STUDY OF SOME INTERESTING TOPICS IN THE THEORY OF GRAPHS

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in

MATHEMATICS

by

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Certificate

This is to certify that the thesis entitled **Study of Some Interesting Topics in The Theory of Graphs** submitted by **Nilesh A. Dani** to **Saurashtra University, RAJKOT (GUJARAT)** for the award of the degree of DOCTOR OF PHILOSOPHY in Mathematics is bonafide record of research work carried out by him under my supervision. The contents embodied in the thesis have not been submitted in part or full to any other Institution or University for the award of any degree or diploma.

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Declaration

I hereby declare that the content embodied in this thesis is the bonafide record of investigations carried out by me under the supervision of **Dr. S. K. Vaidya** in the Department of Mathematics, **Saurashtra University, RAJKOT**. The investigations reported here have not been submitted in part or full for the award of any degree or diploma of any other Institution or University.

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It gives me immense pleasure to submit my Ph. D. thesis on "**STUDY OF SOME INTERESTING TOPICS IN THE THEORY OF GRAPHS**". I have registered my name in 2008 for the research study. Submission of this work within a short span of less than three years was indeed a great task but it became possible with blessings of Almighty ***GOD***, guidance of seniors, support of colleagues and family members as well as best wishes of all near ones and dear ones.

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Nilesh A. Dani

Contents

Dedicated to my adored mother

Chapter 1

Introduction

The theory of graphs mainly evolved with the rise of computer age. This theory has rigorous applications in diversified fields like computer technology, communication networks, electrical networks and social sciences. Graphs have been proved a powerful mathematical tool to explain structure of molecules. It is also possible to explain flow of control with the help of graph structures.

The study of famous Königsberg bridge problem during 1736 by Leonhard Euler is supposed to be the birth of graph theory. In 1847 G. R. Kirchhoff developed the theory of trees for their applications in electrical networks. Ten years later, A. Cayley discovered trees while he was trying to enumerate the isomers of hydrocarbons.

It is believed that A. F. Möbious presented the famous four color problem in one of his lecture in 1840. About ten years later, A. De Morgan discussed this problem with his fellow mathematicians in London. The discussion by De Morgan is regarded as the first systematic representation of four color problem. This problem has accelerated the research in graph theory. The well celebrated four color problem took hundred years for its solution. In 1976 Wolfgang Haken and Kenneth Appel solved this problem.

The first book on graph theory was published in 1936 by D. König. At present thousands of research papers have been published and many titles available by eminent authors like C. Berge, Frank Harary, Paul Erdös, D. B. West, Gross and Yellen.

The later part of the last century has witnessed intense activity in graph theory. Development of computer science and optimization techniques boost up the research work in the field. There are many interesting fields of research in graph theory. Some of them are domination of graphs, decomposition of graphs, algebraic graph theory, topological graph theory and labeling of graphs.

Any field of investigation becomes more interesting when there arise number of problems that pose the challenges to our mind for their eventual solutions, more so when the field it self is just emerging and whole galore of seemingly related or even unrelated open problems provide motivation for research. The problems arising from the study of various labeling techniques is one of such field. The labeling of graphs have become a field of multifaceted applications ranging from social science to neural network and to biotechnology, to mention a few.

Graph labeling were first introduced by A. Rosa during 1960. At present couple of dozens labeling techniques exist and vast amount of literature is available in printed as well as in electronic form on various graph labeling problems.

The present work is aimed to discuss cordial labeling, 3-equitable labeling, strongly multiplicative labeling and product cordial labeling. The content is divided in to six chapters.

This first Chapter is of introductory nature.

The immediate Chapter-2 is intended to provide basic terminology and preliminaries which are necessary for the subsequent chapters.

The penultimate Chapter-3 is targeted to discuss cordial labeling of graphs. Here we report some of the existing results. We contribute fifteen new results to the theory of cordial labeling. The focus of this chapter is to provide a cordial labeling for the larger graphs obtained by some graph operations on standard graphs. We have investigated some results in the context of graph operations namely fusion of two vertices and duplication of vertices. We also introduce new graph operation known as duplication of an arbitrary edge and we prove that graphs obtained by duplication of an arbitrary edge in cycle C_n and wheel W_n admit cordial labeling.

The Chapter-4 is focused on 3-equitable labeling of graphs. We investigate fifteen new results for 3-equitable labeling. All the results are analogous with the results investigated in the context of cordial labeling which are reported in the previous chapter.

A graph H is called a *supersubdivision* of G if H is obtained from G by replacing every edge e_i of G by a complete bipartite graph K_{2,m_i} for some $m_i, 1 \leq i \leq q$ in such a way that the end vertices of each e_i are merged with the two vertices of 2-vertices part of K_{2,m_i} after removing the edge e_i from graph G. A supersubdivision H of G is said an *arbitrary supersubdivision* of G if every edge of G is replaced by an arbitrary $K_{2,m}$ (Here m may vary for each edge arbitrarily).

In Chapter-5 we investigate four results in the context of cordial labeling and arbitrary supersubdivision of graphs. We also contribute five new results which relate strongly multiplicative labeling and arbitrary supersubdivision of graphs.

The last Chapter-6 is aimed to discuss product cordial labeling of graphs. We investigate eleven new results for the product cordial labeling. Here we show that the graph obtained by duplication of apex vertex of wheel W_n is not product cordial. Here we also investigate product cordial labeling for the larger graphs resulted from the graph operations on standard graphs.

Throughout this work we pose some open problems and throw some light on the future scope of research.

The list of symbols and references are listed alphabetically at the end of the thesis.

List of Publications Arising From the Thesis

1. Cordial and 3-equitable labeling for some star related graphs., *International Mathematical Forum*,4(3), 2009, 1543-1553. (http://www.m-hikari.com/ imf.html)
2. Cordial and 3-equitable labeling for some shell related graphs., *Journal of Scientific Research*, 1(3), 2009, 438-449.
(http://www.banglajol.info/index.php/JSR/index)
3. Some wheel related 3-Equitable Graphs in the context of vertex duplication., *Advances Applications in Discrete Mathematics*, 4(1), 2009, 71-85.
(http://www.pphmj.com)
4. Some new star related graphs and their cordial as well as 3-equitable labeling.,*Journal of Science*,1(1),2010, 111-114.
5. Cordial and 3-equitable labeling for some wheel related graphs., Accepted for publication in *International Journal of Applied Mathematics*.

6. Strongly multiplicative labeling in the context of arbitrary supersubdivision., *Journal of Mathematics Research*, 2(2),2010, 28-33.
(http://ccsenet.org/journal/index.php/jmr)

7. Some new product cordial graphs., *Journal of Applied Computer Science & Mathematics*,8(4),2010, 62-65.(http://jacs.usv.ro)

8. Cordial labeling and arbitrary supersubdivision of some graphs., Accepted for publication in *International J. of Information Sc. and Computer Maths*.
(http://pphmj.com/journals/ijiscm.htm)

The reprints/preprints of above papers are provided as an annexure.

Details of the Work Presented in Conferences

1. The paper entitled as "Gracefulness of union of two path graphs with grid graph and complete bipartite graph" in *International Conference on Emerging Technologies and Applications in Engineering, Technology and Sciences* at Saurashtra University, Rajkot during 13-14 January, 2008.

2. The paper entitled as "Product cordial graphs induced by some graph operations on cycle related graphs" in *Fifth Annual Instructional Conference of ADMA & Graph Theory Day V* at Periyar University, Salem (Tamil Nadu) during 8-10 June, 2009.

3. The paper entitled as "Some cordial graphs in the context of fusion and duplication" in *Sixth Annual Instructional Conference of ADMA & Graph Theory Day VI* at College of Engineering, Pune(Maharashtra) during 8-10 June, 2010.

Chapter 2

Basic Terminology and Preliminaries

2.1 Introduction

This chapter is intended to provide all the fundamental terminology and notations which are needed for the present work.

2.2 Basic Definitions

Definition 2.2.1. A *graph* $G = (V(G), E(G))$ consists of two sets, $V(G) = \{v_1, v_2, \ldots\}$ called *vertex set* of G and $E(G) = \{e_1, e_2, \ldots\}$ called *edge set* of G. Sometimes we denote vertex set of G as $V(G)$ and edge set of G as $E(G)$. Elements of $V(G)$ and $E(G)$ are called *vertices* and *edges* respectively.

Definition 2.2.2. An edge of a graph that joins a vertex to itself is called a *loop*. A loop is an edge $e = v_i v_i$.

Definition 2.2.3. If two vertices of a graph are joined by more than one edge then these edges are called *multiple edges*.

Definition 2.2.4. A graph which has neither loops nor parallel edges is called a *simple graph*.

Definition 2.2.5. If two vertices of a graph are joined by an edge then these vertices are called *adjacent vertices*.

Definition 2.2.6. Two vertices of a graph which are adjacent are said to be *neighbours*. The set of all neighbours of a vertex v of G is called the *neighbourhood set* of v. It is denoted by $N(v)$ or $N[v]$ and they are respectively known as open and closed neighbourhood set.

$N(v) = \{u \in V(G)/u \text{ adjacent to } v \text{ and } u \neq v\}$

$N[v] = N(v) \cup \{v\}$

Definition 2.2.7. If two or more edges of a graph have a common vertex then these edges are called *incident edges*.

Definition 2.2.8. *Degree* of a vertex v of any graph G is defined as the number of edges incident on v, counting twice the number of loops. It is denoted by $deg(v)$ or $d(v)$.

Definition 2.2.9. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. Then *cartesian product* of G_1 and G_2 which is denoted by $G_1 \times G_2$ is the graph with vertex set $V = V_1 \times V_2$ consisting of vertices $u = (u_1, u_2)$, $v = (v_1, v_2)$ such that u and v are adjacent in $G_1 \times G_2$ whenever ($u_1 = v_1$ and u_2 adjacent to v_2) or ($u_2 = v_2$ and u_1 adjacent to v_1).

Definition 2.2.10. The *corona* $G_1 \odot G_2$ of two graph G_1 and G_2 is defined as a graph obtained by taking one copy of G_1 (which has p_1 vertices) and p_1 copies of G_2 and attach one copy of G_2 at every vertex of G_1.

Definition 2.2.11. An *armed crown* is a graph in which path P_m is attached at each vertex of cycle C_n. This graph is denoted by $C_n \odot P_m$.

Definition 2.2.12. The *eccentricity* of a vertex u, written $\varepsilon(u)$, is $\max_{v \in V(G)} d(u, v)$.

Definition 2.2.13. Consider a cycle C_m. Let T_i $(i = 1, 2, \ldots, n \leq m)$ be a rooted tree, that is to say, a vertex in T_i is distinguished as the root of T_i. Form a graph G from C_m and the T_i's by identifying the root of each tree T_i with a vertex of C_m so that different roots are identified with different vertices of C_m. Then G is a *unicyclic graph* which will be denoted by $C_m(T_1, T_2, \ldots, T_n)$

Definition 2.2.14. g_n is the graph with $n + 2$ vertices and $3n - 1$ edges obtained by joining all the vertices of P_n to two additional vertices.

Definition 2.2.15. A graph $G = (V(G), E(G))$ is said to be *bipartite* if the vertex set can be partitioned into two disjoint subsets V_1 and V_2 such that for every edge $e_i = v_i v_j$ $\in E(G)$, $v_i \in V_1$ and $v_j \in V_2$.

Definition 2.2.16. A *complete bipartite graph* is a simple bipartite graph such that two vertices are adjacent if and only if they are in different partite sets. If partite sets V_1 and V_2 are having m and n vertices respectively then the related complete bipartite graph is denoted by $K_{m,n}$ and V_1 is called m-vertices part and V_2 is called n-vertices part of $K_{m,n}$.

2.3 Concluding Remarks

This chapter provides basic definitions and terminology required for the advancement of the topic. For all other standard terminology and notations we refer to Harrary[24], West[44], Gross and Yellen[23], Clark and Helton[13].

The next chapter is focused on the cordial labeling of graphs.

Chapter 3

Cordial Labeling of Graphs

3.1 Introduction

In the previous chapter, we have provided all the preliminaries and terminology related to the present work while this chapter is aimed to discuss cordial labeling of graphs in detail.

In the succeeding sections we will provide brief account of the concepts of labeling, Graceful labeling, Harmonious labeling and Cordial labeling.

The problems arising from the study of a variety of labeling techniques of the elements of a graph or of any discrete structure is the potential area of challenge. Graph labeling problems are really not of recent origin. e.g. coloring of the vertices arose in connection with the now well known Four Color Theorem, which remain unsolved for long time and took more than 150 years for its solution in 1976. The problem of enumeration of isomers in the hydrocarbon series C_nH_{2n+2} initiated by the work of Keyley is as old as the map coloring problem. In the late 1960's a problem in radio astronomy led to the assignments of the absolute differences of pairs of numbers occurring on the positions of radio antennae to the links of the lay-out plans of the antennae under the constrains of the optimal layout to scan the visible regions of the celestial dome quickly made its way to formulate more tersk mathematical problem on graph labeling. In the effort to provide the solution for this problem the notion of β-valuation was put forward by A. Rosa[37] in 1967. Independent discovery of β-valuation termed as Graceful labeling by Golomb[21] in 1972 which is now the popular term. He also pointed out the importance of studying Graceful graphs in trying to settle the complex problem of decomposing the complete graph by isomorphic copies of a given tree of the same order.

3.2 Labeling of Graphs

Definition 3.2.1. If the vertices of the graph are assigned values subject to certain conditions is known as *graph labeling*.

Most of the graph labeling problem will have following three common characteristics,

- ⋆ a set of numbers from which vertex labels are chosen;
- ⋆ a rule that assigns a value to each edge;
- ⋆ a condition that theses values must satisfy.

A dynamic survey of graph labeling is regularly updated by Gallian[19] and available online on the web site of the electronics journal of combinatorics.

In the succeeding sections the discussion on various labeling techniques will be carried out in chronological order as they introduced.

3.3 Graceful Labeling of Graphs

Graceful labeling was introduced by Rosa [37] in 1967.

Definition 3.3.1. A function f is called *graceful labeling* of a graph G if $f : V(G) \rightarrow \{0,1,2,\ldots,q\}$ is injective and the induced function $f^* : E(G) \rightarrow \{1,2,\ldots,q\}$ defined as $f^*(e = uv) = |f(u) - f(v)|$ is bijective.

A graph which admits graceful labeling is called *graceful graph*.

Initially Rosa named above defined labeling as $\beta-$ *valuation* but Golomb[21] renamed $\beta-$valuation as graceful labeling.

3.3.1 Some Known Facts About Graceful Labeling

- The graceful labeling of a graph is not unique.
- In any graceful graph the vertices with labels 0 and q are always adjacent.

- If the vertex labels $a_i (i = 1, 2, \ldots, p)$ assigned to the graceful graph then $q - a_i$ will yield another graceful labeling for the same graph.
- Subgraph of a graceful graph need not be a graceful graph.
- Supergraph of a graceful graph need not be a graceful graph.
- All the graphs with $p \leq 5$ are graceful except C_5, K_5 and Bowtie graph.
- There are $q!$ graceful graph with q edges.

3.3.2 Some Known Results

- Rosa[37] proved that an Eulerian graph with $q \equiv 1, 2 (mod 4)$ is not graceful.
- Truszczyński[43] studied unicyclic graphs and conjectured that all unicyclic graph C_n, where $n \equiv 1, 2 (mod 4)$ are graceful. Because of the immense diversity of unicyclic graphs a proof of above conjecture seems to be out of reach in the near future.
- Delorme et al.[14] and Ma and Feng[34] proved that the cycle with one chord is graceful.
- Gracefulness of cycle with k consecutive chord is discussed by Koh et al.[31] and Goh and Lim[20].
- Koh and Rogers[32] conjectured that cycle with triangle is graceful if and only if $n \equiv 0, 1 (mod 4)$.
- Ayel and Favaron[6] proved that helms are graceful.
- Kang et al.[28] proved that web graphs are graceful.
- Seoud and Youssef[40] proved that flowers are graceful.
- Golomb[21] proved that the complete graph K_n is not graceful for $n \geq 5$.
- Frucht[18], Hoede and Kuiper[26] proved that all wheels W_n are graceful.

- Drake and Redl[15] enumerated the non graceful Eulerian graph with $q \equiv 1, 2(mod 4)$ edges.
- Kathiresan[29] has investigated the graceful labeling for subdivision of Ladders.
- Sethuraman and Selvaraju[41] have discussed gracefulness of arbitrary super subdivisions of cycles.

3.3.3 Gracefulness of Trees

The conjecture of Ringel-Kotzig[36] states that "All the trees are graceful." has been the focus of many research papers. Kotzig called the efforts to prove gracefulness of trees as a 'disease'. Among all the trees known to be graceful are caterpillars, paths, olive trees, banana trees etc., Some advance results regarding the gracefulness of trees are listed below.

- Huang et al.[27] proved that trees with at most 4 end vertices are graceful.
- Aldred and Mckey[1] proved that trees with at most 27 vertices are graceful.
- Bermond and Sotteau[9] proved that rooted tree in which every level contains vertices of same degree(symmetric trees) are graceful.
- Pastel and Raynaud[35] proved that rooted trees consisting of k branches where the i^{th} branch is a path of length i (olive trees) are graceful.
- Eshghi and Azimi[17] discussed the programming model for finding graceful labeling of graphs. Using this method, they verified that trees with 30,35 or 40 vertices are graceful.

Despite the efforts of many the graceful tree conjecture remained open and faith in the conjecture is so strong that if a tree without a graceful labeling were indeed found than it is possibly would not be considered a tree!

In the next section we will discuss Harmonious labeling in detail and take up the survey of existing results.

3.4 Harmonious Labeling of Graphs

Graham and Sloane[22] introduced harmonious labeling in 1980 during their study of modular versions of additive bases problems stemming from error correcting codes.

Definition 3.4.1. A function f is called *harmonious labeling* of a graph G if $f : V(G) \rightarrow \{0,1,2,\dots,q-1\}$ is injective and the induced function $f^* : E(G) \rightarrow \{0,1,2,\dots,q-1\}$ defined as $f^*(e=uv) = (f(u)+f(v))(mod q)$ is bijective.

A graph which admits harmonious labeling is called *harmonious graph*.

3.4.1 Some Known Results

- Graham and Sloane[22] conjectured that every tree is harmonious.
- Graham and Sloane[22] proved that
  - ◇ $K_{m,n}$ is harmonious if and only if m or $n = 1$.
  - ◇ wheel is harmonious.
  - ◇ Petersen graph is harmonious.
  - ◇ cycle C_n is harmonious if and only if n is odd.
  - ◇ If a harmonious graph has even number of edges q and degree of every vertex is divisible by $2^{\alpha}(\alpha \geq 1)$ than q is divisible by $2^{\alpha+1}$.
  - ◇ All ladders except L_2 are harmonious.
  - ◇ Friendship graph F_n is harmonious except $n \equiv 2(mod 4)$.
  - ◇ Fan f_n is harmonious.
  - ◇ The graph $g_n(n \geq 2)$ is harmonious.
- Aldred and Mckay[1] provided an algorithm and used computer to show that all trees with at most 26 vertices are harmonious.
- Golomb[21] proved that complete graph is harmonious if and only if $n \leq 4$.

3.5 Cordial Labeling of Graphs and Some Existing Results

In 1987 Cahit[10] introduced the concept of cordial labeling as a weaker version of graceful and harmonious labeling.

Definition 3.5.1. A function $f : V(G) \rightarrow \{0,1\}$ is called *binary vertex labeling* of a graph G and $f(v)$ is called *label of the vertex* v of G under f. For an edge $e = uv$, the induced function $f^* : E(G) \rightarrow \{0,1\}$ is given as $f^*(e = uv) = |f(u) - f(v)|$. Let $v_f(0)$, $v_f(1)$ be number of vertices of G having labels 0 and 1 respectively under f and let $e_f(0)$, $e_f(1)$ be number of edges of G having labels 0 and 1 respectively under f^*. A binary vertex labeling f of a graph G is called *cordial labeling* if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph which admits cordial labeling is called *cordial graph*.

There are three types of problems that can be considered in this area.

1. How cordiality is affected under various graph operations?
2. Construct new families of cordial graph by investigating suitable labeling.
3. Given a graph theoretic property P, characterise the class of graphs with property P that are cordial.

In the above referred seminal paper Cahit investigated some classes of cordial graphs as well as a necessary condition for an Eulerian graph to be cordial graph. Ho et al.[25] have also proved some important results on cordial labeling of graphs. We will report some results from these two papers for ready reference.

Theorem 3.5.2. Every tree is cordial.

Proof. We use induction on n, the number of vertices. The statement is obvious for $n \leq 2$. Now let $n \geq 3$, and assume that all trees with $m < n$ vertices are cordial. Let T be any tree with n vertices, and let w be any end-vertex on maximum length path in T. Let $e_1 = wz$ be the end-edge incident with w. If there exists another end-edge $e_2 = zy$

incident with z, delete from T vertices w,y and edges e_1,e_2. The resulting tree T^* has $n-2$ vertices, and so by induction hypothesis it admits a cordial labeling, say f. Define now a labeling f^* of T by $f^*(x) = f(x)$ for all $x \in V(T^*), f^*(w) = 0, f^*(y) = 1$. Clearly, f^* is a cordial labeling of T.

If there is no such end-edge e_2, there must be an edge $e_3 = zu$ (here u is not and end-vertex). Delete form T vertices w,z and e_1,e_3 obtaining tree T_1. Let f_1 be a cordial labeling to T_1. Define a labeling f_1^* of T by $f_1^*(x) = f_1(x)$ for all $x \in V(T_1)$; if $f_1(u) = 0$ put $f_1^*(z) = 0, f_1^*(w) = 1$ and if $f_1(u) = 1$ put $f_1^*(z) = 1, f_1^*(w) = 0$. Again, f_1^* is a cordial labeling of T, and the proof is complete. □

Theorem 3.5.3. The complete graph K_n is cordial if and only if $n \leq 3$.

Proof. If f is a cordial labeling of K_n then either $v_f(1) = v_f(0) = \frac{n}{2}$, or, if n is odd, $|v_f(1) - v_f(0)| = 1$. In the former case, $e_f(1) = \frac{n^2}{4}, e_f(0) = \frac{n(n-2)}{4}$, and we can have $|e_f(1) - e_f(0)| \leq 1$ only if $n = 2$. In the latter case, $e_f(1) = \frac{n^2-1}{4}, e_f(0) = \frac{(n-1)^2}{4}$ and we have $|e_f(1) - e_f(0)| \leq 1$ only if $n = 1$ or 3. On the other hand, it is trivial to show that there exists a cordial labeling of K_1, K_2 and K_3. □

Theorem 3.5.4. The complete bipartite graph $K_{m,n}$ is cordial for all $m,n \geq 1$.

Proof. Let $V = V_1 \cup V_2$, $|V_1| = m, |V_2| = n$, be the bipartition of $K_{m,n}$. If $m = n$, label $\lceil \frac{m}{2} \rceil$ vertices of V_1 and $\lfloor \frac{m}{2} \rfloor$ vertices of V_2 with 0, and the remaining vertices with 1. If $m \neq n$, we may assume $m > n$, say, $m = n + k$. Label $\lceil \frac{k}{2} \rceil$ of the extra k vertices with 0, and the remaining $\lfloor \frac{k}{2} \rfloor$ extra vertices with 1 (and the other vertices as before). It is easy to verify that we have a cordial labeling of $K_{m,n}$. □

Theorem 3.5.5. If G is an Eulerian graph with q edges where $q \equiv 2(mod 4)$ then G has no cordial labeling.

Proof. In a cordial labeling of a graph G with $q \equiv 2(mod 4)$ edges, exactly $\frac{q}{2} \equiv 1(mod 2)$ edges must have label 1. Thus in at least one component of G the number of edges with label 1 must be odd. In such a competent, a closed Eulerian trail starting at a vertex labeled 0 would have to end at (the same) vertex labeled 1, a contradiction. □

Theorem 3.5.6. The cycle C_n with n vertices is cordial if and only if $n \not\equiv 2(mod4)$.

Proof. Necessity follows from the Theorem 3.5.5. For sufficiency, let $n = 4m + r, r \in \{0,1,3\}$, and let $C_n = (v_1, v_2, \dots v_n)$. For $1 \leq i \leq 4m$, put $f(v_i) = 0$ if $i \equiv 1,2(mod4)$, and $f(v_i) = 1$ if $i \equiv 0,3(mod4)$. Moreover, if $r = 1$ put $f(v_{4m+1}) = 1$, and if $r = 3$ put $f(v_{4m+1}) = f(v_{4m+2}) = 0, f(v_{4m+3}) = 1$. It is straightforward to verify that in each case f is a cordial labeling. □

Theorem 3.5.7. A regular graph of degree 1 on $2n$ vertices denoted by $L(2n)$ is cordial if and only if $n \not\equiv 2(mod4)$.

Proof. Let $n \equiv 2(mod4)$. In a cordial labeling of $L(2n)$, let x_i, $i = 0,1,2$, be the number of edges having i of its vertices labeled with 0. Then $x_0 + x_1 + x_2 = n$ and $x_0 = x_2$ which implies $x_1 \equiv 0(mod2)$. On the other hand, if $n \equiv 2(mod4)$, by counting the total number of zeros, we get $2x_0 + x_1 = \frac{n}{2} \equiv 1(mod2)$ which implies $x_1 \equiv 1(mod2)$, a contradiction. Thus $n \not\equiv 2(mod4)$. To obtain a cordial labeling of $L(2n)$, $n \not\equiv 2(mod4)$ take $x_0 = x_2 = \lfloor \frac{n+1}{4} \rfloor, x_1 = n - 2\lfloor \frac{n+1}{4} \rfloor$. □

Theorem 3.5.8. The wheel W_n is cordial if and only if $n \not\equiv 3(mod4)$.

Proof. For necessity, let $n \equiv 3(mod4)$, let f be a cordial labeling of W_n. We may assume w.l.o.g that the center is labeled 0. Then exactly $\frac{n-1}{2}$ vertices of cycle C_n are labeled with 0 and exactly $\frac{n+1}{2}$ with 1. If the vertices labeled 0 were arranged consecutively they would account for $\frac{n-3}{2}$ edges labeled 0, and similarly, if the vertices labeled 1 were arranged consecutively, they would account for $\frac{n-1}{2}$ edges labeled 0. In addition, there are $\frac{n-1}{2}$ edges incident with the center labeled 0. Thus the total number of edges labeled 0 in such a labeling is $\frac{n-1}{2} + \frac{n-3}{2} + \frac{n-1}{2} = \frac{3n-5}{2} \equiv 0(mod2)$ since $n \equiv 3(mod4)$. It is readily seen that transposing labels of adjacent vertices of the cycle C_n either leaves the number edges labeled 0 unchanged or increases or decreases by two. Thus the number of edges labeled 0 in any cordial labeling of W_n, $n \equiv 3(mod4)$, is even. On the other hand, however, this number must equal n, a contradiction.

For sufficiency: When $n \equiv 0$ or $1(mod4)$, take the cordial labeling of C_n given in the Theorem 3.5.6, and in addition, label the center with 0. This results in a cordial labeling of W_n. When $n \equiv 2(mod4)$, label the center with 0, and the vertices of C_n as follows: $1,1,1,1,0,0,1,1,0,0,1,1,\ldots,0,0$. □

Lemma 3.5.9. Let T be an odd tree of order at least 5. If T has end vertices a and b with a common adjacent vertex c, and if $\deg(c)=3$, then there exists a cordial labeling of T such that

$$f(a)=0, \qquad f(b)=f(c)=1 \text{ and } v_f(0)>v_f(1).$$

Proof. By induction on $|V(T)|$.

(i) When $|V(T)|=5$, then T is the tree shown in FIGURE 3.1. It is cordial and has the stated property as indicated.

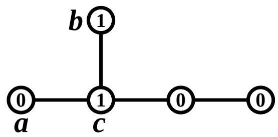

FIGURE 3.1

(ii) Assume that the Lemma 3.5.9 is true for all odd tree T with $5 \le |V(T)| \le 2k+1$, $k \ge 3$, and satisfying the given condition. Let T^* be a tree satisfying the given condition and of order $2k+3$. Since $G = T^* \backslash \{a,b,c\}$ is a tree. G must either two end vertices u and w with a common adjacent vertex, or two adjacent vertices u and w with $\deg(u)=1$ and $\deg(w)=2$. Science the graph $H = T^* \backslash \{u,w\}$ is an odd tree satisfying the given condition and of order $2k+1$, by induction hypothesis there exists a cordial labeling f of H with $f(a)=0$, $f(b)=f(c)=1$ and $v_f(0)>v_f(1)$.

In case that G has two end vertices u and w with a common vertex, then the following binary labeling f^* of T^* is cordial

$$f^*(v) \;=\; \begin{cases} f(v); & v \in V(H) \\ 0; & v = u \\ 1; & v = w \end{cases}$$

In case that G has two other adjacent vertices u and w with $\deg(u) = 1$ and $\deg(w) = 2$, and the other vertex, say z, adjacent to w is labeled 0 in H then

$$f^*(v) = \begin{cases} f(v); & v \in V(H) \\ 1; & v = u \\ 0; & v = w \end{cases}$$

is cordial labeling of T^*. If $f(z) = 1$, then

$$f^*(v) = \begin{cases} f(v); & v \in V(H) \\ 0; & v = u \\ 1; & v = w \end{cases}$$

is cordial. In any case f^* is a cordial labeling of of T^* with the stated property. □

Lemma 3.5.10. The unicyclic graph $G = C_m(T_1, T_2, \ldots, T_m)$, where $m \geq 3$ and $1 \leq n \leq m$ is cordial, if each $T_i (i = 1, 2 \ldots, m)$ is a path of length 1.

Proof. If $m+n$ is odd, then G is a unicyclic graph of odd order, and hence is cordial. Assume that $m+n$ is even.

For $m = 3$, the unicyclic graph $C_3(T_1)$ and $C_3(T_1, T_2, T_3)$, where $T_i (i = 1, 2, 3)$ is a path of length 1, are cordial as shown if FIGURE 3.2 (a) and (b) respectively.

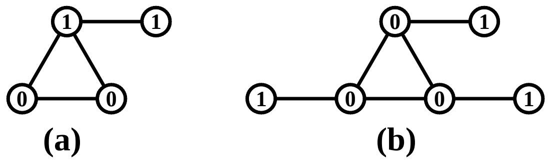

FIGURE 3.2

Now let $m \geq 4$ and $n \geq 1$. Suppose first $m = n$. Then the binary labeling f of G such that

$$f(v) = \begin{cases} 0; & v \in V(C_m) \\ 1; & v \notin V(C_m) \end{cases}$$

is a cordial labeling of G.

Next suppose $m > n$. Let $C_m = [a_1, a_2, \dots, a_m]$. then we must have $\deg(a_j) = 2$ and $\deg(a_{j+1})$=3 for some $j = 1, 2, \dots, m$(here $a_{m+1} = a_1$) without lost of generality we assume that $\deg(a_2) = 2$, $\deg(a_3) = 3$ and b_1 is the vertex adjacent to a_3 but not on the cycle. Two cases to be considered

Case 1: $\deg(a_1) = 2$(see FIGURE 3.3(a))

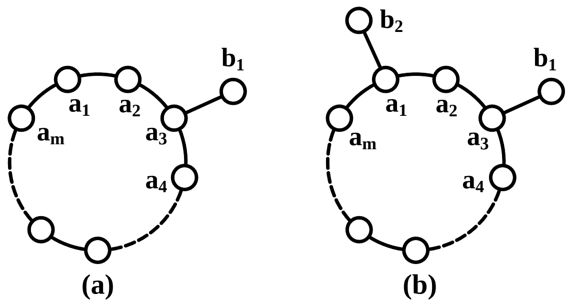

FIGURE 3.3

Then the graph $G_1 = G \backslash \{a_1\}$ is an odd tree satisfying the condition of the Lemma 3.5.9. Let g be the cordial labeling of G_1 such that $g(a_2) = 0, g(b_1) = g(a_3) = 1$ and $v_g(0) > v_g(1)$. If $g(a_m) = 1$, then the binary labeling f of G defined below is easily checked to be cordial.

$$f(v) = \begin{cases} g(v); & v \in V(G_1) \\ 1; & v = a_1 \end{cases}$$

If $g(a_m) = 0$, then the following binary labeling of G is cordial.

$$f(v) = \begin{cases} 1; & v = a_1 \\ 1; & v = a_2 \\ 0; & v = b_1 \\ g(v); & v \in V(G_1 \backslash \{a_2, b_1\}) \end{cases}$$

Case 2: $\deg(a_1) = 3$ (see FIGURE 3.3(b))

Let $G_2 = (G \backslash \{a_1, a_2\}) \backslash \{b_2\}$, that is the graph obtained from G by removing the edge a_1a_2 and the end vertex b_2 adjacent to a_1. Then G_2 is an odd tree satisfying the condition of the Lemma 3.5.9. It follows that there is a cordial labeling h of G_2 with the property that $h(a_2) = 0$, $h(a_3) = h(b_1) = 1$ and $v_h(0) > v_h(1)$

Then the following binary labeling f of G is easily verified to be cordial

$$f(v) = \begin{cases} h(v); & v \in V(G_2) \\ 1; & v = b_2 \end{cases}$$

□

Lemma 3.5.11. The unicyclic graph $G = C_m(T_1)$, where T_1 is a path of length 2 rooted at the center vertex, is cordial for all m.

Proof. When m is odd, then the unicyclic graph G is of odd order and hence is cordial.

Assume that m is even. Two cases to be considered.

Case 1: $m = 4k$. By the Theorem 3.5.6 there exist a cordial labeling f of C_m.

Then the binary labeling f^* of G defined below is cordial.

$$f^*(v) = \begin{cases} f(v); & v \in V(C_m) \\ 1; & v = b_1 \\ 0; & v = b_2 \end{cases}$$

Case 2: $m = 4k+2$. Without loss of generality we can assume that the root of T_1 is identified with the vertex a_1 of the cycle $C_m = [a_1, a_2, \ldots, a_m]$.

Define a binary labeling f^* of G as follows

$$f^*(v) = \begin{cases} 0; & v = a_{2i-1} \\ 0; & v = a_{2i}, \quad i = 2p+1, p = 0,1,2,\ldots,k \\ 1; & v = a_{2j+1} \\ 1; & v = a_{2j+2}, \quad j = 2q+1, q = 0,1,2,\ldots,k-1 \\ 1; & v = b_1 \\ 1; & v = b_2 \end{cases}$$

It is straight forward to check that f^* is a cordial labeling of $G = C_m(T_1)$. □

Lemma 3.5.12. Consider a unicyclic graph $G = C_m(T_1, T_2, \ldots, T_m)$. Suppose some T_i has (i) two end vertices u and w with a common adjacent vertex, or (ii) two adjacent vertices u and w such that $\deg(u) = 1$ and $\deg(w) = 2$, where u and w are not the root of T_i. Then if $G_1 = G \backslash \{u, w\}$ is cordial, so is G.

Proof. Suppose f is a cordial labeling of G_1. In case (i) above the following binary labeling f^* of G is cordial

$$f^*(v) = \begin{cases} f(v); & v \in V(G_1) \\ 0; & v = u \\ 1; & v = w \end{cases}$$

In case (ii) if the other vertex, say z, adjacent to w has label 0, that is, $f(z) = 0$, then

$$f^*(v) = \begin{cases} f(v); & v \in V(G_1) \\ 1; & v = u \\ 0; & v = w \end{cases}$$

is cordial labeling of G. If z has label 1 in G_1, then

$$f^*(v) = \begin{cases} f(v); & v \in V(G) \\ 0; & v = u \\ 1; & v = w \end{cases}$$

is a cordial labeling of G. □

Theorem 3.5.13. A unicyclic graph G is cordial if and only if $G \neq C_{4k-2}$ for all $k \geq 1$.

Proof. Necessity follows from the Theorem 3.5.6.

For sufficiency assume first that the unicyclic graph G is of odd order. Let x be the edge on the cycle of G. Since $G\backslash\{x\}$ is an odd tree, by the Theorem 3.5.2 there exists a cordial labeling f of $G\backslash\{x\}$. As $|E(G)\backslash\{x\}|$ is even, we must have $e_f(0) = e_f(1)$, and hence f is also a cordial labeling of G.

Now assume that the unicyclic graph G is of even order.

Let $m \geq 3$, $n \geq 1$ and $|V(G)|$ is even. From each $T_i (i = 1,2,\ldots,n)$ we repeatedly remove two vertices (not the root) with the stated property (i) or (ii) in the Lemma 3.5.12 until we obtain a unicyclic graph $G_1^* = C_m(T_1^*, T_2^*, \ldots, T_r^*)$ $(r \leq n)$, or $G_2^* = C_m(T^*)$, where each tree $T_i^*, i = 1,2,\ldots,r$ is a path of length 1 and the tree T^* is a path of length 2 rooted at the center vertex. (this can always be achieved). By the Lemma 3.5.10 and Lemma 3.5.11 the above unicyclic graphs G_1^* and G_2^* are both cordial, and by repeated applications of the Lemma 3.5.12 we see that $G = C_m(T_1, T_2, \ldots, T_m)$ is cordial. □

Theorem 3.5.14. The generalized Petersen graph $P(n,k)$ is cordial iff $n \not\equiv 2(mod 4)$.

Proof. Several cases are to be considered.

Case 1: $n = 2m+1$ and $2k+1 \geq m$. Let

$$q = \begin{cases} 2k+1-m; & \text{if } m \text{ is odd.} \\ 2k+2-m; & \text{if } m \text{ is even.} \end{cases}$$

and $q = 2l$. Define a binary labeling f of $P(n,k)$ as follows.

$$f(b_i) = \begin{cases} 1; & i = 0,1,\ldots,m \\ 0; & i = m+1, m+2, \ldots, n-1 \end{cases}$$

$$f(a_i) = \begin{cases} 0; & i = n-l, n-l+1, \dots, n-1, 0, 1, \dots, m-l \\ 1; & i = m-l+1, m-l+2, \dots, n-l-1 \end{cases}$$

Then obviously we have $v_f(0) = v_f(1) = n$. The number of zeros contributed to $e_f(0)$ by edges $b_i b_{i+k}$ is $(m+1-k)+(m-k) = 2m-2k+1$, that by the edges $a_i a_{i+1}$ is $2m-1$, and that by the edges $a_i b_i$ is $2l-2k+1-m$ or $2k+2-m$.

Hence $e_f(0) = 3m+1$ or $3m+2$, and correspondingly $e_f(1) = 3m+2$ or $3m+1$. Hence f is a cordial labeling for $P(n,k)$.

Case 2: $n = 2m+1$ and $2k+1 < m$. Let

$$q = \begin{cases} m-(2k+1); & \text{if } m \text{ is odd.} \\ m-(2k+2); & \text{if } m \text{ is even.} \end{cases}$$

and $q = 2l$. Then by similar argument as in case 1, we can show that the following binary labeling f of $P(n,k)$ is cordial.

$$f(b_i) = \begin{cases} 1; & i = 0, 1, \dots, m \\ 0; & i = m+1, m+2, \dots, n-1 \end{cases}$$

if $l = 0$

$$f(a_i) = \begin{cases} 0; & i = 0, 1, \dots, m \\ 1; & i = m+1, m+2, \dots, n-1 \end{cases}$$

if $l > 0$

$$f(a_i) = \begin{cases} 0; & i = 2p \\ 1; & i = 2p+1 \\ 1; & i = m+2p+1 \\ 0; & i = m+2p+2, \quad p = 0, 1, \dots, l-1 \\ 0; & i = 2l+r, \quad r = 0, 1, \dots, m-2l \\ 1; & i = m+2l+t, \quad t = 0, 1, \dots, n-m-2l-1 \end{cases}$$

Case 3: $n = 4m$ and $k+1 \geq m$. Let $k+1-m = l$. Define a binary labeling f of $P(n,k)$ as follows

$$f(b_i) = \begin{cases} 1; & i = 0,1,\ldots,2m-1 \\ 0; & i = 2m,2m+1,\ldots,n-1 \end{cases}$$

$$f(a_i) = \begin{cases} 0; & i = n-l,n-l+1,\ldots,n-l,0,1,\ldots,2m-l-1 \\ 1; & i = 2m-l,2m-l+1,\ldots,n-l-1 \end{cases}$$

Then as in case 1 we can show that f is a cordial labeling of $P(n,k)$

Case 4: $n = 4m$ and $k+1 < m$. Let $m-(k+1) = l$. Then the following binary labeling f of $P(n,k)$ is cordial

$$f(b_i) = \begin{cases} 1; & i = 0,1,\ldots,2m-1 \\ 0; & i = 2m,2m+1,\ldots,n-1 \end{cases}$$

$$f(a_i) = \begin{cases} 0; & i = 2p \\ 1; & i = 2p+1 \\ 1; & i = 2m+2p \\ 0; & i = 2m+2p+1, \quad p = 0,1,\ldots,l-1 \\ 0; & i = 2l+r, \quad r = 0,1,\ldots,2m-2l-1 \\ 1; & i = 2m+2l+t, \quad t = 0,1,\ldots,n-2m-2l-1 \end{cases}$$

Case 5: $n = 4m+2$. In this case $G = P(n,k)$ is a regular graph of degree 3 with $|V(G)| = 8m+4 \equiv 0(mod4)$ and $|E(G)| = 12m+6 \equiv 2(mod4)$. If G is cordial, then the graph G^* obtained by joining one new vertex to every vertex of G would be cordial. But science the degree of every vertex in G^* is even. G^* is Eulerian and since $|E(G^*)| \equiv 2(mod4)$, it follows from the Theorem 3.5.5 that G^* cannot be cordial. Hence $G = P(n,k)$, with $n = 4m+2$ is not cordial. This completes the proof. □

3.5.1 Some Other Known Results

- Lee and Liu[33], Du[16] proved that complete n-partite graph is cordial if and only if at most three of its partite sets have odd cardinality.
- Seoud and Maqsoud[39] proved that if G is a graph with p vertices and q edges and every vertex has odd degree then G is not cordial when $p+q \equiv 2(mod4)$.
- Andar et al. in [2],[3],[4] and [5] proved that
  - ◇ Multiple shells are cordial.
  - ◇ t-ply graph $P_t(u,v)$ is cordial except when it is Eulerian and the number of edges is congruent to $2(mod4)$.
  - ◇ Helms, closed helms and generalized helms are cordial.
- In [5], Andar et al. showed that a cordial labeling g of a graph G can be extended to a cordial labeling of the graph obtained from G by attaching $2m$ pendant edges at each vertex of G. They also proved that a cordial labeling g of a graph G with p vertices can be extended to a cordial labeling of the graph obtained from G by attaching $2m+1$ pendant edges at each vertex of G if and only if G does not satisfy either of the following conditions:
  (1) G has an even number of edges and $p \equiv 2(mod4)$.
  (2) G has an odd number of edges and either $p \equiv 1(mod4)$ with $e_g(1) = e_g(0) + i(G)$ or $p \equiv 3(mod4)$ with $e_g(0) = e_g(1) + i(G)$, where $i(G) = min\{|e_g(0) - e_g(1)|\}$

In the succeeding sections we will report the results investigated by us.

3.6 Cordial Labeling of Some Star Related Graphs

Definition 3.6.1. Consider two stars $K_{1,n}^{(1)}$ and $K_{1,n}^{(2)}$ then $G = < K_{1,n}^{(1)} : K_{1,n}^{(2)} >$ is the graph obtained by joining apex vertices of stars to a new vertex x.

Here $|V(G)| = 2n+3$ and $|E(G)| = 2n+2$.

Definition 3.6.2. Consider k copies of stars namely $K_{1,n}^{(1)}, K_{1,n}^{(2)}, K_{1,n}^{(3)}, \ldots K_{1,n}^{(k)}$. Then the $G =< K_{1,n}^{(1)} : K_{1,n}^{(2)} : K_{1,n}^{(3)} : \ldots : K_{1,n}^{(k)} >$ is the graph obtained by joining apex vertices of each $K_{1,n}^{(p-1)}$ and $K_{1,n}^{(p)}$ to a new vertex x_{p-1} where $2 \leq p \leq k$.

Here $|V(G)| = k(n+2) - 1$ and $|E(G)| = k(n+2) - 2$.

Definition 3.6.3. Consider two stars $K_{1,n}^{(1)}$ and $K_{1,n}^{(2)}$ then $G =< K_{1,n}^{(1)} \blacktriangle K_{1,n}^{(2)} >$ is the graph obtained by joining apex vertices of stars by an edge as well as to a new vertex x.

Here $|V(G)| = 2n+3$ and $|E(G)| = 2n+3$.

Definition 3.6.4. Consider k copies of stars namely $K_{1,n}^{(1)}, K_{1,n}^{(2)}, K_{1,n}^{(3)}, \ldots\ldots K_{1,n}^{(k)}$. Then the $G =< K_{1,n}^{(1)} \blacktriangle K_{1,n}^{(2)} \blacktriangle K_{1,n}^{(3)} \blacktriangle \ldots \blacktriangle K_{1,n}^{(k)} >$ is the graph obtained by joining apex vertices of each $K_{1,n}^{(p-1)}$ and $K_{1,n}^{(p)}$ by an edge as well as to a new vertex x_{p-1} where $2 \leq p \leq k$.

Here $|V(G)| = k(n+2) - 1$ and $|E(G)| = k(n+3) - 3$.

Theorem 3.6.5. Graph $< K_{1,n}^{(1)} : K_{1,n}^{(2)} >$ is cordial.

Proof. Let $v_1^{(1)}, v_2^{(1)}, v_3^{(1)}, \ldots v_n^{(1)}$ be the pendant vertices $K_{1,n}^{(1)}$ and $v_1^{(2)}, v_2^{(2)}, v_3^{(2)}, \ldots v_n^{(2)}$ be the pendant vertices $K_{1,n}^{(2)}$. Let c_1 and c_2 be the apex vertices of $K_{1,n}^{(1)}$ and $K_{1,n}^{(2)}$ respectively and they are adjacent to a new common vertex x. Let $G =< K_{1,n}^{(1)} : K_{1,n}^{(2)} >$. We define binary vertex labeling $f : V(G) \to \{0,1\}$ as follows.

For any $n \in N$ and $i = 1,2,\ldots n$ where N is set of natural numbers.

In this case we define labeling as follows

Case 1: If n even

$$\left.\begin{array}{rcl} f(v_i^{(j)}) & = & 0; \text{if } 1 \leq i \leq \frac{n}{2} \\ & = & 1; \frac{n+2}{2} \leq i \leq n \end{array}\right\} \text{For } j = 1,2$$

$$f(c_1) = 0;$$
$$f(c_2) = 1;$$
$$f(x) = 0;$$

Case 2: If n odd

$$\left.\begin{aligned} f(v_i^{(j)}) &= 0; \text{if } 1 \le i \le \tfrac{n-1}{2} \\ &= 1; \tfrac{n+1}{2} \le i \le n \end{aligned}\right\} \text{For } j = 1,2$$

$$f(c_1) = f(c_2) = f(x) = 0;$$

The labeling pattern defined above covers all possible arrangement of vertices. The graph G satisfies the conditions $|v_f(0) - v_f(1)| \le 1$ and $|e_f(0) - e_f(1)| \le 1$ as shown in TABLE 3.1. i.e. G admits cordial labeling. □

n	**Vertex Condition**	**Edge Condition**
all	$v_f(0) = v_f(1) + 1 = n+2$	$e_f(0) = e_f(1) = n+1$

TABLE 3.1

Illustration 3.6.6. Consider $G =< K_{1,7}^{(1)} : K_{1,7}^{(2)} >$. Here $n = 7$. The cordial labeling is as shown in FIGURE 3.4.

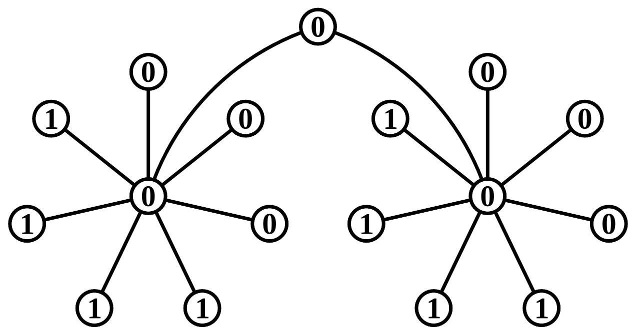

FIGURE 3.4

Above result can be extended for $k-$copies of $K_{1,n}$ as follows.

Theorem 3.6.7. Graph $< K_{1,n}^{(1)} : K_{1,n}^{(2)} : K_{1,n}^{(3)} : \ldots : K_{1,n}^{(k)} >$ is cordial.

Proof. Let $K_{1,n}^{(j)}$ be k copies of star $K_{1,n}$, $v_i^{(j)}$ be the pendant vertices of $K_{1,n}^{(j)}$ and c_j be the apex vertex of $K_{1,n}^{(j)}$ (here $i = 1,2,\ldots n$ and $j = 1,2,\ldots\ k$).Let $x_1, x_2 \ldots\ x_{k-1}$ be the vertices such that c_{p-1} and c_p are adjacent to x_{p-1} where $2 \le p \le k$. Consider $G =< K_{1,n}^{(1)} : K_{1,n}^{(2)} : K_{1,n}^{(3)} : \ldots : K_{1,n}^{(k)} >$. To define binary vertex labeling $f : V(G) \to \{0,1\}$ we consider following cases.

Case 1: $n \in N$ even and k where $k \in N - \{1,2\}$.

In this case we define labeling function f as

For $j = 1, 2, \ldots, k$

$$\begin{aligned} f(v_i^{(j)}) &= 0; \text{ if } 1 \leq i \leq \tfrac{n}{2} \\ &= 1; \text{ if } \tfrac{n+2}{2} \leq i \leq n \\ f(c_j) &= 1; \text{ if } j \text{ even} \\ &= 0; \text{ if } j \text{ odd} \\ f(x_j) &= 1; \text{ if } j \text{ even}, \ j \neq k \\ &= 0; \text{ if } j \text{ odd}, \ j \neq k \end{aligned}$$

Case 2: $n \in N - \{1,2\}$ odd and k where $k \in N - \{1,2\}$.

In this case we define labeling function f as

For $j = 1, 2, \ldots, k$

$$\begin{aligned} f(v_i^{(j)}) &= 0; \text{ if } 1 \leq i \leq \tfrac{n-1}{2} \\ &= 1; \text{ if } \tfrac{n+1}{2} \leq i \leq n \\ f(c_j) &= 1; \text{ if } j \text{ even} \\ &= 0; \text{ if } j \text{ odd} \\ f(x_j) &= 0; \ j \neq k \end{aligned}$$

The labeling pattern defined above covers all the possibilities. In each case, the graph G under consideration satisfies the conditions $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ as shown in TABLE 3.2(where $n = 2a + b$, $k = 2c + d$ and $a \in N \cup \{0\}, c \in N$). i.e. G admits cordial labeling. □

b	***d***	**Vertex Condition**	**Edge Condition**
0	0,1	$v_f(0) = v_f(1) + 1 = \frac{k(n+2)}{2}$	$e_f(0) = e_f(1) = \frac{k(n+2)}{2} - 1$
1	0	$v_f(0) + 1 = v_f(1) = \frac{k(n+2)}{2}$	$e_f(0) = e_f(1) = \frac{k(n+2)}{2} - 1$
	1	$v_f(0) = v_f(1) = \frac{k(n+2)-1}{2}$	$e_f(0) + 1 = e_f(1) = \frac{k(n+2)-1}{2}$

TABLE 3.2

Illustration 3.6.8. Consider $G = < K_{1,6}^{(1)} : K_{1,6}^{(2)} : K_{1,6}^{(3)} >$. Here $n = 6$ and $k = 3$. The cordial labeling is as shown in FIGURE 3.5.

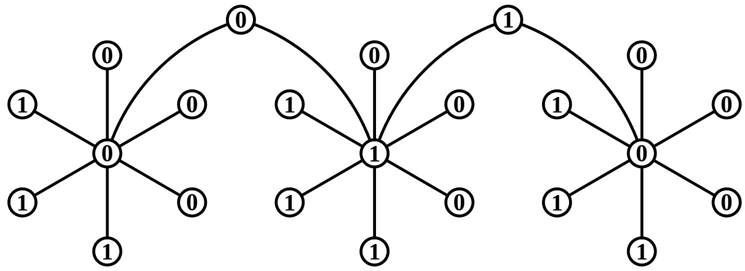

FIGURE 3.5

Theorem 3.6.9. Graph $< K_{1,n}^{(1)} \blacktriangle K_{1,n}^{(2)} >$ is cordial.

Proof. Let $v_1^{(1)}, v_2^{(1)}, v_3^{(1)}, \ldots v_n^{(1)}$ be the pendant vertices $K_{1,n}^{(1)}$ and $v_1^{(2)}, v_2^{(2)}, v_3^{(2)}, \ldots v_n^{(2)}$ be the pendant vertices $K_{1,n}^{(2)}$. Let c_1 and c_2 be the apex vertices of $K_{1,n}^{(1)}$ and $K_{1,n}^{(2)}$ respectively and they are adjacent to a new common vertex x. Let $G =< K_{1,n}^{(1)} \blacktriangle K_{1,n}^{(2)} >$. We define binary vertex labeling $f : V(G) \to \{0,1\}$ as follows.

For any $n \in N$ and $i = 1,2,\ldots n$, we define labeling as follows

Case 1: If n even

$$\left.\begin{array}{lcl} f(v_i^{(j)}) & = & 0;\ \text{if } 1 \leq i \leq \frac{n}{2} \\ & = & 1;\ \text{if } \frac{n+2}{2} \leq i \leq n \\ f(c_j) & = & 1; \end{array}\right\} \text{For } j = 1,2$$

$f(x) = 0;$

Case 2: If n odd

$$\left.\begin{array}{lcl} f(v_i^{(j)}) & = & 0;\ \text{if } 1 \leq i \leq \frac{n-1}{2} \\ & = & 1;\ \text{if } \frac{n+1}{2} \leq i \leq n \\ f(c_j) & = & 0; \end{array}\right\} \text{For } j = 1,2$$

$f(x) = 0;$

The labeling pattern defined above covers all possible arrangement of vertices. The graph G satisfies the conditions $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ as shown in TABLE 3.3. i.e. G admits cordial labeling. □

n	**Vertex Condition**	**Edge Condition**
even	$v_f(0)+1 = v_f(1) = n+2$	$e_f(0)+1 = e_f(1) = n+2$
odd	$v_f(0) = v_f(1)+1 = n+2$	$e_f(0) = e_f(1)+1 = n+2$

TABLE 3.3

Illustration 3.6.10. Consider $G =< K_{1,8}^{(1)} \blacktriangle K_{1,8}^{(2)} >$. Here $n = 8$. The cordial labeling is as shown in FIGURE 3.6.

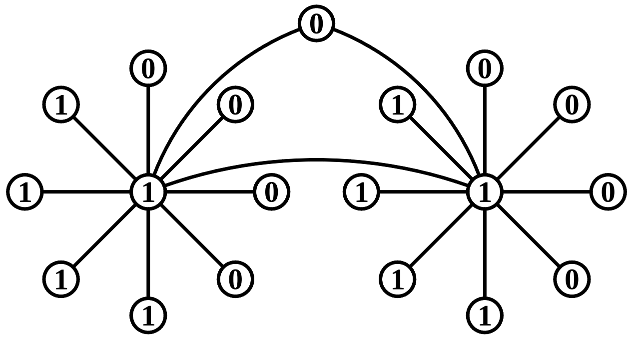

FIGURE 3.6

Theorem 3.6.11. Graph $< K_{1,n}^{(1)} \blacktriangle K_{1,n}^{(2)} \blacktriangle K_{1,n}^{(3)} \blacktriangle \ldots \blacktriangle K_{1,n}^{(k)} >$ is cordial.

Proof. Let $K_{1,n}^{(j)}$ be k copies of star $K_{1,n}$, $v_i^{(j)}$ be the pendant vertices of $K_{1,n}^{(j)}$ and c_j be the apex vertex of $K_{1,n}^{(j)}$ (here $i = 1,2,\ldots n$ and $j = 1,2,\ldots\ k$). Let $x_1, x_2 \ldots\ x_{k-1}$ be the vertices such that c_{p-1} and c_p are adjacent to x_{p-1} where $2 \leq p \leq k$. Consider $G =< K_{1,n}^{(1)} \blacktriangle K_{1,n}^{(2)} \blacktriangle K_{1,n}^{(3)} \blacktriangle \ldots \blacktriangle K_{1,n}^{(k)} >$. To define binary vertex labeling $f : V(G) \to \{0,1\}$ we consider following cases.

Case 1: $n \in N$ even and k where $k \in N - \{1,2\}$

In this case we define labeling function f as

For $j = 1,2,\ldots,k$

$$\left.\begin{array}{lcl} f(v_i^{(j)}) & = & 0;\ \text{if } 1 \le i \le \frac{n}{2} \\ & = & 1;\ \text{if } \frac{n+2}{2} \le i \le n \\ f(c_j) & = & 1; \end{array}\right\} \text{if } j \text{ odd}$$

$$\left.\begin{array}{lcl} f(v_i^{(j)}) & = & 0;\ \text{if } 1 \le i \le \frac{n+2}{2} \\ & = & 1;\ \text{if } \frac{n+4}{2} \le i \le n \\ f(c_j) & = & 0; \end{array}\right\} \text{if } j \text{ even}$$

$f(x_j) = 1$; for all j, $j \neq k$

Case 2: $n \in N - \{1,2\}$ odd and k where $k \in N - \{1,2\}$

In this case we define labeling function f as

For $j = 1,2,\ldots k$

$$\begin{array}{rl} f(v_i^{(j)}) & = 0;\ \text{if } 1 \le i \le \frac{n-1}{2} \\ & = 1;\ \text{if } \frac{n+1}{2} \le i \le n \\ f(c_j) & = 0; \\ f(x_j) & = 1;\ \text{if } j \text{ even} \\ & = 0;\ \text{if } j \text{ odd},\ j \neq k \end{array}$$

The labeling pattern defined above covers all the possibilities. In each case the graph G under consideration satisfies the conditions $|v_f(0) - v_f(1)| \le 1$ and $|e_f(0) - e_f(1)| \le 1$ as shown in TABLE 3.4(where $n = 2a+b$, $k = 2c+d$ and $a \in N \cup \{0\}, c \in N$). i.e. G admits cordial labeling. □

b	d	Vertex Condition	Edge Condition
0	0	$v_f(0) = v_f(1) + 1 = \frac{k(n+2)}{2}$	$e_f(0) = e_f(1) + 1 = \frac{k(n+3)-2}{2}$
	1	$v_f(0) + 1 = v_f(1) = \frac{k(n+2)}{2}$	$e_f(0) = e_f(1) = \frac{k(n+3)-3}{2}$
1	0	$v_f(0) = v_f(1) + 1 = \frac{k(n+2)}{2}$	$e_f(0) = e_f(1) + 1 = \frac{k(n+3)-2}{2}$
	1	$v_f(0) = v_f(1) = \frac{k(n+2)-1}{2}$	$e_f(0) + 1 = e_f(1) = \frac{k(n+3)-2}{2}$

TABLE 3.4

Illustration 3.6.12. Consider $G =< K_{1,6}^{(1)} \blacktriangle K_{1,6}^{(2)} \blacktriangle K_{1,6}^{(3)} >$. Here $n = 6$ and $k = 3$. The cordial labeling is as shown in FIGURE 3.7.

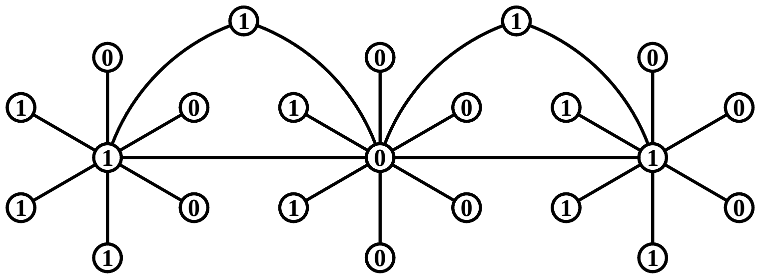

FIGURE 3.7

3.7 Cordial Labeling of Some Shell Related Graphs

Definition 3.7.1. Consider two shells $S_n^{(1)}$ and $S_n^{(2)}$ then graph $G =< S_n^{(1)} : S_n^{(2)} >$ obtained by joining apex vertices of shells to a new vertex x.

Here $|V(G)| = 2n + 1$ and $|E(G)| = 4n - 4$.

Definition 3.7.2. Consider k copies of shells namely $S_n^{(1)}, S_n^{(2)}, S_n^{(3)}, \ldots, S_n^{(k)}$. Then the graph $G =< S_n^{(1)} : S_n^{(2)} : S_n^{(3)} : \ldots : S_n^{(k)} >$ obtained by joining apex vertex of each $S_n^{(p)}$ and apex of $S_n^{(p-1)}$ to a new vertex $x_p(where 2 \leq p \leq k)$.

Here $|V(G)| = k(n + 1) - 1$ and $|E(G)| = k(2n - 1) - 2$.

Theorem 3.7.3. Graph $< S_n^{(1)} : S_n^{(2)} >$ is cordial.

Proof. Let $v_1^{(1)}, v_2^{(1)}, v_3^{(1)}, \ldots, v_n^{(1)}$ be the vertices $S_n^{(1)}$ and $v_1^{(2)}, v_2^{(2)}, v_3^{(2)}, \ldots, v_n^{(2)}$ be the vertices $S_n^{(2)}$. Let $v_1^{(1)}$ and $v_1^{(2)}$ be the apex vertices of $S_n^{(1)}$ and $S_n^{(2)}$ respectively. Let $G =< S_n^{(1)} : S_n^{(2)} >$. We define binary vertex labeling $f : V(G) \to \{0,1\}$ as follows.

$$\left.\begin{array}{rcl} f(v_i^{(j)}) & = & 0;\ \text{if } i \equiv 2,3(mod 4) \\ f(v_i^{(j)}) & = & 1;\ \text{if } i \equiv 0,1(mod 4) \end{array}\right\} \text{For } j = 1,2$$

$$f(x) = 0;\ \text{if } n \equiv 1(mod 4)$$
$$f(x) = 1;\ \text{if } n \equiv 0,2,3(mod 4)$$

The labeling pattern defined above covers all possible arrangement of vertices. The graph G satisfies the conditions $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ as shown in TABLE 3.5(where $n = 4a + b$ and $a \in N \cup \{0\}$). i.e. G admits cordial labeling. □

a	**Vertex Condition**	**Edge Condition**
0,1,2	$v_f(0) + 1 = v_f(1) = n + 1$	$e_f(0) = e_f(1) = 2n - 2$
3	$v_f(0) = v_f(1) + 1 = n + 1$	$e_f(0) = e_f(1) = 2n - 2$

TABLE 3.5

Illustration 3.7.4. Consider a graph $G =< S_7^{(1)} : S_7^{(2)} >$. Here $n = 7$. The cordial labeling is as shown in FIGURE 3.8 .

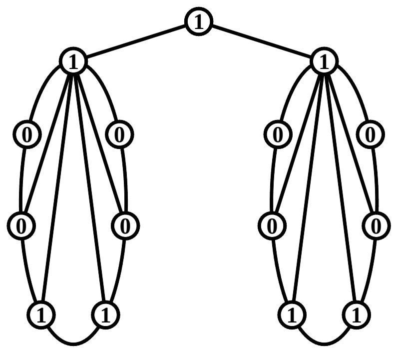

FIGURE 3.8

Theorem 3.7.5. Graph $< S_n^{(1)} : S_n^{(2)} : S_n^{(3)} : \ldots : S_n^{(k)} >$ is cordial.

Proof. Let $S_n^{(j)}$ be the shells. Let $v_i^{(j)}$ be the vertices $S_n^{(j)}$ and $v_1^{(j)}$ be the apex vertices of $S_n^{(j)}$. Let $x_j (j \neq k)$ be the new vertices. Let $G =< S_n^{(1)} : S_n^{(2)} : S_n^{(3)} : \ldots : S_n^{(k)} >$. We define binary vertex labeling $f : V(G) \to \{0,1\}$ as follows.

$$\left.\begin{aligned} f(v_i^{(j)}) &= 0;\ \text{if } i \equiv 2,3(mod4) \\ f(v_i^{(j)}) &= 1;\ \text{if } i \equiv 0,1(mod4) \end{aligned}\right\} \text{For } j \equiv 1,2(mod4)$$

$$\left.\begin{aligned} f(v_i^{(j)}) &= 0;\ \text{if } i \equiv 0,1(mod4) \\ f(v_i^{(j)}) &= 1;\ \text{if } i \equiv 2,3(mod4) \end{aligned}\right\} \text{For } j \equiv 0,3(mod4)$$

$$\left.\begin{aligned} f(x_j) &= 0;\ \text{if } j \equiv 2,3(mod4) \\ f(x_j) &= 1;\ \text{if } j \equiv 0,1(mod4)),\ j \neq k \end{aligned}\right\} \text{For } n \equiv 0,2,3(mod4)$$

$$\left.\begin{aligned} f(x_j) &= 0;\ \text{if } j \equiv 1,2(mod4) \\ f(x_j) &= 1;\ \text{if } j \equiv 0,3(mod4)),\ j \neq k \end{aligned}\right\} \text{For } n \equiv 1(mod4)$$

The labeling pattern defined above covers all possible arrangement of vertices. The graph G satisfies the conditions $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ as shown in TABLE 3.6(where $n = 4a + b$, $k = 4c + d$ and $a, c \in N \cup \{0\}$). i.e. G admits cordial labeling. □

b	d	Vertex Condition	Edge Condition
0,2	0	$v_f(0) = v_f(1)+1 = \frac{k(n+1)}{2}$	$e_f(0) = e_f(1) = \frac{k(2n-1)-2}{2}$
	1,3	$v_f(0) = v_f(1) = \frac{k(n+1)-1}{2}$	$e_f(0)+1 = e_f(1) = \frac{k(2n-1)-1}{2}$
	2	$v_f(0)+1 = v_f(1) = \frac{k(n+1)}{2}$	$e_f(0) = e_f(1) = \frac{k(2n-1)-2}{2}$
1	0	$v_f(0) = v_f(1)+1 = \frac{k(n+1)}{2}$	$e_f(0) = e_f(1) = \frac{k(2n-1)-2}{2}$
	1	$v_f(0)+1 = v_f(1) = \frac{k(n+1)}{2}$	$e_f(0) = e_f(1)+1 = \frac{k(2n-1)-1}{2}$
	2	$v_f(0)+1 = v_f(1) = \frac{k(n+1)}{2}$	$e_f(0) = e_f(1) = \frac{k(2n-1)-2}{2}$
	3	$v_f(0) = v_f(1)+1 = \frac{k(n+1)}{2}$	$e_f(0) = e_f(1)+1 = \frac{k(2n-1)-1}{2}$
3	0,2	$v_f(0) = v_f(1)+1 = \frac{k(n+1)}{2}$	$e_f(0) = e_f(1) = \frac{k(2n-1)-2}{2}$
	1,3	$v_f(0) = v_f(1)+1 = \frac{k(n+1)}{2}$	$e_f(0)+1 = e_f(1) = \frac{k(2n-1)-1}{2}$

TABLE 3.6

Illustration 3.7.6. Consider a graph $G = < S_5^{(1)} : S_5^{(3)} : S_5^{(3)} >$. Here $n = 5$. The cordial labeling is as shown in FIGURE 3.9.

