A STUDY ON SOME OPERATIONS OF GRAPHS

THESIS

SUBMITTED IN THE FULFILLMENT OF THE REQUIREMENT FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY (MATHEMATICS)

DEPARTMENT OF MATHEMATICS

SAURASHTRA UNIVERSITY ---- RAJKOT

SEPTMBER-2008

Researcher

Paras D. Uchat (M.Sc.)

Lecture

VPMP POLYTECHNIC COLLEGE

GANDHINAGER

Guide (Research Supervisor)

Dr. D. K. Thakkar (M.Sc, PhD)

Professor &Head

DEPARTMENT OF MATHEMATICS

SAURASTRA UNIVERSITY ---- RAJKOT

Field of Research

Graph Theory

Area of Research

Algebraic Graph Theory

Mathematics subject classification AMS (2000): 05Cxx, 05Exx

DEDICATED

To

My Mother

REKHA UCHAT

&

My Father

DINESHCHANDRA UCHAT

Who were my inspiration in this research

STATEMENT UNDER O.PH.D.7 OF SAURASTRA UNIVERSITY, RAJKOT.

DECLRATION

I hereby declare that

a) the research work embodied in this thesis on **A STUDY ON SOME OPERATIONS OF GRAPHS** submitted for PhD degree has not been submitted for my other degree of this or any other university on any previous occasion.

b) to the best of my knowledge no work of this type has been reported on the above subject. Since I have discovered new relations of facts, this work can be considered to be contributory to the advancement of knowledge on Graph Theory.

c) all the work presented in the thesis is original and wherever references have been made to the work of others, it has been clearly indicated as such.

(Countersigned by the Guiding Teacher)

Signature of Research Student

Date:

Date:

CERTIFICATE OF APPROVAL

This thesis, directed by the Candidate's guide, has been accepted by the Department of Mathematics, Saurastra University, Rajkot, in the fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY
(MATHEMATICS)

Title : **A STUDY ON SOME OPERATIONS OF GRAPHS**

Candidate: **Paras Dineshchandra Uchat**

Guide
(Dr. D K. Thakkar)

Dr. D K Thakkar
Professor and Head
Department of Mathematics
Saurashtra University
RAJKOT-360001

Date :

Date:

Acknowledgement

I am fortune enough to have an opportunity to work under the able guidance of **Dr. D K Thakkar** (Professor and Head, Department of Mathematics, Saurastra University, Rajkot) for having extended valuable guidance, various reference books and encouragement during and prior to the period of present investigation in spite of his busy schedule. It is due to his kind and scientific direction that the work has taken the present shape. Without his help, this study could not have been possible. I, therefore, owe and enormous debt to him, not for his timely valuable guidance and parental care but for his deep insight and critical outlook with positive attitude. He always guided and suggested to me to mold this thesis in purely logical and mathematical way.

How can I forget the prime motivation of my academic Zeal? I express my deep sense of gratitude to my beloved **parents** who are my founding bricks of my whole education. Without their blessing these task would not have been accomplished. I bow my head with complete dedication at their feet.

I express my deep scene of gratitude from the bottom of my heart to my wife **Vaishali** who had not only taken my all social responsibility during this study, but also helped me in type setting of this thesis. She had also looked after of my lovely son **Bhavya.**

I will never forget love and affection from my sisters **Shripa Parekh** and **Dr. Purvi Modi** for their moral support and inspiration during my course of study.

I convey my heartfelt gratitude to my Parents-in-law **Mr**. **Indubhai Mehta** and **Mrs. Hansaben Mehta** for their constant support.

I extend my thanks to my Brother-in-laws **Mr. Amit Parekh** and **Dr. Kunal Modi** for their kindness and warmly help.

Lastly, I express my sincere gratitude to the **family members** of my guide for providing homely atmosphere.

SANWATSARI

4th September, 2008

INDEX

“*In most sciences one generation tears down what another has built and what one has established another undoes, in Mathematics along each generation built a new story to the old structure”. HERMAN HANKEL*

Review of Related Literature

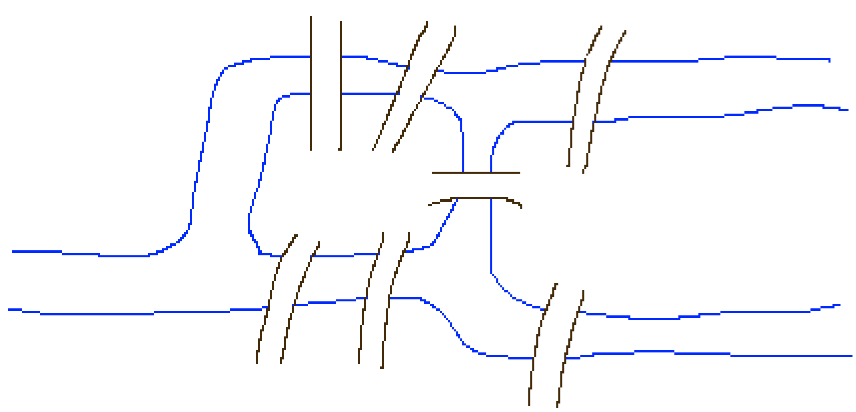
CHAPTER 1

1. A. Historical Status

Graph Theory

It has become necessity to mention that there are applications of graph theory to some areas of Physics, Chemistry, Communication Science, Computer Technology, Electrical Engineering, Architecture, Operation Research, Economics, Sociology, Psychology, and Genetics. The theory is also intimately related to many branches of mathematics, including Group theory, Matrix theory, Numerical analysis, Probability, Topology, and Combinatorics. The fact is that graph theory serves as a mathematical model for any system involving a binary relation. Partly because of their diagrammatic representation, graphs have an intuitive and authentic appeal.

The origin of graph theory can be traced back to Euler's work on the Konigsberg bridges problem (1735), which subsequently led to the concept of an Eulerian graph. Konigsberg was a city in Prussia situated on the Pregel River, which served as the residence of the dukes of Prussia in the 16th century. (Today, the city is named Kaliningrad, and is a major industrial and commercial center of western Russia.) The river Pregel flowed through the town, creating an island, as in the following picture. Seven bridges spanned the various branches of the river, as shown.



A famous problem concerning Konigsberg was whether it was possible to take a walk through the town in such a way as to cross over every bridge once, and only once.

This problem was first solved by the prolific Swiss mathematician Leonhard Euler (pronounced "Oiler"), who invented the branch of mathematics now known as *graph theory* in the process of his solution.

It is no coincidence that graph theory has been independently discovered many times. The study of cycles on polyhedra by the Thomas P. Kirkman (1806 - 95) and William R. Hamilton (1805-65) led to the concept of a Hamiltonian graph.

The concept of a tree, a connected graph without cycles, appeared implicitly in the work of Gustav Kirchhoff (1824-87), who employed graph-theoretical ideas in the calculation of currents in electrical networks or circuits. Later, Arthur Clayey (1821-95), James J. Sylvester (1806-97), George Polya(1887-1985), and others use 'tree' to enumerate chemical molecules.

The study of planar graphs originated in two recreational problems involving the complete graph K_5 and the complete bipartite graph $K_{3,3}$. These graphs proved to be planarity, as was subsequently demonstrated by Kuratowski. First problem was presented by A. F. Mobius around the year 1840 as follows

Once upon a time, there was a king with five sons.
In his will he stated that after his death the sons
should divide the kingdom into five provinces so
that the boundary of each province should have
a frontiers line in common with each of the other
four provinces.

Here the problem is whether one can draw five mutually neighboring regions in the plane.

The king further stated that all five brothers should join the provincial capital by roads so that no two roads intersect.

Here the problem is that deciding whether the graph K_5 is planar.

The origin of second problem is unknown but it is first mentioned by H. Dudeney in 1913 in its present form.

The puzzle is to lay a water, gas, and electricity to each of the three houses without any pipe crossing another.

This problem is that of deciding whether the graph $K_{3,3}$ is planar.

The celebrated four-color problem was first posed by Francis Guthrie in 1852. and a celebrated incorrect "proof" by appeared in 1879 by Alfred B. Kempe. It was proved by Kenneth Appel and Wolfgang Haken in 1976 and a simpler and more systematic proof was produced by Neil Roberton, Daniel Sanders, Paul Seymour, and Robin Thomas in 1994.

Graph Theory was born to study problems of this type. For much more on the history of graph theory, I suggest the book **Graph Theory 1736-1936**, by N.L. Biggs, E.K. Lloyd and R.J. Wilson, Clarenden Press 1986.

Algebraic Graph Theory

Algebraic graph theory is a branch of mathematics in which algebraic methods are applied to problems about graphs. Nowadays it became an established discipline within the field of Graph Theory. In one sense, algebraic graph theory studies graphs in connection with linear algebra. Especially, it studies the spectrum of the adjacency matrix, the Kirchhoff matrix, or the Laplacian matrix of a graph. This part of algebraic graph theory is also called spectral graph theory.

In another sense, algebraic graph theory studies graphs in connection to group theory, particularly automorphism groups and geometric group theory. In the latter, the endomorphism monoid of a graph plays an important role. The focus is placed on various families of symmetric graphs, such as vertex-transitive graphs, edge-transitive graphs, arc-transitive graphs, distance-transitive graphs, clayey graphs, etc.

The aim is to translate properties of graphs into algebraic properties and then, using the results and methods of algebra, to deduce theorem about graphs. The literature of this area itself has grown enormously since 1974. Algebraic Graph Theory uses different branches of algebra to explore various aspects of Graph Theory. These areas have links with other areas of mathematics, such as harmonic analysis, computer networks where symmetry is an important feature.

Homomorphism of Graphs

The study of graph homomorphism is over thirty years old. It was pioneered by G.Sabidussi, Z.hedrlin & A.pulter.Orignally it had come from Algebra. This concept intersects many areas of Graph Theory. The usefulness of the homomorphism perspective in the areas such as Graph reconstruction, Products, Applications in Complexity Theory, Artificial Intelligence & Telecommunication. Recently, graph homomorphism has been found useful to model configurations in Statically Physics. The notion of homomorphism comes in handy in many applications in graph theory and theoretical computer sciences.

At the same time; the homomorphism framework strengthens the link between graph theory and other parts of mathematics, making graph theory more attractive and understandable, to other mathematician. The term homomorphism was initially also used for minors.

A homomorphism between two graphs is an edge preserving mapping of their vertex sets. The theory of homomorphism can be viewed as a generalization of the theorem of graph coloring, as a *m*- coloring of a graph G is exactly a homomorphism of G to the complete graph K_m . It is useful to extend the standard notion of homomorphism to directed graph as well.

Note that homomorphism between two graphs is not as informative as an isomorphism between them and this lack of perfect information is useful in many situations.

Homomorphism provides algebraic treatment to graph theoretic problems. Graph Theorist mainly uses it to characterize the important concepts and to translate the parameters from one graph to another. For that reason it becomes very popular among the researchers. Thus I was affected under the same influence.

1. B. Fundamentals of Graph Theory

In this section I would like to mention some of the basic concepts of graph theory which we had used during our study. Definitions of some of the key words have also been given again in between the chapter whenever they are required.

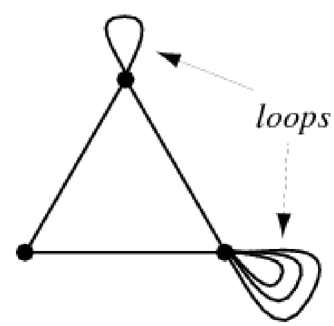
Definition1.1: A **graph** or **undirected graph** G is an ordered pair $G = (V,E)$ that is subject to the following conditions:

- V is a set, whose elements are called **vertices** or **nodes**,
- E is a set of pairs (unordered) of distinct vertices, called **edges** or **lines**.

The vertices belonging to an edge are called the **ends**, **endpoints**, or **end vertices** of the edge. *V* is usually taken to be nonempty set, the **order** of a graph is $|V|$ (the number of vertices). A graph's **size** is $|E|$, the number of edges. The edge set *E* induces a symmetric binary relation ~ on *V* that is called the **adjacency** relation of *G*. Specifically, for each edge $e = \{u, v\}$ the vertices *u* and *v* are said to be **adjacent** to one another, which is denoted as $u \sim v$ & the vertex *u* (or *v*) and edge *e* are said to be **incident** with each other. For an edge $\{u, v\}$, graph theorists usually use the somewhat shorter notation *uv*. The edges *e* & *m* are said to be **adjacent** if they have a vertex in common.

Definition1.2: **Multiple edges** (or parallel edges) are two or more edges connecting the same two vertices in the graph.

Definition1.3: A degenerate edge of a graph which joins a vertex to itself, also called a **self-loop**.



Definition1.4: A graph without multiple edges and self loops is known as **simple graph**.

Definition1.5: A graph whose vertex set V and edge set E are finite then it is known as **finite graph** otherwise it is known as **infinite**.

Definition1.6: A graph without any edges is known as **null graph.**

Definition1.7: The **degree** deg (*v*), of a vertex *v* is the number of edges with which it is incident.

Definition1.8: An **isolated point** of a graph is a vertex of degree 0.

Definition1.9: A **pendent vertex** of a graph is a vertex of degree 1.

Definition1.10: A **walk** is an alternating sequence of vertices and edges, with each edge being incident to the vertices immediately preceding and succeeding it in the sequence. A **trail** is a walk with no repeated edges. A walk is **closed** if the initial vertex is also the terminal vertex, otherwise it is known as **open.**

Definition1.11: A **cycle** is a closed trail with at least one edge and with no repeated vertices except that the initial vertex is the terminal vertex.

A cycle graph C_n, sometimes simply known as a cycle is a graph on n vertices containing a single cycle through all vertices. In other words the graph C_n having vertices 0,1,2,…,n -1 and edges 01,12,23,…, (n-1)1.Cycle graphs can be generated using cycle.

Definition1.12: A graph without cycle is known as **forest.**

Definition1.13: A **path** is a walk with no repeated vertices.

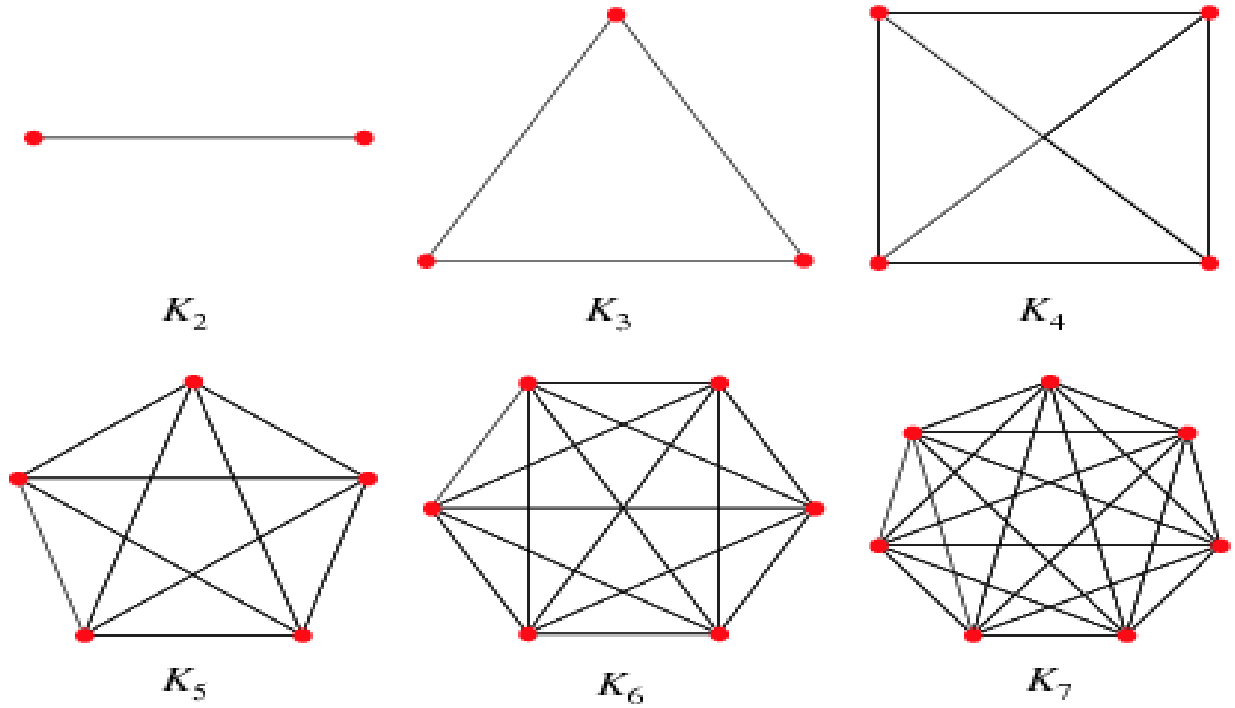
The path graph P_n is a tree with two end vertices having degree 1, and the other n -2 vertices have degree 2. In other words the graph P_n having vertices 0,1,2,…,n and edges 01,12,23,…,(n - 1) n.

The **length** of a walk is the number of edges in the sequence defining the walk. Thus, the length of a path or cycle is also the number of edges in the path or cycle.

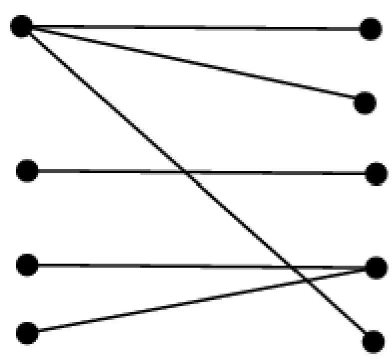
Definition1.14: A graph is **connected** if there is a path between any two distinct vertices; otherwise it is referred as **disconnected.** A connected graph without any cycle is known as **tree.**

Definition1.15: A graph is known as ***k*-regular** if each of its vertex having degree ***k*.**

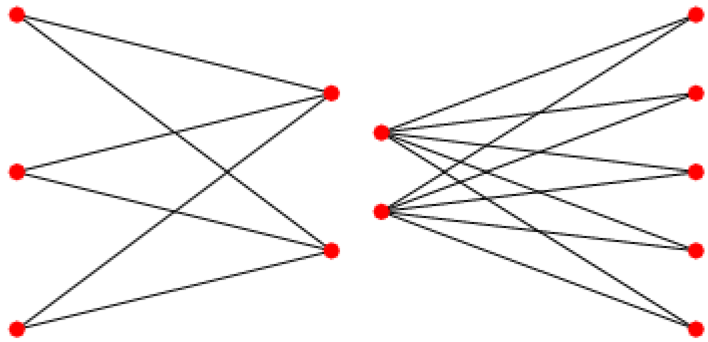
Definition1.16: A **complete graph** is a graph in which each pair of graph vertices is connected by an edge. The complete graph with n vertices is denoted $\mathbf{K}_n$ and has $\frac{n(n-1)}{2}$ edges. In literature, complete graphs are also sometimes called **universal** graphs.



Definition1.17: A **bipartite graph**, also called a bi graph, is a set of graph vertices decomposed into two disjoint sets such that no two graph vertices within the same set are adjacent



Definition1.18: A **complete bipartite graph** is a bipartite graph such that every pair of vertices in the two sets are adjacent. If there are ***p*** and ***q*** graph vertices in the two sets, the complete bipartite graph (sometimes also called a complete bi graph) is denoted $\mathbf{K}_{p,q}$. The figures given below show $K_{3,2}$ **&** $K_{2,5}$.



Definition1.19: The **open neighborhood** of the vertex *v* is the set of vertices which are adjacent to *v*. It is denoted by N (*v*), the degree of a vertex is also the cardinality of its open neighborhood set.

Definition1.20: The **closed neighborhood** of the vertex *v* is the set of vertices which are adjacent to *v* including vertex *v*. It is denoted by N [*v*].
Thus N [*v*] = N (*v*) ∪ {*v*}.

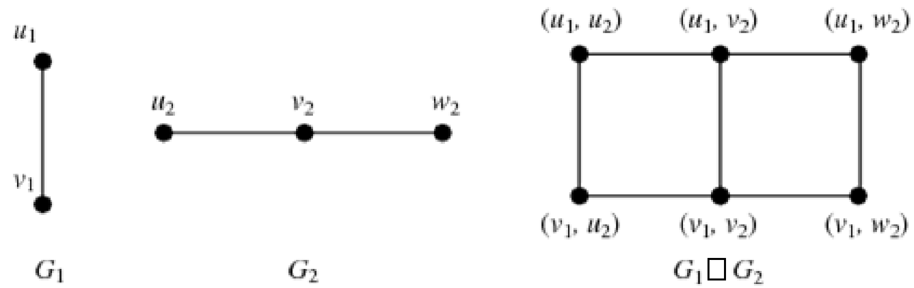
Definition1.21: For any graphs G & H, if $V(H) \subseteq V(G)$ & $E(H) \subseteq E(G)$ and the assignment of end points to edges in H is the same as in G, then graph H is known as **sub graph** of graph G (or G contains H) & conversely graph G is known as **super graph** of H (or H is contained in G). In the case of $V(H) = V(G)$, graph H is known as **spanning sub graph** of graph G. The maximal connected sub graph of a graph G is known as **Component**. A spanning sub graph which is tree is known as **spanning tree**.

Definition1.22: For a set of vertices *X*, we use **G[X]** to denote the **induced sub graph** of *G* whose vertex set is *X* and whose edge set is the subset of *E(G)* consisting of those edges with both ends in *X*.

Definition1.23: The **distance** $d_G(u, v)$ between two vertices *u* and *v* of a finite graph G is the minimum length of the paths connecting them (i.e., the length of a graph geodesic). If no such path exists (i.e., if the vertices lie in different connected components), then the distance is equal to ∞. In an undirected graph, this is obviously a metric.

Definition1.24: An **independent set** of a graph G is a subset of the vertices such that no two distinct vertices in the subset are adjacent in G. The independence number α (G) of a graph is the cardinality of the largest independent set. An independent set is said to be **maximal** if no other vertex can be added without destroying its independence property.

Definition1.25:



The **Cartesian graph product** G = $G_1 \square G_2$, is the graph product of graphs G_1 and G_2 with vertex sets V_1 and V_2 and edge sets X_1 and X_2 , is the graph with vertex set $V_1 \times V_2$ & $u = (u_1, u_2)$ adjacent with $v = (v_1, v_2)$ whenever (i) $u_1 = v_1$ and u_2 is adjacent to v_2 -or- (ii) u_1 is adjacent to v_1 and $u_2 = v_2$.

Definition1.26: The **Categorical graph product** G = $G_1 \times G_2$,(it is also known as **weak graph product ,tensor graph product**) is the graph product of graphs G_1 and G_2 with vertex sets V_1 and V_2 and edge sets X_1 and X_2 , is the graph with vertex set $V_1 \times V_2$ & $u = (u_1, u_2)$ adjacent with $v = (v_1, v_2)$ whenever u_1 is adjacent to v_1 and u_2 is adjacent to v_2.

Definition1.27: The **Strong graph product** G = $G_1 \boxtimes G_2$ is the graph product of two graphs G_1 and G_2 has vertex set $V(G_1) \times V(G_2)$ and two distinct vertices (u_1, u_2) & (v_1, v_2) are adjacent if and only if whenever (i) $u_1 = v_1$ and u_2 is adjacent to v_2 -or- (ii) u_1 is adjacent to v_1 and $u_2 = v_2$ -or- (iii) u_1 is adjacent to v_1 and u_2 is adjacent to v_2.

Definition1.28: The **Lexicographic product** G = $G_1 \circ G_2$,(it is also known as **Composition of graphs**) is the graph product of graphs G_1 and G_2 with vertex sets V_1 and V_2 and edge sets X_1 and X_2 , is the graph with vertex set $V_1 \times V_2$ & $u = (u_1, u_2)$ adjacent with $v = (v_1, v_2)$ whenever (i) $u_1 = v_1$ and u_2 is adjacent to v_2 -or- (ii) u_1 is adjacent to v_1 .

Definition1.29: The **Join** of simple graphs G & H, written G +H, is the graph obtained from the disjoint union of their vertex sets by adding the edges $\{ xy : x \in V(G), y \in V(H)\}$. The join graph $C_n + K_1$ is known as **wheel graph**.

Definition1.30: Let v, k, i be fixed positive integers with $v \geq k \geq i$; let Ω be a fixed set of size v; **Jonson graph** $J(v, k, i)$ define as follows. The vertices of

J (v, k, i) are the subsets with size k, where two subsets are adjacent if there intersection has size i. Therefore J (v, k, i) has $\binom{v}{k}$ vertices, and it is a regular graph with valency $\binom{k}{i}\binom{v-k}{k-i}$.

Definition1.31: Let G be the graph, then **Line graph** of graph G is denoted by L (G) and defined as follows: V (L (G)) = E (G) and two edges are adjacent if and only if they are adjacent in G *i.e.* they have a common vertex.

Definition1.32: The set S ⊂ V (G) is said to be **vertex cover** if it contains at least one vertex of every edge of the graph.

Definition1.33: A subset S of V (G) is said to be a **dominating set** if for each vertex v, $v \in$ S or v is adjacent to some vertex of S.A dominating set S is said to be a **minimal dominating set** if no proper subset of S is a dominating set. The **domination number** γ(G) of graph G is
γ(G) = minimum{ ∣ S ∣ : S is a minimal dominating set}

Definition1.34: A subset S of V (G) is said to be a **totally dominating set** if every vertex x in V (G) is adjacent to some vertex y of S. Obviously every totally dominating set is a dominating set.

Definition1.35: A subset S of V (G) is said to be a **clique** in G, if whenever x and y are distinct vertices in S, they are adjacent.(Used by many author to mean complete graph). Maximum order of a clique in graph G, ω (G) is known as **clique number.**

Definition1.36: The **complement** of simple graph G, denoted as $\overline{\mathrm{G}}$, with same vertex set as G, defined by $uv \in \mathrm{E}(\bar{G})$ if and only if $uv \notin \mathrm{E}\ (\mathrm{G})$.

Definition1.37: For A sub set S ⊂ V(G), $v \in$ S is said to be **enclave** of S if N[v] ⊆ S, and $v \in$ S is an **isolate** of S if N(v) ⊆ V(G) –S.A set is said to be **enclave less** if it does not contain any **enclave**.

Definition1.38: If G and H are graphs and f : V (G) → V (H) is a mapping then f is said to be a **homomorphism** of graphs G & H ,if whenever x and y are adjacent in G, f (x) and f (y) are adjacent in H.

Definition1.39: A mapping $f : V(G) \rightarrow V(H)$ is said to be **strong homomorphism (Full homomorphism)** of graphs G & H, if x is adjacent to y in graph G if and only if $f(x)$ is adjacent to $f(y)$ in graph H.

Definition1.40: A mapping $f : V(G) \rightarrow V(H)$ is said to be **isomorphism** of graphs G & H, if (i) f is bijective & (ii) f is strong homomorphism. In this case graphs G & H are said to be isomorphic to each other and it is denoted by $G \cong H$.

Definition1.41: The **Map Graph** H^G of two graphs G and H as follows: the vertex set of the graph is the set of all functions from V (G) to V (H) & two elements f and g are adjacent if whenever u and v are adjacent vertices of G, $f(u)$ and $g(v)$ are adjacent vertices of H.

Let f be a homomorphism from graphs X to Y. If g is a function from V(Y) to V(F) (for any graph F), then the composition $g \circ f$ is a function from V(X) to V(F). Hence f^* determines a map from the vertices of F^Y to F^X, which we call **adjoint map** to *f*.

Definition1.42: A sub graph H of graph G is said to be **retract** of G, if there is a homomorphism f from G onto H such that $f(x) = x$ for all x in H. The map f is called **retraction**.

Definition1.43: Let G be a graph & for the graph H consider a partition $V_1, V_2 \ldots. V_n$ of the vertex set V(G) as the vertex set of H, & two sub sets V_i & V_j are adjacent if for some $x \in V_i$ & $y \in V_j$ such that x and y are adjacent in G, then graph H is known as **quotient graph** of G.

1. C. Review of related past results

In this section I would like to present the list of all past researches (results) which I had used partially or directly to prove the results of present work.

Result 1.1: (Degree sum formula or Handshaking lemma) for any graph G, sum of all the degree of its vertices is equal to twice the total number of its edges.

Result 1.2: The composition of two homomorphisms, if possible (*i.e.* if the co domain of the first homomorphism is the domain of the second one), is also a homomorphism.

Result 1.3: A function f : V (G) → V(H) is a strong homomorphism and g : V (H) → V(K) is any map . Then g : V (H) → V (K) is homomorphism if and only if $g \circ f$: V(G) → V(K) is homomorphism.

Result 1.4: For any graphs G & H ,every homomorphism f : V (G) → V(H) can be written as $f = i \circ s$ where s is surjective homomorphism and i is injective homomorphism.(With suitable domain & co-domain)

Result 1.5: For any graphs G & H, a bijective homomorphism f : V (G) → V(G) is isomorphism.

Result 1.6: For any graphs G & A, a function f : V (G) → V(A) is a homomorphism and H is a sub graph of G, then the restriction map $f_{/H}$: V(H) → V (A) is also a homomorphism.

Result 1.7: A mapping f : V (P_n) → V (G) is a homomorphism if and only if the sequence $f(0), f(1), f(2)\ldots f(n)$ is a walk in G.

Result 1.8: A mapping f : V (C_n) → V (G) is a homomorphism if and only if the sequence $f(0), f(1), f(2)\ldots f(n)$ is a closed walk in G.

Result 1.9: There exist a homomorphism f : V (C_{2k+1}) → V (C_{2m+1}) if and only if $m \leq k$.

Result 1.10: For any connected graphs G &H, a mapping f : V (G) → V(H) is a homomorphism, then ,$d_H(f(u), f(v)) \leq d_G(u, v)$, $\forall\, u, v$ in G .

Result 1.11: A mapping f : V (G) → V(H) is a homomorphism, then $\chi(G) \leq \chi(H)$, where $\chi(G)$ is the chromatic number of graph G.

Result 1.12: Every quotient of graph G is a homomorphic image of G, and conversely every homomorphic image of G is isomorphic to a quotient of G.

Result 1.13: Let G , H and Z be any graphs. A function f: V(Z) → V(G×H) is a homomorphism if and only if projection maps $\pi_1 \circ f$: V(Z) → V(G) and $\pi_2 \circ f$: V(Z) → V(H) are homomorphism.

Result 1.14: For any graphs G,H and Z , if a mapping f : V (G) → V(H) is a homomorphism, then the mapping f^* : V (Z^H) → V(Z^G) is also a homomorphism, where f^* is adjoint map to f

Result 1.15: For any graphs G,H and Z, $Z^{G+H} \cong Z^G \times Z^H$.

Result 1.16: For any graphs G,H and Z, $Z^{G \times H} \cong (Z^G)^H$.

Result 1.17: For any graphs G,H and Z, $G \times Z \to H$ if and only if $Z \to H^G$.

Result 1.18: The second projection $\pi_2 : V(G \times H^G) \to V(H)$ is a homomorphism.

Result 1.19: Let $G = G_1 \boxtimes G_2 \ldots \boxtimes G_n$ be the strong product of connected graphs G_i. Then $d_G(u,v) = \max_{1 \le i \le n} d_{G_i}(u_i, v_i)$ for $i = \{1,2,\ldots,n\}$

Result 1.20: For any graphs G & H, the weak product G × H is connected if and only if both G and H are connected and at least one of them is non bipartite.

Result 1.21: For any graphs G & H, the strong product $G \boxtimes H$ as well as weak product G × H are associative, commutative & well defined. In addition strong product has unit element K_1 while weak product does not have any unit.

Result 1.22: Let $G = G_1 \boxtimes G_2 \ldots \boxtimes G_n$ be the strong product, then G is connected graph if and only if each factor is connected.

Result 1.23: Every monoid is isomorphic to the endomorphism monoid of a suitable graph G.

“The mathematical experience if the student is incomplete if he never had the opportunity to solve a problem invented by himself. “G.POLYA

Introduction to the Study

CHAPTER 2

2. A. About the present work

Theme

The present work is functioning into three phases which are divided into remaining four chapters. I have proved several varieties of results in each phase.

In the first phase I have introduced some variants of homomorphism and derived the results involving

(i) Their basic properties
(ii) Their relations
(iii) Their limitations
(iv) Effect of some graph operations or graph parameters under them

These results are covered in Chapter 3 and partly Chapter 5.

In the second phase I have introduced some versions of function graphs and derived the results involving

(i) Their basic properties
(ii) Their relations
(iii) Graph identities
(iv) Effect of different variants of homomorphism under them
(v) Validity of different graph parameters under them

These results are covered in Chapter 4 and partly in Chapter 5.

The third phase reflects algebraic aspects of graph theory. I have introduced a modified form of retract of a graph, called quasi retract of a graph. I have also proved the results which involved some basic algebraic concepts (like semi group, monoid) with different variant of homomorphism.

These results are covered in Chapter 6.

Features of the presentation

Examples: I have given and illustrated sufficient number of examples of each new term or key concept.

Counter Examples: I have provided counter example(s) whenever they are needed.

Labeling: Figure of the graph in this study has vertex which is represented by small darken circle and an edge between two vertices is represented by a line or an arc. Generally numbers or alphabets (small) have been used as labels for vertices. However there is no label(s) for the edges in the figure.

In the case of mapping, the vertex(s) of co-domain graph labeled by the set '{}' which contain the label(s) of domain graph. This set represents the pre-image(s) (or fibers) of the vertex in co-domain graph.

Tester: It is one of the interesting features in the present study. On my initial days I have often used graph laboratory for testing the result (s). Tester is just modified form of it. Reader may get satisfaction of the results from the tester.

Key words and symbols: For the purpose of quick reference, the list of all key words and symbols are given in appendix A and appendix B respectively at the end.

Reference: Reference (s) has been shown in square box [ ]. The number in the box represents the number of the references book which is listed at the end. Note that some of the references mentioned in the thesis may not indicate the original source (s).

Type setting: Text document –MS Word 2007, Figures – Paint brush,
Font size- 14 pt, Line spacing -single, Font type –Times New Roman,
Page size-A4 legal, Mathematical font – equation editor software.

2. B. Preliminaries

I have tried to use symbols & notations from standard books & well known reference(s). Notations and symbols of the familiar concepts of graph theory are as usual. However I have used some new notations or symbols which are listed in Appendix - B.

If G is a graph, then V (G) denotes the vertex set & E (G) denotes edge set of the graph G. Let $v \sim u$ denote that vertex *v* is adjacent to vertex *u*.
Let 'd_G (*u*, *v*)' denote the distance between two vertices *u* & *v* in the graph G. If G and H are graphs then G × H will denote the weak product graph, G □H will denote the Cartesian product graph, G ⊠ H will denote strong product graph and G + H will denote join of graphs. Let G ∗ H denote a graph whose vertex set is V (G) × V (H). N[*v*] will denote the closed neighborhood & N (*v*) will denote open

neighborhood of vertex *v* in the graph. Let L (G) denote the line graph of a given graph G. Let $\overline{G}$ denote the complement of graph G. Let 'deg (*v*)' denotes the degree of vertex *v* in the graph. Let $J(v, k, i)$ denote Jonson graph. Let $G \rightarrow H$ denote there is a homomorphism from graph G to H and $G \nrightarrow H$ for not. Let H^G denote the map graph of graphs G & H. The domination number of graph G is denoted by $\gamma(G)$.

2 .C. Abstract of the content

In this section I would like to mention the central result of remaining each chapter. You can get the reference of new term or key worlds which are used in the following results from Appendix –A.

Chapter-3: **Homomorphism of Graphs & its Variants**

In this chapter, I defined some variants of a homomorphism of graphs. Some of the results of this chapter are listed below:

Theorem 3.19: A onto function $f : V(G) \rightarrow V(H)$ is homomorphism if and only if f is a quasi- homomorphism & $f^{-1}(y)$ is an independent set for each y in V(H).

Theorem 3.22: A function $f : V(G) \rightarrow V(H)$ is a homomorphism if and only if $f : V(\overline{G}) \rightarrow V(\overline{H})$ is a quasi complementary homomorphism.

Theorem 3.**33**:A mapping $f : V(G) \rightarrow V(H)$ is quasi homomorphism if and only if $d_H(f(u), f(v)) \leq d_G(u, v)$,$\forall (u, v)$ in G .

Theorem 3.38:A function $f : V(G) \rightarrow V(H)$ is a complementary homomorphism if and only if for every vertex $v \in V(G)$, $f(N[v]) \supset N[f(v)]$.

Chapter-4: **Function Graphs**

In this chapter I have defined some of the versions of function graphs. Some of the results of this chapter are listed below:

Theorem 4.3: Let $|G| = n$. Then for any $f \in V(D(H^G))$, Deg (f) = $\left[\prod_{i=1}^{n} N(f(u_i))\right]$

Theorem 4.7: Let G and H are two graphs with $|G| = n$.
Then Q $(H^G) \cong H^n$ = H ⊠ ... ⊠ H (n times).

Corollary 4.7.2: Let graph Q (H^G) is connected, then for any u_i in graph G
$d_{Q(H^G)}(f, g) = \max_{1 \le i \le n} d_H(f(u_i), g(u_i))$ for $i = \{1,2,\ldots,n\}$

Theorem 4.9: For any graphs X , Y, Z , Q $(Z^{X * Y}) \cong$ Q (Q $(Z^Y)^X) \cong$ Q (Q $(Z^X)^Y)$

Theorem 4.14: Let G, H & Z be any graphs.
Then $D((G \times H)^Z) \cong D(G^Z) \times D(H^Z)$

Theorem 4.19: A Graph G is connected if and only if P $(H^G) \cong$ H.

Chapter 5-**Resultes Involving Parameters**

In this chapter the results which I have included, show the effect of graph parameters under the different (1) variants of a homomorphism (2) versions of function graphs.

The parameters which I have considered for the present study are given below:
(1) Independent set (2) Maximal independent set (3) Dominating set (4) Minimal dominating set (5) Totally dominating set (6) Vertex cover (7) Enclave
(8) Distance (9) Clique. Some of the results of this chapter are listed below:

Theorem 5.1: If f : V (G) → V (H) is an onto quasi-homomorphism then f(S) is a dominating set in H, whenever S is a dominating set in G. Hence
$\gamma(H) \le \gamma(G)$.Converse is not true.

Theorem 5.5: A function f : V (G) → V(H) is an onto strong quasi-homomorphism. Then $S \subset$ V (H) is dominating set if and only if f^{-1}(S) is dominating set in G.

Theorem 5.11: A sub set K ⊂ V(H) is a dominating set if and only if the set
$Q_K = \{f :$ Range of $f \subset K\}$is dominating set in Q(H^G).

Theorem 5.13:A sub set T ⊂ V (H) is a independent set if and only if the set
$Q_T = \{f :$ Range of $f \subset T\}$ is independent set in Q (H^G).

Chapter 6-**Some algebraic properties**

In this chapter I have tested some algebraic concepts (like semi group, monoid) for different variants of homomorphism. I have also defined quasi retract. Some of the results of this chapter are listed below:

Theorem 6.4: If H is quasi retract of G, then there is a spanning sub graph G_1 of G such that H is retract of G_1.

Theorem 6.6: If H_1 is retract of graph H then Q (H_1^G) is retract of Q (H^G).

Theorem 6.9: {Com (G, G), ∘} is monoid.

Theorem 6.13: Every monoid is isomorphic to monoid {Q (G, G), ∘} of a suitable graph G.

Theorem 6.15: P (G, G) is an ideal in semi-group {Q (G, G), ∘}

2. D. Conventions

Unless mention otherwise the following assumption are considered in this thesis

1. All the graphs are assumed to be simple, finite and undirected.
2. All graphs are assumed to be without isolated vertices and having at least an edge.
3. Definition of any symbol has to be consider as per the given symbol index in this thesis.
4. A mapping between two graphs means a mapping between the vertex sets of two graphs.

"Mathematics is the art of saying many things in many different ways."
- *MAXWELL.*

Homomorphism of Graphs & its Variants

CHAPTER 3

3. A. Introduction

In this chapter I am introducing some variants of homomorphism of graphs. Latter on, they play the key role in this present research work. Some basic proprieties and the results involving the relations between them have been derived. Composition and Decomposition of these variants are also proved. The comparative study of these variants of homomorphism is also presented in this chapter.

