, v_{s+r}, v_{s+2r}, \ldots, v_s$ in which two vertices v_i and v_j are joined if one of them is included and either $|j-i| = r$ or $|j - i + n| = r$. Then we call it an $r-$*cycle*.

8.2.2 Example of an $r-$**cycle :** Consider a graph G with 16 vertices $v_1, v_2, \ldots, v_{16}$. Then $v_1, v_5, v_9, v_{13}, v_1$ is a $4-$cycle of the graph G and $v_4, v_{10}, v_{16}, v_6, v_{12}, v_2, v_8, v_{14}, v_4$ is a $6-$cycle of the graph G as shown in following *Figure*-8.1.

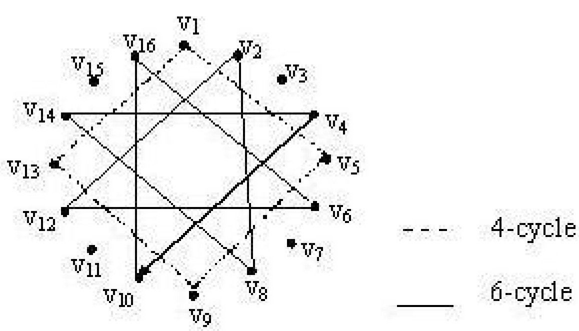


Figure-8.1

Lemma$-8.2.3$ **:** It is always possible to construct precisely (n, r) disjoint $r-$cycles from given n vertices $v_0, v_1, v_2, \ldots, v_{n-1}$, where $(n, r) =$ gcd of n and r.

Proof : Consider $Z_n = \{0, 1, \ldots, n-1\}$. It is obvious that $(n, r)Z_n$ is a subgroup of Z_n and it is same as rZ_n. Because $rZ_n \subseteq (n, r)Z_n$. Moreover

(n,r) is gcd of n and r, so there exist $a,b \in Z$ such that $an+br=(n,r)$.
$\Rightarrow (n,r)Z_n = (an+br)Z_n = brZ_n \subseteq rZ_n$
$\Rightarrow (n,r)Z_n \subseteq rZ_n$
Thus $rZ_n = (n,r)Z_n$.

Now as $\{v_0, v_{(n,r)}, \ldots, v_r, \ldots, v_{n-(n,r)}\} = \{v_{kr(mod \quad n)}/k = 1,2,\ldots,\frac{n}{(n,r)}\}$, then by joining the vertices from above set give rise to an $r-$cycle.

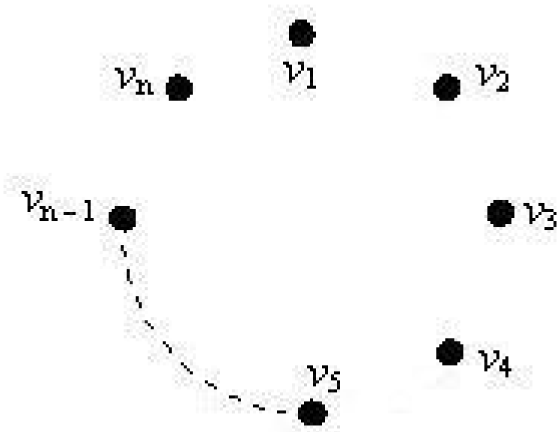
Now $Z_n = \bigcup$ Coset of $(n,r)Z_n = \bigcup$ Coset of rZ_n one can join the vertices of $\{v_i, v_{i+(n,r)}, \ldots, v_{i+r}, \ldots, v_{i+n-(n,r)}\} = \{v_{kr+i(mod \quad n)}/k = 1,2,\ldots,\frac{n}{(n,r)}\}$ to form $r-$cycles, for each $i = 1,2,\ldots,(n,r)-1$, which are in number precisely (n,r).

Therefore one can construct precisely (n,r) $r-$cycles from the n vertices $v_0, v_1, v_2, \ldots, v_{n-1}$. Since the cosets of rZ_n are disjoint, all such (n,r), $r-$cycles are distinct.