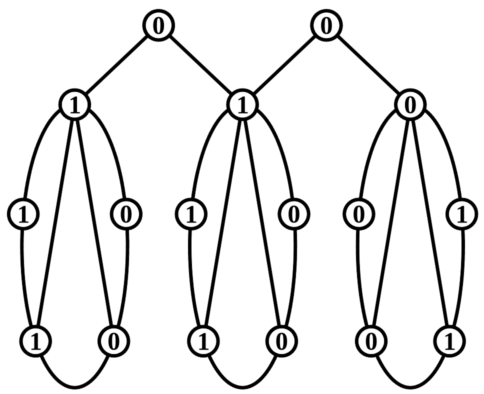

FIGURE 3.9

3.8 Cordial Labeling of Some Wheel Related Graphs

Definition 3.8.1. Consider two wheels $W_n^{(1)}$ and $W_n^{(2)}$ then $G = < W_n^{(1)} : W_n^{(2)} >$ is the graph obtained by joining apex vertices of wheels to a new vertex x.

Here $|V(G)| = 2n+3$ and $|E(G)| = 4n+2$.

Definition 3.8.2. Consider k copies of wheels namely $W_n^{(1)}, W_n^{(2)}, W_n^{(3)}, \dots W_n^{(k)}$. Then the $G =< W_n^{(1)} : W_n^{(2)} : W_n^{(3)} : \dots : W_n^{(k)} >$ is the graph obtained by joining apex vertices of each $W_n^{(p-1)}$ and $W_n^{(p)}$ to a new vertex x_{p-1} where $2 \leq p \leq k$.

Here $|V(G)| = k(n+2) - 1$ and $|E(G)| = 2k(n+1) - 2$.

Theorem 3.8.3. Graph $< W_n^{(1)} : W_n^{(2)} >$ is cordial.

Proof. Let $v_1^{(1)}, v_2^{(1)}, v_3^{(1)}, \dots v_n^{(1)}$ be the rim vertices $W_n^{(1)}$ and $v_1^{(2)}, v_2^{(2)}, v_3^{(2)}, \dots v_n^{(2)}$ be the rim vertices $W_n^{(2)}$. Let c_1 and c_2 be the apex vertices of $W_n^{(1)}$ and $W_n^{(2)}$ respectively and they are adjacent to a new common vertex x. Let $G =< W_n^{(1)} : W_n^{(2)} >$. We define binary vertex labeling $f : V(G) \to \{0, 1\}$ as follows.

For any $n \in N - \{1, 2\}$ and $i = 1, 2, \dots n$ where N is set of natural numbers.
In this case we define labeling as follows

$f(v_i^{(1)}) = 1;$

$f(c_1) = 0;$

$f(v_i^{(2)}) = 0;$

$f(c_2) = 1;$

$f(x) = 1;$

Thus rim vertices of $W_n^{(1)}$ and $W_n^{(2)}$ are labeled with the sequences $1, 1, 1, \dots, 1$ and $0, 0, \dots, 0$ respectively. The common vertex x is labeled with 1 and apex vertices with 0 and 1 respectively.

The labeling pattern defined above covers all possible arrangement of vertices. The graph G satisfies the conditions $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ as shown in TABLE 3.7(where $n \in N - \{1, 2\}$). i.e. G admits cordial labeling. □

n	**Vertex Condition**	**Edge Condition**
all	$v_f(0) + 1 = v_f(1) = n + 2$	$e_f(0) = e_f(1) = 2n + 1$

TABLE 3.7

Illustration 3.8.4. Consider $G =< W_6^{(1)} : W_6^{(2)} >$. Here $n = 6$. The cordial labeling is as shown in FIGURE 3.10.

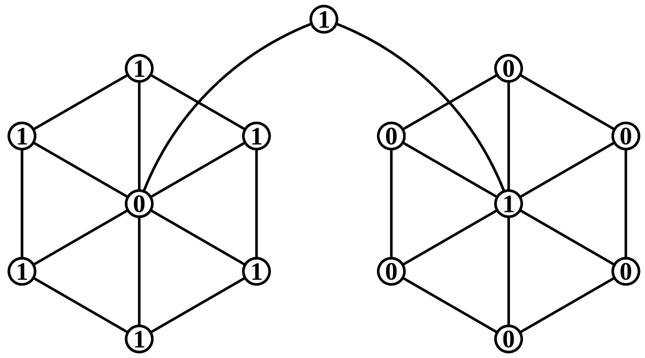

FIGURE 3.10

Theorem 3.8.5. Graph $< W_n^{(1)} : W_n^{(2)} : W_n^{(3)} : \ldots : W_n^{(k)} >$ is cordial.

Proof. Let $W_n^{(j)}$ be k copies of wheel W_n, $v_i^{(j)}$ be the rim vertices of $W_n^{(j)}$ and c_j be the apex vertex of $W_n^{(j)}$ (here $i = 1,2,\ldots n$ and $j = 1,2,\ldots\ k$).Let $x_1,x_2 \ldots\ x_{k-1}$ be the vertices such that c_{p-1} and c_p are adjacent to x_{p-1} where $2 \leq p \leq k$. Consider $G =< W_n^{(1)} : W_n^{(2)} : W_n^{(3)} : \ldots : W_n^{(k)} >$. To define binary vertex labeling $f : V(G) \to \{0,1\}$ we consider following cases.

Case 1: $n \in N - \{1,2\}$ and even k where $k \in N - \{1,2\}$

In this case we define labeling function f as

For $i = 1,2,\ldots n$ and $j = 1,2,\ldots k$

$$
\begin{aligned}
f(v_i^{(j)}) &= 0;\ \text{if } j \text{ even}\\
&= 1;\ \text{if } j \text{ odd}\\
f(c_j) &= 1;\ \text{if } j \text{ even}\\
&= 0;\ \text{if } j \text{ odd}\\
f(x_j) &= 1;\ \text{if } j \text{ even},\ j \neq k\\
&= 0;\ \text{if } j \text{ odd},\ j \neq k
\end{aligned}
$$

Case 2: $n \in N - \{1,2\}$ and odd k where $k \in N - \{1,2\}$

In this case we define labeling function f for first $k-1$ wheels as

For $i = 1,2,\ldots n$ and $j = 1,2,\ldots k-1$

$$
\begin{aligned}
f(v_i^{(j)}) &= 0;\ \text{if } j \text{ even}\\
&= 1;\ \text{if } j \text{ odd}
\end{aligned}
$$

$f(c_j) = 1$; if j even

$= 0$; if j odd

$f(x_j) = 1$; if j even

$= 0$; if j odd

To define labeling function f for k^{th} copy of wheel we consider following subcases

Subcase 1: If $n \equiv 3(mod4)$

$$\left.\begin{array}{rcl} f(v_i^{(k)}) & = & 0; \text{if } i \equiv 0,1(mod4) \\ & = & 1; \text{ if } i \equiv 2,3(mod4) \end{array}\right\} \text{For } 1 \leq i \leq n-1$$

$f(v_n^{(k)}) = 0;$

$f(c_k) = 1;$

Subcase 2: If $n \equiv 0,2(mod4)$

$$\left.\begin{array}{rcl} f(v_i^{(k)}) & = & 0; \text{ if } i \equiv 0,1(mod4) \\ & = & 1; \text{ if } i \equiv 2,3(mod4) \end{array}\right\} \text{For } 1 \leq i \leq n$$

$f(c_k) = 0; n \equiv 0(mod4)$

$f(c_k) = 1; n \equiv 2(mod4)$

Subcase 3: If $n \equiv 1(mod4)$

$$\left.\begin{array}{rcl} f(v_i^{(k)}) & = & 0; \text{ if } i \equiv 0,3(mod4) \\ & = & 1; \text{ if } i \equiv 1,2(mod4) \end{array}\right\} \text{For } 1 \leq i \leq n$$

$f(c_k) = 0;$

The labeling pattern defined above exhaust all the possibilities and in each one the graph G under consideration satisfies the conditions $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ as shown in TABLE 3.8(where $n = 4a+b$ and $a \in N \cup \{0\}$). i.e. G admits cordial labeling. □

k	***b***	**Vertex Condition**	**Edge Condition**
even	0,1,2,3	$v_f(0) = v_f(1) + 1 = \frac{k(n+2)}{2}$	$e_f(0) = e_f(1) = k(n+1) - 1$
odd	0	$v_f(0) = v_f(1) + 1 = \frac{k(n+2)}{2}$	$e_f(0) = e_f(1) = k(n+1) - 1$
	1,3	$v_f(0) = v_f(1) = \frac{k(n+2)-1}{2}$	$e_f(0) = e_f(1) = k(n+1) - 1$
	2	$v_f(0) + 1 = v_f(1) = \frac{k(n+2)}{2}$	$e_f(0) = e_f(1) = k(n+1) - 1$

TABLE 3.8

Illustration 3.8.6.

Example 1: Consider $G =< W_7^{(1)} : W_7^{(2)} : W_7^{(3)} : W_7^{(4)} >$. Here $n = 7$ and $k = 4$ i.e k is even. The cordial labeling is as shown in FIGURE 3.11.

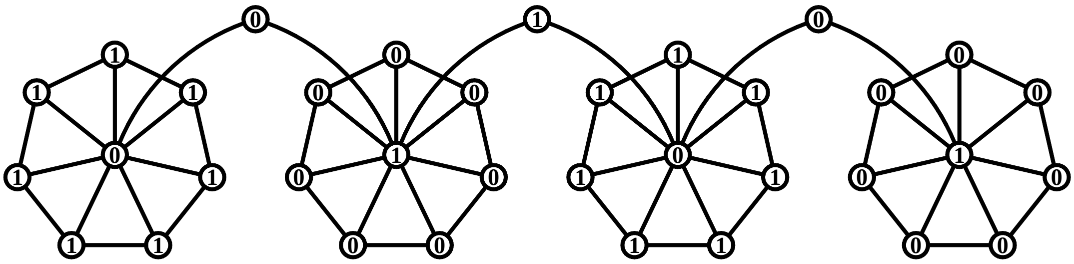

FIGURE 3.11

Example 2: Consider $G =< W_5^{(1)} : W_5^{(2)} : W_5^{(3)} >$. Here $n = 5$ i.e $n \equiv 1(mod4)$ and $k = 3$ i.e k is odd. The cordial labeling is as shown in FIGURE 3.12.

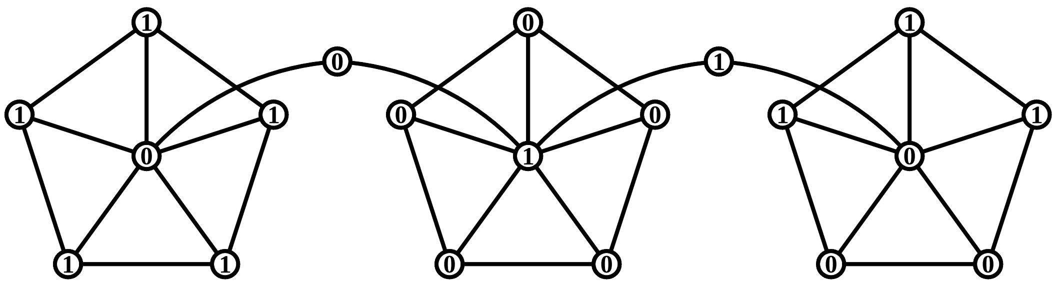

FIGURE 3.12

3.9 Some Graph Operations and Cordial Labeling

Definition 3.9.1. Let u and v be two distinct vertices of a graph G. A new graph G_1 constructed by *fusing* (or *identifying*) two vertices u and v by a single new vertex x such that every edge which was incident with either u or v in G is now incident with x.

Definition 3.9.2. *Duplication* of a vertex v_k of graph G produces a new graph G_1 by adding a vertex v'_k with $N(v'_k) = N(v_k)$.

In other words a vertex v'_k is said to be duplication of v_k if all the vertices which are adjacent to v_k are now adjacent to v'_k also.

Definition 3.9.3. *Duplication* of an edge $e = uv$ of graph G produces a new graph G_1 by adding an edge $e^{'} = u^{'}v^{'}$ such that $N(u) = N(u^{'})$ and $N(v) = N(v^{'})$.

In other words an edge $e^{'}$ is said to be duplication of edge e if all the edges which are incident to e are now incident to $e^{'}$ also.

Theorem 3.9.4. Fusion of two vertices v_i and v_j with $d(v_i, v_j) \geq 3$ of cycle C_n is cordial except $n \equiv 2(mod 4)$.

Proof. Consider cycle C_n with n vertices namely $v_1, v_2, \ldots v_n$. Let the vertex v_1 be fused with v_k and graph $G = C_n - \{v_k\}$. To define binary vertex labeling $f : V(G) \to \{0,1\}$ we consider the following cases.

Case 1: $n \equiv 0,1,3(mod 4)$ and $k \equiv 0,1,2(mod 4)$

In this case we define labeling as follows

$$\left.\begin{aligned} f(v_i) &= 0;\ \text{if } i \equiv 0,1(mod 4) \\ &= 1;\ \text{if } i \equiv 2,3(mod 4) \end{aligned}\right\} \text{For } 1 \leq i < k$$

$$\left.\begin{aligned} f(v_i) &= 0;\ \text{if } i \equiv 1,2(mod 4) \\ &= 1;\ \text{if } i \equiv 0,3(mod 4) \end{aligned}\right\} \text{For } k < i \leq n$$

Case 2: $n \equiv 0(mod4)$ and $k \equiv 3(mod4)$

In this case we define labeling as follows

$$\left.\begin{array}{rcl} f(v_i) & = & 0;\ \text{if } i \equiv 0,3(mod4) \\ & = & 1;\ \text{if } i \equiv 1,2(mod4) \end{array}\right\} \text{For } 1 \leq i < k$$

$$\left.\begin{array}{rcl} f(v_i) & = & 0;\ \text{if } i \equiv 0,1(mod4) \\ & = & 1;\ \text{if } i \equiv 2,3(mod4) \end{array}\right\} \text{For } k < i \leq n$$

Case 3: $n \equiv 1(mod4)$and $k \equiv 3(mod4)$

In this case we define labeling as follows

$$\left.\begin{array}{rcl} f(v_i) & = & 0;\ \text{if } i \equiv 1,2(mod4) \\ & = & 1;\ \text{if } i \equiv 0,3(mod4) \end{array}\right\} \text{For } 1 \leq i < k$$

$$\left.\begin{array}{rcl} f(v_i) & = & 0;\ \text{if } i \equiv 2,3(mod4) \\ & = & 1;\ \text{if } i \equiv 0,1(mod4) \end{array}\right\} \text{For } k < i \leq n$$

Case 4: $n \equiv 2(mod4)$

The graph resulted due to fusion of two vertices is Eulerian which will have number of edges congruent to $2(mod4)$. As we mentioned earlier (Theorem 3.5.5) an Eulerian graph with number of edges congruent to $2(mod4)$ is not cordial.

Case 5: $n \equiv 3(mod4)$and $k \equiv 3(mod4)$

In this case we define labeling as follows

$f(v_1) = 0;$

$$\left.\begin{array}{rcl} f(v_i) & = & 0;\ \text{if } i \equiv 0,3(mod4) \\ & = & 1;\ \text{if } i \equiv 1,2(mod4) \end{array}\right\} \text{For } 2 \leq i < k$$

$$\left.\begin{array}{rcl} f(v_i) & = & 0;\ \text{if } i \equiv 0,1mod4) \\ & = & 1;\ \text{if } i \equiv 2,3(mod4) \end{array}\right\} \text{For } k < i \leq n$$

The labeling pattern defined above covers all possible arrangement of vertices. In each case, the graph G under consideration satisfies the conditions $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ as shown in TABLE 3.9(where $n = 4a + b$ and $a \in N$). i.e. G admits cordial labeling. $\square$

b	**Vertex Condition**	**Edge Condition**
0	$v_f(0)+1 = v_f(1) = \frac{n}{2}$	$e_f(0) = e_f(1) = \frac{n}{2}$
1	$v_f(0) = v_f(1) = \frac{n-1}{2}$	$e_f(0) = e_f(1)+1 = \frac{n+1}{2}$
3	$v_f(0) = v_f(1) = \frac{n-1}{2}$	$e_f(0)+1 = e_f(1) = \frac{n+1}{2}$

TABLE 3.9

Remark 3.9.5. When $d(v_i, v_j) < 3$ the fusion yields a graph which is not simple and cordiality can not be discussed.

Illustration 3.9.6.

Example 1: Consider a graph obtained by fusing two vertices v_1 and v_6 of cycle C_{12}. Here $n = 12$ i.e. $n \equiv 0(mod4)$ and $k = 6$ i.e $k \equiv 2(mod4)$. The cordial labeling is as shown in FIGURE 3.13.

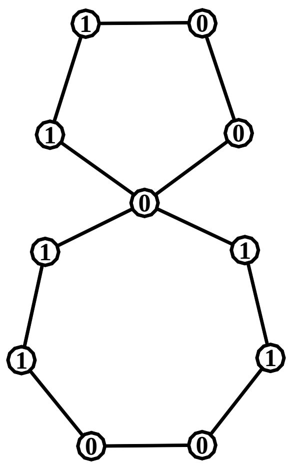

FIGURE 3.13

Example 2: Consider a graph obtained by fusing two vertices v_1 and v_7 of cycle C_{12}. Here $n = 12$ i.e. $n \equiv 0(mod4)$ and $k = 7$ i.e $k \equiv 3(mod4)$. The cordial labeling is as shown in FIGURE 3.14

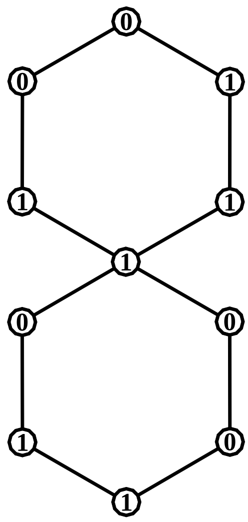

FIGURE 3.14

Example 3: Consider a graph obtained by fusing two vertices v_1 and v_7 of cycle C_{13}. Here $n = 13$ i.e. $n \equiv 1(mod4)$ and $k = 7$ i.e $k \equiv 3(mod4)$. The cordial labeling is as shown in FIGURE 3.15.

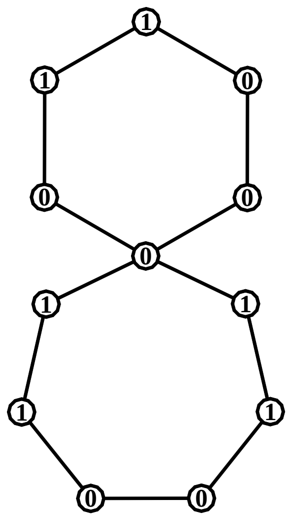

FIGURE 3.15

Example 4: Consider a graph obtained by fusing two vertices v_1 and v_7 of cycle C_{11}. Here $n = 11$ i.e. $n \equiv 3(mod4)$ and $k = 7$ i.e $k \equiv 3(mod4)$. The cordial labeling is shown in FIGURE 3.16.

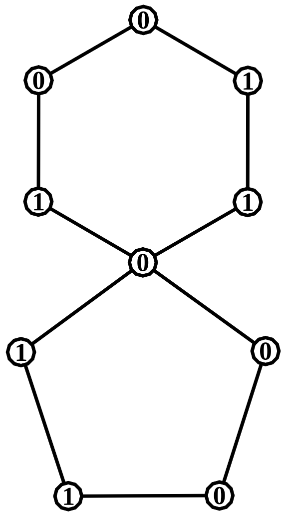

FIGURE 3.16

Theorem 3.9.7. Duplication of arbitrary vertex v_k of cycle C_n produces a cordial graph.

Proof. Let C_n be the cycle with n vertices. Let v_k be the vertex of C_n. Let $v_k^{'}$ be the duplicated vertex of v_k and G be the graph resulted due to duplication. To define binary vertex labeling $f : V(G) \to \{0,1\}$ we consider following cases.

Case 1: $n \equiv 0,3(mod4)$ and $k \in N$, $1 \leq k \leq n$

In this case we define labeling function f as

$$\left.\begin{aligned} f(v_{k+i-1}) &= 0;\text{ if } i \equiv 1,2(mod4) \\ &= 1;\text{ if } i \equiv 0,3(mod4) \end{aligned}\right\} \text{For } 1 \leq i \leq n-k+1$$

$$\left.\begin{aligned} f(v_{k+i-n-1}) &= 0;\text{ if } i \equiv 1,2(mod4) \\ &= 1;\text{ if } i \equiv 0,3(mod4) \end{aligned}\right\} \text{For } n-k+2 \leq i \leq n$$

$$\begin{aligned} f(v_k^{'}) &= 0;\text{ if } i \equiv 1(mod4) \\ &= 1;\text{ if } i \equiv 0(mod4).\ i = n+1 \end{aligned}$$

Case 2: $n \equiv 1,2(mod4)$ and $k \in N$, $1 \leq k \leq n$

In this case we define labeling function f as

$$\left.\begin{array}{rcl} f(v_{k+i-1}) & = & 0;\ \text{if } i \equiv 0,1(mod4) \\ & = & 1;\ \text{if } i \equiv 2,3(mod4) \end{array}\right\} \text{For } 1 \leq i \leq n-k+1$$

$$\left.\begin{array}{rcl} f(v_{k+i-n-1}) & = & 0;\ \text{if } i \equiv 0,1(mod4) \\ & = & 1;\ \text{if } i \equiv 2,3(mod4) \end{array}\right\} \text{For } n-k+2 \leq i \leq n$$

$$f(v_k^{'}) = 1;\ i = n+1$$

The labeling pattern defined above covers all possible arrangement of vertices. In each case, the graph G under consideration satisfies the conditions $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ as shown in TABLE 3.10(where $n = 4a + b$ and $a \in N$). i.e. G admits cordial labeling. □

b	**Vertex Condition**	**Edge Condition**
0	$v_f(0) = v_f(1) + 1 = \frac{n+2}{2}$	$e_f(0) = e_f(1) = \frac{n+2}{2}$
1	$v_f(0) = v_f(1) = \frac{n+1}{2}$	$e_f(0) = e_f(1) + 1 = \frac{n+3}{2}$
2	$v_f(0) + 1 = v_f(1) = \frac{n+2}{2}$	$e_f(0) = e_f(1) = \frac{n+2}{2}$
3	$v_f(0) = v_f(1) = \frac{n+1}{2}$	$e_f(0) + 1 = e_f(1) = \frac{n+3}{2}$

TABLE 3.10

Illustration 3.9.8.

Example 1: Consider a graph obtained by duplicating vertex v_4 of cycle C_7. Here $n = 7$ i.e $n \equiv 3(mod4)$ and $k = 4$ i.e $k \equiv 0(mod4)$. The cordial labeling is as shown in FIGURE 3.17.

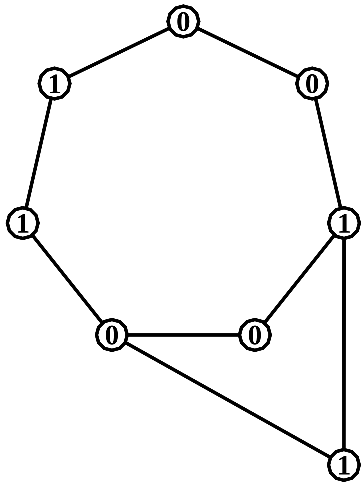

FIGURE 3.17

Example 2: Consider a graph obtained by duplicating vertex v_3 of cycle C_5. Here $n = 5$ i.e $n \equiv 1(mod4)$ and $k = 3$ i.e $k \equiv 3(mod4)$.The cordial labeling is as shown in FIGURE 3.18.

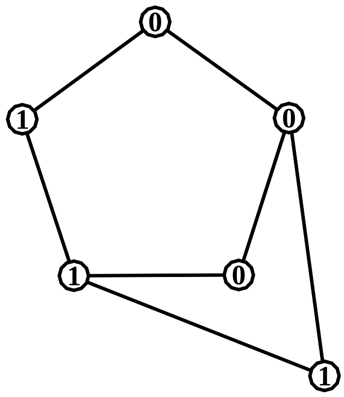

FIGURE 3.18

Theorem 3.9.9. Duplicating vertices of cycle C_n altogether produces a cordial graph except $n \equiv 2(mod4)$.

Proof. Let C_n be the cycle with n vertices and $v_1, v_2, ..., v_n$ be the vertices of C_n moreover G be the graph obtained by duplicating the vertices of C_n altogether and $v'_1, v'_2, ..., v'_n$ be the duplicated vertices of $v_1, v_2, ..., v_n$respectively. To define binary vertex labeling $f : V(G) \to \{0, 1\}$ we consider the following cases.

Case 1: $n \equiv 0,1,3(mod4)$

In this case we define labeling f as:

$$\left.\begin{aligned} f(v_i) &= 0;\ \text{if } i \equiv 0,1(mod4) \\ &= 1;\ \text{if } i \equiv 2,3(mod4) \end{aligned}\right\} \text{for } 1 \leq i \leq n$$

$$\left.\begin{aligned} f(v_i') &= 0;\ \text{if } i \equiv 2,3(mod4) \\ &= 1;\ \text{if } i \equiv 0,1(mod4) \end{aligned}\right\} \text{for } 1 \leq i \leq n$$

Case 2: $n \equiv 2(mod4)$

In this case the graph is an Eulerian graph with number of edges congruent to $2(mod4)$. As we mentioned earlier (Theorem 3.5.5) an Eulerian graph with number of edges congruent to $2(mod4)$ is not cordial.

The labeling pattern defined above covers all possible arrangement of vertices. In each case the graph G under consideration satisfies the conditions $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ as shown in TABLE 3.11(where $n = 4a + b$ and $a \in N \cup \{0\}$). i.e. G admits cordial labeling. □

b	**Vertex Condition**	**Edge Condition**
0	$v_f(0) = v_f(1) = n$	$e_f(0) = e_f(1) = \frac{3n}{2}$
1,3	$v_f(0) = v_f(1) = n$	$e_f(0) = e_f(1) + 1 = \frac{3n+1}{2}$

TABLE 3.11

Illustration 3.9.10. Consider a graph obtained by duplicating vertices of cycle C_5 altogether. Here $n = 5$ i.e $n \equiv 1(mod4)$. The corresponding cordial labeling is shown in FIGURE 3.19.

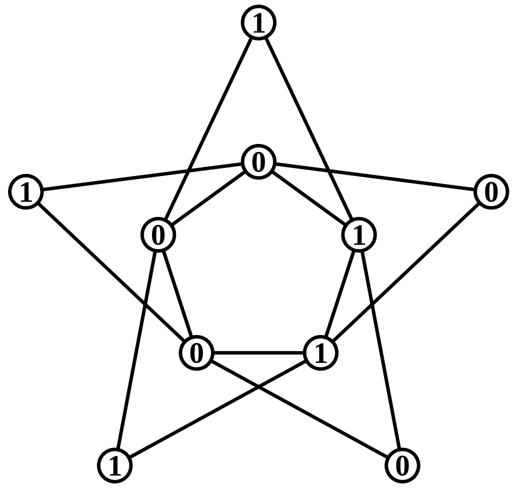

FIGURE 3.19

Theorem 3.9.11. The graph obtained by duplicating arbitrary rim vertex of wheel $W_n = C_n + K_1$ is cordial for all n and duplicating apex vertex is cordial except $n \equiv 2(mod4)$.

Proof. Consider the wheel $W_n = C_n + K_1$. Let $v_1, v_2, ..., v_n$ be the rim vertices of W_n, c_1 be the apex vertex of W_n and G be the graph obtained by duplicating either rim vertex or apex vertex of W_n. Let $v_k^{'}$ be the duplicated vertex of v_k and $c_1^{'}$ be the duplicated vertex of c_1. To define binary vertex labeling $f : V(G) \to \{0, 1\}$ we consider the following cases.

Case 1: Duplication of arbitrary rim vertex v_k , where $k \in N$, $1 \leq k \leq n$

Subcase 1: $n \equiv 0, 1, 3(mod4)$

In this case we define labeling function f as

$$\left.\begin{aligned} f(v_{k+i-1}) &= 0;\text{ if } i \equiv 1,2(mod4) \\ &= 1;\text{ if } i \equiv 0,3(mod4) \end{aligned}\right\} \text{for } 1 \leq i \leq n-k+1$$

$$\left.\begin{aligned} f(v_{k+i-n-1}) &= 0;\text{ if } i \equiv 1,2(mod4) \\ &= 1;\text{ if } i \equiv 0,3(mod4) \end{aligned}\right\} \text{for } n-k+2 \leq i \leq n$$

$$f(v_k^{'}) = 1;$$
$$f(c_1) = 0;$$

Subcase 2: $n \equiv 2(mod4)$

Here $f(c_1) = 1$ and label remaining vertices same as subcase 1

Case 2: Duplication of apex vertex c_1

Subcase 1: $n \equiv 0,1,3(mod4)$

In this case we define labeling f as

$$\left.\begin{aligned} f(v_i) &= 0; \text{ if } i \equiv 0,1(mod4) \\ &= 1; \text{ if } i \equiv 2,3(mod4) \end{aligned}\right\} \text{ for } 1 \leq i \leq n$$

$f(c_1) = 0;$

$f(c_1') = 1;$

Subcase 2: $n \equiv 2(mod4)$

In this case the graph is an Eulerian graph with number of edges congruent to $2(mod4)$. As we mentioned earlier (Theorem 3.5.5) an Eulerian graph with number of edges congruent to $2(mod4)$ is not cordial.

The labeling pattern defined above covers all possible arrangement of vertices. In each case, the graph G under consideration satisfies the conditions $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ as shown in TABLE 3.12(where $n = 4a + b$ and $a, b \in N \cup \{0\}$). i.e. G admits cordial labeling. □

b	**Vertex Condition**	**Edge Condition**
Duplication of a rim vertex		
0,2	$v_f(0) = v_f(1) = \frac{n+2}{2}$	$e_f(0) + 1 = e_f(1) = n + 2$
1,3	$v_f(0) = v_f(1) + 1 = \frac{n+3}{2}$	$e_f(0) + 1 = e_f(1) = n + 2$
Duplication of apex vertex		
0	$v_f(0) = v_f(1) = \frac{n+2}{2}$	$e_f(0) = e_f(1) = \frac{3n}{2}$
1	$v_f(0) = v_f(1) + 1 = \frac{n+3}{2}$	$e_f(0) = e_f(1) + 1 = \frac{3n+1}{2}$
3	$v_f(0) + 1 = v_f(1) = \frac{n+3}{2}$	$e_f(0) + 1 = e_f(1) = \frac{3n+1}{2}$

TABLE 3.12

Illustration 3.9.12.

Example 1: Consider a graph obtained by duplicating vertex v_3 on rim of wheel W_5. Here $n = 5$ i.e $n \equiv 1(mod4)$ and $k = 3$ i.e $k \equiv 3(mod4)$. The cordial labeling is shown in FIGURE 3.20.

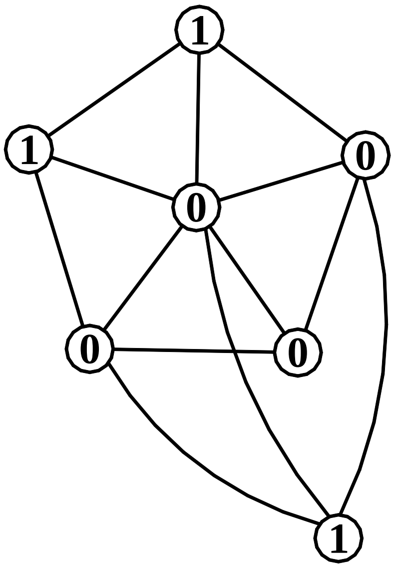

FIGURE 3.20

Example 2: Consider a graph obtained by duplicating apex vertex c_1 of wheel W_5. Here $n = 5$ i.e $n \equiv 1(mod4)$.The cordial labeling is shown in FIGURE 3.21.

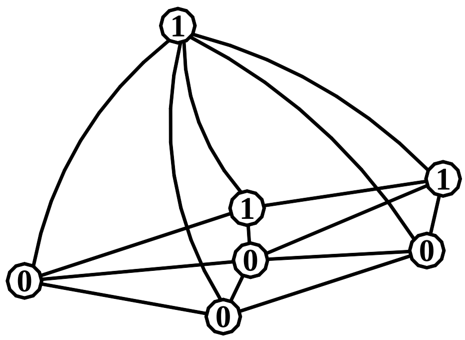

FIGURE 3.21

Theorem 3.9.13. Duplication of the vertices of wheel W_n altogether produces a cordial graph, where $n \in N$.

Proof. Consider the wheel $W_n = C_n + K_1$. Let $v_1, v_2, ..., v_n$ be the rim vertices of W_n, c_1 be the apex vertex of W_n and G be the graph obtained by duplicating vertices altogether moreover $v'_1, v'_2, ..., v'_n$ be the duplicated vertices of $v_1, v_2, ..., v_n$ respectively and $c_1^{'}$ be the duplicated vertex of c_1. To define binary vertex labeling $f : V(G) \to \{0,1\}$ we consider the following cases.

In this case we define labeling f as

$f(v_i) = 0$; for all i, $1 \leq i \leq n$

$f(v_i^{'}) = 1$; for all i, $1 \leq i \leq n$

$f(c_1) = 1$;

$f(c_1^{'}) = 0$;

The labeling pattern defined above covers all possible arrangement of vertices. In each case, the graph G under consideration satisfies the conditions $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ as shown in TABLE 3.13(where $n = 4a + b$ and $a, b \in N \cup \{0\}$). i.e. G admits cordial labeling. □

b	**Vertex Condition**	**Edge Condition**
0,1,2,3	$v_f(0) = v_f(1) = n+1$	$e_f(0) = e_f(1) = 3n$

TABLE 3.13

Illustration 3.9.14. Consider a graph obtained by duplicating vertices of wheel W_3 altogether. Here $n = 3$ i.e $n \equiv 3(mod 4)$. The cordial labeling is shown in FIGURE 3.22.

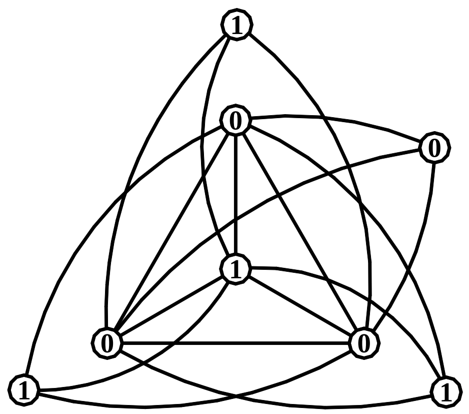

FIGURE 3.22

Theorem 3.9.15. Duplication of arbitrary edge e_k of cycle C_n produces a cordial graph.

Proof. Let C_n be the cycle with n vertices. Let $e_k = v_k v_{k+1}$ be the vertex of C_n. Let $e_k^{'} = v_k^{'} v_{k+1}^{'}$ be the duplicated edge of e_k and G be the graph resulted due to duplication. To define binary vertex labeling $f : V(G) \rightarrow \{0,1\}$ we consider following cases.

Case 1: If $n \equiv 0(mod4)$

$$\left.\begin{aligned} f(v_{k+i-1}) &= 0;\ \text{if } i \equiv 0,3(mod4) \\ &= 1;\ \text{if } i \equiv 1,2(mod4) \end{aligned}\right\} \text{for } 1 \leq i \leq n-k+1$$

$$\left.\begin{aligned} f(v_{k+i-n-1}) &= 0;\ \text{if } i \equiv 0,3(mod4) \\ &= 1;\ \text{if } i \equiv 1,2(mod4) \end{aligned}\right\} \text{for } n-k+2 \leq i \leq n$$

$$\begin{aligned} f(v_k^{'}) &= 0; \\ f(v_{k+1}^{'}) &= 1;\ \text{if } k \neq n \\ f(v_{k-n+1}^{'}) &= 1;\ \text{if } k = n \end{aligned}$$

Case 2: If $n \equiv 1(mod4)$

$$\left.\begin{aligned} f(v_{k+i-1}) &= 0;\ \text{if } i \equiv 0,3(mod4) \\ &= 1;\ \text{if } i \equiv 1,2(mod4) \end{aligned}\right\} \text{for } 1 \leq i \leq n-k+1$$

$$\left.\begin{aligned} f(v_{k+i-n-1}) &= 0;\ \text{if } i \equiv 0,3(mod4) \\ &= 1;\ \text{if } i \equiv 1,2(mod4) \end{aligned}\right\} \text{for } n-k+2 \leq i \leq n-1$$

$$\begin{aligned} f(v_{k-1}) &= 0;\ \text{if } k \neq 1 \\ f(v_{k+n-1}) &= 0;\ \text{if } k = 1 \\ f(v_k^{'}) &= 0; \\ f(v_{k+1}^{'}) &= 1;\ \text{if } k \neq n \\ f(v_{k-n+1}^{'}) &= 1;\ \text{if } k = n \end{aligned}$$

Case 3: If $n \equiv 2(mod4)$

$$f(v_k) = 1;$$
$$f(v_{k+1}) = 0; \text{ if } k \neq n$$
$$f(v_{k-n+1}) = 0; \text{ if } k = n$$

$$\left.\begin{aligned} f(v_{k+i-1}) &= 0;\ \text{if } i \equiv 1,2(mod4) \\ &= 1;\ \text{if } i \equiv 0,3(mod4) \end{aligned}\right\} \text{for } 3 \leq i \leq n-k+1$$

$$\left.\begin{aligned} f(v_{k+i-n-1}) &= 0;\ \text{if } i \equiv 1,2(mod4) \\ &= 1;\ \text{if } i \equiv 0,3(mod4) \end{aligned}\right\} \text{for } n-k+2 \leq i \leq n$$

$$f(v_k^{'}) = 0;$$
$$f(v_{k+1}^{'}) = 1; \text{ if } k \neq n$$
$$f(v_{k-n+1}^{'}) = 1; \text{ if } k = n$$

Case 4: If $n \equiv 3(mod4)$

$$f(v_k) = 1;$$

$$\left.\begin{aligned} f(v_{k+i-1}) &= 0;\ \text{if } i \equiv 2,3(mod4) \\ &= 1;\ \text{if } i \equiv 0,1(mod4) \end{aligned}\right\} \text{for } 2 \leq i \leq n-k+1$$

$$\left.\begin{aligned} f(v_{k+i-n-1}) &= 0;\ \text{if } i \equiv 2,3(mod4) \\ &= 1;\ \text{if } i \equiv 0,1(mod4) \end{aligned}\right\} \text{for } n-k+2 \leq i \leq n-1$$

$$f(v_{k-1}) = 1; \text{ if } k \neq 1$$
$$f(v_{k+n-1}) = 1; \text{ if } k = 1$$
$$f(v_k^{'}) = 1;$$
$$f(v_{k+1}^{'}) = 0; \text{ if } k \neq n$$
$$f(v_{k-n+1}^{'}) = 0; \text{ if } k = n$$

The labeling pattern defined above covers all possible arrangement of vertices. The graph G satisfies the conditions $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ as shown in TABLE 3.14(where $n = 4a+b$ and $a, b \in N \cup \{0\}$). i.e. G admits cordial labeling. □

b	**Vertex Condition**	**Edge Condition**
0,2	$v_f(0) = v_f(1) = \frac{n+2}{2}$	$e_f(0)+1 = e_f(1) = \frac{n+4}{2}$
1	$v_f(0) = v_f(1)+1 = \frac{n+3}{2}$	$e_f(0) = e_f(1) = \frac{n+3}{2}$
3	$v_f(0)+1 = v_f(1) = \frac{n+3}{2}$	$e_f(0) = e_f(1) = \frac{n+3}{2}$

TABLE 3.14

Illustration 3.9.16. Consider C_{10} and duplicate e_2. The cordial labeling is as shown in FIGURE 3.23

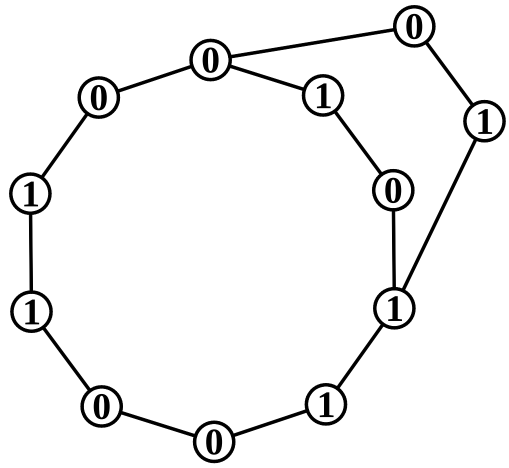

FIGURE 3.23

Theorem 3.9.17. Duplication of arbitrary edge e_k of wheel W_n produces a cordial graph.

Proof. Consider the wheel $W_n = C_n + K_1$. Let $v_1, v_2, \dots, v_n$ be the rim vertices of W_n, c be the apex vertex of W_n and G be the graph obtained by duplicating either rim edge or spoke edge of W_n. Let $e_k^{'}$ be the duplicated edge of e_k. To define binary vertex labeling $f : V(G) \to \{0,1\}$ we consider the following cases.

Case 1: Duplication of arbitrary rim edge e_k, where $k \in N, 1 \le k \le n$

Subcase 1: If $n \equiv 0(mod4)$

$$\left.\begin{aligned} f(v_{k+i-1}) &= 0;\ \text{if } i \equiv 0,1(mod4) \\ &= 1;\ \text{if } i \equiv 2,3(mod4) \end{aligned}\right\} \text{for } 1 \le i \le n-k+1$$

$$\left.\begin{aligned} f(v_{k+i-n-1}) &= 0;\ \text{if } i \equiv 0,1(mod4) \\ &= 1;\ \text{if } i \equiv 2,3(mod4) \end{aligned}\right\} \text{for } n-k+2 \leq i \leq n$$

$$f(c) = 0;$$
$$f(v_k^{'}) = 0;$$
$$f(v_{k+1}^{'}) = 1;\ \text{if } k \neq n$$
$$f(v_{k-n+1}^{'}) = 1;\ \text{if } k = n$$

Subcase 2: If $n \equiv 1,2(mod4)$

$$f(v_k) = 0;$$

$$\left.\begin{aligned} f(v_{k+i-1}) &= 0;\ \text{if } i \equiv 0,1(mod4) \\ &= 1;\ \text{if } i \equiv 2,3(mod4) \end{aligned}\right\} \text{for } 2 \leq i \leq n-k+1$$

$$\left.\begin{aligned} f(v_{k+i-n-1}) &= 0;\ \text{if } i \equiv 0,1(mod4) \\ &= 1;\ \text{if } i \equiv 2,3(mod4) \end{aligned}\right\} \text{for } n-k+2 \leq i \leq n$$

$$f(c) = 0;$$
$$f(v_k^{'}) = 1;$$
$$f(v_{k+1}^{'}) = 1;\ \text{if } k \neq n$$
$$f(v_{k-n+1}^{'}) = 1;\ \text{if } k = n$$

Subcase 3: If $n \equiv 3(mod4)$

$$\left.\begin{aligned} f(v_{k+i-1}) &= 0;\ \text{if } i \equiv 1,2(mod4) \\ &= 1;\ \text{if } i \equiv 0,3(mod4) \end{aligned}\right\} \text{for } 1 \leq i \leq n-k+1$$

$$\left.\begin{aligned} f(v_{k+i-n-1}) &= 0;\ \text{if } i \equiv 1,2(mod4) \\ &= 1;\ \text{if } i \equiv 0,3(mod4) \end{aligned}\right\} \text{for } n-k+2 \leq i \leq n$$

$$f(c) = 0;$$
$$f(v_k^{'}) = 1;$$
$$f(v_{k+1}^{'}) = 1;\ \text{if } k \neq n$$
$$f(v_{k-n+1}^{'}) = 1;\ \text{if } k = n$$

Case 2: Duplication of arbitrary spoke edge $e_k = cv_k$, where $k \in N, n+1 \leq k \leq 2n$

Subcase 1: If $n \equiv 0(mod4)$

$$\left.\begin{aligned} f(v_{k+i-1}) &= 0;\ \text{if } i \equiv 0,3(mod4) \\ &= 1;\ \text{if } i \equiv 1,2(mod4) \end{aligned}\right\} \text{for } 1 \leq i \leq n-k+1$$

$$\left.\begin{aligned} f(v_{k+i-n-1}) &= 0;\ \text{if } i \equiv 0,3(mod4) \\ &= 1;\ \text{if } i \equiv 1,2(mod4) \end{aligned}\right\} \text{for } n-k+2 \leq i \leq n$$

$$f(c) = 0;$$
$$f(c') = 0;$$
$$f(v'_k) = 1;$$

Subcase 2: If $n \equiv 1(mod4)$

$$\left.\begin{aligned} f(v_{k+i-1}) &= 0;\ \text{if } i \equiv 0,3(mod4) \\ &= 1;\ \text{if } i \equiv 1,2(mod4) \end{aligned}\right\} \text{for } 1 \leq i \leq n-k+1$$

$$\left.\begin{aligned} f(v_{k+i-n-1}) &= 0;\ \text{if } i \equiv 0,3(mod4) \\ &= 1;\ \text{if } i \equiv 1,2(mod4) \end{aligned}\right\} \text{for } n-k+2 \leq i \leq n-1$$

$$f(v_{k-1}) = 0;\ \text{if } k \neq 1$$
$$f(v_{k+n-1}) = 0;\ \text{if } k = 1$$
$$f(c) = 0;$$
$$f(c') = 1;$$
$$f(v'_k) = 1;$$

Subcase 3: If $n \equiv 2(mod4)$

$$\left.\begin{aligned} f(v_{k+i-1}) &= 0;\ \text{if } i \equiv 1,2(mod4) \\ &= 1;\ \text{if } i \equiv 0,3(mod4) \end{aligned}\right\} \text{for } 1 \leq i \leq n-k+1$$

$$\left.\begin{aligned} f(v_{k+i-n-1}) &= 0;\ \text{if } i \equiv 1,2(mod4) \\ &= 1;\ \text{if } i \equiv 0,3(mod4) \end{aligned}\right\} \text{for } n-k+2 \leq i \leq n-1$$

$$f(v_{k-1}) = 1; \text{ if } k \neq 1$$
$$f(v_{k+n-1}) = 1; \text{ if } k = 1$$
$$f(c) = 0;$$
$$f(c^{'}) = 1;$$
$$f(v_k^{'}) = 1;$$

Subcase 4: If $n \equiv 3(mod4)$

$$f(v_k) = 0;$$

$$\left.\begin{array}{rcl} f(v_{k+i-1}) & = & 0; \text{ if } i \equiv 0,1(mod4) \\ & = & 1; \text{ if } i \equiv 2,3(mod4) \end{array}\right\} \text{for } 2 \leq i \leq n-k+1$$

$$\left.\begin{array}{rcl} f(v_{k+i-n-1}) & = & 0; \text{ if } i \equiv 0,1(mod4) \\ & = & 1; \text{ if } i \equiv 2,3(mod4) \end{array}\right\} \text{for } n-k+2 \leq i \leq n-1$$

$$f(v_{k-1}) = 0; \text{ if } k \neq 1$$
$$f(v_{k+n-1}) = 0; \text{ if } k = 1$$
$$f(c) = 0;$$
$$f(c^{'}) = 1;$$
$$f(v_k^{'}) = 1;$$

The labeling pattern defined above covers all the possibilities. In each case, the graph G under consideration satisfies the conditions $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ as shown in TABLE 3.15(where $n = 4a + b$ and $a \in N \cup \{0\}$). i.e. G admits cordial labeling. □

b	**Vertex Condition**	**Edge Condition**
Duplication of a rim edge		
0	$v_f(0) = v_f(1) + 1 = \frac{n+4}{2}$	$e_f(0) = e_f(1) + 1 = n + 3$
1,3	$v_f(0) = v_f(1) = \frac{n+3}{2}$	$e_f(0) = e_f(1) + 1 = n + 3$
2	$v_f(0) + 1 = v_f(1) = \frac{n+4}{2}$	$e_f(0) + 1 = e_f(1) = n + 3$
Duplication of a spoke edge		
0	$v_f(0) = v_f(1) + 1 = \frac{n+4}{2}$	$e_f(0) = e_f(1) = \frac{3n+2}{2}$
1,3	$v_f(0) = v_f(1) = \frac{n+3}{2}$	$e_f(0) = e_f(1) + 1 = \frac{3n+3}{2}$
2	$v_f(0) + 1 = v_f(1) = \frac{n+4}{2}$	$e_f(0) = e_f(1) = \frac{3n+2}{2}$

TABLE 3.15

Illustration 3.9.18. Consider W_4 and duplicate spoke edge e_6. The cordial labeling is as shown in FIGURE 3.24

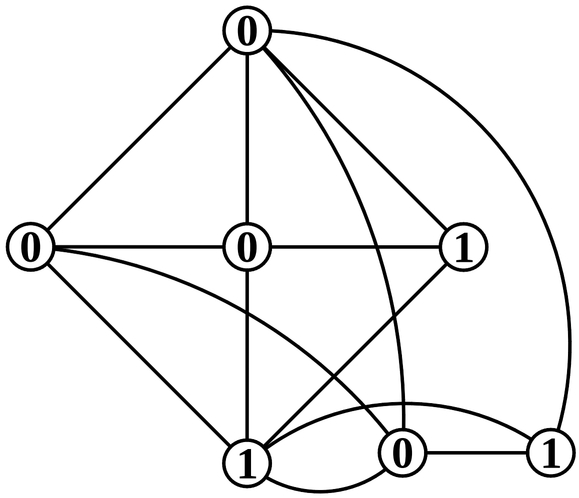

FIGURE 3.24

3.10 Some Open Problems

It is possible to obtain the results similar to that of Section 3.9 using different graph operations as well as various graph labeling techniques.

3.11 Concluding Remarks

This chapter was intended to discuss cordial labeling of graphs. The graceful labeling and harmonious labeling are discussed to prepare a platform for cordial labeling. Some existing results are reported and fifteen new results are investigated.

The penultimate Chapter-4 is targeted to discussed 3-equitable labeling of graphs.

Chapter 4

3-equitable Labeling of Graphs

4.1 Introduction

In 1990 Cahit[12] proposed the idea of distributing the vertex and the edge labels among $\{0,1,2,\dots,k-1\}$ as evenly as possible to obtain a generalization of graceful labeling. A vertex labeling of a graph $G=(V(G),E(G))$ is a function $f:V(G)\rightarrow\{0,1,2,\dots,k-1\}$ and the value $f(u)$ is called label of vertex u. For the vertex labeling function $f:V(G)\rightarrow\{0,1,\dots,k-1\}$, the induced function $f^*:E(G)\rightarrow\{0,1,\dots,k-1\}$ defined as $f^*(e=uv)=|f(u)-f(v)|$ which satisfies the conditions

1. $|v_f(i)-v_f(j)|\leq 1$ and
2. $|e_f(i)-e_f(j)|\leq 1$, $0\leq i,j\leq k-1$,

where $v_f(i)$ and $e_f(i)$ denotes number of vertices and number of edges having label i under f and f^* respectively, $0\leq i\leq k-1$. Such labeling f is called *k-equitable labeling* for the graph G. A graph which admits k-equitable labeling is called *k-equitable graph*. Obviously 2-equitable labeling is a cordial labeling which is already discussed in the previous chapter-3. When $k=3$ the labeling is called 3-equitable labeling. The present chapter is aimed to discuss 3-equitable labeling of graphs.

4.2 3-equitable Labeling of Graphs

Definition 4.2.1. Let $G=(V(G),E(G))$ be a graph. A mapping $f:V(G)\rightarrow\{0,1,2\}$ is called *ternary vertex labeling* of G and $f(v)$ is called *label of the vertex* v of G under f.

For an edge $e=uv$, the induced edge labeling $f^*:E(G)\rightarrow\{0,1,2\}$ is given by $f^*(e)=|f(u)-f(v)|$. Let $v_f(0)$, $v_f(1)$, $v_f(2)$ be the number of vertices of G having labels 0, 1 and 2 respectively under f and let $e_f(0)$, $e_f(1)$, $e_f(2)$ be the number of edges having labels 0, 1 and 2 respectively under f^*.

Definition 4.2.2. A ternary vertex labeling of a graph G is called *3-equitable labeling* if $|v_f(i)-v_f(j)|\leq 1$ and $|e_f(i)-e_f(j)|\leq 1$, $0\leq i,j\leq 2$.

A graph which admits 3-equitable labeling is called *3-equitable graph*.

4.2.1 Some Known Results

- Cahit[11],[12] proved that

  ◇ C_n is 3-equitable if and only if n is not congruent to $3(mod6)$.

  ◇ An Eulerian graph with $q \equiv 3(mod6)$ is not 3-equitable where q is the number of edges.

  ◇ All caterpillars are 3-equitable.

  ◇ *(Conjecture) A triangular cactus with n blocks is 3-equitable if and only if n is even.*

  ◇ Every tree with fewer than five end vertices has a 3-equitable labeling.

- Seoud and Abdel Maqsoud[38] proved that

  ◇ A graph with p vertices and q edges in which every vertex has odd degree is not 3-equitable if $p \equiv 0(mod3)$ and $q \equiv 3(mod6)$.

  ◇ All fans except $P_2 + K_1$ are 3-equitable.

  ◇ P_n^2 is 3-equitable for all n except 3.

  ◇ $K_{m,n}$(where $3 \leq m \leq n$) is 3-equitable if and only if $(m,n) = (4,4)$.

- Bapat and Limaye[7] proved that Helms H_n(where $n \geq 4$) are 3-equitable.

- Youssef[45] proved that $W_n = C_n + K_1$ is 3-equitable for all $n \geq 4$.

In the immediate section we will provide brief account of results investigated by us about 3-equitable labeling of some graphs.

4.3 3-equitable Labeling of Some Star Related Graphs

Theorem 4.3.1. Graph $< K_{1,n}^{(1)} : K_{1,n}^{(2)} >$ is 3-equitable.

Proof. Let $v_1^{(1)}, v_2^{(1)}, v_3^{(1)}, \ldots v_n^{(1)}$ be the pendant vertices $K_{1,n}^{(1)}$ and $v_1^{(2)}, v_2^{(2)}, v_3^{(2)}, \ldots v_n^{(2)}$ be the pendant vertices $K_{1,n}^{(2)}$. Let c_1 and c_2 be the apex vertices of $K_{1,n}^{(1)}$ and $K_{1,n}^{(2)}$ respectively and they are adjacent to a new common vertex x. Let $G =< K_{1,n}^{(1)} : K_{1,n}^{(2)} >$. To define ternary vertex labeling $f : V(G) \to \{0, 1, 2\}$ we consider following cases.

Case 1: $n \equiv 0(mod3)$

In this case we define labeling f as

$$\left.\begin{aligned} f(v_i^{(j)}) &= 0; \text{if } i \equiv 0(mod3) \\ &= 1;\ \text{if } i \equiv 1(mod3) \\ &= 2;\ \text{if } i \equiv 2(mod3) \end{aligned}\right\} \text{ for } 1 \leq i \leq n-1,\ j = 1, 2$$

$$f(v_n^{(1)}) = 1;$$
$$f(v_n^{(2)}) = f(c_1) = f(x) = 0;$$
$$f(c_2) = 2;$$

Case 2: $n \equiv 1(mod3)$

In this case we define labeling f as

$$\left.\begin{aligned} f(v_i^{(j)}) &= 0; \text{if } i \equiv 0(mod3) \\ &= 1;\ \text{if } i \equiv 1(mod3) \\ &= 2;\ \text{if } i \equiv 2(mod3) \end{aligned}\right\} \text{ for } 1 \leq i \leq n,\ j = 1, 2$$

$$f(c_1) = f(x) = 0;$$
$$f(c_2) = 2;$$

Case 3: $n \equiv 2(mod3)$

In this case we define labeling f as

$$\left.\begin{aligned} f(v_i^{(j)}) &= 0; \text{if } i \equiv 0(mod3) \\ &= 1; \text{ if } i \equiv 1(mod3) \\ &= 2; \text{ if } i \equiv 2(mod3) \end{aligned}\right\} \text{ for } 1 \le i \le n,\ j = 1,2$$

$$f(c_1) = f(c_2) = f(x) = 0;$$

The labeling pattern defined above covers all possible arrangement of vertices. In each case, the graph G under consideration satisfies the conditions $|v_f(i) - v_f(j)| \le 1$ and $|e_f(i) - e_f(j)| \le 1$ for all $0 \le i, j \le 2$ as shown in TABLE 4.1(where $n = 3a + b$ and $a \in N \cup \{0\}$). i.e. G admits 3-equitable labeling. □

b	**Vertex Condition**	**Edge Condition**
0	$v_f(0) = v_f(1) = v_f(2) = \frac{2n+3}{3}$	$e_f(0) + 1 = e_f(1) = e_f(2) = \frac{2n+3}{3}$
1	$v_f(0) = v_f(1) = v_f(2) + 1 = \frac{2n+4}{3}$	$e_f(0) + 1 = e_f(1) = e_f(2) + 1 = \frac{2n+4}{3}$
2	$v_f(0) = v_f(1) + 1 = v_f(2) + 1 = \frac{2n+5}{3}$	$e_f(0) = e_f(1) = e_f(2) = \frac{2n+2}{3}$

TABLE 4.1

Illustration 4.3.2. Consider a graph $G =< K_{1,8}^{(1)} : K_{1,8}^{(2)} >$ Here $n = 8$ i.e $n \equiv 2(mod3)$. The corresponding 3-equitable labeling is shown in FIGURE 4.1. It is the case related to case -3

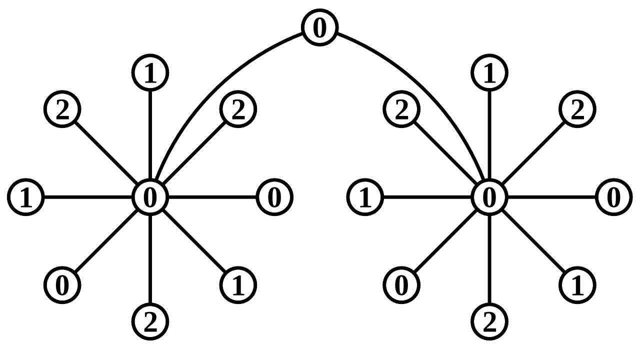

FIGURE 4.1

Above result can be extended for $k-$copies of $K_{1,n}$ as follows.

Theorem 4.3.3. Graph $< K_{1,n}^{(1)} : K_{1,n}^{(2)} : K_{1,n}^{(3)} : \ldots : K_{1,n}^{(k)} >$ is 3-equitable.

Proof. Let $K_{1,n}^{(j)}$, $j = 1,2,\ldots k$ be k copies of star $K_{1,n}$. Let $v_i^{(j)}$ be the pendant vertices of $K_{1,n}^{(j)}$ where $i = 1,2,\ldots n$ and $j = 1,2,\ldots k$. Let c_j be the apex vertex of $K_{1,n}^{(j)}$ where $j = 1,2,\ldots k$. Let $G =< K_{1,n}^{(1)} : K_{1,n}^{(2)} : K_{1,n}^{(3)} : \ldots : K_{1,n}^{(k)} >$ and $x_1, x_2, \ldots, x_{k-1}$ are the vertices as stated in Theorem 2.3. To define ternary vertex labeling $f : V(G) \to \{0,1,2\}$ we consider following cases.

Case 1: For $n \equiv 0(mod3)$

In this case we define labeling function f as follows

Subcase 1: For $k \equiv 0(mod3)$

$$\left.\begin{aligned} f(v_i^{(j)}) &= 0; \text{if } i \equiv 1(mod3) \\ &= 1; \text{ if } i \equiv 2(mod3) \\ &= 2; \text{ if } i \equiv 0(mod3) \end{aligned}\right\} \text{for } i \leq n-1$$

$$\begin{aligned} f(v_n^{(j)}) &= 1; \text{ if } j \equiv 1,2(mod3) \\ &= 2; \text{ if } j \equiv 0(mod3) \\ f(c_j) &= 0; \text{ if } j \equiv 1,2(mod3) \\ &= 2; \text{ if } j \equiv 0(mod3) \\ f(x_j) &= 2; \text{ if } j \leq n-1 \end{aligned}$$

Subcase 2: For $k \equiv 1(mod3)$

$$\begin{aligned} f(v_i^{(1)}) &= 0; \text{ if } i \equiv 1(mod3) \\ &= 1; \text{ if } i \equiv 2(mod3) \\ &= 2; \text{ if } i \equiv 0(mod3) \\ f(c_1) &= 2; \\ f(x_1) &= 0; \end{aligned}$$

For remaining vertices take $j = k-1$ and use the pattern of subcase 1.

Subcase 3: For $k \equiv 2(mod3)$

$$\left.\begin{aligned} f(v_i^{(j)}) &= 0;\text{ if } i \equiv 1(mod3)\\ &= 1;\text{ if } i \equiv 2(mod3)\\ &= 2;\text{ if } i \equiv 0(mod3)\end{aligned}\right\}\text{ for } 1 \leq i \leq n-1,\ j=1,2$$

$$f(v_n^{(1)}) = 1;$$
$$f(v_n^{(2)}) = f(c_2) = f(x_j) = 2;$$
$$f(c_1) = 0;$$

For remaining vertices take $j = k-2$ and use the pattern of subcase 1.

Case 2: For $n \equiv 1(mod3)$

In this case we define labeling function f as follows

Subcase 1: For $k \equiv 0(mod3)$

Subcase 1.1: For $n = 1$

$$\begin{aligned} f(v_1^{(j)}) &= 2;\text{ if } j \equiv 0(mod3)\\ &= 1;\text{ if } j \equiv 1,2(mod3)\\ f(c_j) &= 2;\text{ if } j \equiv 1(mod3)\\ &= 1;\text{ if } j \equiv 2(mod3)\\ &= 0;\text{ if } j \equiv 0(mod3)\\ f(x_j) &= 0;\ j \neq k\end{aligned}$$

Subcase 1.2: For $n > 1$

$$\left.\begin{aligned} f(v_i^{(j)}) &= 0;\text{ if } i \equiv 0(mod3)\\ &= 1;\text{ if } i \equiv 1(mod3)\\ &= 2;\text{ if } i \equiv 2(mod3)\end{aligned}\right\}\text{ for } i \leq n-2$$

$$\begin{aligned} f(v_{n-1}^{(j)}) &= 0;\text{ if } j \equiv 1,2(mod3)\\ &= 2;\text{ if } j \equiv 0(mod3)\\ f(v_n^{(j)}) &= 1;\end{aligned}$$

$$
\begin{aligned}
f(c_j) &= 2;\text{ if } j \equiv 1(mod3)\\
&= 0;\text{ if } j \equiv 0,2(mod3)\\
f(x_j) &= 0;\text{ if } j \equiv 1,2(mod3)\\
&= 2;\text{ if } j \equiv 0(mod3),\ j \neq k
\end{aligned}
$$

Subcase 2: For $k \equiv 1(mod3)$

$$
\begin{aligned}
f(v_i^{(1)}) &= 0;\text{ if } i \equiv 0(mod3)\\
&= 1;\text{ if } i \equiv 1(mod3)\\
&= 2;\text{ if } i \equiv 2(mod3)\\
f(c_1) &= 0;\\
f(x_1) &= 2;
\end{aligned}
$$

For remaining vertices take $j = k-1$ and use the pattern of subcase 1.1 or subcase 1.2 if $n = 1$ or $n > 1$ respectively.

Subcase 3: For $k \equiv 2(mod3)$

$$
\left.
\begin{aligned}
f(v_i^{(j)}) &= 0;\text{ if } i \equiv 0(mod3)\\
&= 1;\text{ if } i \equiv 1(mod3)\\
&= 2;\text{ if } i \equiv 2(mod3)
\end{aligned}
\right\}\text{ for } j = 1,2
$$

$$
\begin{aligned}
f(c_1) &= f(x_2) = 2;\\
f(c_2) &= f(x_1) = 0;\\
f(x_1) &= 2;\text{ if } n = 1\\
f(x_1) &= 0;\text{ if } n > 1
\end{aligned}
$$

For remaining vertices take $j = k-2$ and use the pattern of subcase 1.1 or subcase 1.2 if $n = 1$ or $n > 1$ respectively.

Case 3: For $n \equiv 2(mod3)$

In this case we define labeling function f as follows

Subcase 1: For $k \equiv 0(mod3)$

$$\left.\begin{array}{rcl} f(v_i^{(j)}) & = & 0;\ \text{if } i \equiv 0(mod3) \\ & = & 1;\ \text{if } i \equiv 1(mod3) \\ & = & 2;\ \text{if } i \equiv 2(mod3) \end{array}\right\} \text{for } i \leq n-1$$

$$\begin{aligned} f(v_n^{(j)}) &= 1;\ \text{if } j \equiv 1(mod3) \\ &= 2;\ \text{if } j \equiv 0,2(mod3) \\ f(c_j) &= 2;\ \text{if } j \equiv 1(mod3) \\ &= 0;\ \text{if } j \equiv 0,2(mod3) \\ f(x_j) &= 0;\ \text{if } j \equiv 1,2(mod3) \\ &= 2;\ \text{if } j \equiv 0(mod3) \end{aligned}$$

Subcase 2: For $k \equiv 1(mod3)$

$$\left.\begin{array}{rcl} f(v_i^{(1)}) & = & 0;\ \text{if } i \equiv 0(mod3) \\ & = & 1;\ \text{if } i \equiv 1(mod3) \\ & = & 2;\ \text{if } i \equiv 2(mod3) \end{array}\right\} \text{for } i \leq n$$

$$\begin{aligned} f(c_1) &= 0; \\ f(x_1) &= 2; \end{aligned}$$

For remaining vertices take $j = k-1$ and use the pattern of subcase 1.