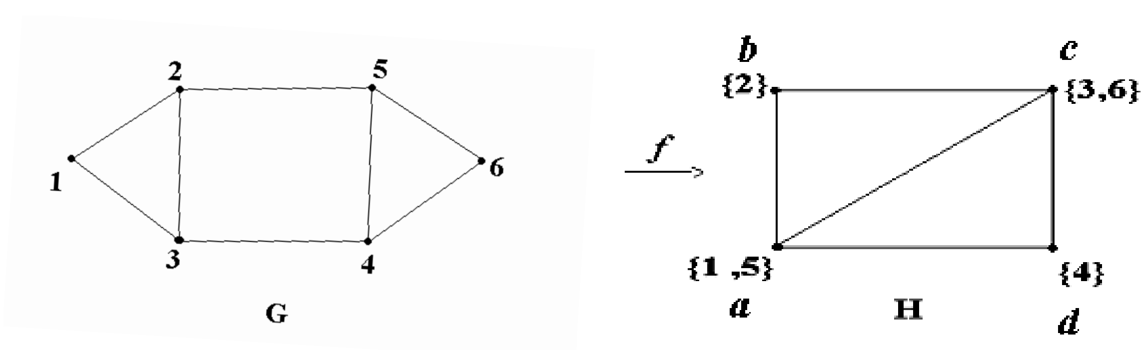
Variant means the form or the version which varies from the original. I am modifying the concept of homomorphism in order to define its variants between the vertex sets of the graphs. But first of all we need to understand the concept of *Homomorphism* of graphs.

Definition 3.1:
A mapping $f : V(G) \to V(H)$ is said to be a *homomorphism* **[18],** if whenever $x \sim y$ in G then $f(x) \sim f(y)$ in graph H.

Adjacency is one of the structural features of the graph and this mapping preserves it from one graph to another. As a result we get an important medium between two graphs.

If there is a homomorphism $f : V(G) \to V(H)$ then we shall write it '$G \to H$' and '$G \nrightarrow H$' when no such homomorphism exists.
If $f : V(G) \to V(H)$ is any mapping then the pre image $f^{-1}(y)$ of each vertex y in H is called a *fibers* of *f.* Clearly each fiber of a homomorphism is an independent set.

Example 3.1:
(1) $P_n \to C_n$
(2)The following mapping from graph G onto graph H is a homomorphism.



(3) $K_m \nrightarrow K_n$ ($m > n$)

Now let's move toward the first variant of homomorphism named '*Quasi Homomorphism of Graphs*'

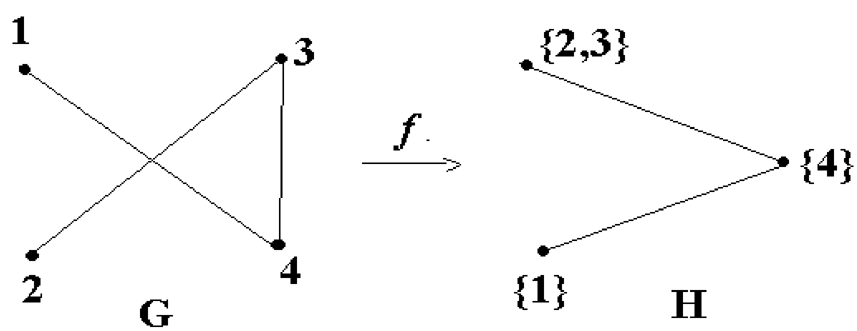
Motivation: The idea of extending the horizon of homomorphism gives rise to this new concept.

Intuition about the word Quasi: The word *'Quasi'* is coming from Latin. Dictionary meaning of the word is 'Seemingly or almost'. The property which I am adding to form the new variant of homomorphism is quite similar to this.

Definition 3.2:
A mapping $f : V(G) \rightarrow V(H)$ is said to be a *quasi Homomorphism* , if whenever $x \sim y$ in G then $f(x) \sim f(y)$ *or* $f(x) = f(y)$ in graph H.

Quasi Homomorphism provides more freedom to the adjacent vertices of the domain. In other word, the restriction on the fibers of *f* is removed in compare to homomorphism of graphs. We will discuss this in latter section of this chapter.

Example 3.2:
(1) The mapping $f : V(K_{m,n}) \rightarrow V(K_2)$ is quasi homomorphism if all *m* and *n* vertices of $K_{m,n}$ goes to different vertices of K_2 .
(2) The following mapping from graph G onto graph H is quasi homomorphism, but not homomorphism.



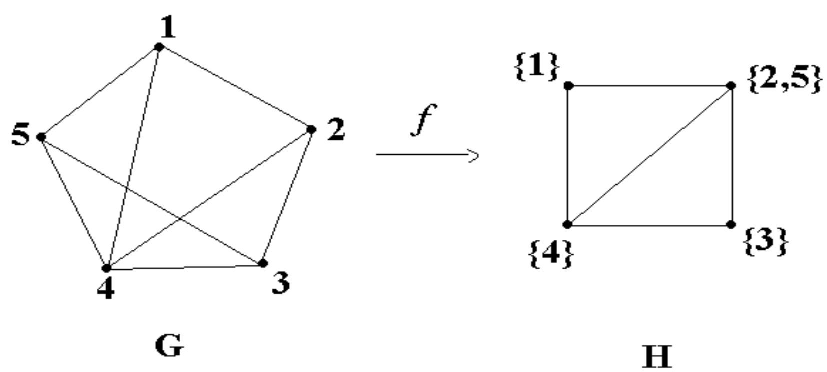
(3) Projection map $\pi_2 : V(G \circ H) \rightarrow V(H)$ is not a quasi-homomorphism. (Lexicographic product)

Existence: Existence of Homomorphism is one of the interesting questions for the graph theorist. In several case there may not exist homomorphism between graphs. But there exists a quasi homomorphism between any two graphs. It is straight forward from its definition so that one can freely study any property of one graph to another graph with help of this medium.

The next variant of homomorphism is '*Strong Homomorphism of Graphs*' which is also referred as *full homomorphism* by some of the graph theorist. It was as old as homomorphism.

Definition 3.3:
A mapping $f : V(G) \to V(H)$ is said to be *strong homomorphism* **[17]**of graphs, whenever $x \sim y$ in graph G if and only if $f(x) \sim f(y)$ in graph H.

Example 3.3:
(1) There is an injective strong homomorphism between P_n to C_{n+1}.
(2) There is no strong homomorphism from K_n to C_n $(n \geq 4)$
Justification: If there is a strong homomorphism from K_n to C_n then it must be a bijection. Therefore there are at least as many edges in C_n as there are edges in K_n . This is not true.
(3) The following mapping from graph G onto graph H is strong homomorphism.



The next on the line is '*Complementary Homomorphism of Graphs*'. This is also an interesting homomorphism. The word itself declares the characteristic of the concept.

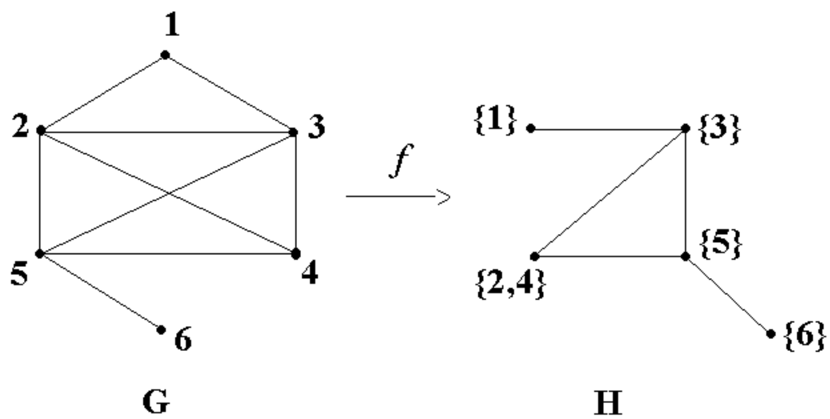
Definition 3.4:
A mapping $f : V(G) \to V(H)$ is said to be *complementary homomorphism* of graphs, if whenever $f(x) \sim f(y)$ in H then $x \sim y$ in graph G.

It is clear that the definition of this mapping put some condition on non adjacency of the domain graph. Actually we are breaking the definition of strong homomorphism into two parts (1) homomorphism (2) complementary homomorphism.

Example 3.4:

(1) There is an injective complementary homomorphism between C_n to P_n

(2) The following mapping from graph G onto graph H is complementary homomorphism.



(3) There is no complementary homomorphism between any graph (which is not complete) to a complete graph.

<u>Justification</u>: Any injective function from non complete graph to complete graph with at least two vertices is not complementary homomorphism because if $u \nsim v$ in G then $f(u) \sim f(v)$ in H.

Now the next variant of homomorphism is '*Pure Quasi Homomorphism of Graphs*'.

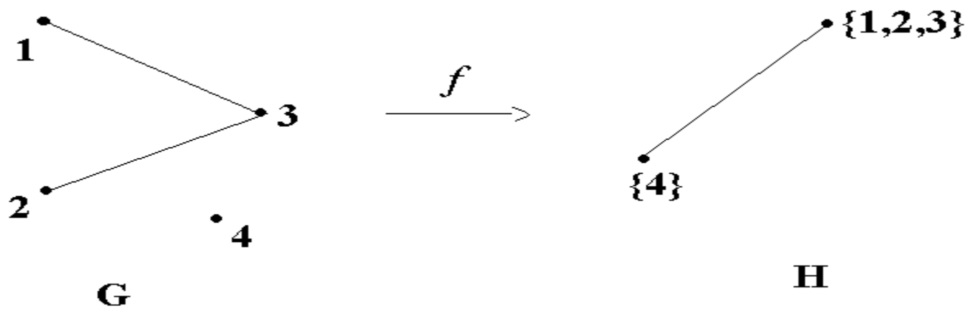
Definition 3.5:

A mapping $f : V(G) \rightarrow V(H)$ is said to be *pure quasi homomorphism* of graphs, if whenever $x \sim y$ in G then $f(x) = f(y)$ in graph H.

Example 3.5:

(1) Any constant function is pure quasi homomorphism.

(2) The following mapping from graph G onto graph H is pure quasi homomorphism.



(3) There is no onto pure quasi homomorphism between two connected graphs (more than two vertices) [Refer theorem 3.15]

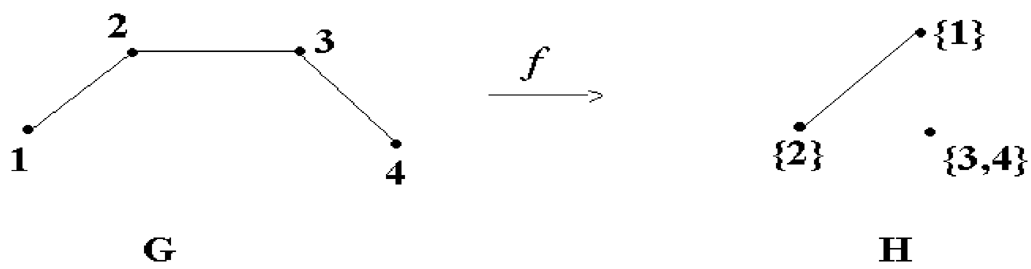
Motivation: Naturally the idea was coming from quasi homomorphism. Removing the adjacency from the co domain.

Another flavor of homomorphism is '*Quasi Complementary Homomorphism of Graphs*', which is the modified form of complementary homomorphism.

Definition 3.6:
A mapping $f : V(G) \rightarrow V(H)$ is said to be *quasi complementary homomorphism* of graphs, for any two distinct vertices x & y in G, if whenever $f(x) \sim f(y)$ *or* $f(x) = f(y)$ in H then $x \sim y$ in graph G.

Clearly fiber of vertices of co-domain graph under this mapping have strong characteristic than complementary homomorphism because non adjacency of domain graph strictly goes to non adjacency of co-domain graph under this mapping.

Example 3.6:
(1) There is an injective Quasi complementary homomorphism between C_n to P_n
(2) The following mapping from graph G onto graph H is quasi complementary homomorphism.

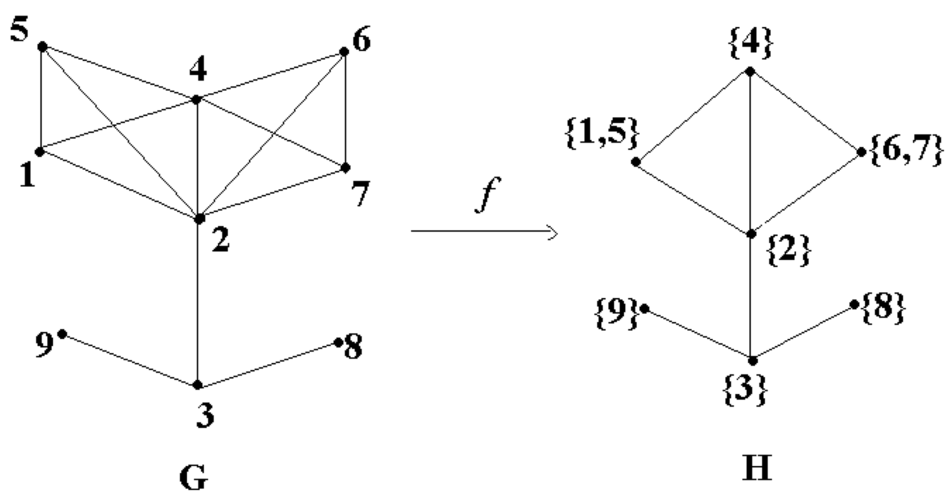


(3)There is no Quasi Complementary homomorphism between null graphs to any connected graph more then two vertices.

Last variant of homomorphism is '*Strong Quasi Homomorphism of Graphs*' which make complete, mirror symmetry of this family of morphism.

Definition 3.7:
A mapping f : V (G) $\rightarrow$ V(H) is said to be *strong quasi homomorphism* of graphs, if (1) it is quasi homomorphism (2) it is quasi complementary homomorphism.

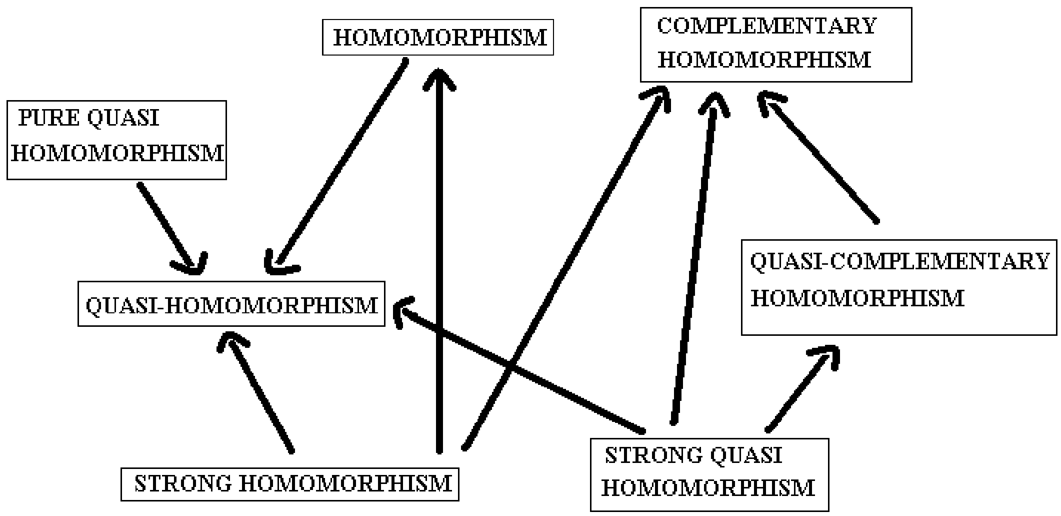
Example 3.7:
(1) There exists a strong quasi homomorphism from K_4 to K_2.
(2) The following mapping from graph G onto graph H is strong quasi homomorphism.



It is clear that non-adjacency of the graph is the first priority under this mapping while adjacency is secondary.

3. B. Empirical relation between the variants of homomorphism

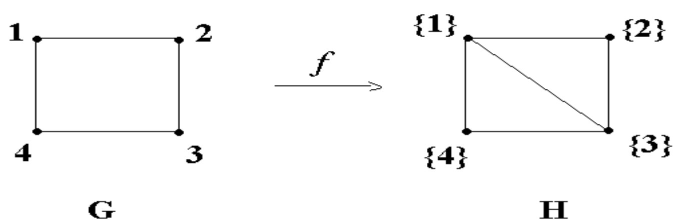
After studying these variants we can observe the empirical relation between them. *e.g.* "Every homomorphism is quasi – homomorphism."



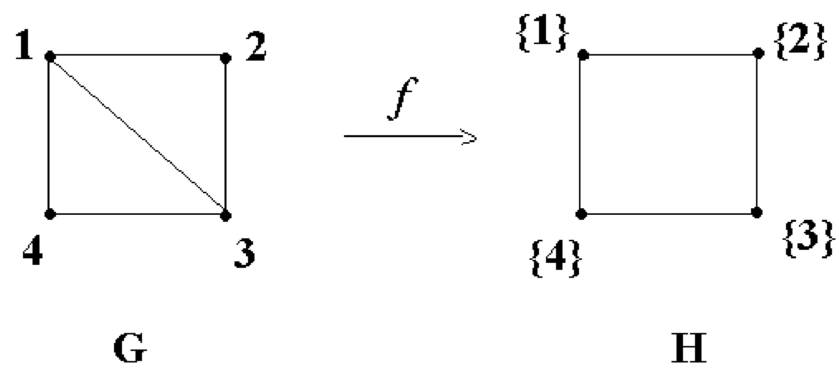
3. C. Counter examples

Example 3.8:
Any pure quasi homomorphism is a quasi homomorphism but it is not homomorphism as well as strong homomorphism.

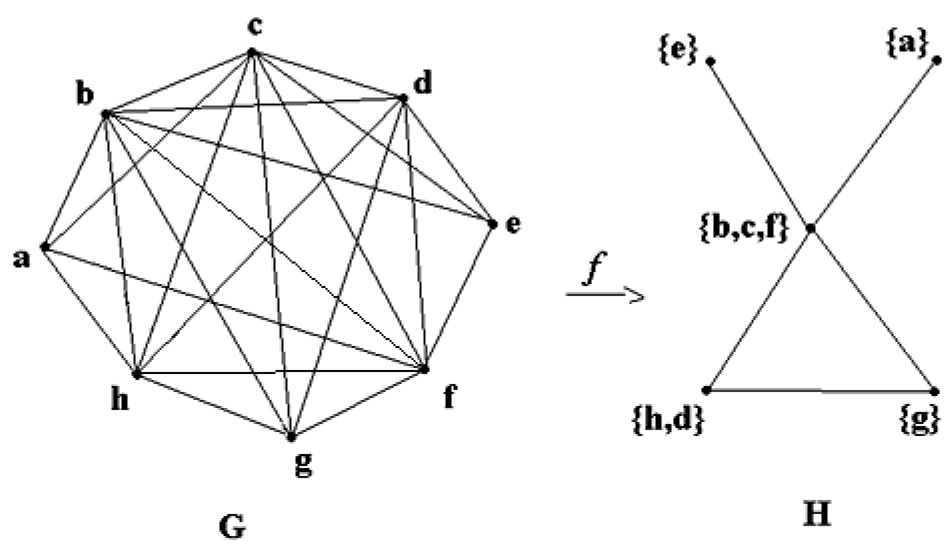
Example 3.9:
The following mapping $f : V(G) \rightarrow V(H)$ is homomorphism (or quasi homomorphism) but it is not(i) complementary homomorphism or (ii) quasi complementary homomorphism or (iii) strong homomorphism or (iv) strong quasi homomorphism or (v) pure quasi homomorphism.



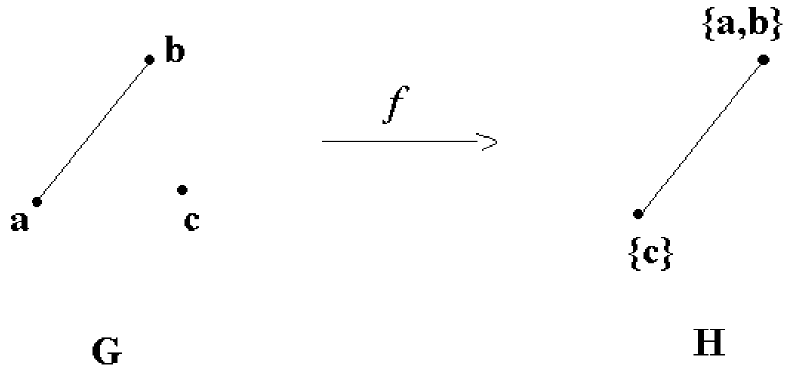
Example 3.10:
The following mapping f : V (G) $\rightarrow$ V(H) is complementary homomorphism but it is not (i)homomorphism or (ii) quasi homomorphism or (iii)strong quasi homomorphism or (iv)pure quasi homomorphism (v) strong homomorphism.



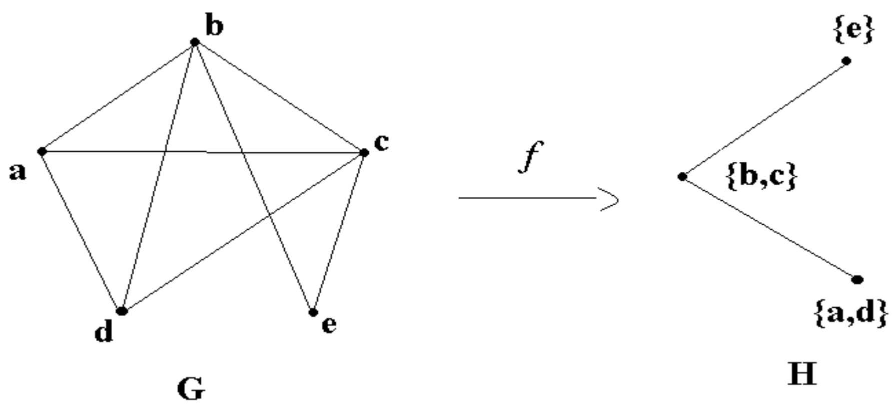
Example 3.11:
The following mapping f : V (G) $\rightarrow$ V(H) is quasi complementary homomorphism but is not (i) homomorphism or (ii) quasi homomorphism or (iii) strong homomorphism or (iv)strong quasi homomorphism or (v) pure quasi homomorphism.



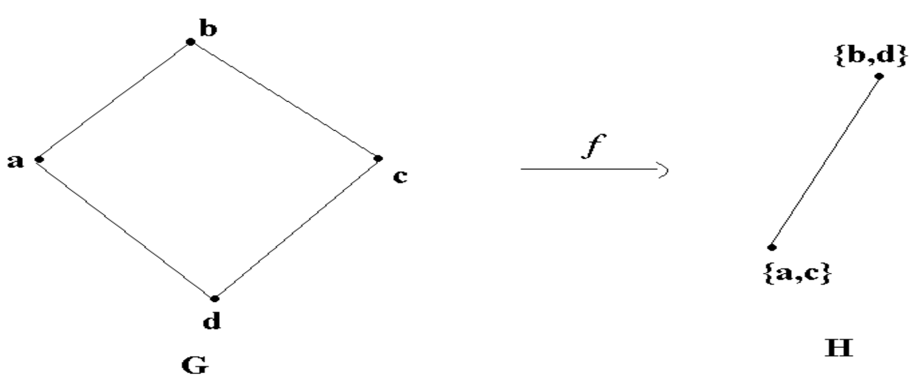
Example 3.12:
The following mapping f : V (G) $\rightarrow$ V(H) is pure quasi homomorphism but it is not (i) homomorphism or (ii) complementary homomorphism or (iii) quasi complementary homomorphism or (iv) strong homomorphism or (v) strong quasi homomorphism.



Example 3.13:
The following mapping f : V (G) → V(H) is strong quasi homomorphism but is not (i) homomorphism or (ii) strong homomorphism or (iii) pure quasi homomorphism.



Example 3.14:
The following mapping f : V (G) → V(H) is strong homomorphism but is not (i) quasi complementary homomorphism or (ii) strong quasi homomorphism or (iii) pure quasi homomorphism.



3. D. Abstract structural impact of graphs under the different variants of homomorphism

No.	Variant of homomorphism	Structural property	
		of domain vertices	of co- domain vertices
1	Homomorphism	Adjacency	Adjacency
2.	Quasi homomorphism	Adjacency	Adjacency -or –equal
3.	Strong homomorphism	Adjacency	Adjacency
		Non Adjacency	Non Adjacency –or- equal
4.	Pure quasi homomorphism	Adjacency(If domain is connected)	Singleton
5.	Complementary homomorphism	Non Adjacency	Non Adjacency –or- equal
6.	Quasi complementary homomorphism	Non Adjacency	Non Adjacency
7.	Strong quasi homomorphism	Adjacency	Adjacency -or- equal
		Non Adjacency	Non Adjacency

Table 3.1

To understand these variants of homomorphism we need to do the groundwork for them. So, I think composition of any two variants is one of the best options.

3. E. Composition of variants of homomorphism

Let f : V (G) $\rightarrow$ V(H) and g : V (H) $\rightarrow$ V(K) are two functions. One of the compositions of two variants has been proved here.

Theorem 3.1:
Composition of any strong quasi homomorphism f : V (G) $\rightarrow$ V(H) and any strong homomorphism g : V(H) $\rightarrow$ V(K) is quasi homomorphism as well as complementary homomorphism (consider Range of f = domain of g)

Proof:
Let f : V (G) $\rightarrow$ V(H) be a strong quasi homomorphism and g : V(H) $\rightarrow$ V(K) be a strong homomorphism.
(i)Let $x \sim y$ in G.

$\Rightarrow f(x) \sim f(y)$ or $f(x) = f(y)$ in H ,since f is a quasi homomorphism.
$\Rightarrow$If $f(x) = f(y)$ in H, then $g(f(x)) = g(f(y))$ in K
& if $f(x) \sim f(y)$ in H, then $g(f(x)) \sim g(f(y))$ in K , since g is a homomorphism.
Thus the mapping $g \circ f$: V(G) $\rightarrow$ V(H) is a quasi homomorphism.
(ii) Let $g(f(x)) \sim g(f(y))$ in K.
$\Rightarrow f(x) \sim f(y)$ in H ,since g is a complementary homomorphism.
$\Rightarrow x \sim y$ in G, since f is a quasi complementary homomorphism & $f(x) \neq f(y)$ in H. Thus the mapping $g \circ f$: V(G) $\rightarrow$ V(H) is a quasi complementary homomorphism. □

Note: Let G be any set & H is a graph. For any mapping f : G $\rightarrow$ V(H) whenever $f(x) \sim f(y)$ or $f(x) = f(y)$ in H then $x \sim y$ in G. Then G is known as weak graph induced by mapping f.

Theorem 3.2:
Suppose G is a weak graph induced by the function f : G $\rightarrow$ V(H). Then a function h : V (G) $\rightarrow$ V(H) is a quasi- homomorphism if and only if $h \circ f$: V (G) $\rightarrow$ V(H) is a quasi- homomorphism.

Proof:
Note that f : V (G) $\rightarrow$ V(H) is a quasi homomorphism. If h is also quasi homomorphism, then obviously $h \circ f$ is quasi homomorphism.[Refer table 3.2]
Conversely suppose $h \circ f$ is quasi homomorphism.
Let $x \sim y$ in G. Then $f(x) \sim f(y)$ or $f(x) = f(y)$ in H.
Since $h \circ f$ is quasi homomorphism, $g(f(x)) = g(f(y))$ or $g(f(x)) \sim g(f(y))$ in K. □

Remark:
(1) In general composition of a quasi homomorphism and a quasi complementary homomorphism may not be quasi homomorphism. [Refer counter example B]
(2) It is clear that composition of pure quasi homomorphism with any other mapping (or variant of homomorphism) is pure quasi homomorphism.
<u>Justification</u>:
Let $x \sim y$ in G.
$\Rightarrow f(x) = f(y)$ in H. Since f is a pure quasi homomorphism.
$\Rightarrow g(f(x)) = g(f(y))$ in K, since g is any map. Thus it is pure quasi homomorphism.(What about converse !!)

The remaining composition between different variant of homomorphism are shown in the following Table-3.2.

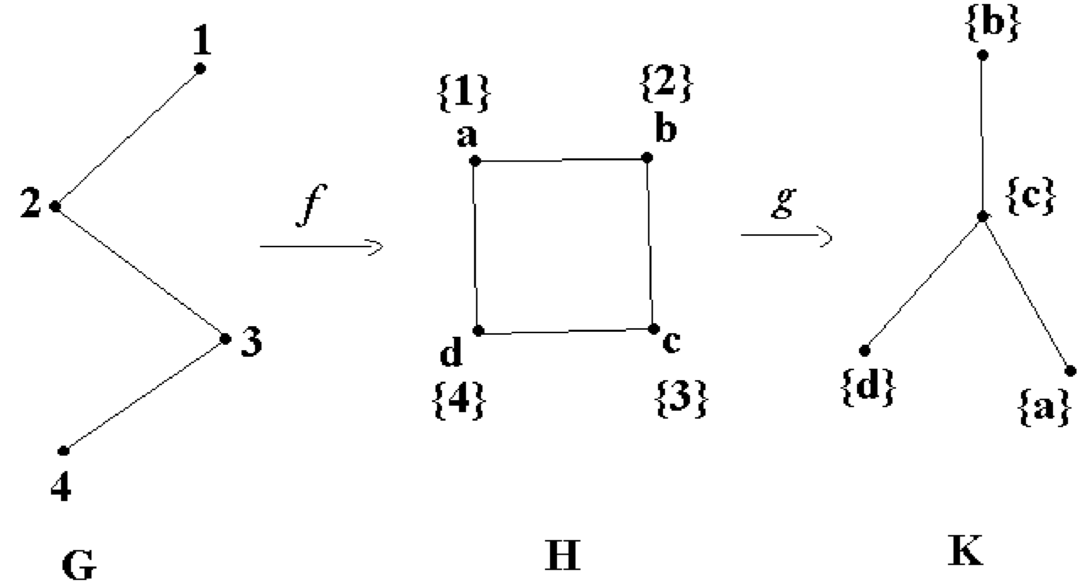
Table-3.2:

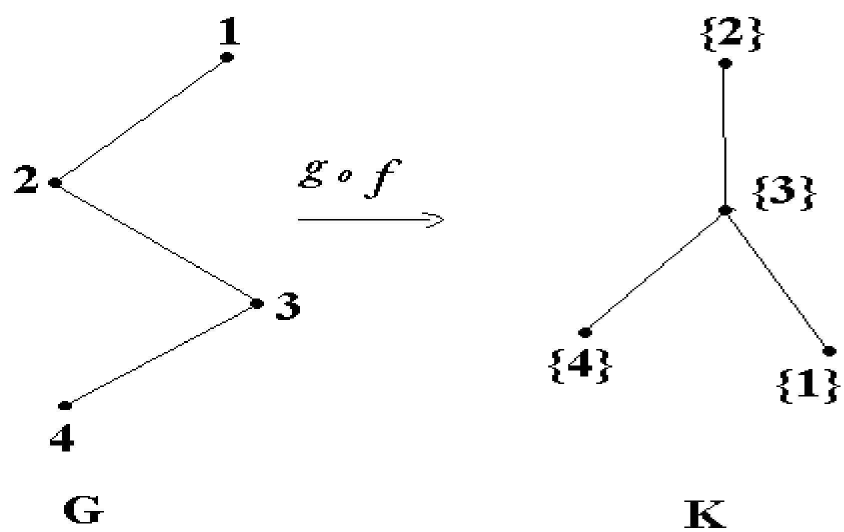
COMPOSITION BETWEEN VARIANTS OF HOMOMORPHISM

$f \rightarrow$ $g \downarrow$ $g \circ f$ ↘	Homomorphism	Quasi Homomorphism	Strong Homomorphism	Pure quasi Homomorphism	Complementary Homomorphism	Quasi complementary Homomorphism	Strong Quasi Homomorphism
Homomorphism	Homomorphism	Quasi Homomorphism	Homomorphism	Pure Quasi Homomorphism	(Example B)	(Example B)	Quasi homomorphism (Example C)
Quasi Homomorphism	Quasi Homomorphism	Quasi homomorphism	Quasi homomorphism	Pure quasi homomorphism	(Example B)	(Example B)	Quasi homomorphism (Example C)
Strong Homomorphism	Homomorphism	Quasi homomorphism	Strong homomorphism	Pure Quasi homomorphism	Complementary homomorphism (Example G)	Complementary homomorphism (Example G)	Complementary and Quasi homomorphism
Pure quasi Homomorphism	Pure Quasi homomorphism	Pure Quasi homomorphism	Pure Quasi homomorphism	Pure Quasi homomorphism	(Example E)	(Example E)	Pure Quasi homomorphism
Complementary Homomorphism	(Example A)	(Example A)	Complementary homomorphism (Example F)	Pure Quasi homomorphism	Complementary homomorphism	Complementary homomorphism	Complementary homomorphism (Example H)
Quasi complementary Homomorphism	(Example A)	(Example A)	Complementary homomorphism (Example F)	Pure Quasi homomorphism	Complementary homomorphism	Quasi complementary homomorphism	Quasi complementary homomorphism
Strong Quasi Homomorphism	Quasi homomorphism (Example D)	Quasi homomorphism (Example D)	Complementary and quasi homomorphism	Pure Quasi homomorphism	Complementary homomorphism (Example I)	Quasi complementary homomorphism	Strong quasi homomorphism

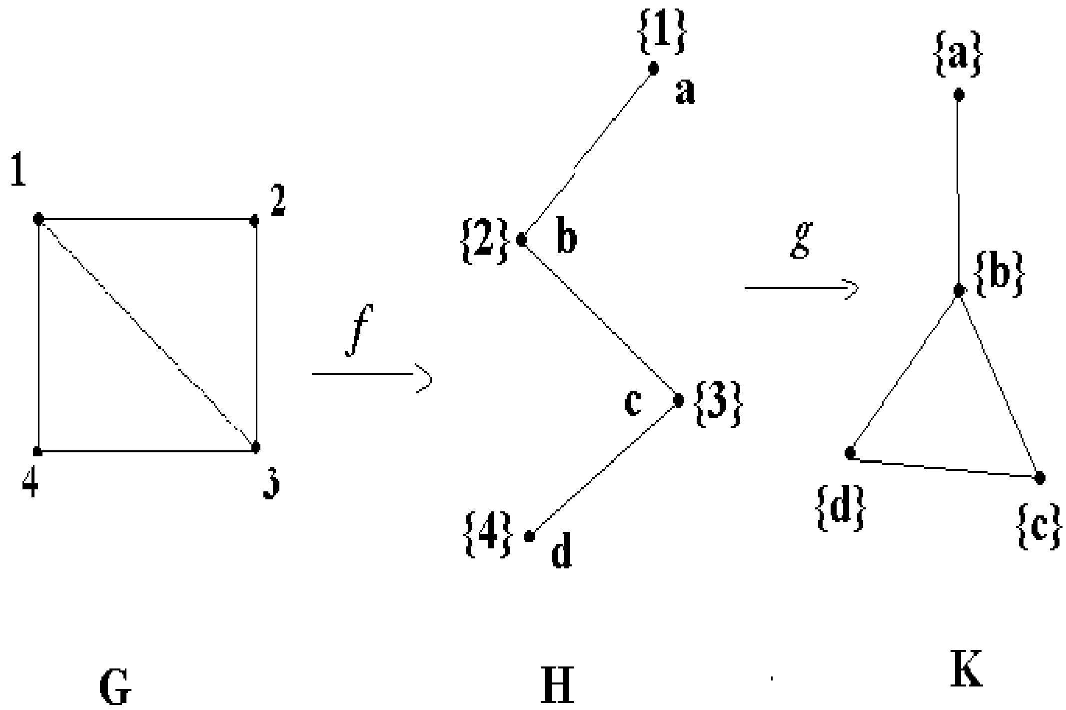
Examples & Counter Examples

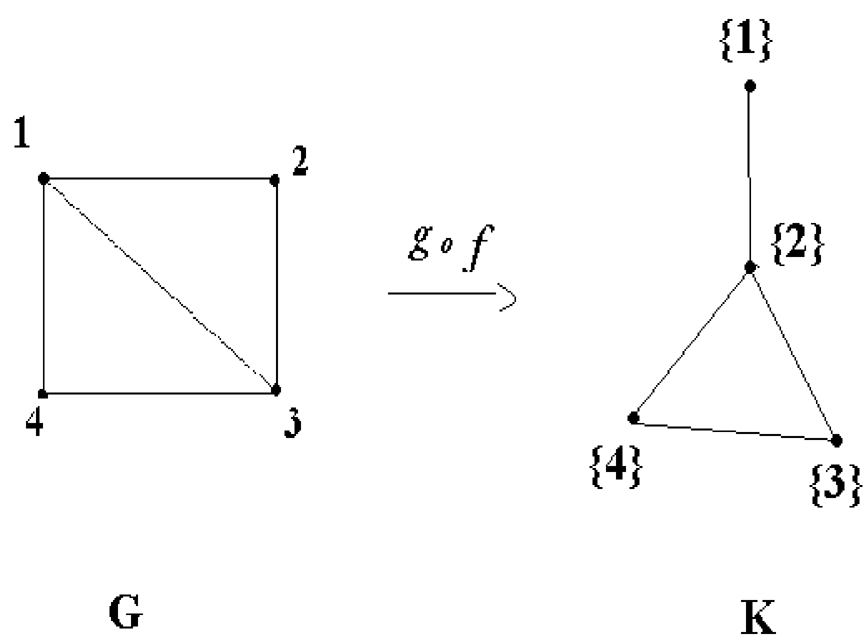
Example A



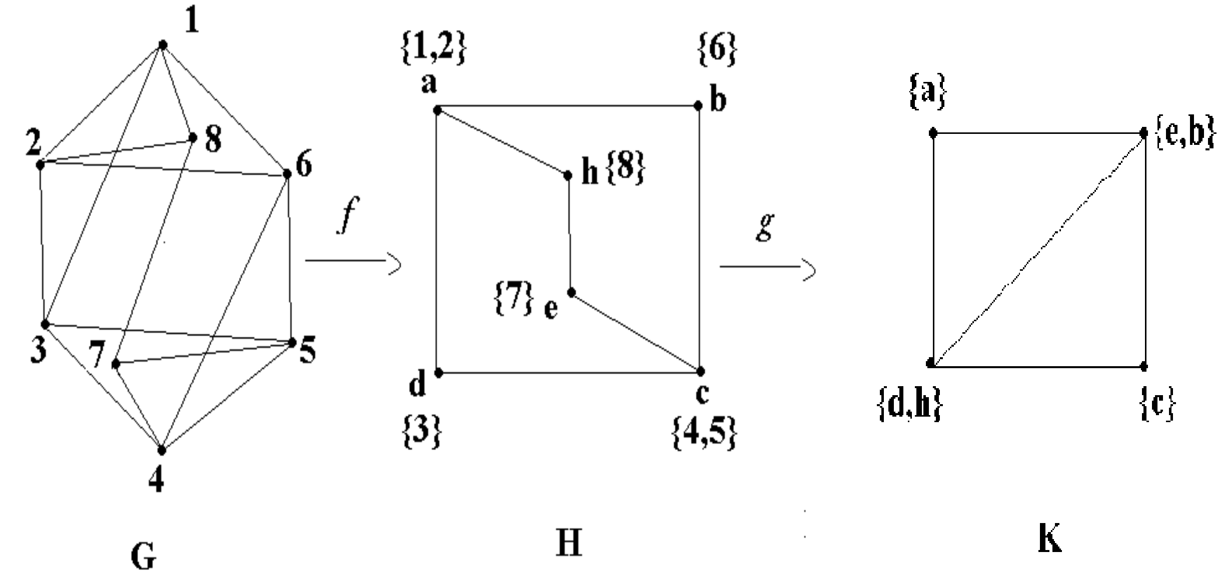


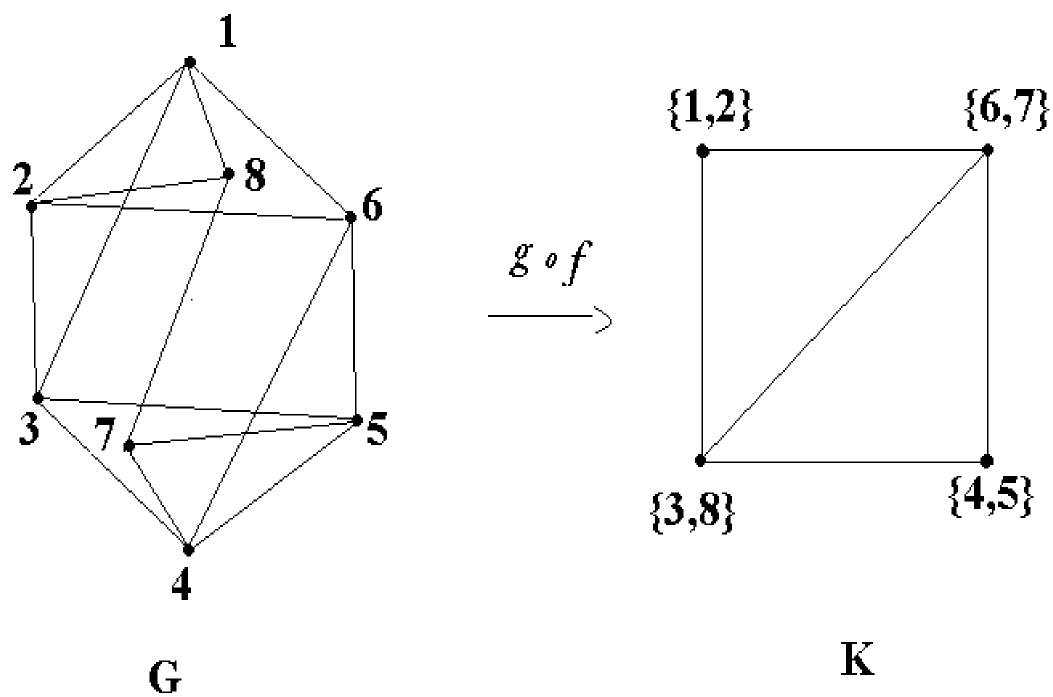
Example B



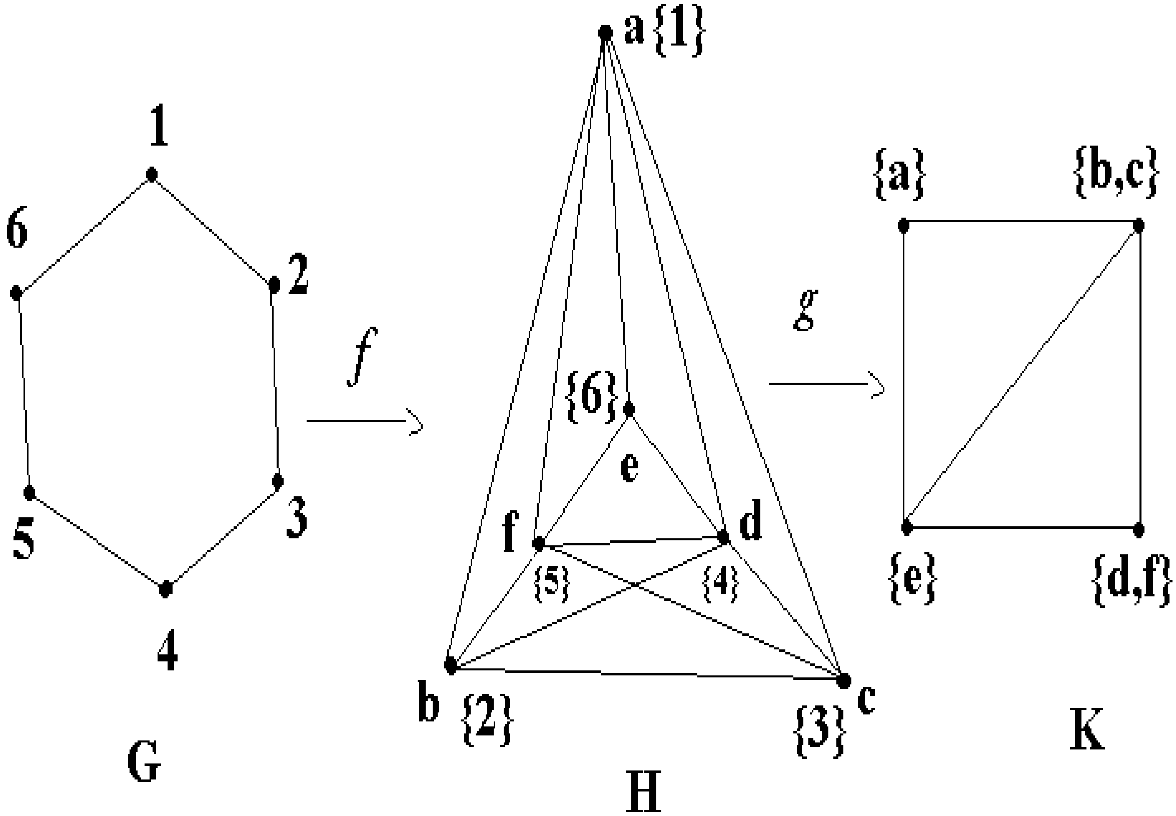


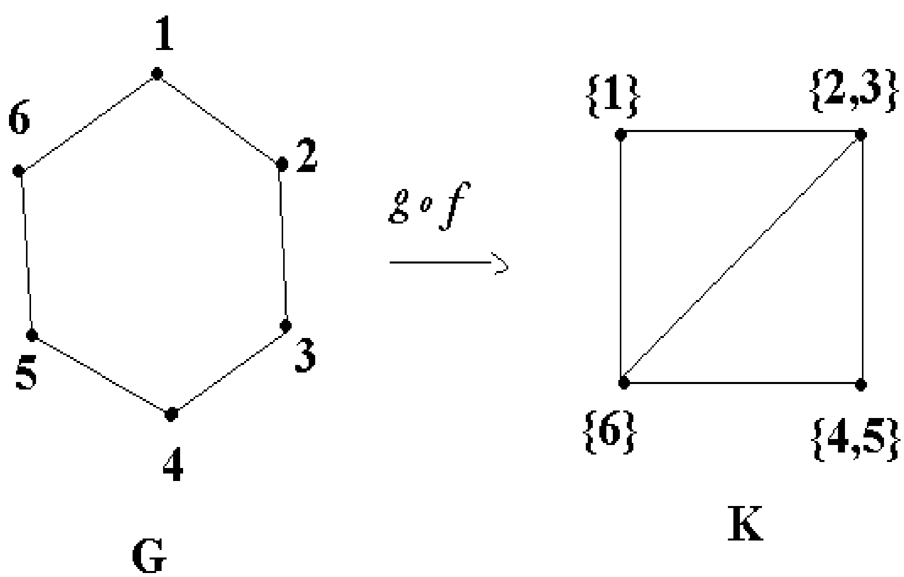
Example C


1
2
8
6
3
7
5
4
G
f
{1,2}
a
{6}
b
h {8}
{7} e
d
{3}
c
{4,5}
H
g
{a}
{e,b}
{d,h}
{c}
K