Remark$-$8.2.4 **:** In above *Example*$-$8.2.2, G has two $6-$cycles which are $v_4, v_{10}, v_{16}, v_6, v_{12}, v_2, v_8, v_{14}, v_4$ and $v_4, v_{10}, v_{16}, v_6, v_{12}, v_2, v_8, v_{14}, v_4$ as $(16,6) = 2$. Also these are $10-$cycles for G. In fact if $r-$cycle and $(n-r)-$cycle for a graph on n vertices, have a common vertex than these cycles are always identical. Thus $1-$cycle, $2-$cycle, ..., $r-$cycle, when $r < \frac{n}{2}$ for a graph G of n vertices are always distinct.

Theorem$-$8.2.5 **:** The complete graph K_n has an even $k-$regular induced connected subgraph for $2 \leq k \leq n-1$.

Proof : We use induction to prove the require result. Without loss of generality we arrange the vertices $v_1, \ldots, v_n$, as shown in following *Figure*$-$8.2.



Figure−8.2

For $k = 2$ **:** One can construct a 1−cycle by joining $v_1, v_2, \ldots, v_n, v_1$, which is 2−regular induced connected subgraph of K_n.

For better understanding of construction we exhibit one more case when $k = 4$.

For $k = 4$ **:** One can construct 2−cycle using previous *Lemma*−8.2.3. The number of such cycles is precisely $(n, 2)$. We can also construct 1−cycle like case when $k = 2$. This together will induce an even 4−regular connected subgraph of K_n.

Now as k is even $k = 2t$ for some $t \in N$ and let us assume that one can construct $(k-2)$−regular graph which is an induced connected subgraph of K_n, by constructing 1−cycle, 2−cycles, $\ldots, (t-1)$−cycles, using Lemma−8.2.3.

For $k = 2t$ **:** Note that $2t = k \leq n - 1 < n \Rightarrow t < \frac{n}{2}$.

Therefore by previous Lemma−8.2.3 one can construct t−cycles which are in number precisely (n, r). These together with 1−cycle, 2−cycles,

$\ldots, (t-1)-$cycles, constructed by assumption form $k-$regular connected induced subgraph of K_n.

8.2.6 Example of a $k-$regular graph : Consider a graph G with $v_1, v_2, \ldots, v_7$ vertices. Then $1-$cycle $C_1 = v_1, v_2, \ldots, v_7, v_1$, $2-$cycle $C_2 = v_1, v_3, v_5, v_7, v_2, v_4, v_6, v_1$ and $3-$cycle $C_3 = v_1, v_4, v_7, v_3, v_6, v_2, v_5, v_1$ forms $2, 4, 6-$regular induced connected subgraphs of K_7 by C_1, $C_1 \bigcup C_2$ and $C_1 \bigcup C_2 \bigcup C_3$ respectively as shown in following *Figure*-8.3.

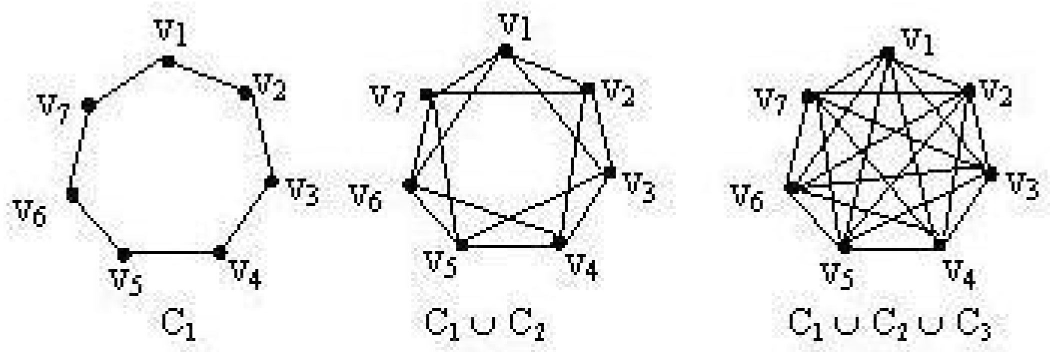


Figure-8.3

In the succeeding *Theorem*$-8.2.7$ the regular subgraph of K_n is enumerated.