Subcase 3: For $k \equiv 2(mod3)$

$$\left.\begin{array}{rcl} f(v_i^{(j)}) & = & 0;\ \text{if } i \equiv 0(mod3) \\ & = & 1;\ \text{if } i \equiv 1(mod3) \\ & = & 2;\ \text{if } i \equiv 2(mod3) \end{array}\right\} \text{for } i \leq n,\ j = 1,2$$

$$\begin{aligned} f(c_1) &= 2; \\ f(c_2) &= f(x_j) = 0; \end{aligned}$$

For remaining vertices take $j = k-2$ and use the pattern of subcase 1.

The labeling pattern defined above covers all possible arrangement of vertices. In each case, the graph G under consideration satisfies the conditions $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $0 \leq i, j \leq 2$ as shown in TABLE 4.2(where $n = 3a+b$, $k = 3c+d$ and $a \in N \cup \{0\}, c \in N$). i.e. G admits 3-equitable labeling. □

b	***d***	**Vertex Condition**	**Edge Condition**
	0	$v_f(0) = v_f(1) = v_f(2)+1 = \frac{k(n+2)}{3}$	$e_f(0)+1 = e_f(1) = e_f(2)+1 = \frac{k(n+2)}{3}$
0	1	$v_f(0) = v_f(1)+1 = v_f(2)+1 = \frac{k(n+2)+1}{3}$	$e_f(0) = e_f(1) = e_f(2) = \frac{k(n+2)-2}{3}$
	2	$v_f(0) = v_f(1) = v_f(2) = \frac{k(n+2)-1}{3}$	$e_f(0) = e_f(1) = e_f(2)+1 = \frac{k(n+2)-1}{3}$
1	0,1,2	$v_f(0) = v_f(1) = v_f(2)+1 = \frac{k(n+2)}{3}$	$e_f(0)+1 = e_f(1) = e_f(2)+1 = \frac{k(n+2)}{3}$
	0	$v_f(0) = v_f(1) = v_f(2)+1 = \frac{k(n+2)}{3}$	$e_f(0)+1 = e_f(1) = e_f(2)+1 = \frac{k(n+2)}{3}$
2	1	$v_f(0) = v_f(1) = v_f(2) = \frac{k(n+2)-1}{3}$	$e_f(0)+1 = e_f(1) = e_f(2) = \frac{k(n+2)-1}{3}$
	2	$v_f(0) = v_f(1)+1 = v_f(2)+1 = \frac{k(n+2)+1}{3}$	$e_f(0) = e_f(1) = e_f(2) = \frac{k(n+2)-2}{3}$

TABLE 4.2

Illustration 4.3.4. Consider a graph $G = < K_{1,5}^{(1)} : K_{1,5}^{(2)} : K_{1,5}^{(3)} : K_{1,5}^{(4)} >$. Here $n = 5$ and $k = 4$. The corresponding 3-equitable labeling is as shown in FIGURE 4.2.

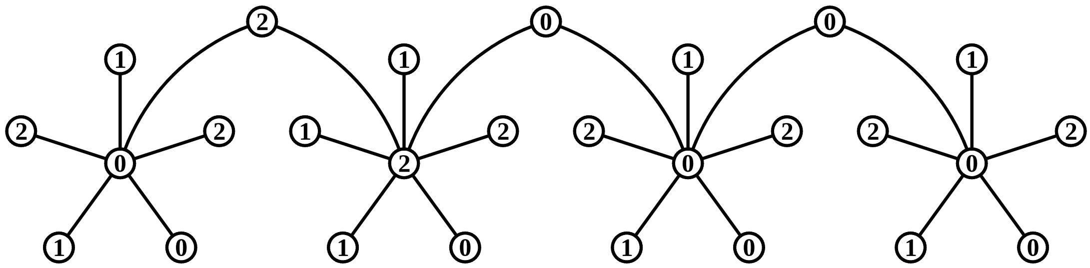

FIGURE 4.2

Theorem 4.3.5. Graph $< K_{1,n}^{(1)} \blacktriangle K_{1,n}^{(2)} >$ is 3-equitable.

Proof. Let $v_1^{(1)}, v_2^{(1)}, v_3^{(1)}, \dots v_n^{(1)}$ be the pendant vertices $K_{1,n}^{(1)}$ and $v_1^{(2)}, v_2^{(2)}, v_3^{(2)}, \dots v_n^{(2)}$ be the pendant vertices $K_{1,n}^{(2)}$. Let c_1 and c_2 be the apex vertices of $K_{1,n}^{(1)}$ and $K_{1,n}^{(2)}$ respectively and they are adjacent to a new common vertex x. Let $G = < K_{1,n}^{(1)} \blacktriangle K_{1,n}^{(2)} >$. To define vertex labeling $f : V(G) \to \{0,1,2\}$ we consider the following cases.

Case 1: $n \equiv 0(mod3)$

In this case we define labeling f as

$$\left.\begin{array}{rcl} f(v_i^{(j)}) & = & 0;\ \text{if } i \equiv 0(mod3) \\ & = & 1;\ \text{if } i \equiv 1(mod3) \\ & = & 2;\ \text{if } i \equiv 2(mod3) \end{array}\right\} \text{for } 1 \leq i \leq n-1,\ j = 1,2$$

$$f(v_n^{(1)}) = 1;$$
$$f(v_n^{(2)}) = f(c_2) = f(x) = 0;$$
$$f(c_1) = 2;$$

Case 2: $n \equiv 1(mod3)$

In this case we define labeling f as

$$\left.\begin{array}{rcl} f(v_i^{(j)}) & = & 0;\ \text{if } i \equiv 0(mod3) \\ & = & 1;\ \text{if } i \equiv 1(mod3) \\ & = & 2;\ \text{if } i \equiv 2(mod3) \end{array}\right\} \text{for } 1 \leq i \leq n,\ j = 1,2$$

$$f(c_1) = f(x) = 0;$$
$$f(c_2) = 2;$$

Case 3: $n \equiv 2(mod3)$

In this case we define labeling f as

$$\left.\begin{array}{rcl} f(v_i^{(j)}) & = & 0;\ \text{if } i \equiv 0(mod3) \\ & = & 1;\ \text{if } i \equiv 1(mod3) \\ & = & 2;\ \text{if } i \equiv 2(mod3) \\ f(c_j) & = & f(x) = 0; \end{array}\right\} \text{for } 1 \leq i \leq n,\ j = 1,2$$

The labeling pattern defined above covers all possible arrangement of vertices. In each case, the graph G under consideration satisfies the conditions $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $0 \leq i, j \leq 2$ as shown in TABLE 4.3(where $n = 3a + b$ and $a \in N \cup \{0\}$). i.e. G admits 3-equitable labeling. □

b	Vertex Condition	Edge Condition
0	$v_f(0)=v_f(1)=v_f(2)=\frac{2n+3}{3}$	$e_f(0)=e_f(1)=e_f(2)=\frac{2n+3}{3}$
1	$v_f(0)=v_f(1)=v_f(2)+1=\frac{2n+4}{3}$	$e_f(0)+1=e_f(1)=e_f(2)=\frac{2n+4}{3}$
2	$v_f(0)=v_f(1)+1=v_f(2)+1=\frac{2n+5}{3}$	$e_f(0)=e_f(1)+1=e_f(2)+1=\frac{2n+5}{3}$

TABLE 4.3

Illustration 4.3.6. Consider a graph $G=<K_{1,8}^{(1)}\blacktriangle K_{1,8}^{(2)}>$ Here $n=8$ i.e $n\equiv 2(mod3)$. The corresponding 3-equitable labeling is shown in FIGURE 4.3. It is the case related to case 3.

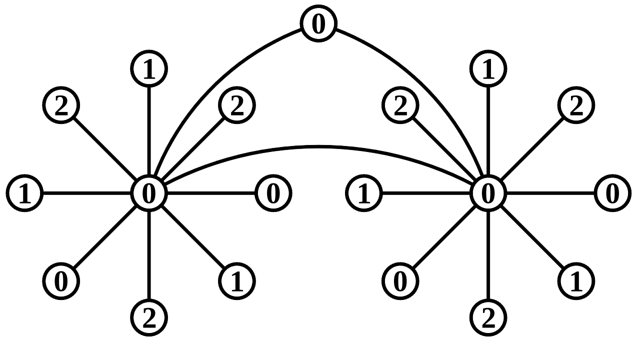

FIGURE 4.3

Theorem 4.3.7. Graph $<K_{1,n}^{(1)}\blacktriangle K_{1,n}^{(2)}\blacktriangle K_{1,n}^{(3)}\blacktriangle\dots\blacktriangle K_{1,n}^{(k)}>$ is 3-equitable.

Proof. Let $K_{1,n}^{(j)}$, $j=1,2,\dots k$ be k copies of star $K_{1,n}$. Let $v_i^{(j)}$ be the pendant vertices of $K_{1,n}^{(j)}$ where $i=1,2,\dots n$ and $j=1,2,\dots k$. Let c_j be the apex vertex of $K_{1,n}^{(j)}$ where $j=1,2,\dots k$. Let $G=<K_{1,n}^{(1)}\blacktriangle K_{1,n}^{(2)}\blacktriangle K_{1,n}^{(3)}\blacktriangle\dots\blacktriangle K_{1,n}^{(k)}>$ and $x_1,x_2,\dots,x_{k-1}$ are the vertices as stated in Theorem 2.3. To define vertex labeling $f:V(G)\to\{0,1,2\}$ we consider following cases.

Case 1: For $n \equiv 0(mod3)$

In this case we define labeling function f as follows

Subcase 1: For $k \equiv 0(mod3)$

$$\left.\begin{aligned} f(v_i^{(j)}) &= 0;\ \text{if } i \equiv 0(mod3),\ j \neq 3 \text{ and } i \neq n \\ &= 1;\ \text{if } i \equiv 1(mod3) \\ &= 2;\ \text{if } i \equiv 2(mod3) \end{aligned}\right\} \text{for } j \equiv 0,1(mod3)$$

$f(v_n^{(3)}) = 1;$

$f(c_j) = 2;$ if $j \equiv 1(mod3)$

$f(c_j) = 0;$ if $j \equiv 0(mod3)$ and $j \neq 3$

$f(c_3) = 2;$

$f(x_j) = 2;$ if $j \equiv 1(mod3)$

$f(x_j) = 0;$ if $j \equiv 0(mod3)$

$$\left.\begin{aligned} f(v_i^{(j)}) &= 0;\ \text{if } i \equiv 2(mod3) \\ &= 1;\ \text{if } i \equiv 1(mod3) \\ &= 2;\ \text{if } i \equiv 0(mod3),\ i \neq n \\ f(v_n^{(j)}) &= 1; \\ f(c_j) &= 2;\ \text{if } j \neq 2 \\ f(x_j) &= 1;\ \text{if } j \neq 2 \end{aligned}\right\} \text{for } j \equiv 2(mod3)$$

$f(c_2) = f(x_2) = 0;$

Subcase 2: For $k \equiv 1(mod3)$

$$\left.\begin{array}{rcl} f(v_i^{(j)}) & = & 0;\text{ if } i \equiv 0(mod3) \\ & = & 1;\text{ if } i \equiv 1(mod3) \\ & = & 2;\text{ if } i \equiv 2(mod3) \\ f(c_j) & = & 0;\text{ if } j \equiv 1(mod3 \text{ and } j \neq 1 \\ f(c_j) & = & 2;\text{ if } j \equiv 2(mod3) \\ f(c_1) & = & 2; \\ f(x_j) & = & 0;\text{ if } j \equiv 1(mod3) \\ f(x_j) & = & 2;\text{ if } j \equiv 2(mod3) \end{array}\right\} \text{for } j \equiv 1,2(mod3)$$

$$\left.\begin{array}{rcl} f(v_i^{(j)}) & = & 0;\text{ if } i \equiv 2(mod3) \\ & = & 1;\text{ if } i \equiv 1(mod3) \\ & = & 2;\text{ if } i \equiv 0(mod3),\ i \neq n \\ f(v_n^{(j)}) & = & f(x_j) = 1; \\ f(c_j) & = & 2; \end{array}\right\} \text{for } j \equiv 0(mod3)$$

Subcase 3: For $k \equiv 2(mod3)$

$$\left.\begin{array}{rcl} f(v_i^{(j)}) & = & 0;\text{ if } i \equiv 0(mod3) \\ & = & 1;\text{ if } i \equiv 1(mod3) \\ & = & 2;\text{ if } i \equiv 2(mod3) \\ f(c_j) & = & 2;\text{ if } j \equiv 0(mod3) \\ f(c_j) & = & 0;\text{ if } j \equiv 2(mod3) \text{ and } j \neq 2 \\ f(x_j) & = & 2;\text{ if } j \equiv 0(mod3) \\ f(x_j) & = & 0;\text{ if } j \equiv 2(mod3) \\ f(c_2) & = & 2; \end{array}\right\} \text{for } j \equiv 0,2(mod3)$$

$$\left.\begin{aligned}
f(v_i^{(j)}) &= 0;\ \text{if } i \equiv 2(mod3)\\
&= 1;\ \text{if } i \equiv 1(mod3)\\
&= 2;\ \text{if } i \equiv 0(mod3),\ i \neq n\\
f(v_n^{(j)}) &= 1;\\
f(c_j) &= 2;\ \text{if } j \neq 1\\
f(c_1) &= 0;\\
f(x_j) &= 1;\ \text{if } j \neq 1\\
f(x_1) &= 2;
\end{aligned}\right\}\ \text{for } j \equiv 1(mod3)$$

Case 2: For $n \equiv 1(mod3)$

In this case we define labeling function f as follows

Subcase 1: For $k \equiv 0(mod3)$

Subcase 1.1: For $n = 1$

$$\begin{aligned}
f(v_1^{(1)}) &= 1;\\
f(v_1^{(2)}) &= f(v_1^{(3)}) = f(c_1) = 2;\\
f(c_2) &= f(c_3) = f(x_2) = 0;\\
f(x_1) &= 1;
\end{aligned}$$

For remaining vertices use the pattern of subcase 1.2.

Subcase 1.2: For $n > 1$

$$\begin{aligned}
f(v_i^{(j)}) &= 0;\ \text{if } i \equiv 0(mod3),\ i \neq n-1 \text{ and } j \neq 3\\
&= 1;\ \text{if } i \equiv 1(mod3),\ i \neq n,\ j = 1\\
&= 2;\ \text{if } i \equiv 2(mod3)\\
f(v_n^{(j)}) &= f(v_{n-1}^{(3)}) = 2 \text{ if } j \neq 1\\
f(c_j) &= 2;\ \text{if } j \equiv 1(mod3)\\
&= 0;\ \text{if } j \equiv 0,2(mod3)\\
f(x_j) &= 2;\ \text{if } j \equiv 1(mod3)\\
&= 0;\ \text{if } j \equiv 0(mod3),\ j \neq k\\
&= 1;\ \text{if } j \equiv 2(mod3),\ j \neq 2\\
f(x_2) &= 0;
\end{aligned}$$

Subcase 2: For $k \equiv 1(mod3)$

$$
\begin{aligned}
f(v_i^{(j)}) &= 0;\ \text{if } i \equiv 0(mod3)\\
&= 1;\ \text{if } i \equiv 1(mod3),\ i \neq n \text{ and } j \equiv 0,1(mod3)\\
&= 2;\ \text{if } i \equiv 2(mod3)\\
f(v_n^{(j)}) &= 2;\ \text{if } j \equiv 2(mod3)\\
f(c_j) &= 0;\ \text{if } j \equiv 0,1(mod3) \text{ and } j \neq 1\\
f(c_j) &= 2;\ \text{if } j \equiv 2(mod3)\\
f(c_1) &= 2;\\
f(x_j) &= 1;\ \text{if } j \equiv 0(mod3)\\
f(x_j) &= 0;\ \text{if } j \equiv 1(mod3)\\
f(x_j) &= 2;\ \text{if } j \equiv 2(mod3)
\end{aligned}
$$

Subcase 3: For $k \equiv 2(mod3)$

$$
\begin{aligned}
f(v_i^{(j)}) &= 0;\ \text{if } i \equiv 0(mod3)\\
&= 1;\ \text{if } i \equiv 1(mod3),\ i \neq n \text{ and } j \equiv 1,2(mod3)\\
&= 2;\ \text{if } i \equiv 2(mod3)\\
f(v_n^{(j)}) &= 2;\ \text{if } j \equiv 0(mod3)\\
f(c_j) &= 0;\ \text{if } j \equiv 1,2(mod3) \text{ and } j \neq 1\\
f(c_j) &= 2;\ \text{if } j \equiv 0(mod3)\\
f(c_1) &= f(x_1) = 2;\\
f(x_j) &= 1;\ \text{if } j \equiv 1(mod3) \text{ and } j \neq 1\\
f(x_j) &= 0;\ \text{if } j \equiv 2(mod3)\\
f(x_j) &= 2;\ \text{if } j \equiv 0(mod3)
\end{aligned}
$$

Case 3: For $n \equiv 2(mod3)$

In this case we define labeling function f as follows

Subcase 1: For $k \equiv 0(mod3)$

$$
\begin{aligned}
f(v_i^{(j)}) &= 0;\ \text{if } i \equiv 0(mod3)\\
&= 1;\ \text{if } i \equiv 1(mod3)\\
&= 2;\ \text{if } i \equiv 2(mod3),\ j \equiv 1,2(mod3) \text{ and } i \neq n
\end{aligned}
$$

$$
\begin{aligned}
f(v_n^{(3)}) &= 1; \\
f(v_n^{(j)}) &= 0; \text{ if } j \equiv 0(mod3) \text{ and } j \neq 3 \\
f(c_j) &= 2; \text{ if } j \equiv 1,2(mod3) \text{ and } j \neq 1,2 \\
&= 0; \text{ if } j \equiv 0(mod3) \text{ and } j \neq 3 \\
f(c_1) &= f(c_2) = 0; \\
f(c_3) &= f(x_2) = 2; \\
f(x_j) &= 0; \text{ if } j \equiv 0,1(mod3) \\
&= 1; \text{ if } j \equiv 2(mod3) \text{ and } j \neq 2
\end{aligned}
$$

Subcase 2: For $k \equiv 1(mod3)$

$$
\begin{aligned}
f(v_i^{(j)}) &= 0; \text{ if } i \equiv 0(mod3) \\
&= 1; \text{ if } i \equiv 1(mod3) \\
&= 2; \text{ if } i \equiv 2(mod3),\ j \equiv 0,2(mod3) \text{ and } i \neq n \\
f(v_n^{(j)}) &= 0; \text{ if } j \equiv 1(mod3) \text{ and } j \neq 1 \\
f(v_n^{(1)}) &= 2; \\
f(c_j) &= 0; \text{ if } j \equiv 1(mod3) \\
f(c_j) &= 2; \text{ if } j \equiv 0,2(mod3) \\
f(x_j) &= 1; \text{ if } j \equiv 0(mod3) \\
f(x_j) &= 0; \text{ if } j \equiv 1,2(mod3)
\end{aligned}
$$

Subcase 3: For $k \equiv 2(mod3)$

$$
\begin{aligned}
f(v_i^{(j)}) &= 0; \text{ if } i \equiv 0(mod3) \\
&= 1; \text{ if } i \equiv 1(mod3) \\
&= 2; \text{ if } i \equiv 2(mod3),\ j \equiv 0,1(mod3) \text{ and } i \neq n \\
f(v_n^{(j)}) &= 0; \text{ if } j \equiv 2(mod3) \text{ and } j \neq 2 \\
f(v_n^{(2)}) &= 2; \\
f(c_j) &= 0; \text{ if } j \equiv 2(mod3) \\
f(c_j) &= 2; \text{ if } j \equiv 0,1(mod3) \text{ and } j \neq 1 \\
f(x_j) &= 1; \text{ if } j \equiv 1(mod3) \text{ and } j \neq 1 \\
f(x_j) &= 0; \text{ if } j \equiv 0,2(mod3) \\
f(c_1) &= f(x_1) = 0;
\end{aligned}
$$

The labeling pattern defined above covers all possible arrangement of vertices. In each case, the graph G under consideration satisfies the conditions $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $0 \leq i, j \leq 2$ as shown in TABLE 4.4(where $n = 3a + b$, $k = 3c + d$ and $a \in N \cup \{0\}$,$c \in N$). i.e. G admits 3-equitable labeling. □

b	***d***	**Vertex Condition**	**Edge Condition**
0	0	$v_f(0)+1 = v_f(1) = v_f(2) = \frac{k(n+2)}{3}$	$e_f(0) = e_f(1) = e_f(2) = \frac{k(n+3)-3}{3}$
	1	$v_f(0)+1 = v_f(1)+1 = v_f(2) = \frac{k(n+2)+1}{3}$	$e_f(0) = e_f(1) = e_f(2) = \frac{k(n+3)-3}{3}$
	2	$v_f(0) = v_f(1) = v_f(2) = \frac{k(n+2)-1}{3}$	$e_f(0) = e_f(1) = e_f(2) = \frac{k(n+3)-3}{3}$
1	0	$v_f(0)+1 = v_f(1) = v_f(2) = \frac{k(n+2)}{3}$	$e_f(0) = e_f(1) = e_f(2) = \frac{k(n+3)-3}{3}$
	1	$v_f(0)+1 = v_f(1) = v_f(2) = \frac{k(n+2)}{3}$	$e_f(0)+1 = e_f(1) = e_f(2)+1 = \frac{k(n+3)-1}{3}$
	2	$v_f(0)+1 = v_f(1) = v_f(2) = \frac{k(n+2)}{3}$	$e_f(0)+1 = e_f(1) = e_f(2) = \frac{k(n+3)-2}{3}$
2	0	$v_f(0)+1 = v_f(1) = v_f(2) = \frac{k(n+2)}{3}$	$e_f(0) = e_f(1) = e_f(2) = \frac{k(n+3)-3}{3}$
	1	$v_f(0) = v_f(1) = v_f(2) = \frac{k(n+2)-1}{3}$	$e_f(0)+1 = e_f(1) = e_f(2) = \frac{k(n+3)-2}{3}$
	2	$v_f(0) = v_f(1)+1 = v_f(2)+1 = \frac{k(n+2)+1}{3}$	$e_f(0) = e_f(1)+1 = e_f(2)+1 = \frac{k(n+3)-1}{3}$

TABLE 4.4

Illustration 4.3.8. Consider a graph $G = < K_{1,5}^{(1)} \blacktriangle K_{1,5}^{(2)} \blacktriangle K_{1,5}^{(3)} \blacktriangle K_{1,5}^{(4)} >$. Here $n = 5$ and $k = 4$. The corresponding 3-equitable labeling is as shown in FIGURE 4.4.

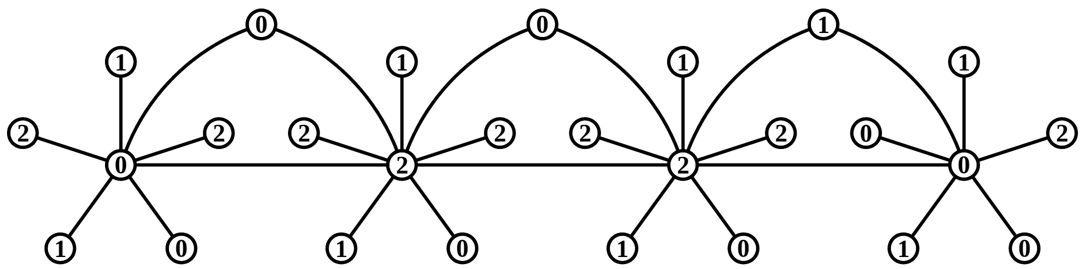

FIGURE 4.4

4.4 3-equitable Labeling of Some Shell Related Graphs

Theorem 4.4.1. Graph $< S_n(1) : S_n(2) >$ is 3-equitable.

Proof. Let $v_1^{(1)}, v_2^{(1)}, \dots, v_n^{(1)}$ be the vertices $S_n^{(1)}$ and $v_1^{(2)}, v_2^{(2)}, v_3^{(2)}, \dots, v_n^{(2)}$ be the vertices $S_n^{(2)}$. Let $v_1^{(1)}$ and $v_1^{(2)}$ be the apex vertices of $S_n^{(1)}$ and $S_n^{(2)}$ respectively. Let $G =< S_n^{(1)} : S_n^{(2)} >$. We define ternary vertex labeling $f : V(G) \to \{0,1,2\}$ as follows.

Case 1: For $n \equiv 0,5(mod6)$

$f(v_i^{(1)}) = 0$; if $i \equiv 1,4(mod6)$
$f(v_i^{(1)}) = 1$; if $i \equiv 0,5(mod6)$
$f(v_i^{(1)}) = 2$; if $i \equiv 2,3(mod6)$
$f(v_i^{(2)}) = 0$; if $i \equiv 0,3(mod6)$
$f(v_i^{(2)}) = 1$; if $i \equiv 4,5(mod6)$
$f(v_i^{(2)}) = 2$; if $i \equiv 1,2(mod6)$
$f(x) = 0$;

Case 2: For $n \equiv 1(mod6)$

$f(v_i^{(1)}) = 0$; if $i \equiv 1,4(mod6), i \neq n$
$f(v_i^{(1)}) = 1$; if $i \equiv 0,5(mod6)$
$f(v_i^{(1)}) = 2$; if $i \equiv 2,3(mod6)$
$f(v_n^{(1)}) = 1$;
$f(v_i^{(2)}) = 0$; if $i \equiv 0,3(mod6)$
$f(v_i^{(2)}) = 1$; if $i \equiv 4,5(mod6)$
$f(v_i^{(2)}) = 2$; if $i \equiv 1,2(mod6)$
$f(x) = 0$;

Case 3: For $n \equiv 2(mod6)$

$f(v_i^{(1)}) = 0$; if $i \equiv 1,4(mod6), i \neq n-1$
$f(v_i^{(1)}) = 1$; if $i \equiv 0,5(mod6)$
$f(v_i^{(1)}) = 2$; if $i \equiv 2,3(mod6), i \neq n$
$f(v_i^{(2)}) = 0$; if $i \equiv 0,3(mod6)$

$f(v_i^{(2)}) = 1$; if $i \equiv 4,5(mod6)$
$f(v_i^{(2)}) = 2$; if $i \equiv 1,2(mod6), i \neq n$
$f(v_{n-1}^{(1)}) = 1$;
$f(v_n^{(1)}) = f(v_n^{(2)}) = 0$;
$f(x) = 2$;

Case 4: For $n \equiv 3(mod6)$

$f(v_i^{(1)}) = 0$; if $i \equiv 1,4(mod6), i \neq n-2$
$f(v_i^{(1)}) = 1$; if $i \equiv 0,5(mod6)$
$f(v_i^{(1)}) = 2$; if $i \equiv 2,3(mod6), i \neq n-1, n$
$f(v_i^{(2)}) = 0$; if $i \equiv 0,3(mod6), i \neq n$
$f(v_i^{(2)}) = 1$; if $i \equiv 4,5(mod6)$
$f(v_i^{(2)}) = 2$; if $i \equiv 1,2(mod6), i \neq n-1, n-2$
$f(v_{n-2}^{(1)}) = f(v_n^{(2)}) = 1$;
$f(v_{n-1}^{(1)}) = f(v_{n-1}^{(2)}) = 2$;
$f(v_n^{(1)}) = f(v_{n-2}^{(2)}) = 0$;
$f(x) = 0$;

Case 5: For $n \equiv 4(mod6)$

$f(v_i^{(1)}) = 0$; if $i \equiv 1,4(mod6)$
$f(v_i^{(1)}) = 1$; if $i \equiv 0,5(mod6)$
$f(v_i^{(1)}) = 2$; if $i \equiv 2,3(mod6)$
$f(v_i^{(2)}) = 0$; if $i \equiv 0,3(mod6), i \neq n-1$
$f(v_i^{(2)}) = 1$; if $i \equiv 4,5(mod6)$
$f(v_i^{(2)}) = 2$; if $i \equiv 1,2(mod6), i \neq n-2$
$f(v_{n-2}^{(2)}) = f(v_{n-1}^{(2)}) = 1$;
$f(x) = 0$;

The labeling pattern defined above covers all possible arrangement of vertices. The graph G satisfies the conditions $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$, where $0 \leq i, j \leq 2$ as shown in TABLE 4.5(where $n = 6a + b$ and $a \in N \cup \{0\}$). i.e. G admits 3-equitable labeling. □

b	Vertex Condition	Edge Condition
0,3	$v_f(0) = v_f(1)+1 = v_f(2)+1 = \frac{2n+3}{3}$	$e_f(0)+1 = e_f(1) = e_f(2) = \frac{4n-3}{3}$
1,4	$v_f(0) = v_f(1) = v_f(2) = \frac{2n+1}{3}$	$e_f(0) = e_f(1) = e_f(2) = \frac{4n-4}{3}$
2,5	$v_f(0) = v_f(1)+1 = v_f(2) = \frac{2n+2}{3}$	$e_f(0)+1 = e_f(1)+1 = e_f(2) = \frac{4n-2}{3}$

TABLE 4.5

Illustration 4.4.2. Consider a graph $G = < S_6^{(1)} : S_6^{(2)} >$. Here $n = 6$. The 3-equitable labeling is as shown in FIGURE4.5.

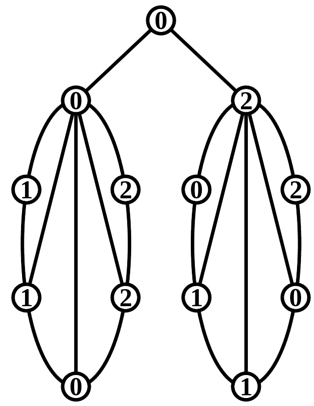

FIGURE 4.5

Theorem 4.4.3. Graph $< S_n^{(1)} : S_n^{(2)} : S_n^{(3)} : \ldots : S_n^{(k)} >$ is 3-equitable.

Proof. Let $S_n^{(j)}$ be the shells. Let $v_i^{(j)}$ be the vertices $S_n^{(j)}$ and $v_1^{(j)}$ be the apex vertices of $S_n^{(j)}$. Let $x_j (j \neq k)$ be the new vertices. Let $G = < S_n^{(1)} : S_n^{(2)} : S_n^{(3)} : \ldots : S_n^{(k)} >$. We define vertex labeling $f : V(G) \to \{0,1,2\}$ as follows.

Case 1: For $n \equiv 0(mod 6)$

Subcase 1: $k \equiv 0(mod 3)$

$$\left.\begin{array}{lcl} f(v_i^{(j)}) & = & 0;\text{ if } i \equiv 1,4(mod6) \\ f(v_i^{(j)}) & = & 1;\text{ if } i \equiv 0,5(mod6) \\ f(v_i^{(j)}) & = & 2;\text{ if } i \equiv 2,3(mod6) \\ f(x_j) & = & 0; \end{array}\right\} \text{for } j \equiv 1(mod3)$$

$$\left.\begin{array}{lcl} f(v_i^{(j)}) & = & 0;\text{ if } i\equiv 0,3(mod6),\ i\neq n\\ f(v_i^{(j)}) & = & 1;\text{ if } i\equiv 4,5(mod6)\\ f(v_i^{(j)}) & = & 2;\text{ if } i\equiv 1,2(mod6)\\ f(v_n^{(j)}) & = & 2;\text{ if } j\equiv 2(mod3)\\ f(v_n^{(j)}) & = & 1;\text{ if } j\equiv 0(mod3)\\ f(x_j) & = & 0; j\neq k \end{array}\right\}\text{ for } j\equiv 0,2(mod3)$$

Subcase 2: $k \equiv 1(mod3)$

For first $k-1$ copies of shells use the pattern of subcase 1 and for k^{th} copy define function as follow.

$$f(v_i^{(k)}) = 0;\text{ if } i\equiv 1,4(mod6)$$
$$f(v_i^{(k)}) = 1;\text{ if } i\equiv 0,5(mod6)$$
$$f(v_i^{(k)}) = 2;\text{ if } i\equiv 2,3(mod6)$$
$$f(x_{k-1}) = 0;$$

Subcase 3: $k \equiv 2(mod3)$

For first $k-2$ copies of shells use the pattern of subcase 1 and for $k-1$ and k^{th} copy define function as follow.

$$f(v_i^{(k-1)}) = 0;\text{ if } i\equiv 1,4(mod6)$$
$$f(v_i^{(k-1)}) = 1;\text{ if } i\equiv 0,5(mod6)$$
$$f(v_i^{(k-1)}) = 2;\text{ if } i\equiv 2,3(mod6)$$
$$f(v_i^{(k)}) = 0;\text{ if } i\equiv 0,3(mod6)$$
$$f(v_i^{(k)}) = 1;\text{ if } i\equiv 4,5(mod6)$$
$$f(v_i^{(k)}) = 2;\text{ if } i\equiv 1,2(mod6)$$
$$f(x_{k-2}) = f(x_{k-1}) = 0;$$

Case 2: For $n \equiv 1(mod6)$

Subcase 1: $k \equiv 0(mod3)$

$$\left.\begin{array}{lcl} f(v_i^{(j)}) & = & 0;\text{ if } i \equiv 1,4(mod6),\ i \neq n \\ f(v_i^{(j)}) & = & 1;\text{ if } i \equiv 0,2,3,5(mod6)\text{ and } j \equiv 1(mod3) \\ f(v_i^{(j)}) & = & 2;\text{ if } i \equiv 0,2,3,5(mod6)\text{ and } j \equiv 2(mod3) \\ f(v_n^{(j)}) & = & 1;\text{ if } j \equiv 1(mod3) \\ f(v_n^{(j)}) & = & 0;\text{ if } j \equiv 2(mod3) \\ f(x_j) & = & 1;\text{ if } j \equiv 1(mod3) \\ f(x_j) & = & 2;\text{ if } j \equiv 2(mod3) \end{array}\right\} \text{for } j \equiv 1,2(mod3)$$

$$\left.\begin{array}{lcl} f(v_i^{(j)}) & = & 0;\text{ if } i \equiv 0,3(mod6) \\ f(v_i^{(j)}) & = & 1;\text{ if } i \equiv 4,5(mod6) \\ f(v_i^{(j)}) & = & 2;\text{ if } i \equiv 1,2(mod6) \\ f(x_j) & = & 0; j \neq k \end{array}\right\} \text{for } j \equiv 0(mod3)$$

Subcase 2: $k \equiv 1(mod3)$

For first $k-1$ copies of shells use the pattern of subcase 1 and for k^{th} copy define function as follow.

$$\begin{array}{l} f(v_i^{(k)}) = 0;\text{ if } i \equiv 1,4(mod6) \\ f(v_i^{(k)}) = 1;\text{ if } i \equiv 0,5(mod6) \\ f(v_i^{(k)}) = 2;\text{ if } i \equiv 2,3(mod6) \\ f(x_{k-1}) = 2; \end{array}$$

Subcase 3: $k \equiv 2(mod3)$

For first $k-2$ copies of shells use the pattern of subcase 1 and for $k-1$ and k^{th} copy define function as follow.

$$\left.\begin{array}{lcl} f(v_i^{(j)}) & = & 0;\ \text{if}\ i \equiv 1,4(mod6)\ \text{and}\ j \neq k,\ i \neq n \\ f(v_i^{(j)}) & = & 1;\ \text{if}\ i \equiv 0,5(mod6) \\ f(v_i^{(j)}) & = & 2;\ \text{if}\ i \equiv 2,3(mod6) \\ f(v_n^{(k)}) & = & 1; \\ f(x_{k-2}) & = & 0; \\ f(x_{k-1}) & = & 2; \end{array}\right\}\ \text{for}\ j = k-1,k$$

Case 3: For $n \equiv 2(mod6)$

Subcase 1: $k \equiv 0(mod3)$

$$\left.\begin{array}{lcl} f(v_i^{(j)}) & = & 0;\ \text{if}\ i \equiv 1,4(mod6) \\ f(v_i^{(j)}) & = & 1;\ \text{if}\ i \equiv 2,3(mod6) \\ f(v_i^{(j)}) & = & 2;\ \text{if}\ i \equiv 0,5(mod6) \\ f(x_j) & = & 2; \end{array}\right\}\ \text{for}\ j \equiv 1(mod3)$$

$$\left.\begin{array}{lcl} f(v_i^{(j)}) & = & 0;\ \text{if}\ i \equiv 1,4(mod6),\ i \neq n-1 \\ f(v_i^{(j)}) & = & 1;\ \text{if}\ i \equiv 0,5(mod6) \\ f(v_i^{(j)}) & = & 2;\ \text{if}\ i \equiv 2,3(mod6),\ i \neq n \\ f(v_{n-1}^{(j)}) & = & 1; \\ f(v_n^{(j)}) & = & 0; \\ f(x_j) & = & 2; \end{array}\right\}\ \text{for}\ j \equiv 2(mod3)$$

$$\left.\begin{array}{lcl} f(v_i^{(j)}) & = & 0;\ \text{if}\ i \equiv 0,5(mod6) \\ f(v_i^{(j)}) & = & 1;\ \text{if}\ i \equiv 2,3(mod6) \\ f(v_i^{(j)}) & = & 2;\ \text{if}\ i \equiv 1,4(mod6) \\ f(x_j) & = & 0; j \neq k \end{array}\right\}\ \text{for}\ j \equiv 0(mod3)$$

Subcase 2: $k \equiv 1(mod3)$

For first $k-1$ copies of shells use the pattern of subcase 1 and for k_{th} copy define function as follow.

$$f(v_i^{(k)}) = 0; \text{ if } i \equiv 1,4(mod6)$$
$$f(v_i^{(k)}) = 1; \text{ if } i \equiv 2,3(mod6)$$
$$f(v_i^{(k)}) = 2; \text{ if } i \equiv 0,5(mod6)$$
$$f(x_{k-1}) = 2;$$

Subcase 3: $k \equiv 2(mod3)$

For first $k-2$ copies of shells use the pattern of subcase 1 and for $k-1$ and k^{th} copy define function as follow.

$$\left.\begin{array}{lcl} f(v_i^{(j)}) & = & 0; \text{ if } i \equiv 1,4(mod6) \text{ and } j \neq k,\, i \neq 1 \\ f(v_i^{(j)}) & = & 1; \text{ if } i \equiv 2,3(mod6) \\ f(v_i^{(j)}) & = & 2; \text{ if } i \equiv 0,5(mod6) \\ f(v_1^{(k)}) & = & 2; \\ f(x_{k-2}) & = & 2; \\ f(x_{k-1}) & = & 0; \end{array}\right\} \text{for } j = k-1, k$$

Case 4: For $n \equiv 3(mod6)$

Subcase 1: $k \equiv 0(mod3)$

$$\left.\begin{array}{lcl} f(v_i^{(j)}) & = & 0; \text{ if } i \equiv 1,4(mod6) \\ f(v_i^{(j)}) & = & 1; \text{ if } i \equiv 0,2,3,5(mod6),\ j \equiv 2(mod3) \\ f(v_i^{(j)}) & = & 2; \text{ if } i \equiv 0,2,3,5(mod6),\ j \equiv 1(mod3) \\ f(x_j) & = & 1; \text{ if } j \equiv 1(mod3) \\ f(x_j) & = & 2; \text{ if } j \equiv 2(mod3) \end{array}\right\} \text{for } j \equiv 1,2(mod3)$$

$$\left.\begin{array}{lcl} f(v_i^{(j)}) & = & 0;\ \text{if } i \equiv 0,5(mod6) \\ f(v_i^{(j)}) & = & 1;\ \text{if } i \equiv 2,3(mod6),\ i \neq n-1 \\ f(v_i^{(j)}) & = & 2;\ \text{if } i \equiv 1,4(mod6),\ i \neq n-2 \\ f(v_{n-2}^{(j)}) & = & 0; \\ f(v_{n-1}^{(j)}) & = & 2; \\ f(x_j) & = & 0; j \neq k \end{array}\right\} \text{for } j \equiv 0(mod3)$$

Subcase 2: $k \equiv 1(mod3)$

For first $k-1$ copies of shells use the pattern of subcase 1 and for k^{th} copy define function as follow.

$$\begin{array}{l} f(v_i^{(k)}) = 0;\ \text{if } i \equiv 1,4(mod6),\ i \neq n-2 \\ f(v_i^{(k)}) = 1;\ \text{if } i \equiv 2,3(mod6),\ i \neq n-1 \\ f(v_i^{(k)}) = 2;\ \text{if } i \equiv 0,5(mod6) \\ f(v_{n-2}^{(k)}) = 2; \\ f(v_{n-1}^{(k)}) = 0; \\ f(x_{k-1}) = 0; \end{array}$$

Subcase 3: $k \equiv 2(mod3)$

For first $k-2$ copies of shells use the pattern of subcase 1 and for $k-1$ and k^{th} copy define function as follow.

$$\left.\begin{array}{lcl} f(v_i^{(j)}) & = & 0;\ \text{if } i \equiv 1,4(mod6),\ i \neq n-2,\ j \neq k-1 \\ f(v_i^{(j)}) & = & 1;\ \text{if } i \equiv 2,3(mod6),\ i \neq n-1 \\ f(v_i^{(j)}) & = & 2;\ \text{if } i \equiv 0,5(mod6) \\ f(v_{n-2}^{(k-1)}) & = & 2; \\ f(v_{n-1}^{(k-1)}) & = & 0; \\ f(v_{n-1}^{(k)}) & = & 2; \\ f(x_{k-2}) & = & f(x_{k-1}) = 0; \end{array}\right\} \text{for } j = k-1,k$$

Case 5: For $n \equiv 4(mod6)$

Subcase 1: $k \equiv 0(mod3)$

$$\left.\begin{array}{lcl} f(v_i^{(j)}) & = & 0;\text{ if } i \equiv 1,4(mod6) \\ f(v_i^{(j)}) & = & 1;\text{ if } i \equiv 0,2,3,5(mod6) \text{ and } j \equiv 2(mod3) \\ f(v_i^{(j)}) & = & 2;\text{ if } i \equiv 0,2,3,5(mod6) \text{ and } j \equiv 1(mod3) \\ f(x_j) & = & 2; \end{array}\right\} \text{for } j \equiv 1,2(mod3)$$

$$\left.\begin{array}{l} \left.\begin{array}{lcl} f(v_i^{(j)}) & = & 0;\text{ if } i \equiv 0,5(mod6) \\ f(v_i^{(j)}) & = & 1;\text{ if } i \equiv 2,3(mod6) \\ f(v_i^{(j)}) & = & 2;\text{ if } i \equiv 1,4(mod6) \end{array}\right\} \text{If } 1 \leq i \leq n-4 \\ f(v_{n-3}^{(j)}) = f(v_{n-2}^{(j)}) = f(v_{n-1}^{(j)}) = 1; \\ f(v_n^{(j)}) = 0; \\ f(x_j) = 2; j \neq k \end{array}\right\} \text{for } j \equiv 0(mod3)$$

Subcase 2: $k \equiv 1(mod3)$

For first $k-1$ copies of shells use the pattern of subcase 1 and for k^{th} copy define function as follow.

$$\begin{array}{l} f(v_i^{(k)}) = 0;\text{ if } i \equiv 1,4(mod6) \text{ and } i \neq n \\ f(v_i^{(k)}) = 1;\text{ if } i \equiv 2,3(mod6) \text{ and } i \neq n-1 \\ f(v_i^{(k)}) = 2;\text{ if } i \equiv 0,5(mod6) \\ f(v_{n-1}^{(k)}) = f(v_n^{(k)}) = 2; \\ f(x_{k-1}) = 2; \end{array}$$

Subcase 3: $k \equiv 2(mod3)$

For first $k-2$ copies of shells use the pattern of subcase 1 and for $k-1$ and k^{th} copy define function as follow.

$$\begin{array}{l} f(v_i^{(k-1)}) = 0;\text{ if } i \equiv 1,4(mod6) \text{ and } i \neq n \\ f(v_i^{(k-1)}) = 1;\text{ if } i \equiv 0,5(mod6) \\ f(v_i^{(k-1)}) = 2;\text{ if } i \equiv 2,3(mod6) \end{array}$$

$f(v_i^{(k)}) = 0$; if $i \equiv 1,4(mod6)$ and $i \neq n$
$f(v_i^{(k)}) = 1$; if $i \equiv 2,3(mod6)$ and $i \neq n-2$
$f(v_i^{(k)}) = 2$; if $i \equiv 0,5(mod6)$
$f(v_{n-2}^{(k)}) = f(x_{k-2}) = 2$;
$f(v_n^{(k)}) = f(v_n^{(k-1)}) = 1$;
$f(x_{k-1}) = 0$;

Case 6: For $n \equiv 5(mod6)$

Subcase 1: $k \equiv 0(mod3)$

$$\left.\begin{array}{lcl} f(v_i^{(j)}) & = & 0;\ \text{if } i \equiv 1,4(mod6) \\ f(v_i^{(j)}) & = & 1;\ \text{if } i \equiv 0,2,3,5(mod6),\ j \equiv 2(mod3) \\ f(v_i^{(j)}) & = & 2;\ \text{if } i \equiv 0,2,3,5(mod6),\ j \equiv 1(mod3) \\ f(x_j) & = & 2;\ \text{if } j \equiv 1(mod3) \\ f(x_j) & = & 0;\ \text{if } j \equiv 2(mod3) \end{array}\right\} \text{for } j \equiv 1,2(mod3)$$

If $1 \leq i \leq n-2$

$$\left.\begin{array}{lcl} f(v_i^{(j)}) & = & 0;\ \text{if } i \equiv 0,5(mod6) \\ f(v_i^{(j)}) & = & 1;\ \text{if } i \equiv 2,3(mod6) \\ f(v_i^{(j)}) & = & 2;\ \text{if } i \equiv 1,4(mod6) \\ f(v_{n-1}^{(j)}) & = & 1; \\ f(v_n^{(j)}) & = & 2; \\ f(x_j) & = & 0;\ j \neq k \end{array}\right\} \text{for } j \equiv 0(mod3)$$

Subcase 2: $k \equiv 1(mod3)$

For first $k-1$ copies of shells use the pattern of subcase 1 and for k^{th} copy define function as follow.

$f(v_i^{(k)}) = 0$; if $i \equiv 1,4(mod6)$
$f(v_i^{(k)}) = 1$; if $i \equiv 2,3(mod6)$
$f(v_i^{(k)}) = 2$; if $i \equiv 0,5(mod6)$
$f(x_{k-1}) = 2$;

Subcase 3: $k \equiv 2(mod3)$

For first $k-2$ copies of shells use the pattern of subcase 1 and for $k-1$ and k^{th} copy define function as follow.

$$f(v_i^{(k-1)}) = 0; \text{ if } i \equiv 1,4(mod6)$$
$$f(v_i^{(k-1)}) = 1; \text{ if } i \equiv 2,3(mod6)$$
$$f(v_i^{(k-1)}) = 2; \text{ if } i \equiv 0,5(mod6)$$
$$f(v_i^{(k)}) = 0; \text{ if } i \equiv 0,5(mod6)$$
$$f(v_i^{(k)}) = 1; \text{ if } i \equiv 2,3(mod6)$$
$$f(v_i^{(k)}) = 2; \text{ if } i \equiv 1,4(mod6)$$
$$f(x_{k-2}) = 2;$$
$$f(x_{k-1}) = 0;$$

The labeling pattern defined above covers all possible arrangement of vertices. The graph G satisfies the conditions $|v_f(i) - v_f(j)| \neq 1$ and $|e_f(i) - e_f(j)| \leq 1$, where $0 \leq i, j \leq 2$ as shown in TABLE 4.6(where $n = 6a+b$, $k = 3c+d$ and $a, c \in N \cup \{0\}$). i.e. G admits 3-equitable labeling. □

b	***d***	**Vertex Condition**	**Edge Condition**
0	0	$v_f(0)+1=v_f(1)=v_f(2)=\frac{k(n+1)}{3}$	$e_f(0)+1=e_f(1)=e_f(2)+1=\frac{k(2n-1)}{3}$
	1	$v_f(0)=v_f(1)=v_f(2)=\frac{k(n+1)-1}{3}$	$e_f(0)=e_f(1)=e_f(2)=\frac{k(2n-1)-2}{3}$
	2	$v_f(0)=v_f(1)+1=v_f(2)+1=\frac{k(n+1)+1}{3}$	$e_f(0)+1=e_f(1)=e_f(2)=\frac{k(2n-1)-1}{3}$
1	0	$v_f(0)+1=v_f(1)=v_f(2)=\frac{k(n+1)}{3}$	$e_f(0)+1=e_f(1)=e_f(2)+1=\frac{k(2n-1)}{3}$
	1	$v_f(0)+1=v_f(1)+1=v_f(2)=\frac{k(n+1)+1}{3}$	$e_f(0)=e_f(1)=e_f(2)+1=\frac{k(2n-1)-1}{3}$
	2	$v_f(0)=v_f(1)=v_f(2)=\frac{k(n+1)-1}{3}$	$e_f(0)=e_f(1)=e_f(2)=\frac{k(2n-1)-2}{3}$
2,5	0,1,2	$v_f(0)+1=v_f(1)=v_f(2)=\frac{k(n+1)}{3}$	$e_f(0)+1=e_f(1)=e_f(2)+1=\frac{k(2n-1)}{3}$
3	0	$v_f(0)+1=v_f(1)=v_f(2)=\frac{k(n+1)}{3}$	$e_f(0)+1=e_f(1)=e_f(2)+1=\frac{k(2n-1)}{3}$
	1	$v_f(0)=v_f(1)=v_f(2)=\frac{k(n+1)-1}{3}$	$e_f(0)=e_f(1)=e_f(2)=\frac{k(2n-1)-2}{3}$
	2	$v_f(0)=v_f(1)+1=v_f(2)+1=\frac{k(n+1)+1}{3}$	$e_f(0)=e_f(1)+1=e_f(2)=\frac{k(2n-1)-1}{3}$
4	$0_{(n=4)}$	$v_f(0)=v_f(1)=v_f(2)+1=\frac{5k}{3}$	$e_f(0)=e_f(1)+1=e_f(2)+1=\frac{7k}{3}$
	$0_{(n\neq 4)}$	$v_f(0)=v_f(1)=v_f(2)+1=\frac{k(n+1)}{3}$	$e_f(0)+1=e_f(1)=e_f(2)+1=\frac{k(2n-1)}{3}$
	1	$v_f(0)+1=v_f(1)+1=v_f(2)=\frac{k(n+1)+1}{3}$	$e_f(0)+1=e_f(1)=e_f(2)=\frac{k(2n-1)-1}{3}$
	2	$v_f(0)=v_f(1)=v_f(2)=\frac{k(n+1)-1}{3}$	$e_f(0)=e_f(1)=e_f(2)=\frac{k(2n-1)-2}{3}$

TABLE 4.6

Illustration 4.4.4. Consider a graph $G=<S_4^{(1)}:S_4^{(2)}:S_4^{(3)}>$. Here $n=4$. The 3-equitable labeling is as shown in FIGURE 4.6.

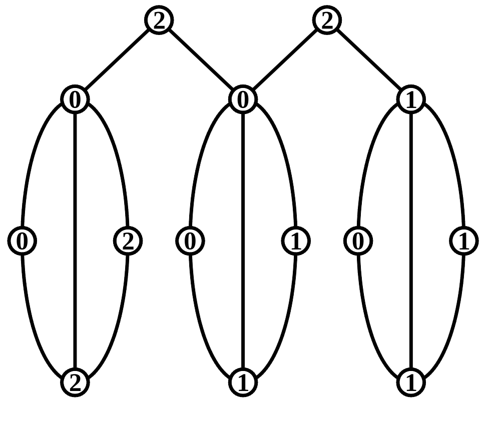

FIGURE 4.6

4.5 3-equitable Labeling of Some Wheel Related Graphs

Theorem 4.5.1. Graph $< W_n^{(1)} : W_n^{(2)} >$ is 3-equitable.

Proof. Let $v_1^{(1)}, v_2^{(1)}, v_3^{(1)}, \dots v_n^{(1)}$ be the rim vertices $W_n^{(1)}$ and $v_1^{(2)}, v_2^{(2)}, v_3^{(2)}, \dots v_n^{(2)}$ be the rim vertices $W_n^{(2)}$. Let c_1 and c_2 be the apex vertices of $W_n^{(1)}$ and $W_n^{(2)}$ respectively and they are adjacent to a new common vertex x. Let $G =< W_n^{(1)} : W_n^{(2)} >$. To define vertex labeling $f : V(G) \to \{0,1,2\}$ we consider the following cases.

Case 1: $n \equiv 0(mod6)$

In this case we define labeling f as

$$\left.\begin{aligned} f(v_i^{(1)}) &= 0;\ \text{if } i \equiv 1,4(mod6) \\ &= 1;\ \text{if } i \equiv 2,3(mod6) \\ &= 2;\ \text{if } i \equiv 0,5(mod6) \end{aligned}\right\} \text{ for } 1 \leq i \leq n$$

$$f(c_1) = 2;$$

$$\left.\begin{aligned} f(v_i^{(2)}) &= 0;\ \text{if } i \equiv 1,4(mod6) \\ &= 2;\ \text{if } i \equiv 2,3(mod6) \\ &= 1;\ \text{if } i \equiv 0,5(mod6) \end{aligned}\right\} \text{ for } 1 \leq i \leq n-3$$

$$f(v_i^{(2)}) = 1;\ i \geq n-2$$
$$f(c_2) = 0;$$
$$f(x) = 0;$$

Case 2: $n \equiv 1(mod6)$

In this case we define labeling f as

$$\left.\begin{aligned} f(v_i^{(1)}) &= 0;\ \text{if } i \equiv 1,4(mod6) \\ &= 1;\ \text{if } i \equiv 2,3(mod6) \\ &= 2;\ \text{if } i \equiv 0,5(mod6) \end{aligned}\right\} \text{ for } 1 \leq i \leq n$$

$$f(c_1) = 2;$$

$$\left.\begin{aligned} f(v_i^{(2)}) &= 0;\ \text{if } i \equiv 1,4(mod6) \\ &= 1;\ \text{if } i \equiv 2,3(mod6) \\ &= 2;\ \text{if } i \equiv 0,5(mod6) \end{aligned}\right\}\ \text{for } 1 \le i \le n$$

$$f(c_2) = 2;$$
$$f(x) = 1;$$

Case 3: $n \equiv 2(mod6)$

In this case we define labeling f as

$$\left.\begin{aligned} f(v_i^{(1)}) &= 0;\ \text{if } i \equiv 1,4(mod6) \\ &= 1;\ \text{if } i \equiv 0,5(mod6) \\ &= 2;\ \text{if } i \equiv 2,3(mod6) \end{aligned}\right\}\ \text{for } 1 \le i \le n-2$$

$$f(v_i^{(1)}) = 1;\ i \ge n-1$$
$$f(c_1) = 0;$$

$$\left.\begin{aligned} f(v_i^{(2)}) &= 0;\ \text{if } i \equiv 1,4(mod6) \\ &= 1;\ \text{if } i \equiv 0,5(mod6) \\ &= 2;\ \text{if } i \equiv 2,3(mod6) \end{aligned}\right\}\ \text{for } 1 \le i \le n-2$$

$$f(v_i^{(2)}) = 2;\ i \ge n-1$$
$$f(c_2) = 0;$$
$$f(x) = 1;$$

Case 4: $n \equiv 3(mod6)$

Subcase 1: $n \neq 3$

$$\left.\begin{aligned} f(v_i^{(1)}) &= 0;\ \text{if } i \equiv 1,4(mod6) \\ &= 1;\ \text{if } i \equiv 0,5(mod6) \\ &= 2;\ \text{if } i \equiv 2,3(mod6) \end{aligned}\right\}\ \text{for } 1 \le i \le n$$

$$f(c_1) = 0;$$

$$\left.\begin{aligned} f(v_i^{(2)}) &= 0;\ \text{if } i \equiv 1,4(mod6) \\ &= 1;\text{if } i \equiv 2,3(mod6) \\ &= 2;\ \text{if } i \equiv 0,5(mod6) \end{aligned}\right\} \text{for } 1 \leq i \leq n-3$$

$$f(v_i^{(2)}) = 1;\ i \geq n-2$$
$$f(c_2) = 0;$$
$$f(x) = 2;$$

Subcase 2: $n = 3$

$$f(v_1^{(1)}) = f(v_1^{(2)}) = f(c_2) = 0;$$
$$f(v_2^{(1)}) = f(v_3^{(1)}) = f(c_1) = 1;$$
$$f(v_2^{(2)}) = f(v_3^{(2)}) = f(x) = 2;$$

Case 5: $n \equiv 4(mod6)$

In this case we define labeling f as

$$\left.\begin{aligned} f(v_i^{(1)}) &= 0;\ \text{if } i \equiv 1,4(mod6) \\ &= 1;\ \text{if } i \equiv 0,5(mod6) \\ &= 2;\ \text{if } i \equiv 2,3(mod6) \end{aligned}\right\} \text{for } 1 \leq i \leq n-3$$

$$f(v_i^{(1)}) = 1;\ i = n-2, n-1$$
$$f(v_i^{(1)}) = 0;\ i = n$$
$$f(c_1) = 2;$$

$$\left.\begin{aligned} f(v_i^{(2)}) &= 0;\ \text{if } i \equiv 1,4(mod6) \\ &= 1;\ \text{if } i \equiv 0,5(mod6) \\ &= 2;\ \text{if } i \equiv 2,3(mod6) \end{aligned}\right\} \text{for } 1 \leq i \leq n$$

$$f(c_2) = 2;$$
$$f(x) = 1;$$

Case 6: $n \equiv 5(mod6)$

In this case we define labeling f as

$$\left.\begin{aligned} f(v_i^{(1)}) &= 0;\ \text{if } i \equiv 1,4(mod6) \\ &= 1;\ \text{if } i \equiv 2,3(mod6) \\ &= 2;\ \text{if } i \equiv 0,5(mod6) \end{aligned}\right\} \text{for } 1 \leq i \leq n-5$$

$$\begin{aligned} f(v_i^{(1)}) &= 1;\ i = n-4, n-3 \\ f(v_i^{(1)}) &= 2;\ i = n-2, n \\ f(v_i^{(1)}) &= 0;\ i = n-1 \\ f(c_1) &= 2; \end{aligned}$$

$$\left.\begin{aligned} f(v_i^{(2)}) &= 0;\ \text{if } i \equiv 1,4(mod6) \\ &= 1;\ \text{if } i \equiv 0,5(mod6) \\ &= 2;\ \text{if } i \equiv 2,3(mod6) \end{aligned}\right\} \text{for } 1 \leq i \leq n-5$$

$$\begin{aligned} f(v_i^{(2)}) &= 0;\ i = n-4, n-1 \\ f(v_i^{(2)}) &= 1;\ i = n-3, n-2 \\ f(v_i^{(2)}) &= 2;\ i = n \\ f(c_2) &= 0; \\ f(x) &= 0; \end{aligned}$$

The labeling pattern defined above covers all the possible arrangement of vertices and in each case the resulting labeling satisfies the conditions $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $0 \leq i, j \leq 2$ as shown in TABLE 4.7(where $n = 6a + b$ and $a \in N \cup \{0\}$). i.e. G admits 3-equitable labeling. □

b	**Vertex Condition**	**Edge Condition**
0	$v_f(0) = v_f(1) = v_f(2) = \frac{2n+3}{3}$	$e_f(0) = e_f(1) = e_f(2) + 1 = \frac{4n+3}{3}$
1,4	$v_f(0) = v_f(1) + 1 = v_f(2) = \frac{2n+4}{3}$	$e_f(0) = e_f(1) = e_f(2) = \frac{4n+2}{3}$
2	$v_f(0)+1 = v_f(1) = v_f(2)+1 = \frac{2n+5}{3}$	$e_f(0) + 1 = e_f(1) = e_f(2) + 1 = \frac{4n+4}{3}$
$3_{(n=3)}$	$v_f(0) = v_f(1) = v_f(2) = 3$	$e_f(0) = e_f(1) + 1 = e_f(2) = 5$
$3_{(n\neq 3)}$	$v_f(0) = v_f(1) = v_f(2) = \frac{2n+3}{3}$	$e_f(0) + 1 = e_f(1) = e_f(2) = \frac{4n+3}{3}$
5	$v_f(0) = v_f(1)+1 = v_f(2)+1 = \frac{2n+5}{3}$	$e_f(0) + 1 = e_f(1) = e_f(2) + 1 = \frac{4n+4}{3}$

TABLE 4.7

Illustration 4.5.2. Consider a graph $G =< W_5^{(1)} : W_5^{(2)} >$ Here $n = 5$ i.e $n \equiv 5(mod6)$. The corresponding 3-equitable labeling is shown in FIGURE 4.7. It is the case related to case -6

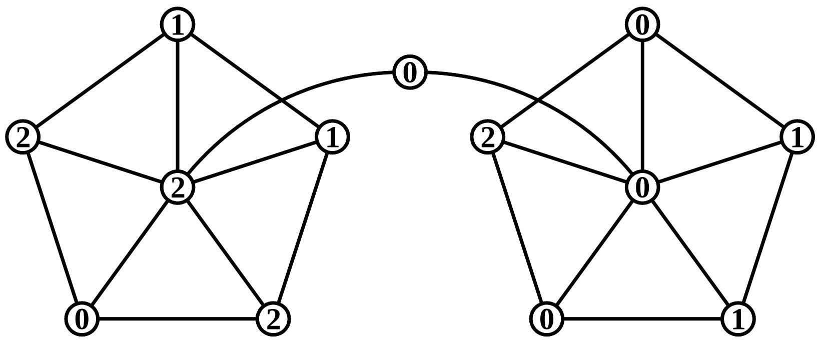

FIGURE 4.7

Theorem 4.5.3. Graph $< W_n^{(1)} : W_n^{(2)} : W_n^{(3)} : \ldots : W_n^{(k)} >$ is 3-equitable.

Proof. Let $W_n^{(j)}$ be k copies of wheel W_n, $v_i^{(j)}$ be the rim vertices of $W_n^{(j)}$ where $i = 1,2,\ldots n$ and $j = 1,2,\ldots k$. Let c_j be the apex vertex of $W_n^{(j)}$. Consider $G =< W_n^{(1)} : W_n^{(2)} : W_n^{(3)} : \ldots : W_n^{(k)} >$ and vertices $x_1, x_2, \ldots x_{k-1}$ as stated in Theorem 2.3. To define vertex labeling $f : V(G) \to \{0,1,2\}$ we consider following cases.

Case 1: For $n \equiv 0(mod6)$

In this case we define labeling function f as follows

Subcase 1: For $k \equiv 0(mod3)$

For $j \equiv 1,2(mod3)$

$$\left.\begin{array}{rcl} f(v_i^{(j)}) &=& 0;\text{ if } i \equiv 1,4(mod6) \\ &=& 1;\text{ if } i \equiv 0,5(mod6) \\ &=& 2;\text{ if } i \equiv 2,3(mod6) \end{array}\right\} \text{for } i \leq n-3$$

$$\begin{aligned} f(v_i^{(j)}) &= 1;\text{ if } i \geq n-2 \\ f(c_j) &= 0; \\ f(x_j) &= 2;\text{ if } j \equiv 1(mod3) \\ &= 0;\text{ if } j \equiv 2(mod3) \end{aligned}$$

For $j \equiv 0(mod3)$

$$\left.\begin{array}{rcl} f(v_i^{(j)}) &=& 0;\text{ if } i \equiv 1,4(mod6) \\ &=& 1;\text{ if } i \equiv 0,5(mod6) \\ &=& 2;\text{ if } i \equiv 2,3(mod6) \end{array}\right\} \text{for } i \leq n$$

$$\begin{aligned} f(c_j) &= 2; \\ f(x_j) &= 0;\ j \neq k \end{aligned}$$

Subcase 2: For $k \equiv 1(mod3)$

$$\left.\begin{array}{rcl} f(v_i^{(1)}) &=& 0;\text{ if } i \equiv 1,4(mod6) \\ &=& 1;\text{ if } i \equiv 0,5(mod6) \\ &=& 2;\text{ if } i \equiv 2,3(mod6) \end{array}\right\} \text{for } i \leq n$$

$$\begin{aligned} f(c_1) &= 2; \\ f(x_1) &= 0; \end{aligned}$$

For remaining vertices take $j = k-1$ and label them as in subcase 1.

Subcase 3: For $k \equiv 2(mod3)$

$$\left.\begin{aligned} f(v_i^{(1)}) &= 0;\ \text{if } i \equiv 1,4(mod6) \\ &= 1;\ \text{if } i \equiv 0,5(mod6) \\ &= 2;\ \text{if } i \equiv 2,3(mod6) \end{aligned}\right\} \text{for } i \leq n$$

$f(c_1) = 0;$
$f(x_1) = 2;$

$$\left.\begin{aligned} f(v_i^{(2)}) &= 0;\ \text{if } i \equiv 1,4(mod6) \\ &= 1;\ \text{if } i \equiv 0,5(mod6) \\ &= 2;\ \text{if } i \equiv 2,3(mod6) \end{aligned}\right\} \text{for } i \leq n-3$$

$f(v_i^{(2)}) = 1;$ if $i \geq n-2$
$f(c_2) = 0;$
$f(x_2) = 0;$

For remaining vertices take $j = k-2$ and label them as in subcase 1.