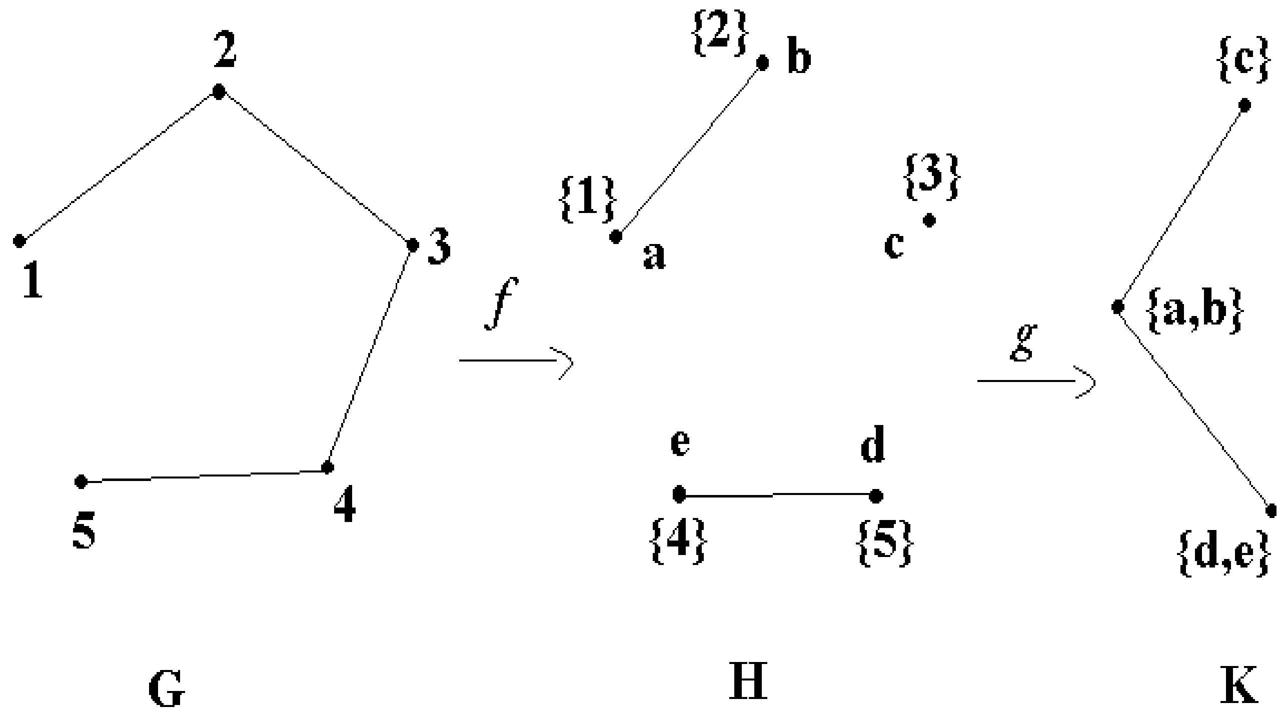

1
2
8
6
3
7
5
4
G
g ∘ f
{1,2}
{6,7}
{3,8}
{4,5}
K

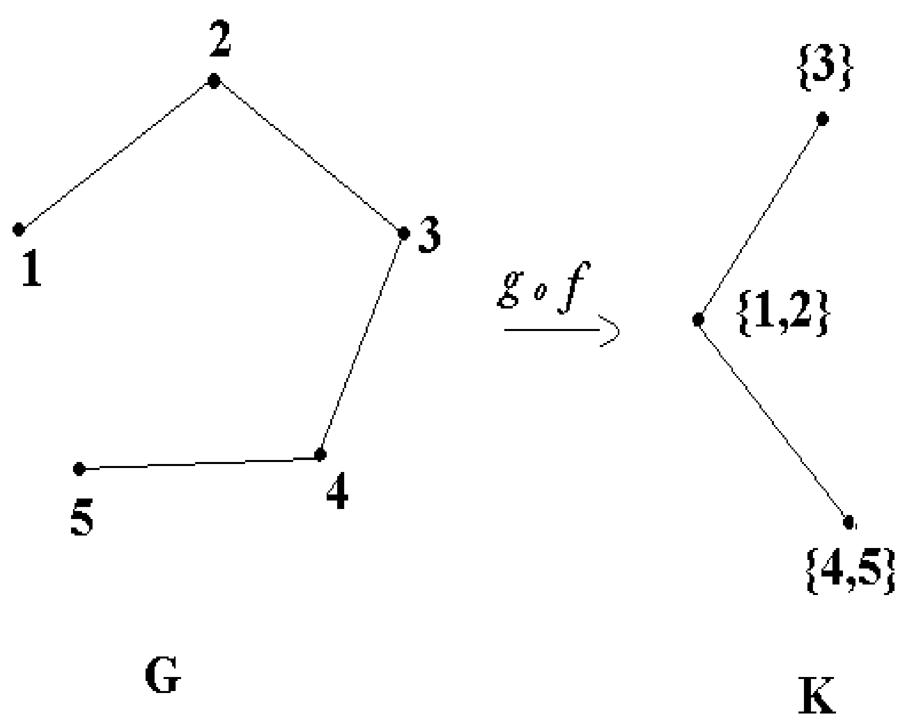
Example D



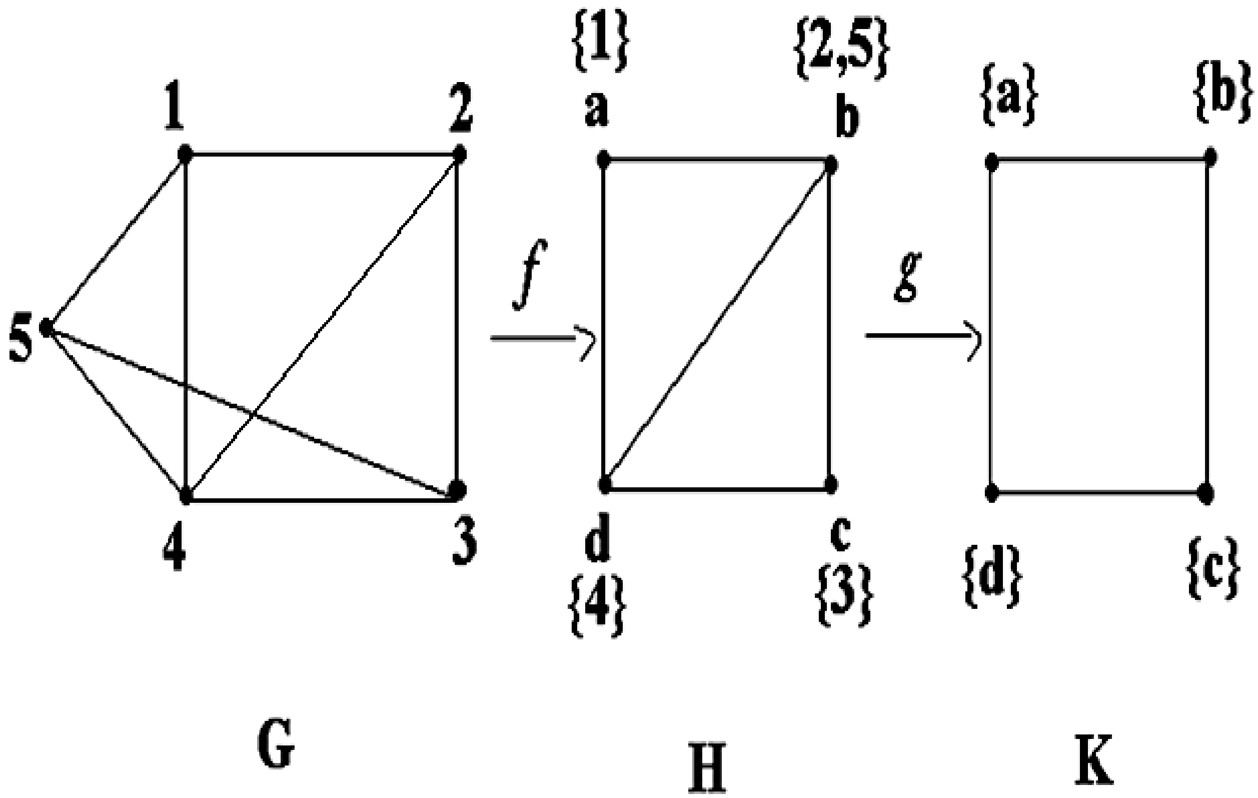


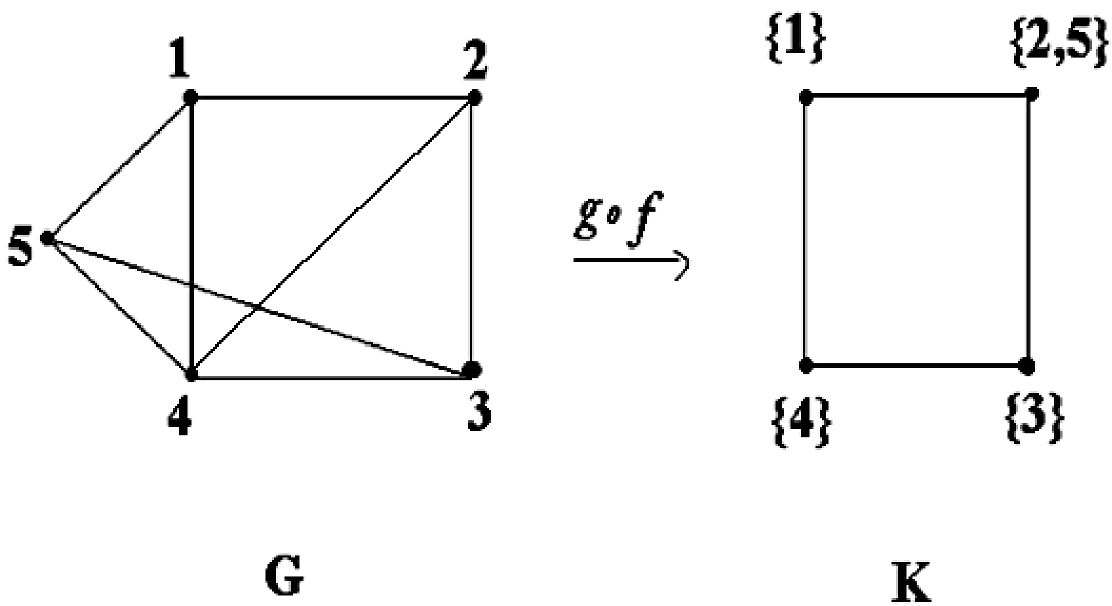
Example E



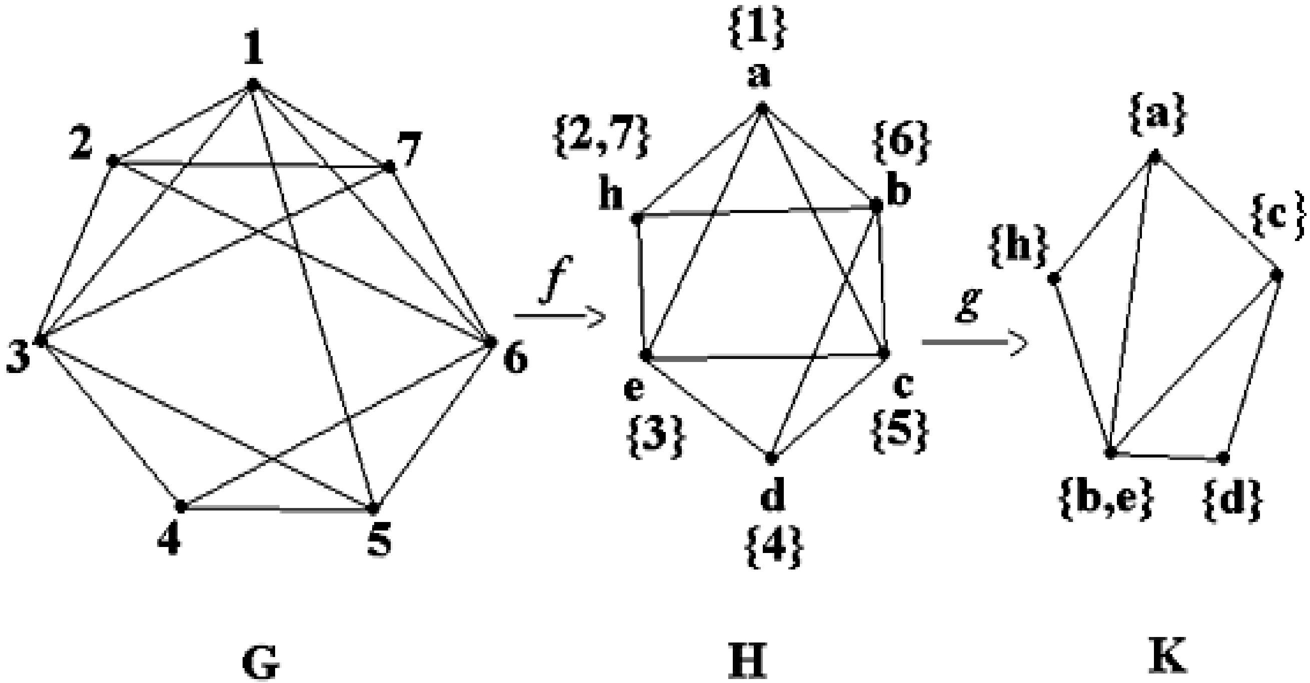


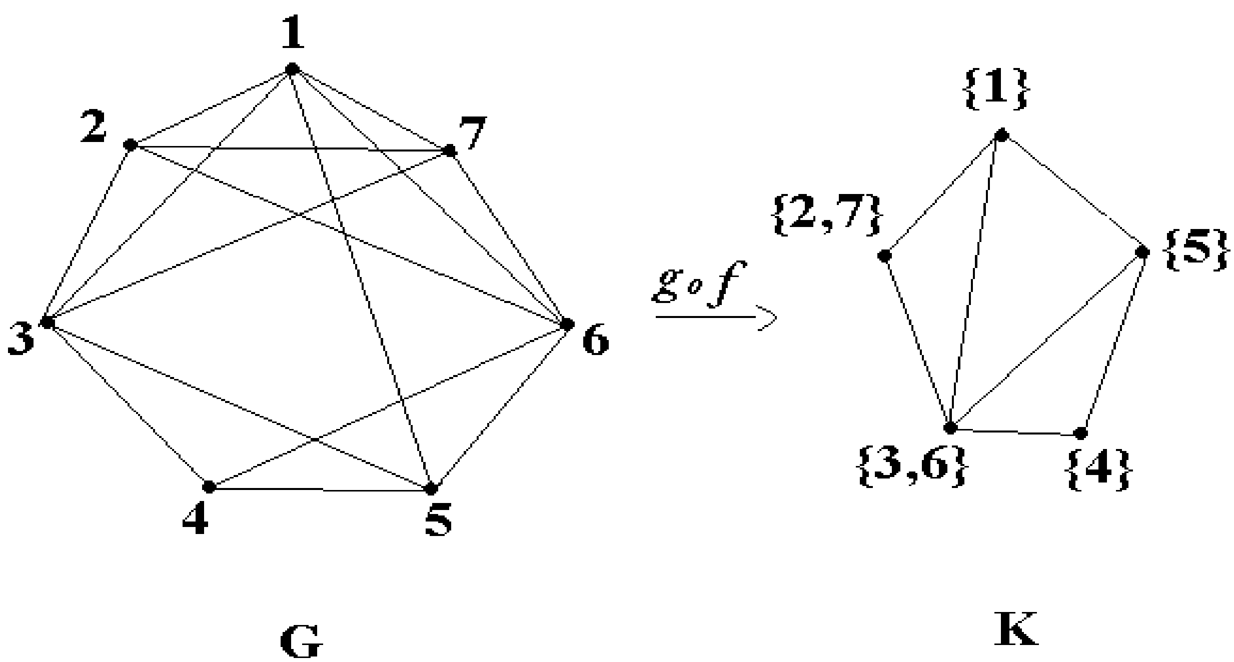
Example F



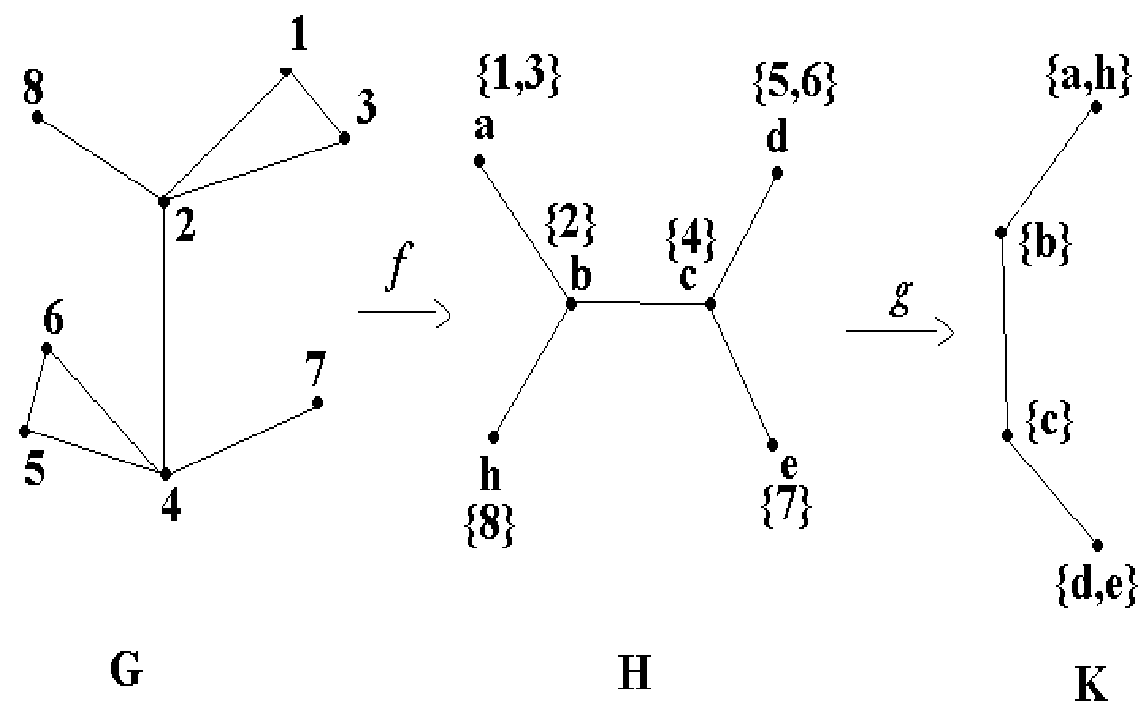


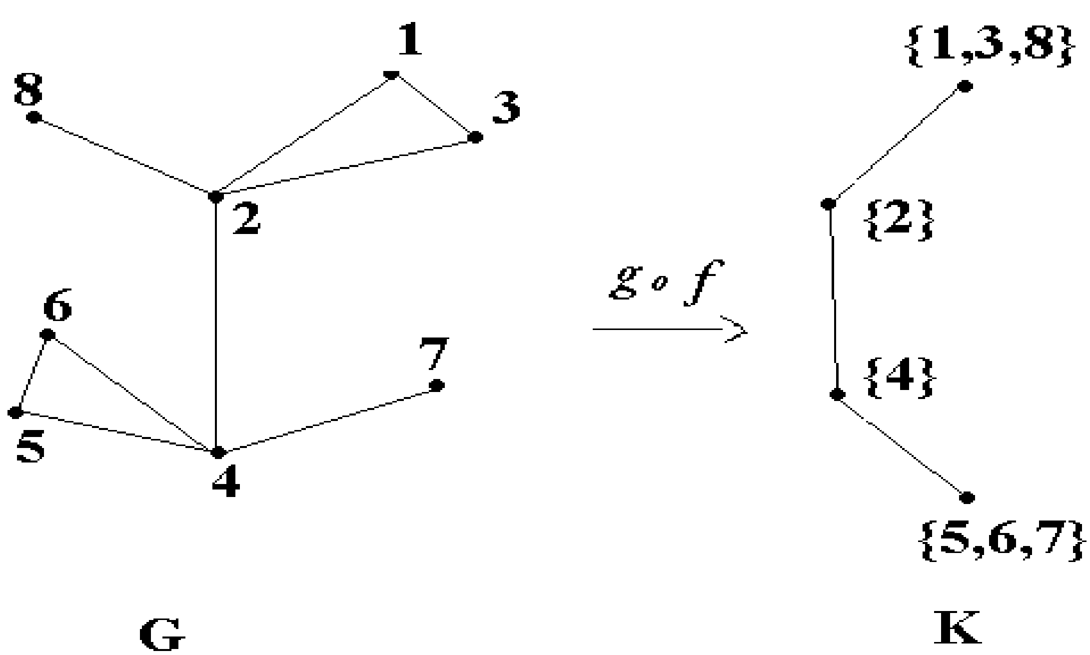
Example G



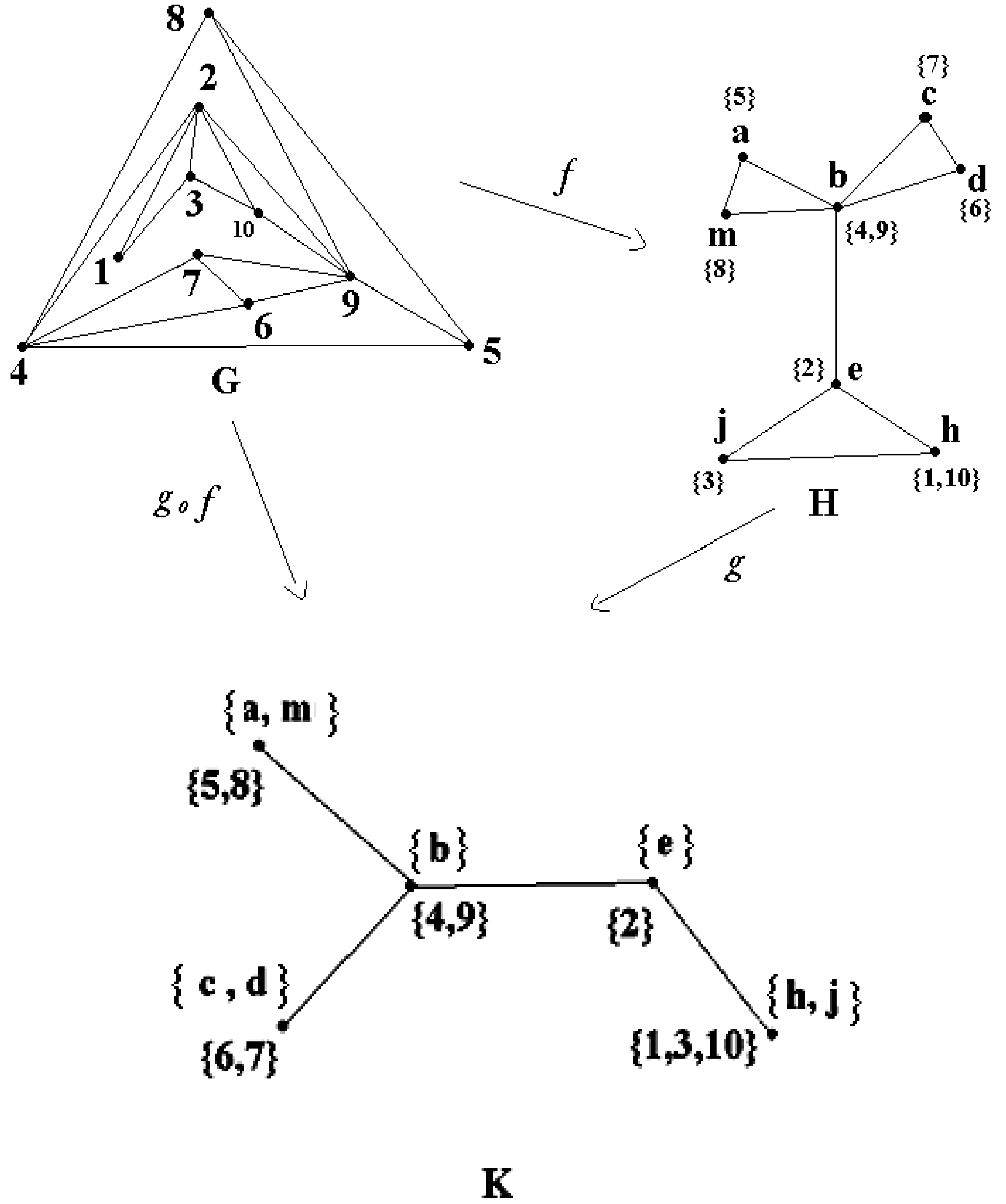


Example H


1
8
3
2
6
7
5
4
f
{1,3}
a
{5,6}
d
{2}
b
{4}
c
h
{8}
e
{7}
g
{a,h}
{b}
{c}
{d,e}
G
H
K


1
8
3
2
6
7
5
4
g o f
{1,3,8}
{2}
{4}
{5,6,7}
G
K

Example I


8
2
3
10
1
7
9
6
4
5
G
f
{5}
a
{7}
c
b
d
{6}
m
{8}
{4,9}
{2}
e
j
h
{3}
{1,10}
H
g
g∘f
{a, m}
{5,8}
{b}
{4,9}
{e}
{2}
{c, d}
{6,7}
{h, j}
{1,3,10}
K

Theorem 3.3:
A function f : V (G) → V(H) is a strong quasi-homomorphism and g : V (H) → V(K) is any map . Then g :V (H) → V(K) is quasi-complementary homomorphism if and only if $g \circ f$: V(G) → V(K) is quasi-complementary homomorphism.

Proof:
If g:V(H)→V(K) is quasi-complementary homomorphism then $g \circ f$: V(G) → V(K) is quasi-complementary homomorphism which is clear from table-3.2.
Conversely suppose $g \circ f$: V(G) → V(K)is a quasi-complementary homomorphism.
Let $u \neq v$ in H. Let $u = f(x)$ & $v = f(y)$ in H.
Case-1: $g(u) = g(v)$ in K.
⇒ $g(f(x)) = g(f(y))$in K.
⇒ $g \circ f(x) = g \circ f(y)$ in K.
⇒ $x \sim y$ in G, since $g \circ f$ is a quasi- complementary homomorphism
⇒ $f(x) \sim f(y)$ or $f(x) = f(y)$ in H, since f is a strong quasi-homomorphism.
⇒ $f(x) \sim f(y)$ in H.(∵ $u \neq v$ in H)
⇒ $u \sim v$ in H.
Case-2: $g(u) \sim g(v)$ in K. By the similar argument, we can prove that $u \sim v$ in H.
□

Theorem 3.4:
A function f : V (G) → V(H) is a strong quasi-homomorphism and g : V (H) → V(K) is any map . Then g :V (H) → V(K) is quasi-homomorphism if and only if $g \circ f$: V(G) → V(K) is quasi-homomorphism.

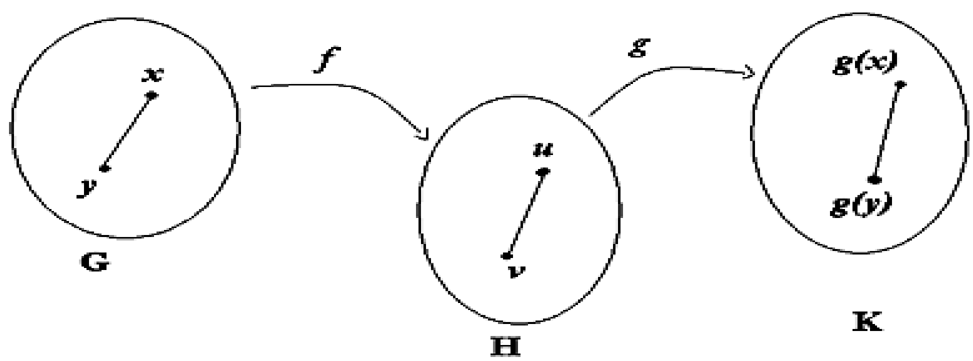
Proof:
If g:V(H)→V(K)is quasi-homomorphism then $g \circ f$: V(G) → V(K)is quasi-homomorphism which is clear from table-3.2.
Conversely suppose $g \circ f$: V(G) → V(K) is quasi-homomorphism.
Let $u \sim v$ in H. Let $u = f(x)$ & $v = f(y)$ in H.
⇒ $f(x) \sim f(y)$ in H.
⇒ $u \sim v$ in G, since f is a strong quasi-homomorphism.
⇒ $g(f(x)) = g(f(x))$ -or - $g(f(x)) \sim g(f(x))$ in K. Since $g \circ f$ is a quasi homomorphism.
⇒ $g(u) = g(v)$ -*or* - $g(u) \sim g(v)$ in K. Thus g is a quasi-homomorphism. □

Similarly we can prove the following results.

Theorem 3.5:
A function f : V (G) → V(H) is a strong homomorphism and g : V (H) → V(K) is any map . Then g :V (H) → V(K) is complementary homomorphism if and only if $g \circ f$: V(G) → V(K)is complementary homomorphism.

Theorem 3.6:
A function f : V (G) → V(H) is a strong homomorphism and g : V (H) → V(K) is any map . Then g :V (H) → V(K) is homomorphism if and only if $g \circ f$: V(G) → V(K)is homomorphism.

A VIEW FOR ABOVE THEOREMS



3. F. Decomposition of variants of homomorphism

Decomposition means expressing a given map as a composition of two or more maps. Any homomorphism can decomposed into injective and surjective homomorphism which is a well known results.

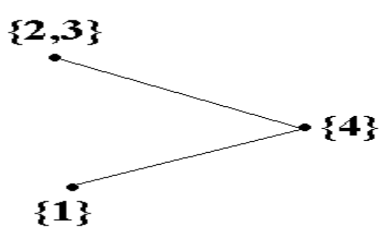
Definition 3.8: Let G be a graph & for the graph H consider a partition V_1, V_2 … V_n of the vertex set V(G) as the vertex set of H, & two sub sets V_i & V_j are adjacent if for some $x \in V_i$ & $y \in V_j$ such that x and y are adjacent in G, then graph H is known as *quotient graph* **[17]**of G.

Notation: Let f : V (G) $\to$ V(H) be any function. Let θ_f = { $f^{-1}(y)$ / y is in range of f }. Then θ_f is a partition of V (G). The quotation graph induced by the partition θ_f is denoted by G/ θ_f.

Theorem 3.7:
Let f : V (G) $\to$ V(H) be a quasi homomorphism. Then there is a graph K and two functions g & h such that g : V (G) $\to$ V(K) surjective quasi homomorphism & h : V (K) $\to$ V(H) injective homomorphism. So, $f = h \circ g$

Proof:
Consider the partition θ_f and let K be equal to quotient graph G/ θ_f. For each $x \in$ V (G). Let $f(x) = y$.
Now define a map g : V (G) $\to$ V(K) by $g(x) = f^{-1}(y) \in \theta_f$ and h : V (K) $\to$ V(H) by $h(f^{-1}(y)) = y$. It can be easily verified that $h \circ g = f$.
Now we prove that g is a surjective quasi homomorphism.
Let $f^{-1}(y) \in$ V(K). Let $x \in f^{-1}(y)$ then $f(x) = y$ therefore $g(x) = f^{-1}(y)$.
Thus g is onto. Let $x_1 \sim x_2$ in G. If $f(x_1) = g(x_2) = y$ then $x_1, x_2 \in f^{-1}(y)$.
Therefore $g(x_1) = g(x_2) = f^{-1}(y)$.
On other hand if $f(x_1) \sim f(x_2)$ & $y_1 = f(x_1)$, $y_2 = f(x_2)$ then $f^{-1}(y_1) \sim f^{-1}(y_2)$ in the graph K= G/ θ_f . *i.e.* $g(x_1) \sim g(x_2)$ in K.
Now we prove that h is an injective homomorphism.
Suppose $f^{-1}(y_1) \neq f^{-1}(y_2)$ .Therefore $f^{-1}(y_1)$ & $f^{-1}(y_2)$ are disjoint sets.
Therefore $y_1 \neq y_2$.Thus $h\left(f^{-1}(y_1)\right) \neq h(f^{-1}(y_2))$. So, h is one-one map.
Suppose $f^{-1}(y_1) \sim f^{-1}(y_2)$ in K= G/ θ_f .
Then there exist $x_1 \in f^{-1}(y_1)$ & $x_2 \in f^{-1}(y_2)$ such that $x_1 \sim x_2$.
Since f is quasi homomorphism, $f(x_1) \sim f(x_2)$ *or* $f(x_1) = f(x_2)$.
i.e. $y_1 \sim y_2$ *or* $y_1 = y_2$. If $y_1 = y_2$ then $f^{-1}(y_1) = f^{-1}(y_2)$, this is not true.
Therefore $y_1 \sim y_2$. *i.e.* $h\left(f^{-1}(y_1)\right) \sim h(f^{-1}(y_2))$ in H. □

Tester:
In Example 3.2 (2) the following graph is the quotient graph of G.