Theorem$-8.2.7$ **:** K_n has atleast $2^{\frac{n-1}{2}}$ regular subgraphs, where n is odd integer.

Proof : For $n = 1$ it is obvious that K_1 is a regular subgraph of itself. Suppose $n = 2t+1$, for some $t \in N$. $\quad \Rightarrow t = \frac{(n-1)}{2}$.

Then by previous *Lemma*$-8.2.3$ K_n contains $1-$cycle, $2-$cycles, $\ldots\ldots$, $t-$cycles, which are $2-$regular subgraphs of K_n. Thus there ${}^tC_1$ $2-$regular subgraph of K_n.

If we take union of any two of above graphs, results into $4-$regular subgraph of K_n. Thus there are ${}^tC_2$, $4-$regular subgraph of K_n.

Proceeding in like way there are ${}^tC_r$, $2r-$regular subgraph of K_n, for $r = 3, 4, \ldots, t$. Moreover null graph on n vertices is also a regular subgraph of K_n.

Therefore K_n has atleast ${}^tC_1 + \text{- - -} + {}^tC_r + \text{- - -} + {}^tC_t + 1 = 2^{\frac{n-1}{2}}$ regular subgraphs.

8.3 Concluding Remarks :

A particular class of graphs namely $k-$regular graphs is considered. We proved that K_n has an even $k-$regular induced connected subgraph for $2 \leq k \leq n-1$. Moreover if K_n having odd number of vertices then the minimum number of regular subgraph of K_n is enumerated. The combination of Number Theory, Group Theory and Graph Theory is the real essence and beauty of these investigations.

The results obtained here are supposed to be new and independent. We have not found result of this nature in the survey of existing literature of Graph Theory. We believe that these results can be applicable to optimization problems of air traffic system, mobile telephone network and radio frequency assignment.

Chapter – 9

Maximal non–Hamiltonian Graphs

9.1 Introduction :

We begin with simple, finite graph $G = (V, E)$ with n vertices. We denote the number of vertices in G by $\mid V \mid$ and any cycle in G by C. We shall denote $w(G)$ the number of components of the graph G and $< U >$ is subgraph of G generated by the vertex subset U of V. We represent the number of partition of integer n with exactly r parts by P_n^r. For standard definitions, existing results and other notations which concern to present chapter we follow Clark and Holtan [39].

In the next section we will give brief summary of definitions and result which are useful for this chapter.

9.2 Some Useful Definitions and Results :

Definition–9.2.1 **(Hamiltonian graph) :** A cycle C in graph G is said to be *Hamiltonian cycle* if it contains all the vertices of the graph G. A graph G having such cycle is called a *Hamiltonian graph*.

Definition–9.2.2 **(Maximal non–Hamiltonian graph) :** A simple graph G is called *maximal non–Hamiltonian* if it is not Hamiltonian, but $G + e$ is Hamiltonian graph, where e is an edge between any two non–adjacent vertices of G.

Theorem$-$9.2.3 (Bondy and Chvatal [26]) **:** Let G be a simple graph with n vertices. Let u and v be two non$-$adjacent vertices in G such that $d(u)+d(v) \geq n$. Let $G+uv$ denote the super graph of G obtained by joining an edge between u and v. Then G is Hamiltonian if and only if $G+uv$ is Hamiltonian.

Definition$-$9.2.4**(Closure of a graph) :** Let G be a simple graph with n vertices. If there are two non$-$adjacent vertices u_1 and v_1 in G such that $d_G(u_1) + d_G(v_1) \geq n$ then join them by an edge to form super graph G_1. Then if there are two non$-$adjacent vertices u_2 and v_2 in G_1 such that $d_G(u_2) + d_G(v_2) \geq n$ then join them by an edge to form the super graph G_2. Continuing recursively joining pairs of non$-$adjacent vertices whose degree sum is greater than or equal to n until no such pair remains. The ultimate super graph obtained is called the *closure of* G and it is denoted by $c(G)$. If such pair does not exists then $c(G)$ is itself G.