Case 2: For $n \equiv 1(mod6)$

In this case we define labeling function f as follows

Subcase 1: For $k \equiv 0(mod3)$

For $1 \leq j \leq k$

$$\left.\begin{aligned} f(v_i^{(j)}) &= 0;\ \text{if } i \equiv 1,4(mod6) \\ &= 1;\ \text{if } i \equiv 0,5(mod6) \\ &= 2;\ \text{if } i \equiv 2,3(mod6) \end{aligned}\right\} \text{for } i \leq n-1$$

$f(v_n^{(j)}) = 0;$ if $j \equiv 1(mod3)$
$f(v_n^{(j)}) = 1;$ if $j \equiv 0,2(mod3)$
$f(c_j) = 2;$ if $j \equiv 1(mod3)$
$f(c_j) = 0;$ if $j \equiv 0,2(mod3)$
$f(x_j) = 1;$ if $j \equiv 1(mod3)$
$= 2;$ if $j \equiv 0,2(mod3),\ j \neq k$

Subcase 2: For $k \equiv 1(mod3)$

$$\left.\begin{array}{rcl} f(v_i^{(1)}) & = & 0;\text{ if } i \equiv 1,4(mod6) \\ & = & 1;\text{ if } i \equiv 0,5(mod6) \\ & = & 2;\text{ if } i \equiv 2,3(mod6) \end{array}\right\} \text{for } i \leq n-1$$

$$f(v_n^{(1)}) = 1;$$
$$f(c_1) = 2;$$
$$f(x_1) = 0;$$

For remaining vertices take $j = k-1$ and label them as in subcase 1

Subcase 3: For $k \equiv 2(mod3)$

For $j = 1,2$

$$\left.\begin{array}{rcl} f(v_i^{(j)}) & = & 0;\text{ if } i \equiv 1,4(mod6) \\ & = & 1;\text{ if } i \equiv 0,5(mod6) \\ & = & 2;\text{ if } i \equiv 2,3(mod6) \end{array}\right\} \text{for } i \leq n-1$$

$$f(v_n^{(j)}) = 1;$$
$$f(c_1) = 0;$$
$$f(c_2) = 2;$$
$$f(x_1) = 2;$$
$$f(x_2) = 0;$$

For remaining vertices take $j = k-2$ and label them as in subcase 1.

Case 3: For $n \equiv 2(mod6)$

Subcase 1: For $k \equiv 0(mod3)$

For $j \equiv 1,2(mod3)$

$$\left.\begin{aligned} f(v_i^{(j)}) &= 0;\ \text{if } i \equiv 1,4(mod6) \\ &= 1;\ \text{if } i \equiv 0,5(mod6) \\ &= 2;\ \text{if } i \equiv 2,3(mod6) \end{aligned}\right\} \text{for } i \leq n-4$$

$$f(v_{n-3}^{(j)}) = 2;$$
$$f(v_i^{(j)}) = 1;\ \text{if } i \geq n-2$$
$$f(c_j) = 0;\ \text{if } j \equiv 1(mod3)$$
$$f(c_j) = 2;\ \text{if } j \equiv 2(mod3)$$
$$f(x_j) = 0;$$

For $j \equiv 0(mod3)$

$$\left.\begin{aligned} f(v_i^{(j)}) &= 0;\ \text{if } i \equiv 1,4(mod6) \\ &= 1;\ \text{if } i \equiv 2,3(mod6) \\ &= 2;\ \text{if } i \equiv 0,5(mod6) \end{aligned}\right\} \text{for } i \leq n-2$$

$$f(v_i^{(j)}) = 1;\ \text{if } i \geq n-1$$
$$f(c_j) = 2;$$
$$f(x_j) = 0;\ j \neq k$$

Subcase 2: For $k \equiv 1(mod3)$

$$\left.\begin{aligned} f(v_i^{(1)}) &= 0;\ \text{if } i \equiv 1,4(mod6) \\ &= 1;\ \text{if } i \equiv 0,5(mod6) \\ &= 2;\ \text{if } i \equiv 2,3(mod6) \end{aligned}\right\} \text{for } i \leq n-2$$

$$f(v_{n-1}^{(1)}) = 2;$$
$$f(v_n^{(1)}) = f(c_1) = 0;$$
$$f(x_1) = 1;$$

For remaining vertices take $j = k-1$ and label them as in subcase 1.

Subcase 3: For $k \equiv 2(mod3)$

For $j = 1,2$

$$\left.\begin{array}{rcl} f(v_i^{(j)}) & = & 0;\ \text{if } i \equiv 1,4(mod6) \\ & = & 1;\ \text{if } i \equiv 0,5(mod6) \\ & = & 2;\ \text{if } i \equiv 2,3(mod6) \end{array}\right\} \text{ for } i \leq n-4$$

$f(v_{n-3}^{(j)}) = 2;$

$f(v_i^{(j)}) = 1;$ if $i \geq n-2$

$f(c_j) = 0;$

$f(x_1) = 1;$

$f(x_2) = 0;$

For remaining vertices take $j = k-2$ and label them as in subcase 1.

Case 4: For $n \equiv 3(mod6)$

In this case we define labeling function f as follows

Subcase 1: For $k \equiv 0(mod3)$

$$\left.\begin{array}{rcl} f(v_i^{(j)}) & = & 0;\ \text{if } i \equiv 1,4(mod6) \\ & = & 1;\ \text{if } i \equiv 0,5(mod6) \\ & = & 2;\ \text{if } i \equiv 2,3(mod6) \end{array}\right\} \text{ for } i \leq n-3$$

If $j \equiv 1(mod3)$

$f(v_i^{(j)}) = 1;$ if $i \geq n-2$

$f(c_j) = 0;$

$f(x_j) = 1;$

If $j \equiv 2(mod3)$

$f(v_{n-2}^{(j)}) = f(c_j) = 0;$

$f(v_{n-1}^{(j)}) = f(x_j) = 2;$

$f(v_n^{(j)}) = 1;$

If $j \equiv 0(mod3)$

$f(v_i^{(j)}) = 0; \, if \, j = n-1, n-2$

$f(v_n^{(j)}) = f(c_j) = 2;$

$f(x_j) = 2; \; j \neq k$

Subcase 2: For $k \equiv 1(mod3)$

$$\left.\begin{aligned} f(v_i^{(1)}) &= 0; \text{ if } i \equiv 1,4(mod6) \\ &= 1; \text{ if } i \equiv 2,3(mod6) \\ &= 2; \text{ if } i \equiv 0,5(mod6) \end{aligned}\right\} \text{ for } i \leq n-3$$

$f(v_i^{(1)}) = 2; \text{ if } i \geq n-2$

$f(c_1) = 0;$

$f(x_1) = 1;$

For remaining vertices take $j = k-1$ and label them as in subcase 1.

Subcase 3: For $k \equiv 2(mod3)$

For $j = 1,2$

$$\left.\begin{aligned} f(v_i^{(j)}) &= 0; \text{ if } i \equiv 1,4(mod6) \\ &= 1; \text{ if } i \equiv 0,5(mod6) \\ &= 2; \text{ if } i \equiv 2,3(mod6) \end{aligned}\right\} \text{ for } i \leq n-3$$

$f(v_i^{(1)}) = 1; \text{ if } i = n-1, n-2$

$f(v_n^{(1)}) = 0;$

$f(v_i^{(2)}) = 2; \text{ if } i \geq n-2$

$f(c_j) = 0;$

$f(x_1) = 1;$

$f(x_2) = 2;$

For $n = 3$ label rim vertices of $W_n^{(1)}$ by $0,1,0$ and apex vertex by 1.

For remaining vertices take $j = k-2$ and label them as in subcase 1.

Case 5: For $n \equiv 4(mod6)$

In this case we define labeling function f as follows

Subcase 1: For $k \equiv 0(mod3)$

For $j \equiv 0,1,2(mod3)$

$$\left.\begin{aligned} f(v_i^{(j)}) &= 0;\ \text{if } i \equiv 1,4(mod6) \\ &= 1;\ \text{if } i \equiv 0,5(mod6) \\ &= 2;\ \text{if } i \equiv 2,3(mod6) \end{aligned}\right\} \text{for } i \leq n-4$$

$$f(v_{n-3}^{(j)}) = 0;\ \text{if } j \equiv 0,1(mod3)$$
$$f(v_{n-3}^{(j)}) = 2;\ \text{if } j \equiv 2(mod3)$$
$$f(v_i^{(j)}) = 1;\ \text{if } j \equiv 1,2(mod3),\ i \geq n-2$$
$$f(v_i^{(j)}) = 2;\ \text{if } j \equiv 0(mod3),\ i \geq n-2$$
$$f(c_j) = 2;\ j \equiv 1,2(mod3)$$
$$f(c_j) = 0;\ j \equiv 0(mod3)$$
$$f(x_j) = 0;\ j \neq k$$

Subcase 2: For $k \equiv 1(mod3)$

$$\left.\begin{aligned} f(v_i^{(1)}) &= 0;\ \text{if } i \equiv 1,4(mod6) \\ &= 1;\ \text{if } i \equiv 0,5(mod6) \\ &= 2;\ \text{if } i \equiv 2,3(mod6) \end{aligned}\right\} \text{for } i \leq n$$

$$f(c_1) = 0;$$
$$f(x_1) = 1;$$

For remaining vertices take $j = k-1$ and label them as in subcase 1.

Subcase 3: For $k \equiv 2(mod3)$

$$\left.\begin{array}{rcl} f(v_i^{(1)}) & = & 0;\ \text{if } i \equiv 1,4(mod6) \\ & = & 1;\ \text{if } i \equiv 2,3(mod6) \\ & = & 2;\ \text{if } i \equiv 0,5(mod6) \\ f(v_i^{(2)}) & = & 0;\ \text{if } i \equiv 1,4(mod6) \\ & = & 1;\ \text{if } i \equiv 0,5(mod6) \\ & = & 2;\ \text{if } i \equiv 2,3(mod6) \end{array}\right\} \text{for } i \leq n$$

$$\begin{aligned} f(c_1) &= 2; \\ f(c_2) &= 0; \\ f(x_1) &= 1; \\ f(x_2) &= 2; \end{aligned}$$

For remaining vertices take $j = k-2$ and label them as in subcase 1.

Case 6: For $n \equiv 5(mod6)$

In this case we define labeling function f as follows

Subcase 1: For $k \equiv 0(mod3)$

For $j \equiv 1,2(mod3)$

$$\left.\begin{array}{rcl} f(v_i^{(j)}) & = & 0;\ \text{if } i \equiv 1,4(mod6) \\ & = & 1;\ \text{if } i \equiv 2,3(mod6) \\ & = & 2;\ \text{if } i \equiv 0,5(mod6) \end{array}\right\} \text{for } i \leq n-2$$

$$\begin{aligned} f(v_{n-1}^{(j)}) &= 1; \\ f(v_n^{(j)}) &= 2;\ \text{if } j \equiv 1(mod3) \\ f(v_n^{(j)}) &= 0;\ \text{if } j \equiv 2(mod3) \\ f(c_j) &= 2;\ \text{if } j \equiv 1(mod3) \\ f(c_j) &= 0;\ \text{if } j \equiv 2(mod3) \\ f(x_j) &= 1;\ \text{if } j \equiv 1(mod3) \\ f(x_j) &= 2;\ \text{if } j \equiv 2(mod3) \end{aligned}$$

For $j \equiv 0(mod3)$

$$\left.\begin{array}{rcl} f(v_i^{(j)}) & = & 0;\ \text{if } i \equiv 1,4(mod6) \\ & = & 1;\ \text{if } i \equiv 0,5(mod6) \\ & = & 2;\ \text{if } i \equiv 2,3(mod6) \end{array}\right\} \text{for } i \leq n-1$$

$$f(v_n^{(j)}) = 2;$$
$$f(c_j) = 0;$$
$$f(x_j) = 2;\ j \neq k$$

Subcase 2: For $k \equiv 1(mod3)$

$$\left.\begin{array}{rcl} f(v_i^{(1)}) & = & 0;\ \text{if } i \equiv 1,4(mod6) \\ & = & 1;\ \text{if } i \equiv 0,5(mod6) \\ & = & 2;\ \text{if } i \equiv 2,3(mod6) \end{array}\right\} \text{for } i \leq n-2$$

$$f(v_i^{(1)}) = 1;\ \text{if } i \geq n-1$$
$$f(c_1) = 0;$$
$$f(x_1) = 2;$$

For remaining vertices take $j = k-1$ and label them as in subcase 1.

Subcase 3: For $k \equiv 2(mod3)$

For $j = 1,2$

$$\left.\begin{array}{rcl} f(v_i^{(j)}) & = & 0;\ \text{if } i \equiv 1,4(mod6) \\ & = & 1;\ \text{if } i \equiv 0,5(mod6) \\ & = & 2;\ \text{if } i \equiv 2,3(mod6) \end{array}\right\} \text{for } i \leq n-2$$

$$f(v_i^{(j)}) = 1;\ i \geq n-1$$
$$f(c_1) = 0;$$
$$f(c_2) = 2;$$
$$f(x_j) = 0;$$

For remaining vertices take $j = k-2$ and label them as in subcase 1.

The labeling pattern defined above covers all possible arrangement of vertices. In each case, the graph G under consideration satisfies the conditions $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $0 \leq i, j \leq 2$ as shown in TABLE 4.8(where $n = 6a + b$, $k = 3c + d$ and $a \in N \cup \{0\}$,$c \in N$). i.e. G admits 3-equitable labeling. $\square$

b	***d***	**Vertex Condition**	**Edge Condition**
0	0	$v_f(0)+1 = v_f(1) = v_f(2) = \frac{k(n+2)}{3}$	$e_f(0)+1 = e_f(1) = e_f(2)+1 = \frac{2k(n+1)}{3}$
	1	$v_f(0)+1 = v_f(1)+1 = v_f(2) = \frac{k(n+2)+1}{3}$	$e_f(0) = e_f(1) = e_f(2) = \frac{2k(n+1)-2}{3}$
	2	$v_f(0) = v_f(1) = v_f(2) = \frac{k(n+2)-1}{3}$	$e_f(0) = e_f(1) = e_f(2)+1 = \frac{2k(n+1)-1}{3}$
1	0	$v_f(0) = v_f(1) = v_f(2)+1 = \frac{k(n+2)}{3}$	$e_f(0)+1 = e_f(1) = e_f(2)+1 = \frac{2k(n+1)}{3}$
	1	$v_f(0) = v_f(1) = v_f(2)+1 = \frac{k(n+2)}{3}$	$e_f(0)+1 = e_f(1) = e_f(2) = \frac{2k(n+1)-1}{3}$
	2	$v_f(0) = v_f(1) = v_f(2)+1 = \frac{k(n+2)}{3}$	$e_f(0) = e_f(1) = e_f(2) = \frac{2k(n+1)-2}{3}$
2	0	$v_f(0)+1 = v_f(1) = v_f(2) = \frac{k(n+2)}{3}$	$e_f(0)+1 = e_f(1) = e_f(2)+1 = \frac{2k(n+1)}{3}$
	1	$v_f(0) = v_f(1) = v_f(2) = \frac{k(n+2)-1}{3}$	$e_f(0)+1 = e_f(1) = e_f(2)+1 = \frac{2k(n+1)}{3}$
	2	$v_f(0)+1 = v_f(1) = v_f(2)+1 = \frac{k(n+2)+1}{3}$	$e_f(0)+1 = e_f(1) = e_f(2)+1 = \frac{2k(n+1)}{3}$
3	0	$v_f(0) = v_f(1) = v_f(2)+1 = \frac{k(n+2)}{3}$	$e_f(0)+1 = e_f(1) = e_f(2)+1 = \frac{2k(n+1)}{3}$
	1	$v_f(0)+1 = v_f(1)+1 = v_f(2) = \frac{k(n+2)+1}{3}$	$e_f(0) = e_f(1) = e_f(2) = \frac{2k(n+1)-2}{3}$
	2	$v_f(0) = v_f(1) = v_f(2) = \frac{k(n+2)-1}{3}$	$e_f(0) = e_f(1)+1 = e_f(2) = \frac{2k(n+1)-1}{3}$
4	0	$v_f(0)+1 = v_f(1) = v_f(2) = \frac{k(n+2)}{3}$	$e_f(0)+1 = e_f(1) = e_f(2)+1 = \frac{2k(n+1)}{3}$
	1	$v_f(0) = v_f(1)+1 = v_f(2) = \frac{k(n+2)}{3}$	$e_f(0) = e_f(1)+1 = e_f(2) = \frac{2k(n+1)-1}{3}$
	2	$v_f(0) = v_f(1)+1 = v_f(2) = \frac{k(n+2)}{3}$	$e_f(0) = e_f(1) = e_f(2) = \frac{2k(n+1)-2}{3}$
5	0	$v_f(0) = v_f(1) = v_f(2)+1 = \frac{k(n+2)}{3}$	$e_f(0)+1 = e_f(1) = e_f(2)+1 = \frac{2k(n+1)}{3}$
	1	$v_f(0) = v_f(1) = v_f(2) = \frac{k(n+2)-1}{3}$	$e_f(0)+1 = e_f(1) = e_f(2)+1 = \frac{2k(n+1)}{3}$
	2	$v_f(0) = v_f(1)+1 = v_f(2)+1 = \frac{k(n+2)+1}{3}$	$e_f(0)+1 = e_f(1) = e_f(2)+1 = \frac{2k(n+1)}{3}$

TABLE 4.8

Illustration 4.5.4. Consider a graph $G = < W_6^{(1)} : W_6^{(2)} : W_6^{(3)} : W_6^{(4)} >$. Here $n = 6$ and $k = 4$. The corresponding 3-equitable labeling is as shown in FIGURE 4.8 .

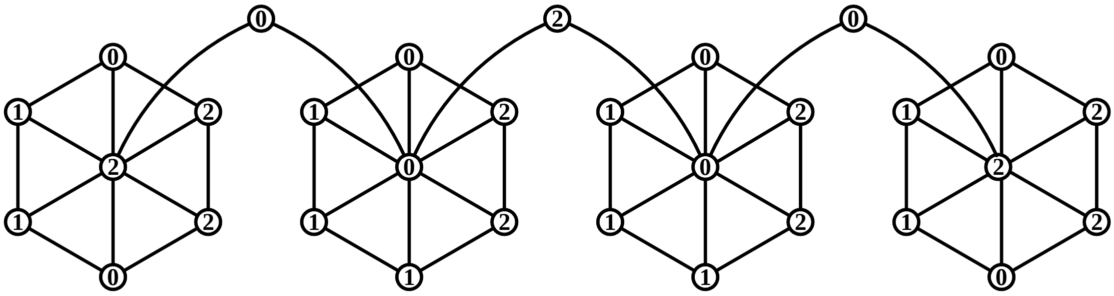

FIGURE 4.8

4.6 Some Graph Operations and 3-equitable Labeling

Theorem 4.6.1. Fusion of two vertices v_i and v_j with $d(v_i, v_j) \geq 3$ of cycle C_n is 3-equitable graph except $n \equiv 3(mod6)$.

Proof. Consider cycle C_n with n vertices namely $v_1, v_2, \ldots v_n$. Let G be the graph obtained by fusion of two vertices v_1 and v_k of cycle C_n. To define vertex labeling $f : V(G) \rightarrow \{0, 1, 2\}$ we consider the following cases.

Case 1: $n \equiv 0, 1(mod6)$

Subcase 1: $k \equiv 0, 3(mod6)$

In this case we define labeling as follows

$$\left.\begin{aligned} f(v_i) &= 0;\ \text{if } i \equiv 0,3(mod6) \\ &= 1;\ \text{if } i \equiv 4,5(mod6) \\ &= 2;\ \text{if } i \equiv 1,2(mod6) \end{aligned}\right\} \text{ for } 1 \leq i < k$$

$$\left.\begin{aligned} f(v_i) &= 0;\ \text{if } i \equiv 1,4(mod6) \\ &= 1;\ \text{if } i \equiv 0,5(mod6) \\ &= 2;\ \text{if } i \equiv 2,3(mod6) \end{aligned}\right\} \text{ for } k < i \leq n-1$$

$$\begin{aligned} f(v_n) &= 1;\ \text{if } n \equiv 0(mod6) \text{ and } k \equiv 0(mod6) \\ &= 0;\ \text{if } n \not\equiv 0(mod6) \text{ and } k \not\equiv 0(mod6) \end{aligned}$$

Subcase 2: $k \equiv 1,2,4,5(mod6)$

In this case we define labeling as follows

$$\left.\begin{aligned} f(v_i) &= 0;\text{ if } i \equiv 1,4(mod6) \\ &= 1;\text{ if } i \equiv 0,5(mod6) \\ &= 2;\text{ if } i \equiv 2,3(mod6) \end{aligned}\right\} \text{for } 1 \le i < k$$

$$\left.\begin{aligned} f(v_i) &= 0;\text{ if } i \equiv 2,5(mod6) \\ &= 1;\text{ if } i \equiv 0,1(mod6) \\ &= 2;\text{ if } i \equiv 3,4(mod6) \end{aligned}\right\} \text{for } k < i \le n$$

Case 2: $n \equiv 2(mod6)$

Subcase 1: $k \equiv 0(mod6)$

In this case we define labeling as follows

$$\left.\begin{aligned} f(v_i) &= 0;\text{ if } i \equiv 0,3(mod6) \\ &= 1;\text{ if } i \equiv 4,5(mod6) \\ &= 2;\text{ if } i \equiv 1,2(mod6) \end{aligned}\right\} \text{for } 1 \le i < k$$

$$\left.\begin{aligned} f(v_i) &= 0;\text{ if } i \equiv 1,4(mod6) \\ &= 1;\text{ if } i \equiv 0,5(mod6) \\ &= 2;\text{ if } i \equiv 2,3(mod6) \end{aligned}\right\} \text{for } k < i \le n$$

Subcase 2: $k \equiv 1,2,4,5(mod6)$

In this case we define labeling as follows

$$\left.\begin{aligned} f(v_i) &= 0;\text{ if } i \equiv 1,4(mod6) \\ &= 1;\text{ if } i \equiv 0,5(mod6) \\ &= 2;\text{ if } i \equiv 2,3(mod6) \end{aligned}\right\} \text{for } 1 \le i < k$$

$$\left.\begin{aligned} f(v_i) &= 0;\text{ if } i \equiv 2,5(mod6) \\ &= 1;\text{ if } i \equiv 0,1(mod6) \\ &= 2;\text{ if } i \equiv 3,4(mod6) \end{aligned}\right\} \text{for } k < i \le n-1$$

$f(v_n) = 2;$

Subcase 3: $k \equiv 3(mod6)$

In this case we define labeling as follows

$$\left.\begin{aligned} f(v_i) &= 0;\text{ if } i \equiv 0,3(mod6) \\ &= 1;\text{ if } i \equiv 4,5(mod6) \\ &= 2;\text{ if } i \equiv 1,2(mod6) \end{aligned}\right\} \text{ for } 1 \leq i < k$$

$$\left.\begin{aligned} f(v_i) &= 0;\text{ if } i \equiv 1,4(mod6) \\ &= 1;\text{ if } i \equiv 0,5(mod6) \\ &= 2;\text{ if } i \equiv 2,3(mod6) \end{aligned}\right\} \text{ for } k < i \leq n-2$$

$$f(v_{n-1}) = 2;$$
$$f(v_n) = 0;$$

Case 3: $n \equiv 3(mod6)$

In this case the graph resulted from the fusion of two vertices is an Eulerian graph which will have the number of edges congruent to $3(mod6)$. As proved by Cahit[11] an Eulerian graph with number of edges congruent to $3(mod6)$ is not 3-equitable.

Case 4: $n \equiv 4(mod6)$

Subcase 1: $k \equiv 0(mod6)$

In this case we define labeling as follows

$$\left.\begin{aligned} f(v_i) &= 0;\text{ if } i \equiv 0,3(mod6) \\ &= 1;\text{ if } i \equiv 4,5(mod6) \\ &= 2;\text{ if } i \equiv 1,2(mod6) \end{aligned}\right\} \text{ for } 1 \leq i < k$$

$$\left.\begin{aligned} f(v_i) &= 0;\text{ if } i \equiv 1,4(mod6) \\ &= 1;\text{ if } i \equiv 0,5(mod6) \\ &= 2;\text{ if } i \equiv 2,3(mod6) \end{aligned}\right\} \text{ for } k < i \leq n-3$$

$$f(v_{n-2}) = 0;$$
$$f(v_{n-1}) = 2;$$
$$f(v_n) = 1;$$

Subcase 2: $k \equiv 1,2,4,5(mod6)$

In this case we define labeling as follows

$$\left.\begin{aligned} f(v_i) &= 0;\ \text{if } i \equiv 1,4(mod6) \\ &= 1;\ \text{if } i \equiv 0,5(mod6) \\ &= 2;\ \text{if } i \equiv 2,3(mod6) \end{aligned}\right\} \text{for } 1 \leq i < k$$

$$\left.\begin{aligned} f(v_i) &= 0;\ \text{if } i \equiv 2,5(mod6) \\ &= 1;\ \text{if } i \equiv 0,1(mod6) \\ &= 2;\ \text{if } i \equiv 3,4(mod6) \end{aligned}\right\} \text{for } k < i \leq n-2$$

$f(v_{n-1}) = 2;$

$f(v_n) = 1;$

Subcase 3: $k \equiv 3(mod6)$

In this case we define labeling as follows

$$\left.\begin{aligned} f(v_i) &= 0;\ \text{if } i \equiv 0,3(mod6) \\ &= 1;\ \text{if } i \equiv 4,5(mod6) \\ &= 2;\ \text{if } i \equiv 1,2(mod6) \end{aligned}\right\} \text{for } 1 \leq i < k$$

$$\left.\begin{aligned} f(v_i) &= 0;\ \text{if } i \equiv 1,4(mod6) \\ &= 1;\ \text{if } i \equiv 0,5(mod6) \\ &= 2;\ \text{if } i \equiv 2,3(mod6) \end{aligned}\right\} \text{for } k < i \leq n-3$$

$f(v_{n-2}) = 1;$

$f(v_{n-1}) = 2;$

$f(v_n) = 0;$

Case 5: $n \equiv 5(mod6)$

Subcase 1: $k \equiv 0,3(mod6)$

In this case we define labeling as follows

$$\left.\begin{aligned} f(v_i) &= 0;\text{ if } i \equiv 0,3(mod6)\\ &= 1;\text{ if } i \equiv 4,5(mod6)\\ &= 2;\text{ if } i \equiv 1,2(mod6)\end{aligned}\right\}\text{ for } 1 \leq i < k$$

$$\left.\begin{aligned} f(v_i) &= 0;\text{ if } i \equiv 1,4(mod6)\\ &= 1;\text{ if } i \equiv 0,5(mod6)\\ &= 2;\text{ if } i \equiv 2,3(mod6)\end{aligned}\right\}\text{ for } k < i \leq n$$

Subcase 2: $k \equiv 1,2,4,5(mod6)$

In this case we define labeling as follows

$$\left.\begin{aligned} f(v_i) &= 0;\text{ if } i \equiv 1,4(mod6)\\ &= 1;\text{ if } i \equiv 0,5(mod6)\\ &= 2;\text{ if } i \equiv 2,3(mod6)\end{aligned}\right\}\text{ for } 1 \leq i < k$$

$$\left.\begin{aligned} f(v_i) &= 0;\text{ if } i \equiv 2,5(mod6)\\ &= 1;\text{ if } i \equiv 0,1(mod6)\\ &= 2;\text{ if } i \equiv 3,4(mod6)\end{aligned}\right\}\text{ for } k < i \leq n-1$$

$f(v_n) = 1;$

The labeling pattern defined above covers all possible arrangement of vertices. In each case, the graph G under consideration satisfies the conditions $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $0 \leq i, j \leq 2$ as shown in TABLE 4.9(where $n = 6a + b$, $k = 6c + d$ and $a, c \in N, b, d \in N \cup \{0\}$). i.e. G admits 3-equitable labeling. □

b	*d*	Vertex Condition	Edge Condition
0	1 to 5	$v_f(0) = v_f(1)+1 = v_f(2) = \frac{n}{3}$	$e_f(0) = e_f(1) = e_f(2) = \frac{n}{3}$
	0	$v_f(0)+1 = v_f(1) = v_f(2) = \frac{n}{3}$	$e_f(0) = e_f(1) = e_f(2) = \frac{n}{3}$
1	1 to 5	$v_f(0) = v_f(1) = v_f(2) = \frac{n-1}{3}$	$e_f(0) = e_f(1)+1 = e_f(2)+1 = \frac{n+2}{3}$
	0	$v_f(0) = v_f(1) = v_f(2) = \frac{n-1}{3}$	$e_f(0)+1 = e_f(1)+1 = e_f(2) = \frac{n+2}{3}$
2	0 to 5	$v_f(0)+1 = v_f(1)+1 = v_f(2) = \frac{n+1}{3}$	$e_f(0) = e_f(1)+1 = e_f(2) = \frac{n+1}{3}$
4	0 to 5	$v_f(0) = v_f(1) = v_f(2) = \frac{n-1}{3}$	$e_f(0)+1 = e_f(1) = e_f(2)+1 = \frac{n+2}{3}$
5	1 to 5	$v_f(0)+1 = v_f(1)+1 = v_f(2) = \frac{n+1}{3}$	$e_f(0) = e_f(1) = e_f(2)+1 = \frac{n+1}{3}$
	0	$v_f(0)+1 = v_f(1)+1 = v_f(2) = \frac{n+1}{3}$	$e_f(0)+1 = e_f(1) = e_f(2) = \frac{n+1}{3}$

TABLE 4.9

Remark 4.6.2. When $d(v_i, v_j) < 3$ the fusion yields a graph which is not simple and 3-equitability can not be discussed.

Illustrations 4.6.3.

Example 1: Consider a graph obtained by fusion of two vertices v_1 and v_6 of cycle C_{11}. This example is related to subcase 1 of case 5. The 3-equitable labeling is as shown in FIGURE 4.9.

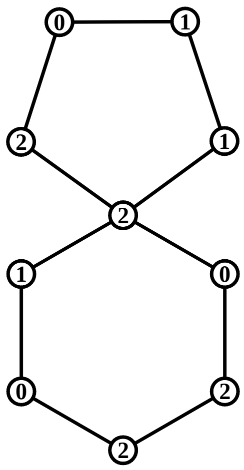

FIGURE 4.9

Example 2: Consider a graph obtained by fusion of two vertices v_1 and v_5 of cycle C_{10}. This example is related to subcase 2 of case 4. The 3-equitable labeling is as shown in FIGURE 4.10.

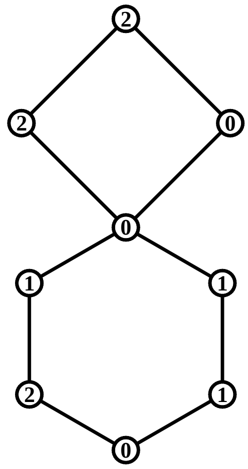

FIGURE 4.10

Theorem 4.6.4. Duplication of arbitrary vertex v_k of cycle C_n produces a 3-equitable graph.

Proof. Let C_n be the cycle with n vertices. Let v_k be the vertex of C_n. Let $v_k^{'}$ be the duplicated vertex of v_k and G be the graph resulted due to duplication. To define vertex labeling $f : V(G) \rightarrow \{0,1,2\}$ we consider following cases.

Case 1: $n \equiv 0,3,4,5(mod6)$ and $k \in N$, $1 \leq k \leq n$

In this case we define labeling function f as

$$\left.\begin{aligned} f(v_{k+i-1}) &= 0;\text{ if } i \equiv 1,4(mod6) \\ &= 1;\text{ if } i \equiv 0,5(mod6) \\ &= 2;\text{ if } i \equiv 2,3(mod6) \end{aligned}\right\}\text{ for } 1 \leq i \leq n-k+1$$

$$\left.\begin{aligned} f(v_{k+i-n-1}) &= 0;\text{ if } i \equiv 1,4(mod6) \\ &= 1;\text{ if } i \equiv 0,5(mod6) \\ &= 2;\text{ if } i \equiv 2,3(mod6) \end{aligned}\right\}\text{ for } n-k+2 \leq i \leq n$$

$$f(v_k^{'}) = 0;\text{ if } n \equiv 0(mod6)$$
$$f(v_k^{'}) = 1;\text{ if } n \equiv 3,4,5(mod6)$$

Case 2: $n \equiv 1(mod6)$ and $k \in N$, $1 \leq k \leq n$

In this case we define labeling function f as

$$\left.\begin{aligned} f(v_{k+i-1}) &= 0;\ \text{if } i \equiv 1,4(mod6) \\ &= 1;\ \text{if } i \equiv 2,3(mod6) \\ &= 2;\ \text{if } i \equiv 0,5(mod6) \end{aligned}\right\} \text{for } 1 \leq i \leq n-k+1$$

$$\left.\begin{aligned} f(v_{k+i-n-1}) &= 0;\ \text{if } i \equiv 1,4(mod6) \\ &= 1;\ \text{if } i \equiv 2,3(mod6) \\ &= 2;\ \text{if } i \equiv 0,5(mod6) \end{aligned}\right\} \text{for } n-k+2 \leq i \leq n$$

$$f(v_k^{'}) = 2;$$

Case 3: $n \equiv 2(mod6)$ and $k \in N$, $1 \leq k \leq n$

In this case we define labeling function f as

Subcase 1: if $k \leq 2$

$$\left.\begin{aligned} f(v_{k+i-1}) &= 0;\ \text{if } i \equiv 0,3(mod6) \\ &= 1;\ \text{if } i \equiv 4,5(mod6) \\ &= 2;\ \text{if } i \equiv 1,2(mod6) \end{aligned}\right\} \text{for } 1 \leq i \leq n-2$$

$$\left.\begin{aligned} f(v_{n-1}) &= 1; \\ f(v_n) &= 2; \end{aligned}\right\} \text{if } k = 1$$

$$\left.\begin{aligned} f(v_1) &= 2; \\ f(v_n) &= 1; \end{aligned}\right\} \text{if } k = 2$$

$$f(v_k^{'}) = 0;$$

Subcase 2: if $k \geq 3$

$$\left.\begin{aligned} f(v_{k+i-1}) &= 0;\ \text{if } i \equiv 0,3(mod6) \\ &= 1;\ \text{if } i \equiv 4,5(mod6) \\ &= 2;\ \text{if } i \equiv 1,2(mod6) \end{aligned}\right\} \text{for } 1 \leq i \leq n-k+1$$

$$\left.\begin{aligned} f(v_{k+i-n-1}) &= 0;\ \text{if } i \equiv 0,3(mod6) \\ &= 1;\ \text{if } i \equiv 4,5(mod6) \\ &= 2;\ \text{if } i \equiv 1,2(mod6) \end{aligned}\right\} \text{for } n-k+2 \leq i < n-1$$

$$f(v_{k-2}) = 1;$$
$$f(v_{k-1}) = 2;$$
$$f(v_k^{'}) = 0;$$

The labeling pattern defined above covers all possible arrangement of vertices. In each case, the graph G under consideration satisfies the conditions $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $0 \leq i, j \leq 2$ as shown in TABLE 4.10(where $n = 6a + b$ and $a, b \in N \cup \{0\}$). i.e. G admits 3-equitable labeling. □

b	**Vertex Condition**	**Edge Condition**
0	$v_f(0) = v_f(1) + 1 = v_f(2) + 1 = \frac{n+3}{3}$	$e_f(0) + 1 = e_f(1) = e_f(2) = \frac{n+3}{3}$
1,4	$v_f(0) = v_f(1) + 1 = v_f(2) = \frac{n+2}{3}$	$e_f(0) = e_f(1) = e_f(2) = \frac{n+2}{3}$
2,5	$v_f(0) = v_f(1) = v_f(2) = \frac{n+1}{3}$	$e_f(0) + 1 = e_f(1) = e_f(2) + 1 = \frac{n+4}{3}$
3	$v_f(0) + 1 = v_f(1) + 1 = v_f(2) = \frac{n+3}{3}$	$e_f(0) + 1 = e_f(1) = e_f(2) = \frac{n+3}{3}$

TABLE 4.10

Illustrations 4.6.5. Consider a graph obtained by duplicating vertex v_1 of cycle C_8. This is example of subcase 1 of case 3. The 3-equitable labeling is as shown in Figure 4.11.

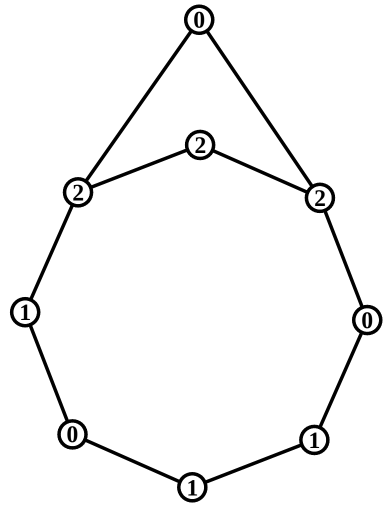

FIGURE 4.11

Theorem 4.6.6. The graph resulted form the duplication of the vertices of cycle C_n altogether is 3-equitable for even n and not 3-equitable for odd n.

Proof. Let C_n be the cycle with n vertices and $v_1, v_2, ..., v_n$ be the vertices of C_n. Let G be the graph obtained by duplicating the vertices of C_n altogether and $v'_1, v'_2, ..., v'_n$ be the duplicated vertices corresponding to $v_1, v_2, ..., v_n$ respectively. To define vertex labeling $f : V(G) \to \{0,1,2\}$ we consider the following cases.

Case 1: $n \equiv 0(mod6)$

In this case we define labeling f as

$$\left.\begin{aligned} f(v_i) &= 0;\ \text{if } i \equiv 1,4(mod6)\\ &= 1;\ \text{if } i \equiv 2,3(mod6)\\ &= 2;\ \text{if } i \equiv 0,5(mod6)\\ f(v'_i) &= 0;\ \text{if } i \equiv 1,4(mod6)\\ &= 1;\ \text{if } i \equiv 2,3(mod6)\\ &= 2;\ \text{if } i \equiv 0,5(mod6) \end{aligned}\right\} \text{for } 1 \leq i \leq n$$

Case 2: $n \equiv 2(mod6)$

In this case we define labeling f as

$$\left.\begin{aligned} f(v_i) &= 0;\ \text{if } i \equiv 1,4(mod6) \\ &= 1;\ \text{if } i \equiv 2,3(mod6) \\ &= 2;\ \text{if } i \equiv 0,5(mod6) \end{aligned}\right\} \text{for } 1 \leq i < n$$

$$f(v_n) = 2;$$

$$\left.\begin{aligned} f(v'_i) &= 0;\ \text{if } i \equiv 1,4(mod6) \\ &= 1;\ \text{if } i \equiv 2,3(mod6) \\ &= 2;\ \text{if } i \equiv 0,5(mod6) \end{aligned}\right\} \text{for } 1 \leq i \leq n-2$$

$$f(v'_{n-1}) = 1;$$
$$f(v'_n) = 0;$$

Case 3: $n \equiv 4(mod6)$

In this case we define labeling f as

$$f(v_1) = f(v_4) = 0;$$
$$f(v_2) = f(v_3) = 2;$$

$$\left.\begin{aligned} f(v_i) &= 0;\ \text{if } i \equiv 2,5(mod6) \\ &= 1;\ \text{if } i \equiv 0,1(mod6) \\ &= 2;\ \text{if } i \equiv 3,4(mod6) \end{aligned}\right\} \text{for } 5 \leq i \leq n$$

$$f(v'_1) = 0;$$
$$f(v'_2) = f(v'_3) = 1;$$
$$f(v'_4) = 2;$$

$$\left.\begin{aligned} f(v'_i) &= 0;\ \text{if } i \equiv 2,5(mod6) \\ &= 1;\ \text{if } i \equiv 0,1(mod6) \\ &= 2;\ \text{if } i \equiv 3,4(mod6) \end{aligned}\right\} \text{for } 5 \leq i \leq n$$

Case 4: For odd n

In this case the graph obtained is an Eulerian graph with number of edges congruent to $3(mod 6)$. Such graphs are not 3-equitable as proved by Cahit [11].

The labeling pattern defined above covers all possible arrangement of vertices. In each case the graph G under consideration satisfies the conditions $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $0 \leq i, j \leq 2$ as shown in TABLE 4.11(where $n = 6a + b$ and $a, b \in N \cup \{0\}$). i.e. G admits 3-equitable labeling. □

b	**Vertex Condition**	**Edge Condition**
0	$v_f(0) = v_f(1) = v_f(2) = \frac{2n}{3}$	$e_f(0) = e_f(1) = e_f(2) = n$
2	$v_f(0) = v_f(1) + 1 = v_f(2) + 1 = \frac{2n+2}{3}$	$e_f(0) = e_f(1) = e_f(2) = n$
4	$v_f(0) = v_f(1) + 1 = v_f(2) = \frac{2n+1}{3}$	$e_f(0) = e_f(1) = e_f(2) = n$

TABLE 4.11

Illustrations 4.6.7. Consider a graph obtained by duplicating vertices of cycle C_6 altogether. This is an example related to case 1. The corresponding 3-equitable labeling is shown in FIGURE 4.12.

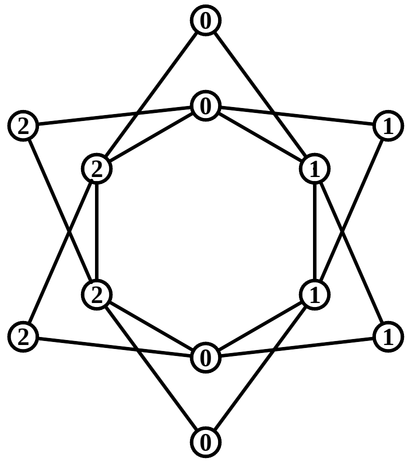

FIGURE 4.12

Theorem 4.6.8. The graph obtained by duplication of arbitrary rim vertex of wheel $W_n = C_n + K_1$ is 3-equitable for $n \geq 5$ while duplication of apex vertex is 3-equitable for even n and not 3-equitable for odd n, $n \geq 5$.

Proof. Consider the wheel $W_n = C_n + K_1$. Let $v_1, v_2, ..., v_n$ be the rim vertices of W_n, c_1 be the apex vertex of W_n and G be the graph obtained by duplicating either rim vertex or apex vertex of W_n. Let $v_k^{'}$ be the duplicated vertex of v_k and $c_1^{'}$ be the duplicated vertex of c_1. To define vertex labeling $f : V(G) \rightarrow \{0,1,2\}$ we consider the following cases.

Case 1: Duplication of arbitrary rim vertex v_k , where $k \in N$, $1 \leq k \leq n$

Subcase 1: $n \equiv 0,1(mod6)$

In this case we define labeling function f as

$$\left.\begin{aligned} f(v_{k+i-1}) &= 0;\ \text{if } i \equiv 1,4(mod6) \\ &= 1;\ \text{if } i \equiv 2,3(mod6) \\ &= 2;\ \text{if } i \equiv 0,5(mod6) \end{aligned}\right\} \text{for } 1 \leq i \leq n-k+1$$

$$\left.\begin{aligned} f(v_{k+i-n-1}) &= 0;\ \text{if } i \equiv 1,4(mod6) \\ &= 1;\ \text{if } i \equiv 2,3(mod6) \\ &= 2;\ \text{if } i \equiv 0,5(mod6) \end{aligned}\right\} \text{for } n-k+2 \leq i \leq n$$

$$f(v_k^{'}) = 2;\ \text{if } n \equiv 0(mod6)$$
$$f(v_k^{'}) = 1;\ \text{if } n \equiv 1(mod6)$$
$$f(c_1) = 0;\ \text{if } n \equiv 0(mod6)$$
$$f(c_1) = 2;\ \text{if } n \equiv 1(mod6)$$

Subcase 2: $n \equiv 2,5(mod6)$

In this case we define labeling function f as

$$\left.\begin{aligned} f(v_{k+i-1}) &= 0;\ \text{if } i \equiv 0,3(mod6) \\ &= 1;\ \text{if } i \equiv 4,5(mod6) \\ &= 2;\ \text{if } i \equiv 1,2(mod6) \end{aligned}\right\} \text{for } 1 \leq i \leq n-k+1$$

$$\left.\begin{aligned} f(v_{k+i-n-1}) &= 0;\ \text{if } i \equiv 0,3(mod6) \\ &= 1;\ \text{if } i \equiv 4,5(mod6) \\ &= 2;\ \text{if } i \equiv 1,2(mod6) \end{aligned}\right\} \text{for } n-k+2 \leq i \leq n$$

$f(v_k^{'}) = 1$; if $n \equiv 2(mod6)$

$f(v_k^{'}) = 2$; if $n \equiv 5(mod6)$

$f(c_1) = 0$;

Subcase 3: $n \equiv 3,4(mod6)$

In this case we define labeling function f as

Subcase 3.1: if $k \leq 2$

$$\left.\begin{array}{rcl} f(v_{k+i-1}) &=& 0;\ \text{if } i \equiv 1,4(mod6) \\ &=& 1;\ \text{if } i \equiv 0,5(mod6) \\ &=& 2;\ \text{if } i \equiv 2,3(mod6) \end{array}\right\} \text{for } 1 \leq i \leq n-3$$

$$\left.\begin{array}{rcl} f(v_{n-2}) &=& 0;\ \text{if } n \equiv 3(mod6) \\ f(v_{n-2}) &=& 1;\ \text{if } n \equiv 4(mod6) \\ f(v_{n-1}) &=& 1; \\ f(v_n) &=& 2; \end{array}\right\} \text{if } k = 1$$

$$\left.\begin{array}{rcl} f(v_{n-1}) &=& 0;\ \text{if } n \equiv 3(mod6) \\ f(v_{n-1}) &=& 1;\ \text{if } n \equiv 4(mod6) \\ f(v_n) &=& 1; \\ f(v_1) &=& 2; \end{array}\right\} \text{if } k = 2$$

$f(v_k^{'}) = 2$;

$f(c_1) = 0$;

Subcase 3.2: if $k \geq 3$

$$\left.\begin{array}{rcl} f(v_{k+i-1}) &=& 0;\ \text{if } i \equiv 1,4(mod6) \\ &=& 1;\ \text{if } i \equiv 0,5(mod6) \\ &=& 2;\ \text{if } i \equiv 2,3(mod6) \end{array}\right\} \text{for } 1 \leq i \leq n-k+1$$

$$\left.\begin{array}{rcl} f(v_{k+i-n-1}) &=& 0;\ \text{if } i \equiv 1,4(mod6) \\ &=& 1;\ \text{if } i \equiv 0,5(mod6) \\ &=& 2;\ \text{if } i \equiv 2,3(mod6) \end{array}\right\} \text{for } n-k+2 \leq i \leq n-3$$

$f(v_{k-3}) = 0$; if $n \equiv 3(mod6)$

$f(v_{k-3}) = 1$; if $n \equiv 4(mod6)$

$f(v_{k-2}) = 1$;

$f(v_{k-1}) = f(v_k^{'}) = 2$;

$f(c_1) = 0$;

Case 2: Duplication of apex vertex c_1

Subcase 1: $n \equiv 0(mod6)$

In this case we define labeling f as

$$\left.\begin{aligned} f(v_i) &= 0;\ \text{if } i \equiv 1,4(mod6) \\ &= 1;\ \text{if } i \equiv 0,5(mod6) \\ &= 2;\ \text{if } i \equiv 2,3(mod6) \end{aligned}\right\} \text{for } 1 \leq i \leq n$$

$f(c_1) = 0$;

$f(c_1') = 2$;

Subcase 2: $n \equiv 2(mod6)$

In this case we define labeling f as

$$\left.\begin{aligned} f(v_i) &= 0;\ \text{if } i \equiv 1,4(mod6) \\ &= 1;\ \text{if } i \equiv 0,5(mod6) \\ &= 2;\ \text{if } i \equiv 2,3(mod6) \end{aligned}\right\} \text{for } 1 \leq i \leq n-2$$

$f(v_{n-1}) = 1$;

$f(v_n) = 0$;

$f(c_1) = f(c_1') = 2$;

Subcase 3: $n \equiv 4(mod6)$

In this case we define labeling f as

$$\left.\begin{aligned} f(v_i) &= 0;\text{if } i \equiv 1,4(mod6) \\ &= 1;\text{ if } i \equiv 0,5(mod6) \\ &= 2;\text{ if } i \equiv 2,3(mod6) \end{aligned}\right\} \text{for } 1 \leq i \leq n-4$$

$$f(v_{n-3}) = f(v_{n-2}) = 1;$$
$$f(v_{n-1}) = f(c_1) = 0;$$
$$f(v_n) = f(c_1') = 2;$$

Subcase 4: $n \equiv 1(mod6)$

To satisfy the vertex condition it is essential to label $\frac{n+2}{3}$ vertices with 1. It is obvious that any edge will have label 1 if it is incident to one vertex with label 1. As G has $\frac{n+2}{3}$ vertices with label 1 and all the rim vertices are of degree 4 implies that there are at least $3(\frac{n+2}{3}-3)+8 = n+1$ edges with label 1. As the number of edges in $G = 3n$ and in order to satisfy the edge conditions number of edges with label 1 must be exactly n. Thus edge condition is violated and G is not 3-equitable.

Subcase 5: $n \equiv 3(mod6)$

To satisfy the vertex condition it is essential to label $\frac{n}{3}$ vertices with label 1. It is obvious that any edge will have label 1 if it is incident to one vertex with label 1. As G has $\frac{n}{3}$ vertices with label 1 and all the rim vertices are of degree 4 implies that either $3(\frac{n}{3}-3)+8 = n-1$ or $3(\frac{n}{3}-1)+4 = n+1$ edges with label 1. As the number of edges in $G = 3n$ and in order to satisfy the edge conditions number of edges with label 1 must be exactly n. Thus edge condition is violated and G is not 3-equitable.

Subcase 6: $n \equiv 5(mod6)$

To satisfy vertex condition it is essential to label $\frac{n+1}{3}$ vertices with label 1. It is obvious that any edge will have label 1 if it is incident to one vertex with label 1. As G has $\frac{n+1}{3}$ vertices with label 1 and all the rim vertices are of degree 4, it has either $3(\frac{n+1}{3}-4)+10 = n-1$ or $3(\frac{n+1}{3}) = n+1$ edges with label 1. As the number of edges in $G = 3n$ and in order to satisfy the edge conditions number of edges with label 1 must be exactly n. Thus edge condition is violated and G is not 3-equitable.

The labeling pattern defined above covers all possible arrangement of vertices. In each case, the graph G under consideration satisfies the conditions $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_j(1)| \leq 1$ for all $0 \leq i, j \leq 2$ as shown in TABLE 4.12(where $n = 6a + b$, $k \in N$ and $1 \leq k \leq n$, $a \in N \cup \{0\}$). i.e. G admits 3-equitable labeling. □

b	**Vertex Condition**	**Edge Condition**
	Duplication of a rim vertex	
0,3	$v_f(0) = v_f(1) + 1 = v_f(2) = \frac{n+3}{3}$	$e_f(0) = e_f(1) = e_f(2) = \frac{2n+3}{3}$
1,4	$v_f(0) = v_f(1) = v_f(2) = \frac{n+2}{3}$	$e_f(0) = e_f(1) = e_f(2) + 1 = \frac{2n+4}{3}$
2,5	$v_f(0) + 1 = v_f(1) + 1 = v_f(2) = \frac{n+4}{3}$	$e_f(0) + 1 = e_f(1) = e_f(2) + 1 = \frac{2n+5}{3}$
	Duplication of apex vertex	
0	$v_f(0) = v_f(1) + 1 = v_f(2) = \frac{n+3}{3}$	$e_f(0) = e_f(1) = e_f(2) = n$
2	$v_f(0) + 1 = v_f(1) + 1 = v_f(2) = \frac{n+4}{3}$	$e_f(0) = e_f(1) = e_f(2) = n$
4	$v_f(0) = v_f(1) = v_f(2) = \frac{n+2}{3}$	$e_f(0) = e_f(1) = e_f(2) = n$

TABLE 4.12

Illustrations 4.6.9.

Example 1: Consider a graph obtained by duplicating vertex v_2 on rim of wheel W_5. This is the example related to subcase 2 of case 1. The 3-equitable labeling is shown in FIGURE 4.13.

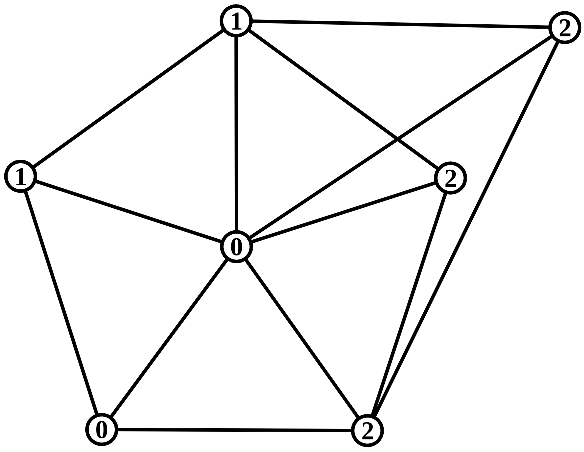

FIGURE 4.13

Example 2: Consider a graph obtained by duplicating apex vertex c_1 of wheel W_6. This is the example related to subcase 1 of case 2. The 3-equitable labeling is shown in FIGURE 4.14.

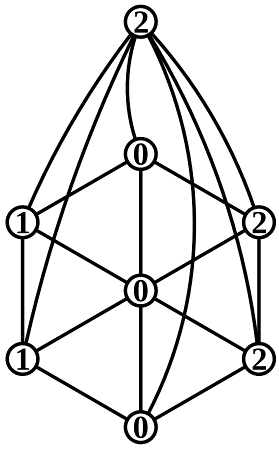

FIGURE 4.14

Theorem 4.6.10. Duplication of the vertices of wheel W_n altogether produces a 3-equitable graph for $n \neq 5$, where $n \in N$.

Proof. Consider the wheel $W_n = C_n + K_1$. Let $v_1, v_2, ..., v_n$ be the rim vertices of W_n, c_1 be the apex vertex of W_n and G be the graph obtained by duplicating vertices altogether moreover $v'_1, v'_2, ..., v'_n$ be the duplicated vertices of $v_1, v_2, ..., v_n$ respectively and c'_1 be the duplicated vertex of c_1. To define vertex labeling $f : V(G) \rightarrow \{0,1,2\}$ we consider the following cases.

Case 1: $n \equiv 0(mod6)$

In this case we define labeling f as

$$\left.\begin{aligned} f(v_i) &= 0;\text{ if } i \equiv 1,4(mod6)\\ &= 1;\text{ if } i \equiv 0,5(mod6)\\ &= 2;\text{ if } i \equiv 2,3(mod6)\\ f(v_i') &= 0;\text{ if } i \equiv 1,4(mod6)\\ &= 1;\text{ if } i \equiv 0,5(mod6)\\ &= 2;\text{ if } i \equiv 2,3(mod6) \end{aligned}\right\}\text{ for all } i,\ 1 \leq i \leq n$$

$f(c_1) = 0;$

$f(c_1') = 2;$

Case 2: $n \equiv 1(mod6)$

In this case we define labeling f as

$$\left.\begin{aligned} f(v_i) &= 0;\text{ if } i \equiv 1,4(mod6)\\ &= 1;\text{ if } i \equiv 0,5(mod6)\\ &= 2;\text{ if } i \equiv 2,3(mod6) \end{aligned}\right\}\text{ for for all } i,\ 1 \leq i \leq n-1$$

$f(v_n) = 1;$

$$\left.\begin{aligned} f(v_i') &= 0;\text{ if } i \equiv 1,4(mod6)\\ &= 1;\text{ if } i \equiv 0,5(mod6)\\ &= 2;\text{ if } i \equiv 2,3(mod6) \end{aligned}\right\}\text{ for for all } i,\ 1 \leq i \leq n-1$$

$f(v_n') = f(c_1') = 2;$

$f(c_1) = 0;$

Case 3: $n \equiv 2(mod6)$

In this case we define labeling f as

$$\left.\begin{array}{rcl} f(v_i) & = & 0;\ \text{if } i \equiv 1,4(mod6) \\ & = & 1;\ \text{if } i \equiv 0,5(mod6) \\ & = & 2;\ \text{if } i \equiv 2,3(mod6) \end{array}\right\} \text{for for all } i,\ 1 \leq i \leq n-2$$

$$f(v_{n-1}) = f(v_n) = 0;$$

$$\left.\begin{array}{rcl} f(v'_i) & = & 0;\ \text{if } i \equiv 1,4(mod6) \\ & = & 1;\ \text{if } i \equiv 0,5(mod6) \\ & = & 2;\ \text{if } i \equiv 2,3(mod6) \end{array}\right\} \text{for for all } i,\ 1 \leq i \leq n-2$$

$$f(v'_{n-1}) = f(v'_n) = 1;$$
$$f(c_1) = f(c'_1) = 2;$$

Case 4: $n \equiv 3(mod6)$

In this case we define labeling f as

$$f(v_1) = f(v_2) = 2;$$
$$f(v_3) = 0;$$

$$\left.\begin{array}{rcl} f(v_i) & = & 0;\ \text{if } i \equiv 1,4(mod6) \\ & = & 1;\ \text{if } i \equiv 2,3(mod6) \\ & = & 2;\ \text{if } i \equiv 0,5(mod6) \end{array}\right\} \text{for } 4 \leq i \leq n$$

$$f(v'_1) = 0;$$
$$f(v'_2) = f(v'_3) = 1;$$

$$\left.\begin{array}{rcl} f(v'_i) & = & 0;\ \text{if } i \equiv 1,4(mod6) \\ & = & 1;\ \text{if } i \equiv 2,3(mod6) \\ & = & 2;\ \text{if } i \equiv 0,5(mod6) \end{array}\right\} \text{for } 4 \leq i \leq n$$

$f(c_1) = 2;$

$f(c'_1) = 0;$ if $n \neq 3$

$f(c_1) = 0;$

$f(c'_1) = 2;$ if $n = 3$

Case 5: $n \equiv 4(mod 6)$

In this case we define labeling f as

$f(v_1) = 0;$

$f(v_2) = f(v_4) = 2;$

$f(v_3) = 1;$

$$\left.\begin{array}{rcl} f(v_i) & = & 0;\ \text{if } i \equiv 2,5(mod6) \\ & = & 1;\ \text{if } i \equiv 3,4(mod6) \\ & = & 2;\ \text{if } i \equiv 0,1(mod6) \end{array}\right\} \text{for } 5 \leq i \leq n$$

$f(v'_1) = 0;$

$f(v'_2) = f(v'_4) = 1;$

$f(v'_3) = 2;$

$$\left.\begin{array}{rcl} f(v'_i) & = & 0;\ \text{if } i \equiv 2,5(mod6) \\ & = & 1;\ \text{if } i \equiv 3,4(mod6) \\ & = & 2;\ \text{if } i \equiv 0,1(mod6) \end{array}\right\} \text{for } 5 \leq i \leq n$$

$f(c_1) = 0;$

$f(c'_1) = 2;$

Case 6: $n \equiv 5(mod6)$

In this case we define labeling f as

$f(v_1) = f(v_4) = 0;$
$f(v_2) = f(v_3) = 1;$
$f(v_5) = 2;$

$$\left.\begin{array}{rcl} f(v_i) & = & 0;\ \text{if } i \equiv 0,3(mod6) \\ & = & 1;\ \text{if } i \equiv 4,5(mod6) \\ & = & 2;\ \text{if } i \equiv 1,2(mod6) \end{array}\right\} \text{for } 6 \leq i \leq n$$

$f(v'_1) = f(v'_4) = 1;$
$f(v'_2) = f(v'_3) = 2;$
$f(v'_5) = 0;$

$$\left.\begin{array}{rcl} f(v'_i) & = & 0;\ \text{if } i \equiv 0,3(mod6) \\ & = & 1;\ \text{if } i \equiv 4,5(mod6) \\ & = & 2;\ \text{if } i \equiv 1,2(mod6) \end{array}\right\} \text{for } 6 \leq i \leq n$$

$f(c_1) = 0;$
$f(c'_1) = 2;$

Case 7: $n = 5$

G contains 12 vertices. In order to satisfy vertex condition 4 vertices must be labeled one. It is obvious that any edge will have label 1 if it is incident to one vertex with label 1. All the rim vertices are of degree 6 and duplicated vertices are of degree 3. Assign label one to v_1, v'_n, v'_1 and v'_2. It results minimum 11 edges with label one. As number of edges in W_5 is 30 edge condition is not satisfied. Therefore for $n = 5$ graph G is not 3-equitable.

The labeling pattern defined above covers all possible arrangement of vertices. In each case, the graph G under consideration satisfies the conditions $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $0 \leq i, j \leq 2$ as shown in TABLE 4.13(where $n = 6a + b$ and $a \in N \cup \{0\}$). i.e. G admits 3-equitable labeling. □

b	**Vertex Condition**	**Edge Condition**
0,3	$v_f(0) = v_f(1) + 1 = v_f(2) = \frac{2n+3}{3}$	$e_f(0) = e_f(1) = e_f(2) = 2n$
1,4	$v_f(0) + 1 = v_f(1) + 1 = v_f(2) = \frac{2n+4}{3}$	$e_f(0) = e_f(1) = e_f(2) = 2n$
2,5	$v_f(0) = v_f(1) = v_f(2) = \frac{2n+2}{3}$	$e_f(0) = e_f(1) = e_f(2) = 2n$

TABLE 4.13

Illustration 4.6.11. Consider a graph obtained by duplicating vertices of wheel W_4 altogether. This is example of case 5. The 3-equitable labeling is shown in FIGURE 4.15.

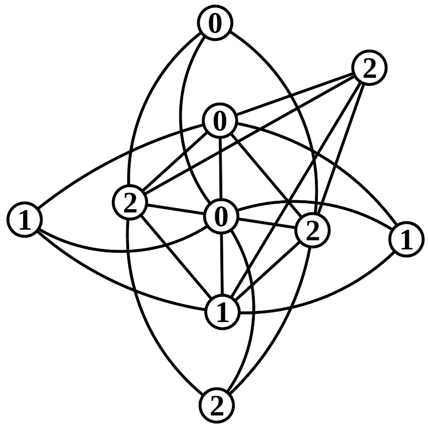

FIGURE 4.15

Theorem 4.6.12. Duplication of arbitrary edge e_k of cycle C_n produces a 3-equitable graph.

Proof. Let C_n be the cycle with n vertices. Let $e_k = v_k v_{k+1}$ be the vertex of C_n. Let $e_k^{'} = v_k^{'} v_{k+1}^{'}$ be the duplicated edge of e_k and G be the graph resulted due to duplication. To define ternary vertex labeling $f : V(G) \to \{0,1,2\}$ we consider following cases.

Case 1: If $n \equiv 1,2,3,4(mod6)$

$$\left.\begin{aligned} f(v_{k+i-1}) &= 0;\ \text{if } i \equiv 0,3(mod6) \\ &= 1;\ \text{if } i \equiv 1,2(mod6) \\ &= 2;\ \text{if } i \equiv 4,5(mod6) \end{aligned}\right\} \text{ for } 1 \leq i \leq n-k+1$$

$$\left.\begin{aligned} f(v_{k+i-n-1}) &= 0;\ \text{if } i \equiv 0,3(mod6) \\ &= 1;\ \text{if } i \equiv 1,2(mod6) \\ &= 2;\ \text{if } i \equiv 4,5(mod6) \end{aligned}\right\} \text{for } n-k+2 \le i \le n-1$$

$$\begin{aligned} f(v_{k-1}) &= 0;\ \text{if } k \neq 1 \\ f(v_{k+n-1}) &= 0;\ \text{if } k = 1 \\ f(v_k^{'}) &= 2;\ \text{if } n \equiv 2,3,4(mod6) \\ f(v_k^{'}) &= 1;\ \text{if } n \equiv 1(mod6) \\ f(v_{k+1}^{'}) &= 2;\ \text{if } k \neq n \\ f(v_{k-n+1}^{'}) &= 2;\ \text{if } k = n \end{aligned}$$

Case 2: If $n \equiv 0(mod6)$

$$\left.\begin{aligned} f(v_{k+i-1}) &= 0;\ \text{if } i \equiv 1,4(mod6) \\ &= 1;\ \text{if } i \equiv 2,3(mod6) \\ &= 2;\ \text{if } i \equiv 0,5(mod6) \end{aligned}\right\} \text{for } 1 \le i \le n-k+1$$

$$\left.\begin{aligned} f(v_{k+i-n-1}) &= 0;\ \text{if } i \equiv 1,4(mod6) \\ &= 1;\ \text{if } i \equiv 2,3(mod6) \\ &= 2;\ \text{if } i \equiv 0,5(mod6) \end{aligned}\right\} \text{for } n-k+2 \le i \le n$$

$$\begin{aligned} f(v_k^{'}) &= 0; \\ f(v_{k+1}^{'}) &= 1;\ \text{if } k \neq n \\ f(v_{k-n+1}^{'}) &= 1;\ \text{if } k = n \end{aligned}$$

Case 3: If $n \equiv 5(mod6)$

$$\left.\begin{aligned} f(v_{k+i-1}) &= 0;\ \text{if } i \equiv 2,5(mod6) \\ &= 1;\ \text{if } i \equiv 0,1(mod6) \\ &= 2;\ \text{if } i \equiv 3,4(mod6) \end{aligned}\right\} \text{for } 1 \le i \le n-k+1$$

$$\left.\begin{aligned} f(v_{k+i-n-1}) &= 0;\ \text{if } i \equiv 2,5(mod6) \\ &= 1;\ \text{if } i \equiv 0,1(mod6) \\ &= 2;\ \text{if } i \equiv 3,4(mod6) \end{aligned}\right\} \text{for } n-k+2 \le i \le n-1$$

$$f(v_{k-1}) = 1; \text{ if } k \neq 1$$
$$f(v_{k+n-1}) = 1; \text{ if } k = 1$$
$$f(v_k^{'}) = 0;$$
$$f(v_{k+1}^{'}) = 0; \text{ if } k \neq n$$
$$f(v_{k-n+1}^{'}) = 0; \text{ if } k = n$$

The labeling pattern defined above covers all possible arrangement of vertices. In each case, the graph G under consideration satisfies the conditions $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $0 \leq i, j \leq 2$ as shown in TABLE 4.14(where $n = 6a + b$ and $a \in N \cup \{0\}$). i.e. G admits 3-equitable labeling. □

b	**Vertex Condition**	**Edge Condition**
0	$v_f(0) = v_f(1) = v_f(2) + 1 = \frac{n+3}{3}$	$e_f(0) = e_f(1) = e_f(2) = \frac{n+3}{3}$
1	$v_f(0) = v_f(1) = v_f(2) = \frac{n+2}{3}$	$e_f(0) + 1 = e_f(1) = e_f(2) + 1 = \frac{n+5}{3}$
2	$v_f(0) + 1 = v_f(1) + 1 = v_f(2) = \frac{n+4}{3}$	$e_f(0) + 1 = e_f(1) = e_f(2) = \frac{n+4}{3}$
3	$v_f(0) + 1 = v_f(1) = v_f(2) = \frac{n+3}{3}$	$e_f(0) = e_f(1) = e_f(2) = \frac{n+3}{3}$
4	$v_f(0) = v_f(1) = v_f(2) = \frac{n+2}{3}$	$e_f(0) = e_f(1) + 1 = e_f(2) + 1 = \frac{n+5}{3}$
5	$v_f(0) = v_f(1) + 1 = v_f(2) + 1 = \frac{n+4}{3}$	$e_f(0) = e_f(1) = e_f(2) + 1 = \frac{n+4}{3}$

TABLE 4.14

Illustration 4.6.13. Consider W_9 and duplicate edge e_3. The corresponding 3-equitable labeling is shown in FIGURE 4.16

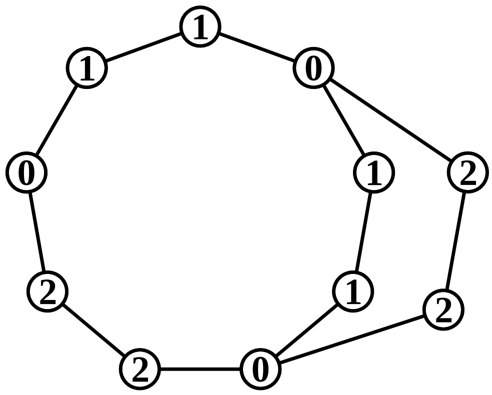

FIGURE 4.16

Theorem 4.6.14. Duplication of arbitrary edge e_k of wheel W_n produces a 3-equitable graph.