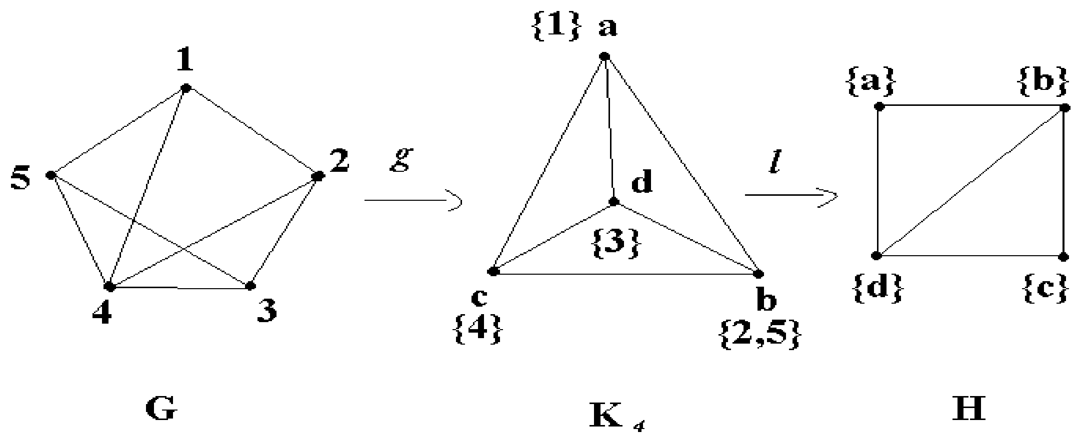


T = G/θ_f

Theorem 3.8:
Let f : V (G) → V(H) be a strong homomorphism. Then there is a graph K and two functions g & h such that g : V (G) → V(K) surjective homomorphism & h : V (K) → V(H) injective complementary homomorphism. So, $f = h \circ g$

Proof:
A function f : V (G) → V(H) is a strong homomorphism. Consider the partition θ_f which consist of the pre images of the mapping f .i.e. collection of fibers of the mapping f .
Assume that there are m sets in the partition θ_f .
Clearly the mapping g : V (G) → V(K_m) defined by $g(x) = f^{-1}(y)$, if $f(x) = y$ is a surjective homomorphism.
Also the mapping l : V (K_m) → V(H) defined by $l(f^{-1}(y)) = y$ is an injective complementary homomorphism. □

Tester:
In example 3.3 (2) fibers of the mapping f are;
$f^{-1}(a) = \{1\}$, $f^{-1}(b) = \{2,5\}$, $f^{-1}(c) = \{3\}$, $f^{-1}(d) = \{4\}$.Then $m = 4$



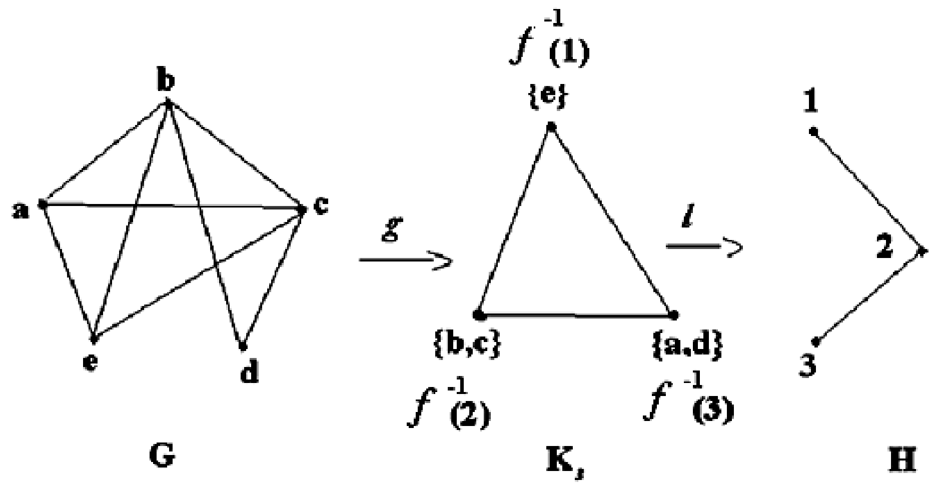
Similarly we can prove the following Results.

Theorem 3.9:
Let f : V (G) → V(H) be a strong quasi homomorphism. Then there is a graph K and two functions g & l such that g : V (G) → V(K) surjective quasi homomorphism & l : V (K) → V(H) injective quasi complementary homomorphism. So, $f = l \circ g$

Tester:

Example 3.15:

Consider example 3.13, the given mapping g is surjective quasi homomorphism and mapping l is injective quasi complementary homomorphism such that f is strong homomorphism.



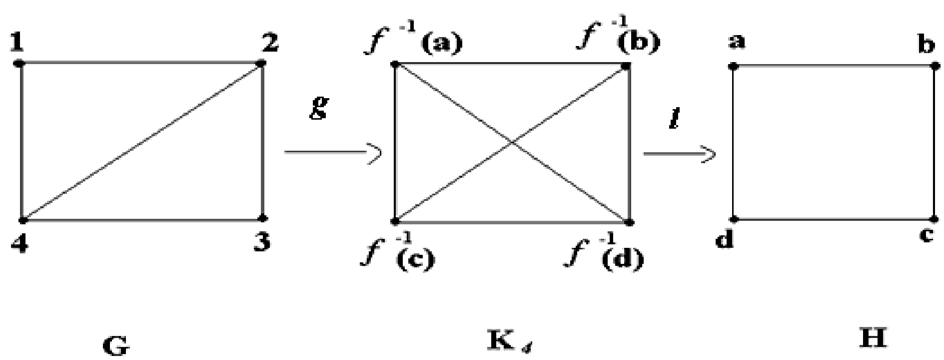
Theorem 3.10:

Let $f : V(G) \rightarrow V(H)$ be a complementary homomorphism. Then there is a graph K and two functions g & l such that $g : V(G) \rightarrow V(K)$ surjective homomorphism & $l : V(K) \rightarrow V(H)$ injective complementary homomorphism. So, $f = l \circ g$

Tester:

Example 3.16:

Consider example 3.10. In the given mapping g is surjective homomorphism and mapping l is injective complementary homomorphism such that $f = l \circ g$ is complementary homomorphism.



Theorem 3.11:
Let f : V (G) → V(H) be a quasi complementary homomorphism. Then there is a graph K and two functions g & l such that g : V (G) → V(K) surjective homomorphism & l : V (K) → V(H) injective quasi complementary homomorphism. So, $f = l \circ g$

Tester:
Example 3.17:
For the example 3.11 intermediate quotient graph is K_5.

Theorem 3.12:
An onto function f : V (G) → V(H) is a strong homomorphism. There is a quotient graph T such that which is isomorphic to graph H.

Proof:
Consider the partition θ_f which consist of the fibers of the mapping f.
Now, let's define the set T.
V(T) = { $f^{-1}(y)$ / $y \in$ V(H) } & E(T) = { $f^{-1}(y_1) \sim f^{-1}(y_2)$ / if $y_1 \sim y_2$ in H}
Now, consider the mapping,
F: V(T) → V(H) defined as follows, $F(f^{-1}(y)) = y \quad \forall\, y \in$ V(H).
Clearly F is bijective map.
Now, we have to show that F is a strong homomorphism.
$F(f^{-1}(y_1)) \sim F(f^{-1}(y_2))$ in H.
⇔ $y_1 \sim y_2$ in H.
⇔ $f^{-1}(y_1) \sim f^{-1}(y_2)$ in graph T. Since f is a strong homomorphism.
Thus F is an isomorphism. □

Similarly we can prove

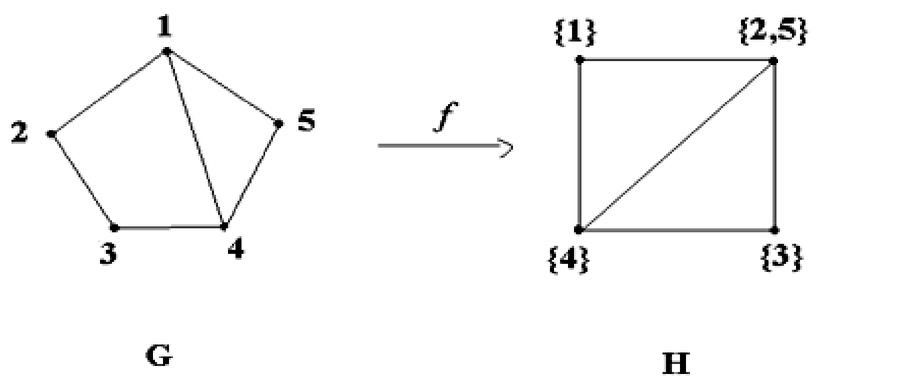
Theorem 3.13:
A function f : V (G) → V(H) is a strong quasi homomorphism. There is a quotient graph T which is isomorphic to graph H.

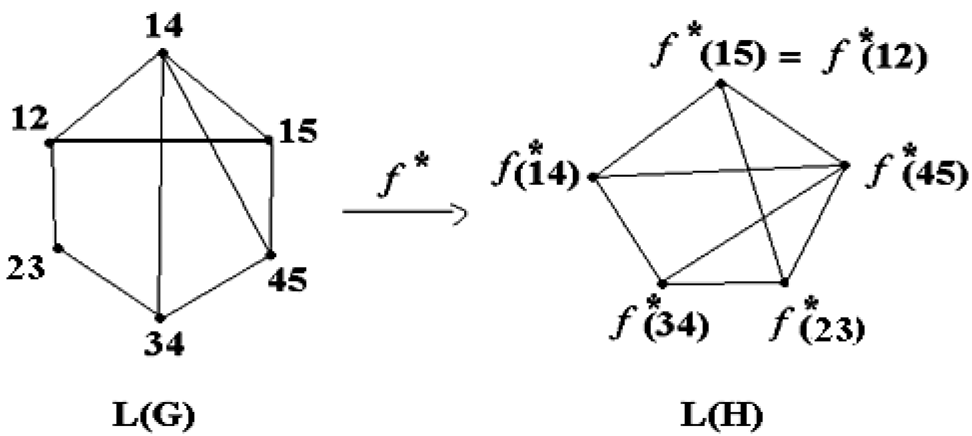
3. G. Basic properties of the variants of homomorphism

Theorem 3.14:
If a function f : V (G) → V (H) is a homomorphism then the function $f^\star : \mathrm{V}\left(\mathrm{L(G)}\right) \to \mathrm{V(L(H))}$ defined by $f^\star(uv) = f(u)f(v)$ is a quasi-homomorphism.

Proof:
Let a function f : V (G) → V (H) be a homomorphism. Let *uv* & *uw* are two vertices in L(G)(Where *uv* & *uw* are two edges in graph G with common vertex *u*). Since f is a homomorphism, $f(u)\,f(v)$ & $f(u)\,f(w)$ are two edges in H.
If $f(w) = f(v)$ then $f^*(uv) = f^*(uw)$.If $f(w) \neq f(v)$ then $f^*(uv) \sim f^*(uw)$ in L(H).Thus the induced mapping f^* is a quasi-homomorphism. □

Tester:
Example 3.18:
In the following mapping f is homomorphism.



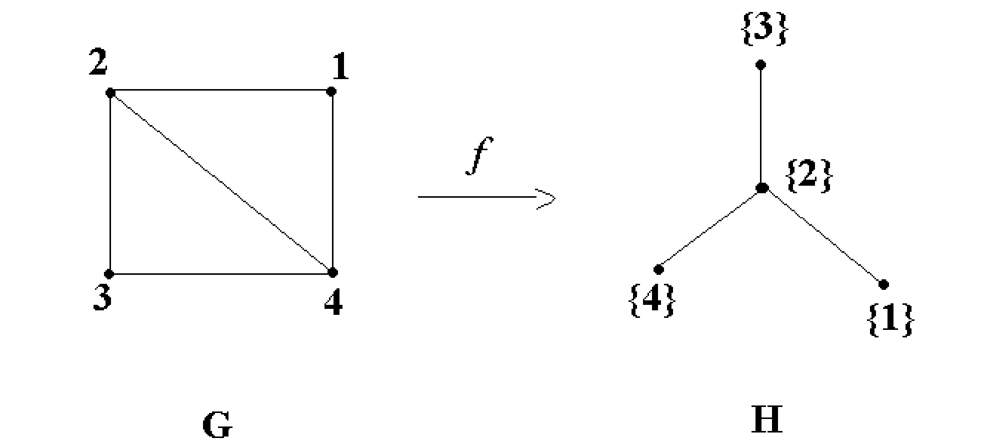


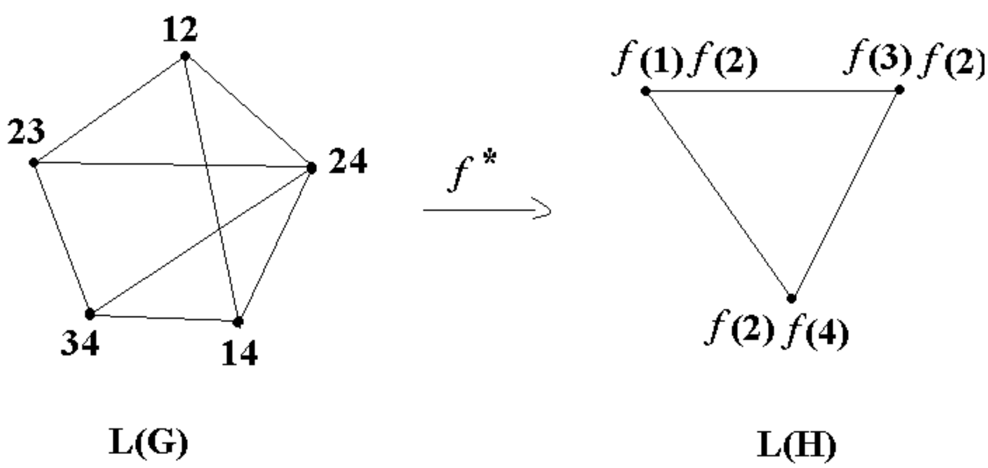
Clearly f^* is a quasi homomorphism from graph L(G) to graph L(H).

Remark:

Counter example 3.19:

This result may not be true for complementary homomorphism or quasi complementary homomorphism. In the following mapping f is complementary (or quasi complementary) homomorphism.





(14) is vertex in L (G) but vertex $f(1)\,f(4)$ is not exists in L (H). Thus the mapping f^* can not be defined.

Theorem 3.15:

If G is a connected graph , H is a graphs and $f : V(G) \to V(H)$ is a pure quasi homomorphism then f must be constant.

Proof:
We prove the Theorem by mathematical induction on the number of vertices n of the graph G. If G has n =2 vertices say x and y ,then $x \sim y$ and so $f(x) = f(y)$. Suppose G has n vertices and assume that the Theorem is true for all connected graphs with m vertices , $m < n$.
Let T be a spanning tree of the graph G and u be a vertex of T whose degree in T is 1. Then T – u is a tree with n -1 vertices and so by induction hypothesis f is constant on the vertex set of T – u. That is f is constant on the vertex set of G –u. Thus there is a vertex t in H such that $f(v) = t$ for all vertices v of G – u. Since u is adjacent at least one vertex v of G – u, and f is a pure quasi homomorphism, $f(u) = f(v) = t$. Thus $f(x)= t$ for all vertices x of G. Thus f is constant . □

Notation: Let P (G, H) denote the set of all pure quasi homomorphiams from graph G to graph H.

Corollary 3.15.1:
If a graph G has k components and graph H has n vertices, then $|\,\mathrm{P}\,(\mathrm{G},\mathrm{H})| = n^k$

Proof:
Let G_i be a component of the graph G, $\forall\, i \in \{1,..,k\}$and suppose f :V(G)$\rightarrow$V(H) be a pure quasi homomorphism. For each component G_i the restriction of the function on V(G_i)is a constant function and so it can take values in n ways. Since there are k components , then the function f itself can take $n \cdot n \cdots n$ (k times) $= n^k$ values in H. □

3. H. Relation between the variants of homomorphism

In this section we would like to establish the inter relation between the variants of homomorphism.

Theorem 3.16:
An injective quasi-homomorphism of graphs is a homomorphism.

Proof:
Let a function f : V (G) $\rightarrow$ V(H) be an injective quasi homomorphism.
Let $x \sim y$ in G.
$\Rightarrow f(x) \sim f(y)$ or $f(x) = f(y)$ in H. Since f is a quasi-homomorphism.

As f is an injective map then $f(x) \neq f(y)$ in H. So, $f(x) \sim f(y)$ in H.
Thus f is a homomorphism. □

Theorem 3.17:
An injective complementary homomorphism of graphs is quasi complementary homomorphism.

Proof:
Let a function $f : V(G) \rightarrow V(H)$ be an injective complementary homomorphism.
The map is an injective. So, each fiber of f is singleton. Then clearly f is a quasi complementary homomorphism. □

Similarly we can prove the following results.

Theorem 3.18:
A mapping $f : V(G) \rightarrow V(H)$ is an injective mapping. Then f is a strong quasi homomorphism if and only if it is strong homomorphism.

Thus one-oneness play very crucial role between the variants of homomorphism.

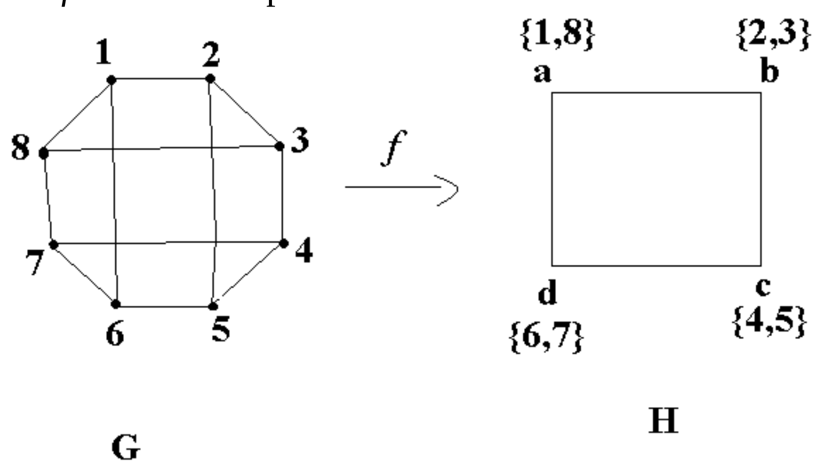
Theorem 3.19:
An onto function $f : V(G) \rightarrow V(H)$ is homomorphism if and only if f is a quasi homomorphism & $f^{-1}(y)$ is an independent set for each y in V(H).

Proof:
Suppose f is onto homomorphism. Let $z \in V(H)$ and $x, y \in f^{-1}(z)$ and $x \neq y$.
If $x \sim y$ then $f(x) \sim f(y)$.But $f(x) = f(y) = z$ .This is a contradiction. Thus $x \nsim y$.
This implies that $f^{-1}(z)$ is independent.
For Converse, suppose $x \sim y$ in G. If $f(x) = f(y) = z$ then $x, y \in f^{-1}(z)$ which implies that $f^{-1}(z)$ is not independent. Thus, $f(x) \sim f(y)$ in H. Thus f is a homomorphism. □

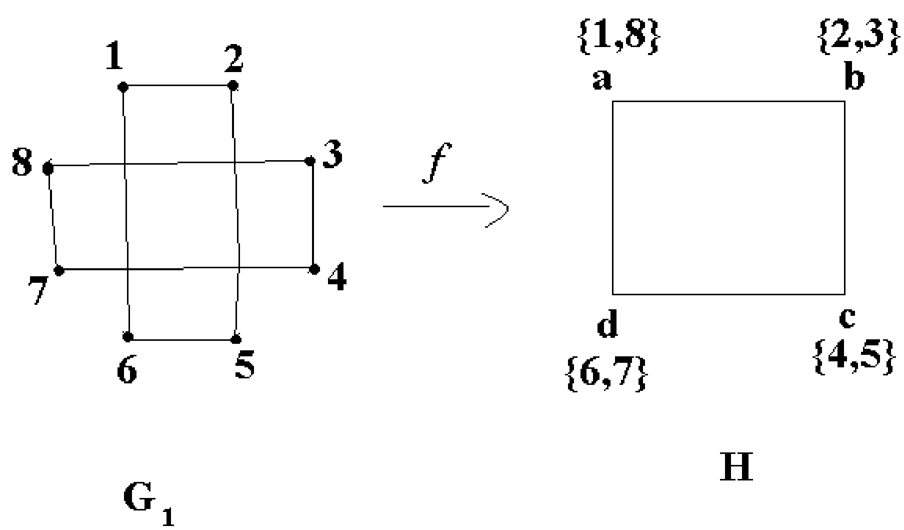
Corollary 3.19.1:
If f is an onto quasi-homomorphism from G onto H, then there is a spanning sub graph G_1 of G such that f is a homomorphism from G_1 to H.

Proof:
Suppose function $f : V(G) \to V(H)$ is an onto quasi-homomorphism. For each $y \in V(H)$ consider $f^{-1}(y)$,which is subset of V(G). Consider the graph G_1 obtained by removing edges of $f^{-1}(y)$ for every y in V (H). The resulting graph G_1 has same vertices as G but possibly less edges then G.
Now $f^{-1}(y)$ is an independent set in the new graph G_1,for each y in V(H).So, by theorem3.19 , f is a homomorphism from G_1 onto H. □

Tester:
Example 3.20:
In the following a function f is quasi homomorphism but not homomorphism because fibers of f are not independent set.



Consider the spanning sub graph G_1 of G as defined in the corollary 3.19.1 than mapping f is a homomorphism.



Corollary 3.19.2:
If f :V(G) → V(G) is a bijective quasi-homomorphism then it is an isomorphism.

Proof:
By above theorem 3.16, f is a homomorphism. Since f is an injective function from V(G) to V(G) and a homomorphism, there is an injective function f^* from E(G) to E(G) defined by $f^*(uv) = f(u)f(v)$.This function f^* takes adjacent edges to adjacent edges. Also note that f^* is injective because it is surjective. Thus f^* : E(G) → E(G) is a bijection.

Suppose $f(u) \sim f(v)$ in G. Let $rs \in$ E(G) such that $f^*(rs) = f(u) f(v)$.Then $\{ f(r), f(s)\} = \{ f(u), f(v)\}$.Since f is an injection $u = r$ and $v = s$(or $u = s$ and $v = r$).Thus $uv = rs$ (or sr) and hence uv is an edge in G. Therefore f is an isomorphism.[Refer result 1.5] □

Now, similar result can be proving for complementary homomorphism.

Theorem 3.20:
An onto function f :V (G) → V (H) is a quasi complementary homomorphism if and only if f is a complementary homomorphism such that each fiber of f is a clique.

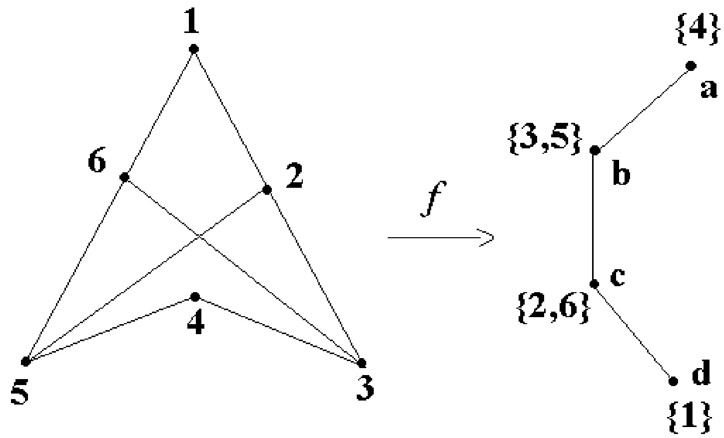
Proof:
Let a function f :V (G) → V (H) be a quasi complementary homomorphism. Clearly it is a complementary homomorphism. Assume that $f^{-1}(z)$ is not a clique , for some z in H. Then for some distinct vertices x & y in $f^{-1}(z)$ $x \nsim y$ in G. But $f(x) = f(y) = z$ in H. Then $x \sim y$ in G, since f is a quasi complementary homomorphism. This makes contradiction.

Conversely suppose the function f :V (G) → V (H) is a complementary homomorphism such that each fiber of f is clique set.
Let $x, y \in$ V (G) such that $f(x) = f(y) or f(x) \sim f(y)$
<u>Case-1:</u> $f(x) = f(y) = z$ in H.
⇒ $x, y \in f^{-1}(z)$.But $f^{-1}(z)$ is a clique, for every z in H.
⇒ $x \sim y$ in G.
<u>Case-2:</u> $f(x) \sim f(y)$ in H.
⇒ $x \sim y$ in G, since f is a complementary homomorphism.
Thus, f is a quasi complementary homomorphism. □

Similarly we can prove the following Result.

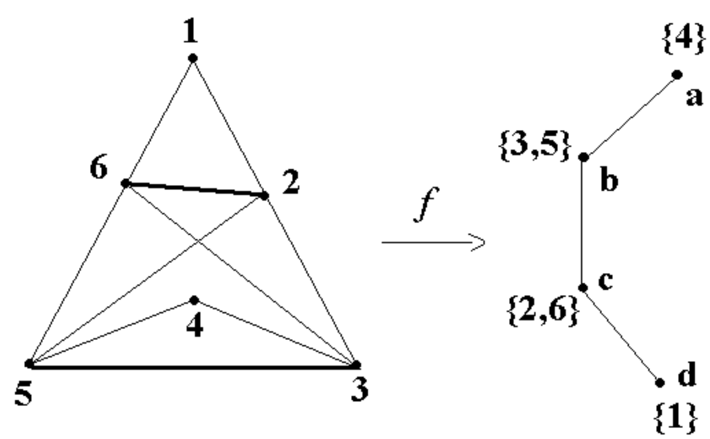
Corollary 3.20.1:
If *f* is an onto complementary homomorphism from G onto H, then there is a spanning super graph G_1 of G such that *f* is a quasi complementary homomorphism from G_1 to H.

Tester:
Example 3.21:
The following mapping *f* is a complementary homomorphism but not quasi complementary homomorphism because fibers of *f* are not clique sets.



G H

Now consider the spanning super graph G_1 of G by adding edges between all vertices in each fiber of *f*, then mapping *f* from G_1 to H is quasi complementary homomorphism.



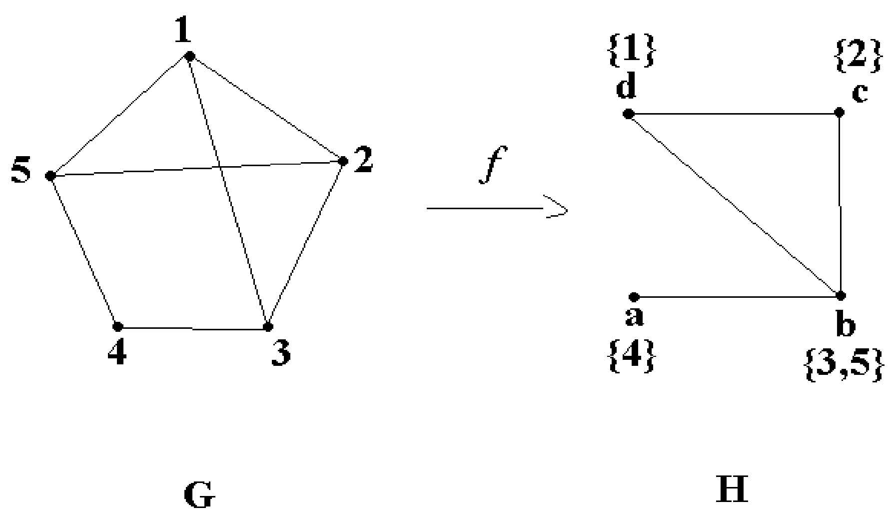
G_1 H

Similarly we can prove the following result.

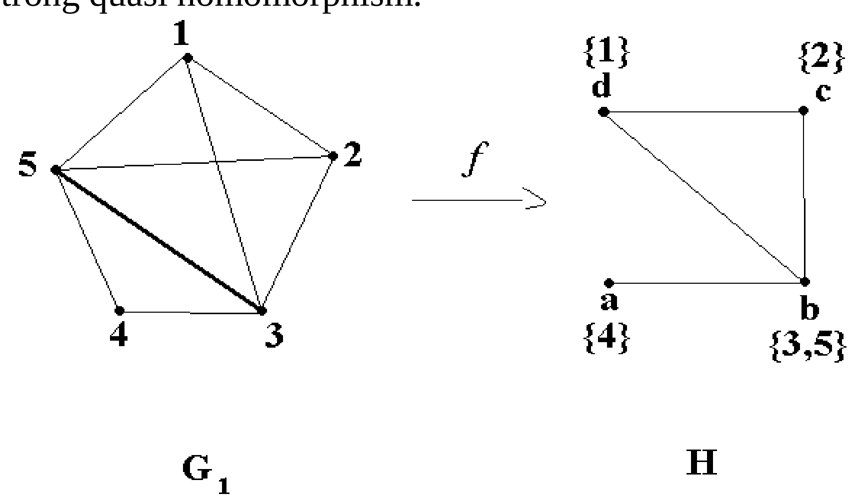
Theorem 3.21:
An onto function f :V (G) $\to$ V (H) is strong homomorphism such that each fiber of f is clique set, then it is strong quasi homomorphism.

Corollary 3.21.1:
An onto function f :V (G) $\to$ V (H) is strong homomorphism, then there is a spanning super graph G_1 of G such that mapping f from G_1 to H is strong quasi homomorphism.

Tester:
Example 3.22:
In the following mapping f is a strong homomorphism but it is not strong quasi homomorphism.



Consider the spanning super graph G_1 of G, then the mapping f from G_1 to H is become strong quasi homomorphism.



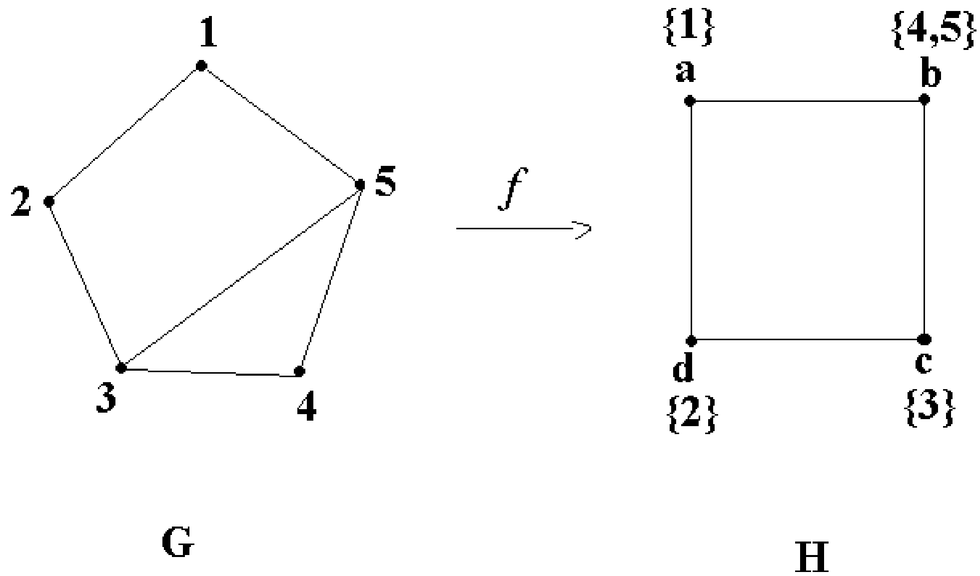
Theorem 3.22:
A function f : V (G) → V(H) is a homomorphism if and only if
f : V ($\overline{\text{G}}$) → V($\overline{\text{H}}$) is a quasi complementary homomorphism.

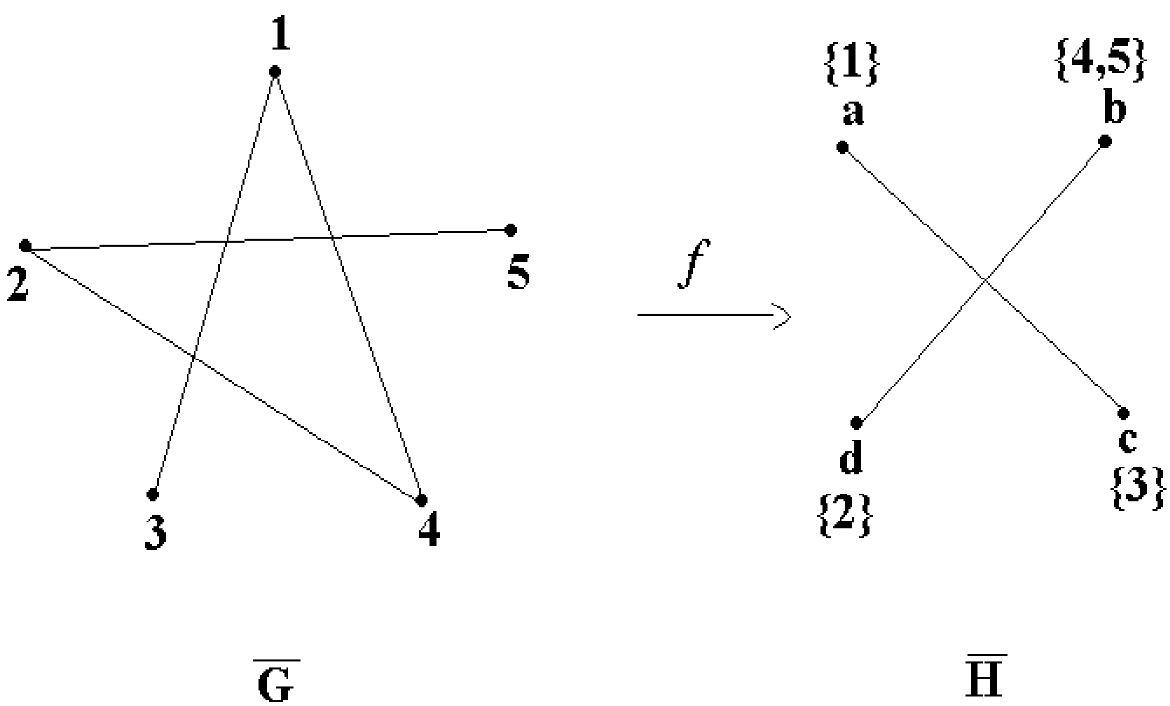
Proof:
Let a function f : V (G) → V(H) be a homomorphism. Let $x \neq y$ in G.
Suppose $f(x) = f(y)$ or $f(x) \sim f(y)$ in $\overline{\text{H}}$.
<u>Case-1:</u> $f(x) = f(y) = z$ in $\overline{\text{H}}$.
⇒ $f(x) = f(y) = z$ in H.
⇒ $x, y \in f^{-1}(z)$ for some z in H.
We know that each fiber of homomorphism is an independent set. Since f is a homomorphism from graph G to graph H.[Refer theorem 3.19]
⇒x ≁ y in G.
⇒x ∼ y in $\overline{\text{G}}$.
<u>Case-2:</u> $f(x) \sim f(y)$ in $\overline{\text{H}}$.
⇒$f(x) \nsim f(y)$ in H.
⇒ $x \nsim y$ in G, since f is a homomorphism.
⇒ $x \sim y$ in $\overline{\text{G}}$.Thus f : V ($\overline{\text{G}}$) → V($\overline{\text{H}}$) is a quasi complementary homomorphism.
Conversely suppose $f(x) \nsim f(y)$ in H.
⇒$f(x) \sim f(y)$ in $\overline{H}$.
⇒ $x \sim y$ in $\overline{\text{G}}$,since f is a quasi complementary homomorphism.
⇒ $x \nsim y$ in G. Thus f : V (G) → V(H) is a homomorphism. □

Theorem 3.23:
A function f : V (G) → V(H) is a quasi homomorphism if and only if
f : V ($\overline{\text{G}}$) → V($\overline{\text{H}}$) is a complementary homomorphism.

Proof: Let a function f : V (G) → V(H) be a quasi homomorphism.
Let $x \nsim$ y in $\bar{G}$.
⇒$x \sim y$ in G.
⇒ $f(x) \sim f(y)$or $f(x) = f(y)$ in H,since f is a quasi-homomorphism.
⇒ $f(x) \nsim f(y)$ or $f(x) = f(y)$ in $\overline{\text{H}}$.
⇒ $f(x) \nsim f(y)$ in $\overline{\text{H}}$.Thus f : V ($\overline{\text{G}}$) → V($\overline{\text{H}}$) is a complementary homomorphism.
Conversely suppose f : V ($\overline{\text{G}}$) → V($\overline{\text{H}}$) is a complementary homomorphism.
Let $x \sim y$ in G.
⇒$x \nsim$ y in $\overline{\text{G}}$.
⇒ $f(x) \nsim f(y)$ in $\overline{\text{H}}$, ,since f is a complementary homomorphism.
⇒ $f(x) \sim f(y)$ in H. Thus f : V (G) → V(H) is a quasi homomorphism □

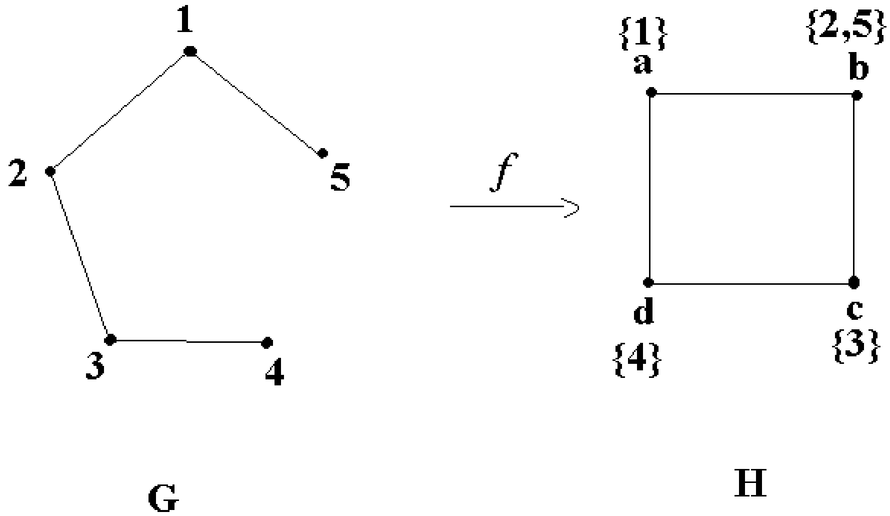
Tester:
Example 3.23:
(1) In the following mapping f from G to H is quasi homomorphism

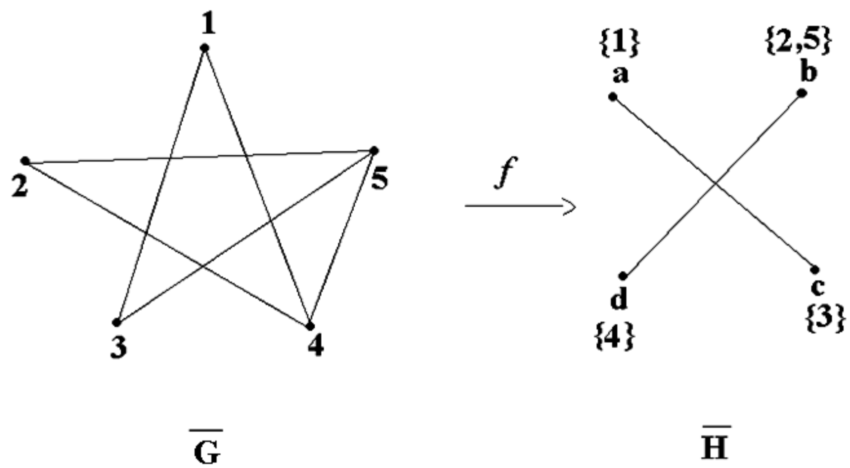




Clearly $f : V(\overline{G}) \rightarrow V(\overline{H})$ is complementary homomorphism.

(2) In the given example, mapping *f* from G to H is the homomorphism





Clearly $f : V(\overline{G}) \rightarrow V(\overline{H})$ is quasi complementary homomorphism.

Similarly we can prove the following

Theorem 3.24:
A function $f : V(G) \rightarrow V(H)$ is a strong homomorphism if and only if $f : V(\overline{G}) \rightarrow V(\overline{H})$ is a strong quasi homomorphism.

At the end of the section you may feel that these variants have mirror reflective properties. But that is not exactly true which you can see in letter sections.

3. I. Comparative study of the variants of homomorphism

Restriction Map

Homomorphism of graphs can hold the restriction map. *i.e.* f : V (G) → V(A) is a homomorphism and H is a sub graph of G than $f_{/H}$: V(H) → V (A) is also homomorphism. Now let's check for other variants.

Theorem 3.25:
A function f : V (G) → V(A) is a quasi-homomorphism and H is a sub graph of G then the restriction map $f_{/H}$: V(H) → V (A) is also a quasi-homomorphism.

Proof :
It is quite obvious. □

Theorem 3.26:
A function f : V (G) → V(A) is a complementary homomorphism and H is an induced sub graph of G then the restriction map $f_{/H}$: V(H) → V (A) is also a complementary homomorphism.

Proof:
Let $f_{/H}(x) \sim f_{/H}(x)$ in A.
⇒ $f(x) \sim f(y)$ in A.
⇒ $x \sim y$ in G, since f is a complementary homomorphism.
⇒ $x \sim y$ in H, since H is an induced sub graph of G. □

Similarly we can prove the following results.

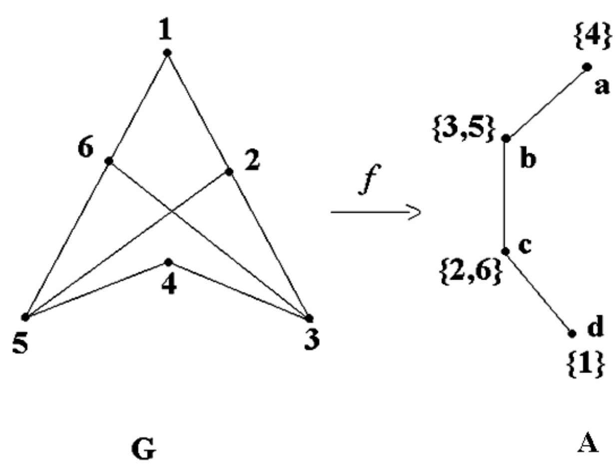
Theorem 3.27:
A function f : V (G) → V(A) is a pure quasi homomorphism and H is any sub graph of G then the restriction map $f_{/H}$: V(H) → V (A) is also a pure quasi homomorphism.