9.3 Main Results :

Theorem$-$9.3.1 **:** Let $G = (V, E)$ be a graph on n vertices, $n \geq 3$. Let $V = B \cup V_0 \cup \ldots \cup V_k$ be a partition of vertex set V such that $< B \cup V_i >$ is a complete graph, $0 \leq i \leq k$, $B = \{v_1, v_2, \ldots, v_k\}$ and there is no edge with one end vertex in V_i and other end vertex in V_j, where $i, j \in \{0, 1, \ldots, k\}, i \neq j$. Then G is a maximal non$-$Hamiltonian graph.

Proof : Suppose each V_i consist t_i vertices $\forall\ i = 0, 1, \ldots, k$ and $n = k + t_0 +$ - - - $+ t_k$ in graph G such graph is shown following *Figure*$-$9.1.

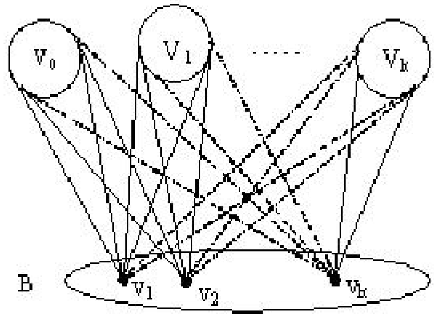


Figure-9.1

As $w(G\backslash B) > \mid B \mid$, it is obvious that G is non$-$Hamiltonian graph.

Now we will show that $G + e$ (where e is an edge between any two non$-$adjacent vertices of G) is a Hamiltonian graph. Since all the vertices of B are adjacent to all other vertices of graph G, e must be the edge whose one end vertex is in V_i and other end vertex is in V_j for some $i, j \in \{0, 1, \ldots, k\}, i \neq j$, as each $< V_s >$ is a complete graph, for $s = 0, 1, \ldots, k$.

Suppose $e = w_i w_j$, for some $w_i \in V_i$ and $w_j \in V_j$. Then $C = v_1 - V_0 - v_2 - \ldots - v_{i+1} - V_i\backslash\{w_i\} - w_i - w_j - V_j\backslash\{w_j\} - v_{i+2} - V_{i+1} - \ldots - v_j - V_{j-1} - v_{j+1} - V_{j+1} - v_{j+2} - \ldots - v_k - V_k - v_1$, is a Hamiltonian cycle for $G+e$. Thus G is a maximal non$-$Hamiltonian graph.

Theorem$-9.3.2$ **:** Let G be a maximal non$-$Hamiltonian graph on n vertices discussed in *Theorem*$-9.3.1$. Then $c(G) = G$.

Proof : To construct $c(G)$ we have to join non$-$adjacent vertices u and v of G by an edge e such that $d_G(u) + d_G(v) \geq n$. In G all the vertices of B are adjacent to all other vertices of G. Moreover all the vertices of $V_i, i = 0, 1, \ldots, k$ are adjacent to each other. Thus $u \in V_i$ and $v \in V_j$ for some $i, j \in \{0, 1, \ldots, k\}, i \neq j$.

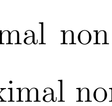
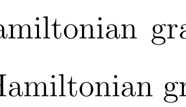
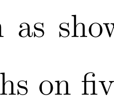
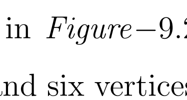
$$\begin{aligned}
\text{Now } d_G(u) + d_G(v) &= k + t_i - 1 + k + t_j - 1 \\
&= 2k + t_i + t_j - 2 \\
&< t_i + t_j + k + k - 1 \\
&< t_i + t_j + k + \sum_{s \in A} 1 \\
&< t_i + t_j + k + \sum_{s \in A} t_s = n, \text{ where } A = \{0, 1, \ldots, k\} \backslash \{i, j\}
\end{aligned}$$

i.e. $d_G(u) + d_G(v) < n$.