Proof. Consider the wheel $W_n = C_n + K_1$. Let $v_1, v_2, \ldots, v_n$ be the rim vertices of W_n, c be the apex vertex of W_n and G be the graph obtained by duplicating either rim edge or spoke edge of W_n. Let $e_k^{'}$ be the duplicated edge of e_k. To define ternary vertex labeling $f : V(G) \to \{0,1,2\}$ we consider following cases.

Case 1: Duplication of arbitrary rim edge e_k, where $k \in N, 1 \leq k \leq n$

Subcase 1: If $n \equiv 0,5(mod6)$

$$\left.\begin{aligned} f(v_{k+i-1}) &= 0;\ \text{if } i \equiv 1,4(mod6) \\ &= 1;\ \text{if } i \equiv 2,3(mod6) \\ &= 2;\ \text{if } i \equiv 0,5(mod6) \end{aligned}\right\} \text{for } 1 \leq i \leq n-k+1$$

$$\left.\begin{aligned} f(v_{k+i-n-1}) &= 0;\ \text{if } i \equiv 1,4(mod6) \\ &= 1;\ \text{if } i \equiv 2,3(mod6) \\ &= 2;\ \text{if } i \equiv 0,5(mod6) \end{aligned}\right\} \text{for } n-k+2 \leq i \leq n$$

$$f(c) = 0;$$
$$f(v_k^{'}) = 2;$$

$$\left.\begin{aligned} f(v_{k+1}) &= 1;\ \text{if } n \equiv 0(mod6) \\ &= 2;\ \text{if } n \equiv 5(mod6) \end{aligned}\right\} \text{for } k \neq n$$

$$\left.\begin{aligned} f(v_{k-n+1}) &= 1;\ \text{if } n \equiv 0(mod6) \\ &= 2;\ \text{if } n \equiv 5(mod6) \end{aligned}\right\} \text{for } k = n$$

Subcase 2: If $n \equiv 1,2,3,4(mod6)$

$$\left.\begin{aligned} f(v_{k+i-1}) &= 0;\ \text{if } i \equiv 0,3(mod6) \\ &= 1;\ \text{if } i \equiv 1,2(mod6) \\ &= 2;\ \text{if } i \equiv 4,5(mod6) \end{aligned}\right\} \text{for } 1 \leq i \leq n-k+1$$

$$\left.\begin{aligned} f(v_{k+i-n-1}) &= 0;\ \text{if } i \equiv 0,3(mod6)\\ &= 1;\ \text{if } i \equiv 1,2(mod6)\\ &= 2;\ \text{if } i \equiv 4,5(mod6)\end{aligned}\right\} \begin{array}{l}\text{for } n-k+2 \le i \le n-1 \text{ and } n \equiv 1(mod6)\\ \text{for } n-k+2 \le i \le n \text{ and } n \equiv 2,3,4(mod6)\end{array}$$

$$f(v_{k-1}) = 0;\ \text{if } n \equiv 1(mod6) \text{ and } k \neq 1$$
$$f(v_{k+n-1}) = 0;\ \text{if } n \equiv 1(mod6) \text{ and } k = 1$$
$$f(c) = 0;\ \text{if } n \equiv 2,4(mod6)$$
$$f(c) = 2;\ \text{if } n \equiv 1,3(mod6)$$
$$f(v_k^{'}) = 0;\ \text{if } n \equiv 2(mod6)$$
$$f(v_k^{'}) = 1;\ \text{if } n \equiv 1(mod6)$$
$$f(v_k^{'}) = 2;\ \text{if } n \equiv 3,4(mod6)$$

$$\left.\begin{aligned} f(v_{k+1}) &= 0;\ \text{if } n \equiv 3,4(mod6)\\ &= 2;\ \text{if } n \equiv 1,2(mod6)\end{aligned}\right\} \text{for } k \neq n$$

$$\left.\begin{aligned} f(v_{k-n+1}) &= 0;\ \text{if } n \equiv 3,4(mod6)\\ &= 2;\ \text{if } n \equiv 1,2(mod6)\end{aligned}\right\} \text{for } k = n$$

Case 2: Duplication of arbitrary spoke edge $e_{n+k} = cv_k$, where $k \in N, 1 \le k \le n$

Subcase 1: If $n \equiv 0,5(mod6)$

$$\left.\begin{aligned} f(v_{k+i-1}) &= 0;\ \text{if } i \equiv 0,3(mod6)\\ &= 1;\ \text{if } i \equiv 1,2(mod6)\\ &= 2;\ \text{if } i \equiv 4,5(mod6)\end{aligned}\right\} \text{for } 1 \le i \le n-k+1$$

$$\left.\begin{aligned} f(v_{k+i-n-1}) &= 0;\ \text{if } i \equiv 0,3(mod6)\\ &= 1;\ \text{if } i \equiv 1,2(mod6)\\ &= 2;\ \text{if } i \equiv 4,5(mod6)\end{aligned}\right\} \text{for } n-k+2 \le i \le n$$

$$f(c) = 0;$$
$$f(c^{'}) = 2;$$
$$f(v_k^{'}) = 1;\ \text{if } n \equiv 0(mod6)$$
$$f(v_k^{'}) = 0;\ \text{if } n \equiv 5(mod6)$$

Subcase 2: If $n \equiv 1(mod6)$

$$\left.\begin{array}{rcl} f(v_{k+i-1}) & = & 0; \text{if } i \equiv 1,4(mod6) \\ & = & 1; \text{ if } i \equiv 2,3(mod6) \\ & = & 2; \text{ if } i \equiv 0,5(mod6) \end{array}\right\} \text{for } 1 \leq i \leq n-k+1$$

$$\left.\begin{array}{rcl} f(v_{k+i-n-1}) & = & 0; \text{ if } i \equiv 1,4(mod6) \\ & = & 1; \text{ if } i \equiv 2,3(mod6) \\ & = & 2; \text{ if } i \equiv 0,5(mod6) \end{array}\right\} \text{for } n-k+2 \leq i \leq n$$

$$f(c) = 2;$$
$$f(c^{'}) = 2;$$
$$f(v_k^{'}) = 1;$$

Subcase 3: If $n \equiv 2(mod6)$

$$\left.\begin{array}{rcl} f(v_{k+i-1}) & = & 0; \text{ if } i \equiv 1,4(mod6) \\ & = & 1; \text{ if } i \equiv 0,5(mod6) \\ & = & 2; \text{ if } i \equiv 2,3(mod6) \end{array}\right\} \text{for } 1 \leq i \leq n-k+1$$

$$\left.\begin{array}{rcl} f(v_{k+i-n-1}) & = & 0; \text{ if } i \equiv 1,4(mod6) \\ & = & 1; \text{ if } i \equiv 0,5(mod6) \\ & = & 2; \text{ if } i \equiv 2,3(mod6) \end{array}\right\} \text{for } n-k+2 \leq i \leq n$$

$$f(c) = 0;$$
$$f(c^{'}) = 2;$$
$$f(v_k^{'}) = 1;$$

Subcase 4: If $n \equiv 3(mod6), n \neq 3$

$$f(v_k) = 1;$$
$$f(v_{k+1}) = 0; \text{ if } k+1 \leq n$$
$$f(v_{k-n+1}) = 0; \text{ if } k+1 > n$$
$$f(v_{k+2}) = 2; \text{ if } k+2 \leq n$$
$$f(v_{k-n+2}) = 2; \text{ if } k+2 > n$$

$$\left.\begin{aligned} f(v_{k+i-1}) &= 0;\ \text{if } i \equiv 1,4(mod6)\\ &= 1;\ \text{if } i \equiv 2,3(mod6)\\ &= 2;\ \text{if } i \equiv 0,5(mod6)\end{aligned}\right\} \text{for } 4 \leq i \leq n-k+1$$

$$\left.\begin{aligned} f(v_{k+i-n-1}) &= 0;\ \text{if } i \equiv 1,4(mod6)\\ &= 1;\ \text{if } i \equiv 2,3(mod6)\\ &= 2;\ \text{if } i \equiv 0,5(mod6)\end{aligned}\right\} \text{for } n-k+2 \leq i \leq n$$

$$f(c) = 0;$$
$$f(c^{'}) = 2;$$
$$f(v_k^{'}) = 1;$$

If $n = 3$ the labeling starting from v_k is $0,2,2$ for rim vertices, labeling of apex vertex 0 and labeling of vertices $v_k^{'}$ and $c^{'}$ is 1.

Subcase 5: If $n \equiv 4(mod6), n \neq 4$

$$\left.\begin{aligned} f(v_{k+i-1}) &= 0;\ \text{if } i \equiv 0,3(mod6)\\ &= 1;\ \text{if } i \equiv 1,2(mod6)\\ &= 2;\ \text{if } i \equiv 4,5(mod6)\end{aligned}\right\} \text{for } 4 \leq i \leq n-k+1$$

$$\left.\begin{aligned} f(v_{k+i-n-1}) &= 0;\ \text{if } i \equiv 0,3(mod6)\\ &= 1;\ \text{if } i \equiv 1,2(mod6)\\ &= 2;\ \text{if } i \equiv 4,5(mod6)\end{aligned}\right\} \text{for } n-k+2 \leq i \leq n-4$$

$$f(v_{k+i-n-1}) = 2;\ \text{if } n-3, n-2$$
$$f(v_{k+i-n-1}) = 1;\ \text{if } n-1, n$$
$$f(c) = 0;$$
$$f(c^{'}) = 0;$$
$$f(v_k^{'}) = 1;$$

If $n = 4$ the labeling starting from v_k is $0,2,2,0$ for rim vertices, labeling of apex vertex 2 and labeling of vertices $v_k^{'}$ and $c^{'}$ is 1.

The labeling pattern defined above covers all possible arrangement of vertices. In each case, the graph G under consideration satisfies the conditions $|v_f(i) - v_f(j)| \leq 1$

and $|e_f(i)-e_f(j)| \leq 1$ for all $0 \leq i,j \leq 2$ as shown in TABLE 4.15(where $n = 6a+b$ and $a \in N \cup \{0\}$). i.e. G admits 3-equitable labeling. $\square$

b	**Vertex Condition**	**Edge Condition**
	Duplication of a rim edge	
0	$v_f(0)=v_f(1)=v_f(2)=\frac{n+3}{3}$	$e_f(0)=e_f(1)=e_f(2)+1=\frac{2n+6}{3}$
1	$v_f(0)+1=v_f(1)+1=v_f(2)=\frac{n+5}{3}$	$e_f(0)+1=e_f(1)=e_f(2)+1=\frac{2n+7}{3}$
2	$v_f(0)=v_f(1)=v_f(2)+1=\frac{n+4}{3}$	$e_f(0)=e_f(1)=e_f(2)=\frac{2n+5}{3}$
3	$v_f(0)=v_f(1)=v_f(2)=\frac{n+3}{3}$	$e_f(0)+1=e_f(1)=e_f(2)=\frac{2n+6}{3}$
4	$v_f(0)=v_f(1)+1=v_f(2)+1=\frac{n+5}{3}$	$e_f(0)=e_f(1)+1=e_f(2)+1=\frac{2n+7}{3}$
5	$v_f(0)=v_f(1)+1=v_f(2)=\frac{n+4}{3}$	$e_f(0)=e_f(1)=e_f(2)=\frac{2n+5}{3}$
	Duplication of a spoke edge	
0	$v_f(0)=v_f(1)=v_f(2)=\frac{n+3}{3}$	$e_f(0)=e_f(1)=e_f(2)+1=n+1$
1	$v_f(0)+1=v_f(1)+1=v_f(2)=\frac{n+5}{3}$	$e_f(0)=e_f(1)=e_f(2)+1=n+1$
2	$v_f(0)=v_f(1)+1=v_f(2)=\frac{n+4}{3}$	$e_f(0)+1=e_f(1)=e_f(2)=n+1$
3	$v_f(0)=v_f(1)=v_f(2)=\frac{n+3}{3}$	$e_f(0)=e_f(1)+1=e_f(2)=n+1$
4	$v_f(0)+1=v_f(1)=v_f(2)+1=\frac{n+5}{3}$	$e_f(0)=e_f(1)+1=e_f(2)=n+1$
5	$v_f(0)=v_f(1)+1=v_f(2)=\frac{n+4}{3}$	$e_f(0)+1=e_f(1)=e_f(2)=n+1$

TABLE 4.15

Illustration 4.6.15. Consider W_5 and duplicate edge e_1. The corresponding 3-equitable labeling is shown in FIGURE 4.17

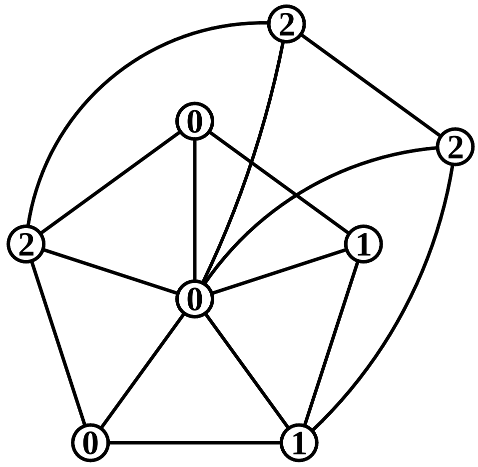

FIGURE 4.17

4.7 Some Open Problems

It is possible to derive similar results using different graph labeling schemes and in the context of various graph families. The results reported in this chapter can be extended for *k*-equitable labeling.

4.8 Concluding Remarks

This chapter was aimed to discuss 3-equitable labeling of graphs. Fifteen new results are investigated and labeling patterns are demonstrated by means of several examples.

The investigations reported in chapter-3 and 4 give rise to following research papers.

- Cordial and 3-equitable labeling for some star related graphs., *International Mathematical Forum*,4(3), 2009, 1543-1553. (http://www.m-hikari.com/ imf.html)
- Cordial and 3-equitable labeling for some shell related graphs., *Journal of Scientific Research*, 1(3), 2009, 438-449. (http://www.banglajol.info/index.php/JSR/index)

- Some wheel related 3-equitable Graphs in the context of vertex duplication., *Advances Applications in Discrete Mathematics*, 4(1), 2009, 71-85. (http://www.pphmj.com)

- Some new star related graphs and their cordial as well as 3-equitable labeling.,*Journal of Science*,1(1),2010, 111-114.

- Cordial and 3-equitable labeling for some wheel related graphs., Accepted for publication in *International Journal of Applied Mathematics*.

The reprints/preprint of above research papers are given in Annexure.

The next chapter is targeted to discuss arbitrary supersubdivision and some graph labeling problems.

Chapter 5

Arbitrary Supersubdivision and Graph Labeling

5.1 Introduction

This chapter is focused on arbitrary supersubdivision of graphs and some graph labeling schemes. We investigate nine results corresponding to this concept.

5.2 Arbitrary Supersubdivision and Graceful Labeling of Some Graphs

Sethuraman and Selvaraju[41] introduced a new method of construction called supersubdivision of graph.

Definition 5.2.1. Let G be a graph with q edges. A graph H is called a *supersubdivision* of G if H is obtained from G by replacing every edge e_i of G by a complete bipartite graph K_{2,m_i} for some $m_i, 1 \leq i \leq q$ in such a way that the end vertices of each e_i are identified with the two vertices of 2-vertices part of K_{2,m_i} after removing the edge e_i from graph G. If m_i is varying arbitrarily for each edge e_i then supersubdivision is called *arbitrary supersubdivision* which is denoted by $SS(G)$.

- In the same paper Sethuraman and Selvaraju proved that arbitrary supersubdivisions of any path are graceful.
- They also proved that arbitrary supersubdivisions cycle C_n are graceful.
- Kathiresan and Amutha[30] proved that arbitrary supersubdivisions of any star are graceful.

In the present work we discuss cordial labeling and strongly multiplicative labeling in the context of arbitrary supersubdivision of graph.

5.3 Arbitrary Supersubdivision and Cordial Labeling of Some Graphs

Theorem 5.3.1. Arbitrary supersubdivision of tree T is cordial.

Proof. Let T be the tree with n vertices and $v_i (1 \leq i \leq n)$ be the vertices of T. Arbitrary supersubdivision of T is obtained by replacing every edge of tree with K_{2,m_i} and we denote this graph by G. Let $\alpha = \sum_{1}^{n-1} m_i$. Let u_j be the vertices of m_i-vertices part where $1 \leq j \leq \alpha$. Denote the vertex with minimum eccentricity as v_1 and n_1 and n_2 be the number of vertices which are at odd and even distance respectively form v_1 in T. Here $|V(G)| = \alpha + n$ and $|E(G)| = 2\alpha$. We define binary vertex labeling $f : V(G) \to \{0,1\}$ as follows.

$$\left.\begin{array}{rcl} f(v_1) & = & 0; \\ f(v_i) & = & 1; \text{if } d(v_1, v_i) \text{ in } T \text{ is odd} \\ & = & 0; \text{if } d(v_1, v_i) \text{ in } T \text{ is even} \end{array}\right\} \text{for } 2 \leq i \leq n$$

$$\left.\begin{array}{rcl} f(u_i) & = & 0; \text{If } n_1 \geq n_2 \\ & = & 1; \text{If } n_1 < n_2 \end{array}\right\} \text{for } 1 \leq i \leq |n_1 - n_2|$$

$$\left.\begin{array}{rcl} f(u_i) & = & 0;\ \text{If } |n_1 - n_2| + 1 \leq i \leq \lfloor \frac{\alpha + |n_1 - n_2|}{2} \rfloor \\ & = & 1;\ \text{If } \lceil \frac{\alpha + |n_1 - n_2|}{2} \rceil \leq i \leq \alpha \end{array}\right\} \text{for } i > |n_1 - n_2|$$

In view of the above defined labeling pattern we have the followings.

- When $\alpha + n$ is even
  $v_f(0) = v_f(1) = \frac{\alpha+n}{2}$; $e_f(0) = e_f(1) = \alpha$
- When $\alpha + n$ is odd
  $v_f(0) = v_f(1) + 1 = \frac{\alpha+n+1}{2}$; $e_f(0) = e_f(1) = \alpha$

Thus the graph G satisfies the conditions $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. That is, G admits cordial labeling. □

Remarks 5.3.2. In the FIGURE 5.1 to 5.10 the dark vertices correspond to 2-vertices part while hollow vertices correspond to m_i-vertices part.

Illustration 5.3.3. Consider $G = \mathrm{SS}(T)$. Here $n = 12$ and $\alpha = 24$. The cordial labeling is as shown in FIGURE 5.1

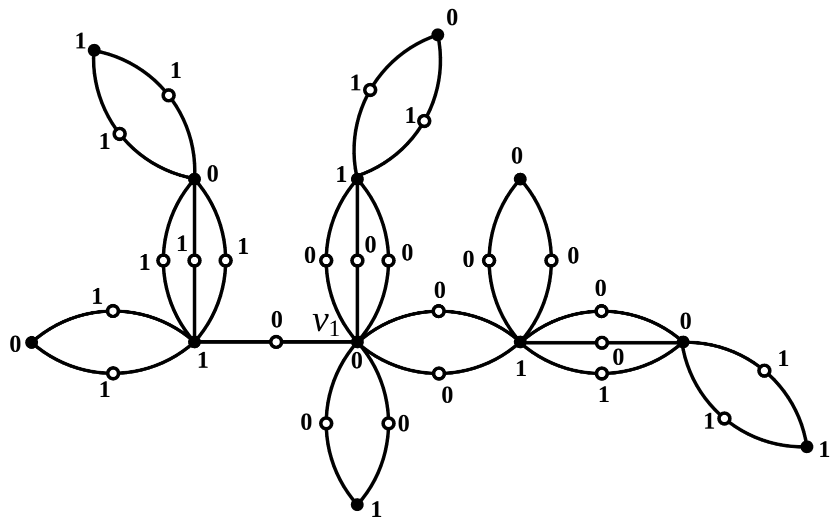

FIGURE 5.1

Theorem 5.3.4. Arbitrary supersubdivision of complete bipartite graph $K_{m,n}$ is cordial.

Proof. Let $v_1, v_2, v_3, \ldots v_m$ be the vertices of m-vertices part and $v_{m+1}, v_{m+2}, v_{m+3}, \ldots v_{m+n}$ be the vertices of n-vertices part of $K_{m,n}$. Arbitrary supersubdivision of $K_{m,n}$ is obtained by replacing every edge of $K_{m,n}$ with K_{2,m_i} and we denote this graph by G. Let $\alpha = \sum_{1}^{mn} m_i$. Let u_j be the vertices which are used for arbitrary supersubdivision, where $1 \leq j \leq \alpha$. Note that $|V(G)| = \alpha + m + n$, $|E(G)| = 2\alpha$. We define binary vertex labeling $f : V(G) \to \{0, 1\}$ as follows.

$$
\begin{aligned}
f(v_i) &= 0; \text{ if } 1 \leq i \leq m \\
&= 1; \text{ if } m+1 \leq i \leq m+n
\end{aligned}
$$

$$
\left.
\begin{aligned}
f(u_i) &= 1; \text{ if } m \geq n \\
&= 0; \text{ if } m < n
\end{aligned}
\right\} \text{ for } 1 \leq i \leq |m-n|
$$

$$\left.\begin{array}{rcl} f(u_i) & = & 0;\text{ if } |m-n|+1 \leq i \leq \lfloor \frac{\alpha+|m-n|}{2} \rfloor \\ & = & 1;\text{ if } \lceil \frac{\alpha+|m-n|}{2} \rceil \leq i \leq \alpha \end{array}\right\} \text{for } i > |m-n|$$

Above defined function f is cordial labeling for the graph under consideration because

- $v_f(0) = v_f(1) = \frac{\alpha+m+n}{2}$; $e_f(0) = e_f(1) = \alpha$ (When $\alpha+m+n$ is even)
- $v_f(0)+1 = v_f(1) = \frac{\alpha+m+n+1}{2}$; $e_f(0) = e_f(1) = \alpha$ (When $\alpha+m+n$ is odd)

That is, G admits cordial labeling. □

Illustration 5.3.5. Consider $G = SS(K_{2,2})$. Here $m = 2$, $n = 2$ and $\alpha = 12$. The cordial labeling is as shown in FIGURE 5.2

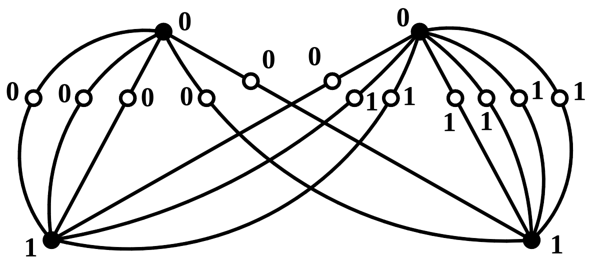

FIGURE 5.2

Theorem 5.3.6. Arbitrary supersubdivision of grid graph $P_n \times P_m$ is cordial.

Proof. Let v_{ij} be the vertices of $P_n \times P_m$, where $1 \leq i \leq n$ and $1 \leq j \leq m$. Arbitrary supersubdivision of $P_n \times P_m$ is obtained by replacing every edge of grid graph with K_{2,m_i} and we denote the resultant graph by G. Let $\alpha = \sum_{1}^{2mn-m-n} m_i$. Let u_j be the vertices of m_i-vertices part of K_{2,m_i} supersubdivision, where $1 \leq j \leq \alpha$. Here $|V(G)| = \alpha + mn$, $|E(G)| = 2\alpha$. We define binary vertex labeling $f : V(G) \to \{0,1\}$ as follows.

$$\left.\begin{array}{rcl} f(v_{ij}) & = & 0;\text{ if } i \text{ and } j \text{ both are even or } i \text{ and } j \text{ both are odd} \\ & = & 1;\text{ if } i \text{ is even and } j \text{ is odd or } i \text{ is odd and } j \text{ is even} \end{array}\right\} \begin{array}{l}\text{Where } 1 \leq i \leq n \text{ and} \\ 1 \leq j \leq m\end{array}$$

$$\begin{array}{rcl} f(u_j) & = & 0;\text{ if } 1 \leq j \leq \lfloor \frac{\alpha}{2} \rfloor \\ & = & 1;\text{ if } \lceil \frac{\alpha}{2} \rceil \leq j \leq \alpha \end{array}$$

Above defined function f is cordial labeling for the graph under consideration because

- $v_f(0) = v_f(1) = \frac{\alpha+mn}{2}$; $e_f(0) = e_f(1) = \alpha$ (When $\alpha + mn$ is even)
- $v_f(0)+1 = v_f(1) = \frac{\alpha+mn+1}{2}$; $e_f(0) = e_f(1) = \alpha$ (When α odd and mn is even)
- $v_f(0) = v_f(1)+1 = \frac{\alpha+mn+1}{2}$; $e_f(0) = e_f(1) = \alpha$ (When α even and mn is odd)

That is, f is a cordial labeling for the G. Hence the result. □

Illustration 5.3.7. Consider $G =$SS$(P_3 \times P_3)$. Here $n = 3$, $m = 3$ and $\alpha = 29$. The corresponding cordial labeling is shown in FIGURE 5.3

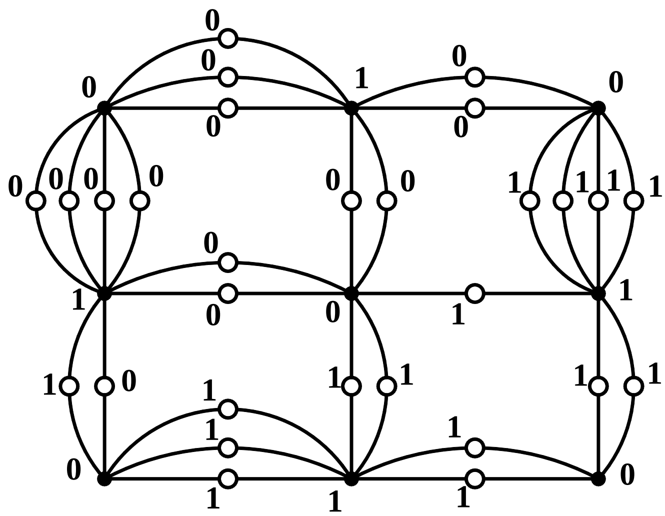

FIGURE 5.3

Theorem 5.3.8. Arbitrary supersubdivision of armed crown $C_n \odot P_m$ is cordial except $m_i (1 \le i \le n)$ are odd, $m_i (n+1 \le i \le nm)$ are even and n is odd.

Proof. Let $v_1, v_2, v_3, \dots v_n$ be the vertices of C_n and $v_{ij} (1 \le i \le n, 2 \le j \le m)$ be the vertices of paths. Arbitrary supersubdivision of $C_n \odot P_m$ is obtained by replacing every edge of $C_n \odot P_m$ with K_{2,m_i} and we denote this graph by G. Let $\alpha = \sum_{1}^{mn} m_i$ and u_j be the vertices of m_i-vertices part of K_{2,m_i}, where $1 \le j \le \alpha$. Here $|V(G)| = \alpha + mn$, $|E(G)| = 2\alpha$. To define binary vertex labeling $f : V(G) \to \{0,1\}$ we consider following cases.

Case 1: For n even

$$\left.\begin{aligned} f(v_i) &= 0;\text{ if } i \text{ is odd} \\ &= 1;\text{ if } i \text{ is even} \\ f(v_{ij}) &= 0;\text{ if } i \text{ and } j \text{ both are even or } i \text{ and } j \text{ both are odd} \\ &= 1;\text{ if } i \text{ is even and } j \text{ is odd or } i \text{ is odd and } j \text{ is even} \end{aligned}\right\} \begin{array}{l}\text{for } 1 \le i \le n \text{ and} \\ 2 \le j \le m\end{array}$$

$$\left.\begin{aligned} f(u_j) &= 0;\text{ if } 1 \le j \le \lfloor \tfrac{\alpha}{2} \rfloor \\ &= 1;\text{ if } \lceil \tfrac{\alpha}{2} \rceil \le j \le \alpha \end{aligned}\right\} \text{for } 1 \le j \le \alpha$$

Case 2: For n odd and at least one $m_i(1 \le i \le n)$ is even and at least one $m_i(n+1 \le i \le mn)$ is odd

Without loss of generality we assume that m_1 is even.

$$\left.\begin{aligned} f(v_1) &= 0; \\ f(v_i) &= 0;\text{ if } i \text{ is even} \\ &= 1;\text{if } i \text{ is odd} \\ f(v_{1j}) &= 0;\text{ if } j \text{ is odd} \\ &= 1;\text{ if } j \text{ is even} \\ f(v_{ij}) &= 0;\text{ if } i \text{ is even and } j \text{ is odd or } i \text{ is odd and } j \text{ is even} \\ &= 1;\text{ if } i \text{ and } j \text{ both are even or } i \text{ and } j \text{ both are odd} \end{aligned}\right\} \begin{array}{l}\text{for } 2 \le i \le n \text{ and} \\ 2 \le j \le m\end{array}$$

$$\left.\begin{aligned} f(u_j) &= 0;\text{ if } 1 \le j \le \tfrac{m_1}{2} \\ &= 1;\text{ if } \tfrac{m_1}{2}+1 \le j \le m_1 \\ f(u_j) &= 0;\text{ if } m_1+1 \le j \le \lfloor \tfrac{\alpha+m_1}{2} \rfloor \\ &= 1;\text{ if } \lceil \tfrac{\alpha+m_1}{2} \rceil \le j \le \alpha \end{aligned}\right\} \text{for } 1 \le j \le \alpha$$

In view of the above two cases graph G satisfies the following conditions.

- $v_f(0) = v_f(1) = \frac{\alpha+mn}{2}$; $e_f(0) = e_f(1) = \alpha$ (When $\alpha + mn$ is even)
- $v_f(0)+1 = v_f(1) = \frac{\alpha+mn+1}{2}$; $e_f(0) = e_f(1) = \alpha$ (When α odd and mn is even)
- $v_f(0) = v_f(1)+1 = \frac{\alpha+mn+1}{2}$; $e_f(0) = e_f(1) = \alpha$ (When α even and mn is odd)

That is, f is a cordial labeling for G and consequently G is a cordial graph.

Case 3: If n is odd number with $m_i(1 \leq i \leq n)$ are odd and $m_i(n+1 \leq i \leq nm)$ are even

In this case G is an Eulerian graph with number of edges congruent to $2(mod4)$. As we mentioned earlier (Theorem 3.5.5) an Eulerian graph with number of edges congruent to $2(mod4)$ is not cordial.

Hence from the case 1 to 3 we have the required result. □

Illustration 5.3.9. Consider $G = SS(C_4 \odot P_3)$. Here $n = 4$, $m = 3$ and $\alpha = 29$. The corresponding cordial labeling is as shown in FIGURE 5.4

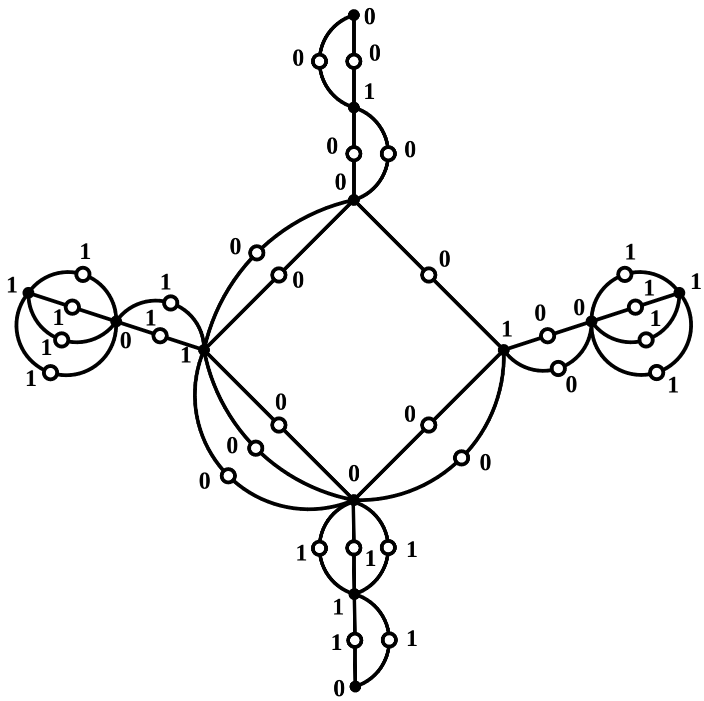

FIGURE 5.4

5.4 Strongly Multiplicative Labeling

Definition 5.4.1. A graph $G = (V(G), E(G))$ with p vertices is said to be multiplicative if the vertices of G can be labeled with p distinct positive integers such that label induced on the edges by the product of labels of end vertices are all distinct.

The concept of multiplicative labeling was introduced by Beineke and Hedge[8]. In the same paper they shown that every graph admits multiplicative labeling and they defined strongly multiplicative labeling as follows.

Definition 5.4.2. A graph $G = (V(G), E(G))$ with p vertices is said to be strongly multiplicative if the vertices of G can be labeled with p distinct integers $1, 2, ...p$ such that label induced on the edges by the product of labels of the end vertices are all distinct.

5.4.1 Some Known Results

Beineke and Hedge[8] have proved the following results.

- Every cycle C_n is strongly multiplicative.
- Every wheel W_n is strongly multiplicative.
- Complete graph K_n is strongly multiplicative if and only if $n \leq 5$.
- Complete bipartite graph $K_{n,n}$ is strongly multiplicative if and only if $n \leq 4$.
- Every spanning subgraph of a strongly multiplicative graph is strongly multiplicative.
- Every graph is an induced subgraph of a strongly multiplicative graph .

5.5 Arbitrary Supersubdivision and Strongly Multiplicative Labeling of Some Graphs

Theorem 5.5.1. Arbitrary supersubdivision of tree T is strongly multiplicative.

Proof. Let T be the tree with n vertices. Arbitrary supersubdivision $SS(T)$ of tree T obtained by replacing every edge of tree with K_{2,m_i} and we denote such graph by G. Let $\alpha = \sum m_i$ $(1 \leq i \leq n-1)$. Let v_j $(1 \leq j \leq \alpha + n)$ be the vertices of G. Denote the vertex with minimum eccentricity as v_1. Then v_2 will be the vertex which is at 1- distance apart from v_1. If there are more than one such vertices then throughout the work we will follow one of the direction (clockwise or anticlockwise) and denote them as $v_3, v_4, \ldots$. Next consider the vertices which are at 2- distance apart from v_1, 3- distance apart from v_1 and so on. (e.g. if there are seven vertices and two vertices are at distance 1- apart, one vertex is at distance 2- apart and three vertices are at distance 3- apart respectively form v_1. In this situation the vertices which are at 1- distance apart from v_1 will be identified as v_2 and v_3, the vertex which is at distance 2- apart will be identified as v_4 and the vertices which are at distance 3- apart will be identified as v_5, v_6 and v_7.) We define vertex labeling $f : V(G) \to \{1, 2 \ldots \alpha + n\}$ as follows.

For any $1 \leq i \leq n + \alpha$ define

$f(v_i) = i$;

Then the above defined function f is strongly multiplicative labeling for the graph G. That is, the graph G under consideration admits strongly multiplicative labeling. □

Illustration 5.5.2. In FIGURE 5.6 strongly multiplicative labeling of $SS(T)$ corresponding to tree T of FIGURE 5.5 is shown where $n = 13$ and $\alpha = 26$.

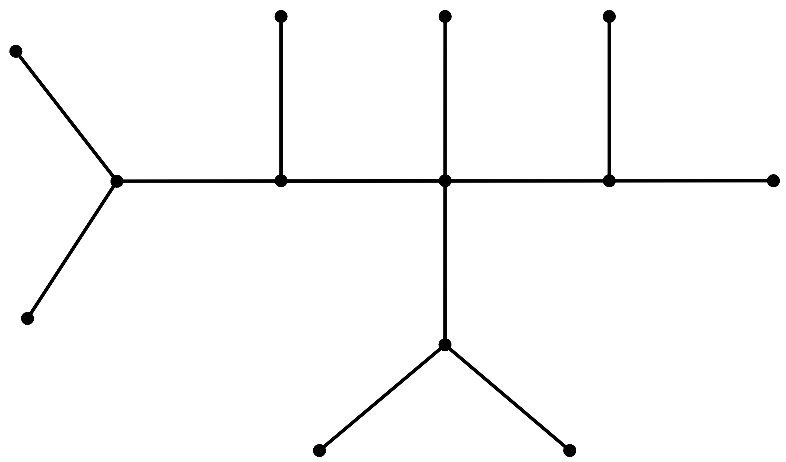

FIGURE 5.5

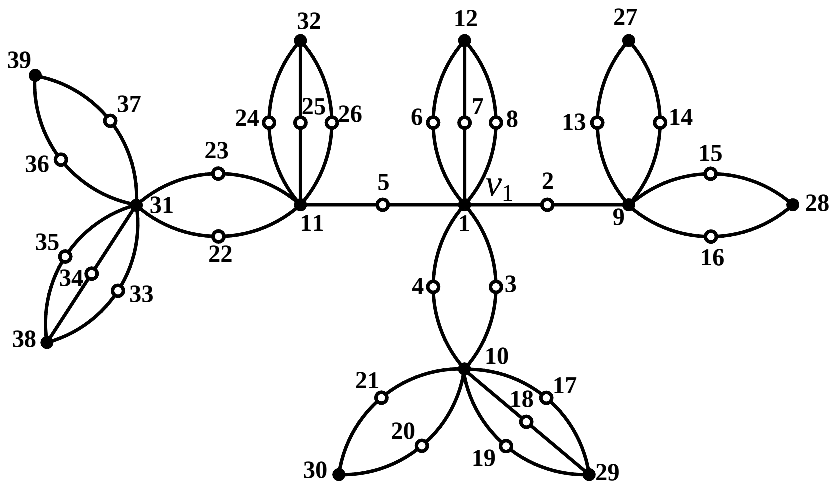

FIGURE 5.6

Theorem 5.5.3. Arbitrary supersubdivision of complete bipartite graph $K_{m,n}$ is strongly multiplicative.

Proof. Let $v_1, v_2, v_3, \dots v_m$ be the vertices of m-vertices part and $v_{m+1}, v_{m+2}, v_{m+3}, \dots v_{m+n}$ be the vertices of n-vertices part of $K_{m,n}$. Arbitrary supersubdivision $\mathrm{SS}(K_{m,n})$ of $K_{m,n}$ obtained by replacing every edge of $K_{m,n}$ with K_{2,m_i} and we denote such graph by G. Let $\alpha = \sum m_i$ $(1 \leq i \leq mn)$. Let u_j be the vertices which are used for arbitrary supersubdivision, where $1 \leq j \leq \alpha$. We denote vertices by u_j which are used for supersubdivision of edges $v_1v_{m+1}, v_1v_{m+2}, \ldots v_1v_{m+n}, v_2v_{m+1}, \ldots v_nv_{m+n}$. Let p_o be the highest prime less than $\alpha+m+n$. We define vertex labeling $f: V(G) \to \{1,2\ldots\alpha+m+n\}$ as follows.

If $p_o \leq \alpha+m$

$$f(v_i) = \begin{cases} i; \quad if \quad 1 \leq i \leq m \\ \alpha+i; \quad if \quad m+2 \leq i \leq m+n \end{cases}$$

$f(v_{m+1}) = p_o;$

$$f(u_j) = \begin{cases} m+j; \quad if \quad 1 \leq j < p_o \\ m+j+1; \quad if \quad p_o \leq j \leq \alpha \end{cases}$$

If $p_o > \alpha+m$

$$f(v_i) = \begin{cases} i; \quad if \quad 1 \leq i \leq m, \\ \alpha+i-1; \quad if \quad m+2 \leq i < p_o, \\ \alpha+i; \quad if \quad p_o \leq i \leq m+n \end{cases}$$

$f(v_{m+1}) = p_o;$

$f(u_j) = m+j; \quad where \quad 1 \leq j \leq \alpha$

Then in each possibilities described above the function f is strongly multiplicative labeling for the graph G. That is, the graph G under consideration admits strongly multiplicative labeling. □

Illustration 5.5.4. Consider $SS(K_{2,3})$. Here $m = 2$, $n = 3$ and $\alpha = 14$. The strongly multiplicative labeling is as shown in FIGURE 5.7.

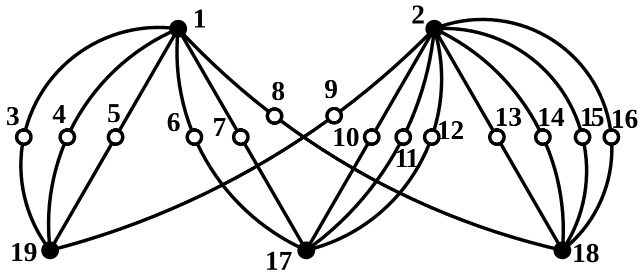

FIGURE 5.7

Theorem 5.5.5. Arbitrary supersubdivision of grid graph $P_n \times P_m$ is strongly multiplicative.

Proof. Arbitrary supersubdivision $\mathrm{SS}(P_n \times P_m)$ of $P_n \times P_m$ obtained by replacing every edge of grid graph with K_{2,m_i} and we denote such graph by G. Let $\alpha = \sum m_i$ $(1 \le i \le mn)$. Let v_i $(1 \le i \le mn+\alpha)$ be the vertices of G. Denote the vertex of left upper corner with v_1. Here we designate vertices by v_i $(2 \le i \le mn+\alpha)$ according to the procedure described in Theorem 5.5.1 We define vertex labeling $f : V(G) \to \{1, 2, \ldots, mn+\alpha\}$

$f(v_i) = i; \quad where \quad 1 \le i \le mn+\alpha$

Then the above defined function f is strongly multiplicative labeling for the graph G. That is, the graph G under consideration admits strongly multiplicative labeling. □

Illustration 5.5.6. Consider $\mathrm{SS}(P_4 \times P_3)$. Here $n = 4$, $m = 3$ and $\alpha = 41$. The corresponding strongly multiplicative labeling is shown in FIGURE 5.8.

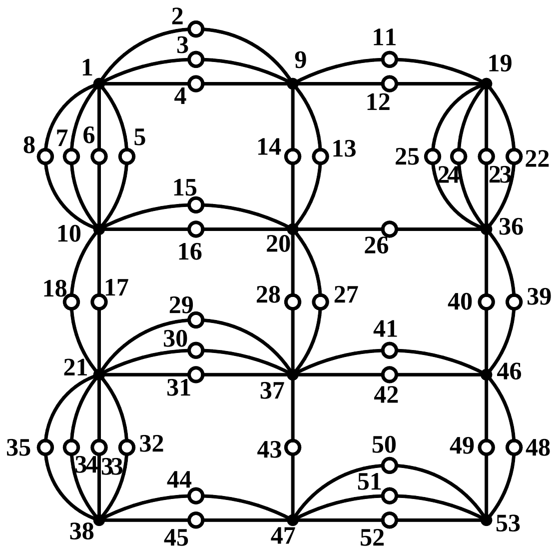

FIGURE 5.8

Theorem 5.5.7. Arbitrary supersubdivision of armed crown $C_n \odot P_m$ is strongly multiplicative.

Proof. Arbitrary supersubdivision $SS(C_n \odot P_m)$ of $C_n \odot P_m$ obtained by replacing every edge of $C_n \odot P_m$ with K_{2,m_i} and we denote such graph by G. Let $\alpha = \sum m_i$ $(1 \leq i \leq mn)$. Let v_i $(1 \leq i \leq mn+\alpha)$ be the vertices of G. Designate arbitrary vertex of C_n as v_1 and employing the scheme used in Theorem 5.5.1 the remaining vertices will receive labels $v_2, v_3, \ldots, v_{mn+\alpha}$. We define vertex labeling $f : V(G) \to \{1, 2, \ldots, mn+\alpha\}$ as follows.

$$f(v_i) = i; \quad where \quad 1 \leq i \leq mn+\alpha$$

Then the above defined function f is strongly multiplicative labeling for the graph G. That is, the graph G under consideration admits strongly multiplicative labeling. □

Illustration 5.5.8. Consider $SS(C_5 \odot P_3)$. Here $n = 5$, $m = 3$ and $\alpha = 37$. The corresponding strongly multiplicative labeling is as shown in FIGURE 5.9.

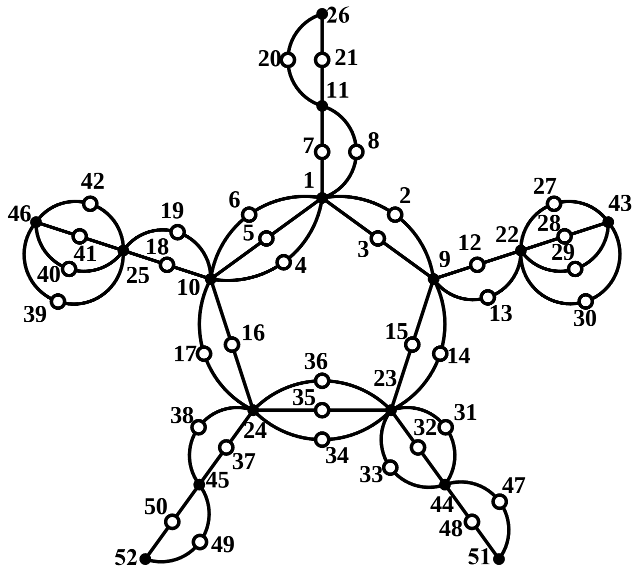

FIGURE 5.9

Theorem 5.5.9. Arbitrary supersubdivision of $C_n^{(m)}$ is strongly multiplicative.

Proof. Arbitrary supersubdivision of $C_n^{(m)}$ is obtained by replacing every edge of $C_n^{(m)}$ with K_{2,m_i} and we denote this graph by G. Let $\alpha = \sum m_i$. Let $v_i(1 \leq i \leq m(n-1)+\alpha+1$ be the vertices of G. Denote the common vertex of cycles by v_1. According to the procedure followed in previous results the remaining vertices will be designated as v_i $(2 \leq i \leq m(n-1)+\alpha+1)$. We define vertex labeling $f : V(G) \to \{1, 2, \ldots, m(n-1)+\alpha+1\}$ as follows.

For any $1 \leq i \leq m(n-1)+\alpha+1$ we define

$f(v_i) = i$;

Then the above defined function f is strongly multiplicative labeling for the graph G. That is, the graph G under consideration admits strongly multiplicative labeling. □

Illustration 5.5.10. Consider $\mathrm{SS}(C_4^{(3)})$. Here $n = 4$, $m = 3$ and $\alpha = 26$. The strongly multiplicative labeling is as shown in FIGURE 5.10.

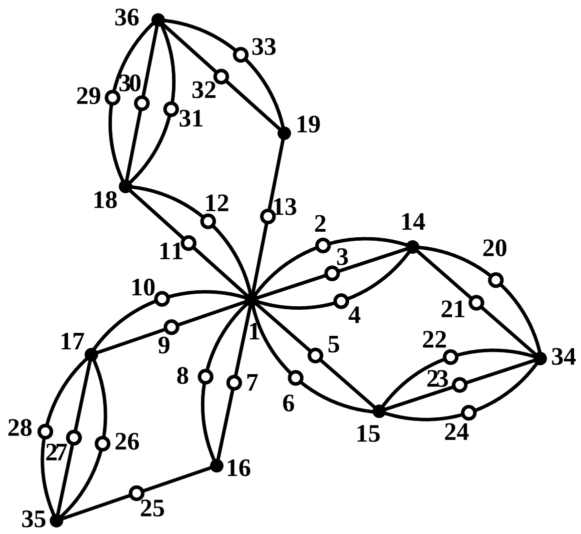

FIGURE 5.10

5.6 Some Open Problems

- It is possible to investigate some result corresponding to different graph labeling techniques.
- Try to find out some characterisation for strongly multiplicative labeling.

5.7 Concluding Remarks

It is very interesting to investigate graph or families of graph which admits particular type of labeling. Here we discuss cordial and strongly multiplicative labeling in the context of arbitrary supersubdivision of some graphs.

The content of this chapter give rise to the following two research papers.

1. Strongly multiplicative labeling in the context of arbitrary supersubdivision., *Journal of Mathematics Research*, 2(2),2010, 28-33.
2. Cordial labeling and arbitrary supersubdivision of some graphs., Accepted for publication in *International J. of Information Sc. and Computer Maths.* (http://pphmj.com/journals/ijiscm.htm)

The reprint/preprint of above research papers are given in Annexure.

The last Chapter-6 is intended to discuss product cordial labeling of graphs.

Chapter 6

Product Cordial Labeling of Graphs

6.1 Introduction

In cordial labeling the induced edge labels are absolute difference of vertex labels while in product cordial labeling the induced edge labels are product of vertex labels. In the present chapter we contribute eleven new results corresponding to product cordial labeling.

6.2 Product Cordial Labeling

Definition 6.2.1. Let $G = (V(G), E(G))$ be a graph. A mapping $f : V(G) \longrightarrow \{0,1\}$ is called *binary vertex labeling* of G and $f(v)$ is called the *label* of vertex v of G under f.

For an edge $e = uv$, the induced edge labeling $f^* : E(G) \rightarrow \{0,1\}$ is given by $f^*(e) = f(u)f(v)$. Let $v_f(0), v_f(1)$ be the number of vertices of G having labels 0 and 1 respectively under f and let $e_f(0), e_f(1)$ be the number of edges of G having labels 0 and 1 respectively under f^*

Definition 6.2.2. A binary vertex labeling of graph G is called a *product cordial labeling* if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph G is *product cordial* if it admits product cordial labeling.

6.2.1 Some Known Results

The concept of product cordial labeling was introduced by Sundaram et al.[42]. They proved that

- All trees are product cordial.
- Unicyclic graphs of odd order are product cordial.
- triangular snakes are product cordial .
- dragons are product cordial.

- helms are product cordial.
- union of two path graphs are product cordial.
- A graph with p vertices and q edges with $p \geq 4$ is product cordial then $q < \frac{p^2-1}{4}$.

6.3 Some New Product Cordial Graphs

Theorem 6.3.1. The graph obtained by fusion of two vertices v_i and v_j with $d(v_i, v_j) \geq 3$ of cycle C_n is product cordial.

Proof. Let C_n be any cycle. $v_1, v_2, v_3, \ldots, v_n$ be the vertices of C_n. G is the graph produced by fusion of v_1 with v_k. To define binary vertex labeling $f : V(G) \rightarrow \{0, 1\}$ we consider following cases.

Case 1: For any odd n and $k \leq \frac{n+1}{2}$

$$\begin{aligned} f(v_i) &= 1; \text{ if } 1 \leq i \leq \tfrac{n+1}{2} \text{ and } i \neq k \\ &= 0; \text{ if } \tfrac{n+3}{2} \leq i \leq n \end{aligned}$$

Case 2: For any odd n and $k > \frac{n+1}{2}$

$$\begin{aligned} f(v_1) &= 1; \\ f(v_i) &= 0; \text{ if } 2 \leq i \leq \tfrac{n+1}{2} \\ &= 1; \text{ if } \tfrac{n+3}{2} \leq i \leq n \text{ and } i \neq k \end{aligned}$$

Case 3: For any even n and $k \leq \frac{n+2}{2}$

$$\begin{aligned} f(v_i) &= 1; \text{ if } 1 \leq i \leq \tfrac{n+2}{2} \text{ and } i \neq k \\ &= 0; \text{ if } \tfrac{n+4}{2} \leq i \leq n \end{aligned}$$

Case 4: For any even n and $k > \frac{n+2}{2}$

$$\begin{aligned} f(v_1) &= 1; \\ f(v_i) &= 0; \text{ if } 2 \leq i \leq \tfrac{n}{2} \\ &= 1; \text{ if } \tfrac{n+2}{2} \leq i \leq n \text{ and } i \neq k \end{aligned}$$

The labeling pattern defined above includes all possible arrangement of vertices. In each case the graph G under consideration satisfies the conditions for product cordiality as shown in TABLE 6.1(where $n = 2a + b$ and $a \in N$). i.e. G is product cordial graph. □

b	**Vertex Condition**	**Edge Condition**
0	$v_f(0)+1 = v_f(1) = \frac{n}{2}$	$e_f(0) = e_f(1) = \frac{n}{2}$
1	$v_f(0) = v_f(1) = \frac{n-1}{2}$	$e_f(0) = e_f(1)+1 = \frac{n+1}{2}$

TABLE 6.1

Remarks 6.3.2. If $d(v_i, v_j) < 3$ the graph obtained by fusion is not simple and product cordiality can not be discussed.

Illustrations 6.3.3. Consider a graph obtained by fusing two vertices, v_1 and v_7 of cycle C_{11}. Here $n = 11$ i.e. n is odd and $k = 7$. Here $k > \frac{n+1}{2}$. The product cordial labeling is as shown in FIGURE 6.1.

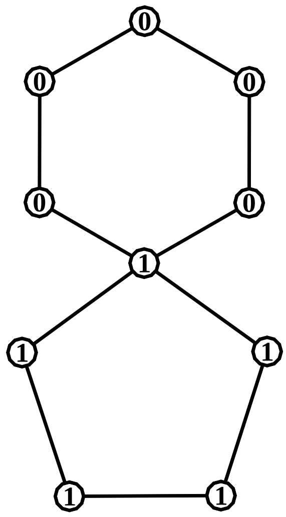

FIGURE 6.1

Theorem 6.3.4. Duplication of arbitrary vertex v_k of cycle C_n with $n \geq 6$ produces product cordial graph.

Proof. Let C_n be cycle with n vertices, where $n \geq 6$. Let v_k be arbitrary vertex of C_n. Let G be the graph obtained by duplicating vertex v_k of cycle C_n. Let v_k' be duplicated vertex of v_k. To define binary vertex labeling $f : V(G) \longrightarrow \{0,1\}$. We consider following cases.

Case 1: For even n

$$f(v_{k+i-1}) = 1; \text{ if } 1 \leq i \leq \tfrac{n-2}{2} \text{ and } k+i-1 \leq n$$
$$f(v_{k+i-n-1}) = 1; \text{ if } 1 \leq i \leq \tfrac{n-2}{2} \text{ and } k+i-1 > n$$
$$f(v_{k+i-1}) = 0; \text{ if } \tfrac{n}{2} \leq i < n \text{ and } k+i-1 \leq n$$
$$f(v_{k+i-n-1}) = 0; \text{ if } \tfrac{n}{2} \leq i < n \text{ and } k+i-1 > n$$
$$f(v_n) = 1; \text{ if } k = 1$$
$$f(v_{k-1}) = 1; \text{ if } k > 1$$
$$f(v_k^{'}) = 1;$$

Case 2: For odd n

$$f(v_{k+i-1}) = 1; \text{ if } 1 \leq i \leq \tfrac{n-3}{2} \text{ and } k+i-1 \leq n$$
$$f(v_{k+i-n-1}) = 1; \text{ if } 1 \leq i \leq \tfrac{n-3}{2} \text{ and } k+i-1 > n$$
$$f(v_{k+i-1}) = 0; \text{ if } \tfrac{n-1}{2} \leq i < n \text{ and } k+i-1 \leq n$$
$$f(v_{k+i-n-1}) = 0; \text{ if } \tfrac{n-1}{2} \leq i < n \text{ and } k+i-1 > n$$
$$f(v_n) = 1; \text{ if } k = 1$$
$$f(v_{k-1}) = 1; \text{ if } k > 1$$
$$f(v_k^{'}) = 1;$$

The above defined labeling pattern includes all possible arrangement of vertices. In each case 1 and case 2 the conditions for product cordiality is satisfied as shown in TABLE 6.2(where $n = 2a+b$ and $a \in N$).

b	**Vertex Condition**	**Edge Condition**
0	$v_f(0)+1 = v_f(1) = \frac{n+2}{2}$	$e_f(0) = e_f(1) = \frac{n+2}{2}$
1	$v_f(0) = v_f(1) = \frac{n+1}{2}$	$e_f(0) = e_f(1)+1 = \frac{n+3}{2}$

TABLE 6.2

Case 3 : For $n = 3, 4$

The graph G with p vertices and q edges does not satisfy the condition $q < \frac{p^2-1}{4}$ hence G is not product cordial as stated by Sundaram et al.[42]

Case 4 : For $n = 5$

To satisfy vertex condition it is essential to label 3 vertices with label 0. It is obvious that any edge will have label 0 if it is incident to vertex with label 0. As G has 3 vertices with label zero and minimum degree of the vertices are of 2, it has at least $3 \times 2 - 1 = 5$ edges with label 0 and at most $7 - 5 = 2$ edges with label 1. Here $|e_f(0) - e_f(1)| = |5 - 2| = 3$. Thus edge condition is not satisfied. Hence G is not product cordial.

Thus from the case 1 to 4 we conclude that the Duplication of arbitrary vertex v_k of cycle C_n with $n \geq 6$ produces product cordial graph. □

Illustration 6.3.5. Consider a graph obtained by duplicating vertex v_3 of cycle C_8. Here $n = 8$ i.e. n is even. The product cordial labeling is as shown in FIGURE 6.2.

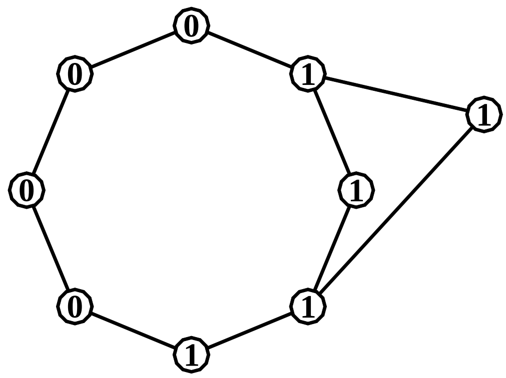

FIGURE 6.2

Theorem 6.3.6. The graph obtained by duplication of arbitrary rim vertex of wheel W_n is product cordial for odd n and not product cordial for even n, where $n \geq 6$.

Proof. Consider the wheel. Let $v_1, v_2, v_3, \ldots, v_n$ be the rim vertices of wheel and let c_1 be the apex vertex. Let G be the graph obtained by duplicating arbitrary rim vertex v_k

of wheel. Let $v_k^{'}$ be duplicated vertex of v_k. The following function $f : V(G) \longrightarrow \{0,1\}$ gives product cordial labeling for the following case.

Case 1 : For odd n

$$
\begin{aligned}
f(v_{k+i-1}) &= 1; \text{ if } 1 \le i \le \tfrac{n-3}{2} \text{ and } k+i-1 \le n\\
f(v_{k+i-n-1}) &= 1; \text{ if } 1 \le i \le \tfrac{n-3}{2} \text{ and } k+i-1 > n\\
f(v_{k+i-1}) &= 0; \text{ if } \tfrac{n-1}{2} \le i < n \text{ and } k+i-1 \le n\\
f(v_{k+i-n-1}) &= 0; \text{ if } \tfrac{n-1}{2} \le i < n \text{ and } k+i-1 > n\\
f(v_n) &= 1; \text{ if } k = 1\\
f(v_{k-1}) &= 1; \text{ if } k > 1\\
f(v_k^{'}) &= 1;\\
f(c_1) &= 1;
\end{aligned}
$$

The above defined labeling pattern includes all possible arrangement of vertices. The following TABLE 6.3(where $n = 2a + b$ and $a \in N$) show the conditions of product cordiality for the above defined function is satisfied by G.

b	**Vertex Condition**	**Edge Condition**
$a \geq 3$	$v_f(0)+1 = v_f(1) = \frac{n+3}{2}$	$e_f(0) = e_f(1)+1 = n+2$

TABLE 6.3

Case 2 : For even n

If n is even $n-1$ is odd. According to case 1 duplication of arbitrary rim vertex of W_{n-1} is product cordial and satisfy vertex condition $v_f(0)+1 = v_f(1)$. W_n contains one more vertex than W_{n-1}. In order to satisfy vertex condition this vertex must have label 0 which forces us to assign 0 labels to two edges. i.e. $e_f(0) = e_f(1)+3$. Therefore the graph obtained by duplication of arbitrary rim vertex of W_n is not product cordial for even n.

Case 3 : For $n = 3,4,5$

The graph G with p vertices and q edges does not satisfy the condition $q < \frac{p^2-1}{4}$ hence G is not product cordial as proved by Sundaram et al.[42].

Thus from the case 1 to 3 we conclude that the graph obtained by duplication of arbitrary rim vertex of wheel W_n is product cordial for odd n and not product cordial for even n, where $n \geq 6$. □

Illustration 6.3.7. Consider a graph obtained by duplication of rim vertex v_4 of wheel W_9. Here $n = 9$. The product cordial labeling is as shown in FIGURE 6.3.

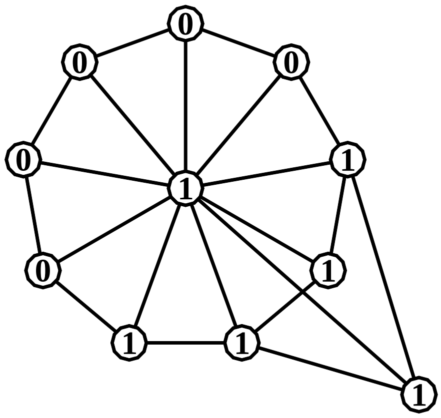

FIGURE 6.3

Theorem 6.3.8. The graph obtained by duplication of apex vertex of wheel W_n is not product cordial graph.

Proof. Consider the wheel. Let G be the graph obtained by duplication of apex vertex c_1 of wheel. Let $c_1^{'}$ be duplicated vertex of c_1. Graph G contains $n+2$ vertices and $3n$ edges. Degree of each rim vertex is 4 and degree of apex vertex and its duplicated vertex is n. Vertex label of c_1 and $c_1^{'}$ must be 1 because label 0 will give rise to $2n$ edges with label 0 which will violate edge condition.

Case 1 : For even n

To satisfy vertex condition it is essential to label $\frac{n+2}{2}$ vertices with label 0. It is obvious that any edge will have label 0 if it is incident to vertex with label 0. As G has $\frac{n+2}{2}$ vertices with label zero and all the rim vertices are of degree 4, it has at least $\frac{3(n+2)}{2} + 1$ edges with label 0 and at most $3n - \frac{3(n+2)}{2} - 1$, i.e $\frac{3n-8}{2}$ edges with label 1. Here $|e_f(0) - e_f(1)| = |\frac{3(n+2)}{2} + 1 - \frac{3n-8}{2}| = 8$. Thus edge condition is not satisfied. Hence G is not product cordial graph.

Case 2 : For odd n

To satisfy vertex condition it is essential to label $\frac{n+1}{2}$ vertices with label 0. It is obvious that any edge will have label 0 if it is incident to vertex with label 0. As G has $\frac{n+1}{2}$ vertices with label zero and all the rim vertices are of degree 4, it has at least $\frac{3(n+1)}{2}+1$ edges with label 0 and at most $3n-\frac{3(n+1)}{2}-1$, i.e $\frac{3n-5}{2}$ edges with label 1. Here $|e_f(0)-e_f(1)| = |\frac{3(n+1)}{2}+1-\frac{3n-5}{2}| = 5$. Thus edge condition is not satisfied. Hence G is not product cordial. □

Definition 6.3.9. A *vertex switching* G_v of a graph G is obtained by taking a vertex v of G, removing all edges incidence to v and adding edges joining v to every vertex not adjacent to v in G.

Theorem 6.3.10. Vertex switching of cycle C_n is product cordial.

Proof. Let $G = C_n$ and $v_1, v_2, \ldots, v_n$ be successive vertices of C_n. G_{v_k} denotes the vertex switching of G with respect to the vertex v_k of G. To define binary vertex labeling $f : V(G_{v_k}) \longrightarrow \{0,1\}$ we consider following.

$$
\begin{aligned}
f(v_{k+i-1}) &= 1; \text{ if } 1 \le i \le \lceil \tfrac{n+2}{2} \rceil \text{ and } k+i-1 \le n \text{ and } i \ne 2\\
f(v_{k+i-n-1}) &= 1; \text{ if } 1 \le i \le \lceil \tfrac{n+2}{2} \rceil \text{ and } k+i-1 > n \text{ and } i \ne 2\\
f(v_{k+i-1}) &= 0; \text{ if } \lceil \tfrac{n+4}{2} \rceil \le i < n \text{ and } k+i-1 \le n\\
f(v_{k+i-n-1}) &= 0; \text{ if } \lceil \tfrac{n+4}{2} \rceil \le i < n \text{ and } k+i-1 > n\\
f(v_{k+1}) &= 0; \text{ if } k \ne n\\
f(v_1) &= 0; \text{ if } k = n
\end{aligned}
$$

The above defined labeling pattern includes all possible arrangement of vertices. The following TABLE 6.4(where $n = 2a+b$ and $a \in N$) show the conditions of product cordiality for the above defined function is satisfied by G_{v_k}. □

b	**Vertex Condition**	**Edge Condition**
0	$v_f(0)+1 = v_f(1) = \frac{n+1}{2}$	$e_f(0)+1 = e_f(1) = n-2$
1	$v_f(0) = v_f(1) = \frac{n}{2}$	$e_f(0) = e_f(1)+1 = n-2$

TABLE 6.4

Illustration 6.3.11. Consider a graph obtained by vertex switching of v_4 of wheel C_9. Here $n = 9$. The product cordial labeling is as shown in FIGURE 6.4.