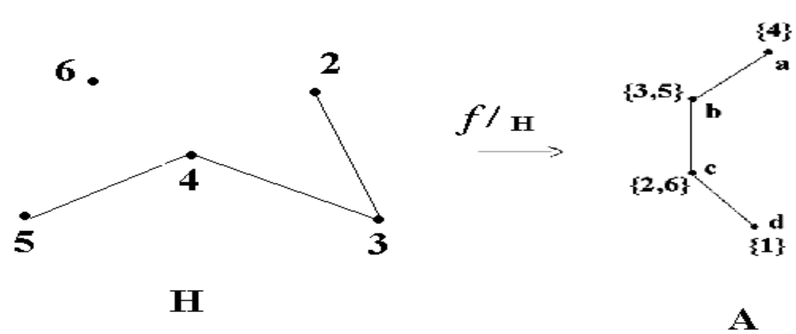
Theorem 3.28:
A function f : V (G) → V(A) is a quasi complementary homomorphism and H is an induced sub graph of G, then the restriction map $f_{/H}$: V(H) → V (A) is also a quasi complementary homomorphism.

Theorem 3.29:
A function f : V (G) → V(A) is a strong homomorphism and H is an induced sub graph of G, then the restriction map $f_{/H}$: V(H) → V (A) is also a strong homomorphism.

Theorem 3.30:
If a function f : V (G) → V(A) is a strong quasi homomorphism and H is a connected induced sub graph of G then the restriction map $f_{/H}$: V(H) → V (A) is also a strong quasi homomorphism.

Remark cum Tester:
Example 3.24:
Why induced sub graph is added as a condition? Let's answer the question in the following example. Here it is clear that f : V (G) → V(A) is complementary homomorphism but $f_{/H}$: V(H) → V (A) is not as $f(5) \sim f(2)$ in A ⟹ $5 \nsim 2$ in H. Because H is not induced sub graph of G.





Walk or Path

It can be easily proved that the image of a path under a homomorphism is a walk of the same length. However this is not the case for quasi – homomorphism which can be verified by simple example. Consider the graph G is connected in the following results.

Theorem 3.31:
Let G and H are graphs. Let f from G to H is mapping in which image of every path is a path of same length than f is an injective homomorphism.

Proof:
Suppose $x \neq y$ & $f(x) = f(y)$ in H.
$\Rightarrow x$ & y are vertices of the connected graph G.
$\Rightarrow$ There is a path $P_k = x\, u_1 ... u_{k-1}\, y$ of length k.
Then $f(x)\, f(u_1)...f(u_{k-1})\, f(y) = f\,(P_k)$ is also a path of same length k. But $f(x) = f(y)$ in H. This makes a contradiction to the path $f\,(P_k)$ .Thus the mapping f is injective.
Let $x \sim y$ in G. Then it is a path P_2 of length 2. Then $f\,(P_2)$ is also path length 2 .
So, $f(x) \sim f(y)$ in H. Thus f is a homomorphism. □

Tester:
In example 3.1 (2), Consider the path between vertices 6 and 1 is under the given homomorphism.P_3 (1,6) = $1 \sim 2 \sim 5 \sim 6$ is a path of length 3.
While $W_3\,(f\,(1), f\,(6)) = a \sim b \sim a \sim c$ is a walk of length 3.

Similarly we can prove the following results.

Corollary 3.31.1:
A mapping $f : V(G) \rightarrow V(H)$ is an onto complementary homomorphism such that image of every path is a path of same length if and only if f is an isomorphism.

Corollary 3.31.2:
A mapping $f : V(G) \rightarrow V(H)$ is a quasi homomorphism such that image of every path is a walk of same length if and only if f is a homomorphism.

Theorem 3.32:
Let G and H are graphs. Let f from G to H is mapping in which whenever P_k is path in H, $f^{-1}(P_k)$ is also a path in G of same length if and only if f is an onto complementary homomorphism.

Since a homomorphism preserves the connectedness we have the following. "There is no onto homomorphism from connected graph to disconnected graph".

Similarly we found the non existence of complementary homomorphism .

Theorem 3.33:
There is no complementary homomorphism from disconnected graph to connected graph

Proof:
Let a mapping $f : V(G) \rightarrow V(H)$ be a complementary homomorphism. Assume that graph G is disconnected and graph H is connected. Let G_1 & G_2 be two components of graph G. Then there are the vertices u_1 of G_1 & u_2 of G_2 such that there is no path between them in the graph G. But there is a path between $f(u_1)$ to $f(u_2)$ in graph H, as H is connected. This makes a contradiction with theorem 3.32. □

Distance

Next of our study is the distance between two vertices under the different variants of homomorphism. We have a nice result about distance in homomorphism that homomorphism are non-expensive map *i.e.* a mapping $f : V(G) \rightarrow V(H)$ is a homomorphism ,then
$d_H(f(u), f(v)) \leq d_G(u, v)$, $\forall\, u, v$ in G.[Refer result 1.10]

Theorem 3.34:
A mapping $f : V(G) \rightarrow V(H)$ is quasi homomorphism if and only if
$d_H(f(u), f(v)) \leq d_G(u, v)$, $\forall\, u, v$ in G.

Proof:
First part is clear from above result & relation between quasi homomorphism and homomorphism.
Conversely suppose $d_H(f(u), f(v)) \leq d_G(u, v)$, $\forall\, (u, v)$ in G .

Let $x \sim y$ in G.

$\Rightarrow d_G(x, y) = 1$. $f(x) \sim f(y)$ in A, since f is a complementary homomorphism.

$\Rightarrow d_H(f(x), f(y)) \leq 1 = d_G(x, y)$.

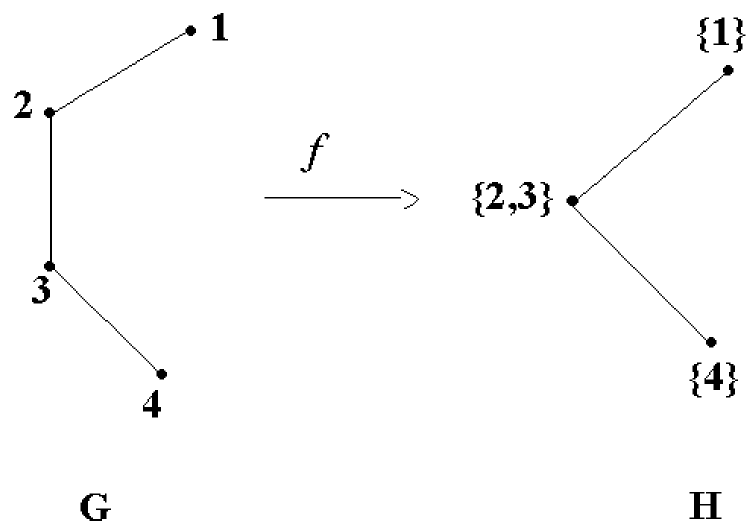
$\Rightarrow d_H(f(x), f(y)) = 1$ or $d_H(f(x) f(y)) = 0$.

$\Rightarrow f(x) \sim f(y)$ or $f(x) = f(y)$ in H. Thus f is a quasi homomorphism. □

Remark:

Counter Example 3.25:

Converse of above result is not true in homomorphism. In the following mapping f from G onto H . It is clear that $d_H(f(u), f(v)) \leq d_G(u, v)$, $\forall\, u, v$ in G.
But it is not homomorphism because $2 \sim 3$ in G $\not\Rightarrow f(2) \sim f(3)$ in H.



Theorem 3.35:

A mapping $f : V(G) \rightarrow V(H)$ is a complementary homomorphism if and only if $d_G(x, y) \leq d_H(a, b)$, $\forall\, a, b \in f(G) \subset$ H & $\forall\, x \in f^{-1}(a)$, $y \in f^{-1}(b)$.

Proof:

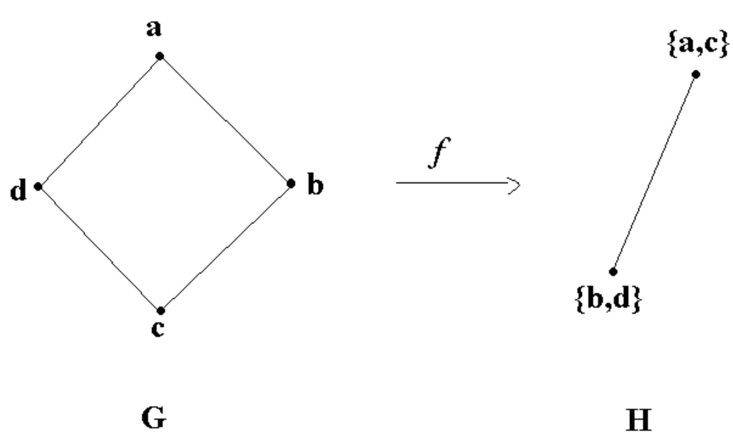
Let a mapping $f : V(G) \rightarrow V(H)$ is a complementary homomorphism.
Let $d_H(a, b) = k$. Let $P_k = a\ u_1 \ldots u_{k-1}\ b$ is a shortest path of length k in f(G).
$\Longrightarrow f^{-1}(P_k) = x \cdot m_1 \cdots m_{k-1} \cdot y$ is also a path of same length k in G,
$\forall\, x \in f^{-1}(a)$, $m_1 \in f^{-1}(u_1), \ldots, m_{k-1} \in f^{-1}(u_{k-1})$, $y \in f^{-1}(b)$, by theorem 3.32.
Thus $d_G(x, y) \leq k = d_H(a, b)$, $\forall\, a, b \in f(G) \subset$ H & $\forall\, x \in f^{-1}(a)$, $y \in f^{-1}(b)$.
Conversely suppose $d_G(x, y) \leq d_H(a, b)$, $\forall\, a, b \in f(G) \subset$ H & $\forall\, x \in f^{-1}(a)$, $y \in f^{-1}(b)$. Let $f(x) = a$ & $f(y) = b$ in H such that $f(x) \sim f(y)$. Cleary if $a \neq b$ in H, then $x \neq y$ in G.

Then $d_H(a, b) = 1$.
$\Rightarrow d_G(x, y) \leq d_H(a, b)$,$\forall\ a, b \in f(G) \subset H$ & $\forall\ x \in f^{-1}(a),\ y \in f^{-1}(b)$.
$\Rightarrow d_G(x, y) = 0$ -or- $d_G(x, y) = 1$ in G, $\forall\ x \in f^{-1}(a),\ y \in f^{-1}(b)$.
$\Rightarrow$ $x = y$ -or- $x \sim y$ in G. But $x \neq y$ because $a \neq b$ in H.
So, $x \sim y$ in G, $\forall\ x \in f^{-1}(a),\ y \in f^{-1}(b)$.
$\Rightarrow$ $u \sim v$ in G. Thus the mapping f is a complementary homomorphism. □

Similarly we can prove the following,

Corollary 3.35.1:
If a mapping $f : V(G) \rightarrow V(H)$ is a quasi complementary homomorphism, then $d_G(x, y) \leq d_H(a, b)$,$\forall\ a, b \in f(G) \subset H$ & $\forall\ x \in f^{-1}(a),\ y \in f^{-1}(b)$.

Remark:
Counter example 3.26:
Converse of this corollary 3.35.1 is not true. In the following mapping $f : V(G) \rightarrow V(H)$, it is clear that $d_G(x, y) \leq d_H(a, b)$,$\forall\ a, b \in f(G) \subset H$ & $\forall\ x \in f^{-1}(a),\ y \in f^{-1}(b)$. Here $f(a) = f(c) = 1$ in H. But $a \nsim c$ in G.

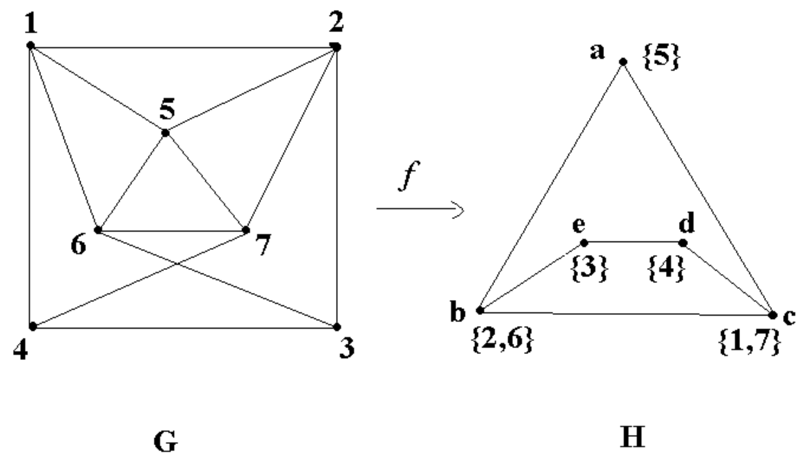


Some parallel results we can draw by above Results.

Theorem 3.36:
A mapping $f : V(G) \rightarrow V(H)$ is a strong homomorphism(or strong quasi homomorphism) than then
(**A**) $d_H(f(u), f(v)) \leq d_G(u, v)$,$\forall\ u, v$ in G.
(**B**) $d_G(x, y) \leq d_H(a, b)$,$\forall\ a, b \in f(G) \subset H$ & $\forall\ x \in f^{-1}(a),\ y \in f^{-1}(b)$.

You may feel that in strong homomorphism (or strong quasi homomorphism) $d_H(f(x), f(y)) = d_G(x, y)$, $\forall\, x, y$ in G.
But it is not true.

Tester:
Example 3.27: In the following mapping f is a strong homomorphism.



Now consider all the distance for non adjacent vertices from above mapping ,

Vertices in G $(u,,v)$	$d_G(u,v)$ (I)	$d_H(f(u),f(v))$ (II)	Result	Vertices in H (a,b)	$d_H(a,b)$ (III)	$d_G(x,y)$ (IV)	Result
(1,3)	2	2		(a,e)	2	$d_G(5,3) = 2$	
(1,7)	2	0	(I) ≥ (II)	(a,d)	2	$d_G(5,4) = 2$	
(2,6)	2	0		(b,d)	2	$d_G(2,4) = 2$	(III) ≥ IV) **B**
(2,4)	2	2				$d_G(6,4) = 2$	
(3,5)	2	2	**A**	(e,c)	2	$d_G(3,1) = 2$	
(3,7)	2	2				$d_G(3,7) = 2$	
(4,6)	2	2					
(4,5)	2	2					

Table-3.3

Neighborhood

Adjacency relation is a core feature of any neighborhood of a vertex. We have a nice result for this concept in homomorphism.
(1) A function $f : V(G) \rightarrow V(H)$ is a homomorphism if and only if for every vertex $v \in V(G)$, $f(N(v)) \subset N(f(v))$.
(2) A function $f : V(G) \rightarrow V(H)$ is a homomorphism then for every vertex $v \in V(G)$, $f(N[v]) \subset N[f(v)]$.

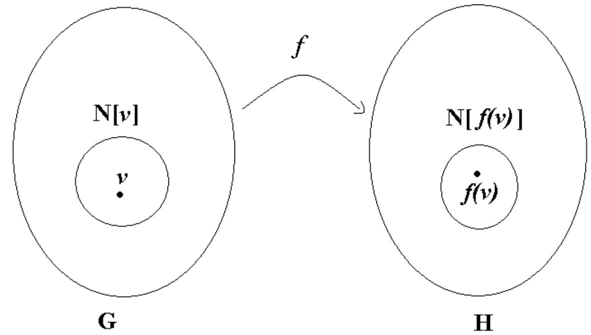
The result (1) about open neighborhood is not true for quasi-homomorphism.
Justification: Suppose $y \sim v$ in G such that $f(y) = f(v)$ in H. Then $f(y) \notin N(f(v))$ & $f(y) \in f(N(v))$.Which is also explained in the remark next to the theorem 3.37

And for the closed neighborhood quasi-homomorphism have opposite nature.

Theorem 3.37:
A function $f : V(G) \to V(H)$ is a quasi-homomorphism if and only if for every vertex $v \in V(G)$, $f(N[v]) \subset N[f(v)]$.

Proof:
Suppose $f : V(G) \to V(H)$ is a quasi-homomorphism.
Let $x \in f(N[v])$
$\Rightarrow x = f(y)$ for some $y \in N[v]$
$\Rightarrow y \sim v$ or $y = v$ in graph G.
$\Rightarrow f(y) = f(v)$ -or- $f(y) \sim f(v)$ in H ,as f is quasi-homomorphism.
$\Rightarrow f(y) \in N[f(v)]$
$\Rightarrow x \in N[f(v)]$
Thus, $f(N[v]) \subset N[f(v)]$.
Conversely suppose $f(N[v]) \subset N[f(v)]$ for every vertex $v \in V(G)$.
Let $x \sim v$ in graph G.
$\Rightarrow x \in N[v]$
$\Rightarrow f(x) \in f(N[v])$
$\Rightarrow f(x) \in N[f(v)]$ as $f(N[v]) \subset N[f(v)]$
$\Rightarrow f(x) = f(v)$ or $f(x) \sim f(v)$ in graph H.
Thus f is a quasi-homomorphism. □

ABSTRACT VIEW



Remark:

Counter example 3.28:

Converse of this theorem3.37 is not true for homomorphism. In the following mapping $f : V(G) \rightarrow V(H)$, it is clear that $f(N[v]) \subset N[f(v)]$,for every vertex $v \in V(G)$ which is shown in table-3.4.

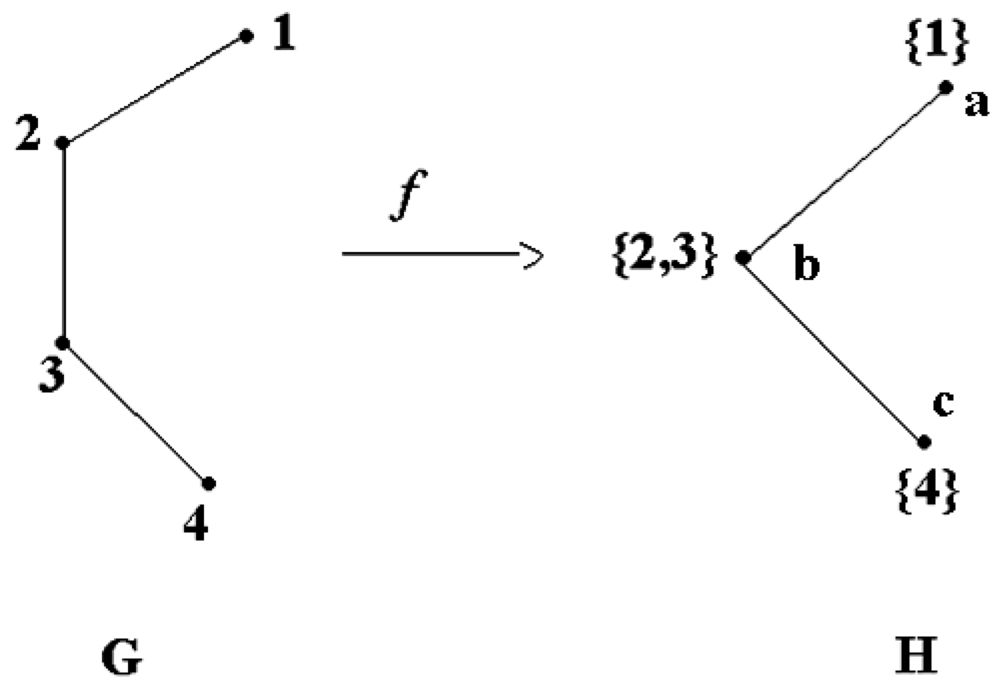


Table-3.4

Vertex v in G	N [v] in graph G	f(N [v]) in graph H	N[f(v)] in graph H	Result
1	{1,2}	{a,b}	{a,b}	$f(N[v]) \subset N[f(v)]$
2	{1,2,3}	{a,b}	{a,b,c}	
3	{2,3,4}	{c,b}	{a,b,c}	
4	{3,4}	{c,b}	{c,b}	

But it is not homomorphism because $2 \sim 3$ in G $\nRightarrow f(2) \sim f(3)$ in H.

Counter example 3.29:

For the open neighborhood, quasi-homomorphism is fail, In example 3.28 , f is quasi homomorphism but $f(N(v)) \not\subset N(f(v))$ for every vertex $v \in V(G)$ which is shown in table-3.5.

Table-3.5

Vertex v in G	N (v) in graph G	f(N (v)) in graph H	N(f(v)) in graph H	Result
1	{2}	{b}	{b}	$f(N(v)) \subset N(f(v))$
2	{1,3}	{a,b}	{a,c}	$f(N(v)) \not\subset N(f(v))$
3	{2,4}	{c,b}	{a,c}	$f(N(v)) \not\subset N(f(v))$
4	{3}	{b}	{b}	$f(N(v)) \subset N(f(v))$

Theorem 3.38:
A function $f : V(G) \to V(H)$ is such that for every vertex $v \in V(G)$,
$f(N(v)) \subset N(f(v))$ then it is quasi-homomorphism.

Proof: Let $x \sim v$ in graph G.
$\Rightarrow x \in N(v)$
$\Rightarrow f(x) \in f(N(v))$
$\Rightarrow f(x) \in N(f(v))$ as $f(N(v)) \subset N(f(v))$
$\Rightarrow f(x) \sim f(v)$ in graph H.
Thus f is a quasi-homomorphism. □

Similarly we can get the results for quasi complementary homomorphism & complementary homomorphism.

Theorem 3.39:
A function $f : V(G) \to V(H)$ is a complementary homomorphism
if and only if for every vertex $v \in V(G)$, $f(N[v]) \supset N[f(v)]$.

Proof:
Suppose $f : V(G) \to V(H)$ is a complementary homomorphism.
Let $f(x) \notin f(N[v])$
$\Rightarrow x \notin N[v]$
$\Rightarrow x \nsim v$ & $x \neq v$ in graph G.
$\Rightarrow x \nsim v$ in graph G.
$\Rightarrow f(x) \nsim f(v)$ in graph H, since f is complementary homomorphism.
$\Rightarrow f(x) \notin N[f(v)]$.Thus $f(N[v]) \supset N[f(v)]$.
Conversely suppose $f(N[v]) \supset N[f(v)]$ for every vertex $v \in V(G)$.
Let $f(x) \sim f(v)$ in graph H.
$\Rightarrow f(x) \in N[f(v)] \subset f(N[v])$
$\Rightarrow f(x) \in f(N[v])$ in H.
$\Rightarrow x \in N[v]$ in G.

$\Longrightarrow x = v$ -or- $x \sim v$ in G. But $x = v$ is not possible as graph H is simple.
Thus f is a complementary homomorphism. □

Theorem 3.40:
A function $f : V(G) \to V(H)$ is a complementary homomorphism
if and only if for every vertex $v \in V(G)$, $f(N(v)) \supset N(f(v))$

Proof:
Let a function $f : V(G) \to V(H)$ is a complementary homomorphism.
Let $f(x) \in N(f(v))$ in H.
$\Longrightarrow f(x) \sim f(v)$ in graph H.
$\Longrightarrow x \sim v$ in G, since f is complementary homomorphism.
$\Longrightarrow x \in N(v)$ in G.
$\Longrightarrow f(x) \in f(N(v))$ in H. Thus $f(N(v)) \supset N(f(v))$.
Conversely suppose $f(N(v)) \supset N(f(v))$ for every vertex $v \in V(G)$.
Let $f(x) \sim f(v)$ in graph H.
$\Longrightarrow f(x) \in N(f(v)) \subset f(N(v))$ in H.
$\Longrightarrow f(x) \in f(N(v))$ in H.
$\Longrightarrow x \in N(v)$ in G.
$\Longrightarrow x \sim v$ in G. Thus is a complementary homomorphism. □

By similar kind of argument gives us another fact.

Theorem 3.41:
A function $f : V(G) \to V(H)$ is a quasi-complementary homomorphism, then
for every vertex $v \in V(G)$, $f(N[v]) \supset N[f(v)]$.

Theorem 3.42:
A function $f : V(G) \to V(H)$ is a quasi-complementary homomorphism, then
for every vertex $v \in V(G)$, $f(N(v)) \supset N(f(v))$.

Remark:
Counter example 3.30:
Converse of above result theorem3.41 is not true for quasi complementary homomorphism. i.e. If $f : V(G) \to V(H)$ is any mapping such that, for every vertex $v \in V(G)$, $f(N[v]) \supset N[f(v)]$, then f need not be quasi complementary homomorphism. It is shown in the following mapping and table 3.6.

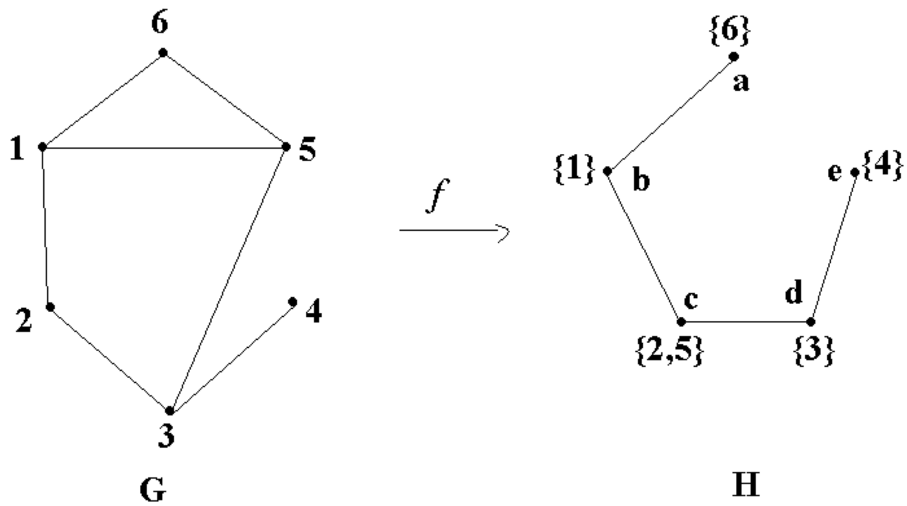


Table-3.6

Vertex v in G	N[v] in G	f(N [v]) in H	N[f(v)] in H	Result
1	{1,6,5,2}	{a,b,c}	{a,b,c}	
2	{1,2,3}	{b,c,d}	{b,c,d}	
3	{2,3,4,5}	{c,d,e}	{c,d,e}	
4	{3,4}	{d,e}	{d,e}	f(N [v]) ⊃ N[f(v)]
5	{1,6,5,3}	{a,b,c,d}	{b,c,d}	
6	{1,5,6}	{a,b,c}	{a,b}	

Here , f is not quasi complementary homomorphism as $f(2) = f(5)$ in H but $2 \nsim 5$ in G. But it is complementary homomorphism.

Counter example 3.31:

Similarly it can easily verify for the converse of theorem 3.42 from the following figure. In the following mapping $f : V(G) \rightarrow V(H)$ .It is clear that for every vertex $v \in V(G)$, $f(N(v)) \supset N(f(v))$, which is also verified in the given table-3.7

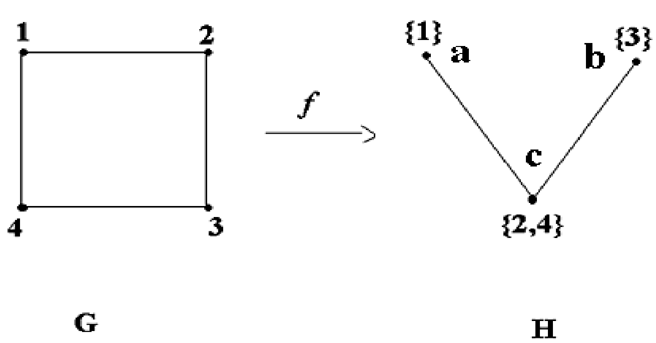


Table-3.7

Vertex v in G	N(v)in G	f(N(v)) in H	N(f(v)) in H	Result
1	{2,4}	{b}	{b}	
2	{1,3}	{c,a}	{c,a}	
3	{2,4}	{b}	{b}	f(N (v)) ⊃ N(f(v))
4	{3,1}	{b,c}	{a,c}	

But f is not quasi complementary homomorphism as $f(2) = f(4)$ in H but $2 \nsim 4$ in G.
Now we can clearly state the following.

Theorem 3.43:
An onto function $f : V(G) \rightarrow V(H)$ is a strong homomorphism if and only if for every vertex $v \in V(G)$, $f(N(v)) = N(f(v))$.

Tester:
Example 3.32:
The following mapping f is strong homomorphism from graph G onto graph H. Clearly for every vertex $v \in V(G)$, $f(N(v)) = N(f(v))$. Above result is tested in table 3.8

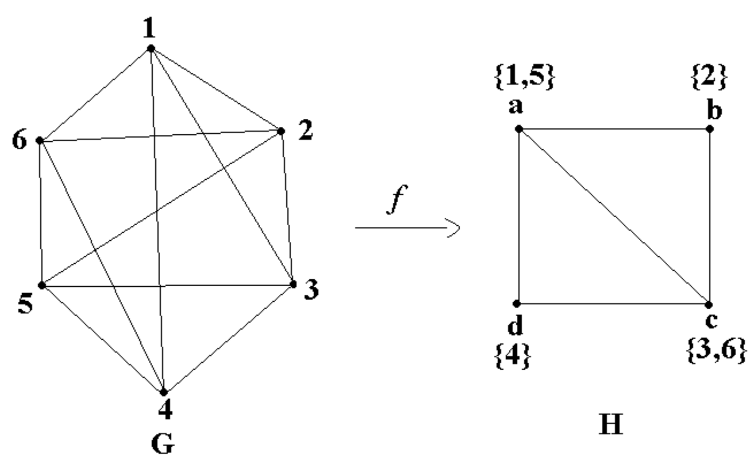


Table-3.8

Vertex v in G	N(v) in G	f(N (v)) in H	N(f(v)) in H	Result
1	{2,3,4,6}	{d,b,c}	{d,b,c}	
2	{1,3,5,6}	{a,c}	{a,c}	
3	{1,2,4,5}	{b,d,a}	{b,d,a}	f(N(v)) = N(f(v))
4	{1,3,5,6}	{a,c}	{a,c}	
5	{,3,4,6}	{b,c,d}	{b,c,d}	
6	{1,2,4,5}	{a,b,d}	{a,b,d}	

Projection Map

Projection map are very natural mapping. It is also very interesting because domain send all the images to one of its component. Sometimes it is very easy to study the properties of one compound graph to its single part. Obviously anyone get influenced by its nature.

In the following table 3.9, the effect of variants of homomorphism under different projection maps on the various graph products (defined on the vertex set) are shown. It is also clear that most of the variants don't have projective nature except quasi- homomorphism.

Projection Map	Homo-morphism	Quasi Homo-morphism	Strong homomorphism	Complementary homomorphism	Quasi-complementary homomorphism	Strong -quasi homomorphism
$\pi_1(\pi_2)$:G×H →G(H)	Yes	Yes	No	No	No	No
$\pi_1(\pi_2)$:G□H →G(H)	No	Yes	No	No	No	No
$\pi_1(\pi_2)$:G⊠H →G(H)	No	Yes	No	No	No	No
π_1:G∘H →G	No	Yes	No	Yes	No	No
π_2 : G∘H → H	No	No	No	No	No	No

Table-3.9: Projections and variants of homomorphism

Our specific interest in this study is "By which condition on the graph we can change this NO into YES". Infect we are able to justify it in the following theorems.

Theorem 3.44:
Projection map π_2 :V(G∘H) →V(H) is (1) Quasi homomorphism, if H is complete graph (2) Quasi complimentary homomorphism, if G is complete graph.

Proof:
First we prove that projection map π_2 :V(G∘H) →V(H) is quasi homomorphism.
Consider graph H as complete graph. Let $(u_1, v_1) \sim (u_2, v_2)$ in G∘H.
Case-1: Let $u_1 \sim u_2$ in G.Then there are two possibilities between vertices v_1 &v_2 in H.(1) $v_1 = v_2$ (2) $v_1 \sim v_2$.(If $v_1 \neq v_2$ then $v_1 \sim v_2$ in the complete graph H)
$\Rightarrow$ $\pi_2(u_1, v_1) \sim \pi_2(u_2, v_2)$ -or- $\pi_2(u_1, v_1) = \pi_2(u_2, v_2)$ in H.
Case-2: $u_1 = u_2$ in G & $v_1 \sim v_2$ in H.
$\Rightarrow$ $\pi_2(u_1, v_1) \sim \pi_2(u_2, v_2)$ in H. Thus projection map π_2: V(G∘H) →V(H) is quasi homomorphism.
Now, Consider graph G as complete graph. Let $(u_1, v_1) \neq (u_2, v_2)$ in G∘H.

Case-1: $\pi_2 (u_1, v_1) = \pi_2 (u_2, v_2)$ in H.
$\Rightarrow v_1 = v_2$ in H.
$\Rightarrow u_1 \neq u_2$ in G ,as $(u_1, v_1) \neq (u_2, v_2)$
$\Rightarrow u_1 \sim u_2$ in G. For any u_1 & u_2 in the complete graph G.
$\Rightarrow (u_1, v_1) \sim (u_2, v_2)$ in G∘H.
Case-2: $\pi_2 (u_1, v_1) \sim \pi_2 (u_2, v_2)$ in H.
$\Rightarrow v_1 \sim v_2$ in H.
$\Rightarrow (u_1, v_1) \sim (u_2, v_2)$ in G∘H. For any u_1 & u_2 in complete graph G.
Thus projection map π_2 :V(G∘H) →V(H) is quasi complimentary homomorphism.

□

Theorem 3.45:
Projection map $\pi_1(\pi_2)$:V(G⊠H) →V(G){or V(H)} is quasi complimentary homomorphism, if H (or G) is complete graph.

Proof:
Let's prove the result for first projection. Consider graph H as complete graph.
Let $(u_1, v_1) \neq (u_2, v_2)$ in G⊠H.
Case-1: $\pi_1 (u_1, v_1) = \pi_1 (u_2, v_2)$ in G.
Then $u_1 = u_2$ in G and so $v_1 \sim v_2$ as H is complete.
$\Rightarrow (u_1, v_1) \sim (u_2, v_2)$ in G⊠H.
Case-2: $\pi_1 (u_1, v_1) \sim \pi_1 (u_2, v_2)$ in G.
$\Rightarrow u_1 \sim u_2$ in G.
$\Rightarrow$ Then there are two possibilities between vertices v_1 &v_2 in H.
(1) $v_1 = v_2$ (2) $v_1 \sim v_2$.(If $v_1 \neq v_2$ then $v_1 \sim v_2$ in the complete graph H)
$\Rightarrow (u_1, v_1) \sim (u_2, v_2)$ in G⊠H. For any v_1 & v_2 in complete graph H.
Thus projection map π_1 :V(G⊠H)→V(H) is quasi complimentary homomorphism.
Similarly we can prove for second projection. □

Similarly we can prove the following result.

Theorem 3.46:
Projection map $\pi_1(\pi_2)$:V(G⊠H) →V(G){or V(H)} is strong quasi homomorphism, if H (or G) is complete graph.

Theorem 3.47:
Projection map π_1:V(G∘H) →V(G) is quasi complimentary homomorphism, if H is complete graph.

Theorem 3.48:
Projection map π_1:V(G∘H) →V(G) is strong quasi homomorphism, if H is complete graph.

Theorem 3.49:
Projection map π_2; V(G∘H) →V(H) is strong quasi homomorphism, if G and H both are complete graphs.

Tester:
Example 3.33: In the following example graph H =K_2 which is a complete graph.

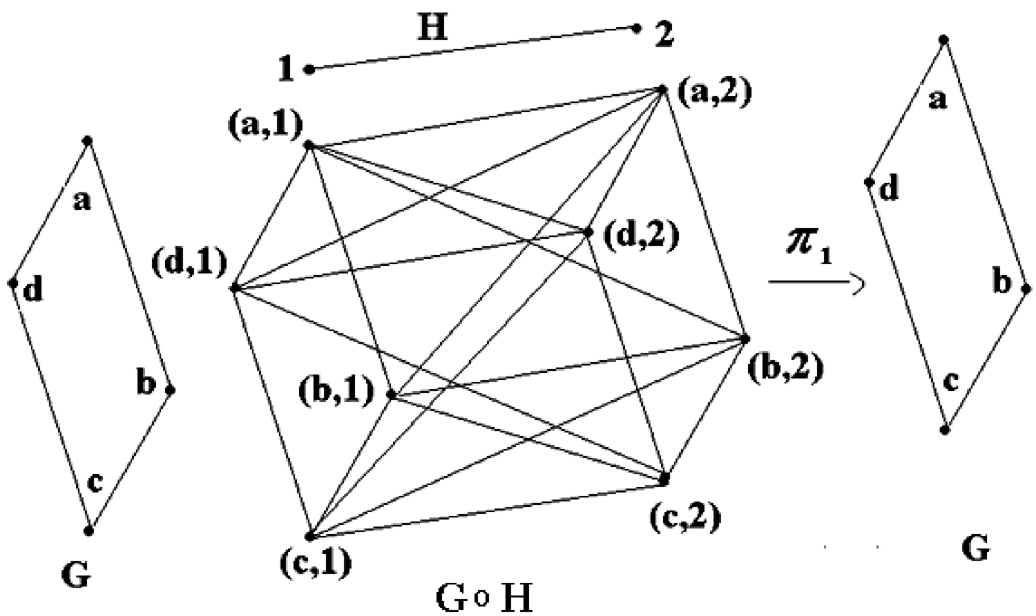


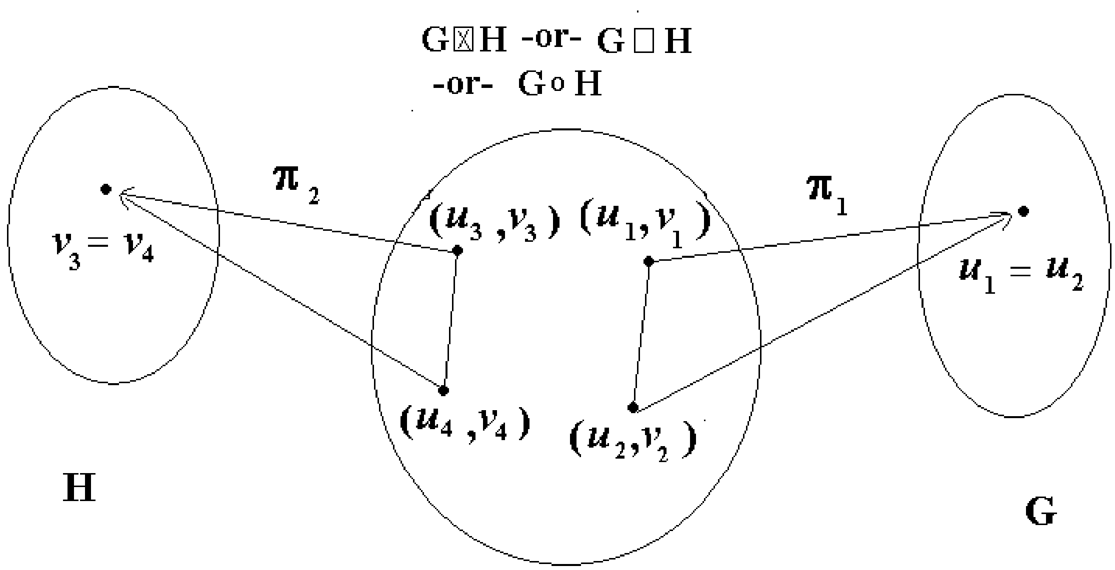
Table-3.10

Projection Map π_1:G∘H →G	Projection Map π_2:V(G∘H) →V(H)
$\pi_1(a\ ,\alpha,)$→a	$\pi_2(a\ ,\alpha,)\rightarrow \alpha$
$\pi_1(a\ ,\beta,)$→a	$\pi_2(a\ ,\beta,)\rightarrow \beta$
$\pi_1(b,\alpha,)$→ b	$\pi_2(b,\alpha,)\rightarrow \alpha$
$\pi_1(b\ \ ,\beta,)$→b	$\pi_2(b\ \ ,\beta,)\rightarrow \beta$
$\pi_1(c\ ,\alpha,)$→c	$\pi_2(c\ ,\alpha,)\rightarrow \alpha$
$\pi_1(c\ ,\beta,)$→b	$\pi_2(c\ ,\beta,)\rightarrow \beta$
$\pi_1(d\ ,\alpha,)$→d	$\pi_2(d\ ,\alpha,)\rightarrow \alpha$
π_1 (d, β) $\rightarrow d$	π_2 (d, β) $\rightarrow \beta$

It is clear from the above table 3.10 that the projection map π_1:V(G∘H) →V(G) is (1) Complimentary homomorphism (2) Quasi complimentary homomorphism (3) Quasi homomorphism. Similarly projection map π_2 :V(G∘H) →V(H) is quasi homomorphism.

Remark:
(1)Projection map never become homomorphism under Cartesian product or Lexicographic product or Strong product.
Justification:
In the case of (i)$u_1 = u_2$ in G & $v_1 \sim v_2$ in H, then $(u_1, v_1) \sim (u_2, v_2)$ in G□H –or- in G∘H. First projections are equal.(ii) $u_3 \sim u_4$ in G & $v_3 = v_4$ in H, then $(u_3, v_3) \sim (u_4, v_4)$ in G⊠H. Second projections are equal.