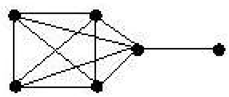
Thus there is no pair of vertices for which $d_G(u) + d_G(v) \geq n$. Therefore one can not add any edge in G to construct $c(G)$. Therefore $c(G) = G$.

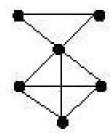
9.3.3 Discussion on maximal non−Hamiltonian graph :

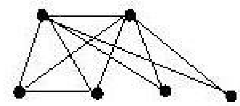
By *Theorem*−9.3.1 one can see that $| B | \leq (n-1)/2$ as $V_0, V_1, \ldots, V_k$ are nonempty sets. Thus one can obtain maximal non−Hamiltonian graphs for various values of n as follows. When $n = 3$, there is only one maximal non−Hamiltonian graph which is P_3. When $n = 4$, then also there is only one maximal non−Hamiltonian graph as shown in *Figure*−9.2. There are three maximal non−Hamiltonian graphs on five and six vertices each, which are shown in *Figures*−9.3 to 9.8

Figure−9.2 *Figure*−9.3 *Figure*−9.4 *Figure*−9.5







Figure−9.6 *Figure*−9.7 *Figure*−9.8

For $n = 7$, $n = 8$ and $n = 11$ one can obtain total number of distinct maximal non−Hamiltonian graphs $P_6^2 + P_5^3 + P_4^4 = 6$, $P_7^2 + P_6^3 + P_5^4 = 7$ and $P_{10}^2 + P_9^3 + P_8^4 + P_7^5 + P_6^6 = 20$ respectively. These are the only maximal non−Hamiltonian graphs for given number of vertices. Thus number of partitions are useful to enumerate the maximal non−Hamiltonian graphs for given number of vertices. Above results and discussion gives following indications which are believed to be true.

Conjecture−9.3.4 **:** Any maximal non−Hamiltonian graph G must be of the type as discussed in *Theorem*−9.3.1.

Conjecture−9.3.5 **:** For any maximal non−Hamiltonian graph G, $c(G) = G$.

Conjecture−9.3.6 **:** The number of maximal non−Hamiltonian graphs on n vertices are precisely $P_{n-1}^2 + P_{n-2}^3 +$ - - - $+ P_{(n/2)+1}^{n/2}$ (For n even) and $P_n^2 + P_{n-2}^3 +$ - - - $+ P_{(n+1)/2}^{(n+1)/2}$ (For n odd).

9.3.7 Algorithm for the construction of maximal non−Hamiltonian graphs :

Input - Vertex set V, with $\mid V \mid = n$.

Step - 1 Choose $B \subset V$ such that $\mid B \mid = k \leq (n-1)/2$.

Step - 2 Choose a partition of integer $n - k$ with $k + 1$ parts say $t_0 + t_1 +$ - - - $+ t_k = n - k$.

Step - 3 Construct complete graphs $< B >$ and $< V_i >$, where $| V_i |= t_i$, $i = 0, 1, \ldots, k$.

Step - 4 Make all the vertices of $V_0 \cup V_1 \cup \ldots \cup V_k$ adjacent to each vertex of B.

Output - $G = \cup < B \cup V_i >$ is a maximal non$-$Hamiltonian graph.

9.3.8 Application of above algorithm :

Input - Let $V = \{v_1, v_2, \ldots, v_{10}\}$.

Step - 1 Choose $B = \{v_1, v_2, v_3\}$, so that $| B |= 3 \leq (10 - 1)/2$.

Step - 2 Choose a partition of integer $10 - 3 = 7$ with four parts e.g. $1 + 1 + 2 + 3$.

Step - 3 Construct complete graphs $< B >$ and $< V_i >$, $i = 0, 1, 2, 3$ and $V_0 = \{v_4\}$,

$V_1 = \{v_5\}$, $V_2 = \{v_6, v_7\}$ and $V_3 = \{v_8, v_9, v_{10}\}$.