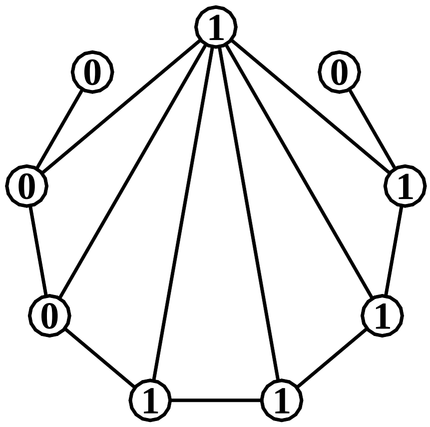

FIGURE 6.4

Theorem 6.3.12. Graph $< S_n^{(1)} : S_n^{(2)} >$ is product cordial.

Proof. Let $v_1^{(1)}, v_2^{(1)}, v_3^{(1)}, \dots, v_n^{(1)}$ be the vertices $S_n^{(1)}$ and $v_1^{(2)}, v_2^{(2)}, v_3^{(2)}, \dots, v_n^{(2)}$ be the vertices $S_n^{(2)}$. Let $v_1^{(1)}$ and $v_1^{(2)}$ be the apex vertices of $S_n^{(1)}$ and $S_n^{(2)}$ respectively which are joined to a vertex x. For $G =< S_n^{(1)} : S_n^{(2)} >$. We define binary vertex labeling $f : V(G) \to \{0,1\}$ as follows.

$$\left.\begin{array}{rcl} f(v_i^{(1)}) & = & 1; \\ f(v_i^{(2)}) & = & 0; \end{array}\right\} \text{For } 1 \le i \le n$$

$$f(x) = 1;$$

Thus vertices of $S_n^{(1)}$ are labeled with 1 and vertices of $S_n^{(2)}$ are labeled with 0 while the vertex x is labeled with 1. Consequently $v_f(0) = n, v_f(1) = n+1$ and $e_f(0) = e_f(1) = 2n-2$. Thus the graph G satisfies the conditions for product cordial graph. That is, G admits product cordial labeling. □

Illustration 6.3.13. Consider a graph $G =< S_8^{(1)} : S_8^{(2)} >$. Here $n = 8$. The product cordial labeling is as shown in FIGURE 6.5 .

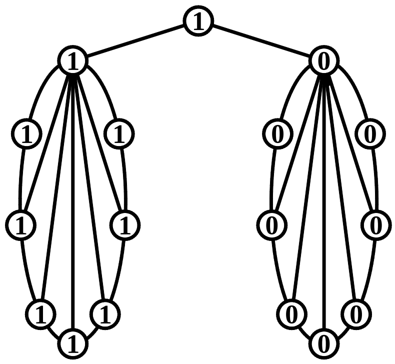

FIGURE 6.5

Theorem 6.3.14. Graph $< S_n^{(1)} : S_n^{(2)} : \ldots : S_n^{(k)} >$ is product cordial except k odd and n even.

Proof. Let $S_n^{(j)}$ be the shells. Let $v_i^{(j)}$ be the vertices $S_n^{(j)}$ and $v_1^{(j)}$ be the apex vertices of $S_n^{(j)}$. Let $x_j (j \neq k)$ be the new vertices where $1 \leq j \leq k$. Let $G =< S_n^{(1)} : S_n^{(2)} : \ldots : S_n^{(k)} >$. For $1 \leq i \leq n$ we define binary vertex labeling $f : V(G) \to \{0,1\}$ as follows.

Case 1: k even

$f(v_i^{(j)}) = 1$; if $j \leq \frac{k}{2}$

$f(v_i^{(j)}) = 0$; if $j > \frac{k}{2}$

$f(x_j) = 1$; if $j \leq \frac{k}{2}$

$f(x_j) = 0$; if $\frac{k}{2} < j \leq k-1$

Case 2: both k and n odd

$f(v_i^{(j)}) = 1$; if $j \leq \frac{k-1}{2}$

$f(v_i^{(j)}) = 1$; if $j = \frac{k+1}{2}$ and $i \leq \frac{n+1}{2}$

$f(v_i^{(j)}) = 0$; if $j = \frac{k+1}{2}$ and $i > \frac{n+1}{2}$

$f(v_i^{(j)}) = 0$; if $j > \frac{k+1}{2}$

$f(x_j) = 1$; if $j \leq \frac{k-1}{2}$

$f(x_j) = 0$; if $\frac{k-1}{2} < j \leq k-1$

In both the cases described above the graph G satisfies the vertex condition $v_f(0) + 1 = v_f(1) = \frac{k(n+1)}{2}$ and edge condition $e_f(0) = e_f(1) + 1 = \frac{k(2n-1)-1}{2}$.

Case 3 : k odd and n even

We assign label 1 to all the vertices of first copies of shells and assign label 0 to all the vertices of last copies of shells. This will provide equal number of vertices and edges with label 0 and 1. Now our task is to label n vertices of a shell (i.e. vertices of $(\frac{k+1}{2})^{\text{th}}$ copy). In order to satisfy vertex condition for product cordiality $\frac{n}{2}$ vertices must be labeled with 0. Then at least n edges will get label 0. Consequently the number of edges with label 1 is $(2n-3)-(n) = n-3$ because $|S_n(E)| = 2n-3$. Hence $|e_f(0) - e_f(1)| = |n-(n-3)| = 3$. Thus edge condition is not satisfied. i.e. G is not product cordial graph.

Thus from the case 1 to 3 we conclude that graph $< S_n^{(1)} : S_n^{(2)} : \ldots : S_n^{(k)} >$ is product cordial except k odd and n even. $\square$

Illustration 6.3.15. Consider a graph $G =< S_7^{(1)} : S_7^{(2)} : S_7^{(3)} >$. Here $n = 7$. The product cordial labeling is as shown in FIGURE 6.6.

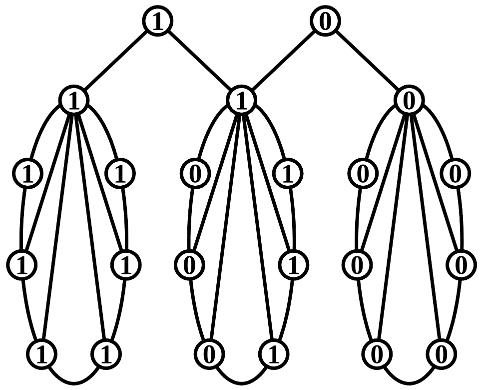

FIGURE 6.6

Theorem 6.3.16. Graph $< K_{1,n}^{(1)} : K_{1,n}^{(2)} >$ is product cordial.

Proof. Let $v_1^{(1)}, v_2^{(1)}, \ldots, v_n^{(1)}$ and $v_1^{(2)}, v_2^{(2)}, \ldots, v_n^{(2)}$ be the pendant vertices of $K_{1,n}^{(1)}$ and $K_{1,n}^{(2)}$ respectively. Let c_1 and c_2 be the apex vertices of $K_{1,n}^{(1)}$ and $K_{1,n}^{(2)}$ respectively which are adjacent to a common vertex x. Let $G =< K_{1,n}^{(1)} : K_{1,n}^{(2)} >$. We define binary vertex labeling $f : V(G) \to \{0,1\}$ as follows.

$$\left.\begin{array}{rcl} f(v_i^{(1)}) & = & 1; \\ f(v_i^{(2)}) & = & 0; \end{array}\right\} \text{For } 1 \le i \le n$$

$$f(x) = 1;$$

In view of the above defined labeling pattern $v_f(0) = e_f(0) = e_f(1) = n+1$ and $v_f(1) = n+2$. Thus the graph G satisfies the vertex condition and edge condition because $v_f(0)+1 = v_f(1)$ and $e_f(0) = e_f(1)$. That is, G admits product cordial labeling. □

Illustration 6.3.17. Consider a graph $G = < K_{1,8}^{(1)} : K_{1,8}^{(2)} >$. Here $n = 8$. The product cordial labeling is as shown in FIGURE 6.7.

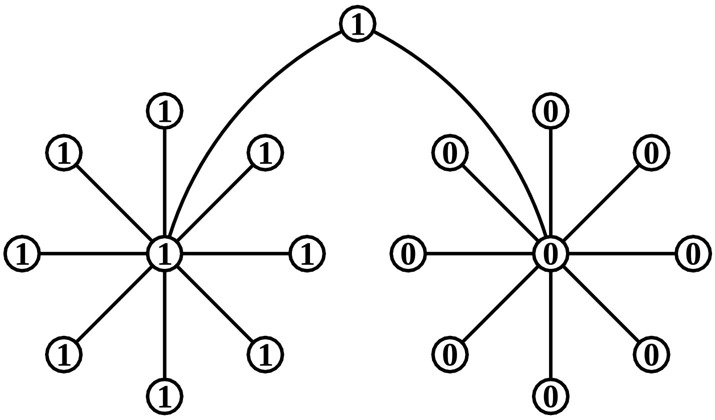

FIGURE 6.7

Theorem 6.3.18. Graph $< K_{1,n}^{(1)} : K_{1,n}^{(2)} : K_{1,n}^{(3)} : \ldots : K_{1,n}^{(k)} >$ is product cordial.

Proof. Let $v_i^{(j)}$ be the pendant vertices of $K_{1,n}^{(j)}$ and c_j be the apex vertices of $K_{1,n}^{(j)}$. Let $x_j (j \neq k)$ be the new vertices where Let$G = < K_{1,n}^{(1)} : K_{1,n}^{(2)} : K_{1,n}^{(3)} : \ldots : K_{1,n}^{(k)} >$. We define binary vertex labeling $f : V(G) \to \{0,1\}$ as follows.

Case 1: k even

$$\left.\begin{array}{rcl} f(v_i^{(j)}) & = & 1; \text{ if } 1 \le j \le \frac{k}{2} \\ f(v_i^{(j)}) & = & 0; \text{ if } \frac{k+2}{2} \le j \le k \end{array}\right\} \text{For } 1 \le i \le n$$

$f(c_j) = 1$; if $1 \leq j \leq \frac{k}{2}$

$f(c_j) = 0$; if $\frac{k+2}{2} \leq j \leq k$

$f(x_j) = 1$; if $1 \leq j \leq \frac{k}{2}$

$f(x_j) = 0$; if $\frac{k+2}{2} \leq j \leq k-1$

Case 2: k odd

Subcase 1: n even

$$\left.\begin{array}{rcl} f(v_i^{(j)}) & = & 1; \text{ if } 1 \leq j \leq \frac{k-1}{2} \\ f(v_i^{(j)}) & = & 0; \text{ if } \frac{k+3}{2} \leq j \leq k \end{array}\right\} \text{For } 1 \leq i \leq n$$

$f(c_j) = 1$; if $1 \leq j \leq \frac{k+1}{2}$

$f(c_j) = 0$; if $\frac{k+3}{2} \leq j \leq k$

$f(x_j) = 1$; if $1 \leq j \leq \frac{k-1}{2}$

$f(x_j) = 0$; if $\frac{k+1}{2} \leq j \leq k-1$

$$\left.\begin{array}{rcl} f(v_i^{(j)}) & = & 1; \text{ if } 1 \leq i \leq \frac{n}{2} \\ f(v_i^{(j)}) & = & 0; \text{ if } \frac{n+2}{2} \leq i \leq n \end{array}\right\} \text{For } j = \frac{k+2}{2}$$

Subcase 2: n odd

$$\left.\begin{array}{rcl} f(v_i^{(j)}) & = & 1; \text{ if } 1 \leq j \leq \frac{k-1}{2} \\ f(v_i^{(j)}) & = & 0; \text{ if } \frac{k+3}{2} \leq j \leq k \end{array}\right\} \text{For } 1 \leq i \leq n$$

$f(c_j) = 1$; if $1 \leq j \leq \frac{k+1}{2}$

$f(c_j) = 0$; if $\frac{k+3}{2} \leq j \leq k$

$f(x_j) = 1$; if $1 \leq j \leq \frac{k-1}{2}$

$f(x_j) = 0$; if $\frac{k+1}{2} \leq j \leq k-1$

$$\left.\begin{array}{rcl} f(v_i^{(j)}) & = & 1; \text{ if } 1 \leq i \leq \frac{n-1}{2} \\ f(v_i^{(j)}) & = & 0; \text{ if } \frac{n+1}{2} \leq i \leq n \end{array}\right\} \text{For } j = \frac{k+2}{2}$$

The labeling pattern defined above exhaust all the possibilities for n and k and in each cases the graph G satisfies the conditions $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$

as shown in TABLE 6.5(where $n = 2a+b$, $k = 2c+d$ and $a,c \in N$). That is, G admits product cordial labeling. □

d	b	Vertex Condition	Edge Condition
0	0,1	$v_f(0)+1 = v_f(1) = \frac{k(n+2)}{2}$	$e_f(0) = e_f(1) = \frac{k(n+2)-2}{2}$
1	0	$v_f(0)+1 = v_f(1) = \frac{k(n+2)}{2}$	$e_f(0) = e_f(1) = \frac{k(n+2)-2}{2}$
	1	$v_f(0) = v_f(1) = \frac{k(n+2)-1}{2}$	$e_f(0) = e_f(1)+1 = \frac{k(n+2)-1}{2}$

TABLE 6.5

Illustration 6.3.19. Consider a graph $G =< K_{1,5}^{(1)} : K_{1,5}^{(2)} : K_{1,5}^{(3)} >$. Here $n = 5$. The product cordial labeling is as shown in FIGURE 6.8.

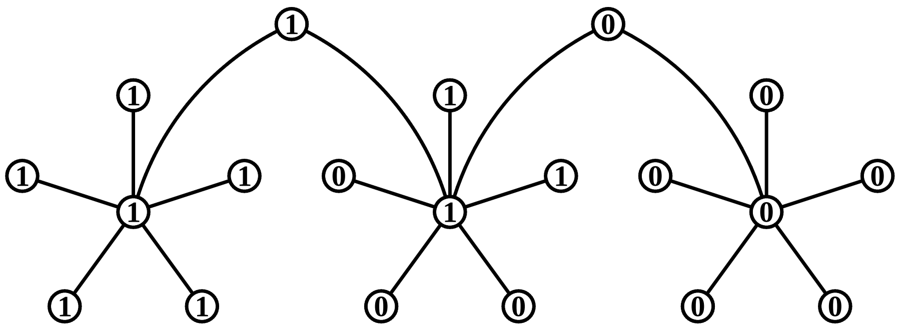

FIGURE 6.8

Theorem 6.3.20. Graph $< W_n^{(1)} : W_n^{(2)} >$ is product cordial.

Proof. Let $v_1^{(1)}, v_2^{(1)}, \ldots, v_n^{(1)}$ and $v_1^{(2)}, v_2^{(2)}, \ldots, v_n^{(2)}$ be the rim vertices of $W_n^{(1)}$ and $W_n^{(2)}$ respectively. Let c_1 and c_2 be the apex vertices of $W_n^{(1)}$ and $W_n^{(2)}$ respectively which are adjacent to a common vertex x. Let $G =< W_n^{(1)} : W_n^{(2)} >$. We define binary vertex labeling $f : V(G)\{0,1\}$ as follows.

$$\left.\begin{array}{rcl} f(v_i^{(1)}) & = & 1; \\ f(v_i^{(2)}) & = & 0; \end{array}\right\} \text{For } 1 \leq i \leq n$$

$$f(x) = 1;$$

Then the graph G satisfies the vertex condition $v_f(0)+1 = v_f(1) = n+2$ and edge condition $e_f(0) = e_f(1) = 2n+1$. That is, G admits product cordial labeling. □

Illustration 6.3.21. Consider a graph $G =< W_7(1) : W_7(2) >$. Here $n = 7$. The product cordial labeling is as shown in FIGURE 6.9.

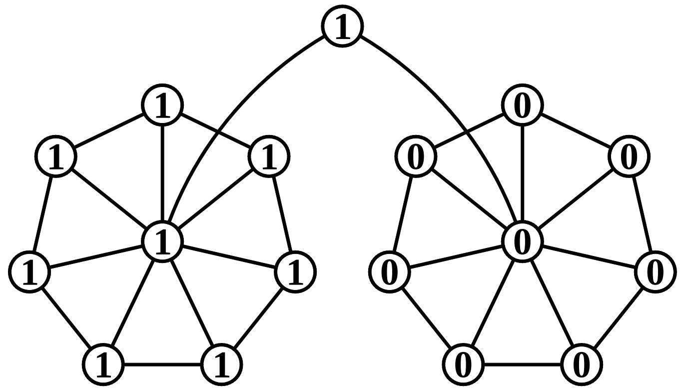

FIGURE 6.9

Theorem 6.3.22. Graph $< W_n^{(1)} : W_n^{(2)} : W_n^{(3)} : ... : W_n^{(k)} >$ is product cordial (i)for k even and n even or odd (ii)for k odd and n even with $k > n$ and (iii) not product cordial otherwise.

Proof. Let $v_i^{(j)}$ be the rim vertices $W_n^{(j)}$ and c_j be the apex vertices of $W_n^{(j)}$. Let $x_j (j \neq k)$ be the new vertices. Let $G =< W_n^{(1)} : W_n^{(2)} : W_n^{(3)} : ... : W_n^{(k)} >$. We define binary vertex labeling $f : V(G) \to \{0, 1\}$ as follows.

Case 1: k even

$$\left.\begin{array}{rcl} f(v_i^{(j)}) &=& 1;\text{ if } 1 \leq j \leq \frac{k}{2} \\ f(v_i^{(j)}) &=& 0;\text{ if } \frac{k+2}{2} \leq j \leq k \end{array}\right\} \text{For } 1 \leq i \leq n$$

$f(c_j) = 1;$ if $1 \leq j \leq \frac{k}{2}$
$f(c_j) = 0;$ if $\frac{k+2}{2} \leq j \leq k$
$f(x_j) = 1;$ if $1 \leq j \leq \frac{k}{2}$
$f(x_j) = 0;$ if $\frac{k+2}{2} \leq j \leq k-1$

Case 2: k odd, n even with $k > n$

$$\left.\begin{array}{rcl} f(v_i^{(j)}) &=& 1;\text{ if } 1 \leq j \leq \frac{k+1}{2} \\ f(v_i^{(j)}) &=& 0;\text{if } \frac{k+3}{2} \leq j \leq k \end{array}\right\} \text{For } 1 \leq i \leq n$$

$f(c_j) = 1$; if $1 \leq j \leq \frac{k+1}{2}$
$f(c_j) = 0$; if $\frac{k+3}{2} \leq j \leq k$
$f(x_j) = 1$; if $1 \leq j \leq \frac{k-n-1}{2}$
$f(x_j) = 0$; if $\frac{k-n+1}{2} \leq j \leq k-1$

In both the cases described above the graph G satisfies the vertex condition as $v_f(0)+1 = v_f(1) = \frac{k(n+2)}{2}$ and edge condition as $e_f(0) = e_f(1) = k(n+1)-1$. i.e. G admits product cordial labeling.

Thus we proved (i) and (ii) while to prove (iii) we have to consider following two cases.

Case 3: k and n odd

We assign label 1 to all the vertices of first $\frac{k-1}{2}$ copies of wheels and assign label 0 to all the vertices of last $\frac{k-1}{2}$ copies of wheels. This will provide equal number of vertices and edges with label 0 and 1. Now our task is to label $n+1$ vertices of a wheel (i.e. vertices of $(\frac{k+1}{2})^{\text{th}}$ copy). In order to satisfy vertex condition for product cordiality $\frac{n+1}{2}$ vertices must be labeled with 0. Then at least $n+2$ edges will get label 0. Consequently the number of edges with label 1 is $(2n)-(n+2) = n-2$ because $|W_n(E)| = 2n$. Hence $|e_f(0) - e_f(1)| = |n+2-(n-2)| = 4$. Thus edge condition is not satisfied. i.e. G is not product cordial graph.

Case 4: For k odd and n even with $n \geq k$

If $\frac{k+1}{2}$ copies of wheel are labeled with 1 then vertex condition is not satisfied as $n \geq k$. Then arguing as in case 3 the graph G does not admit product cordial labeling.

Thus from case 1 to 4 we have the required result. □

Illustration 6.3.23. Consider a graph $G = < W_6^{(1)} : W_6^{(2)} : W_6^{(3)} : W_6^{(4)} >$. Here $n = 6$. The product cordial labeling is as shown in FIGURE 6.10.

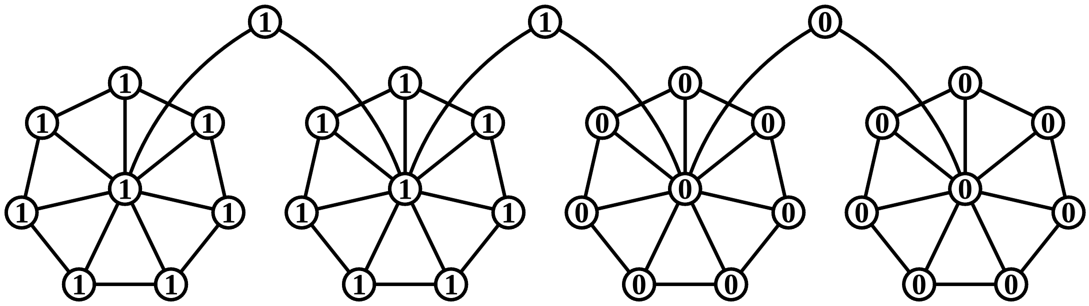

FIGURE 6.10

6.4 Open Problems

It is always interesting to investigate a particular type of labeling for a larger graph resulted from some graph operations on standard graphs. It is possible to derive results corresponding to various graph operations and in the context of different graph labeling assignment.

6.5 Concluding Remarks

We have investigated product cordial labeling for the graph resulted due to graph operations like fusion, duplication and switching of vertex. In addition to this we derive some results for wheel, star and shell related graph.

The results reported here are published in the following research paper.

1. Some new product cordial graphs.,*Journal of Applied Computer Science & Mathematics*, 8(4),2010, 62-65.(http://jacs.usv.ro)

The reprint of the above research paper is provided in Annexure.

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List of Symbols

$\|B\|$	Cardinality of set B.
CH_n	Closed helm on n vertices.
C_n	Cycle with n vertices.
$E(G)$ or E	Edge set of graph G.
F_n	Fan on n vertices.
$G \times H$	Cartesian product of graphs G and H.
$G = (V(G), E(G))$	A graph G with vertex set $V(G)$ and edge set $E(G)$.
$G - e$	Graph G with one edge deleted.
$G - v$	Graph G with one vertex deleted.
H_n	Helm on n vertices.
K_n	Complete graph on n vertices.
$K_{m,n}$	Complete bipartite graph.
$N(v)$	Open neighbourhood of vertex v.
$N[v]$	Closed neighbourhood of vertex v.
P_n	Path graph on n vertices.
S_n	Shell on n vertices.
T	Tree.
$T(G)$	Spanning tree of graph G.
$V(G)$ or V	Vertex set of graphs G.
W_n	Wheel on n vertices.
$d(v)$ or $d_G(v)$	Degree of a vertex v of graph G.
$e_f(n)$	Number of edges with edge label n.
$\lceil n \rceil$	Least integer not less than real number n (Ceiling of n).

$\lfloor n \rfloor$	Greatest integer not greater than real number n (Floor of n).
(p,q)	A graph with order p and size q.
$v_f(n)$	Number of vertices with vertex label n.

Annexure

Publications

Available Online

J. Sci. Res. **1** (3), 438-449 (2009)

JOURNAL OF SCIENTIFIC RESEARCH

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Cordial and 3-Equitable Labeling for Some Shell Related Graphs

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Abstract

We present here cordial and 3-equitable labeling for the graphs obtained by joining apex vertices of two shells to a new vertex. We extend these results for k copies of shells.

Keywords: Cordial labeling; 3-equitable labeling; Shell.

DOI: 10.3329/jsr.v1i3.2227 J. Sci. Res. **1** (3), 438-449 (2009)

1. Introduction

We begin with simple, finite, connected and undirected graph $G = (V, E)$. For all standard terminology and notations we follow Harary [1]. We will give brief summary of definitions which are useful for the present investigations.

Definition 1.1 A *shell* S_n is the graph obtained by taking n-3 concurrent chords in a cycle C_n on n vertices. The vertex at which all the chords are concurrent is called the *apex vertex*. The shell is also called fan F_{n-1}. *i.e.* $S_n=F_{n-1}=P_{n-1}+K_1$.

Definition 1.2: Consider two shells $S_n^{(1)}$ and $S_n^{(2)}$ then graph $G =< S_n^{(1)}: S_n^{(2)} >$ obtained by joining apex vertices of shells to a new vertex x.

Definition 1.3: Consider k copies of shells namely $S_n^{(1)}, S_n^{(2)}, S_n^{(3)}, \ldots, S_n^{(k)}$. Then the graph $G =< S_n^{(1)}: S_n^{(2)}: S_n^{(3)}: \ldots: S_n^{(k)}>$ obtained by joining apex vertex of each $S_n^{(p)}$ and apex of $S_n^{(p-1)}$ to a new vertex x_{p-1} where $2 \le p \le k$.

Definition 1.4: If the vertices of the graph are assigned values subject to certain conditions then it is known as *graph labeling*.

Most interesting graph labeling problems have following three important ingredients.

(i) a set of numbers from which the labels are chosen;
(ii) a rule that assigns a value to each edges;
(iii) a condition that these values must satisfy.

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For detail survey on graph labeling one can refer Gallian [2]. Vast amount of literature is available on different types of graph labeling. According to Beineke and Hegde [3] graph labeling serves as a frontier between number theory and structure of graphs. Labeled graph have variety of applications in coding theory, particularly for missile guidance codes, design of good radar type codes and convolution codes with optimal autocorrelation properties. Labeled graph plays vital role in the study of X-ray crystallography, communication network and to determine optimal circuit layouts. A detail study of variety of applications of graph labeling is given by Bloom and Golomb[4].

Definition 1.5: Let $G = (V, E)$ be a graph. A mapping $f: V(G) \rightarrow \{0, 1\}$ is called binary vertex labeling of G and $f(v)$ is called the label of the vertex v of G under f.

For an edge $e = uv$, the induced edge labeling $f^*: E(G) \rightarrow \{0, 1\}$ is given by $f^*(e) = |f(u) - f(v)|$. Let $v_f(0)$, $v_f(1)$ be the number of vertices of G having labels 0 and 1 respectively under f while $e_f(0)$, $e_f(1)$ be the number of edges having labels 0 and 1 respectively under f^*.

Definition 1.6: A binary vertex labeling of a graph G is called a cordial labeling if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph G is cordial if it admits cordial labeling.

The concept of cordial labeling was introduced by Cahit [5]. Many researchers have studied cordiality of graphs. Cahit [5] proved that tree is cordial. In the same paper he proved that K_n is cordial if and only if $n \leq 3$. Ho *et al.* [6] proved that unicyclic graph is cordial unless it is C_{4k+2}. Andar *et al.* [7] have discussed the cordiality of multiple shells. Vaidya *et al.* [8, 9, 10] have also discussed the cordiality of various graphs.

Definition 1.7: Let $G = (V, E)$ be a graph. A mapping $f: V(G) \rightarrow \{0, 1, 2\}$ is called ternary vertex labeling of G and $f(v)$ is called the label of the vertex v of G under f.

For an edge $e = uv$, the induced edge labeling $f^*: E(G) \rightarrow \{0, 1, 2\}$ is given by $f^*(e) = |f(u) - f(v)|$. Let $v_f(0)$, $v_f(1)$ and $v_f(2)$ be the number of vertices of G having labels 0,1 and 2 respectively under f while $e_f(0)$, $e_f(1)$ and $e_f(2)$ be the number of edges having labels 0,1 and 2 respectively under f^*.

Definition 1.8: A vertex labeling of a graph G is called a 3-equitable labeling if $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $0 \leq i, j \leq 2$. A graph G is 3-equitable if it admits 3-equitable labeling.

The concept of 3-equitable labeling was introduced by Cahit [11]. Many researchers have studied 3-equitability of graphs. For example Cahit [11] proved that C_n is 3-equitable except $n \equiv 3(mod\,6)$. In the same paper he proved that an Eulerian graph with number of edges congruent to $3(mod\,6)$ is not 3-equitable. Youssef [12] proved that W_n is 3-equitable for all $n \leq 4$. In the present investigations we prove that graphs $< S_n^{(1)}: S_n^{(2)} >$ and $< S_n^{(1)}: S_n^{(2)} : S_n^{(3)}: \ldots : S_n^{(k)} >$ cordial as well as 3-equitable.

2. Main Results

Theorem 2.1: Graph $< S_n^{(1)}: S_n^{(2)} >$ is cordial.

Proof: Let $v_1^{(1)}, v_2^{(1)}, v_3^{(1)}, \ldots v_n^{(1)}$ be the vertices $S_n^{(1)}$ of $v_1^{(2)}, v_2^{(2)}, v_3^{(2)}, \ldots v_n^{(2)}$ be the vertices of $S_n^{(2)}$. Let $v_1^{(1)}$ and $v_1^{(2)}$ be the apex vertices of $S_n^{(1)}$ and $S_n^{(2)}$, respectively. Let $G = < S_n^{(1)}: S_n^{(2)} >$. We define binary vertex labeling f$: V(G) \rightarrow \{0, 1\}$ as follows.

For j=1, 2
$f(v_i^{(j)}) = 0$; if $i \equiv 2,3 (mod 4)$
$f(v_i^{(j)}) = 1$; if $i \equiv 0,1 (mod 4)$
$f(x) = 0$; if $n \equiv 1 (mod 4)$
$f(x) = 1$; if $n \equiv 0,2,3 (mod 4)$

The labeling pattern defined above covers all possible arrangement of vertices. The graph G satisfies the conditions $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. as shown in Table 1. i.e. G admits cordial labeling. □

Let $n=4a+b$

Table 1. Table showing vertex and edge conditions.

b	Vertex condition	Edge condition
0,1,2	$v_f(0)+1=v_f(1)$	$e_f(0)=e_f(1)$
3	$v_f(0) = v_f(1) + 1$	$e_f(0)=e_f(1)$

Illustration 2.2: Consider a graph $G = < S_7^{(1)} : S_7^{(2)} >$. Here $n = 7$. The cordial labeling is as shown in Fig. 1.

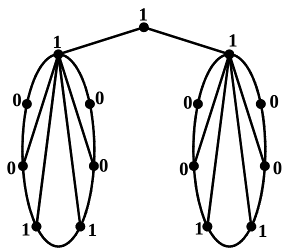

Fig. 1. Cordial labeling of the graph G.

Theorem 2.3: Graph $< S_n^{(1)}: S_n^{(2)}: S_n^{(3)}: \ldots: S_n^{(k)} >$ is cordial.

Proof: Let $S_n^{(j)}$ be the shells. Let $v_i^{(j)}$ be the vertices of $S_n^{(j)}$ and $v_1^{(j)}$ be the apex vertices of $S_n^{(j)}$. Let x_j $(j \neq k)$ be the new vertices. Let $G = < S_n^{(1)}: S_n^{(2)}: S_n^{(3)}: \ldots: S_n^{(k)} >$. We define binary vertex labeling $f: V(G) \rightarrow \{0, 1\}$ as follows.

For $j \equiv 1, 2 (mod 4)$
$f(v_i^{(j)}) = 0$; if $i \equiv 2,3 (mod 4)$
$f(v_i^{(j)}) = 1$; if $i \equiv 0,1 (mod 4)$
For $j \equiv 0, 3 (mod 4)$
$f(v_i^{(j)}) = 0$; if $i \equiv 0,1 (mod 4)$

$f(v_i^{(j)}) = 1;$ if $i \equiv 2,3 (mod 4)$
For $n \equiv 0,2,3 (mod 4)$
$f(x_j) = 0;$ if $j \equiv 2,3 (mod 4)$
$f(x_j) = 1;$ if $j \equiv 0,1 (mod 4), j \neq k$
For $n \equiv 1 (mod 4)$
$f(x_j) = 0;$ if $j \equiv 1,2 (mod 4)$
$f(x_j) = 1;$ if $j \equiv 0,3 (mod 4), j \neq k$

The labeling pattern defined above covers all possible arrangement of vertices. The graph G satisfies the conditions $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ as shown in Table 2. i.e. G admits cordial labeling. □

Let $n=4a+b$, $k=4c+d$

Table 2. Table showing vertex and edge conditions.

b	d	Vertex condition	Edge condition
0	0	$v_f(0) = v_f(1) + 1$	$e_f(0) = e_f(1)$
	1,3	$v_f(0) = v_f(1)$	$e_f(0)+1 = e_f(1)$
	2	$v_f(0)+1 = v_f(1)$	$e_f(0) = e_f(1)$
1	0	$v_f(0) = v_f(1) + 1$	$e_f(0) = e_f(1)$
	1	$v_f(0)+1 = v_f(1)$	$e_f(0) = e_f(1)+1$
	2	$v_f(0)+1 = v_f(1)$	$e_f(0) = e_f(1)$
	3	$v_f(0) = v_f(1) + 1$	$e_f(0) = e_f(1)+1$
2	0	$v_f(0) = v_f(1) + 1$	$e_f(0) = e_f(1)$
	1,3	$v_f(0) = v_f(1)$	$e_f(0) + 1 = e_f(1)$
	2	$v_f(0)+1 = v_f(1)$	$e_f(0) = e_f(1)$
3	0,2	$v_f(0) = v_f(1) + 1$	$e_f(0) = e_f(1)$
	1,3	$v_f(0) = v_f(1) + 1$	$e_f(0) + 1 = e_f(1)$

Illustration 2.4: Consider a graph $G = < S_5^{(1)} : S_5^{(2)} : S_5^{(3)} >$. Here $n = 5$. The cordial labeling is as shown in Fig. 2.

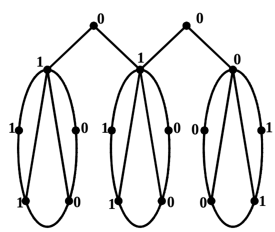

Fig. 2. Cordial labeling of the graph G.

Theorem 2.5: Graph $< S_n^{(1)}: S_n^{(2)} >$ is 3-equitable.

Proof: Let $v_1^{(1)}, v_2^{(1)}, \ldots v_n^{(1)}$ be the vertices $S_n^{(1)}$ and $v_1^{(2)}, v_2^{(2)}, v_3^{(2)}, \ldots v_n^{(2)}$ be the vertices $S_n^{(2)}$. Let $v_1^{(1)}$ and $v_1^{(2)}$ be the apex vertices of $S_n^{(1)}$ and $S_n^{(2)}$, respectively. Let $G =< S_n^{(1)}: S_n^{(2)} >$. We define ternary vertex labeling $f: V(G) \to \{0, 1, 2\}$ as follows.

Case-1: For $n \equiv 0, 5(mod 6)$

$f(v_i^{(1)}) = 0;$ if $i \equiv 1,4(mod 6)$
$f(v_i^{(1)}) = 1;$ if $i \equiv 0,5(mod 6)$
$f(v_i^{(1)}) = 2;$ if $i \equiv 2,3(mod 6)$
$f(v_i^{(2)}) = 0;$ if $i \equiv 0,3(mod 6)$
$f(v_i^{(2)}) = 1;$ if $i \equiv 4,5(mod 6)$
$f(v_i^{(2)}) = 2;$ if $i \equiv 1,2(mod 6)$
$f(x) = 0;$

Case-2: For $n \equiv 1(mod 6)$

$f(v_i^{(1)}) = 0;$ if $i \equiv 1,4(mod 6), i \neq n$
$f(v_i^{(1)}) = 1;$ if $i \equiv 0,5(mod 6)$
$f(v_i^{(1)}) = 2;$ if $i \equiv 2,3(mod 6)$
$f(v_n^{(1)}) = 1;$
$f(v_i^{(2)}) = 0;$ if $i \equiv 0,3(mod 6)$
$f(v_i^{(2)}) = 1;$ if $i \equiv 4,5(mod 6)$
$f(v_i^{(2)}) = 2;$ if $i \equiv 1,2(mod 6)$
$f(x) = 0;$

Case-3: For $n \equiv 2 (mod 6)$

$f(v_i^{(1)}) = 0;$ if $i \equiv 1,4(mod 6), i \neq n-1$
$f(v_i^{(1)}) = 1;$ if $i \equiv 0,5(mod 6)$
$f(v_i^{(1)}) = 2;$ if $i \equiv 2,3(mod 6), i \neq n$
$f(v_i^{(2)}) = 0;$ if $i \equiv 0,3(mod 6)$
$f(v_i^{(2)}) = 1;$ if $i \equiv 4,5(mod 6)$
$f(v_i^{(2)}) = 2;$ if $i \equiv 1,2(mod 6), i \neq n$
$f(v_{n-1}^{(1)}) = 1;$
$f(v_n^{(1)}) = f(v_n^{(2)}) = 0;$
$f(x) = 2;$

Case-4: For $n \equiv 3(mod 6)$

$f(v_i^{(1)}) = 0;$ if $i \equiv 1,4(mod 6), i \neq n-2$
$f(v_i^{(1)}) = 1;$ if $i \equiv 0,5(mod 6)$
$f(v_i^{(1)}) = 2;$ if $i \equiv 2,3(mod 6), i \neq n-1, n$
$f(v_i^{(2)}) = 0;$ if $i \equiv 0,3(mod 6), i \neq n$
$f(v_i^{(2)}) = 1;$ if $i \equiv 4,5(mod 6)$
$f(v_i^{(2)}) = 2;$ if $i \equiv 1,2(mod 6), i \neq n-1, n-2$
$f(v_{n-2}^{(1)}) = f(v_n^{(2)}) = 1;$
$f(v_{n-1}^{(1)}) = f(v_{n-1}^{(2)}) = 2;$
$f(v_n^{(1)}) = f(v_{n-2}^{(2)}) = 0;$
$f(x) = 0;$

Case-5: For $n \equiv 4 (mod 6)$

$f(v_i^{(1)}) = 0;$ if $i \equiv 1,4(mod 6)$
$f(v_i^{(1)}) = 1;$ if $i \equiv 0,5(mod 6)$
$f(v_i^{(1)}) = 2;$ if $i \equiv 2,3(mod 6)$
$f(v_i^{(2)}) = 0;$ if $i \equiv 0,3(mod 6), i \neq n-1$
$f(v_i^{(2)}) = 1;$ if $i \equiv 4,5(mod 6)$
$f(v_i^{(2)}) = 2;$ if $i \equiv 1,2(mod 6), i \neq n-2$
$f(v_{n-2}^{(2)}) = f(v_{n-1}^{(2)}) = 1;$
$f(x) = 0;$

The labeling pattern defined above covers all possible arrangement of vertices. The graph G satisfies the conditions $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$, where $0 \leq i, j \leq 2$ as shown in Table 3. i.e. G admits 3-equitable labeling. □

Let $n=6a+b$

Table 3. Table showing vertex and edge conditions.

b	Vertex condition	Edge condition
0,3	$v_f(0) = v_f(1) + 1 = v_f(2) + 1$	$e_f(0)+1=e_f(1) = e_f(2)$
1,4	$v_f(0) = v_f(1) = v_f(2)$	$e_f(0)=e_f(1) = e_f(2)$
2,5	$v_f(0) = v_f(1) + 1 = v_f(2)$	$e_f(0)+1=e_f(1)+1 = e_f(2)$

Illustration 2.6: Consider a graph $G = < S_6^{(1)} : S_6^{(2)} >$. Here $n = 6$. The 3-equitable labeling is as shown in Fig. 3.

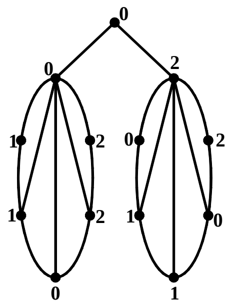

Fig. 3. 3-equitable labeling of the graph G.

Theorem-2.7: Graph $< S_n^{(1)} : S_n^{(2)} : S_n^{(3)} : \ldots : S_n^{(k)} >$ is 3-equitable.

Proof: Let $S_n^{(j)}$ be the shells. Let $v_i^{(j)}$ be the vertices $S_n^{(j)}$ and $v_1^{(j)}$ be the apex vertices of $S_n^{(j)}$. Let x_j $(j \neq k)$ be the new vertices. Let $G = < S_n^{(1)} : S_n^{(2)} : S_n^{(3)} : \ldots : S_n^{(k)} >$. We define vertex labeling $f : V(G) \rightarrow \{0, 1, 2\}$ as follows.

Case-1: For $n \equiv 0 \ (mod 6)$

Subcase 1.1: $k \equiv 0 \ (mod 3)$

For $j \equiv 1 \ (mod 3)$

$f(v_i^{(j)}) = 0$; if $i \equiv 1,4 (mod 6)$

$f(v_i^{(j)}) = 1$; if $i \equiv 0,5 (mod 6)$

$f(v_i^{(j)}) = 2$; if $i \equiv 2,3 (mod 6)$

$f(x_j) \ = 0$;

For $j \equiv 0,2 \ (mod 3)$

$f(v_i^{(j)}) = 0$; if $i \equiv 0,3 (mod 6)$ $i \neq n$

$f(v_i^{(j)}) = 1$; if $i \equiv 4,5 (mod 6)$

$f(v_i^{(j)}) = 2$; if $i \equiv 1,2 (mod 6)$

$f(v_n^{(j)}) = 2$; if $j \equiv 2 (mod 3)$

$f(v_n^{(j)}) = 1$; if $j \equiv 0 (mod 3)$

$f(x_j) \ = 0$; $j \neq k$

Subcase 1.2: $k \equiv 1 \ (mod 3)$

For first k-1 copies of shells use the pattern of subcase 1.1 and for k^{th} copy define function as follow.

$f(v_i^{(k)}) = 0$; if $i \equiv 1,4 (mod 6)$

$f(v_i^{(k)}) = 1$; if $i \equiv 0,5 (mod 6)$

$f(v_i^{(k)}) = 2$; if $i \equiv 2,3 (mod 6)$

$f(x_{k-1}) = 0;$

Subcase 1.3: $k\equiv 2\ (mod 3)$

For first k-2 copies of shells use the pattern of subcase 1.1 and for k-1 and k^{th} copy define function as follow:

$f(v_i^{(k-1)}) = 0;$ if $i\equiv 1,4(mod 6)$
$f(v_i^{(k-1)}) = 1;$ if $i\equiv 0,5(mod 6)$
$f(v_i^{(k-1)}) = 2;$ if $i\equiv 2,3(mod 6)$
$f(v_i^{(k)}) = 0;$ if $i\equiv 0,3(mod 6)$
$f(v_i^{(k)}) = 1;$ if $i\equiv 4,5(mod 6)$
$f(v_i^{(k)}) = 2;$ if $i\equiv 1,2(mod 6)$
$f(x_{k-2}) = f(x_{k-1}) = 0;$

Case-2: For $n\equiv 1\ (mod 6)$

Subcase 2.1: $k\equiv 0\ (mod 3)$

For $j\equiv 1,2\ (mod 3)$
$f(v_i^{(j)}) = 0;$ if $i\equiv 1,4(mod 6), i\neq n$
$f(v_i^{(j)}) = 1;$ if $i\equiv 0,2,3,5(mod 6)$ and $j\equiv 1(mod 3)$
$f(v_i^{(j)}) = 2;$ if $i\equiv 0,2,3,5(mod 6)$ and $j\equiv 2(mod 3)$
$f(v_n^{(j)}) = 1;$ if $j\equiv 1\ (mod 3)$
$f(v_n^{(j)}) = 0;$ if $j\equiv 2\ (mod 3)$
$f(x_j) = 0;$ if $j\equiv 1(mod 3)$
$f(x_j) = 2;$ if $j\equiv 2(mod 3)$

For $j\equiv 0\ (mod 3)$
$f(v_i^{(j)}) = 0;$ if $i\equiv 0,3(mod 6)$
$f(v_i^{(j)}) = 1;$ if $i\equiv 4,5(mod 6)$
$f(v_i^{(j)}) = 2;$ if $i\equiv 1,2(mod 6)$
$f(x_j) = 0; j\neq k$

Subcase 2.2: $k\equiv 1\ (mod 3)$

For first k-1 copies of shells use the pattern of subcase 1.1 and for k^{th} copy define function as follow:

$f(v_i^{(k)}) = 0;$ if $i\equiv 1,4(mod 6)$
$f(v_i^{(k)}) = 1;$ if $i\equiv 0,5(mod 6)$
$f(v_i^{(k)}) = 2;$ if $i\equiv 2,3(mod 6)$
$f(x_{k-1}) = 2;$

Subcase 2.3: $k\equiv 2\ (mod 3)$

For first k-2 copies of shells use the pattern of subcase 1.1 and for k-1 and k^{th} copy define function as follow:

For $j=k$-1, $k;$
$f(v_i^{(j)}) = 0;$ if $i\equiv 1,4(mod 6)$ and $j\neq k, i\neq n$
$f(v_i^{(j)}) = 1;$ if $i\equiv 0,5(mod 6)$
$f(v_i^{(j)}) = 2;$ if $i\equiv 2,3(mod 6)$
$f(v_n^{(k)}) = 1;$
$f(x_{k-2}) = 0;$
$f(x_{k-1}) = 2;$

Case-3: For $n\equiv 2\ (mod 6)$

Subcase 3.1: $k\equiv 0\ (mod 3)$

For $j \equiv 1\ (mod3)$
$f(v_i^{(j)}) = 0;$ if $i \equiv 1,4(mod6)$
$f(v_i^{(j)}) = 1;$ if $i \equiv 2,3(mod6)$
$f(v_i^{(j)}) = 2;$ if $i \equiv 0,5(mod6)$
$f(x_j) = 2;$

For $j \equiv 2\ (mod3)$
$f(v_i^{(j)}) = 0;$ if $i \equiv 1,4(mod6), i \neq n-1$
$f(v_i^{(j)}) = 1;$ if $i \equiv 0,5(mod6)$
$f(v_i^{(j)}) = 2;$ if $i \equiv 2,3(mod6), i \neq n$
$f(v_{n-1}^{(j)}) = 1;$ if $j \equiv 1\ (mod3)$
$f(v_n^{(j)}) = 0;$
$f(x_j) = 2;$

For $j \equiv 0\ (mod3)$
$f(v_i^{(j)}) = 0;$ if $i \equiv 0,5(mod6)$
$f(v_i^{(j)}) = 1;$ if $i \equiv 2,3(mod6)$
$f(v_i^{(j)}) = 2;$ if $i \equiv 1,4(mod6)$
$f(x_j) = 0; j \neq k$

Subcase 3.2: $k \equiv 1\ (mod3)$
For first k-1 copies of shells use the pattern of subcase 1.1 and for k^{th} copy define function as follow:
$f(v_i^{(k)}) = 0;$ if $i \equiv 1,4(mod6)$
$f(v_i^{(k)}) = 1;$ if $i \equiv 2,3(mod6)$
$f(v_i^{(k)}) = 2;$ if $i \equiv 0,5(mod6)$
$f(x_{k-1}) = 2;$

Subcase 3.3: $k \equiv 2\ (mod3)$
For first k-2 copies of shells use the pattern of subcase 1.1 and for k-1 and k^{th} copy define function as follow:
For $j=k-1, k;$
$f(v_i^{(j)}) = 0;$ if $i \equiv 1,4(mod6)$ and $j \neq k, i \neq 1$
$f(v_i^{(j)}) = 1;$ if $i \equiv 2,3(mod6)$
$f(v_i^{(j)}) = 2;$ if $i \equiv 0,5(mod6)$
$f(v_1^{(k)}) = 2;$
$f(x_{k-2}) = 2;$
$f(x_{k-1}) = 0;$

Case-4: For $n \equiv 3\ (mod6)$

Subcase 4.1: $k \equiv 0\ (mod3)$
For $j \equiv 1, 2\ (mod3)$
$f(v_i^{(j)}) = 0;$ if $i \equiv 1,4(mod6)$
$f(v_i^{(j)}) = 1;$ if $i \equiv 0,2,3,5(mod6)$ and $j \equiv 2\ (mod3)$
$f(v_i^{(j)}) = 2;$ if $i \equiv 0,2,3,5(mod6)$ and $j \equiv 1\ (mod3)$
$f(x_j) = 1;$ if $j \equiv 1\ (mod3)$
$f(x_j) = 2;$ if $j \equiv 1\ (mod3)$

For $j \equiv 0\ (mod3)$
$f(v_i^{(j)}) = 0;$ if $i \equiv 0,5(mod6)$
$f(v_i^{(j)}) = 1;$ if $i \equiv 2,3(mod6)$ and $i \neq n-1$
$f(v_i^{(j)}) = 2;$ if $i \equiv 1,4(mod6)$ and $i \neq n-2$
$f(v_{n-2}^{(j)}) = 0;$
$f(v_{n-1}^{(j)}) = 2;$
$f(x_j) = 0; j \neq k$

Subcase 4.2: $k \equiv 1\ (mod3)$

For first k-1 copies of shells use the pattern of subcase 1.1 and for k^{th} copy define function as follow:
$f(v_i^{(k)}) = 0;$ if $i \equiv 1,4(mod 6)$ and $i \neq n-2$
$f(v_i^{(k)}) = 1;$ if $i \equiv 2,3(mod 6)$ and $i \neq n-1$
$f(v_i^{(k)}) = 2;$ if $i \equiv 0,5(mod 6)$
$f(v_{n-2}^{(k)}) = 2;$
$f(v_{n-1}^{(k)}) = 0;$
$f(x_{k-1}) = 0;$

Subcase 4.3: $k \equiv 2\ (mod 3)$
For first k-2 copies of shells use the pattern of subcase 1.1 and for k-1 and k^{th} copy define function as follow:
For $j=k-1, k;$
$f(v_i^{(j)}) = 0;$ if $i \equiv 1,4(mod 6)$ and $i \neq n-2, j \neq k-1$
$f(v_i^{(j)}) = 1;$ if $i \equiv 2,3(mod 6)$ and $i \neq n-1$
$f(v_i^{(j)}) = 2;$ if $i \equiv 0,5(mod 6)$
$f(v_{n-2}^{(k-1)}) = 2;$
$f(v_{n-1}^{(k-1)}) = 0;$
$f(v_{n-1}^{(k)}) = 2;$
$f(x_{k-2}) = f(x_{k-1}) = 0;$

Case-5: For $n \equiv 4\ (mod 6)$

Subcase 5.1: $k \equiv 0\ (mod 3)$
For $j \equiv 1, 2\ (mod 3)$
$f(v_i^{(j)}) = 0;$ if $i \equiv 1,4(mod 6)$
$f(v_i^{(j)}) = 1;$ if $i \equiv 0,2,3,5(mod 6)$ and $j \equiv 2\ (mod 3)$
$f(v_i^{(j)}) = 2;$ if $i \equiv 0,2,3,5(mod 6)$ and $j \equiv 1\ (mod 3)$
$f(x_j) = 2;$

For $j \equiv 0\ (mod 3)$
If $1 \leq i \leq n-4$
$f(v_i^{(j)}) = 0;$ if $i \equiv 0,5(mod 6)$
$f(v_i^{(j)}) = 1;$ if $i \equiv 2,3(mod 6)$
$f(v_i^{(j)}) = 2;$ if $i \equiv 1,4(mod 6)$
$f(v_{n-3}^{(j)}) = f(v_{n-2}^{(j)}) = f(v_{n-1}^{(j)}) = 1;$
$f(v_n^{(j)}) = 0;$
$f(x_j) = 2; j \neq k$

Subcase 5.2: $k \equiv 1\ (mod 3)$
For first k-1 copies of shells use the pattern of subcase 1.1 and for k^{th} copy define function as follow:
$f(v_i^{(k)}) = 0;$ if $i \equiv 1,4(mod 6)$ and $i \neq n$
$f(v_i^{(k)}) = 1;$ if $i \equiv 2,3(mod 6)$
$f(v_i^{(k)}) = 2;$ if $i \equiv 0,5(mod 6)$
$f(v_n^{(k)}) = 1;$
$f(x_{k-1}) = 2;$

Subcase 5.3: $k \equiv 2\ (mod 3)$
For first k-2 copies of shells use the pattern of subcase 1.1 and for k-1 and k^{th} copy define function as follow:
$f(v_i^{(k-1)}) = 0;$ if $i \equiv 1,4(mod 6)$ and $i \neq n$
$f(v_i^{(k-1)}) = 1;$ if $i \equiv 0,5(mod 6)$

$f(v_i^{(k-1)}) = 2$; if $i \equiv 2,3 (mod 6)$
$f(v_i^{(k)}) = 0$; if $i \equiv 1,4 (mod 6)$ and $i \neq n$
$f(v_i^{(k)}) = 1$; if $i \equiv 2,3 (mod 6)$ and $i \neq n-2$
$f(v_i^{(k)}) = 2$; if $i \equiv 0,5 (mod 6)$
$f(v_{n-2}^{(k)}) = f(x_{k-2}) = 2$;
$f(v_n^{(k)}) = f(v_n^{(k-1)}) = 1$;
$f(x_{k-1}) = 0$;

Case-6: For $n \equiv 5 (mod 6)$

Subcase 6.1: $k \equiv 0 (mod 3)$

For $j \equiv 1, 2 (mod 3)$
$f(v_i^{(j)}) = 0$; if $i \equiv 1,4 (mod 6)$
$f(v_i^{(j)}) = 1$; if $i \equiv 0,2,3,5 (mod 6)$ and $j \equiv 2 (mod 3)$
$f(v_i^{(j)}) = 2$; if $i \equiv 0,2,3,5 (mod 6)$ and $j \equiv 1 (mod 3)$
$f(x_j) = 2$; if $j \equiv 1 (mod 3)$
$f(x_j) = 0$; if $j \equiv 2 (mod 3)$

For $j \equiv 0 (mod 3)$
If $1 \leq i \leq n-2$
$f(v_i^{(j)}) = 0$; if $i \equiv 0,5 (mod 6)$
$f(v_i^{(j)}) = 1$; if $i \equiv 2,3 (mod 6)$
$f(v_i^{(j)}) = 2$; if $i \equiv 1,4 (mod 6)$
$f(v_{n-1}^{(j)}) = 1$;
$f(v_n^{(j)}) = 2$;
$f(x_j) = 0$; $j \neq k$

Subcase 6.2: $k \equiv 1 (mod 3)$

For first k-1 copies of shells use the pattern of subcase 1.1 and for k^{th} copy define function as follow:

$f(v_i^{(k)}) = 0$; if $i \equiv 1,4 (mod 6)$
$f(v_i^{(k)}) = 1$; if $i \equiv 2,3 (mod 6)$
$f(v_i^{(k)}) = 2$; if $i \equiv 0,5 (mod 6)$
$f(x_{k-1}) = 2$;

Subcase 6.3: $k \equiv 2 (mod 3)$

For first k-2 copies of shells use the pattern of subcase 1.1 and for k-1 and k^{th} copy define function as follow:

$f(v_i^{(k-1)}) = 0$; if $i \equiv 1,4 (mod 6)$
$f(v_i^{(k-1)}) = 1$; if $i \equiv 2,3 (mod 6)$
$f(v_i^{(k-1)}) = 2$; if $i \equiv 0,5 (mod 6)$
$f(v_i^{(k)}) = 0$; if $i \equiv 0,5 (mod 6)$
$f(v_i^{(k)}) = 1$; if $i \equiv 2,3 (mod 6)$
$f(v_i^{(k)}) = 2$; if $i \equiv 1,4 (mod 6)$
$f(x_{k-2}) = 2$;
$f(x_{k-1}) = 0$;

The labeling pattern defined above covers all possible arrangement of vertices. The graph G satisfies the conditions $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$, where $0 \leq i, j \leq 2$ as shown in *Table* 4. i.e. G admits 3-equitable labeling. □

Let n=6a+b, and k=3c+d

Table 4. Vertex and edge conditions.

b	d	Vertex condition	Edge condition
0	0	$v_f(0)+1=v_f(1)=v_f(2)$	$e_f(0)+1=e_f(1)=e_f(2)+1$
	1	$v_f(0)=v_f(1)=v_f(2)$	$e_f(0)=e_f(1)=e_f(2)$
	2	$v_f(0)=v_f(1)+1=v_f(2)+1$	$e_f(0)+1=e_f(1)=e_f(2)+1$
1	0	$v_f(0)+1=v_f(1)=v_f(2)$	$e_f(0)+1=e_f(1)=e_f(2)+1$
	1	$v_f(0)+1=v_f(1)+1=v_f(2)$	$e_f(0)=e_f(1)=e_f(2)+1$
	2	$v_f(0)=v_f(1)=v_f(2)$	$e_f(0)=e_f(1)=e_f(2)$
2	0,1,2	$v_f(0)+1=v_f(1)=v_f(2)$	$e_f(0)+1=e_f(1)=e_f(2)+1$
3	0	$v_f(0)+1=v_f(1)=v_f(2)$	$e_f(0)+1=e_f(1)=e_f(2)+1$
	1	$v_f(0)=v_f(1)=v_f(2)$	$e_f(0)=e_f(1)=e_f(2)$
	2	$v_f(0)=v_f(1)+1=v_f(2)+1$	$e_f(0)=e_f(1)+1=e_f(2)$
4	0($n\neq4$)	$v_f(0)=v_f(1)=v_f(2)+1$	$e_f(0)+1=e_f(1)=e_f(2)+1$
	0(n=4)	$v_f(0)=v_f(1)=v_f(2)+1$	$e_f(0)=e_f(1)+1=e_f(2)+1$
	1	$v_f(0)+1=v_f(1)+1=v_f(2)$	$e_f(0)+1=e_f(1)=e_f(2)$
	2	$v_f(0)=v_f(1)=v_f(2)$	$e_f(0)=e_f(1)=e_f(2)$
5	0,1,2	$v_f(0)+1=v_f(1)=v_f(2)$	$e_f(0)+1=e_f(1)=e_f(2)+1$

Illustration 2.8: Consider a graph $G=\langle S_4^{(1)} : S_4^{(2)} : S_4^{(3)} \rangle$. Here n = 4. The 3-equitable labeling is as shown in Fig. 4.

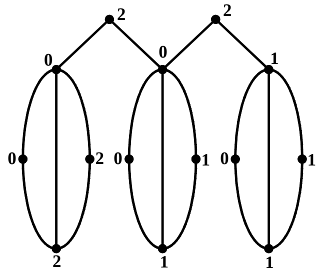

Fig. 4. 3-equitable labeling of the graph G.

3. Concluding Remarks

Labeled graph is the topic of current interest for many researchers as it has diversified applications. We discuss here cordial labeling and 3-equitable labeling of some shell related graphs. This approach is novel and contributes four new results. The derived labeling pattern is demonstrated by means of elegant illustrations which provide better understanding of the derived results. The results reported here are new and expected to add new dimension to the theory of cordial and 3-equitable graphs.

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Cordial and 3-Equitable Labeling for Some Star Related Graphs

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Abstract

We present here cordial and 3-equitable labeling for the graphs obtained by joining apex vertices of two stars to a new vertex. We extend these results for k copies of stars.

Mathematics Subject Classification: 05C78

Keywords: Cordial labeling, 3-equitable labeling

1. Introduction

We begin with simple, finite, connected, undirected graph $G = (V, E)$. In the present work $K_{1,n}$ denote the star. Vertex corresponds to K_1 is called an apex vertex. For all other terminology and notations we follow Harary[7].

We will give brief summary of definitions which are useful for the present investigations.

Definition 1.1 Consider two stars $K_{1,n}^{(1)}$ and $K_{1,n}^{(2)}$ then $G =< K_{1,n}^{(1)} : K_{1,n}^{(2)} >$ is the graph obtained by joining apex vertices of stars to a new vertex x.

Note that G has $2n + 3$ vertices and $2n + 2$ edges.

Definition 1.2 Consider k copies of stars namely $K_{1,n}^{(1)}, K_{1,n}^{(2)}, K_{1,n}^{(3)}, \ldots K_{1,n}^{(k)}$. Then the $G =< K_{1,n}^{(1)} : K_{1,n}^{(2)} : K_{1,n}^{(3)} : \ldots : K_{1,n}^{(k)} >$ is the graph obtained by joining apex vertices of each $K_{1,n}^{(p-1)}$ and $K_{1,n}^{(p)}$ to a new vertex x_{p-1} where $2 \leq p \leq k$.

Note that G has $k(n + 2) - 1$ vertices and $k(n + 2) - 2$ edges.

Definition 1.3 If the vertices of the graph are assigned values subject to certain conditions is known as *graph labeling*.

Most interesting graph labeling problems have three important characteristics.

1. a set of numbers from which the labels are chosen.

2. a rule that assigns a value to each edge.

3. a condition that these values must satisfy.

For detail survey on graph labeling one can refer Gallian[6]. Vast amount of literature is available on different types of graph labeling. According to Beineke and Hegde[2] graph labeling serves as a frontier between number theory and structure of graphs.

Labeled graph have variety of applications in coding theory, particularly for missile guidance codes, design of good radar type codes and convolution codes with optimal autocorrelation properties. Labeled graph plays vital role in the study of X-Ray crystallography, communication network and to determine optimal circuit layouts. A detail study of variety of applications of graph labeling is given by Bloom and Golomb[3].

Definition 1.4 Let $G = (V, E)$ be a graph. A mapping $f : V(G) \rightarrow \{0,1\}$ is called *binary vertex labeling* of G and $f(v)$ is called the *label* of the vertex v of G under f.

For an edge $e = uv$, the induced edge labeling $f^* : E(G) \rightarrow \{0, 1\}$ is given by $f^*(e)=|f(u)-f(v)|$. Let $v_f(0)$, $v_f(1)$ be the number of vertices of G having labels 0 and 1 respectively under f and let $e_f(0)$,$e_f(1)$ be the number of edges having labels 0 and 1 respectively under f^*.

Definition 1.5 A binary vertex labeling of a graph G is called a *cordial* *labeling* if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph G is *cordial* if it admits cordial labeling.

The concept of cordial labeling was introduced by Cahit[4].

Many researchers have studied cordiality of graphs. e.g.Cahit [4] proved that tree is cordial. In the same paper he proved that K_n is cordial if and only if $n \leq 3$. Ho et al.[8] proved that unicyclic graph is cordial unless it is C_{4k+2}. Andar et al.[1] discussed cordiality of multiple shells. Vaidya et al.[9],[10],[11] have also discussed the cordiality of various graphs.

Definition 1.6 Let $G = (V, E)$ be a graph. A mapping $f : V(G) \rightarrow \{0, 1, 2\}$ is called *ternary vertex labeling* of G and $f(v)$ is called *label of the vertex* v of G under f.

For an edge $e = uv$, the induced edge labeling $f^* : E(G) \rightarrow \{0, 1, 2\}$ is given by $f^*(e) = |f(u) - f(v)|$. Let $v_f(0)$, $v_f(1)$, $v_f(2)$ be the number of vertices of G having labels 0, 1, 2 respectively under f and $e_f(0)$, $e_f(1)$, $e_f(2)$ be the number of edges having labels 0, 1, 2 respectively under f^*.

Definition 1.7 A ternary vertex labeling of a graph G is called a *3-equitable labeling* if $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $0 \leq i, j \leq 2$. A graph G is *3-equitable* if it admits 3-equitable labeling.

The concept of 3-equitable labeling was introduced by Cahit[5]. Many researchers have studied 3-equatability of graphs. e.g.Cahit [5] proved that C_n is 3-equitable except $n \equiv 3(mod 6)$. In the same paper he proved that an Eulerian graph with number of edges congruent to $3(mod 6)$ is not 3-equitable. Youssef[12] proved that W_n is 3-equitable for all $n \geq 4$.

In the present investigations we prove that graphs $< K_{1,n}^{(1)} : K_{1,n}^{(2)} >$ and $< K_{1,n}^{(1)} : K_{1,n}^{(2)} : K_{1,n}^{(3)} : \ldots : K_{1,n}^{(k)} >$ are cordial as well as 3-equitable.

2. Main Results

Theorem-2.1: Graph $< K_{1,n}^{(1)} : K_{1,n}^{(2)} >$ is cordial.

Proof: Let $v_1^{(1)}, v_2^{(1)}, v_3^{(1)}, \ldots v_n^{(1)}$ be the pendant vertices $K_{1,n}^{(1)}$ and $v_1^{(2)}, v_2^{(2)}, v_3^{(2)}$, $\ldots v_n^{(2)}$ be the pendant vertices $K_{1,n}^{(2)}$. Let c_1 and c_2 be the apex vertices of $K_{1,n}^{(1)}$ and $K_{1,n}^{(2)}$ respectively and they are adjacent to a new common vertex x. Let $G =< K_{1,n}^{(1)} : K_{1,n}^{(2)} >$. We define binary vertex labeling $f : V(G) \rightarrow \{0, 1\}$ as follows.

For any $n \in N$ and $i = 1, 2, \ldots n$ where N is set of natural numbers.

In this case we define labeling as follows

Case 1: If n even

For $j = 1, 2$

$$f(v_i^{(j)}) = 0; \text{ if } 1 \leq i \leq \tfrac{n}{2}$$
$$= 1; \tfrac{n+2}{2} \leq i \leq n$$
$$f(c_1) = 0;$$
$$f(c_2) = 1;$$

$f(x) = 0;$

Case 2: If n odd

For $j = 1, 2$

$$f(v_i^{(j)}) = 0; \text{ if } 1 \leq i \leq \tfrac{n-1}{2}$$
$$= 1; \tfrac{n+1}{2} \leq i \leq n$$
$$f(c_1) = f(c_2) = f(x) = 0;$$

The labeling pattern defined above covers all possible arrangement of vertices. The graph G satisfies the conditions $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ as shown in *Table 1.* i.e. G admits cordial labeling.

n	Vertex Condition	Edge Condition
$n \in N$	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$

Table 1

For better understanding of the above defined labeling pattern, consider following illustration.

Illustration 2.2 Consider $G =< K_{1,7}^{(1)} : K_{1,7}^{(2)} >$. Here $n = 7$. The cordial labeling is as shown in *Figure 1*.

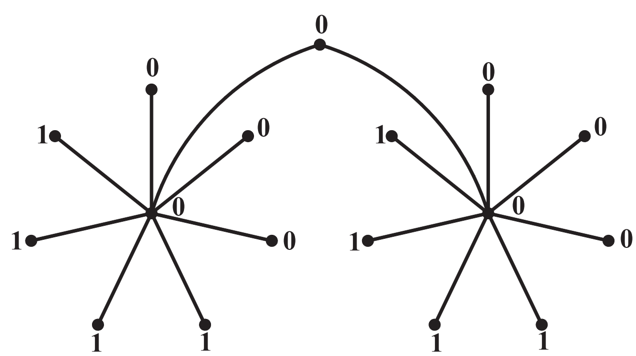

Figure 1

Above result can be extended for $k-$copies of $K_{1,n}$ as follows.