ABSTRACT VIEW

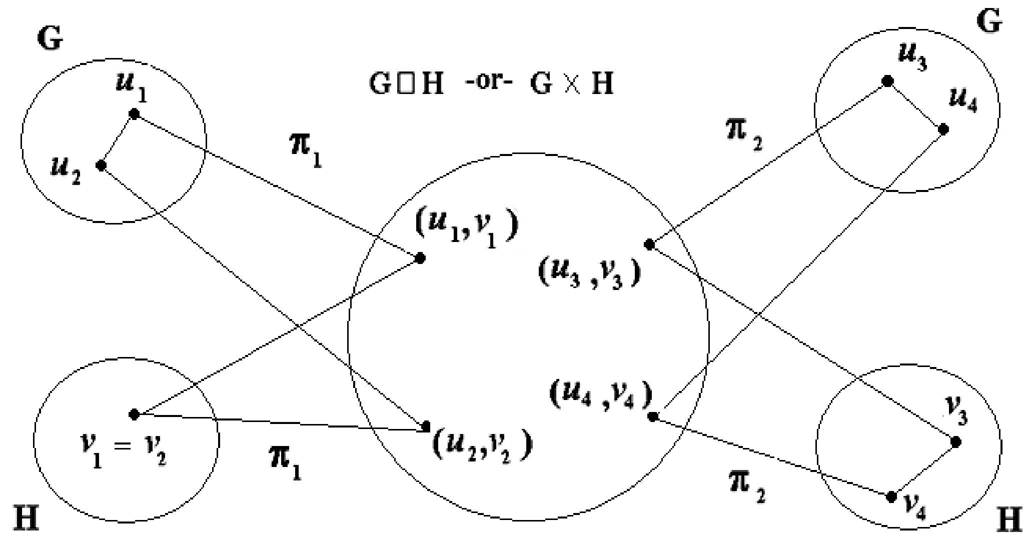


(2) None of the projection map is pure quasi homomorphism for any graph product.
Justification: Let π_1 : V(G*H) →V(G) is the first projection map for any graph product (G*H).
Let $(u_1, v_1) \sim (u_2, v_2)$ in G*H.
Then $\pi_1 (u_1, v_1) = u_1 \neq u_2 = \pi_1 (u_2, v_2)$, for any (u_1, v_1) & (u_2, v_2) in G*H.

(3) Similarly projection map never become complimentary homomorphism or quasi complimentary homomorphism for weak product or Cartesian product.

Justification:
In the case of (I) $u_1 \sim u_2$ in G & $v_1 = v_2$ in H, then $(u_1, v_1) \nsim (u_2, v_2)$ in G×H .
(II) $u_3 \sim u_4$ in G & $v_1 \sim v_4$ in H, then $(u_3, v_3) \nsim (u_4, v_4)$ in G□H.

ABSTRACT VIEW



Composition of Projection Map with Variants of Homomorphism

In this section we will study the composition of projection map with different variants of homomorphism for various graph products.

Theorem 3.50:
A function f: V(Z) → V(G×H) is a pure quasi homomorphism if and only if $\pi_1 \circ f$: V(Z) → V(G) and $\pi_2 \circ f$: V(Z) → V(H) are pure quasi homomorphisms.

Proof:
Suppose f:V(Z) →V(G×H) is a pure quasi homomorphism. Let $z_1 \sim z_2$ in Z.
Let $f(z_1) = (g_1,h_1) = (g_2,h_2) = f(z_2)$ in G×H.
Therefore $\pi_1 \circ f(z_1) = g_1 = g_2 = \pi_2 \circ f(z_2)$ in G &
$\pi_1 \circ f(z_1) = h_1 = h_2 = \pi_2 \circ f(z_2)$ in H.
Conversely suppose $\pi_1 \circ f$: V(Z) → V(G) and $\pi_2 \circ f$: V(Z) → V(H) are pure quasi homomorphisms. Let $z_1 \sim z_2$ in Z. Let $f(z_1) = (g_1,h_1)$ & $f(z_2) = (g_2,h_2)$ in G×H. Then $\pi_1 \circ f(z_1) = g_1 = g_2 = \pi_2 \circ f(z_2)$ in G & $\pi_1 \circ f(z_1) = h_1 = h_2 = \pi_2 \circ f(z_2)$ in H. Hence $f(z_1) = (g_1,h_1) = (g_2,h_2) = f(z_2)$ in G×H.
Thus, f: V(Z) → V(G×H) is a pure quasi homomorphism. □

Theorem 3.51:
A function f: V(Z) → V(G⊠H) is a quasi-homomorphism if and only if
$\pi_1 \circ f : \mathrm{V(Z)} \rightarrow \mathrm{V(G)}$ and $\pi_2 \circ f : \mathrm{V(Z)} \rightarrow \mathrm{V(H)}$ are quasi-homomorphisms.

Proof :
Let f: V(Z) → V(G⊠H) be quasi-homomorphism. Let $x \sim y$ in Z .
$\Rightarrow f(x) \sim f(y)$ –or- $f(x) = f(y)$ in G⊠H.
Let $f(x) = (g_1, h_1)$ and $f(y) = (g_2, h_2)$ in G⊠H.
$\Rightarrow (g_1, h_1) \sim (g_2, h_2)$ –or- $(g_1, h_1) = (g_2, h_2)$ in G⊠H.
$\Rightarrow g_1 = g_2$ -or- $g_1 \sim g_2$ in G.
$\Rightarrow \pi_1 o f(x) = \pi_1(g_1, \mathrm{h}_1) = g_1$ and $\pi_1 o f(y) = \pi_1(g_2, h_2) = g_2$.
$\Rightarrow \pi_1 o f(x) = \pi_1 o f(y)$ -or- $\pi_1 o f(x) \sim \pi_1 o f(y)$ in G.
Thus $\pi_1 o f$:Z → G is a quasi-homomorphism. Similarly $\pi_2 o f$:Z → H is a quasi-homomorphism.
Conversely, suppose $\pi_1 \circ f : \mathrm{V(Z)} \rightarrow \mathrm{V(G)}$ and $\pi_2 \circ f : \mathrm{V(Z)} \rightarrow \mathrm{V(H)}$ are quasi-homomorphism.
Let $x \sim y$ in Z. Let $f(x) = (g_1, h_1)$ and $f(y) = (g_2, h_2)$ in G⊠H.
Now, $\pi_1 o f(x) = \pi_1 o f(y)$ -or- $\pi_1 o f(x) \sim \pi_1 o f(y)$ in G &
$\pi_2 o f(x) = \pi_2 o f(y)$ -or- $\pi_2 o f(x) \sim \pi_2 o f(y)$ in H.
Case 1 : $\pi_1 o f(x) \sim \pi_1 o f(y)$ in G & $\pi_2 o f(x) \sim \pi_2 o f(y)$ in H.
$\Rightarrow g_1 \sim g_2$ in G & $h_1 \sim h_2$ in H.
$\Rightarrow (g_1, h_1) \sim (g_2, h_2)$ in G⊠H.
$\Rightarrow f(x) \sim f(y)$ in G⊠H.
Case 2 : $\pi_1 o f(x) \sim \pi_1 o f(y)$ in G & $\pi_2 o f(x) = \pi_2 o f(y)$ in H.
$\Rightarrow g_1 \sim g_2$ in G & $h_1 = h_2$ in H.
$\Rightarrow (g_1, h_1) \sim (g_2, h_2)$ in G⊠H.
$\Rightarrow f(x) \sim f(y)$ in G⊠H.
Case 3: $\pi_1 o f(x) = \pi_1 o f(y)$ & $\pi_2 o f(x) = \pi_2 o f(y)$
$\Rightarrow g_1 = g_2$ in G & $h_1 = h_2$ in H.
$\Rightarrow (g_1, h_1) = (g_2, h_2)$ in G⊠H.
$\Rightarrow f(x) = f(y)$ in G⊠H.
Case 4 : $\pi_1 o f(x) = \pi_1 o f(y)$ & $\pi_2 o f(x) \sim \pi_2 o f(y)$
$\Rightarrow g_1 = g_2$ in G & $h_1 \sim h_2$ in H.
$\Rightarrow (g_1, h_1) \sim (g_2, h_2)$ in G⊠H.
$\Rightarrow f(x) \sim f(y)$ in G⊠H.
Thus f :V(Z) → V(G⊠H) is a quasi-homomorphism. □

Remark:
A function $f:V(Z) \to V(G\times H)$ is a homomorphism if and only if
$\pi_1 \circ f : V(Z) \to V(G)$ and $\pi_2 \circ f : V(Z) \to V(H)$ are homomorphism.
[Refer result 1. 13**]**

Theorem 3.52:
A function $f:V(Z) \to V(G\times H)$ is a complimentary homomorphism , if
$\pi_1 \circ f : V(Z) \to V(G)$ and $\pi_2 \circ f : V(Z) \to V(H)$ are complimentary homomorphisms.

Proof :
Suppose $\pi_1 \circ f : V(Z) \to V(G)$ and $\pi_2 \circ f : V(Z) \to V(H)$ are complimentary homomorphisms.
Let $f(x) = (g_1, h_1)$ and $f(y) = (g_2, h_2)$ in G×H, such that $f(x) \sim f(y)$.
$\Rightarrow (g_1, h_1) \sim (g_2, h_2)$ in G×H.
$\Rightarrow g_1 \sim g_2$ in G & $h_1 \sim h_2$ in H.
$\Rightarrow \pi_1 o\, f(x) \sim \pi_1 o\, f(y)$ in G & $\pi_2\, o\, f(x) \sim \pi_2\, o\, f(y)$ in H.
$\Rightarrow x \sim y$ in Z , since $\pi_1\, o\, f$ & $\pi_2\, o\, f$ are complimentary homomorphism.
Thus f is complimentary homomorphism. □

Theorem 3.53:
A function f: $V(Z) \to V(G\boxtimes H)$ is a quasi- complimentary homomorphism , if
$\pi_1 \circ f : V(Z) \to V(G)$ and $\pi_2 \circ f : V(Z) \to V(H)$ are quasi- complimentary homomorphisms.

Proof :
Suppose $\pi_1 o f : V(Z) \to V(G)$ and $\pi_2 o f : V(Z) \to V(H)$ are quasi- complimentary homomorphisms.
Let $f(x) = (g_1, h_1)$ and $f(y) = (g_2, h_2)$ in $G\boxtimes H$.
Let $x \neq y$ in Z.
<u>Case-1:</u> $f(x) = f(y)$ in $G\boxtimes H$.
$\Rightarrow (g_1, h_1) = (g_2, h_2)$ in $G\boxtimes H$.
$\Rightarrow \pi_1 o\, f(x) = \pi_1\, o\, f(y)$ in G & $\pi_2\, o\, f(x) = \pi_2\, o\, f(y)$ in H.
$\Rightarrow x \sim y$ in Z, as $\pi_1\, o\, f(x)$ & $\pi_2\, o\, f(x)$ are quasi-complimentary homomorphism.
<u>Case-2:</u> $f(x) \sim f(y)$ in $G\boxtimes H$.
$\Rightarrow (g_1, h_1) \sim (g_2, h_2)$ in $G\boxtimes H$.
Then there are three possibilities (i) $g_1 \sim g_2$ in G & $h_1 = h_2$ in H(ii) $g_1 = g_2$ in G & $h_1 \sim h_2$ in H (iii) $g_1 \sim g_2$ in G & $h_1 \sim h_2$ in H.

Clearly in all three possibilities $x \sim y$ in Z, as $\pi_1 o f(x)$ & $\pi_2 o f(x)$ are quasi-complimentary homomorphism. Thus f is a quasi-complimentary homomorphism.

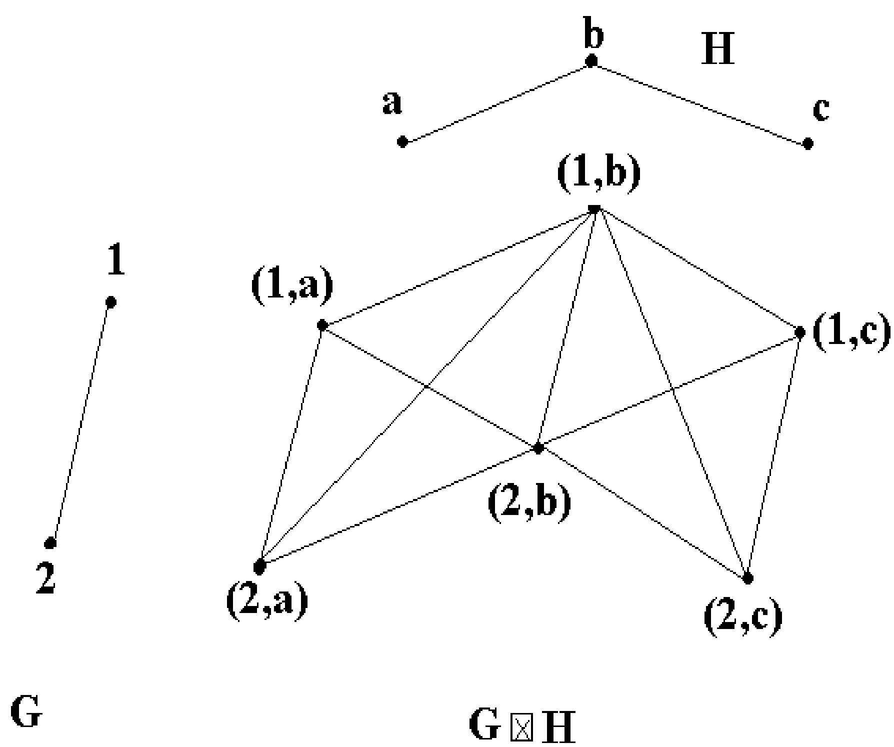
□

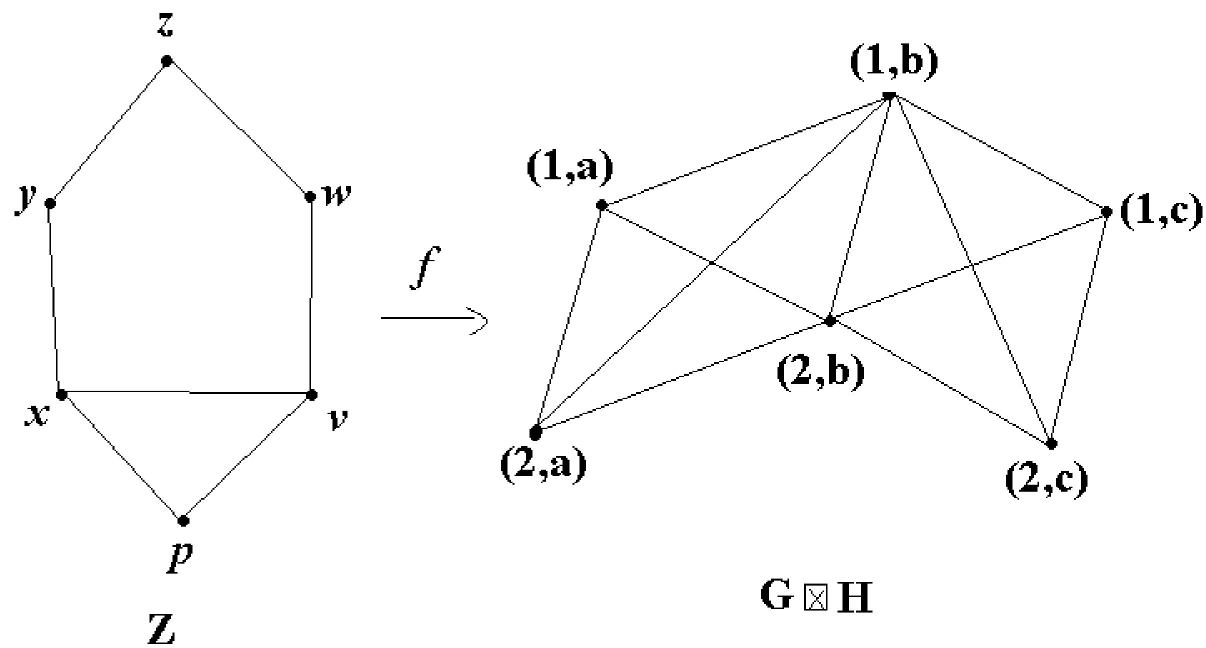
Similarly we can prove the following result.

Theorem 3.54:
A function f: V(Z) → V(G⊠H) s a strong quasi homomorphism , if
$\pi_1 \circ f$: V(Z) → V(G) and $\pi_2 \circ f$: V(Z) → V(H) are strong quasi homomorphisms.

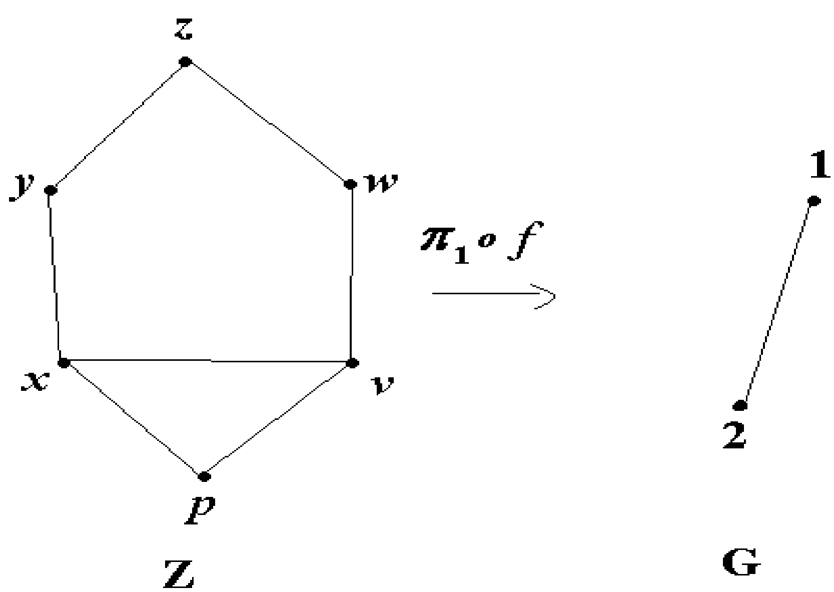
Theorem 3.55:
A function f :V(Z) → V(G×H)is a strong homomorphism , if
$\pi_1 \circ f$: V(Z) → V(G) and $\pi_2 \circ f$: V(Z) → V(H) are strong homomorphisms.

Tester:
Example 3.34:
In the following example function f: V(Z) → V(G⊠H) is a homomorphism but
$\pi_1 \circ f$: V(Z) → V(G) and $\pi_2 \circ f$: V(Z) → V(H) are not homomorphism. This is also verified.





A function f: V(Z) → V(G⊠H) defined as follow:
$f(x) = (1, a)$, $f(y) = (1, b)$, $f(z) = (1, c)$, $f(p) = (2, a)$, $f(v) = (2, b)$ &
$f(w) = (2, c)$. Then f is homomorphism.



Here $\pi_1 o\, f(v) = 2 = \pi_1 o f(p)$ in G.
So, $v \sim p$ in Z but π_1 o $f(x) \nsim \pi_1$ o $f(y)$ in G. Thus $\pi_1 \circ f$: V(Z) → V(G) is not homomorphism, but it is a quasi homomorphism

Similarly we can observe for second projection map.

Remark:
Some of the above theorems can be extended to product of more then two graphs.

(1)A function f: $V(Z) \rightarrow V(G_1 \times \ldots \times G_n)$is a pure quasi homomorphism if and only if $\pi_i \circ f : V(Z) \rightarrow V(G_i)$ are pure quasi homomorphism $\forall\, i \in \{1,..,n\}$.

(2)A function f: $V(Z) \rightarrow V(G_1 \times \ldots \times G_n)$ is a quasi homomorphism if and only if $\pi_i \circ f : V(Z) \rightarrow V(G_i)$ are quasi homomorphism $\forall\, i \in \{1,..,n\}$.

(3) A function f: $V(Z) \rightarrow V(G_1 \boxtimes \ldots \boxtimes G_n)$ is strong quasi homomorphism if and only if $\pi_i \circ f : V(Z) \rightarrow V(G_i)$ are strong quasi homomorphism $\forall\, i \in \{1,..,n\}$.

<u>Product Function</u>

In this section we study the product of different variant of homomorphism for various graph products. Let's understand these concepts with help of its informal definition.

Let $f : A \rightarrow B$ & $g : C \rightarrow D$ are two maps. Then *product of the function* is $f \times g$ of f & g is the function $f \times g : A \times C \rightarrow B \times D$ defined as $f \times g\,(x, y) = (f(x), g(y))\ \forall\ (x, y) \in A \times C$.

Consider the map $i : V(L) \rightarrow V(L)$ as identity function of any graph L, for the following results.

Theorem 3.56:
A function $f : V(G) \rightarrow V(H)$ is a quasi-homomorphism if and only if the product function $f \times i : V(G \square L) \rightarrow V(H \square L)$ is a quasi-homomorphism from $G \square L$ to $H \square L$, for any graph L.

Proof:
Suppose f is a quasi-homomorphism. Let $(u_1, v_1) \sim (u_2, v_2)$ in $G \square L$.
If $u_1 \sim u_2$,then $v_1 = v_1$. If $f(u_1) \sim f(u_2)$ then, $(f(u_1), i(v_1)) \sim (f(u_2), i(v_2))$ in $H \square L$.
If $f(u_1) = f(u_2)$, then $(f(u_1), i(v_1)) = (f(u_2), i(v_2))$ in $H \square L$.
On other hand,
If $u_1 = u_2$,then $v_1 \sim v_2$. Hence $i(v_1) \sim i(v_2)$ in L.

Thus $(f(u_1), i(v_1)) \sim (f(u_2), i(v_2))$ in H □ L .
Conversely suppose $u_1 \sim u_2$ in G. If v is any vertex of L then $(u_1,v) \sim (u_2,v)$ in G □ L.
$\Longrightarrow (f(u_1),i(v)) \sim (f(u_2),i(v))$ –or- $(f(u_1),i(v)) = (f(u_2),i(v))$ in H×L ,since $f \times i$ is a quasi-homomorphism.
$\Longrightarrow f(u_1) \sim f(u_2)$ –or- $f(u_1) = f(u_2)$ in H.
Thus f is a quasi-homomorphism. □

Theorem 3.57:
A function f : V (G) → V(H) is a quasi- complimentary homomorphism, then the product function $f \times i$: V (G⊠ L)→ V(H⊠L) is a quasi- complimentary homomorphism from G⊠L to H⊠L.

Proof :
Let f: V(Z) → V(H) be quasi- complimentary homomorphism. Let $(u_1, v_1) \neq (u_2, v_2)$
<u>Case-1:</u> $f \times i\ (u_1, v_1) = f \times i\ (u_2,v_2)$ in H⊠L.
$\Rightarrow (f(u_1), i(v_1)) = (f(u_2), i(v_2))$ in H⊠L.
$\Rightarrow i(v_1) = i(v_2)$ in L & $f(u_1) = f(u_2)$ in H.
$\Rightarrow u_1 \sim u_2$ in G & $v_1 = v_1$ in H, since f is a quasi- complimentary homomorphism.
$\Rightarrow (u_1, v_1) \sim (u_2,v_2)$ in G⊠L.
<u>Case-2:</u> $f \times i\ (u_1, v_1) \sim f \times i\ (u_2,v_2)$ in H⊠L.
$\Rightarrow (f(u_1)\ i(v_1)) \sim (f(u_2)\ i(v_2))$ in H⊠L.
There are three possibilities,
(1)$i(v_1) = i(v_2)$ in L & $f(u_1) \sim f(u_2)$ in H (2) $i(v_1) \sim i(v_2)$ in L & $f(u_1) \sim f(u_2)$ in H (3) $i(v_1) \sim i(v_2)$ in L & $f(u_1) = f(u_2)$ in H.
Clearly we get $u_1 \sim u_2$ in G & $v_1 = v_2$ -or- $v_1 \sim v_2$ in H in all three possibilities , since f is a quasi- complimentary homomorphism & any identity map is injective.
$\Rightarrow (u_1, v_1) \sim (u_2,v_2)$ in G⊠L.
Thus the product function $f \times i$: V (G⊠ L)→ V(H⊠L) is quasi- complimentary homomorphism. □

Theorem 3.58:
A function f : V (G) → V(H) is a homomorphism if and only if the product function $f \times i$: V (G⊠ L)→ V(H⊠L) is a homomorphism from G⊠L to H⊠L.

Proof :
Suppose f: V(Z) → V(H) be homomorphism.
Let $(u_1, v_1) \sim (u_2,v_2)$ in G⊠L.
<u>Case-1:</u> $u_1 \sim u_2$ in G & $v_1 = v_2$ in L.

$\Rightarrow i(v_1) = i(v_2)$ in L & $f(u_1) \sim f(u_2)$ in H, as f is homomorphism.
$\Rightarrow (f(u_1), i(v_1)) \sim (f(u_2), i(v_2))$ in $H \boxtimes L$.
<u>Case-2:</u> $u_1 \sim u_2$ in G & $v_1 \sim v_2$ in H.
$\Rightarrow i(v_1) \sim i(v_2)$ in L & $f(u_1) \sim f(u_2)$ in H, as f & i are homomorphisms.
$\Rightarrow (f(u_1), i(v_1)) \sim (f(u_2), i(v_2))$ in $H \boxtimes L$.
<u>Case-3:</u> $u_1 = u_2$ in G & $v_1 \sim v_2$ in H.
$\Rightarrow i(v_1) \sim i(v_2)$ in L & $f(u_1) = f(u_2)$ in H.
$\Rightarrow (f(u_1), i(v_1)) \sim (f(u_2), i(v_2))$ in $H \boxtimes L$.
Thus $f \times i : V(G \boxtimes L) \rightarrow V(H \boxtimes L)$ is a homomorphism.
Conversely suppose $u_1 \sim u_2$ in G.
Let $v \in V(L)$ such that $(u_1, v) \sim (u_2, v)$ in $G \boxtimes L$.
$\Rightarrow (u_1, v) \sim (u_2, v)$ in $G \boxtimes L$.
$\Rightarrow f \times i\, (u_1, v) \sim f \times i\, (u_2, v)$ in $H \boxtimes L$, as $f \times i$ is homomorphism.
$\Rightarrow (f(u_1), i(v)) \sim (f(u_2), i(v))$ in $H \boxtimes L$.
$\Rightarrow f(u_1) \sim f(u_2)$ in H. By the definition of strong product.
Thus f is homomorphism. □

Similarly we can prove the following results.

Theorem 3.59:
A function $f : V(G) \rightarrow V(H)$ is a homomorphism if and only if the product function $f \times i : V(G \times L) \rightarrow V(H \times L)$ is a homomorphism from $G \times L$ to $H \times L$.

Theorem 3.60:
A function $f : V(G) \rightarrow V(H)$ is a strong quasi homomorphism ,then the product function $f \times i : V(G \boxtimes L) \rightarrow V(H \boxtimes L)$ is a strong homomorphism from $G \boxtimes L$ to $H \boxtimes L$.

Combination of different product of graph & different variants of homomorphism can also possible for such product function, which you can observe in the following results.

Theorem 3.61:
Suppose G_1&G_2 are two graphs each having at least one edge and $f_2 : G_2 \rightarrow H_2$ is a homomorphism. Then $f_1 : G_1 \rightarrow H_1$ is a pure quasi homomorphism if and only if $f_1 \times f_2 : G_1 \times G_2 \rightarrow H_1 \square H_2$ is a homomorphism .

Proof:
Suppose $f_1 : G_1 \rightarrow H_1$ is a pure quasi homomorphism & $f_2 : G_2 \rightarrow H_2$ is a homomorphism.

Let $(u_1, u_2) \sim (v_1, v_2)$ in the graph $G_1 \times G_2$.
$\Rightarrow u_i \sim v_i$ in G_i, $\forall\, i = 1,2$.
$\Rightarrow f_1(u_1) = f_1(v_1)$ in H_1 & $f_2(u_2) \sim f_2(v_2)$ in H_2, as f_1 is pure quasi homomorphism & f_2 is a homomorphism.
$\Rightarrow (f_1(u_1), f_2(u_2)) \sim (f_1(v_1), f_2(v_2))$ in $H_1 \square H_2$.
Thus $f_1 \times f_2$: $G_1 \times G_2 \rightarrow H_1 \square H_2$ is a homomorphism.
Conversely suppose $u_1 \sim v_1$ in G_1. Let $u_2 v_2$ be an edge in G_2.
$\Rightarrow (u_1, u_2) \sim (v_1, v_2)$ in $G_1 \times G_2$.
$\Rightarrow (f_1(u_1), f_2(u_2)) \sim (f_1(v_1), f_2(v_2))$ in $H_1 \square H_2$, as $f_1 \times f_2$ is a homomorphism .
$\Rightarrow f_2(u_2) \sim f_2(v_2)$ in H_2 ,since $f_2 : G_2 \rightarrow H_2$ is a homomorphism,
$\Rightarrow f_1(u_1) = f_1(v_1)$ in H_1.Thus $f_1 : G_1 \rightarrow H_1$ is a pure quasi homomorphism. $\square$

Theorem 3.62:
Suppose G_1&G_2 are two graphs each having at least one edge. Then $f_1 : G_1 \rightarrow H_1$ and $f_2 : G_2 \rightarrow H_2$ are pure quasi homomorphism if and only if $f_1 \times f_2 : G_1 \times G_2 \rightarrow H_1 \times H_2$ is a pure quasi homomorphism.

Proof:
Suppose $f_1 : G_1 \rightarrow H_1$ and $f_2 : G_2 \rightarrow H_2$ are pure quasi homomorphisms.
Let $(u_1, u_2) \sim (v_1, v_2)$ in $G_1 \times G_2$.
$\Rightarrow u_i \sim v_i$ in G_i, $\forall\, i \in \{1,2\}$.
$\Rightarrow f_i(u_i) = f_i(v_i)$ in H_i ,as f_i are pure quasi homomorphism , $\forall\, i = 1,2$.
So, $(f_1(u_1), f_2(u_2)) = (f_1(v_1), f_2(v_2))$ in $H_1 \times H_2$ .Thus $f_1 \times f_2$: $G_1 \times G_2 \rightarrow H_1 \times H_2$ is a pure quasi homomorphism
Conversely suppose $u_1 \sim v_1$ in G_1.
Let $u_2 v_2$ be an edge in G_2.
$\Rightarrow (u_1, u_2) \sim (v_1, v_2)$ in $G_1 \times G_2$.
$\Rightarrow (f_1(u_1), f_2(u_2)) = (f_1(v_1), f_2(v_2))$ in $H_1 \times H_2$,as $f_1 \times f_2$ is a pure quasi homomorphism.
$\Rightarrow f_1(u_1) = f_1(v_1)$ in H_1.Thus f_1 is pure quasi homomorphism.
Similarly $f_2 : G_2 \rightarrow H_2$ is a pure quasi homomorphism. $\square$

“The moving power of Mathematics invention is not reasoning but imagination.”
A . DEMORGAN

Function Graphs

CHAPTER 4

4. A. Introduction

Function graph is a very natural and well known concept in Algebraic Graph Theory. Functions between two graphs are the vertices in such a graph but the different way of defining the adjacency between two functions give rise to different variant of function graphs. So we can access the properties of both graphs on larger scale.

In this chapter we define two variants of map graph and we investigate some properties. We find that certain natural functions associated with this graphs are variants of homomorphism. We further find that these graphs are related to the co domain graph of the mapping. Using these results we have proved graphical properties like connectedness, distance, completeness, .. Etc.

Notation: Let $\mathfrak{R}$ (G, H) denote the set of all functions from V (G) to V (H). P(G,H) will denote the set of all pure quasi homomorphisms from V(G) to V(H) & Q(G,H) denote the set of all quasi- homomorphisms from V(G) to V(H).

Definition 4.1: *Map Graph* **[18]** H^G of two graphs G and H defined as follows: Two elements *f* and *g* of $\mathfrak{R}$ (G, H) are said to be adjacent if whenever *u* and *v* are adjacent vertices of G, *f* (*u*) and *g* (*v*) are adjacent vertices of H.
Let P (H^G) denote the induced sub graph on P(G,H) in the map graph H^G.

You can observe that Map graph is a reasonable function graph because we need to consider the adjacency of both the graphs. Even domain graph is strongly associated in this graph.

Now we introduce a new concept called '*Quasi map graph'* which does not consider the structure of domain graph. However it has also interesting properties.

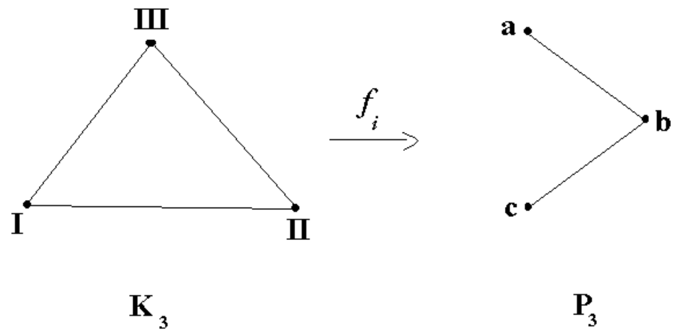
Definition 4.2: *Quasi-Map graph* 'Q (H^G)' is defined as follows: Two distinct elements *f* and *g* of $\mathfrak{R}$ (G, H) are said to be adjacent if *f*(*u*) is adjacent to *g*(*u*) or *f*(*u*) = *g*(*u*) in H, for each vertex *u* in G.

Now I am introducing another variant of function graph namely *'Direct Map Graph'*. This concept also does not consider the structure of the domain graph.

Definition 4.3: *Direct Map graph* 'D (H^G)' is defined as follows: Two distinct elements *f* and *g* of $\mathfrak{R}$ (G, H) are said to be adjacent if *f*(*u*) is adjacent to *g*(*u*) in H, for each vertex *u* in G.
Remark: $Q(H^{K_1}) = D(H^{K_1}) = H^{K_1} = H$.

One straight forward observation from the above definitions is "D (H ^{G}) is spanning sub graph of Q (H ^{G})".

Example 4.1: Consider Graphs G = K_3 & H = P_3.



Now, $|H|^{|G|} = 3^3 = 27$. Thus all possible 27 functions between them are given below in table 4.1.

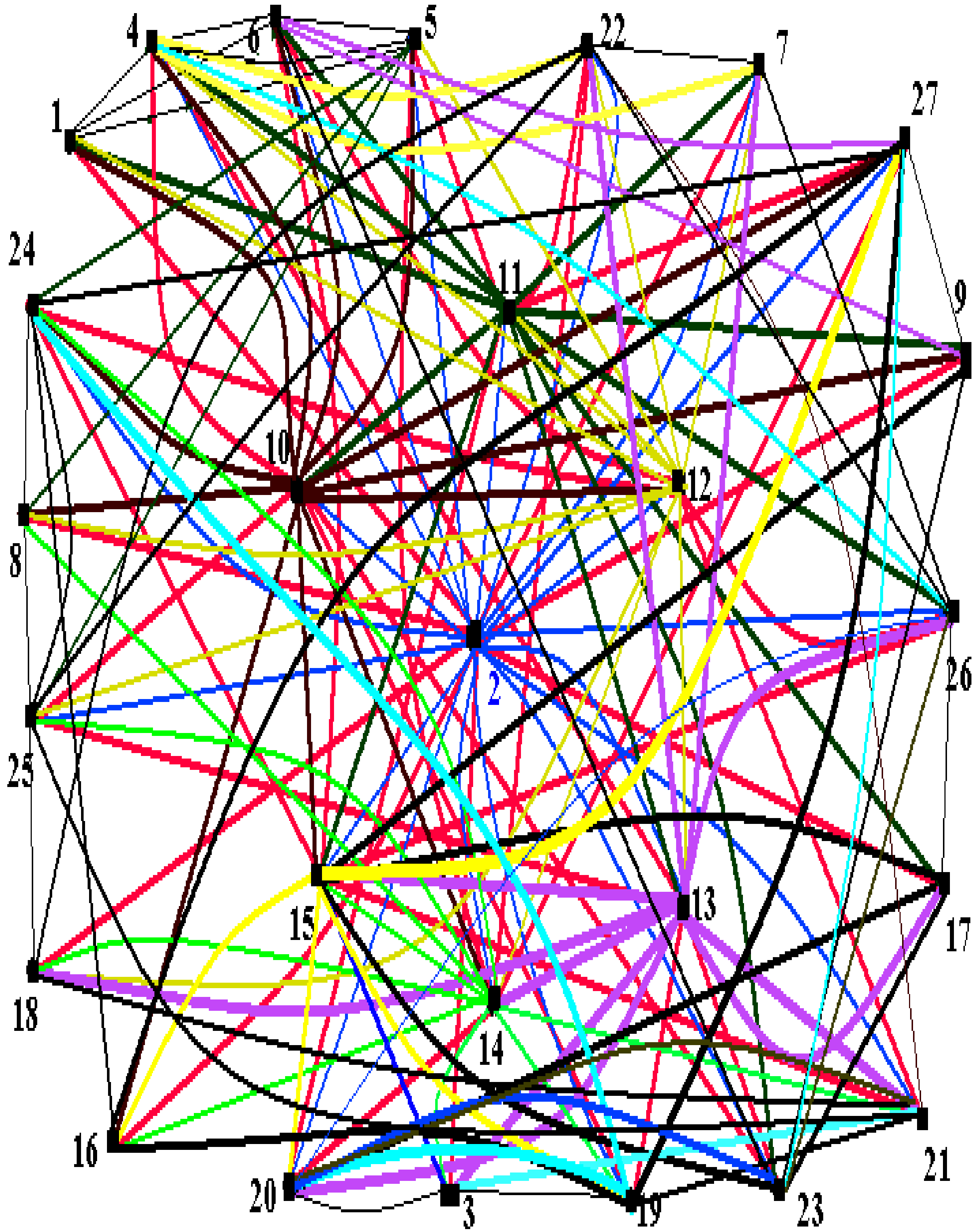
Table 4.1: The mapping from V (G) to V (H) for the graphs Q (H^G) & D (H^G)

function	Corresponding vertex in Q(H^G) or D (H^G)	Mapping from G to H			
		I	II	III	**}** vertices of G
f_1	1	a	a	a	
f_2	2	b	b	b	
f_3	3	c	c	c	
f_4	4	a	a	b	
f_5	5	a	b	a	
f_6	6	b	a	a	
f_7	7	a	a	c	
f_8	8	a	c	a	
f_9	9	c	a	a	
f_{10}	10	b	b	a	
f_{11}	11	b	a	b	
f_{12}	12	a	b	b	
f_{13}	13	b	b	c	}Vertices of H
f_{14}	14	b	c	b	
f_{15}	15	c	b	b	
f_{16}	16	c	c	a	
f_{17}	17	c	a	c	
f_{18}	18	a	c	c	
f_{19}	19	c	c	b	
f_{20}	20	c	b	c	
f_{21}	21	b	c	c	

***f*₂₂**	22	a	b	c	}Vertices of H
***f*₂₃**	23	c	a	b	
***f*₂₄**	24	b	c	a	
***f*₂₅**	25	a	c	b	
***f*₂₆**	26	b	a	c	
***f*₂₇**	27	c	b	a	

Table 4.2: Adjacency for the graphs Q (H^G) & D (H^G)

Vertex *u*	**N(*u*) in Q(H^G)**	**Degree in Q(H^G)**	**N(*u*) in D(H^G)**	**Degree in D(H^G)**
1	2,4,5,6,10,11,12	7	2	1
2	All other vertices	26	1,3,7,8,9,16,17,18	8
3	2,13,14,15,19,20,21	7	2	1
4	1,2,5,6,7,10,11,12,13,22,26	11	13,10	2
5	1,2,4,6,8,10,11,12,14,24,25	11	11,14	2
6	1,2,4,5,9,10,11,12,15,23,27	11	12,15	2
7	2,4,11,12,13,22,26	7	2	1
8	2,5,10,12,14,24,25	7	2	1
9	2,6,10,11,15,23,27	7	2	1
10	1,2,4,5,6,8,9,11,12,14,15,16,19,23,24,25,27	17	4,19,23,25	4
11	1,2,4,5,6,7,9,10,12,13,15,17,20,22,23,26,27	17	5,20,22,27	4
12	1,2,4,5,6,7,8,10,11,13,14,18,21,22,24,25,26	17	6,21,24,26	4
13	2,4,11,12,22,7,26,17,21,23,19,3,20,14,18,15,25	17	4,19,23,25	4
14	2,3,5,22,12,27,13,21,19,20,16,18,25,15,8,24,10	17	5,20,22,27	4
15	2,3,6,11,10,9,26,17,13,21,23,19,14,20,16,24,27	17	6,21,24,26	4
16	2,24,10,15,27,14,19	7	2	1
17	26,11,2,15,13,20,23	7	2	1
18	25,22,12,2,13,21,14	7	2	1
19	3,20,16,14,15,24,10,2,13,27,21	11	10,13	2
20	15,11,2,14,13,26,17,21,23,19,3	11	11,14	2
21	22,12,25,2,13,15,18,14,20,3,19	11	15,12	2
22	1,2,25,18,11,14,12,13,21,26,7	11	11,14	2
23	20,15,13,2,10,11,6,27,9,26,17	11	10,13	2
24	8,25,16,15,14,2,10,12,27,5,19	11	12,15	2
25	8,24,5,22,10,12,2,21,13,14,18	11	10,13	2
26	7,22,4,11,2,15,13,20,23,17,12	11	12,15	2
27	6,24,11,10,2,15,16,14,19,23,9	11	11,14	2



$Q(P_3^{K_3})$

Note: In above figure, consider all dark red lines with all vertices as the graph of $D(P_3^{K_3})$

In this chapter we mainly discuss about these two variants of function graphs.