Step - 4 Make all the vertices of $V_0 \cup V_1 \cup V_2 \cup V_3$ adjacent to each vertex of $B = \{v_1, v_2, v_3\}$.

Output - The graph shown in following *Figure*-9.9 is maximal non$-$Hamiltonian.

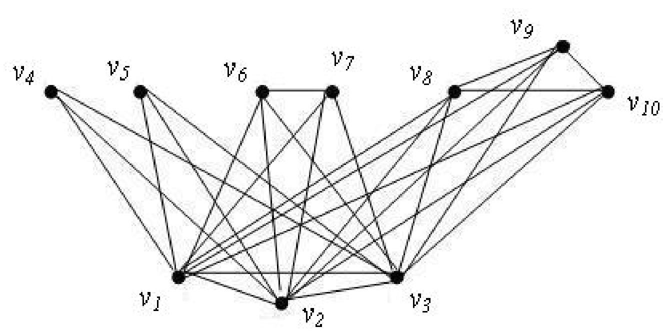


Figure-9.9

Remark$-$9.3.9 **:** There are precisely $P_7^4 = 3$ maximal non$-$Hamiltonian graphs on ten vertices, where $\mid B \mid = 3$.

9.4 Concluding Remarks :

A very general class of maximal non$-$Hamiltonian graphs is investigated. Moreover it is also shown that the closure $c(G)$ of such graph is itself. This work gives powerful indication for the construction of maximal non$-$Hamiltonian graphs as we pose three strong conjectures. In addition to this an algorithm is also developed for the construction of maximal non$-$Hamiltonian graphs. Investigations contained in this chapter are new and we hope this work will set a milestone in the field of Graph Theory.

LIST OF SYMBOLS

$\|B\|$	Cardinality of set B.
CH_n	Closed helm on n vertices.
C_n	Cycle with n vertices.
C_n^*	Star of cycles.
$E(G)$ or E	Edge set of graph G.
F_n	Fan on n vertices.
$\overline{G}$	Complement of G.
$G \cup H$	Union of graphs G and H.
$G \cap H$	Intersection of graphs G and H.
$G \times H$	Cartesian product of graphs G and H.
$G + H$	Join of graphs G and H.
$G \cong H$	G is isomorphic to H.
$G = (V, E)$	A graph G with vertex set V and edge se E.
$G + e$	Super graph of G by adding an edge e in the graph G.
$G + uv$	Super graph of G by adding an edge between vertices u and v in G.
$G + v$	Suspension of graph G and vertex v.
$G * e$	Contraction of edge e in graph G.
$G - e$	Graph G with one edge deleted.
$G - v$	Graph G with one vertex deleted.
H_n	Helm on n vertices.
K_n	Complete graph on n vertices.

$K_{m,n}$	Complete bipartite graph.
$N(v)$	Neighbourhood of vertex v.
P_n	Path graph on n vertices.
P_n^r	Number of partitions of integer n with exactly r parts.
S_n	Shell on n vertices.
T	Tree.
$T(G)$	Spanning tree of graph G.
$< U >$	Induced subgraph of a graph G generated by the vertex set $U \subset V$.
$V(G)$ or V	Vertex set of graphs G.
W_n	Wheel on n vertices.
(a, b)	Greatest Common Divisor of integers a and b.
$c(G)$	The closure graph of G.
$d(v)$	Degree of vertex v.
$d_G(v)$	Degree of vertex v in graph G.
$\triangle(G)$	Maximum degree of a vertex in graph G.
$\delta(G)$	Minimum degree of a vertex in graph G.
$e_f(n)$	Number of edges with edge label n.
${}_nC_r$ or ${}^nC_r$	r Combinations of an n objects.
$\lceil n \rceil$	Least integer not less than real number n (Ceiling of n).
$\lfloor n \rfloor$	Greatest integer not greater than real number n (Floor of n).
(p, q)	A graph with order p and size q.
$v_f(n)$	Number of vertices with vertex label n.
$w(G)$	The number of components in the graph G.

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