Theorem 2.3 Graph $< K_{1,n}^{(1)} : K_{1,n}^{(2)} : K_{1,n}^{(3)} : \ldots : K_{1,n}^{(k)} >$ is cordial.

Proof: Let $K_{1,n}^{(j)}$ be k copies of star $K_{1,n}$, $v_i^{(j)}$ be the pendant vertices of $K_{1,n}^{(j)}$ and c_j be the apex vertex of $K_{1,n}^{(j)}$ (here $i = 1, 2, \ldots n$ and $j = 1, 2, \ldots\ k$).Let $x_1, x_2 \ldots\ x_{k-1}$ be the vertices such that c_{p-1} and c_p are adjacent to x_{p-1} where $2 \leq p \leq k$. Consider $G =< K_{1,n}^{(1)} : K_{1,n}^{(2)} : K_{1,n}^{(3)} : \ldots : K_{1,n}^{(k)} >$. To define binary vertex labeling $f : V(G) \to \{0, 1\}$ we consider following cases.

Case 1: $n \in N$ even and k where $k \in N - \{1, 2\}$.

In this case we define labeling function f as

For $j = 1, 2, \ldots k$

$$f(v_i^{(j)}) = 0; \text{ if } 1 \leq i \leq \tfrac{n}{2}.$$

$= 1$; if $\frac{n+2}{2} \leq i \leq n$.
$f(c_j) = 1$; if j even.
$= 0$; if j odd.
$f(x_j) = 1$; if j even, $j \neq k$.
$= 0$; if j odd, $j \neq k$.

Case 2: $n \in N - \{1, 2\}$ odd and k where $k \in N - \{1, 2\}$.
In this case we define labeling function f as

For $j = 1, 2, \ldots k$
$f(v_i^{(j)}) = 0$; if $1 \leq i \leq \frac{n-1}{2}$.
$= 1$; if $\frac{n+1}{2} \leq i \leq n$.
$f(c_j) = 1$; if j even.
$= 0$; if j odd.
$f(x_j) = 0$, $j \neq k$.

The labeling pattern defined above covers all the possibilities. In each case, the graph G under consideration satisfies the conditions $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ as shown in *Table 2*. i.e. G admits cordial labeling.

Let $n = 2a + b$ and $k = 2c + d$ where $a \in N \cup \{0\}, c \in N$

b	d	Vertex Condition	Edge Condition
0	0,1	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$
1	0	$v_f(0)+1=v_f(1)$	$e_f(0)=e_f(1)$
	1	$v_f(0)=v_f(1)$	$e_f(0)+1=e_f(1)$

Table 2

For better understanding of the above defined labeling pattern, consider following illustration.

Illustration 2.4 Consider $G =< K_{1,6}^{(1)} : K_{1,6}^{(2)} : K_{1,6}^{(3)} >$. Here $n = 6$ and $k = 3$. The cordial labeling is as shown in *Figure 2*. It is the case 1 of Theorem 2.3.

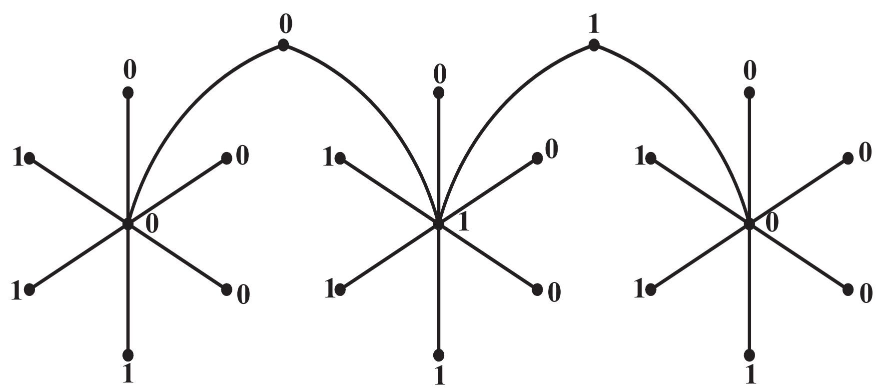

Figure 2

Theorem 2.5 Graph $< K_{1,n}^{(1)} : K_{1,n}^{(2)} >$ is 3-equitable.

Proof: Let $v_1^{(1)}, v_2^{(1)}, v_3^{(1)}, \ldots v_n^{(1)}$ be the pendant vertices $K_{1,n}^{(1)}$ and $v_1^{(2)}, v_2^{(2)}, v_3^{(2)}$, $\ldots v_n^{(2)}$ be the pendant vertices $K_{1,n}^{(2)}$. Let c_1 and c_2 be the apex vertices of $K_{1,n}^{(1)}$

and $K_{1,n}^{(2)}$ respectively and they are adjacent to a new common vertex x. Let $G =< K_{1,n}^{(1)} : K_{1,n}^{(2)} >$. To define ternary vertex labeling $f : V(G) \to \{0, 1, 2\}$ we consider the following cases.

Case 1: $n \equiv 0(mod3)$

In this case we define labeling f as

For $j = 1, 2$

$$f(v_i^{(j)}) = 0;\ i \equiv 0(mod3)$$
$$= 1;\ i \equiv 1(mod3)$$
$$= 2;\ i \equiv 2(mod3),\ 1 \leq i \leq n-1$$
$$f(v_n^{(1)}) = 1;$$
$$f(v_n^{(2)}) = f(c_1) = f(x) = 0;$$
$$f(c_2) = 2;$$

Case 2: $n \equiv 1(mod3)$

In this case we define labeling f as:

For $j = 1, 2$

$$f(v_i^{(j)}) = 0;\ i \equiv 0(mod3)$$
$$= 1;\ i \equiv 1(mod3)$$
$$= 2;\ i \equiv 2(mod3)$$
$$f(c_1) = f(x) = 0;$$
$$f(c_2) = 2;$$

Case 3: $n \equiv 2(mod3)$

In this case we define labeling f as

For $j = 1, 2$

$$f(v_i^{(j)}) = 0;\ i \equiv 0(mod3)$$
$$= 1;\ i \equiv 1(mod3)$$
$$= 2;\ i \equiv 2(mod3)$$
$$f(c_1) = f(c_2) = f(x) = 0;$$

The labeling pattern defined above covers all possible arrangement of vertices. In each case, the graph G under consideration satisfies the conditions $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $0 \leq i, j \leq 2$ as shown in *Table 3*. i.e. G admits 3-equitable labeling.

Let $n = 3a + b$ and $a \in N \cup \{0\}$

b	Vertex Condition	Edge Condition
0	$v_f(0)=v_f(1)=v_f(2)+1$	$e_f(0)+1=e_f(1)=e_f(2)+1$
1	$v_f(0)=v_f(1)+1=v_f(2)+1$	$e_f(0)=e_f(1)=e_f(2)$
2	$v_f(0)=v_f(1)=v_f(2)$	$e_f(0)=e_f(1)=e_f(2)+1$

Table 3

For better understanding of the above defined labeling pattern, consider following illustration.

Illustration 2.6 Consider a graph $G =< K_{1,8}^{(1)} : K_{1,8}^{(2)} >$ Here $n = 8$ i.e $n \equiv 2(mod3)$. The corresponding 3-equitable labeling is shown in *Figure 3*. It is the case related to case -3

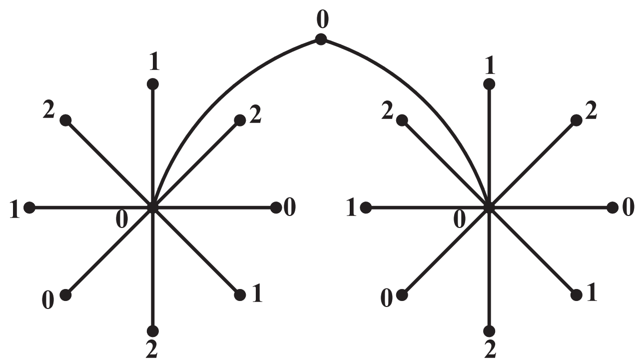

Figure 3

Above result can be extended for $k-$copies of $K_{1,n}$ as follows.

Theorem 2.7 Graph $< K_{1,n}^{(1)} : K_{1,n}^{(2)} : K_{1,n}^{(3)} : \ldots : K_{1,n}^{(k)} >$ is 3-equitable.

Proof: Let $K_{1,n}^{(j)}$, $j = 1, 2, \ldots k$ be k copies of star $K_{1,n}$. Let $v_i^{(j)}$ be the pendant vertices of $K_{1,n}^{(j)}$ where $i = 1, 2, \ldots n$ and $j = 1, 2, \ldots k$. Let c_j be the apex vertex of $K_{1,n}^{(j)}$ where $j = 1, 2, \ldots k$. Let $G =< K_{1,n}^{(1)} : K_{1,n}^{(2)} : K_{1,n}^{(3)} : \ldots : K_{1,n}^{(k)} >$ and $x_1, x_2, \ldots , x_{k-1}$ are the vertices as stated in Theorem 2.3. To define ternary vertex labeling $f : V(G) \to \{0, 1, 2\}$ we consider following cases.

Case 1: For $n \equiv 0(mod3)$

In this case we define labeling function f as follows

Subcase 1: For $k \equiv 0(mod3)$

$$\begin{aligned} f(v_i^{(j)}) &= 0; \text{ if } i \equiv 1(mod3) \\ &= 1; \text{ if } i \equiv 2(mod3) \\ &= 2; \text{ if } i \equiv 0(mod3),\ i \leq n-1 \\ f(v_n^{(j)}) &= 1; \text{ if } j \equiv 1, 2(mod3) \\ &= 2; \text{ if } j \equiv 0(mod3) \\ f(c_j) &= 0; \text{ if } j \equiv 1, 2(mod3) \\ &= 2; \text{ if } j \equiv 0(mod3) \\ f(x_j) &= 2; \text{ if } j \leq n-1 \end{aligned}$$

Subcase 2: For $k \equiv 1(mod3)$

$$\begin{aligned} f(v_i^{(1)}) &= 0; \text{ if } i \equiv 1(mod3) \\ &= 1; \text{ if } i \equiv 2(mod3) \\ &= 2; \text{ if } i \equiv 0(mod3) \\ f(c_1) &= 2 \\ f(x_1) &= 0 \end{aligned}$$

For remaining vertices take $j = k - 1$ and use the pattern of subcase 1.

Subcase 3: For $k \equiv 2(mod3)$

For $j = 1, 2$

$$f(v_i^{(j)}) = 0; \text{ if } i \equiv 1(mod3)$$

$= 1$; if $i \equiv 2(mod3)$
$= 2$; if $i \equiv 0(mod3)$, $1 \leq i \leq n-1$
$f(v_n^{(1)}) = 1$
$f(v_n^{(2)}) = f(c_2) = f(x_j) = 2$
$f(c_1) = 0$

For remaining vertices take $j = k - 2$ and use the pattern of subcase 1.

Case 2: For $n \equiv 1(mod3)$

In this case we define labeling function f as follows

Subcase 1: For $k \equiv 0(mod3)$

Subcase 1.1: For $n = 1$

$f(v_1^{(j)}) = 2$; if $j \equiv 0(mod3)$
$= 1$; if $j \equiv 1, 2(mod3)$
$f(c_j) = 2$; if $j \equiv 1(mod3)$
$= 1$; if $j \equiv 2(mod3)$
$= 0$; if $j \equiv 0(mod3)$
$f(x_j) = 0$; $j \neq k$

Subcase 1.2: For $n > 1$

$f(v_i^{(j)}) = 0$; if $i \equiv 0(mod3)$
$= 1$; if $i \equiv 1(mod3)$
$= 2$; if $i \equiv 2(mod3)$, $i \leq n-2$
$f(v_{n-1}^{(j)}) = 0$; if $j \equiv 1, 2(mod3)$
$= 2$; if $j \equiv 0(mod3)$
$f(v_n^{(j)}) = 1$
$f(c_j) = 2$; if $j \equiv 1(mod3)$
$= 0$; if $j \equiv 0, 2(mod3)$
$f(x_j) = 0$; if $j \equiv 1, 2(mod3)$
$= 2$; if $j \equiv 0(mod3)$, $j \neq k$

Subcase 2: For $k \equiv 1(mod3)$

$f(v_i^{(1)}) = 0$; if $i \equiv 0(mod3)$
$= 1$; if $i \equiv 1(mod3)$
$= 2$; if $i \equiv 2(mod3)$
$f(c_1) = 0$
$f(x_1) = 2$

For remaining vertices take $j = k - 1$ and use the pattern of subcase 1.1 or subcase 1.2 if $n = 1$ or $n > 1$ respectively.

Subcase 3: For $k \equiv 2(mod3)$.

For $j = 1, 2$

$f(v_i^{(j)}) = 0$; if $i \equiv 0(mod3)$
$= 1$; if $i \equiv 1(mod3)$
$= 2$; if $i \equiv 2(mod3)$
$f(c_1) = f(x_2) = 2$

$f(c_2) = f(x_1) = 0$
$f(x_1) = 2$; if $n = 1$
$f(x_1) = 0$; if $n > 1$

For remaining vertices take $j = k - 2$ and use the pattern of subcase 1.1 or subcase 1.2 if $n = 1$ or $n > 1$ respectively.

Case 3: For $n \equiv 2(mod3)$.

In this case we define labeling function f as follows

Subcase 1: For $k \equiv 0(mod3)$

$f(v_i^{(j)}) = 0$; if $i \equiv 0(mod3)$
$= 1$; if $i \equiv 1(mod3)$
$= 2$; if $i \equiv 2(mod3)$, $i \leq n - 1$
$f(v_n^{(j)}) = 1$; if $j \equiv 1(mod3)$
$= 2$; if $j \equiv 0, 2(mod3)$
$f(c_j) = 2$; if $j \equiv 1(mod3)$
$= 0$; if $j \equiv 0, 2(mod3)$
$f(x_j) = 0$; if $j \equiv 1, 2(mod3)$
$= 2$; if $j \equiv 0(mod3)$

Subcase 2: For $k \equiv 1(mod3)$

$f(v_i^{(1)}) = 0$; if $i \equiv 0(mod3)$
$= 1$; if $i \equiv 1(mod3)$
$= 2$; if $i \equiv 2(mod3)$, $i \leq n$
$f(c_1) = 0$
$f(x_1) = 2$

For remaining vertices take $j = k - 1$ and use the pattern of subcase 1.

Subcase 3: For $k \equiv 2(mod3)$

For $j = 1, 2$

$f(v_i^{(j)}) = 0$; if $i \equiv 0(mod3)$
$= 1$; if $i \equiv 1(mod3)$
$= 2$; if $i \equiv 2(mod3)$, $i \leq n$
$f(c_1) = 2$.
$f(c_2) = f(x_j) = 0$.

For remaining vertices take $j = k - 2$ and use the pattern of subcase 1.

The labeling pattern defined above covers all possible arrangement of vertices. In each case, the graph G under consideration satisfies the conditions $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $0 \leq i, j \leq 2$ as shown in *Table 4*. i.e. G admits 3-equitable labeling.

Let $n = 3a + b$ and $k = 3c + d$ where $a \in N \cup \{0\}$,$c \in N$.

b	d	Vertex Condition	Edge Condition
0	0	$v_f(0)=v_f(1)=v_f(2)+1$	$e_f(0)+1=e_f(1)=e_f(2)+1$
	1	$v_f(0)=v_f(1)+1=v_f(2)+1$	$e_f(0)=e_f(1)=e_f(2)$
	2	$v_f(0)=v_f(1)=v_f(2)$	$e_f(0)=e_f(1)=e_f(2)+1$
1	0	$v_f(0)=v_f(1)=v_f(2)+1$	$e_f(0)+1=e_f(1)=e_f(2)+1$
	1	$v_f(0)=v_f(1)=v_f(2)+1$	$e_f(0)+1=e_f(1)=e_f(2)+1$
	2	$v_f(0)=v_f(1)=v_f(2)+1$	$e_f(0)+1=e_f(1)=e_f(2)+1$
2	0	$v_f(0)=v_f(1)=v_f(2)+1$	$e_f(0)+1=e_f(1)=e_f(2)+1$
	1	$v_f(0)=v_f(1)=v_f(2)$	$e_f(0)+1=e_f(1)=e_f(2)$
	2	$v_f(0)=v_f(1)+1=v_f(2)+1$	$e_f(0)=e_f(1)=e_f(2)$

Table 4

For better understanding of the above defined labeling pattern, consider following illustration.

Illustration 2.8 Consider a graph $G =< K_{1,5}^{(1)} : K_{1,5}^{(2)} : K_{1,5}^{(3)} : K_{1,5}^{(4)} >$. Here $n = 5$ and $k = 4$. The corresponding 3-equitable labeling is as shown in *Figure 4*.

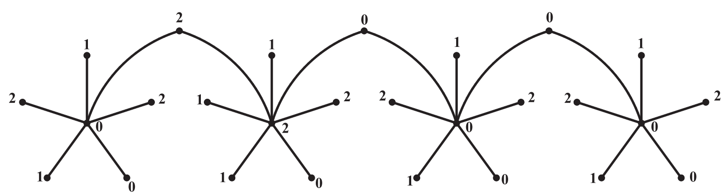

Figure 4

3. Concluding Remarks

Labeled graph is the topic of current interest for many researchers as it has diversified applications. We discuss here cordial labeling and 3-equitable labeling of some star related graphs. This approach is novel and contribute two new graphs to the theory of cordial graphs as well as 3-equitable graphs. The derived labeling pattern is demonstrated by means of elegant illustrations which provides better understanding of the derived results. The results reported here are new and will add new dimension in the theory of cordial and 3-equitable graphs.

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SOME WHEEL RELATED 3-EQUITABLE GRAPHS IN THE CONTEXT OF VERTEX DUPLICATION

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Abstract

In the present investigations, we prove that the graph obtained by duplication of arbitrary rim vertex of wheel W_n and duplication of apex vertex of wheel W_n for even n is 3-equitable and not 3-equitable for odd n, where $n \geq 5$. In addition to this we prove that duplication of vertices of wheel W_n altogether is 3-equitable except $n = 5$.

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1. Introduction

We begin with simple, finite and undirected graph $G = (V, E)$. In the present work, $W_n = C_n + K_1$ $(n \geq 3)$ denotes the wheel. In W_n vertices corresponding to C_n are called *rim vertices* and vertex corresponding to K_1 is called the *apex vertex*. Here $N(v)$ denotes the set of all neighboring vertices of v. For all other terminology and notations we follow Harary [5]. We will give brief summary of definitions which are useful for the present investigations.

Definition 1.1. Duplication of a vertex v_k of graph G produces a new graph G_1 by adding a vertex v'_k with $N(v'_k) = N(v_k)$.

In other words, a vertex v'_k is said to be *duplication* of v_k if all the vertices which are adjacent to v_k are now adjacent to v'_k also.

Definition 1.2. If the vertices of the graph are assigned values subject to certain conditions is known as *graph labeling*.

Most interesting graph labeling problems have three important ingredients as follows:

(1) A set of numbers from which the vertex labels are chosen.

(2) A rule that assigns a value to each edge.

(3) A condition that these values must satisfy.

Labeled graph has variety of applications in coding theory, particularly for missile guidance codes, design of good radar type codes and convolution codes with optimal autocorrelation properties. Labeled graph plays vital role in the study of X-Ray crystallography, communication network and to determine optimal circuit layouts. A detail study on applications of graph labeling is reported in Bloom and Golomb [2].

For extensive survey on graph labeling one can refer Gallian [4]. Vast amount of literature is available on different types of graph labeling and good number of research papers has been published so far in past three decades. According to Beineke and Hegde [1] graph labeling serves as a frontier between number theory and structure of graphs.

There are three types of problems that can be considered in this area.

(1) How 3-equitability is affected under various graph operations?

(2) Construct new families of 3-equitable graph by finding suitable labeling.

(3) Given a graph theoretic property P, characterize the class of graphs with property P that are 3-equitable.

This work is aimed to discuss the problems of the first kind.

Definition 1.3. Let $G = (V, E)$ be a graph. A mapping $f : V(G) \to \{0, 1, 2\}$ is called *ternary vertex labeling* of G and $f(v)$ is called the *label* of the vertex v of G under f.

For an edge $e = uv$, the induced edge labeling $f^* : E(G) \to \{0, 1, 2\}$ is given by $f^*(e) = |f(u) - f(v)|$. Let $v_f(0)$, $v_f(1)$ and $v_f(2)$ be the number of vertices of G having labels 0, 1 and 2, respectively, under f and let $e_f(0)$, $e_f(1)$ and $e_f(2)$ be the number of edges having labels 0, 1 and 2, respectively, under f^*.

Definition 1.4. A ternary vertex labeling of a graph G is called a *3-equitable labeling* if $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $0 \leq i, j \leq 2$. A graph G is *3-equitable* if it admits 3-equitable labeling.

The concept of 3-equitable labeling was introduced by Cahit [3]. Many researchers have studied 3-equitability of graphs, e.g., Cahit [3] proved that C_n is 3-equitable except $n \equiv 3 \pmod 6$. In the same paper he proved that an Eulerian graph with number of edges congruent to 3 (mod 6) is not 3-equitable. Youssef [6] proved that W_n is 3-equitable for all $n \geq 4$.

In the present work, we prove that duplication of arbitrary rim vertex of wheel W_n $(n \geq 5)$ and duplication of apex vertex of wheel W_n for even n $(n \geq 5)$ is 3-equitable and not 3-equitable for odd n $(n \geq 5)$. In addition to this we also prove that duplication of vertices of wheel W_n altogether is 3-equitable except for $n = 5$.

2. Main Results

Theorem 2.1. *The graph obtained by duplication of arbitrary rim vertex of wheel W_n is 3-equitable for $n \geq 5$ while duplication of apex vertex is 3-equitable for even n and not 3-equitable for odd n, $n \geq 5$.*

Proof. Consider the wheel W_n. Let $v_1, v_2, \ldots, v_n$ be the rim vertices of W_n, c_1 be the apex vertex of W_n and G be the graph obtained by duplicating either rim vertex or apex vertex of W_n. Let v_k' be the duplicated vertex of v_k and c_1' be the duplicated vertex of c_1. To define vertex labeling $f : V(G) \to \{0, 1, 2\}$, we consider the following cases.

Case A. Duplication of arbitrary rim vertex v_k, where $k \in N$, $1 \le k \le n$.

Subcase 1. $n \equiv 0, 1 \pmod 6$.

In this case, we define labeling function f as

$$
\begin{aligned}
f(v_{k+i-1}) &= 0; \text{ if } i \equiv 1, 4 \pmod 6.\\
&= 1; \text{ if } i \equiv 2, 3 \pmod 6.\\
&= 2; \text{ if } i \equiv 0, 5 \pmod 6, 1 \le i \le n-k+1.\\
f(v_{k+i-n-1}) &= 0; \text{ if } i \equiv 1, 4 \pmod 6.\\
&= 1; \text{ if } i \equiv 2, 3 \pmod 6.\\
&= 2; \text{ if } i \equiv 0, 5 \pmod 6, n-k+2 \le i \le n.\\
f(v_k') &= 2; \text{ if } n \equiv 0 \pmod 6.\\
f(v_k') &= 1; \text{ if } n \equiv 1 \pmod 6.\\
f(c_1) &= 0; \text{ if } n \equiv 0 \pmod 6.\\
f(c_1) &= 2; \text{ if } n \equiv 1 \pmod 6.
\end{aligned}
$$

Subcase 2. $n \equiv 2, 5 \pmod 6$.

In this case, we define labeling function f as

$$
\begin{aligned}
f(v_{k+i-1}) &= 0; \text{ if } i \equiv 0, 3 \pmod 6.\\
&= 1; \text{ if } i \equiv 4, 5 \pmod 6.\\
&= 2; \text{ if } i \equiv 1, 2 \pmod 6, 1 \le i \le n-k+1.\\
f(v_{k+i-n-1}) &= 0; \text{ if } i \equiv 0, 3 \pmod 6.\\
&= 1; \text{ if } i \equiv 4, 5 \pmod 6.\\
&= 2; \text{ if } i \equiv 1, 2 \pmod 6, n-k+2 \le i \le n.
\end{aligned}
$$

$$f(v_k') = 1; \text{ if } n \equiv 2 \pmod 6.$$

$$f(v_k') = 2; \text{ if } n \equiv 5 \pmod 6.$$

$$f(c_1) = 0.$$

Subcase 3. $n \equiv 3, 4 \pmod 6$.

In this case, we define labeling function f as:

Subcase 3.1. If $k \leq 2$,

$$\begin{aligned} f(v_{k+i-1}) &= 0; \text{ if } i \equiv 1, 4 \pmod 6. \\ &= 1; \text{ if } i \equiv 0, 5 \pmod 6. \\ &= 2; \text{ if } i \equiv 2, 3 \pmod 6, 1 \leq i \leq n-2. \end{aligned}$$

$$f(v_{n-1}) = 1;$$

$$f(v_n) = 2; \text{ if } k = 1.$$

$$f(v_1) = 2;$$

$$f(v_n) = 1; \text{ if } k = 2.$$

$$f(v_k') = 2;$$

$$f(c_1) = 0.$$

Subcase 3.2. If $k \geq 3$,

$$\begin{aligned} f(v_{k+i-1}) &= 0; \text{ if } i \equiv 1, 4 \pmod 6. \\ &= 1; \text{ if } i \equiv 0, 5 \pmod 6. \\ &= 2; \text{ if } i \equiv 2, 3 \pmod 6, 1 \leq i \leq n-k+1. \end{aligned}$$

$$\begin{aligned} f(v_{k+i-n-1}) &= 0; \text{ if } i \equiv 1, 4 \pmod 6. \\ &= 1; \text{ if } i \equiv 0, 5 \pmod 6. \\ &= 2; \text{ if } i \equiv 2, 3 \pmod 6, n-k+2 \leq i \leq n-2. \end{aligned}$$

$$f(v_{k-1}) = 1;$$

$$f(v_k) = f(v_k') = 2;$$

$$f(c_1) = 0.$$

Case B. Duplication of apex vertex c_1.

Subcase 1. $n \equiv 0 \pmod 6$.

In this case, we define labeling f as:

$$f(v_i) = 0; \text{ if } i \equiv 1, 4 \pmod 6.$$

$$= 1; \text{ if } i \equiv 0, 5 \pmod 6.$$

$$= 2; \text{ if } i \equiv 2, 3 \pmod 6, 1 \le i \le n.$$

$$f(c_1) = 0;$$

$$f(c_1') = 2.$$

Subcase 2. $n \equiv 2 \pmod 6$.

In this case, we define labeling f as:

$$f(v_i) = 0; \text{ if } i \equiv 1, 4 \pmod 6.$$

$$= 1; \text{ if } i \equiv 0, 5 \pmod 6.$$

$$= 2; \text{ if } i \equiv 2, 3 \pmod 6, 1 \le i \le n-2.$$

$$f(v_{n-1}) = 1;$$

$$f(v_n) = 0;$$

$$f(c_1) = f(c_1') = 2.$$

Subcase 3. $n \equiv 4 \pmod 6$.

In this case, we define labeling f as:

$$f(v_i) = 0; \text{ if } i \equiv 1, 4 \pmod 6.$$

$$= 1; \text{ if } i \equiv 0, 5 \pmod 6.$$

$$= 2; \text{ if } i \equiv 2, 3 \pmod 6, 1 \le i \le n-4.$$

$$f(v_{n-3}) = f(v_{n-2}) = f(v_{n-1}) = 1.$$

$$f(v_n) = f(c_1) = 0.$$

$$f(c_1') = 2.$$

Subcase 4. $n \equiv 1 \pmod 6$.

To satisfy the vertex condition it is essential to label $\frac{n+2}{3}$ vertices with 1. It is obvious that any edge will have label 1 if it is incident to the vertex with label 1. As G has $\frac{n+2}{3}$ vertices with label 1 and all the rim vertices are of degree 4 implies that there are at least $3\left(\frac{n+2}{3}-3\right)+8 = n+1$ edges with label 1. As the number of edges in $G = 3n$ and in order to satisfy the edge conditions number of edges with label 1 must be exactly n. Thus edge condition is violated and G is not 3-equitable.

Subcase 5. $n \equiv 3 \pmod 6$.

To satisfy vertex condition it is essential to label $\frac{n}{3}$ vertices with label 1. It is obvious that any edge will have label 1 if it is incident to the vertex with label 1. As G has $\frac{n}{3}$ vertices with label one and all the rim vertices are of degree 4, it has either $3\left(\frac{n}{3}-3\right)+8$, i.e., $n-1$ or $3\left(\frac{n}{3}-1\right)+4$, i.e., $n+1$ edges with label one. As G contains $3n$ edges so number of edges with label one should be exactly n. Thus edge condition is not satisfied. Hence G is not 3-equitable.

Subcase 6. $n \equiv 5 \pmod 6$.

To satisfy vertex condition it is essential to label $\frac{n+1}{3}$ vertices with label 1. It is obvious that any edge will have label 1 if it is incident to the vertex with label 1. As G has $\frac{n+1}{3}$ vertices with label one and all the rim vertices are of degree 4, it has either $3\left(\frac{n+1}{3}-4\right)+10$, i.e., $n-1$ or $3\left(\frac{n+1}{3}\right)$, i.e., $n+1$ edges with label one. As G contains $3n$ edges so number of edges with label one should be exactly n. Thus edge condition is not satisfied. Hence G is not 3-equitable.

The labeling pattern defined above covers all possible arrangement of vertices. In each case, the graph G under consideration satisfies the conditions $|v_f(i)-v_f(j)| \le 1$ and $|e_f(i)-e_f(j)| \le 1$ for all $0 \le i,\ j \le 2$ as shown in Table 1 and Table 2, i.e., G admits 3-equitable labeling.

Case A. Let $n = 6a + b$ and $k \in N,\ 1 \le k \le n,\ a \in N \cup \{0\}$.

Table 1

b	Vertex Condition	Edge Condition
0, 3	$v_f(0) = v_f(1) + 1 = v_f(2)$	$e_f(0) = e_f(1) = e_f(2)$
1, 4	$v_f(0) = v_f(1) = v_f(2)$	$e_f(0) = e_f(1) = e_f(2) + 1$
2, 5	$v_f(0) + 1 = v_f(1) + 1 = v_f(2)$	$e_f(0) + 1 = e_f(1) = e_f(2) + 1$

Case B. Let $n = 6a + b,\ a \in N \cup \{0\}$.

Table 2

b	Vertex Condition	Edge Condition
0	$v_f(0) = v_f(1) + 1 = v_f(2)$	$e_f(0) = e_f(1) = e_f(2)$
2	$v_f(0) + 1 = v_f(1) + 1 = v_f(2)$	$e_f(0) = e_f(1) = e_f(2)$
4	$v_f(0) = v_f(1) = v_f(2)$	$e_f(0) = e_f(1) = e_f(2)$

Remark 2.2 (For the duplication of rim vertex).

◊ For $n = 3$, $|V(G)| = 5$ and $|E(G)| = 9$. In order to satisfy the vertex conditions it is essential to label two vertices with the same labels, other two vertices with the same labels but with the label different than the label which is used earlier. The label which is spared after the labeling of above referred two pairs of vertices will be the label of the remaining one. For example, if we label two vertices with 0, two vertices with 1, then the remaining vertex will receive the label 2. Such labeling will give rise to exactly two edges with label 0. On the other hand, in order to satisfy the edge conditions at least four edges with label 1 are needed. Thus G fails to satisfy the edge condition to be the 3-equitable graph.

◊ For $n = 4$, as $|V(G)| = 6$ it is essential to label two vertices with label 1 to satisfy the vertex conditions. This constraint will give rise to at least five edges with label 1 because G contains the vertices with degrees 3 and 4. On the other hand, in order to satisfy the edge conditions the number of edges with label 1 should be at most four as $|E(G)| = 11$. Thus G fails to satisfy edge conditions to be the 3-equitable graph.

Remark 2.3 (For the duplication of an apex vertex). For $n = 4$, in order to satisfy the vertex conditions it is essential to label exactly two vertices with label 1 as $|V(G)| = 6$. This constraint will give rise to at least six edges with label 1 as G contains vertices with degree four. On the other hand, in order to satisfy edge conditions it is essential to have exactly four edges with label 1. Thus edge conditions for 3-equitable graph is violated.

For better understanding of the above Theorem 2.1 let us consider few examples:

Illustrations 2.4.

Example 1. Consider a graph obtained by duplicating the vertex v_2 of W_5. This is the example related to Subcase 2 of Case A. The 3-equitable labeling is shown in Figure 1.

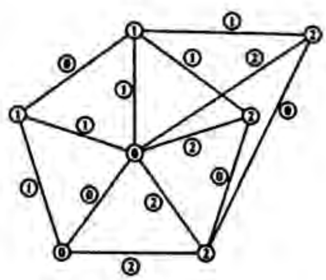

Figure 1

Example 2. Consider a graph obtained by duplicating apex vertex c_1 of W_6. This is the example related to Subcase 1 of Case B. The 3-equitable labeling is shown in Figure 2.

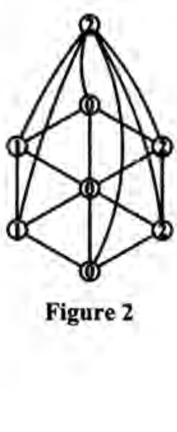

Figure 2

Theorem 2.5. *Duplication of the vertices of wheel W_n altogether produces a 3-equitable graph except for $n = 5$, where $n \in N$.*

Proof. Consider the wheel $W_n = C_n + K_1$. Let $v_1, v_2, \ldots, v_n$ be the rim vertices of W_n, c_1 be the apex vertex of W_n and G be the graph obtained by duplicating vertices altogether. Moreover, $v_1', v_2', \ldots, v_n'$ be the duplicated vertices of $v_1, v_2, \ldots, v_n$ respectively and c_1' be the duplicated vertex of c_1. To define vertex labeling $f : V(G) \to \{0, 1, 2\}$, we consider the following cases.

Case 1. $n \equiv 0 \pmod 6$.

In this case, we define labeling f as:

$$f(v_i) = 0;\ i \equiv 1, 4 \pmod 6.$$
$$= 1;\ i \equiv 0, 5 \pmod 6.$$
$$= 2;\ i \equiv 2, 3 \pmod 6 \text{ for all } i,\ 1 \le i \le n.$$
$$f(v_i') = 0;\ i \equiv 1, 4 \pmod 6.$$
$$= 1;\ i \equiv 0, 5 \pmod 6.$$
$$= 2;\ i \equiv 2, 3 \pmod 6 \text{ for all } i,\ 1 \le i \le n.$$
$$f(c_1) = 0;$$
$$f(c_1') = 2.$$

Case 2. $n \equiv 1 \pmod 6$.

In this case, we define labeling f as:

$$f(v_i) = 0;\ i \equiv 1, 4 \pmod 6.$$
$$= 1;\ i \equiv 0, 5 \pmod 6.$$
$$= 2;\ i \equiv 2, 3 \pmod 6 \text{ for all } i,\ 1 \le i \le n - 1.$$
$$f(v_n) = 1;$$
$$f(v_i') = 0;\ i \equiv 1, 4 \pmod 6.$$
$$= 1;\ i \equiv 0, 5 \pmod 6.$$
$$= 2;\ i \equiv 2, 3 \pmod 6 \text{ for all } i,\ 1 \le i \le n - 1.$$
$$f(v_n') = f(c_1') = 2;$$
$$f(c_1) = 0.$$

Case 3. $n \equiv 2 \pmod 6$.

In this case, we define labeling f as:

$$f(v_i) = 0;\ i \equiv 1, 4 \pmod 6.$$
$$= 1;\ i \equiv 0, 5 \pmod 6.$$
$$= 2;\ i \equiv 2, 3 \pmod 6 \text{ for all } i,\ 1 \le i \le n-2.$$
$$f(v_{n-1}) = f(v_n) = 0;$$
$$f(v_i') = 0;\ i \equiv 1, 4 \pmod 6.$$
$$= 1;\ i \equiv 0, 5 \pmod 6.$$
$$= 2;\ i \equiv 2, 3 \pmod 6 \text{ for all } i,\ 1 \le i \le n-2.$$
$$f(v_{n-1}') = f(v_n') = 1;$$
$$f(c_1) = f(c_1') = 2.$$

Case 4. $n \equiv 3 \pmod 6$.

In this case, we define labeling f as:

$$f(v_1) = f(v_2) = 2;$$
$$f(v_3) = 0;$$
$$f(v_i) = 0;\ i \equiv 1, 4 \pmod 6.$$
$$= 1;\ i \equiv 2, 3 \pmod 6.$$
$$= 2;\ i \equiv 0, 5 \pmod 6),\ 4 \le i \le n.$$
$$f(v_1') = 0;$$
$$f(v_2') = f(v_3') = 1;$$
$$f(v_i') = 0;\ i \equiv 1, 4 \pmod 6.$$
$$= 1;\ i \equiv 2, 3 \pmod 6.$$
$$= 2;\ i \equiv 0, 5 \pmod 6),\ 4 \le i \le n.$$
$$f(c_1) = 2;$$
$$f(c_1') = 0, \text{ if } n \ne 3.$$
$$f(c_1) = 0;$$
$$f(c_1') = 2, \text{ if } n = 3.$$

Case 5. $n \equiv 4 \pmod 6$.

In this case, we define labeling f as:

$$f(v_1) = 0;$$

$$f(v_2) = f(v_4) = 2;$$

$$f(v_3) = 1;$$

$$\begin{aligned} f(v_i) &= 0;\ i \equiv 2, 5 \pmod 6. \\ &= 1;\ i \equiv 3, 4 \pmod 6. \\ &= 2;\ i \equiv 0, 1 \pmod 6,\ 5 \le i \le n. \end{aligned}$$

$$f(v_1') = 0;$$

$$f(v_2') = f(v_4') = 1;$$

$$f(v_3') = 2;$$

$$\begin{aligned} f(v_i') &= 0;\ i \equiv 2, 5 \pmod 6. \\ &= 1;\ i \equiv 3, 4 \pmod 6. \\ &= 2;\ i \equiv 0, 1 \pmod 6,\ 5 \le i \le n. \end{aligned}$$

$$f(c_1) = 0;$$

$$f(c_1') = 2.$$

Case 6. $n \equiv 5 \pmod 6$.

In this case, we define labeling f as:

$$f(v_1) = f(v_4) = 0;$$

$$f(v_2) = f(v_3) = 1;$$

$$f(v_5) = 2;$$

$$\begin{aligned} f(v_i) &= 0;\ i \equiv 0, 3 \pmod 6. \\ &= 1;\ i \equiv 4, 5 \pmod 6. \\ &= 2;\ i \equiv 1, 2 \pmod 6,\ 6 \le i \le n. \end{aligned}$$

$$f(v_1') = f(v_4') = 1;$$

$$f(v_2') = f(v_3') = 2;$$

$$f(v_5') = 0;$$

$$\begin{aligned} f(v_i') &= 0;\ i \equiv 0,\ 3 \pmod 6. \\ &= 1;\ i \equiv 4,\ 5 \pmod 6. \\ &= 2;\ i \equiv 1,\ 2 \pmod 6),\ 6 \le i \le n. \end{aligned}$$

$$f(c_1) = 0;$$

$$f(c_1') = 2.$$

Case 7. $n = 5$.

W_5 contains 12 vertices. In order to satisfy vertex condition 4 vertices must be labeled one. It is obvious that any edge will have label 1 if it is incident to the vertex with label 1. All the rim vertices are of degree 6 and duplicated vertices are of degree 3. Assign label one to v_1, v_n', v_1' and v_2'. It results minimum 11 edges with label one. As number of edges in W_5 is 30, edge condition is not satisfied. Therefore, for $n = 5$ graph G is not 3-equitable.

The labeling pattern defined above covers all possible arrangement of vertices. In each case, the graph G under consideration satisfies the conditions $|v_f(i) - v_f(j)| \le 1$ and $|e_f(i) - e_f(j)| \le 1$ as shown in Table 3, i.e., G admits 3-equitable labeling.

Let $n = 4a + b$ and $a \in N \cup \{0\}$.

Table 3

b	Vertex Condition	Edge Condition
0, 3	$v_f(0) = v_f(1) + 1 = v_f(2)$	$e_f(0) = e_f(1) = e_f(2)$
1	$v_f(0) = v_f(1) + 1 = v_f(2) + 1$	$e_f(0) = e_f(1) = e_f(2)$
2, 5	$v_f(0) = v_f(1) = v_f(2)$	$e_f(0) = e_f(1) = e_f(2)$
4	$v_f(0) + 1 = v_f(1) + 1 = v_f(2)$	$e_f(0) = e_f(1) = e_f(2)$

For better understanding of above defined labeling pattern let us consider following illustration:

Illustration 2.6. Consider a graph obtained by duplicating vertices of wheel W_4 altogether. This is example of Case 5. The 3-equitable labeling is shown in Figure 3.

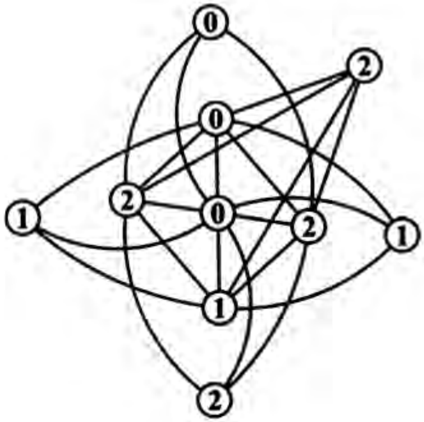

Figure 3

3. Concluding Remarks

Labeled graph is the topic of current interest for many researchers as it has diversified applications. We discuss here 3-equitable labeling for duplication of vertices which is one of the graph operations. This approach is novel and contributes two new graphs to the theory of 3-equitable graphs. The derived results are demonstrated by means of sufficient illustrations which provides better understanding. The results reported here are new and will add new dimension to the theory of 3-equitable graphs.

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Some New Star Related Graphs and Their Cordial as well as 3-equitable Labeling

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Abstract

This paper is in connection with our earlier paper [11]. We present here cordial and 3-equitable labeling for the graphs obtained by joining apex vertices of k copies of stars by an edge as well as to a new vertex.

Keywords : *Cordial labeling, 3-equitable labeling, Star graph.*

INTRODUCTION

We begin with finite undirected graph $G = (V, E)$ without loops and multiple edges. Vertex corresponds to K_1 in star $K_{1,n}$ is called the apex vertex. For all standard terminology and notations we follow Gross and Yellen[6]. We will give brief account of definitions which are useful for the present investigations.

Definition 1.1: Consider k copies of stars namely $K_{1,n}^{(1)}$, $K_{1,n}^{(2)}$, $K_{1,n}^{(3)}$, ... $K_{1,n}^{(k)}$. Then the $G = < K_{1,n}^{(1)} \blacktriangle K_{1,n}^{(2)} \blacktriangle K_{1,n}^{(3)} \blacktriangle \ . \ . \ . \ \blacktriangle K_{1,n}^{(k)}>$ is the graph obtained by joining apex vertices of each $K_{1,n}^{(p-1)}$ and $K_{1,n}^{(p)}$ by an edge as well as to a new vertex x_{p-1} where $2 \leq p \leq k$.

Note that for this new graph G, $|V| = k(n + 2) - 1$ and $|E| = k(n + 3) - 3$.

Definition 1.2: If the vertices of the graph are assigned values subject to certain conditions then it is known as *graph labeling*.

Vast amount of literature is available on different types of graph labeling in printed and electronic form. For detail survey on graph labeling one can refer to Gallian [5] which is updated regularly.

Labeled graph have many diversified applications. A detail study on variety of applications of graph labeling is reported in Bloom and Golomb[2].

Definition 1.3: Let $G = (V, E)$ be a graph. A mapping $f : V(G) \rightarrow \{0,1\}$ is called *binary vertex labeling* of G and $f(v)$ is called the *label* of the vertex v of G under f.

For an edge $e = uv$, the induced edge labeling $f^* : E(G) \rightarrow \{0, 1\}$ is given by $f^*(e)=|f(u) - f(v)|$. Let $v_f(0)$, $v_f(1)$ be the number of vertices of G having labels 0 and 1 respectively under f and let $e_f(0)$, $e_f(1)$ be the number of edges having labels 0 and 1 respectively under f^*.

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Definition 1.4: A binary vertex labeling of a graph G is called a *cordial labeling* if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph G is *cordial* if it admits cordial labeling.

The concept of cordial labeling was introduced by Cahit[3]. Many researchers have studied cordiality of graphs. e.g.Cahit [3] proved that tree is cordial. In the same paper he proved that K_n is cordial if and only if $n \leq 3$. Ho et al.[7] proved that unicyclic graph is cordial unless it is C_{4k+2}. Andar et al.[1] has discussed cordiality of multiple shells. Vaidya et al.[8, 9, 10] have also discussed the cordiality of various graphs.

Definition 1.5: Let $G = (V, E)$ be a graph. A mapping $f:V(G) \rightarrow \{0,1,2\}$ is called ternary vertex labeling of G and $f(v)$ is called label of the vertex v of G under f.

For an edge $e = uv$, the induced edge labeling $f^*:E(G) \rightarrow \{0,1,2\}$ is given by $f^*(e) = |f(u) - f(v)|$. Let $v_f(0)$, $v_f(1)$, $v_f(2)$ be the number of vertices of G having labels 0, 1, 2 respectively under f and $e_f(0)$, $e_f(1)$, $e_f(2)$ be the number of edges having labels 0, 1, 2 respectively under f^*.

Definition 1.6 : A vertex labeling of a graph G is called a *3-equitable labeling* if $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $0 \leq i, j \leq 2$. A graph G is *3-equitable* if it admits 3-equitable labeling.

The concept of 3-equitable labeling was also introduced by Cahit[4] and good number of research papers are available. Many researchers have studied 3-equatability of graphs. e.g.Cahit [4] proved that C_n is 3-equitable except $n \equiv 3(mod 6)$. In the same paper he proved that an Eulerian graph with number of edges congruent to $3(mod 6)$ is not 3-equitable. Youssef[14] proved that W_n is 3-equitable for all $n \geq 4$.

Vaidya et al [12] have discussed 3-equitable labeling in the context of duplication of vertex. The present work is in the sequence of our earlier paper [11]. In that paper we had discussed cordial and 3-equitable labeling of some star related graphs. There we join apex vertices with a new vertex and apex vertices are not adjacent while in this present work the respective apex vertices are also adjacent. Here we prove that the graph $<K_{1,n}^{(1)} \blacktriangle K_{1,n}^{(2)} \blacktriangle K_{1,n}^{(3)} \blacktriangle \ . \ . \ . \ \blacktriangle K_{1,n}^{(k)}>$ is cordial as well as 3-equitable.

MAIN RESULTS

Theorem 2.1: Graph $<K_{1,n}^{(1)} \blacktriangle K_{1,n}^{(2)} \blacktriangle K_{1,n}^{(3)} \blacktriangle \ldots \blacktriangle K_{1,n}^{(k)}>$ is cordial.

Proof: Let $K_{1,n}^{(j)}$ be k copies of star $K_{1,n}$, $v_i^{(j)}$ be the pendant vertices of $K_{1,n}^{(j)}$ and c_j be the apex vertex of $K_{1,n}^{(j)}$ (here $i = 1, 2, \dots n$ and $j = 1, 2, \dots k$). Let $x_1, x_2 \dots x_{k-1}$ be the vertices such that c_{p-1} and c_p are adjacent with them selves as well as to a new common vertex x_{p-1} where $2 \le p \le k$. Consider $G = < K_{1,n}^{(1)} \blacktriangle K_{1,n}^{(2)} \blacktriangle K_{1,n}^{(3)} \blacktriangle \dots \blacktriangle K_{1,n}^{(k)} >$. To define binary vertex labeling $f:V(G)\rightarrow\{0,1\}$ we consider following cases where $n,k \in N$ and $j=1, 2, \dots, k$.

Case 1: n even.

If j odd

$f(v_i^{(j)}) = 0$; if $1 \le i \le \frac{n}{2}$

$= 1$; if $\frac{n+2}{2} \le i \le n$.

$f(c_j) = 1$;

If j even.

$f(v_i^{(j)}) = 0$; if $1 \le i \le \frac{n+2}{2}$

$= 1$; if $\frac{n+4}{2} \le i \le n$.

$f(c_j) = 0$;

$f(x_j) = 1$; for all j, $j \ne k$.

Case 2: n odd.

$f(v_i^{(j)}) = 0$; if $1 \le i \le \frac{n-1}{2}$

$= 1$; if $\frac{n+1}{2} \le i \le n$.

$f(c_j) = 0$;

$f(x_j) = 1$; if j even.

$= 0$; if j odd, $j \ne k$.

The labeling pattern defined above covers all the possibilities. In each case the graph G under consideration satisfies the conditions $|v_f(0) - v_f(1)| \le 1$ and $|e_f(0) - e_f(1)| \le 1$ as shown in *Table 1*. i.e. G admits cordial labeling.

Let $n = 2a + b$ and $k = 2c + d$ where $a \in N \cup \{0\}$, $c \in N$

Table 1: Table showing vertex and edge conditions

b	*d*	Vertex Condition	Edge Condition
0	0	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)+1$
	1	$v_f(0)+1=v_f(1)$	$e_f(0)=e_f(1)$
1	0	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)+1$
	1	$v_f(0)=v_f(1)$	$e_f(0)+1=e_f(1)$

Illustration 2.2: Consider $G = <K_{1,7}^{(1)} \blacktriangle K_{1,7}^{(2)} \blacktriangle K_{1,7}^{(3)} \blacktriangle K_{1,7}^{(4)}>$. Here n=7 and k=4. The cordial labeling is as shown in *Figure 1*. It is the case 2 of Theorem 2.1

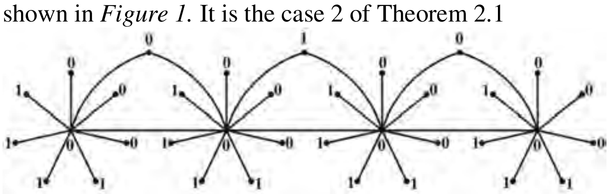

Figure 1: Cordial labeling for graph G

Theorem 2.3: Graph $<K_{1,n}^{(1)} \blacktriangle K_{1,n}^{(2)} \blacktriangle K_{1,n}^{(3)} \blacktriangle \dots \blacktriangle K_{1,n}^{(k)}>$ is 3-equitable.

Proof: Let $K_{1,n}^{(j)}$ be k copies of star $K_{1,n}$, $v_i^{(j)}$ be the pendant vertices of $K_{1,n}^{(j)}$ and c_j be the apex vertex of $K_{1,n}^{(j)}$ (here $i = 1, 2, \dots n$ and $j = 1, 2, \dots k$). Let $G = < K_{1,n}^{(1)} \blacktriangle K_{1,n}^{(2)} \blacktriangle K_{1,n}^{(3)} \blacktriangle \dots \blacktriangle K_{1,n}^{(k)} >$ and $x_1, x_2 \dots x_{k-1}$ are the vertices as stated in Theorem 2.1. To define vertex labeling $f:V(G)\rightarrow\{0,1,2\}$ we consider following cases.

Case 1: For $n \equiv 0(mod3)$

Subcase 1: For $k \equiv 0(mod3)$

For $j \equiv 0, 1(mod3)$

$f(v_i^{(j)}) = 0$; if $i \equiv 0(mod3)$, $j \ne 3$ and $i \ne n$

$= 1$; if $i \equiv 1(mod3)$

$= 2$; if $i \equiv 2(mod3)$

$f(v_n^{(3)}) = 1$;

$f(c_j) = 2$; if $j \equiv 1(mod3)$

$f(c_j) = 0$; if $j \equiv 0(mod3)$ and $j \ne 3$

$f(c_3) = 2$;

$f(x_j) = 2$ if $j \equiv 1(mod3)$

$f(x_j) = 0$ if $j \equiv 0(mod3)$

For $j \equiv 2(mod3)$

$f(v_i^{(j)}) = 0$; if $i \equiv 2(mod3)$

$= 1$; if $i \equiv 1(mod3)$

$= 2$; if $i \equiv 0(mod3)$, $i \ne n$

$f(v_n^{(j)}) = 1$;

$f(c_j) = 2$; if $j \ne 2$

$f(c_2) = f(x_2) = 0$;

$f(x_j) = 1$; if $j \ne 2$

Subcase 2: For $k \equiv 1(mod3)$

For $j \equiv 1, 2(mod3)$

$f(v_i^{(j)}) = 0$; if $i \equiv 0(mod3)$

$= 1$; if $i \equiv 1(mod3)$

$= 2$; if $i \equiv 2(mod3)$

$f(c_j) = 0$; if $j \equiv 1(mod3)$ and $j \ne 1$

$f(c_j) = 2$; if $j \equiv 2(mod3)$

$f(c_1) = 2$;

$f(x_j) = 0$ if $j \equiv 1(mod3)$

$f(x_j) = 2$ if $j \equiv 2(mod3)$

For $j \equiv 0(mod3)$

$f(v_i^{(j)}) = 0$; if $i \equiv 2(mod3)$

$= 1$; if $i \equiv 1(mod3)$

$= 2$; if $i \equiv 0(mod3)$, $i \ne n$

$f(v_n^{(j)}) = f(x_j) = 1$;

$f(c_j) = 2$;

Subcase 3: For $k \equiv 2(mod3)$

For $j \equiv 0, 2(mod3)$

$f(v_i^{(j)}) = 0$; if $i \equiv 0(mod3)$

$= 1$; if $i \equiv 1(mod3)$

$= 2$; if $i \equiv 2(mod3)$

$f(c_j) = 2$; if $j \equiv 0(mod3)$

$f(c_j) = 0$; if $j \equiv 2(mod3)$ and $j \ne 2$

$f(c_2) = 2$;

$f(x_j) = 2$ if $j \equiv 0(mod3)$

$f(x_j) = 0$ if $j \equiv 2(mod3)$

For $j \equiv 1(mod3)$

$f(v_i^{(j)}) = 0$; if $i \equiv 2(mod3)$

$= 1$; if $i \equiv 1(mod3)$

$= 2$; if $i \equiv 0(mod3)$, $i \ne n$

$f(v_n^{(j)}) = 1;$
$f(c_j) = 2$ if $j \neq 1$
$f(c_1) = 0;$
$f(x_1) = 2;$
$f(x_j) = 1$ if $j \neq 1$

Case 2: For $n \equiv 1(mod3)$

Subcase 1: For $k \equiv 0(mod3)$

Subcase 1.1: For $n = 1$

$f(v_1^{(1)}) = 1;$
$f(v_1^{(2)}) = f(v_1^{(3)}) = f(c_1) = 2;$
$f(c_2) = f(c_3) = f(x_2) = 0;$
$f(x_2) = 1;$

For remaining vertices use the pattern of subcase 1.2.

Subcase 1.2: For $n > 1$

$f(v_i^{(j)}) = 0$; if $i \equiv 0(mod3)$, $i \neq n\text{-}1$ and $j \neq 3$
$= 1$; if $i \equiv 1(mod3)$, $i \neq n$, $j=1$
$= 2$; if $i \equiv 2(mod3)$
$f(v_n^{(j)}) = f(v_{n-1}^{(3)}) = 2$; if $j \neq 1$
$f(c_j) = 2$; if $j \equiv 1(mod3)$
$= 0$; if $j \equiv 0, 2(mod3)$
$f(x_j) = 2$; if $j \equiv 1(mod3)$
$= 0$; if $j \equiv 0(mod3)$, $j \neq k$
$= 1$; if $j \equiv 2(mod3)$, $j \neq 2$
$f(x_2) = 0;$

Subcase 2: For $k \equiv 1(mod3)$

$f(v_i^{(j)}) = 0$; if $i \equiv 0(mod3)$
$= 1$; if $i \equiv 1(mod3)$, $i \neq n$, $j \equiv 0, 1(mod3)$
$= 2$; if $i \equiv 2(mod3)$
$f(v_n^{(j)}) = 2$; if $j \equiv 2(mod3)$
$f(c_j) = 0$; if $j \equiv 0, 1(mod3)$ and $j \neq 1$
$f(c_j) = 2$; if $j \equiv 2(mod3)$
$f(c_1) = 2;$
$f(x_j) = 1$; if $j \equiv 0(mod3)$
$f(x_j) = 0$; if $j \equiv 1(mod3)$
$f(x_j) = 2$; if $j \equiv 2(mod3)$

Subcase 3: For $k \equiv 2(mod3)$

$f(v_i^{(j)}) = 0$; if $i \equiv 0(mod3)$
$= 1$; if $i \equiv 1(mod3)$, $i \neq n$, $j \equiv 1, 2(mod3)$
$= 2$; if $i \equiv 2(mod3)$
$f(v_n^{(j)}) = 2$; if $j \equiv 0(mod3)$
$f(c_j) = 0$; if $j \equiv 1, 2(mod3)$ and $j \neq 1$
$f(c_j) = 2$; if $j \equiv 0(mod3)$
$f(c_1) = f(x_1) = 2;$
$f(x_j) = 1$; if $j \equiv 1(mod3)$ and $j \neq 1$
$f(x_j) = 0$; if $j \equiv 2(mod3)$
$f(x_j) = 2$; if $j \equiv 0(mod3)$

Case 3: For $n \equiv 2(mod3)$

Subcase 1: For $k \equiv 0(mod3)$

$f(v_i^{(j)}) = 0$; if $i \equiv 0(mod3)$
$= 1$; if $i \equiv 1(mod3)$
$= 2$; if $i \equiv 2(mod3)$, $i \neq n$, $j \equiv 1, 2(mod3)$
$f(v_n^{(3)}) = 1;$
$f(v_n^{(j)}) = 0$; if $j \equiv 0(mod3)$ and $j \neq 3$
$f(c_j) = 2$; if $j \equiv 1, 2(mod3)$ and $j \neq 1,2$
$f(c_j) = 0$; if $j \equiv 0(mod3)$ and $j \neq 3$
$f(c_1) = f(c_2) = 0;$
$f(c_3) = f(x_2) = 2;$
$f(x_j) = 0$; if $j \equiv 0,1(mod3)$
$= 1$; if $j \equiv 2(mod3)$ and $j \neq 2$

Subcase 2: For $k \equiv 1(mod3)$

$f(v_i^{(j)}) = 0$; if $i \equiv 0(mod3)$
$= 1$; if $i \equiv 1(mod3)$
$= 2$; if $i \equiv 2(mod3)$, $i \neq n$, $j \equiv 0, 2(mod3)$
$f(v_n^{(1)}) = 2;$
$f(v_n^{(j)}) = 0$; if $j \equiv 1(mod3)$ and $j \neq 1$
$f(c_j) = 0$; if $j \equiv 1\ (mod3)$
$f(c_j) = 2$; if $j \equiv 0, 2\ (mod3)$
$f(x_j) = 0$; if $j \equiv 1,2(mod3)$
$= 1$; if $j \equiv 0(mod3)$

Subcase 3: For $k \equiv 2(mod3)$

$f(v_i^{(j)}) = 0$; if $i \equiv 0(mod3)$
$= 1$; if $i \equiv 1(mod3)$
$= 2$; if $i \equiv 2(mod3)$, $i \neq n$, $j \equiv 0, 1(mod3)$
$f(v_n^{(2)}) = 2;$
$f(v_n^{(j)}) = 0$; if $j \equiv 2(mod3)$ and $j \neq 2$
$f(c_j) = 0$; if $j \equiv 2(mod3)$
$f(c_j) = 2$; if $j \equiv 0, 1\ (mod3)$ and $j \neq 1$
$f(x_j) = 0$; if $j \equiv 0,2(mod3)$
$= 1$; if $j \equiv 1(mod3)$ and $j \neq 1$
$f(c_1) = f(x_1) = 0;$

The labeling pattern defined above covers all possible arrangement of vertices. In each case, the graph G under consideration satisfies the conditions $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $0 \leq i, j \leq 2$ as shown in *Table 2*. i.e. G admits 3-equitable labeling.

Let $n = 3a + b$ and $k = 3c + d$ where $a\ 0N \cup \{0\}$, $c\ 0N$.

Table 2 : Table showing vertex and edge conditions

b	*d*	**Vertex Condition**	**Edge Condition**
0	0	$v_f(0)+1=v_f(1)=v_f(2)$	$e_f(0)=e_f(1)=e_f(2)$
	1	$v_f(0)+1=v_f(1)+1=v_f(2)$	$e_f(0)=e_f(1)=e_f(2)$
	2	$v_f(0)=v_f(1)=v_f(2)$	$e_f(0)=e_f(1)=e_f(2)$
1	0	$v_f(0)+1=v_f(1)=v_f(2)$	$e_f(0)=e_f(1)=e_f(2)$
	1	$v_f(0)+1=v_f(1)=v_f(2)$	$e_f(0)+1=e_f(1)=e_f(2)+1$
	2	$v_f(0)+1=v_f(1)=v_f(2)$	$e_f(0)+1=e_f(1)=e_f(2)$
2	0	$v_f(0)+1=v_f(1)=v_f(2)$	$e_f(0)=e_f(1)=e_f(2)$
	1	$v_f(0)=v_f(1)=v_f(2)$	$e_f(0)+1=e_f(1)=e_f(2)$
	2	$v_f(0)=v_f(1)+1=v_f(2)+1$	$e_f(0)=e_f(1)+1=e_f(2)+1$

Illustration 2.4: Consider a graph $G=< K_{1,10}^{(1)} \blacktriangle K_{1,10}^{(2)} \blacktriangle K_{1,10}^{(3)} \blacktriangle K_{1,10}^{(4)}>$. Here $n = 10$ and $k = 4$. The corresponding 3-equitable labeling is as shown in *Figure 2*.

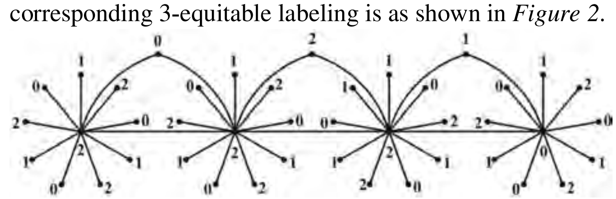

Figure 2: 3-equitable labeling for graph G

CONCLUDING REMARKS

We discuss here cordial labeling and 3-equitable labeling of some star related graphs. The derived labeling pattern is demonstrated by means of elegant illustrations which provide better understanding of the results. We have also investigated similar results for shell related graphs which is the extension of earlier published work by Vaidya et al [13] but for the sake of brevity they are not reported here.

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Some New Product Cordial Graphs

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***Abstract*-We present here product cordial labeling for the graphs obtained by joining apex vertices of two stars, shells and wheels to a new vertex. We extend these results for *k* copies of stars, shells and wheels.**

Keywords: **Graph Labeling, Cordial graphs, Product Cordial graphs.**

I. INTRODUCTION

We begin with simple, finite, connected and undirected graph $G = (V, E)$. In the present work $K_{1,n}$ and $W_n = C_n + K_1$ $(n \geq 3)$ denote the star and wheel respectively. For all other standard terminology and notations we follow Harary [1]. We will give brief summary of definitions which are useful for the present investigations.

Definition 1.1 **:** A *shell* S_n is the graph obtained by taking n-3 concurrent chords in a cycle C_n of n vertices. The vertex at which all the chords are concurrent is called the *apex vertex*. The shell is also called fan F_{n-1}. *i.e.* $S_n=F_{n-1}=P_{n-1}+K_1$.

Definition 1.2 : Consider two shells $S_n^{(1)}$ and $S_n^{(2)}$. Then, the graph $G =< S_n^{(1)}: S_n^{(2)} >$ is obtained by joining apex vertices of shells to a new vertex x. Similar constructions may be operated for two wheels and stars.

Definition 1.3 : Consider k copies of shells namely $S_n^{(1)}, S_n^{(2)}, S_n^{(3)}, \ldots, S_n^{(k)}$. Then, the graph $G =< S_n^{(1)}: S_n^{(2)}: \ldots: S_n^{(k)}>$ is obtained by joining apex vertex of each $S_n^{(p)}$ and apex of $S_n^{(p-1)}$ to a new vertex x_{p-1} where $2 \leq p \leq k$.

The graphs corresponding to $K_{1,n}$ and W_n can be constructed similarly.

Definition 1.4 : If the vertices of the graph are assigned values subject to certain conditions then it is known as *graph labeling*.

For a detailed survey on graph labeling see Gallian [2].

The most interesting graph labeling to be considered has three important characteristics:

(i) a set of numbers from which the labels are chosen;
(ii) a rule that assigns a value to each edge;
(iii) a condition that these values must satisfy.

The present work is intended to discuss one such labeling known as product cordial labeling defined as follows.

Definition 1.5 : Let $G = (V, E)$ be a graph. A mapping $f: V(G) \rightarrow \{0, 1\}$ is called *binary vertex labeling* of G and $f(v)$ is called the label of the vertex v of G under f.

For an edge $e = uv$, the induced edge labeling $f^*: E(G) \rightarrow \{0, 1\}$ is given by $f^*(e) = f(u)f(v)$. Let $v_f(0)$, $v_f(1)$ be the number of vertices of G having labels 0 and 1 respectively under f while $e_f(0)$, $e_f(1)$ be the number of edges having labels 0 and 1 respectively under f^*.

Definition 1.6 : A binary vertex labeling of a graph G is called a *product cordial labeling* if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph G is *product cordial* if it admits product cordial labeling.

The concept of product cordial labeling was introduced by Sundaram et al.[3]. They proved that trees, unicyclic graphs of odd order, triangular snakes, dragons, helms and union of two path graphs are product cordial. They also proved that a graph with p vertices ($p \geq 4$) and q edges is product cordial then $4q < p^2 - 1$.

In the present investigations we prove that the graphs $<S_n^{(1)}: S_n^{(2)}>$, $<K_{1,n}^{(1)}:K_{1,n}^{(2)}>$, $<K_{1,n}^{(1)}:K_{1,n}^{(2)}:K_{1,n}^{(3)}: \ldots: K_{1,n}^{(k)}>$ and $< W_n^{(1)}: W_n^{(2)} >$ are product cordial. We also prove that graph $< S_n^{(1)}: S_n^{(2)} : S_n^{(3)}: \ldots: S_n^{(k)}>$ is product cordial except k odd and n even. Further we prove that graph $< W_n^{(1)}: W_n^{(2)} : W_n^{(3)}: \ldots: W_n^{(k)}>$ is product cordial for (i) k even and n even or odd (ii) k odd and even n with $k>n$ and (iii) not product cordial otherwise.