4. B. Basic properties of function graphs

The initial question comes in our mind is "How to find the degree of a vertex (function) in such graph?"

Here we found the degree formula of these variants of function graph.

Theorem 4.1:

Let $|G| = n$ .Then for any $f \in V(Q(H^G))$, Deg (f) = $\left[\prod_{i=1}^{n} N[f(u_i)]\right] - 1$

Proof:

Let $f \in V(Q(H^G))$ such that $f = (f(u_1), f(u_2), \ldots, f(u_n))$.

It is clear that i^{th} co-ordinate of f can be adjacent to N [$f(u_i)$] vertices in the graph H. Hence by the counting principle, f can adjacent to $\left[\prod_{i=1}^{n} N[f(u_i)]\right] - 1$ vertices in $Q(H^G)$ □

Tester: In above example 4.1, degree of vertex 8 can be calculating as follow:

$$\begin{aligned} deg(8) &= |N[f(\mathrm{I})]| \cdot |N[f(\mathrm{II})]| \cdot |N[f(\mathrm{III})]| - 1 \\ &= |N[a]| \cdot |N[b]| \cdot |N[c]| - 1 \\ &= 2 \cdot 2 \cdot 2 - 1 = 7 \end{aligned}$$

Corollary 4.1.1:{Degree sum Formula for Q (H^G)}

If $|G| = n$ and $|H|^{|G|} = m$ in the graph $Q(H^G)$, then

$$e = \frac{1}{2}\left\{ \sum_{j=1}^{m}\left[\prod_{i=1}^{n}\left|N[f_j(u_i)]\right|\right] - m\right\}$$

Proof: It is clear from degree sum formula.[Refer result 1.1] □

Corollary 4.1.2:

If $|G| = n$ and graph H is k –regular then graph Q (H ^{G}) is $\{(k+1)^n - 1\}$-regular.

Proof:

By Theorem 1, deg (f) = $\left[\prod_{i=1}^{n} N[f(u_i)]\right] - 1$

$= [(k+1)\cdot(k+1)\cdots(k+1)(n\ Times)] - 1$

$= (k+1)^n - 1$ □

Theorem 4.2: {Degree sum Formula for D (H^{G})}

Let $|G| = n$.Then for any $f \in$ V (D (H ^{G})), Deg (f) = $\left[\prod_{i=1}^{n} N(f(u_i))\right]$

Proof:

Let $f \in$ V (D (H ^{G})) such that f = $(f(u_1), f(u_2), \ldots, f(u_n))$.

It is clear that i^{th} co-ordinate of f can be adjacent to N ($f(u_i)$) vertices in the graph H. Hence by the counting principle, f can adjacent to $\left[\prod_{i=1}^{n} N(f(u_i))\right]$ vertices in D (H ^{G}). □

Tester: In above example 4.1, degree of vertex 8 can be calculating as follow:

$$deg(8) = |N(f(I))| \cdot |N(f(II))| \cdot |N(f(III))|$$
$$= |N(a)| \cdot |N(b)| \cdot |N(c)|$$
$$= 1 \cdot 1 \cdot 1 = 1$$

Corollary 4.2.1:

If $|G| = n$ and $|H|^{|G|} = m$ in the graph D (H^{G}), then

$$e = \frac{1}{2}\{\sum_{j=1}^{m}\left[\prod_{i=1}^{n}\left|N(f_j(u_i))\right|\right]\}$$

Proof: It is clear from degree sum formula. □

Corollary 4.2.2:

If $|G| = n$ and graph H is k –regular then graph D (H ^{G}) is k^n-regular.

Proof: By the similar argument of corollary 4.1.2 we can prove this. □

The question arise is “Can the function graph become complete for some graphs G and H?” We have an affirmative answer in the following:

Theorem 4.3:
Graph Q (H^G) is complete if and only if graph H is complete.

Proof:
Let H be a complete graph. Let $f, g \in V(Q(H^G))$.
Then $f(u_i) = f(u_i)$ *or* $f(u_i) \sim f(u_i)$ (for any u_i in G) in the complete graph H.
Thus $f \sim g$ in Q (H^G).
Conversely Suppose Q (H^G) is a complete graph. Let $x, y \in V(H)$. Consider $f \neq g$, $f, g \in V(Q(H^G))$ defined by $f(u) = x$ & $g(u) = y$ for each u in G.
But $f \sim g$ in Q (H^G) then $f(u) = g(u)$ *or* $f(u) \sim g(u)$ in graph H (for any u in G).
Here $f(u) \neq g(u)$because f, g are the constant functions & $f \neq g$. Thus $x \sim y$ in H.
□

Theorem 4.4:
Graph D (H^G) is not complete, for any graphs G & H.

Proof:
Let $V(G) = \{u_1, u_2 \ .... \ u_n\}$. Consider $f, g \in V(D(H^G))$ ($f \neq g$) are defined as follow $f\{u_1, u_2 \ .... \ u_n\} = (w_1, w_2 \ .... \ w_n)$ & $g\{u_1, u_2 \ .... \ u_n\} = (w_1, m_2, m_3 \ .... \ m_n)$, where w_i , $m_j \in V(H)$.Then $w_i \sim m_i$ in H, for $i = \{2, 3, \ldots, n\}$ in the case of H is complete. But the first co ordinates are equal. So, $f \nsim g$ in D (H^G). □
Similarly we can prove that

Theorem 4.5:
Graph H^G is not complete, for any graphs G & H.

4. C. Core results

Theorem 4.6:
Let G and H are two graphs with $|G| = n$,
then Q (H^G) ≅ H^n =H ⊠ ... ⊠ H (n times).

Proof:
Let $\{u_1, u_2, ..., u_n\}$ be the vertex set of graph G.
A function T: V (Q (H^G)) →V (H^n) defined as follows:

T (f) = ($f(u_1), f(u_2),\ldots, f(u_n)$) .It is clear that the function T is bijective. So, it is suffices to prove that T is strong homomorphism.

Let $f \sim g$ in Q (H ^{G})

$\Leftrightarrow f(u_i) = g(u_i)$ or $f(u_i) \sim g(u_i)$ in H (for any u_i in G)

$\Leftrightarrow$ ($f(u_1), f(u_2),\ldots, f(u_n)$) ~ ($g(u_1), g(u_2),\ldots, g(u_n)$) in H^n = H ⊠…⊠H

(by the definition of generalize strong product)

$\Leftrightarrow$ T (f) ~ T (g) in H^n □

Tester: You can verify this result by drawing the graph P_3 ⊠ P_3 ⊠ P_3 which is isomorphic to the graph given in Example 4.1.

From theorem 4.3 we can say that if $|G| = m$ & H = K_n ,then Q (H^G) = $K_{m \cdot n}$

Corollary 4.6.1:

Graph Q (H ^{G}) is connected if and only if graph H is connected.

Proof: Proof is straight forward by theorem 4.6. [Refer result1.22] □

Corollary 4.6.2:

Let graph Q (H ^{G}) is connected, then for any u_i in graph G

$$d_{Q(H^G)}(f, g) = \max_{1 \le i \le n} d_H(f(u_i), g(u_i)) \text{ for } i = \{1,2,\ldots,n\}$$

Proof: Proof is straight forward. [Refer result1.19] □

Tester: In example 4.1,

$d_{Q(P_3{}^{K_3})}(16,26) = \max_{1 \le i \le n} d_H(f_{16}(u_i), f_{26}(u_i))$

$= \max \{ d_H(f_{16}(\text{I}), f_{26}(\text{I})), d_H(f_{16}(\text{II}), f_{26}(\text{II})), d_H(f_{16}(\text{III}), f_{26}(\text{III})) \}$

$= \max \{ d_H(c, b), d_H(c, a), d_H(a, c)\}$

$= \max \{ 1, 2, 2 \} = 2$

Theorem 4.7:

Let G and H be two graphs with $|G| = n$,
then D (H ^{G}) ≅ H^n =H× …×H (n times).

Proof: It is similar to Theorem 4.6. □

Tester: You can verify this result by drawing the graph $P_3 \times P_3 \times P_3$ which is isomorphic to the graph given in example 4.1 with dark red lines & all vertices.

Corollary 4.7.1:
Graph D (H^G) is connected if and only if graph H is connected & non bipartite.

Proof: Proof is straight forward. [Refer result1.20] □

Tester: In example 4.1, d(1,4) = ∞ in D (H^G).

4. D. Identities in Function graphs

In this section we have proved some basic identities in Function graphs which are similar to laws of exponents of numbers.

In the exponential numbers we know that power of the exponential number is the multiplication of two powers. The following result is similar to this.

Note : For any graph G, $x \in$ G means $x \in$ V(G)

Theorem 4.8:
For any graphs X, Y, Z : $Q(Z^{X*Y}) \cong Q(Q(Z^Y)^X) \cong Q(Q(Z^X)^Y)$

Proof:
First we prove that $Q(Z^{X*Y}) \cong Q(Q(Z^Y)^X)$.
Let $f \in Q(Z^{X\times Y})$.
For each x in X, let the map $f_x : V(Y) \rightarrow V(Z)$ be defined as
$f_x(y) = f(x, y), \forall y \in V(Y)$.Then $f_x \in Q(Z^Y)$.
Let the map $t_f : V(X) \rightarrow V(Q(Z^Y))$ be defined as
$t_f(x) = f_x, \forall x \in V(X)$.Then $t_f \in Q(Q(Z^Y)^X)$.
Consider the map, T: $V(Q(Z^{X\times Y})) \rightarrow V(Q(Q(Z^Y)^X))$
Defined by $T(f) = t_f$.
Clearly map T is well defined and bijective map.
Now,let's prove that T is strong homomorphism.
Let $f \sim g$ in $Q(Z^{X\times Y})$.
To prove $T(f) \sim T(g)$, it is sufficient to prove that $t_f \sim t_g$ in $Q(Q(Z^Y)^X)$.
Let $x \in$ X.
It is enough to prove that,
$t_f(x) \sim t_g(x)$ -or- $t_f(x) = t_g(x)$ in $Q(Z^Y)$.

Thus it is enough to prove that $f_x \sim g_x$ -or- $f_x = g_x$ in Q (Z^Y).
Suppose $f_x \neq g_x$. Let $y \in$ Y.
Now $f(x, y) = g(x, y)$ -or- $f(x, y) \sim g(x, y)$ in Z, as $f \sim g$ in Q ($Z^{X \times Y}$), for any $(x, y) \in X * Y$.
So $f_x(y) = g_x(y)$ -or- $f_x(y) \sim g_x(y)$ in Z, $\forall\, y \in V(Y)$.
Thus $f_x \sim g_x$ in Q (Z^Y), $\forall\, x \in V(X)$.
Similarly it can be prove that
(I)T^{-1}: V (Q (Q (Z^Y) ^{X})) $\rightarrow$ V (Q ($Z^{X \times Y}$)) is a homomorphism. Thus T is an isomorphism & (II) Q ($Z^{X * Y}$) $\cong$ Q (Q (Z^X) ^{Y}) □

Similarly we can prove

Theorem 4.9:
For any graphs X, Y, Z : D ($Z^{X * Y}$) $\cong$ D (Q (Z^Y) ^{X}) $\cong$ D (Q (Z^X) ^{Y})

Now the next law of exponents says "In the multiplication of two exponential numbers if bases are same then it is similar to the exponential number with same base and addition of two powers"

Let's compare this with the following result.

Theorem 4.10:
For any graphs X, Y: $Q(H^{X+Y}) \cong Q(H^X) \boxtimes Q(H^Y)$

Proof:
Let X and Y be two graphs such that $|X| = m$ & $|Y| = n$.
Then by theorem 4.6, $Q(H^{X+Y}) \cong H^{m+n}$ =H $\boxtimes$ ... $\boxtimes$ H ($m + n$ times).
Now the remaining proof is straight forward as strong product is associative.[Refer result 1.21] □

Similarly we can prove

Theorem 4.11:
For any graphs X, Y : $D(H^{X+Y}) \cong D(H^X) \times D(H^Y)$

Another law of exponent says "In the multiplication of two exponential numbers if powers are same then it is similar to the exponential number with same power and multiplication of two bases". Let's compare this with the following result.

Theorem 4.12:
G, H & Z are any graphs, then $Q((G \boxtimes H)^{Z}) \cong Q(G^{Z}) \boxtimes Q(H^{Z})$

Proof: Proof is straight forward by theorem 4.6 as strong product is associative.[Refer result 1.21] □

Similarly we can prove

Theorem 4.13:
G, H & Z are any graphs, then $D((G \times H)^{Z}) \cong D(G^{Z}) \times D(H^{Z})$

Proof:
Proof is straight forward by theorem 4.7 as weak product is associative [Refer result 1.21] □

Cancellation law of exponents also exists in function graphs.

Theorem 4.14:
G, H & Z are any graphs, then G ≅ H if and only if $Q(G^{Z}) \cong Q(H^{Z})$

Proof:
Necessary part is straight forward by theorem 4.6 .For the sufficient part, it is clear that G ≅ H if and only if $G \boxtimes G \cong H \boxtimes H$ □

Theorem 4.15:
G, H & Z are any graphs, then G ≅ H if and only if $D(G^{Z}) \cong D(H^{Z})$

Proof: Proof is straight forward by theorem 4.7 □

Theorem 4.16:
G, H & Z are any graphs, if G ≅ H then $Q(Z^{G}) \cong Q(Z^{H})$

Proof:
Proof is straight forward by theorem 4.6. □

Theorem 4.17:
G, H & Z are any graphs, if G ≅ H then $D(Z^{G}) \cong D(Z^{H})$

Proof: Proof is straight forward by theorem 4.7□

Counter Example 4.2: Converse of above results 4.16 & 4.17 are not true:
Let G = K_2 & H = Null graph with two vertices. Then for any graph Z,
$Q(Z^G) \cong Q(Z^H)$ & $D(Z^G) \cong D(Z^H)$ but G $\ncong$ H.

4. E. Effect of variant of homomorphism in function graph.

Notation: Let $P(H^G)$ denote the induced sub graph on P(G,H) in the graph H^G.

Theorem 4.18:
A Graph G is connected if and only if $P(H^G) \cong H$.

Proof:
Suppose G is connected. Then every pure quasi homomorphism is constant function [Refer theorem3.15]. Let P(G,H) = $\{f_x : x \in H\}$ where f_x is defined as
$f_x(u) = x, \forall u \in V(G)$.
Define T: $V(P(H^G)) \rightarrow V(H)$ as follows $T(f_x) = x \ \forall f_x \in P(H^G)$
It is obvious that T is a bijective function.
To prove T is a homomorphism, let $f_x \sim f_y$ in $P(H^G)$.If $u \sim v$ in G, $f_x(u) \sim f_y(v)$ in H.
i.e. $x \sim y$ in H.
This means that $T(f_x) \sim T(f_y)$ in H.
It can be easily prove that if $T(f_x) \sim T(f_y)$ in H then $f_x \sim f_y$ in $P(H^G)$.*i.e.* T is complementary homomorphism.
Thus $P(H^G) \cong H$.
Conversely suppose $P(H^G) \cong H$.
Then $|P(H^G)| = |H| = n = n^k$ [corollary3.15.1].
Therefore $k = 1$ .Thus graph G has only one component. □

For each u in G, consider the function Fu : $\Re(G, H) \rightarrow V(H)$ defined as Fu (f) = $f(u)$ for all functions f in $\Re(G, H)$.

Theorem 4.19:
The function $F_u : V(Q(H^G)) \rightarrow V(H)$ is a quasi homomorphism for each u in V(G).

Proof:

Let $f \sim g$ in Q (H^G) $(f \neq g)$ $\Longrightarrow$ $f(u) = g(u)$ or $f(u) \sim g(u)$ for any u in G.

$\Longrightarrow$ $F_u(f) = F_u(g)$ or $F_u(f) \sim F_u(g)$ in H. □

Similarly we can prove

Theorem 4.20:

The function $F_u : V(D(H^G)) \rightarrow V(H)$ is a homomorphism for each u in V(G).

A mapping $h : V(G) \rightarrow V(H)$ is a quasi-homomorphism from G to H then for any graph K and for any quasi-homomorphism $f : V(H) \rightarrow V(K)$, $f \circ h$ is a quasi-homomorphism from G to K [Refer table 3.2].Thus we have a function

$$h^* : Q(H,K) \rightarrow Q(G,K) \text{ defined as } h^*(f) = f \circ h$$

For the following results consider $Q(G,H) \subset V(Q(H^G))$,where Q(G,H) is the set of all quasi-homomorphisms from G to H .

Theorem 4.21:

The function $h^* : Q(H,K) \rightarrow Q(G,K)$ is a quasi-homomorphism.

Proof:

suppose $f, g \in Q(H,K)$ such that $f \sim g$. Let $u \in V(G)$, Now h(u)is an element of V(H).Since $f \sim g$, $f(h(u)) = g(h(u))$ or $f(h(u)) \sim g(h(u))$ in K.

i.e. $h^*(f)(u) = h^*(g)(u)$ or $h^*(f)(u) \sim h^*(g)(u)$.Thus $h^*(f) \sim h^*(g)$ in Q(G,K). □

Theorem 4.22:

The evaluation map $e : G \,\square\, Q(G, H) \rightarrow H$ defined as $e(u, f) = f(u)$ is quasi homomorphism**.**

Proof:

Suppose $(u, f) \sim (v, g)$ in $G \,\square\, Q(G, H)$. If $u = v$ then $f \sim g$.

Hence $f(u) = g(u)$ or $f(u) \sim g(u)$ in H.

If $u \sim v$ then $f = g$. Since f and g are quasi homomorphism, $f(u) = g(u)$ or $f(u) \sim (u)$ in H. □

If $m : V(G) \rightarrow V(H)$ is a homomorphism from G to H then for any graph K and for any homomorphism $f : V(H) \rightarrow V(K)$, $f \circ m$ is a homomorphism from G to K .Thus we have a function

$$m^* : \mathrm{Hom}(H,K) \to \mathrm{Hom}(G,K) \text{ defined as } h^*(f) = f \circ m$$

For the following results consider Hom (G, H) ⊂ V (D (H ^{G})), where Hom (G, H) is the set of all homomorphisms from G to H.

Then similarly we can prove the following.

Theorem 4.23:
The function m^* : Hom(H,K)→ Hom(G,K) is a homomorphism.

Theorem 4.24:
The evaluation map e : G Hom (G, H) → H defined as e(u, f) = f (u) is homomorphism**.**

“The essence of Mathematics lies in its freedom.” --- *CANTOR*

Results Involving Parameters

CHAPTER 5

5. A. Introduction

Parameters are one of the important tools in the Graph Theory. Graphs can be studied by their properties. Some parameters are certain types of numbers associated with properties of graphs like Bondage, Dominating set, etc. Some parameters show the internal stability of the graph like Independent set, Clique set, etc. They are also useful in studying hereditary properties of the graph.

In the present work we have introduced different variants of homomorphism and function graphs. So it is natural to check the validity of different graph parameters to enrich the development of these new graphs and different variants of homomorphism.

This chapter is mainly framed in two parts. One is effect of variants of homomorphism on graph parameters and second is validity of the graph parameters in function graphs. We consider the following graph parameters for our study. In fact there are quite a huge number of graph parameters.

Graph Parameters
(1) Dominating set (2) Total dominating set (3) Independent set (4) Maximal independent set (5) clique (6) Vertex cover (7) Enclave (8) Enclave less

5. B. Testing of parameters under different variants of homomorphism

Definition 5.1:
A subset S of V(G) is said to be a dominating set if for each vertex *v*, $v \in$ S or *v* is adjacent to some vertex of S.

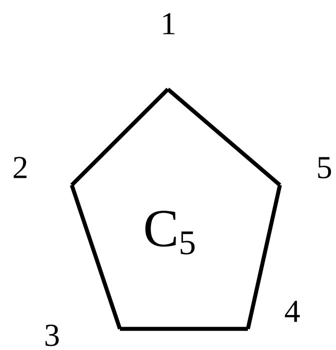
Theorem 5.1:
If f : V (G) $\rightarrow$ V (H) is an onto quasi-homomorphism, then f(S) is a *dominating set* in H, whenever S is a dominating set in G. Hence γ(H) $\leq$ γ(G).

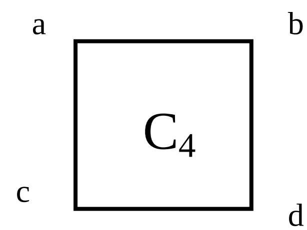
Proof:
Let f : V (G) $\rightarrow$ V (H) be an onto quasi-homomorphism.
If $f(t) \notin f(S)$ then $t \notin S$.
Let $x \in S$ such that $x \sim t$, then $f(x) \sim f(t)$, as f is quasi-homomorphism &
$f(t) \neq f(x)$.

Thus f(S) is dominating set in H.
Now, it follows that $\gamma(H) \leq \gamma(G)$. □

Note that the above result is also true for totally dominating set .

Counter example 5.1: Converse of above theorem is not true. Here we give an example of a function which takes dominating sets to dominating sets but it is not a quasi-homomorphism.





Here $f : V(C_5) \rightarrow V(C_4)$ is onto map defined by $f(1) = a$, $f(2) = a$, $f(3) = b$, $f(4) = c$, $f(5) = d$ .Then image of every dominating set is a dominating set but the map is not a quasi-homomorphism.

Definition 5.2:
A subset S of V (G) is said to be a *clique* in G, if whenever x and y are distinct vertices in S, they are adjacent.

Theorem 5.2 :
An onto function $f : V(G) \rightarrow V(H)$ is a quasi-homomorphism if and only if whenever K is a clique in G, f(K) is a clique in H.

Proof:
Suppose f is a quasi homomorphism. Let K be a clique in the graph G. Let $f(u)$ and $f(v)$ be distinct elements of f(K).Then u and v are distinct elements of K and therefore they are adjacent in graph G. Since f is a quasi homomorphism, $f(u) \sim f(v)$ in graph H. This proves that f(K) is clique in H.
Conversely, suppose $u \sim v$ in G .Then $\{u, v\}$ is a clique in G, as $\{f(u), f(v)\}$ is also clique, $f(u) = f(v)$ or $f(u) \sim f(v)$ in H . □

Theorem 5.3:
Let graph G and H are without isolated vertices. A function f : V (G) → V (H) is a quasi- complementary homomorphism if and only if for any subset K ⊂ V(G) , f (K) is a clique in H then K is a clique in G.

Proof:
Let f : V (G) → V (H) is a quasi- complementary homomorphism. Let K ⊂ V(G) such that f (K) is a clique in H.
Let x , y ∈ K. $\Rightarrow f(x), f(y) \in f$(K)
$\Rightarrow f(x) \sim f(y)$ or $f(x) = f(y)$ in H.
$\Rightarrow x \sim y$ in G as f is a quasi- complementary homomorphism.
Thus K ⊂ V(G) is a clique in G.
Conversely suppose for any subset K ⊂ V(G) , f (K) is a clique in H then K is a clique in G.
Let x , y ∈ V(G) such that $x \neq y$.
<u>Case 1:</u> $f(x) = f(y) = w$ in H.
Let $w \sim f(t)$ in H.
$\Rightarrow \{ f(x), f(y), f(t)\} = f$(I) is a clique in H.
$\Rightarrow \{x, y, t\} =$ I is also a clique in G.
Thus $x \sim y$ in G.
<u>Case 2:</u> $f(x) \sim f(y)$ in H.
$\Rightarrow \{ f(x), f(y) \} = f$(I)is a clique in H.
$\Rightarrow \{x, y\} =$ I is also a clique in G.
Thus $x \sim y$ in G.
Hence f : V (G) → V (H) is a quasi- complementary homomorphism. □

Tester: Consider example 3.11. The clique graph K_4 = {b,d,g,h} of graph G whose image is also graph K_3.

Similarly we can prove the following,

Theorem 5.4:
Let graph G and H are without isolated vertices. A function f : V (G) → V (H) is a map such that for any subset K ⊂ V (G), K is a clique in G whenever f (K) is a clique in H, then f is complementary homomorphism.

Counter example 5.2: Converse of above theorem 5.4 is not true. *i.e* if

f : V (G) $\rightarrow$ V(H) is a complementary homomorphism such that for any subset K $\subset$ V(G) , f (K) is a clique in H then K need not be clique in G which you can observe from the following mapping.
Here, f : V (P_3) $\rightarrow$ V(K_2) is a complementary homomorphism such that
f (a) = f (c)=1 , f (b) = 2.

Theorem 5.5:
A function f : V (G) $\rightarrow$ V(H) is an onto strong quasi-homomorphism then $S \subset$ V (H)is dominating set if and only if f^{-1}(S) is dominating set in G

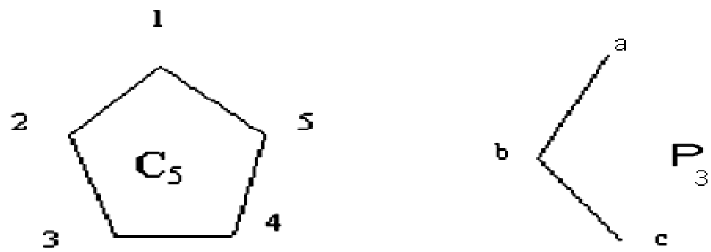
Proof:
Let f : V (G) $\rightarrow$ V(H) be an onto strong quasi-homomorphism. Let $S \subset$ V (H) be a dominating set in graph H.
Let $x \notin f^{-1}$(S) then $f(x) \notin$ S. Let $f(y) \in$ S such that $f(x) \sim f(y)$.
Then $x \sim y$ in G as f is quasi –complementary homomorphism.
Thus f^{-1}(S) is dominating set in G.
Conversely suppose f^{-1}(S) is dominating set in G.
Let $x \notin f^{-1}$(S) & $t \in f^{-1}$(S) such that $x \sim t$.
Then $f(x) \sim f(t)$ or $f(x) = f(t)$ in H, as f is quasi- homomorphism.
Clearly $f(x) \neq f(t)$ in H, so $f(x) \sim f(t)$. □

Definition 5.3: The set S $\subset$ V (G) is said to be *vertex cover* if it contains at least one vertex of every edge of the graph.

Theorem 5.6:
If a function f : V (G) $\rightarrow$ V(H) is an onto strong homomorphism and V $\subset$ V (G) is a vertex cover, then f (V) is also vertex cover in H.

Proof:
Let $f(u) f(u)$ be an edge in H. Since f is complementary, uv is an edge in G. Since V ⊂ V (G) is a vertex cover so $u \in$ V or $v \in$ V. Then $f(u)$ or $f(v) \in f$ (V) □

Counter example 5.3: Converse of above theorem 5.6 is not true. *i.e* if f : V (G) → V(H) is an onto map such that image of every vertex cover in G is vertex cover in H, then f need not be strong homomorphism which you can observe for the following mapping.
Here, f : V (C_5) → V(P_3) is an onto map such that $f(1) = f(3) = a$, $f(2) = f(5) = b$, $f(4) = c$.

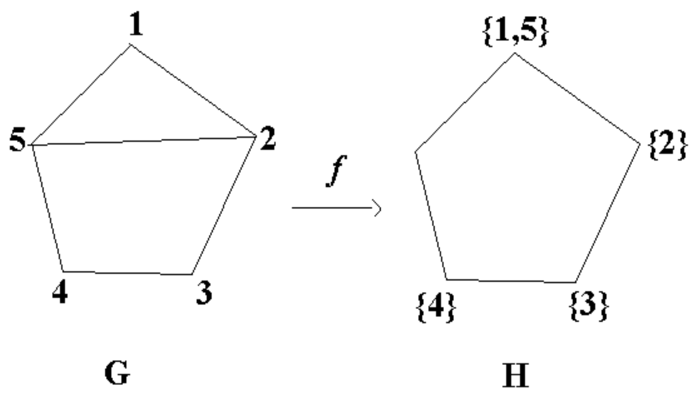


Definition 5.4 :
A subset T of V (G) is said to be an *independent* set if any two distinct vertices of T are non-adjacent.

Theorem 5.7:
A function f : V (G) → V(H) is a strong homomorphism, then
Image of every maximal independent set in G is maximal independent set in H.

Proof:
It is straightforward to prove that every maximal independent set is independent set as well as dominating set. Image of every dominating set under homomorphism is dominating set. Then by theorem 5.1, the image of a maximal independent set is independent and dominating set and thus a maximal independent set. □

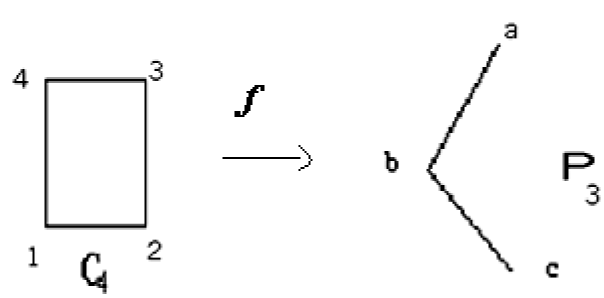
Counter example 5.4: Converse of above theorem 5.7 is not true. *i.e* if f : V (G) → V(H) is a map such that image of every maximal independent set in G is maximal independent set in H , then f need not be strong homomorphism which you can observe from the following mapping, f : V (G) → V(H) .



Theorem 5.8:
A function $f : V(G) \to V(H)$ is a complementary homomorphism, if and only if image of every independent set in G is an independent set in H.

Proof:
Let a function $f : V(G) \to V(H)$ be a complementary homomorphism. Let $I \subset V(G)$ be an independent set. Suppose $f(I)$ is not an independent set in H. Then $f(u), f(v) \in f(I)$ such that $f(u) \sim f(v)$ in H. Hence $u \sim v$ in G. Thus $u, v \in I$ so that $u \sim v$ in G ,which is contradiction.
Conversely suppose image of every independent set in G is an independent set in H under the mapping $f : V(G) \to V(H)$.
Let $u \nsim v$ in G, then $\{u, v\} = I$ be an independent set in G. So, $\{f(u)\ f(v)\} = f(I)$ is also an independent set in H. Thus $f(u) \nsim f(v)$ in H. □

Counter example 5.5: Converse of above theorem 5.8 is not true for quasi-complementary homomorphism. *i.e* if $f : V(G) \to V(H)$ is a map such that image of every independent set in G is an independent set in H , then f need not be quasi- complementary homomorphism which is clear from the following mapping.



Here f : V (C_4) → V(P_3) is a map such that f(1) = f(3)= b , f(2) = a, f(4) = c.

Definition 5.5: For A sub set S ⊂ V(G), v ∈ S is said to be *enclave* of S if N[v] ⊆ S, and v ∈ S is an *isolate* of S if N(v) ⊆ V(G) –S.A set is said to be *enclave less* if it does not contain any enclave.

Theorem 5.9: A function f : V (G) → V(H) is a quasi-complementary homomorphism. Then if a sub set S ⊂ V(G) is enclave less then f(S) is also enclave less in H.

Proof:

Let the function f : V (G) → V(H) is a quasi-complementary homomorphism.
Let S ⊂ V(G) enclave less. Suppose f(S) is not enclave less in H.
Then there exist $f(u) \in f(S)$ such that $N[f(u)] \subset f(S)$. Let $f(v) \in N[f(u)]$ so $f(u) \sim f(v)$ in H. Then $u \sim v$ in G, since f is quasi-complementary homomorphism and $f(v) \in f(S)$.
Thus $\{v/v \sim u\} \cup \{u\} \subset S$. Hence N[$u$] ⊂ S. This is contradiction. □

5. C. Existence of different graph parameters in function graphs.

In this section we will check the validity of different graph parameters in function graphs. In other word we will find the existence of the graph parameters in such graphs.

Theorem 5.10:

A sub set K ⊂ V(H) is a dominating set if and only if the set Q_K = {f **:** Range of f ⊂ K}is dominating set in Q(H^G).

Proof:

Let K ⊂ V(H) is a dominating set .Let $f \notin Q_K$. Let the map g : V(G) → K be defined as follows :(1) $g(u)$ = $f(u)$ if $f(u)$ is in K (2) $g(u)$ = t if $f(u)$ is not in K and $f(u) \sim t$. It is clear that $f(u) = g(u) or\ f(u) \sim g(u)$ in graph H (for any u in G).Therefore $f \sim g$ in Q (H^G).
Thus Q_K is dominating set in Q (H^G).
Conversely suppose the set Q_K = {f **:** Range of f ⊂ K}is dominating set in Q(H^G).
Let $x \in \overline{K}$, Consider the constant function f_x : V (G) → V(H) defined as follow:

$f_x(t) = x$, for all t in V(G).Then $f_x \notin Q_K$. Since Q_K is dominating set ,there exist $g \in Q_K$ *i.e.* $g : V(G) \to V(K)$ such that $g \sim f_x$ in Q (H ^{G}) .Since $g \neq f_x$ there exist a vertex $t \in V(G)$ such that $g(t) \neq f_x(t)$.Since $g \sim f_x$ in Q (H ^{G}) then $g(t) \sim f_x(t)$,*i.e.* $g(t) \sim x$ in H.
Now $g(t) \in K$. Hence $g(t) = v \in K$ such that $v \sim x$ in H.
Thus $K \subset V(H)$ is a dominating set. □

Similarly we can prove the following,

Theorem 5.11:
A sub set $K \subset V(H)$ is a dominating set if and only if the set $D_K = \{f : \text{Range of } f \subset K\}$is dominating set in Q (H ^{G}).

Theorem 5.12:
A sub set $T \subset V(H)$ is a independent set if and only if the set $D_T = \{f : \text{Range of } f \subset T\}$ is an independent set in D (H ^{G}).

Proof:
Let $T \subset V(H)$ is a independent set in H. Let $f, g \in Q_T$ ($f \neq g$).
Then for some u in V(G), $f(u) \neq g(u)$.
Since $f(u), g(u) \in T$, $f(u)$ & $g(u)$ are not adjacent in H .
Thus f can not be adjacent to g in Q (H ^{G}).Thus Q_T is independent set in Q (H ^{G}).
Conversely suppose the set $Q_T = \{f : \text{Range of } f \subset T\}$ is independent set in Q (H ^{G}). Let $x, y \in T$ such that $x \neq y$.
Consider the constant function $f_x : V(G) \to V(H)$ & $f_y : V(G) \to V(H)$ defined as follow : $f_x(t) = x$, & $f_y(t) = y$, for all t in V(H).
Then $f_x , f_y \in Q_T$ so $f_x \nsim f_y$ in Q (H ^{G}).Thus $x \nsim y$ in H. □

Similarly we can prove the following,

Theorem 5.13:
If $T \subset V(H)$ is a independent set if and only if the set $Q_T = \{f : \text{Range of } f \subset T\}$ is independent set in Q (H ^{G}).

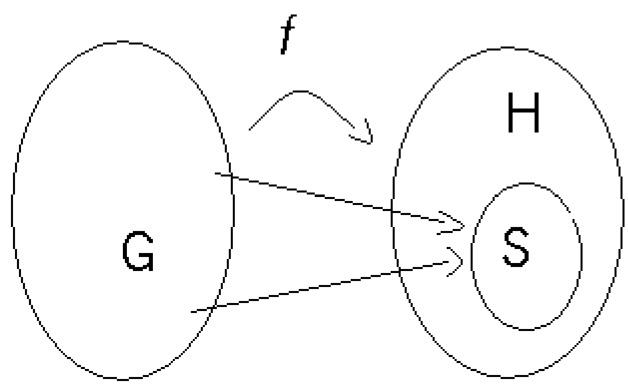
Theorem 5.14:
A sub set $S \subset V(H)$ is a clique if and only if the set $Q_S = \{f : \text{Range of } f \subset S\}$ is a clique in Q(H ^{G}).

Proof:

Let the sub set $S \subset V(H)$ is a clique in H. Let $f, g \in Q_S$. Then $f(u)$ & $g(u) \in S$ (for any u in G).So $f(u) = g(u)$ *or* $f(u) \sim g(u)$ in graph H(for any u in G) as $S \subset V(H)$ is a clique. Therefore $f \sim g$ in $Q(H^G)$.Thus Q_S is clique in $Q(H^G)$. Similarly we can prove the converse. □

Clearly the above theorem 5.14 is not true for the direct map graph $D(H^G)$. Let G & H are any graphs; with $S \subset V(H)$ is a clique in H.
Let $f = (x, x, x \ldots. y, y, y \ldots.)$ & $g_x = (x, x, x \ldots. x, x, \ldots.)$ where $x, y \in S$.
Then $f \nsim g_x$ in $D(H^G)$.

A view of above theorems



Theorem 5.15:
The set $\Re(G,H) \backslash P(G,H)$ is a dominating set in the graph $Q(H^G)$.

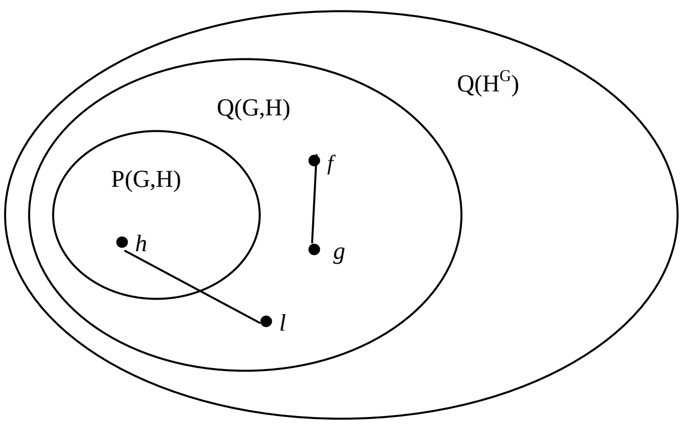
Proof:
Let $f \in P(G,H)$.Let u be any vertex of G. Let w be vertex of H which is adjacent to $f(u)$. Define $g : V(G) \rightarrow V(H)$ as follows : (i) $g(u) = w$ (ii) $g(t) = f(t)$ if $t \neq u$.
It is clear that $f \neq g$. First we prove that g is not a pure quasi homomorphism.
Let $t \sim u$ in G, then $g(t) = f(t) = f(u)$ in H , as $f \in P(G,H)$& $g(u) = w \neq f(u)$.
Thus $g \notin P(G,H)$.
Now we prove that $f \sim g$ in $Q(H^G)$. Let $x \in V(G)$.If $x \neq u$,then $g(x) = f(x)$.
If $x = u$, then $g(x) = w$, $f(x) = f(u)$ & $w \sim f(u)$.
Thus $g(x) \sim f(x)$ -or- $f(x) = g(x)$ in H, $\forall x \in V(G)$.
Hence g is adjacent to f in $Q(H^G)$. □

Definition 5.6:
A subset S of V (G) is said to be a *totally dominating set* if for each vertex $v \in$ G

v is adjacent to some vertex of S.

Theorem 5.16:
The set Q (G, H) \ P (G ,H) is a totally dominating set in the induced sub graph Q(G,H){induced from $Q(H^G)$}.

Proof:
Let $f \in$ Q(G,H) \ P(G,H).Then there are vertices u and v in G ,such that $f(u) \sim f(v)$ in H.
Let g :V (G) →V(H) be defined as follows:
(i) $g(u) = f(v)$ (ii) $g(v) = f(u)$(iii) $g(t) = f(u) \forall\ t \in V(G) \setminus \{u,v\}$.
Then it is clear that $f \neq g$ and g is not a pure quasi-homomorphism because $u \sim v$ in G,
$g(u) \sim g(v)$ in H. Thus, $g \in$ Q(G,H) \ P(G,H).
It is clear from the definition that $f(x) \sim g(x)$ -or- $f(x) = g(x)$ in H, $\forall\, x \in V(G)$.
So for any $f \in$ Q(G,H) \ P(G,H) $\subset$ Q(G,H) there exists $g \in$ Q(G,H)\ P(G,H) such that $f \sim g$ in $Q(H^G)$.



Now let $h \in$ P(G,H) & $z \in$ V(G) such that $h(z) \sim w$ for some w in V(H).
Let l : V (G) →V(H) be defined as follows :(i) $l(t) = h(t)$ if $t \neq z$ (ii) $l(z) = w$.
It is clear that l is a quasi-homomorphism but not a pure quasi-homomorphism.
So, $l \in$ Q(G,H) \ P(G,H) such that $l \sim h$ .Thus for any $h \in$ P(G,H) $\subset$ Q(G,H) there exists $l \in$ Q(G,H) \ P(G,H) $\subset$ Q(G,H) such that $l \sim h$ in $Q(H^G)$.
Thus, Q(G,H)\ P(G,H) is a totally dominating set in the induced sub graph Q(G,H) {induced from $Q(H^G)$}. □

Corollary 5.16.1:
If a function f : V(G)→ V(H) is quasi homomorphism but not pure quasi homomorphism then f is not an isolated vertex in $Q(H^G)$.

Proof: It is clear from above theorem5.16. □

Notation: Let C (G , H) be the set of all constant functions from graph G to graph H.

Theorem 5.17:
If vertex $u \in$ V (H) is pendent vertex, then the constant function f_u is an enclave of C (G, H) in the map graph H^G.

Proof:
Let the vertex $u \in$ V (H) is pendent vertex, then there is only vertex $v \in$ V (H) such that $u \sim v$ in H. Then $f_u \sim f_v$ in H^G, where $f_u , f_v \in$ C (G, H). Suppose any map $g \in$ V (H^G) such that $f_u \sim g$ in H^G .Then $f_u(x) \sim g(y)$ in H^G whenever $x \sim y$ in G. So $u \sim g(y)$ in H for any y in G. Therefore $g = f_v$.
Thus N [f_u] = { f_u , f_v } $\subset$ C (G, H) in the map graph H^G. □

“Where a Mathematical reasoning can be had, it is as great a folly to make use of any other as to grope for a think in the dark, when you have a candle in your hand.” JOHN ARBUTHNOT

Some Algebraic Properties

CHAPTER 6

6. A. Introduction

Finally we would like to present some algebraic aspects of graph theory which are fairly connected with our theme of the research.

In this chapter we mainly focus on two topics. One is retract of the graph and second is aspect of monoid of graphs. We have also proved some miscellaneous results at the end.

Retract is one of the important notion which is closely associate with homomorphism of graphs. It has various properties as well as lots many utility in algebraic graph theory. Here we will find the retract in function graphs. We are also introducing *Quasi-retract* as modified version of retract of a graph.

Monoid is very fundamental and well established notion in Algebra. It has many applications in different branches of Mathematics. Here we will present some primary properties on monoid of graph.

6. B. Retract of a graph

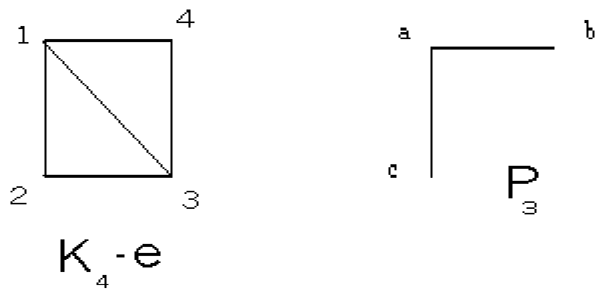
Definition 6.1:
A sub graph H of graph G is said to be *retract* **[18]** of G, if there is a homomorphism f from G onto H such that $f(x) = x$ for all x in H. The map f is called retraction.

Now I am introducing the new version of retract *i.e.*Quasi-retract.

Definition 6.2:
A sub graph H of graph G is said to be *quasi-retract* of G, if there is a quasi-homomorphism f from G onto H such that $f(x) = x$ for all x in H. The map f is called quasi-retraction.

We can easily form the statement that "every retract is quasi-retract" from the relation between homomorphism and quasi- homomorphism of graphs. However the converse is not true which is justified in the following example.

Example 6.1: It is clear that there is no homomorphism from K_3 to P_3.
So f : V (K_4-e) → V(P_3) is not homomorphism and thus P_3 is not retract of K_4-e .
Consider a map f : V (K_4-e) → V(P_3) defined as $f(1) = f(3) =$ a , $f(2) =$ b,
$f(4) =$ c. We know that $P_3 \subset (K_4 - e)$,Clearly f is quasi-homomorphism and the restriction of f : V (P_3) → V(P_3) is identity map. But it is not retract.

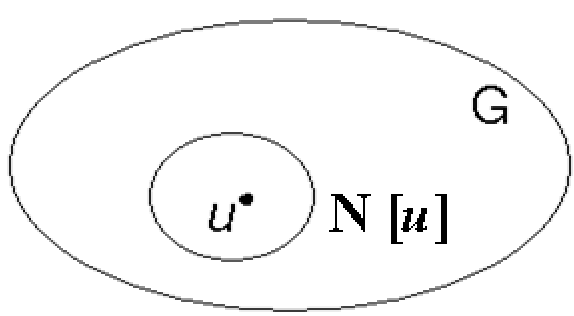


Example 6.2:
(1) An edge in bipartite graph is a quasi-retract.
(2) An edge joining center to any vertex of a wheel graph is quasi-retract.
(3) Any edge in K_3 is a quasi-retract but not a retract.
(4) Any vertex in the complete graph K_n is a quasi-retract but not a retract.

Theorem 6.1:
Every closed neighborhood of a vertex in a simple graph G is quasi retract.

Proof:
Let H be the sub graph induced by $N[u]$ for some $u \in V(G)$.
Define f : V (G) → V (H) as follows:
$(1) f(v) = v$ if $v \sim u$ $(2) f(x) = u$ if $x \notin N[u]$ $(3) f(u) = u$.
It can be easily prove that f is quasi- retraction. Thus H is quasi- retract of G.
□

A view of above theorem



Remark: There are at least as many quasi-retract as there are distinct close neighborhoods in the given graph.

Theorem 6.2:
Let G be a graph without isolated verities. H is a retract of G than H is not independent set.

Proof:
Assume that H is independent set. Let $v \in$ V(H) and $v \sim u$,
for some $u \in \mathrm{V(G)} - \mathrm{V(H)}$.
Consider the retraction f : V (G) → V (H). Clearly $f(u) \in$ V(H) & $f(v) = v$ such that $f(u) \neq f(v)$ as f is homomorphism. Thus $f(u) \sim v$ in H. This is contradiction. □

Theorem 6.3:
Let G be a graph without isolated verities. H is a proper retract of G such that H is dominating set than H is total dominating set.

Proof:
Let $v \in$ V(H) and $v \sim u$ for some $u \in \mathrm{V(G)} - \mathrm{V(H)}$.
Consider the retraction f: V (G) → V (H).
Clearly $f(u) \in$ V(H) & $f(v) = v \in$ V(H).Thus for given vertex $v \in$ V(H) ,there exist $f(u) \in$ V(H) such that $f(u) \sim v$ in V(H). Thus if H is dominating then it is also totally dominating set. □

Theorem 6.4:
If H is quasi retract of G then there is a spanning sub graph G_1 of G such that H is retract of G_1.

Proof:
Let H be a quasi retract of G and f : V (G) → V (H) is quasi-retraction. For each $y \in$ V(H) consider fiber of y *i.e.* $f^{-1}(y)$, which is subset of V (G). Remove all edges between distinct vertices of $f^{-1}(y)$.The resulting graph G_1 has same vertices as G but possibly less edges then G. Note that no edges between vertices of H have been removed because f : V (H) → V (H) is identity map. Clearly H is also sub graph of G_1 and f : V (G_1) → V (H) is a homomorphism. [Refer Corollary 3.19.1]. Thus H is retract of G_1. □

We find the retract of function graph in the following results.

Theorem 6.5:
If H_1 is retract of graph H, then Q (H_1^G) is retract of Q (H^G).

Proof:
Clearly Q (H_1^G) is sub graph of Q (H^G).
Let *h:* V (H) → V (H_1) be the retraction.
Define the function,
F: V (Q (H^G)) → V (Q (H_1^G)) as follows
F (*f*) = $h \circ f$, ∀ f ∈ V (Q (H^G)).
Let *g* ∈ V (Q (H_1^G)).
Then for any *x* in G, $h \circ g(x) = h(g(x)) = g(x)$, since *h* is a retraction.
So $h \circ g$ = F (*g*) = *g*, if *g* ∈ V (Q (H_1^G)). Thus F is a retraction.
Let's prove that F is a homomorphism.
Now $f \sim g$ in Q (H^G).
⇒$f(u) = g(u)$ -or- $f(u) \sim g(u)$ in H, ∀ *u* ∈V (G).
If $f(u) = g(u)$ in H, then $h(f(u)) = h(g(u))$ in H_1 &
if $f(u) \sim g(u)$ in H, then $h(f(u)) \sim h(g(u))$ in H_1,as H_1 is retract of H.
So $h \circ f(u) \sim h \circ g(u)$ -or- $h \circ f(u) = h \circ g(u)$ in H_1, ∀ *u* ∈V (G).
Thus $h \circ f \sim h \circ g$ in Q (H_1^G).
Hence F (*f*) ~ F (*g*) in Q (H_1^G). □

Tester: Consider example 4.1, graph K_2 is the retract of K_3. Then it is clear that Q ($K_2^{K_3}$) = K_8 (by theorem 4.3) which is retract of the graph Q($P_3^{K_3}$).

Corollary 6.5.1:
If G_1, H_1 are retracts of graphs G and H respectively, then $G_1 \boxtimes H_1$ is also retract of graph $G \boxtimes H$.

Proof: Proof of this result follows from theorem 4.6 & theorem 6.5. □

Similarly we can prove the following results.

Theorem 6.6:
If H_1 is retract of graph H then D (H_1^G) is retract of D (H^G).

Corollary 6.6.1:
If G_1, H_1 are retracts of graphs G and H respectively, then $G_1 \times H_1$ is also a retract of graph $G \times H$.

6. C. Monoid of a graph

Definition 6.3:
Let M be a set and ∘ be a binary operation on M. Then {M, ∘} is called *semi-group* **[26]** under ∘ if
(1) for any $x, y \in M$ $x \circ y \in M$ (2) for any $x, y, z \in M$, $x \circ (y \circ z) = (x \circ y) \circ z$.

Definition 6.4:
Let {M, ∘} be a semi-group. If there is an element $1 \in M$ such that
$x \circ 1 = 1 \circ x = x$, for all x. Then {M, ∘} is called *monoid* **[22]** under ∘.

Terminology:
(1) We have the concept of endomorphism (*i.e.* homomorphism of graph to itself), similarly you can say about quasi-endomorphism, complementary endomorphism & quasi- complementary endomorphism.
(2) Q(G,G) is the set of all quasi-endomorphism, Hom(G,G) is the set of all endomorphism , Com(G,G) is the set of all complementary endomorphism & QC(G,G) is refer as set of all quasi- complementary endomorphism of the graph G under composition (of mapping) operation.

Theorem 6.7:
Q (G, G) is monoid under composition.

Proof:
Let G be a graph, then Q (G, G) is the collection of all quasi- endomorphism. We have the result that "Composition of any two quasi-homomorphisms is quasi-homomorphism" [Refer table 3.2]. Now it is clear that composition is (i) closed under composition (ii) associative & (iii) the identity map itself is identity element in Q(G,G) . Thus {Q(G,G), ∘}is monoid. □

Similarly we can prove the following results.

Theorem 6.8:
{Com (G, G), ∘} is monoid.

Theorem 6.9:
{QC (G, G), ∘} is monoid.

Remark:

(1) If G is simple graph and G* is the graph obtained by adding loops at every vertex of G ,then every quasi homomorphism from G to G is homomorphism from G* to G*. Conversely every homomorphism from G* to G* is quasi homomorphism from G to G.

(2) If G is simple graph and G_1 is the super graph of G obtained by adding the edges between any two elements of $f^{-1}(y)$ for every vertex y of G. Than every complementary homomorphism from G to G is quasi- complementary homomorphism from G_1 to G_1 . Conversely every quasi- complementary homomorphism from G_1 to G_1 is complementary homomorphism from G to G.

(3) Monoid {Q (G, G), ∘} contain the monoid {Hom (G, G), ∘}.

(4) Monoid {QC (G, G), ∘} contain the monoid {Com (G, G), ∘}.

Consider G be any graph, G_1 and G* are defined as in above remark, then the following results can be proved by simple algebraic argument.

Theorem 6.10:

Monoid {Q (G, G), ∘} ≅ Monoid {Hom (G^*, G^*), ∘}

Theorem 6.11:

Monoid {Com (G, G), ∘} ≅ Monoid {QC (G_1, G_1), ∘}

This proves that the monoid of all quasi- (or complementary) endomorphism of graph G is isomorphic to the monoid of all (or complementary) endomorphism of graph G^***.**

Theorem 6.12:

Every monoid is isomorphic to monoid {Q (G, G), ∘} of a suitable graph G.

Proof:

Let S be a monoid. We have result "Every monoid is isomorphic to the endomorphism monoid of a suitable graph G" [Refer result 1.23].

i.e. S ≅ Monoid {Hom (G^*, G^*), ∘} of a suitable graph G^*.

Let G be the graph obtain by removing all loops from G^*.

Then by theorem 6.10: "Monoid {Q (G, G), ∘} ≅ Monoid {Hom (G^*, G^*), ∘}"

So by transitive rule of isomorphism S ≅ Monoid {Q (G, G), ∘} □

Definition 6.5:

Let {S, ⋆} be any semi-group, then $x \in S$ is said to be *idempotent* **[26]** if $x^2 = x$.

Definition 6.6:
Let {S, ⋆} be any semi-group and I $\subset S$ is said to be *ideal* **[26]** if
(1) I is semi-group under the binary operation ⋆ (2) $x \star$ I $\subset$ I & I $\star\, x \subset$ I for every $x \in$ S

Theorem 6.13:
Every quasi-homomorphism from G and G ,is an idempotent in semi group of all quasi-endomorphisms of graph G.

Proof: Proof is quite obvious. □

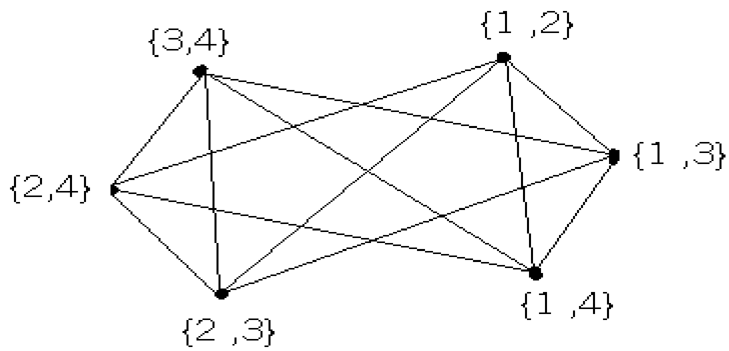
Theorem 6.14:
P (G, G) is an ideal in semi-group {Q (G, G), ∘}

Proof:
We have the result that "Composition of any two pure quasi-homomorphisms is pure quasi-homomorphism" [Refer table 3.2]. Now it is clear that {P (G, G), ∘} is semi-group. Similarly we have another result that "Composition of any pure quasi-homomorphism and quasi-homomorphism is pure quasi-homomorphism. Also Composition of any quasi-homomorphism and pure quasi-homomorphism is also pure quasi-homomorphism" [Refer table 3.2].
Thus P (G, G) is an ideal in semi-group {Q (G, G), ∘} □

4. D. Miscellaneous Results

Definition 6.7: Let *v*, *k*, *i* be fixed positive integers with $v \geq k \geq i$; let Ω be a fixed set of size *v*; *Jonson graph* *J*(*v*, *k*, *i*) **[18]** define as follows. The vertices of *J* (*v*, *k*, *i*) are the subsets with size *k*, where two subsets are adjacent if there inter section has size *i*.

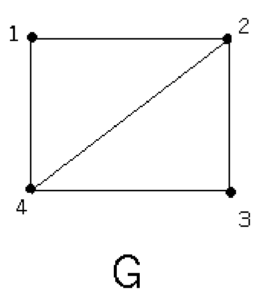
Example 6.3: Consider the Jonson graph *J* (4, 2, 1), which has $\binom{4}{2}$ = 6 vertices & it is $\binom{2}{1}\binom{4-2}{2-1}$ = 4 - regular graph.

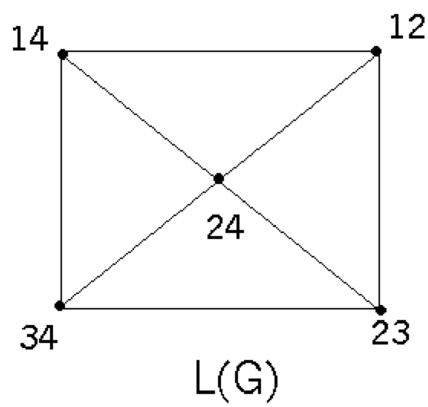


Definition 6.8:
Let G be the graph, then *Line graph* **[4]** of graph G is denoted by L (G) and defined as follows: V (L (G)) = E (G) and two edges are adjacent in L(G) if they are adjacent in G.

Remark: Let G be (*n*, *m*) graph with V (G) = $\{v_1, v_2, \ldots, v_n\}$, then |E (L (G))|= $\frac{1}{2}\sum_{i=1}^{n}(d(v_i))^2 - 2m$.

Example 6.4: Consider the graph G = $K_4 - e$ and its line graph L (G).





Theorem 6.15:
Let G be a simple graph with *n* vertices. Then there exists an induced sub graph T of Jonson graph *J* (*n*, 2, 1) such that L (G) ≅ T.

Proof:
Let G be a simple graph with V (G) = $\{v_1, v_2, \ldots, v_n\}$.Consider the sub graph T of

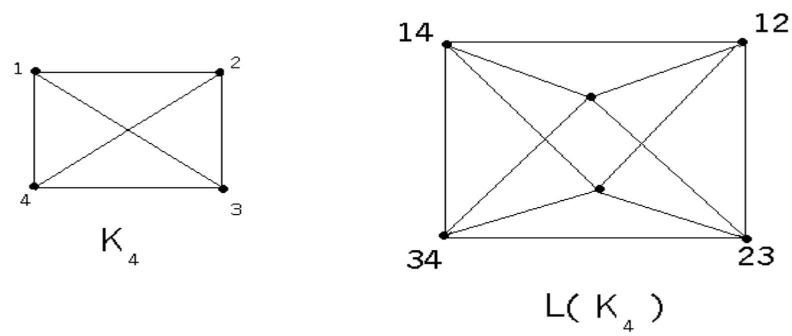
J (*n*, 2, 1) whose vertices are all two sub sets {*p*, *r*} such that $v_p v_r$ ∈ E (G). Then the function *f* : V (L (G)) → V(T) ruled by : *f* ($v_p v_r$) = {*p*, *r*} is well defined map and it is also bijective.
By the definition of adjacency in Jonson graph as well as Line graph it is clear that *f* is strong homomorphism. Thus L (G) ≅ T. □

Tester: Compare Jonson graph *J* (4, 2, 1) and graph L (G) in above examples 6.3 & 6.4.

Corollary 6.15.1:
If G be a complete graph with *n* vertices, then L (G) ≅ *J* (*n*, 2, 1).

Proof:
Arguments are similar to theorem 6.15. □

Tester: Consider the following complete graph K_4 and its line graph L (K_4)



Cleary by example 6.3, *J* (4, 2, 1) ≅ L (K_4).

Corollary 6.15.2:
For any positive integer $n \geq 2$ there exists Jonson graph *J* (*m*, 2, 1) ($m > n$) containing K_n .

Proof: Arguments are similar to theorem 6.15. □

Definition 6.9:
Let *f* : V (G) → V(H) be any function, then the set S_f = {(*u*, *f* (*u*)) / *u* ∈ G} is known as *function set*. Clearly S_f ⊂ V (G) × V(H).

Suppose X and Y are two sets & *f* : X → Y, define T : X → X × Y as

$T(x) = (x, f(x))$, for any x in X.

Theorem 6.16:

A function $f : V(G) \to V(H)$ is pure quasi-homomorphism if and only if $T : V(G) \to S_f \subset V(G \square H)$ is an isomorphism.
i.e. $f \in P(G, H) \Leftrightarrow G \cong S_f$, where $S_f \subset V(G \square H)$

Proof:

Let a function $f : V(G) \to V(H)$ is pure quasi-homomorphism and $S_f = \{(u, f(u)) / u \in G\}$ is function set, where $S_f \subset V(G \square H)$.
Consider the function $T : V(G) \to S_f \subset V(G \square H)$ defined as $T(u) = (u, f(u))$
Then it is clear that the map T is bijective.
Then $x \sim y$ in G $\Rightarrow$ $f(x) = f(y)$ in H, since f is a pure quasi homomorphism.
$\Rightarrow (x, f(x)) \sim (y, f(y))$ in $G \square H$.
$\Rightarrow T(x) \sim T(y)$ in H.
Similarly we can prove that T is complementary homomorphism. Thus, T is strong homomorphism.
Conversely suppose $T : V(G) \to S_f$ is an isomorphism.
Let $x \sim y$ in G $\Rightarrow T(x) \sim T(y)$ in S_f, where $S_f \subset V(G \square H)$.
$\Rightarrow (x, f(x)) \sim (y, f(y))$ in $G \square H$.
$\Rightarrow$ By the adjacency of Cartesian product, $f(x) = f(y)$ in H.
Thus, f is a pure quasi homomorphism. $\square$

Similarly we can prove the following results:

Theorem 6.17:

A function $f : V(G) \to V(H)$ is homomorphism if and only if $T : V(G) \to S_f \subset V(G \times H)$ is isomorphism.(Weak product)
i.e. $f \in \text{Hom}(G, H) \Leftrightarrow G \cong S_f$, where $S_f \subset V(G \times H)$

Theorem 6.18:

A function $f : V(G) \to V(H)$ is quasi-homomorphism if and only if $T : V(G) \to S_f \subset V(G \boxtimes H)$ is isomorphism.(Strong product)
i.e. $f \in Q(G, H) \Leftrightarrow G \cong S_f$, where $S_f \subset V(G \boxtimes H)$

Theorem 6.19:

Any sub set $V(G) \times V(H)$ in $G \square H$ is dominating set then either it is in the form $\pi_1^{-1}(A)$, where A is dominating set in G -or- in the form $\pi_2^{-1}(B)$, where B is dominating set in H.

Proof: Let A be a dominating set in G. Consider the first projection map π_1: V (G□H) → V (G). Clearly $\pi_1^{-1}(A) = A \times$ V (H) . Let $(a, b) \notin A \times$ V (H) then $a \notin A$. Since A is a dominating set in G, there exist $x \in A$ such that $x \sim a$ in G. Then $(x, b) \in A \times$ V (H) such that $(a, b) \sim (x, b)$ in G□H.
Thus $\pi_1^{-1}(A)$ is dominating set in G□H.
Similarly we can prove $\pi_2^{-1}(B)$ where B is dominating set in H.

Conversely suppose $A \times B$ is dominating set in G□H. Then we prove that either A = V (G) and B is dominating set in H -or- B = V (H) and A is dominating set in G.
Suppose $A \neq$ V (G). Let $x \in$ V (G) $- A$, then for any y ∈ V (H), $(x, y) \notin A \times B$.
Since $A \times B$ is dominating set, then there exist $(a, b) \in A \times B$ such that $(x, y) \sim (a, b)$ in G□H.
Now $a \neq x$ because $a \in A$ and $x \notin A$. Then $a \sim x$ & $b = y$ by the adjacency in G□H. Therefore $y \in B$. So, B = V (H). Thus $A \times B = A \times$ V (H).
Now we will show that A is dominating set in G.
Let $r \notin A$ and for any $s \in$ V (H) , then $(r, s) \notin A \times B = A \times$ V (H) which is dominating set in G□H. Therefore there exist $(p, q) \in A \times V(H)$ such that $(r, s) \sim (p, q)$ in G□H. Now $p \neq r$ because $p \in A$ and $r \notin A$. Then $p \sim r$ in A by the adjacency in G□H. Hence A is dominating set in G.
Similarly we can prove the case $B \neq$ V (H). □

"Mathematics is a most exact size and its conclusions are capable of absolute proofs." –C.P.STEINMETZ

Concluding remarks

Here I have identified some of the area where these results may be utilized.

<u>Significant of the study</u>

In the present work, I have established five new models for homomorphism of graphs and proved several basic results related to them. These results may be useful in the following areas.

1. Advance study of retract or absolute retract.
2. Various Coloring problems of graphs
3. Problems on counting variants of homomorphism.
4. Problems on core of graph.
5. Problems on categorical aspect of graph theory.
6. Problems on mathematical modeling (like problems of assignments and schedules)

I have also presented two new models of function graphs and proved several results related to them. These results may be useful in the following areas.

1. Problems on multiplicative graph
2. Problems on projective graph or polymorphism of graphs
3. Problems on homomorphism between any two function graphs

<u>Recommendation for further research</u>

In chapter -5, I have tested some graph parameters to check its effect under different variants of homomorphism as well as to check its validity under different function graphs. But we have large number of graph parameters. So we can investigate the parallel results. Several of the *graph parameters* which we have not tested in the present study are as follow:

1. Connected dominating set
2. Matching set
3. Irredundant set
4. Cut set
5. Bond
6. Packing
7. Arboricity
8. Centre of the graph
9. Private neighbor with respect to some subset of vertex set.
10. Edge cover

Further research is possible for the following *graph invariants* with regard to function graphs in particular.

1. Vertex connectivity or edge connectivity
2. Bandwidth
3. Domination number
4. Independence number
5. Diameter
6. Radius
7. Chromatic number
8. Wiener index
9. Circuit rank
10. Balaban index

In the presence study, I had considered few graph operations. Further research may be possible for the following *graph operations*.

1. Graph joins
2. Graph intersection
3. Graph Union
4. Graph difference
5. Graph power
6. Vertex contraction or edge contraction.
7. Sub division of an edge
8. Modular product (According to the survey, there are 256 graph products are exist)

9. Graph sum
10. Amalgamation of graphs

Thus there is a large area of research which can be explored.

This is end of my research work. I hope the presented work meets the standards of research work. I look forward to continue my research in Graph Theory & related areas.

In particular, I apologize in advance for the corrections of errors if any will be made between printings of this thesis.

PARAS DINESHCHANDRA UCHAT

References

Reference Books

Books on Graph Theory

[1] Diestel R. "**Graph Theory"**, Springer – verlag, 2000 ,ISBN 0-387-95014-1.

[2] Bollobás B. "**Modern Graph Theory"**. Spring GTM 184 New York 1998.

[3] Clark J. & Holton D.A. "**A First Look at Graph theory**" World Sckientific Publishing,1991.

[4] West D. B. " **Introduction to Graph theory** " Prentice- Hall Inc , 2002 , ISBN -81-203-2142-1

[5] Wilson R. J., "**Introduction to Graph theory** " , Pearson Educationpre.Ltd.,2002 ,ISBN 81-7808-635-2

[6] Harary F."***Graph Theory",*** Narosa Publishing House, 2000 , ISBN 81-85015-55-4.

[7] Gross J. & Yellen J. "**Graph Theory and Its Applications" ,** CRC Press,1999 ,ISBN 0- 8493-3982-0

[8] Chartrand G. & Lesniak L., **"** ***Graphs and Digraphs*****",** Champman And Hall/CRC- A CRC Press Cpmpany, 2004, ISBN 1-58488-390-1.

[9] Deo N. & Ghosh A.K. "**Graph Theory - with Applications to Engineering and Computer Science",** Prentice Hall of India , 2001 , ISBN-81-203-0145-5.

[10] Aldous J.M. & Wilson R.J." **Graphs and Applications – An Introductory Approach",** Springer Verlag , 2000, ISBN 1 - 85233-259-X

[11] Gross J. & Yellen J., "**Hand book of Graph Theory**" CRC Press , 2004 , ISBN 1-58488-090-2

[12] Imrich W. & Klavzar S., **"Product Graph- Structure and Recognition",** John Wiley and Sons, Inc., 2000, ISBN 0-471-37039-8

[13] Haynes T.W., Hedetniemi S.T.& Slater P.J., **Dominations in Graphs"- Advanced Topics,** Marcel Dekker, Inc ,1998 , ISBN 0- 8247- 034-1.

[14] Haynes T.W., Hedetniemi S.T.& Slater P.J., "**Fundamentals of Dominations in Graphs" ,** Marcel Dekker, Inc , 1998, ISBN 0-8247 – 0033-3

Books on Algebraic Graph Theory

[15] Beineke L.W.& WilsonR.J.,"**Topics in Algebraic Graph Theory**" **,** Cambridge University Press , 2004 , ISBN 0- 521-80197-4

[16] Biggs N., "**Algebraic Graph Theory**", Cambridge University , 1993 , ISBN 0 -521 45897 8

[17] Hell P. & Nesetrril J. "**Graphs and Homomorphisms**", Oxford University Press,2004, ISBN 0 19-852817 5

[18] Godsil C. & Royle G. , "**Algebraic Graph Theory**", springer-verlag 2001 ,ISBN 0-387-95241-1

[19] Hahn G. & Tardif C.**, "Graph homomorphisms: Structure and symmetry",** Kluwer Acad.Publ.,1997 **,** ISBN 0-8246-0031-4

Books on Algebra & Combinatorics

[20] Anderson I., **"A first Course in Discrete mathematics",** Springer Verlag, 2000, ISBN 1- 85233-236-0.

[21] Stoll R. R., **"Set Theory and Logic",** Eurasia Publishing House, 1963.

[22] Gopalkrishnan N.S. , **" University Algebra",** New Age International Publishers (Formerly Wiley Eastern Limited) , 2006, ISBN 0 -85226-338-4.

[23] Rosen K. H., " **Handbook of Discrete and Combinatorial Mathematics**", CRC Press, 2000, ISBN 0- 8493-0149-1.

[24] Lyons L., **"Mathematics for Science Students",** Cambridge University Press , 2000, ISBN 0-521-78615-0.

[25] Okninski J. , "**Semigroup Algebras",** Marcel Dekker, Inc. , 1990, ISBN- 0- 8247-8356-5.

[26] Herstain I. N., "**Topics in Algebra**"-, John Wiley & Sons, 2000 ISBN 9971-512-53-X.

Communicated Papers

[p.1] "**Quasi Map Graph**" accepted for publication in Mathematics Today, Ahmadabad (Volume No.24, June 2008)

[p.2] "**Quasi-homomorphism of Graphs**" communicated with Allahabad Mathematical Socity,Allahabad (February 2008)

[p.3] "**Pure Quasi homomorphism of graphs**" communicated with International journal of pure and applied mathematical sciences, Iran (May 2008)

[p.4] "**Quasi Map Graph and Related Results**" under revision of International conference proceeding will be publishing by Rumanian Mathematical Society (June 2008)

e **– References**

e – Materials

[m.1] Graph homomorphism, *Peter Cameron*, Combinatorics Study group notes, September 2008

[m.2] **Graph homomorphism**: **Computational aspects & infinite Graphs,** G.hahn & G.Mac Gillivary A survey for Graph homomorphism, June 2002

[m.3] Graph homomorphism: **Tutorial Rick Brewster ,** Field 's institute covering arrays workshops -2006

[m.4] Research Notes: homomorphism of complete *n* partite graphs , *Robert D Girse,* April 1985.

[m.5] Counting homomorphism,*Christian Borgs &Laszlo Lovasz*,February 2006.

[m.6] Homomorphisms **and edge colorings of planner graphs ,** *Reza Naserasr.*

[m.7] Harmonic Homomorphisms , *NoahGiansiracusa* , August 2004.

[m.8] H**omomorphism in Graph property testing –A survey**, *Noga Alon & Asaf Shapira.*

[m.9] Homomorphicity & Non Coreness of labeled posets are NP Compelete, *Erkko Lehtonen,* October 2006.

Web Links

[w.1] http://www.graphtheory.com

[w.2] http://www.cs.sfu.ca/people/Faculty/Profile/pavol.html

[w.3] http://kam.mff.cuni.cz/~nesetril/cz/

[w.4] http://quoll.uwaterloo.ca/~godsil

[w.5] www.google.com

[w.6] http://www.personal.kent.edu/~rmuhamma/GraphTheory.html

[w.7] http://en.wikipedia.org

[w.8] http://www.math.fau.edu/locke/GRAPHTHE.HTM

[w.9] https://cpi.utc.edu/oneweb/~Christopher-Mawata/petersen/

[w.10] http://www-leibniz.imag.fr/GRAPH/english/welcome.html

[w.11] http://mathworld.wolfram.com

“Mathematics is the indispensable instrument of all physical research.”-
BERTHELOT

Glossary of Terms

Appendix - A

In this section I would like to mention, definitions of all new terms which I have introduced during the study so that reader gets the quick reference whenever required. Let's recall that all are assumed to be simple.

Definition 3.2:

A mapping $f : V(G) \rightarrow V(H)$ is said to be *Quasi Homomorphism* of Graphs, if $x \sim y$ in G then $f(x) \sim f(y)$ *or* $f(x) = f(y)$ in graph H.

Definition 3.4:

A mapping $f : V(G) \rightarrow V(H)$ is said to be *complementary homomorphism* of graphs, if $f(x) \sim f(y)$ in H then $x \sim y$ in graph G.

Definition 3.5:

A mapping $f : V(G) \rightarrow V(H)$ is said to be *pure quasi homomorphism* of graphs, if $x \sim y$ in G then $f(x) = f(y)$ in graph H.

Definition 3.6:

A mapping $f : V(G) \rightarrow V(H)$ is said to *be quasi complementary homomorphism* of graphs, for any two distinct vertices x & y in G, if $f(x) \sim f(y)$ *or* $f(x) = f(y)$ in H then $x \sim y$ in graph G.

Definition 3.7:

A mapping $f : V(G) \rightarrow V(H)$ is said to be *strong quasi homomorphism* of graphs, if (1) it is quasi homomorphism (2) it is quasi complementary homomorphism.

Definition 4.2:

Quasi-Map graph Q (H^G) is defined as follows: Two elements f and g of $\Re$ (G, H) are said to be adjacent if $f(u)$ is adjacent to $g(u)$ or $f(u) = g(u)$ in H,for each vertex u in G.

Definition 4.3:

Direct Map graph D (H^G) is defined as follows: Two elements *f* and *g* of $\Re$ (G, H) are said to be adjacent if *f*(*u*) is adjacent to *g*(*u*) in H,for each vertex *u* in G.

Definition 6.2:

A sub graph H of graph G is said to be *quasi-retract* of G, if there is a quasi-homomorphism *f* from G onto H such that *f* (*x*) = *x* for all *x* in H. The map *f* is called *quasi-retraction*.

"One should study Mathematics because it is only through Mathematics that nature can be conceived in harmonious form." BIRKHOFF

Symbol Index

Appendix – B

In this section I would like to mention the list of all symbols or notations which I have used during the study so that reader may get the quick reference whenever required.

Symbol	**Definition / Meaning**
V(G)	Vertex set of graph G
E(G)	Edge set of graph G
$\|S\|$	Number of elements in set S
$\|G\|$	Number of vertices of graph G ; $\|V(G)\|$
γ(G)	Domination number of graph G
H ⊂ G	Graph H is sub graph of graph G *or* graph G is super graph of graph H
G ⊠ H	Strong graph product of graphs G & H
G ∘ H	Lexicographic graph product of graphs G & H
G × H	Weak(Categorical) graph product of graphs G & H
G □ H	Cartesian graph product of graphs G & H
G ∗ H	The graph whose vertex set is V (G) × V (H)
G + H	The join of graphs G & H
N[*u*]	Closed neighborhood of vertex *u*
N(*u*)	open neighborhood of vertex *u*
deg(*u*)	Degree of vertex *u*
$d_G(u, v)$	Distance between vertices *u* & *v* in graph G
$\Re$ (G, H)	Set of all functions from graph G to H
P(G ,H)	Set of all pure quasi homomorphisms from graph G to H
Q(G ,H)	Set of all quasi homomorphisms from graph G to H
Hom(G ,H)	Set of all homomorphisms from graph G to H
Com(G ,H)	Set of all complementary homomorphisms from graph G to H
QC(G ,H)	Set of all quasi complementary homomorphisms from graph G to H
C(G , H)	Set of all constant functions from graph G to H
H^G	Map graph of graphs G &H
Q (H^G)	Quasi map graph of graphs G & H
D (H^G)	Direct power map graph of graphs G & H
$v \sim u$	Vertex *v* is adjacent to vertex *u*
$v \nsim u$	Vertex *v* is not adjacent to vertex *u*
L(G)	The line graph of a given graph G

$\overline{G}$	The complement of graph G
G/θ_f	Quotient graph of graph G with respect to partition θ_f
$G \rightarrow H$	There is a homomorphism from graph G to H
$G \nrightarrow H$	There is no homomorphism from graph G to H
$J(v, k, i)$	Jonson graph for integers v, i, k
$G \cong H$	graphs G & H are isomorphic to each other
K_n	Complete graph with n vertices
C_n	Cycle graph with n vertices
$K_n - e$	Edge deleted K_n
P_n	Path graph with n vertices
$K_{m,n}$	Complete bipartite graph with m & n number of vertices

“There is no permanent place in the world for ugly Mathematics… . It may be very hard to define mathematical beauty but that is just as a true of beauty of any kind, we may not know quite what we main by a beautiful poem, but that does not prevent as from recognizing one when we read it”. G.H.HARDY

Some Notable Graph Theorists

Appendix – C

In this section I would like to mention biographies of some notable graph theorists whose contributions and imaginations made this area of graph theory so enriched & beautiful.

Jaroslav Nesetril

Born, March 13, 1946, Czechslovakia.

Education

Diploma in mathematics at Charles University, Prague, 1964.
M.Sc. at Mc Master University, Canada, 1969.
RNDr.degree at Charles University, Prague, 1970.
DrSc. degree at Charles University, Prague, 1988.

Work Profile

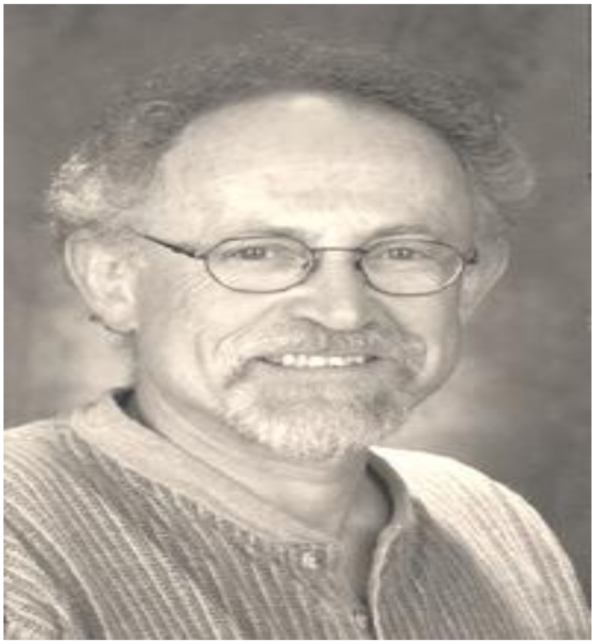
Professor**,** Department of Applied Mathematics, Faculty of Mathematics and Physics, Charles University**,** Prague**. Director,** Institute of Theoretical Computer Science, Charles University, Prague. DIMATIA Centre**,** Prague

Research Interest: Combinatorics (Ramsay Theory), Algebra (Categories, Homomorphism), Graph Theory (Coloring Problems), Computer Science (NP-Completeness).

Publications: Over 250 publications including several books.

Editor of: Mathematics of Ramsay Theory, Topological , Combinatorial & Algebraic Structure

Pavol Hell



Education

M.Sc. 1970, McMaster University
Ph.D. 1973, Université de Montréal
The sequence of ancestors in the Mathematics Genealogy Project

Work Profile

Professor in the School of Computing Science at Simon Fraser University

Research Interests

Algorithmic Graph Theory
Complexity of Algorithms
Combinatorics of Networks

Publications
Over 200 publications including several books.

Chris Godsil

Born, Bendigo (Victoria, Australia)

Education

He had completed his graduated from the University of Melbourne in 1969 with a degree in Biochemistry & completed his Ph. D in 1979.

Work Profile

Professor**,** Department of Combinatorics and Optimization at University of Waterloo.

Research Interests

He is interested in the interaction between algebra and combinatorics. In particular application of algebraic techniques to graphs ,designs & codes.

Publications
More then 70 papers in the area of Algebraic Graph Theory.

Editor of: Journal of Algebraic Combinatorics.