II MAIN RESULTS

Theorem-2.1: Graph $< S_n^{(1)}: S_n^{(2)} >$ is product cordial.

Proof: Let $v_1^{(1)}, v_2^{(1)}, v_3^{(1)}, \ldots v_n^{(1)}$ be the vertices $S_n^{(1)}$ and $v_1^{(2)}, v_2^{(2)}, v_3^{(2)}, \ldots v_n^{(2)}$ be the vertices $S_n^{(2)}$. Let $v_1^{(1)}$ and $v_1^{(2)}$ be the apex vertices of $S_n^{(1)}$ and $S_n^{(2)}$ respectively which are joined to a vertex x.

For $G =< S_n^{(1)}: S_n^{(2)} >$. We define binary vertex labeling $f: V(G) \rightarrow \{0, 1\}$ as follows.

$f(v_i^{(1)}) = 1;$
$f(v_i^{(2)}) = 0;$ where $1 \leq i \leq n$
$f(x) \quad = 1;$

Thus vertices of $S_n^{(1)}$ are labeled with 1 and vertices of $S_n^{(2)}$ are labeled with 0 while the vertex x is labeled with 1. Consequently $v_f(0) = n$, $v_f(1) = n+1$ and $e_f(0) = e_f(1) = 2n$-2. Thus the graph G satisfies the conditions for product cordial graph. i.e. G admits product cordial labeling.

Illustration 2.2 : Consider a graph $G =< S_8^{(1)} : S_8^{(2)} >$. Here

$n=8$. The product cordial labeling is as shown in *Figure 1*.

Theorem-2.3: Graph $< S_n^{(1)}: S_n^{(2)}: \ldots .: S_n^{(k)} >$ is product cordial except k odd and n even.

Proof: Let $S_n^{(j)}$ be the shells. Let $v_i^{(j)}$ be the vertices $S_n^{(j)}$ and $v_1^{(j)}$ be the apex vertices of $S_n^{(j)}$. Let x_j $(j \neq k)$ be the new vertices where $1 \leq j \leq k$. Let $G = <S_n^{(1)}: S_n^{(2)}: S_n^{(3)}: . . .: S_n^{(k)}>$. For $1 \leq i \leq n$ we define binary vertex labeling $f : V(G) \rightarrow \{0, 1\}$ as follows.

*Case-1***:** k even

$f(v_i^{(j)}) = 1;$ if $j \leq \frac{k}{2}$

$f(v_i^{(j)}) = 0;$ if $j > \frac{k}{2}$

$f(x_j) = 1;$ if $j \leq \frac{k}{2}$

$f(x_j) = 0;$ if $\frac{k}{2} < j \leq k-1$

*Case-2***:** both k and n odd

$f(v_i^{(j)}) = 1;$ if $j \leq \frac{k-1}{2}$

$f(v_i^{(j)}) = 1;$ if $j = \frac{k+1}{2}$ and $i \leq \frac{n+1}{2}$

$f(v_i^{(j)}) = 0;$ if $j = \frac{k+1}{2}$ and $i > \frac{n+1}{2}$

$f(v_i^{(j)}) = 0;$ if $j > \frac{k+1}{2}$

$f(x_j) = 1;$ if $j \leq \frac{k-1}{2}$

$f(x_j) = 0;$ if $\frac{k-1}{2} < j \leq k-1$

In both the cases described above the graph G satisfies the vertex condition $v_f(0)+1=v_f(1)$ and edge condition $e_f(0)=e_f(1)+1$. i.e. G admits product cordial labeling.

Case-3 : k odd and n even

We assign label 1 to all the vertices of first $\frac{k-1}{2}$ copies of shells and assign label 0 to all the vertices of last $\frac{k-1}{2}$ copies of shells. This will provide equal number of vertices and edges with label 0 and 1. Now our task is to label n vertices of a shell (i.e. vertices of $\left(\frac{k+1}{2}\right)^{th}$ copy). In order to satisfy vertex condition for product cordiality $\frac{n}{2}$ vertices must be labeled with 0.

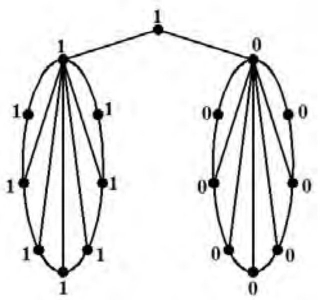

Fig. 1. Product cordial labeling of the graph G.

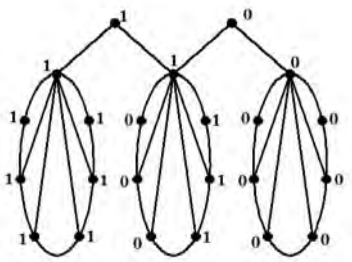

Fig. 2. Product cordial labeling of the graph G.

Then at least n edges will get label 0. Consequently the number of edges with label 1 is $(2n-3)-(n)=n-3$ because $|S_n(E)|=2n-3$.

Hence $|e_f(0)-e_f(1)|=|n-(n-3)|=3$. Thus edge condition is not satisfied. i.e. G is not product cordial.

Illustration 2.4 : Consider a graph $G =< S_7^{(1)}: S_7^{(2)} : S_7^{(3)} >$. Here $n = 7$. The product cordial labeling is as shown in *Figure 2*.

Theorem-2.5: Graph $< K_{1,n}^{(1)}: K_{1,n}^{(2)} >$ is product cordial.

Proof: Let $v_1^{(1)}, v_2^{(1)}, . . . v_n^{(1)}$ and $v_1^{(2)}, v_2^{(2)}, . . . v_n^{(2)}$ be the pendant vertices of $K_{1,n}^{(1)}$ and $K_{1,n}^{(2)}$ respectively. Let c_1 and c_2 be the apex vertices of $K_{1,n}^{(1)}$ and $K_{1,n}^{(2)}$ respectively which are adjacent to a common vertex x. Let $G =<K_{1,n}^{(1)}: K_{1,n}^{(2)}>$. We define binary vertex labeling $f: V(G) \rightarrow \{0, 1\}$ as follows.

$f(v_i^{(1)}) = 1;$

$f(v_i^{(2)}) = 0;$ where $1 \leq i \leq n$

$f(x) = 1;$

In view of the above defined labeling pattern $v_f(0)= e_f(0)= e_f(1)=n+1$ and $v_f(1)=n+2$. Thus the graph G satisfies the vertex condition and edge condition because $v_f(0)+1=v_f(1)$ and $e_f(0)= e_f(1)$. i.e. G admits product cordial labeling.

Illustration 2.6 : Consider a graph $G =< K_{1,8}^{(1)} : K_{1.8}^{(2)} >$. Here $n = 8$. The product cordial labeling is as shown in *Figure 3*.

Theorem-2.7: Graph $< K_{1,n}^{(1)}: K_{1,n}^{(2)} : K_{1,n}^{(3)}: . . .: K_{1,n}^{(k)}>$ is product cordial.

Proof : Let $v_i^{(j)}$ be the pendant vertices of $K_{1,n}^{(j)}$ and c_j be the apex vertices of $K_{1,n}^{(j)}$. Let x_j $(j \neq k)$ be the new vertices where $1 \leq j \leq k$. Let $G =< K_{1,n}^{(1)}: K_{1,n}^{(2)} : K_{1,n}^{(3)}: . . . .: K_{1,n}^{(k)}>$. We define binary vertex labeling f $: V(G) \rightarrow \{0, 1\}$ as follows.

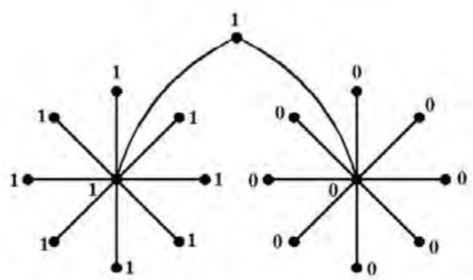

Fig. 3. Product cordial labeling of the graph G.

Case-1: k even.

$f(v_i^{(j)}) = 1;$ if $1 \le j \le \frac{k}{2}$

$f(v_i^{(j)}) = 0;$ if $\frac{k+2}{2} \le j \le k$, where $1 \le i \le n$

$f(c_j) = 1;$ if $1 \le j \le \frac{k}{2}$

$f(c_j) = 0;$ if $\frac{k+2}{2} \le j \le k$

$f(x_j) = 1;$ if $1 \le j \le \frac{k}{2}$

$f(x_j) = 0;$ if $\frac{k+2}{2} \le j \le k-1$

Case-2: k odd.

Subcase 2.1: n even and $1 \le i \le n$

$f(v_i^{(j)}) = 1;$ if $1 \le j \le \frac{k-1}{2}$

$f(v_i^{(j)}) = 0;$ if $\frac{k+3}{2} \le j \le k$

$f(c_j) = 1;$ if $1 \le j \le \frac{k+1}{2}$

$f(c_j) = 0;$ if $\frac{k+3}{2} \le j \le k$

$f(x_j) = 1;$ if $1 \le j \le \frac{k-1}{2}$

$f(x_j) = 0;$ if $\frac{k+1}{2} \le j \le k-1$

For $j = \frac{k+1}{2}$

$f(v_i^{(j)}) = 1;$ if $1 \le i \le \frac{n}{2}$

$f(v_i^{(j)}) = 0;$ if $\frac{n+2}{2} \le i \le n$

Subcase 2.2: n odd and $1 \le i \le n$

$f(v_i^{(j)}) = 1;$ if $1 \le j \le \frac{k-1}{2}$

$f(v_i^{(j)}) = 0;$ if $\frac{k+3}{2} \le j \le k$

$f(c_j) = 1;$ if $1 \le j \le \frac{k+1}{2}$

$f(c_j) = 0;$ if $\frac{k+3}{2} \le j \le k$

$f(x_j) = 1;$ if $1 \le j \le \frac{k-1}{2}$

$f(x_j) = 0;$ if $\frac{k+1}{2} \le j \le k-1$

For $j = \frac{k+1}{2}$

$f(v_i^{(j)}) = 1;$ if $1 \le i \le \frac{n-1}{2}$

$f(v_i^{(j)}) = 0;$ if $\frac{n+1}{2} \le i \le n$

The labeling pattern defined above exhaust all the possibilities for n and k and in each cases the graph G satisfies the conditions $|v_f(0) - v_f(1)| \le 1$ and $|e_f(0) - e_f(1)| \le 1$ as shown in *Table 1*(where $n=2a+b$, $k=2c+d$ and $a,c \in N$). i.e. G admits product cordial labeling. *Illustration 2.8* : Consider a graph $G = \langle K_{1,5}^{(1)}:K_{1,5}^{(2)}:K_{1,5}^{(3)} \rangle$. Here $n = 5$. The product cordial labeling is as shown in Fig. 4.

TABLE: 1 TABLE SHOWING VERTEX AND EDGE CONDITIONS.

d	b	Vertex Condition	Edge Condition
0	*0,1*	$v_f(0)+1=v_f(1)$	$e_f(0)=e_f(1)$
1	*0*	$v_f(0)+1=v_f(1)$	$e_f(0)=e_f(1)$
	1	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$

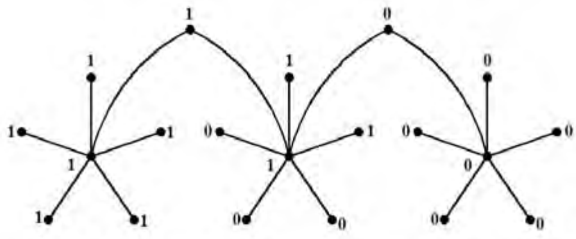

Fig. 4. Product cordial labeling of the graph G.

Theorem-2.9: Graph $\langle W_n^{(1)}: W_n^{(2)} \rangle$ is product cordial.
Proof: Let $v_1^{(1)}, v_2^{(1)}, \ldots v_n^{(1)}$ and $v_1^{(2)}, v_2^{(2)}, \ldots v_n^{(2)}$ be the rim vertices of $W_n^{(1)}$ and $W_n^{(2)}$ respectively. Let c_1 and c_2 be the apex vertices of $W_n^{(1)}$ and $W_n^{(2)}$ respectively which are adjacent to a common vertex x. Let $G = \langle W_n^{(1)}: W_n^{(2)} \rangle$. We define binary vertex labeling $f: V(G) \rightarrow \{0, 1\}$ as follows.

$f(v_i^{(1)}) = 1;$

$f(v_i^{(2)}) = 0;$ where $1 \le i \le n$

$f(x) = 1;$

Then the graph G satisfies the vertex condition $v_f(0)+1=v_f(1)$ and edge condition $e_f(0)= e_f(1)$. i.e. G admits product cordial labeling.

Illustration 2.10 : Consider a graph $G = \langle W_7^{(1)} : W_7^{(2)} \rangle$. Here $n = 7$. The product cordial labeling is as shown in *Figure 5*.

Theorem -2.11: Graph $\langle W_n^{(1)}: W_n^{(2)} : W_n^{(3)}: \ldots : W_n^{(k)} \rangle$ is product cordial for (i) k even and n even or odd (ii) k odd and n even with $k>n$ and (iii) not product cordial otherwise.

Proof: Let $v_i^{(j)}$ be the rim vertices $W_n^{(j)}$ and c_j be the apex vertices of $W_n^{(j)}$. Let x_j $(j \ne k)$ be the new vertices. Let $G = \langle W_n^{(1)}: W_n^{(2)} : W_n^{(3)}: \ldots : W_n^{(k)} \rangle$. We define binary vertex labeling $f : V(G) \rightarrow \{0, 1\}$ as follows.

Case-1: k even and $1 \le i \le n$

$f(v_i^{(j)}) = 1;$ if $1 \le j \le \frac{k}{2}$

$f(v_i^{(j)}) = 0;$ if $\frac{k+2}{2} \le j \le k$

$f(c_j) = 1;$ if $1 \le j \le \frac{k}{2}$

$f(c_j) = 0;$ if $\frac{k+2}{2} \le j \le k$

$f(x_j) = 1;$ if $1 \le j \le \frac{k}{2}$

$f(x_j) = 0;$ if $\frac{k+2}{2} \le j \le k-1$

Case-2: k odd, n even with $k>n$ and $1 \le i \le n$

$f(v_i^{(j)}) = 1;$ if $1 \le j \le \frac{k+1}{2}$

$f(v_i^{(j)}) = 0;$ if $\frac{k+3}{2} \le j \le k$

$f(c_j) = 1;$ if $1 \le j \le \frac{k+1}{2}$

$f(c_j) = 0;$ if $\frac{k+3}{2} \le j \le k$

$f(x_j) = 1;$ if $1 \le j \le \frac{k-n-1}{2}$

$f(x_j) = 0;$ if $\frac{k-n+1}{2} \le j \le k-1$

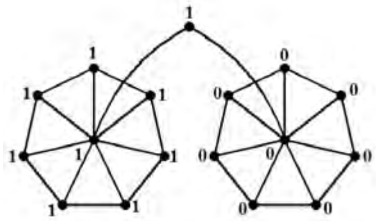

Fig. 5. Product cordial labeling of the graph G.

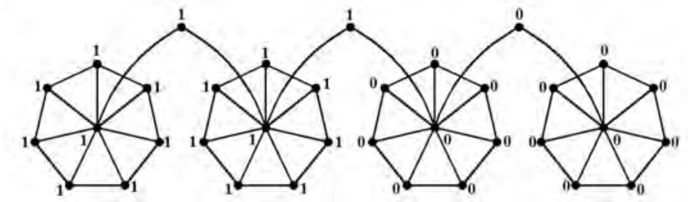

Fig. 6. Product cordial labeling of the graph *G*.

In both the cases described above the graph G satisfies the vertex condition as $v_f(0)+1=v_f(1)$ and edge condition as $e_f(0)= e_f(1)$. i.e. G admits product cordial labeling .

Thus we proved (i) and (ii) while to prove (iii) we have to consider following two cases.

Case-3: k and n odd.

We assign label *1* to all the vertices of first $\frac{k-1}{2}$ copies of wheels and assign label *0* to all the vertices of last $\frac{k-1}{2}$ copies of wheels. This will provide equal number of vertices and edges with label *0* and *1*. Now our task is to label $n+1$ vertices of a wheel (i.e. vertices of $\left(\frac{k+1}{2}\right)^{th}$ copy). In order to satisfy vertex condition for product cordiality $\frac{n+1}{2}$ vertices must be labeled with *0.* Then at least $n+2$ edges will get label *0*. Consequently the number of edges with label 1 is $(2n)-(n+2) = n-2$ because $|W_n(E)|=2n$. Hence $|e_f(0)-e_f(1)| =| n+2-(n-2)|=4$. Thus edge condition is not satisfied. i.e. G is not product cordial.

Case-4: For k odd and n even with $n \geq k$.

If $\frac{k+1}{2}$ copies of wheel are labeled with *1* then vertex condition is not satisfied as $n \geq k$.

Then arguing as in Case-3 the graph G does not admit product cordial labeling.

Illustration 2.12 : Consider a graph $G =< W_6^{(1)}: W_6^{(2)} : W_6^{(3)} : W_6^{(4)} >$. Here $n = 6$. The product cordial labeling is as shown in *Figure 6*.

III. CONCLUDING REMARKS

We derive six new results for product cordial labeling. The defined labeling pattern is demonstrated by means of enough illustrations which will provide better understanding of the derived results. It is also possible to investigate similar results corresponding to other graph families and for different graph labeling techniques.

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Strongly Multiplicative Labeling in the Context of Arbitrary Supersubdivision

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Abstract

We investigate some new results for strongly multiplicative labeling of graph. We prove that the graph obtained by arbitrary supersubdivision of tree T, grid graph $P_n \times P_m$, complete bipartite graph $K_{m,n}$, $C_n \odot P_m$ and one-point union of m cycle of length n are strongly multiplicative.

Keywords: Strongly multiplicative labeling, Strongly multiplicative graphs, Arbitrary supersubdivision

1. Introduction

We begin with simple, finite, undirected and connected graph $G = (V, E)$. In the present work T, $P_n \times P_m$ and $K_{m,n}$ denote the tree, grid graph, and complete bipartite graph respectively. $C_n \odot P_m$ is the graph obtained by identifying an end point of P_m with every vertex of cycle C_n. One point union of m cycles of length n denoted as $C_n^{(m)}$ is the graph obtained by identifying one vertex of each cycles. If V_1 and V_2 are two partitions correspond to complete bipartite graph $K_{m,n}$ then V_1 is called m-vertices part and V_2 is called n-vertices part of $K_{m,n}$. In the graph G eccentricity of a vertex u is $\max_{v \in V(G)} d(u, v)$. For all other terminology and notations we refer to (Harary, F., 1972). We will give brief summary of definitions and other information which are useful for the present investigations.

Definition 1.1 Let G be a graph with q edges. A graph H is called a supersubdivision of G if H is obtained from G by replacing every edge e_i of G by a complete bipartite graph K_{2,m_i} for some m_i, $1 \leq i \leq q$ in such a way that the end vertices of each e_i are merged with the two vertices of 2-vertices part of K_{2,m_i} after removing the edge e_i from graph G.

A supersubdivision H of G is said to be an arbitrary supersubdivision of G if every edge of G is replaced by an arbitrary $K_{2,m}$ (m may vary for each edge arbitrarily). Arbitrary supersubdivision of G is denoted by $SS(G)$.

Definition 1.2 If the vertices of the graph are assigned values subject to certain conditions then it is known as graph labeling.

Most interesting graph labeling problems have following three important characteristics.

1. a set of numbers from which the labels are chosen;

2. a rule that assigns a value to each edges;

3. a condition that these values must satisfy.

For detail survey on graph labeling one can refer to (Gallian, J., 2009). Vast amount of literature is available on different types of graph labeling. According to (Beineke, L., 2001, p.63-75) graph labeling serves as a frontier between number theory and structure of graphs.

Labeled graph have variety of applications in coding theory, particularly for missile guidance codes, design of good radar type codes and convolution codes with optimal autocorrelation properties. Labeled graph plays vital role in the study of X-ray crystallography, communication network and to determine optimal circuit layouts. A systematic study on applications of graph labeling is reported in (Bloom, G., 1977, p. 562-570).

Definition 1.3 A graph $G = (V, E)$ with p vertices is said to be multiplicative if the vertices of G can be labeled with p distinct positive integers such that label induced on the edges by the product of labels of end vertices are all distinct.

Multiplicative labeling was introduced in (Beineke, L., 2001, p.63-75) where it is shown that every graph G admits multiplicative labeling and strongly multiplicative labeling is defined as follows.

Definition 1.4 A graph $G = (V, E)$ with p vertices is said to be strongly multiplicative if the vertices of G can be labeled with p distinct integers $1, 2, ...p$ such that label induced on the edges by the product of labels of the end vertices are all distinct.

In the present investigations we prove that the graphs obtained by arbitrary supersubdivision of tree T, grid graph $P_n \times P_m$, complete bipartite graph $K_{m,n}$, $C_n \odot P_m$ and $C_n^{(m)}$ are strongly multiplicative for all n and m.

2. Main Results

Theorem-2.1: Arbitrary supersubdivisions of tree T are strongly multiplicative.

Proof: Let T be the tree with n vertices. Arbitrary supersubdivision SS(T) of tree T obtained by replacing every edge of tree with K_{2,m_i} and we denote such graph by G. Let $K = \sum m_i$ $(1 \leq i \leq n-1)$. Let v_j $(1 \leq j \leq K+n)$ be the vertices of G. Denote the vertex with minimum eccentricity as v_1. Then v_2 will be the vertex which is at 1- distance apart from v_1. If there are more than one such vertices then throughout the work we will follow one of the direction (clockwise or anticlockwise) and denote them as $v_3, v_4, \ldots$. Next consider the vertices which are at 2- distance apart from v_1, 3- distance apart from v_1 and so on. (e.g. if there are seven vertices and two vertices are at distance 1- apart, one vertex is at distance 2- apart and three vertices are at distance 3- apart respectively form v_1. In this situation the vertices which are at 1- distance apart from v_1 will be identified as v_2 and v_3, the vertex which is at distance 2- apart will be identified as v_4 and the vertices which are at distance 3- apart will be identified as v_5, v_6 and v_7.) We define vertex labeling $f : V(G) \rightarrow \{1, 2 \ldots K+n\}$ as follows.

For any $1 \leq i \leq n+K$ define

$$f(v_i) = i$$

Then the graph G under consideration admits strongly multiplicative labeling.

Illustration 2.2: In *Fig.2* strongly multiplicative labeling of SS(T) corresponding to tree T of *Fig.1* is shown where $n = 13$ and $K = 26$.

Theorem 2.3: Arbitrary supersubdivisions of complete bipartite graph $K_{m,n}$ are strongly multiplicative.
Proof: Let $v_1, v_2, v_3, \ldots v_m$ be the vertices of m-vertices part and $v_{m+1}, v_{m+2}, v_{m+3}, \ldots v_{m+n}$ be the vertices of n-vertices part of $K_{m,n}$. Arbitrary supersubdivision SS$(K_{m,n})$ of $K_{m,n}$ obtained by replacing every edge of $K_{m,n}$ with K_{2,m_i} and we denote such graph by G. Let $K = \sum m_i$ $(1 \leq i \leq mn)$. Let u_j be the vertices which are used for arbitrary supersubdivision, where $1 \leq j \leq K$. We denote vertices by u_j which are used for supersubdivision of edges $v_1v_{m+1}, v_1v_{m+2}, \ldots v_1v_{m+n}, v_2v_{m+1}, \ldots v_nv_{m+n}$. Let p_o be the highest prime less than $K+m+n$. We define vertex labeling $f : V(G) \rightarrow \{1, 2 \ldots K+m+n\}$ as follows.

If $p_o \leq K+m$

$$\begin{aligned}
f(v_i) &= \begin{cases} i; & if \quad 1 \leq i \leq m, \\ k+i; & if \quad m+2 \leq i \leq m+n \end{cases} \\
f(v_{m+1}) &= p_o; \\
f(u_j) &= \begin{cases} m+j; & if \quad 1 \leq j < p_o, \\ m+j+1; & if \quad p_o \leq j \leq K \end{cases}
\end{aligned}$$

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If $p_o > K + m$

$$\begin{aligned} f(v_i) &= \begin{cases} i; \quad if \quad 1 \le i \le m, \\ k+i-1; \quad if \quad m+2 \le i < p_o, \\ k+i; \quad if \quad p_o \le i \le m+n \end{cases} \\ f(v_{m+1}) &= p_o; \\ f(u_j) &= m+j; \quad where \quad 1 \le j \le K \end{aligned}$$

Then in each possibilities described above the graph G under consideration admits strongly multiplicative labeling.

Illustration 2.4: Consider $SS(K_{2,3})$. Here $m = 2$, $n = 3$ and $K = 14$. The strongly multiplicative labeling is as shown in *Fig.3*.

Theorem 2.5: Arbitrary supersubdivisions of grid graph $P_n \times P_m$ are strongly multiplicative.

Proof: Arbitrary supersubdivision SS($P_n \times P_m$) of $P_n \times P_m$ obtained by replacing every edge of grid graph with K_{2,m_i} and we denote such graph by G. Let $K = \sum m_i$ $(1 \le i \le mn)$. Let v_i $(1 \le i \le mn + K)$ be the vertices of G. Denote the vertex of left upper corner with v_1. Here we designate vertices by v_i $(2 \le i \le mn + K)$ according to the procedure described in Theorem 2.1. We define vertex labeling $f : V(G) \to \{1, 2, \ldots, mn + K\}$

$$f(v_i) = i; \quad where \quad 1 \le i \le mn + K$$

Then the graph G under consideration admits strongly multiplicative labeling.

Illustration 2.6: Consider SS($P_4 \times P_3$). Here $n = 4$, $m = 3$ and $K = 41$. The corresponding strongly multiplicative labeling is shown in *Fig.4*.

Theorem 2.7: Arbitrary supersubdivisions of $C_n \odot P_m$ are strongly multiplicative.

Proof: Arbitrary supersubdivision SS($C_n \odot P_m$) of $C_n \odot P_m$ obtained by replacing every edge of $C_n \odot P_m$ with K_{2,m_i} and we denote such graph by G. Let $K = \sum m_i$ $(1 \le i \le mn)$. Let v_i $(1 \le i \le mn + K)$ be the vertices of G. Designate arbitrary vertex of C_n as v_1 and employing the scheme used in Theorem 2.1 the remaining vertices will receive labels $v_2, v_3, \ldots, v_{mn+K}$. We define vertex labeling $f : V(G) \to \{1, 2, \ldots, mn + K\}$ as follows.

$$f(v_i) = i; \quad where \quad 1 \le i \le mn + K$$

Then the graph G under consideration admits strongly multiplicative labeling.

Illustration 2.8: Consider $SS(C_5 \odot P_3)$. Here $n = 5$, $m = 3$ and $K = 37$. The corresponding strongly multiplicative labeling is as shown in *Fig.5*.

Theorem 2.9: Arbitrary supersubdivisions of $C_n^{(m)}$ are strongly multiplicative.

Proof: Arbitrary supersubdivision of $C_n^{(m)}$ is obtained by replacing every edge of $C_n^{(m)}$ with K_{2,m_i} and we denote this graph by G. Let $K = \sum m_i$. Let $v_i(1 \le i \le m(n-1) + K + 1$ be the vertices of G. Denote the common vertex of cycles by v_1. According to the procedure followed in previous results the remaining vertices will be designated as v_i $(2 \le i \le m(n-1) + K + 1)$. We define vertex labeling $f : V(G) \to \{1, 2, \ldots, m(n-1) + K + 1\}$ as follows.

For any $1 \le i \le m(n-1) + K + 1$ we define

$$f(v_i) = i;$$

Then the graph G under consideration admits strongly multiplicative labeling.

Illustration 2.10: Consider SS($C_4^{(3)}$). Here $n = 4$, $m = 3$ and $K = 26$. The strongly multiplicative labeling is as shown in *Fig.6*.

3. Concluding Remarks And Open Problem

Labeled graph is the topic of current interest for many researchers as it has diversified applications. It is also very interesting to investigate graph or families of graph which admits particular type of labeling. In (Sethuraman, G., 2001 p.1059-1064) and (Kathiresan, K., 2004 p.81-84) graceful labeling in the context of arbitrary supersubdivision is discussed while we discuss here strongly multiplicative labeling in the context of arbitrary supersubdivision. We consider five different graph families and investigate their strongly multiplicative labeling. This work is a nice combination of combinatorial number theory and graph theory which will provide enough motivation to any researcher.

Open Problems:

• Similar investigations are possible for other graph families.

• Parallel results can be investigated corresponding to other graph labeling techniques.

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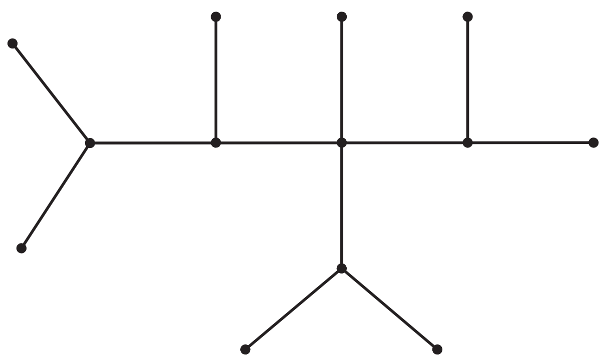

Figure 1. Tree T before arbitrary supersubdivision

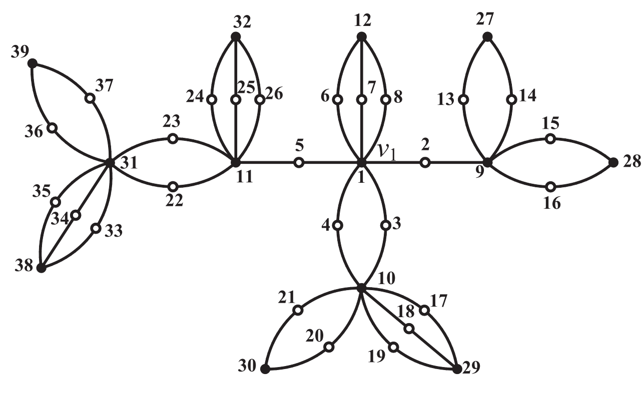

Figure 2. Strongly multiplicative labeling of SS(T)

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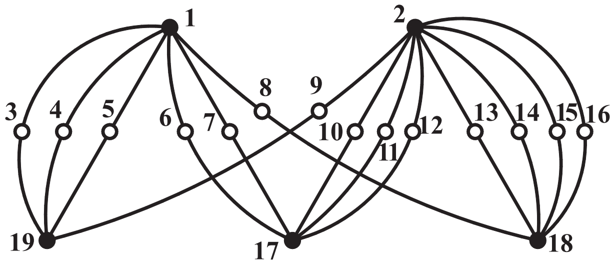

Figure 3. Strongly multiplicative labeling of $SS(K_{2,3})$

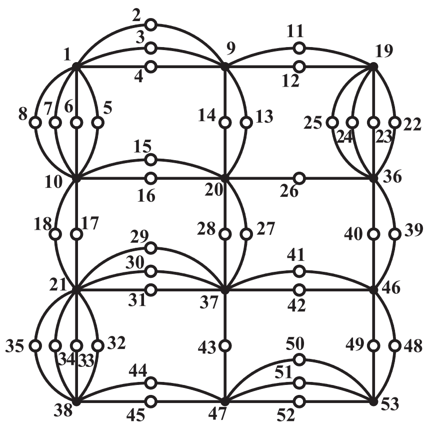

Figure 4. Strongly multiplicative labeling of SS($P_4 \times P_3$)

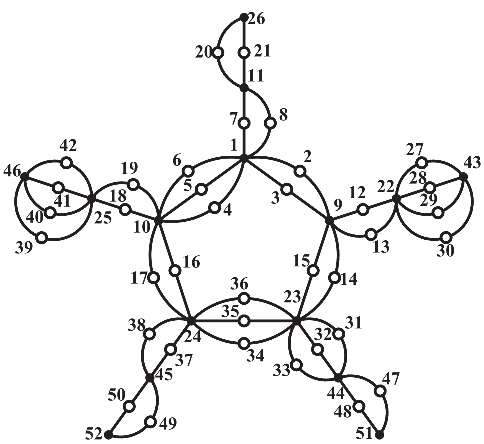

Figure 5. Strongly multiplicative labeling of $SS(C_5 \odot P_3)$

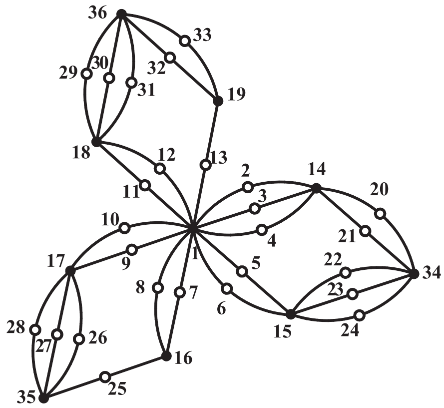

Figure 6. Strongly multiplicative labeling of SS($C_4^{(3)}$)

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To

Dr. S. K. Vaidya
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June 18, 2010

Dear Dr. Vaidya,

I am happy to inform you that Professor Kewen Zhao, Chief Editor of the International Journal of Information Science and Computer Mathematics has recommended and submitted your paper entitled "*Cordial labeling and arbitrary super subdivisions of some graphs*" jointly written with N. A. Dani in the International Journal of Information Science and Computer Mathematics. Accordingly the Editorial Board is pleased to accept it for publication in the International Journal of Information Science and Computer Mathematics.

Yours sincerely

(ARUN AZAD)

Cordial Labeling and Arbitrary SuperSubdivision of Some Graphs

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Abstract

Some new results for cordial labeling of graphs are investigated. We prove that the graphs obtained by arbitrary supersubdivision of tree, grid graph and complete bipartite graph are cordial. We also discuss cordial labeling for the graph obtained by arbitrary supersubdivision of $C_n \odot P_m$.

Key words : Cordial labeling, Cordial graphs, Arbitrary supersubdivision.

AMS **Subject classification number(2000):** 05C78.

1. Introduction

We begin with simple, finite, undirected and connected graph $G = (V(G), E(G))$. In the present work T, $P_n \times P_m$ and $K_{m,n}$ denote the tree, grid graph, and complete bipartite graph respectively. The graph $C_n \odot P_m$ is obtained by identifying an end point of P_m with every vertex of C_n. If V_1 and V_2 are two partitions correspond to complete bipartite graph $K_{m,n}$ then V_1 is called m-vertices part and V_2 is called n-vertices part of $K_{m,n}$. For the graph G eccentricity of a vertex u is $\max_{v \in V(G)} d(u,v)$. For all other terminology and notations we follow Gross and Yellen[4]. Given below are some definitions useful for the present investigations.

Definition 1.1 Let G be a graph with q edges. A graph H is called a *supersubdivision* of G if H is obtained from G by replacing every edge e_i of G by a complete bipartite graph K_{2,m_i} for some $m_i, 1 \leq i \leq q$ in such a way that the end vertices of each e_i are identified with the two vertices of 2-vertices part of K_{2,m_i} after removing the edge e_i from graph G. If m_i is varying arbitrarily for each edge e_i then supersubdivision is called *arbitrary supersubdivision* which is denoted by $SS(G)$.

Definition 1.2 If the vertices of the graph are assigned values subject to certain conditions then it is known as *graph labeling*.

Vast amount of literature is available on different types of graph labeling. For detailed survey on graph labeling we refer to *A Dynamic Survey of Graph Labeling* by Gallian[3].

Definition 1.3 Let $G = (V(G), E(G))$ be a graph. A mapping $f : V(G) \rightarrow \{0,1\}$ is called *binary vertex labeling* of G and $f(v)$ is called the *label* of the vertex v of G under f.

For an edge $e = uv$, the induced edge labeling $f^* : E(G) \to \{0, 1\}$ is given by $f^*(e)$=$|f(u) - f(v)|$. Let $v_f(0)$, $v_f(1)$ be the number of vertices of G having labels 0 and 1 respectively under f and let $e_f(0)$,$e_f(1)$ be the number of edges having labels 0 and 1 respectively under f^*.

Definition 1.4 A binary vertex labeling of a graph G is called a *cordial labeling* if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph G is *cordial* if it admits cordial labeling.

The concept of cordial labeling was introduced by Cahit[2]. Many researchers have studied cordiality of graphs. e.g.Cahit [2] proved that tree is cordial. In the same paper he proved that K_n is cordial if and only if $n \leq 3$. Ho et al.[5] proved that unicyclic graph is cordial unless it is C_{4k+2}. Andar et al.[1] have discussed cordiality of multiple shells. Vaidya et al.[8, 9, 10] have also discussed the cordiality of various graphs.

In the present investigations we prove that the graphs obtained by arbitrary supersubdivision of tree, grid graph, complete bipartite graph are cordial. We also prove that arbitrary supersubdivision of $C_n \odot P_m$ is cordial except $m_i(1 \leq i \leq n)$ are odd and $m_i(n + 1 \leq i \leq nm)$ are even with n is odd.

2. Main Results

Theorem-2.1: Arbitrary supersubdivision of tree T is cordial.

Proof: Let T be the tree with n vertices and $v_i(1 \leq i \leq n)$ be the vertices of T. Arbitrary supersubdivision of T is obtained by replacing every edge of tree with K_{2,m_i} and we denote this graph by G. Let $\alpha = \displaystyle\sum_{1}^{n-1} m_i$. Let u_j be the vertices of m_i-vertices part where $1 \leq j \leq \alpha$. Denote the vertex with minimum eccentricity as v_1 and n_1 and n_2 be the number of vertices which are at odd and even distance respectively form v_1 in T. Here $|V(G)| = \alpha + n$

and $|E(G)| = 2\alpha$. We define binary vertex labeling $f : V(G) \rightarrow \{0, 1\}$ as follows.

$$\left.\begin{array}{rcl} f(v_1) & = & 0; \\ f(v_i) & = & 1; \text{if } d(v_1, v_i) \text{ in } T \text{ is odd} \\ & = & 0; \text{if } d(v_1, v_i) \text{ in } T \text{ is even} \end{array}\right\} \; 1 \leq i \leq n$$

$$\left.\begin{array}{rcl} f(u_i) & = & 0; \text{If } n_1 \geq n_2 \\ & = & 1; \text{If } n_1 < n_2 \end{array}\right\} \; 1 \leq i \leq |n_1 - n_2|$$

$$\left.\begin{array}{rcl} f(u_i) & = & 0;\ \text{If } |n_1 - n_2| + 1 \leq i \leq \lfloor \frac{\alpha+|n_1-n_2|}{2} \rfloor \\ & = & 1;\ \text{If } \lceil \frac{\alpha+|n_1-n_2|}{2} \rceil \leq i \leq \alpha \end{array}\right\} \; i > |n_1 - n_2|$$

In view of the above defined labeling pattern we have the followings.

- When $\alpha + n$ is even

  $v_f(0) = v_f(1) = \frac{\alpha+n}{2}$; $e_f(0) = e_f(1) = \alpha$

- When $\alpha + n$ is odd

  $v_f(0) = v_f(1) + 1 = \frac{\alpha+n+1}{2}$; $e_f(0) = e_f(1) = \alpha$

Thus the graph G satisfies the conditions $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. That is, G admits cordial labeling.

Illustration 2.2: Consider G =SS(T). Here $n = 13$ and $\alpha = 26$. The cordial labeling is as shown in *Fig.1*.

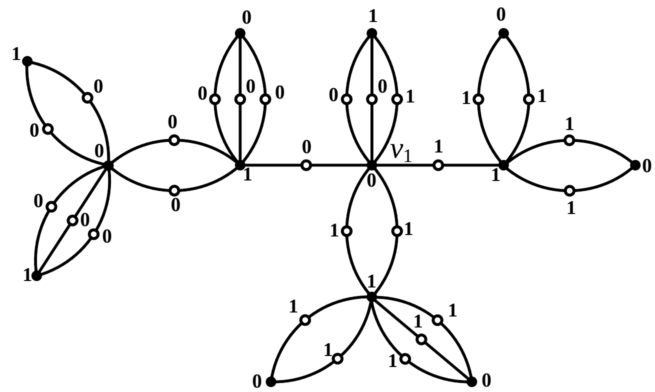

Fig.1 Cordial labeling of $SS(T)$

Theorem 2.3: Arbitrary supersubdivision of complete bipartite graph $K_{m,n}$ is cordial.

Proof: Let $v_1, v_2, v_3, \ldots v_m$ be the vertices of m-vertices part and v_{m+1}, $v_{m+2}, v_{m+3}, \ldots v_{m+n}$ be the vertices of n-vertices part of $K_{m,n}$. Arbitrary supersubdivision of $K_{m,n}$ is obtained by replacing every edge of $K_{m,n}$ with K_{2,m_i} and we denote this graph by G. Let $\alpha = \sum_{1}^{mn} m_i$. Let u_j be the vertices which are used for arbitrary supersubdivision, where $1 \leq j \leq \alpha$. Note that $|V(G)| = \alpha + m + n$, $|E(G)| = 2\alpha$. We define binary vertex labeling $f : V(G) \rightarrow \{0, 1\}$ as follows.

$$\begin{aligned} f(v_i) &= 0; \text{ if } 1 \leq i \leq m \\ &= 1; \text{ if } m+1 \leq i \leq m+n \end{aligned}$$

$$\left.\begin{aligned} f(u_i) &= 1; \text{ if } m \geq n \\ &= 0; \text{ if } m < n \end{aligned}\right\} \quad 1 \leq i \leq |m-n|$$

$$\left.\begin{aligned} f(u_i) &= 0; \text{ if } |m-n|+1 \leq i \leq \lfloor \tfrac{\alpha+|m-n|}{2} \rfloor \\ &= 1; \text{ if } \lceil \tfrac{\alpha+|m-n|}{2} \rceil \leq i \leq \alpha \end{aligned}\right\} \quad i > |m-n|$$

Above defined function f is cordial labeling for the graph under consideration because

- $v_f(0) = v_f(1) = \frac{\alpha+m+n}{2}$; $e_f(0) = e_f(1) = \alpha$ (When $\alpha + m + n$ is even)
- $v_f(0) + 1 = v_f(1) = \frac{\alpha+m+n+1}{2}$; $e_f(0) = e_f(1) = \alpha$ (When $\alpha + m + n$ is odd)

That is, G admits cordial labeling.

Illustration 2.4: Consider $G = SS(K_{2,3})$. Here $m = 2$, $n = 3$ and $\alpha = 14$. The cordial labeling is as shown in *Fig.2*.

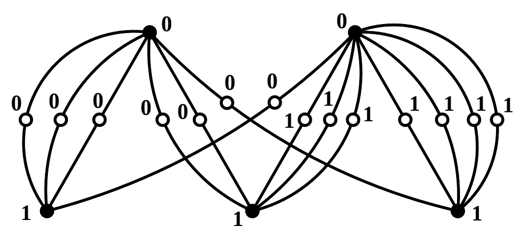

Fig.2 Cordial labeling of $SS(K_{2,3})$

Theorem 2.5: Arbitrary supersubdivision of grid graph $P_n \times P_m$ is cordial.

Proof: Let v_{ij} be the vertices of $P_n \times P_m$, where $1 \leq i \leq n$ and $1 \leq j \leq m$. Arbitrary supersubdivision of $P_n \times P_m$ is obtained by replacing every edge of grid graph with K_{2,m_i} and we denote the resultant graph by G. Let $\alpha = \sum\limits_{1}^{2mn-m-n} m_i$. Let u_j be the vertices of m_i-vertices part of K_{2,m_i} supersubdivision, where $1 \leq j \leq \alpha$. Here $|V(G)| = \alpha + mn$, $|E(G)| = 2\alpha$. We define binary vertex labeling $f : V(G) \to \{0, 1\}$ as follows.

For $1 \leq i \leq n$ and $1 \leq j \leq m$

$$f(v_{ij}) = 0; \text{ if } i \text{ and } j \text{ both are even or } i \text{ and } j \text{ both are odd}$$
$$= 1; \text{ if } i \text{ is even and } j \text{ is odd or } i \text{ is odd and } j \text{ is even}$$

$f(u_j) = 0$; if $1 \leq j \leq \lfloor \frac{\alpha}{2} \rfloor$
$\quad\quad = 1$; if $\lceil \frac{\alpha}{2} \rceil \leq j \leq \alpha$

Above defined function f is cordial labeling for the graph under consideration because

- $v_f(0) = v_f(1) = \frac{\alpha+mn}{2}$; $e_f(0) = e_f(1) = \alpha$ (When $\alpha + mn$ is even)
- $v_f(0)+1 = v_f(1) = \frac{\alpha+mn+1}{2}$; $e_f(0) = e_f(1) = \alpha$ (When α odd and mn is even)
- $v_f(0) = v_f(1) + 1 = \frac{\alpha+mn+1}{2}$; $e_f(0) = e_f(1) = \alpha$ (When α even and mn is odd)

That is, f is a cordial labeling for G. Hence the result.

Illustration 2.6: Consider G =SS$(P_4 \times P_3)$. Here $n = 4$, $m = 3$ and $\alpha = 41$. The corresponding cordial labeling is shown in *Fig.3*.

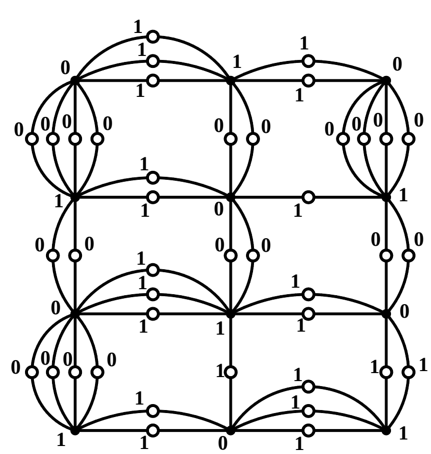

Fig.3 Cordial labeling of SS$(P_4 \times P_3)$

Theorem 2.7: Arbitrary supersubdivision of $C_n \odot P_m$ is cordial except $m_i (1 \leq i \leq n)$ are odd, $m_i (n+1 \leq i \leq nm)$ are even and n is odd.

Proof: Let $v_1, v_2, v_3, \dots v_n$ be the vertices of C_n and $v_{ij} (1 \leq i \leq n, 2 \leq j \leq m)$ be the vertices of paths. Arbitrary supersubdivision of $C_n \odot P_m$ is obtained by replacing every edge of $C_n \odot P_m$ with K_{2,m_i} and we denote this graph by G. Let $\alpha = \sum_1^{mn} m_i$ and u_j be the vertices of m_i-vertices part of K_{2,m_i}, where $1 \leq j \leq \alpha$. Here $|V(G)| = \alpha + mn$, $|E(G)| = 2\alpha$. To define binary vertex labeling $f : V(G) \to \{0, 1\}$ we consider following cases.

Case 1: For n even.

for $1 \leq i \leq n$ and $2 \leq j \leq m$

$f(v_i) = 0$; if i is odd

$= 1$; if i is even

$f(v_{ij}) = 0$; if i and j both are even or i and j both are odd

$= 1$; if i is even and j is odd or i is odd and j is even

$$\left.\begin{array}{rcl} f(u_j) & = & 0;\ \text{if } 1 \leq j \leq \lfloor \frac{\alpha}{2} \rfloor \\ & = & 1;\ \text{if } \lceil \frac{\alpha}{2} \rceil \leq j \leq \alpha \end{array}\right\} 1 \leq j \leq \alpha$$

Case 2: For n odd and at least one $m_i (1 \leq i \leq n)$ is even and at least one $m_i (n+1 \leq i \leq mn)$ is odd. Without loss of generality we assume that m_1 is even.

For $2 \leq i \leq n$ and $2 \leq j \leq m$

$f(v_1) = 0$;

$f(v_i) = 0$; if i is even

$= 1$; if i is odd

$f(v_{1j}) = 0$; if j is odd

$= 1$; if j is even

$f(v_{ij}) = 0$; if i is even and j is odd or i is odd and j is even

$= 1$; if i and j both are even or i and j both are odd

$$\left.\begin{array}{rcl} f(u_j) & = & 0;\ \text{if } 1 \leq j \leq \frac{m_1}{2} \\ & = & 1;\ \text{if } \frac{m_1}{2}+1 \leq j \leq m_1 \\ f(u_j) & = & 0;\ \text{if } m_1+1 \leq j \leq \lfloor\frac{\alpha+m_1}{2}\rfloor \\ & = & 1;\ \text{if } \lceil\frac{\alpha+m_1}{2}\rceil \leq j \leq \alpha \end{array}\right\} 1 \leq j \leq \alpha$$

In view of the above two cases graph G satisfies the following conditions.

- $v_f(0) = v_f(1) = \frac{\alpha+mn}{2}$; $e_f(0) = e_f(1) = \alpha$ (When $\alpha + mn$ is even)
- $v_f(0)+1 = v_f(1) = \frac{\alpha+mn+1}{2}$; $e_f(0) = e_f(1) = \alpha$ (When α odd and mn is even)
- $v_f(0) = v_f(1) + 1 = \frac{\alpha+mn+1}{2}$; $e_f(0) = e_f(1) = \alpha$ (When α even and mn is odd)

That is, f is a cordial labeling for G and consequently G is a cordial graph.

Case 3: If n is odd number with $m_i(1 \leq i \leq n)$ are odd and $m_i(n+1 \leq i \leq nm)$ are even.

In this case G is an Eulerian graph with number of edges congruent to $2(mod4)$ then G is not cordial as proved by Cahit[2].

Hence from the Case 1 to 3 we have the required result.

Illustration 2.8: Consider $G = SS(C_5 \odot P_3)$. Here $n = 5$, $m = 3$ and $\alpha = 37$. The corresponding cordial labeling is as shown in *Fig.4*.

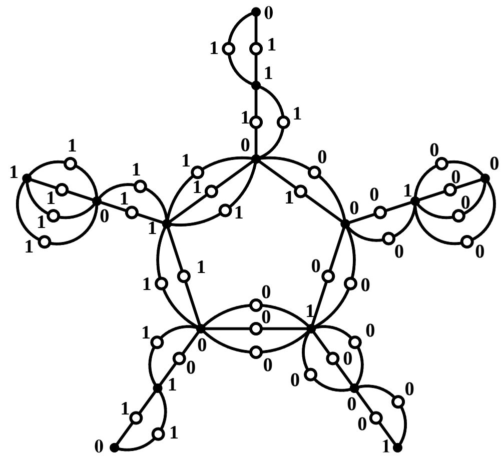

Fig.4 Cordial labeling of $SS(C_5 \odot P_3)$

3. Conclusion and Scope

Sethuraman and Selvaraju[7] and Kathiresan and Amutha[6] have discussed graceful labeling in the context of arbitrary supersubdivision of some graphs while we discuss cordial labeling in the context of arbitrary supersubdivision of some graphs. Similar investigations can be carried out for other graph families as well as in the context of different labeling problems is an open area of research.

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Cordial and 3-equitable Labeling for Some Wheel Related Graphs

S K Vaidya*, N A Dani†, K K Kanani‡, P L Vihol§

Abstract—**We present here cordial and 3-equitable labeling for the graphs obtained by joining apex vertices of two wheels to a new vertex. We extend these results for k copies of wheels.**

Keywords: Cordial graph, Cordial labeling, 3-equitable graph, 3-equitable labeling

AMS **Subject classification number(2000):** 05C78.

1 Introduction

We begin with simple, finite and undirected graph $G = (V, E)$. In the present work $W_n = C_n + K_1$ $(n \geq 3)$ denotes the wheel and in W_n vertices correspond to C_n are called rim vertices and vertex which corresponds to K_1 is called an apex vertex. For all other terminology and notations we follow Harary[7]. We will give brief summary of definitions which are useful for the present investigations.

Definition 1.1 Consider two wheels $W_n^{(1)}$ and $W_n^{(2)}$ then $G =< W_n^{(1)} : W_n^{(2)} >$ is the graph obtained by joining apex vertices of wheels to a new vertex x.
Note that G has $2n + 3$ vertices and $4n + 2$ edges.

Definition 1.2 Consider k copies of wheels namely $W_n^{(1)}, W_n^{(2)}, W_n^{(3)}, \dots W_n^{(k)}$. Then the $G =< W_n^{(1)} : W_n^{(2)} : W_n^{(3)} : \dots : W_n^{(k)} >$ is the graph obtained by joining apex vertices of each $W_n^{(p-1)}$ and $W_n^{(p)}$ to a new vertex x_{p-1} where $2 \leq p \leq k$.
Note that G has $k(n + 2) - 1$ vertices and $2k(n + 1) - 2$ edges.

Definition 1.3 If the vertices of the graph are assigned values subject to certain conditions then it is known as *graph labeling*.
According to Hegde[8] most interesting graph labeling problems have following three important characteristics.

1. a set of numbers from which the labels are chosen;
2. a rule that assigns a value to each edge;
3. a condition that these values must satisfy.

The recent survey on graph labeling can be found in Gallian[6]. Vast amount of literature is available on different types of graph labeling. According to Beineke and Hegde[2] graph labeling serves as a frontier between number theory and structure of graphs.
Labeled graph have variety of applications in coding theory, particularly for missile guidance codes, design of good radar type codes and convolution codes with optimal autocorrelation properties. Labeled graph plays vital role in the study of X-Ray crystallography, communication network and to determine optimal circuit layouts. A detail study of variety of applications of graph labeling is carried out by Bloom and Golomb[3].

Definition 1.4 Let $G = (V, E)$ be a graph. A mapping $f : V(G) \rightarrow$\{0,1\} is called *binary vertex labeling* of G and $f(v)$ is called the *label* of the vertex v of G under f.
For an edge $e = uv$, the induced edge labeling $f^* : E(G) \rightarrow \{0, 1\}$ is given by $f^*(e)=|f(u) - f(v)|$. Let $v_f(0)$, $v_f(1)$ be the number of vertices of G having labels 0 and 1 respectively under f and let $e_f(0)$,$e_f(1)$ be the number of edges having labels 0 and 1 respectively under f^*.

Definition 1.5 A binary vertex labeling of a graph G is called a *cordial labeling* if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph G is *cordial* if it admits cordial labeling.
The concept of cordial labeling was introduced by Cahit[4].
Many researchers have studied cordiality of graphs. e.g.Cahit [4] proved that tree is cordial. In the same paper he proved that K_n is cordial if and only if $n \leq 3$. Ho et al.[9] proved that unicyclic graph is cordial unless it is C_{4k+2}. Andar et al.[1] discussed cordiality of multiple shells. Vaidya et al.[10], [11], [12] have also discussed the cordiality of various graphs.

Definition 1.6 A vertex labeling of a graph G is called a *3-equitable labeling* if $|v_f(i) - v_f(j)| \leq 1$ and

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$|e_f(i) - e_f(j)| \leq 1$ for all $0 \leq i, j \leq 2$. A graph G is *3-equitable* if it admits 3-equitable labeling.
The concept of 3-equitable labeling was introduced by Cahit[5]. Many researchers have studied 3-equitability of graphs. e.g.Cahit [5] proved that C_n is 3-equitable except $n \equiv 3(mod6)$. In the same paper he proved that an Eulerian graph with number of edges congruent to $3(mod6)$ is not 3-equitable. Youssef[16] proved that W_n is 3-equitable for all $n \geq 4$. Several results on 3-equitable labeling for some wheel related graphs in the context of vertex duplication are reported in Vaidya et al.[13].
In the present investigations we prove that graphs $< W_n^{(1)} : W_n^{(2)} >$ and $< W_n^{(1)} : W_n^{(2)} : W_n^{(3)} : \ldots : W_n^{(k)} >$ are cordial as well as 3-equitable.

2 Main Results

Theorem-2.1 Graph $< W_n^{(1)} : W_n^{(2)} >$ is cordial.

Proof Let $v_1^{(1)}, v_2^{(1)}, v_3^{(1)}, \ldots v_n^{(1)}$ be the rim vertices $W_n^{(1)}$ and $v_1^{(2)}, v_2^{(2)}, v_3^{(2)}, \ldots v_n^{(2)}$ be the rim vertices $W_n^{(2)}$. Let c_1 and c_2 be the apex vertices of $W_n^{(1)}$ and $W_n^{(2)}$ respectively and they are adjacent to a new common vertex x. Let $G =< W_n^{(1)} : W_n^{(2)} >$. We define binary vertex labeling $f : V(G) \to \{0, 1\}$ as follows.

For any $n \in N - \{1, 2\}$ and $i = 1, 2, \ldots n$ where N is set of natural numbers.
In this case we define labeling as follows
$f(v_i^{(1)}) = 1;$
$f(c_1) = 0;$
$f(v_i^{(2)}) = 0;$
$f(c_2) = 1;$
$f(x) = 1;$
Thus rim vertices of $W_n^{(1)}$ and $W_n^{(2)}$ are labeled with the sequences $1, 1, 1, \ldots, 1$ and $0, 0, \ldots, 0$ respectively. The common vertex x is labeled with 1 and apex vertices with 0 and 1 respectively.
The labeling pattern defined above covers all possible arrangement of vertices. The graph G satisfies the vertex condition $v_f(0) + 1 = v_f(1)$ and edge condition $e_f(0) = e_f(1)$. i.e. G admits cordial labeling.

Illustration 2.2 Consider $G =< W_6^{(1)} : W_6^{(2)} >$. Here $n = 6$. The cordial labeling is as shown in Figure 1.

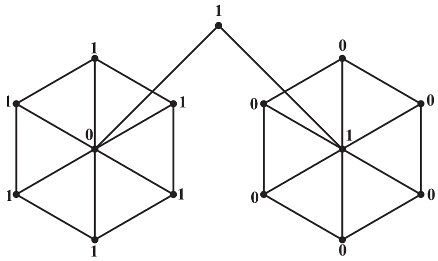

Figure 1: Cordial labeling of graph G.

Theorem 2.3 Graph $< W_n^{(1)} : W_n^{(2)} : W_n^{(3)} : \ldots : W_n^{(k)} >$ is cordial.

Proof Let $W_n^{(j)}$ be k copies of wheel W_n, $v_i^{(j)}$ be the rim vertices of $W_n^{(j)}$ and c_j be the apex vertex of $W_n^{(j)}$ (here $i = 1, 2, \ldots n$ and $j = 1, 2, \ldots\ k$).Let $x_1, x_2 \ldots\ x_{k-1}$ be the vertices such that c_{p-1} and c_p are adjacent to x_{p-1} where $2 \leq p \leq k$. Consider $G =< W_n^{(1)} : W_n^{(2)} : W_n^{(3)} : \ldots : W_n^{(k)} >$. To define binary vertex labeling $f : V(G) \to \{0, 1\}$ we consider following cases.

Case 1: $n \in N - \{1, 2\}$ and even k where $k \in N - \{1, 2\}$.
In this case we define labeling function f as
For $i = 1, 2, \ldots n$ and $j = 1, 2, \ldots k$
$f(v_i^{(j)}) = 0$; if j even.
$= 1$; if j odd.
$f(c_j) = 1$; if j even.
$= 0$; if j odd.
$f(x_j) = 1$; if j even, $j \neq k$.
$= 0$; if j odd, $j \neq k$.
Case 2: $n \in N - \{1, 2\}$ and odd k where $k \in N - \{1, 2\}$.
In this case we define labeling function f for first $k - 1$ wheels as
For $i = 1, 2, \ldots n$ and $j = 1, 2, \ldots k - 1$
$f(v_i^{(j)}) = 0$; if j even.
$= 1$; if j odd.
$f(c_j) = 1$; if j even.
$= 0$; if j odd.
$f(x_j) = 1$; if j even.
$= 0$; if j odd.
To define labeling function f for k^{th} copy of wheel we consider following subcases
Subcase 1: If $n \equiv 3(mod4)$.
For $1 \leq i \leq n - 1$
$f(v_i^{(k)}) = 0$; if $i \equiv 0, 1(mod4)$.
$= 1$; if $i \equiv 2, 3(mod4)$.
$f(v_n^{(k)}) = 0;$
$f(c_k) = 1;$
Subcase 2: If $n \equiv 0, 2(mod4)$.
$f(v_i^{(k)}) = 0$; if $i \equiv 0, 1(mod4)$.
$= 1$; if $i \equiv 2, 3(mod4)$.
$f(c_k) = 0$; $n \equiv 0(mod4)$
$f(c_k) = 1$; $n \equiv 2(mod4)$
Subcase 3: If $n \equiv 1(mod4)$.
$f(v_i^{(k)}) = 0$; if $i \equiv 0, 3(mod4)$.
$= 1$; if $i \equiv 1, 2(mod4)$.
$f(c_k) = 0;$
The labeling pattern defined above exhaust all the possibilities and in each one the graph G under consideration satisfies the conditions $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ as shown in Table 1. i.e. G admits cordial labeling.
(In Table 1 $n = 4a + b$ and $a \in N \cup \{0\}$)

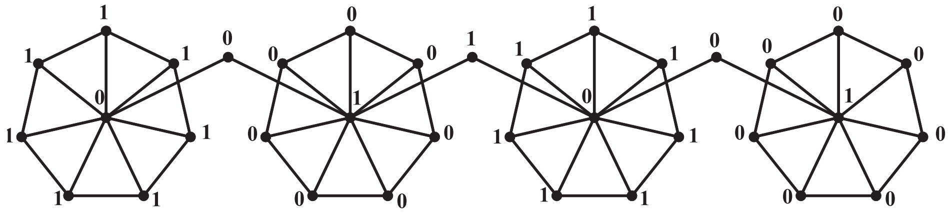

Figure 2: Cordial labeling of graph G.

Table 1: Vertex and Edge conditions for f

k	b	Vertex Condition	Edge Condition
even	0,1,2,3	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$
odd	0	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$
	1,3	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)$
	2	$v_f(0)+1=v_f(1)$	$e_f(0)=e_f(1)$

Let us understand the labeling pattern with some examples given below.

Illustrations 2.4
Example 1: Consider $G =< W_7^{(1)} : W_7^{(2)} : W_7^{(3)} : W_7^{(4)} >$. Here $n = 7$ and $k = 4$ i.e k is even. The cordial labeling is as shown in Figure 2.
Example 2: Consider $G =< W_5^{(1)} : W_5^{(2)} : W_5^{(3)} >$. Here $n = 5$ i.e $n \equiv 1(mod4)$ and $k = 3$ i.e k is odd. The cordial labeling is as shown in Figure 3.

Theorem 2.5 Graph $< W_n^{(1)} : W_n^{(2)} >$ is 3-equitable.

Proof Let $v_1^{(1)}, v_2^{(1)}, v_3^{(1)}, \dots v_n^{(1)}$ be the rim vertices $W_n^{(1)}$ and $v_1^{(2)}, v_2^{(2)}, v_3^{(2)}, \dots v_n^{(2)}$ be the rim vertices $W_n^{(2)}$. Let c_1 and c_2 be the apex vertices of $W_n^{(1)}$ and $W_n^{(2)}$ respectively and they are adjacent to a new common vertex x. Let $G =< W_n^{(1)} : W_n^{(2)} >$. To define vertex labeling $f : V(G) \to \{0, 1, 2\}$ we consider the following cases.

Case 1: $n \equiv 0(mod6)$
In this case we define labeling f as:
$f(v_i^{(1)}) = 0;\ i \equiv 1, 4(mod6)$

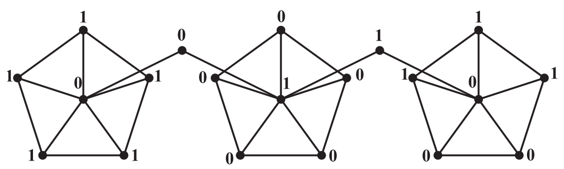

Figure 3: Cordial labeling of graph G.

$= 1;\ i \equiv 2, 3(mod6)$
$= 2;\ i \equiv 0, 5(mod6),\ 1 \le i \le n$
$f(c_1) = 2;$
$f(v_i^{(2)}) = 0;\ i \equiv 1, 4(mod6)$
$= 2;\ i \equiv 2, 3(mod6)$
$= 1;\ i \equiv 0, 5(mod6),\ 1 \le i \le n - 3$
$= 1;\ i \ge n - 2$
$f(c_2) = 0;$
$f(x) = 0;$
Case 2: $n \equiv 1(mod6)$
In this case we define labeling f as:
$f(v_i^{(1)}) = 0;\ i \equiv 1, 4(mod6)$
$= 1;\ i \equiv 2, 3(mod6)$
$= 2;\ i \equiv 0, 5(mod6),\ 1 \le i \le n$
$f(c_1) = 2;$
$f(v_i^{(2)}) = 0;\ i \equiv 1, 4(mod6)$
$= 1;\ i \equiv 2, 3(mod6)$
$= 2;\ i \equiv 0, 5(mod6),\ 1 \le i \le n$
$f(c_2) = 2;$
$f(x) = 1;$
Case 3: $n \equiv 2(mod6)$
In this case we define labeling f as:
$f(v_i^{(1)}) = 0;\ i \equiv 1, 4(mod6)$
$= 1;\ i \equiv 0, 5(mod6)$
$= 2;\ i \equiv 2, 3(mod6),\ 1 \le i \le n - 2$
$= 1;\ i \ge n - 1$
$f(c_1) = 0;$
$f(v_i^{(2)}) = 0;\ i \equiv 1, 4(mod6)$
$= 1;\ i \equiv 0, 5(mod6)$
$= 2;\ i \equiv 2, 3(mod6),\ 1 \le i \le n - 2$
$= 2;\ i \ge n - 1$
$f(c_2) = 0;$
$f(x) = 1;$
Case 4: $n \equiv 3(mod6)$
Subcase 1: $n \neq 3$
In this case we define labeling f as:
$f(v_i^{(1)}) = 0;\ i \equiv 1, 4(mod6)$
$= 1;\ i \equiv 0, 5(mod6)$
$= 2;\ i \equiv 2, 3(mod6),\ 1 \le i \le n$
$f(c_1) = 0;$
$f(v_i^{(2)}) = 0;\ i \equiv 1, 4(mod6)$
$= 1;\ i \equiv 2, 3(mod6)$

$= 2;\ i \equiv 0, 5(mod6),\ 1 \leq i \leq n-3$
$= 1;\ i \geq n-2$
$f(c_2) = 0;$
$f(x) = 2;$
Subcase 2: $n = 3$
$f(v_1^{(1)}) = f(v_1^{(2)}) = f(c_2) = 0;$
$f(v_2^{(1)}) = f(v_3^{(1)}) = f(c_1) = 1;$
$f(v_2^{(2)}) = f(v_3^{(2)}) = f(x) = 2;$
Case 5: $n \equiv 4(mod6)$
In this case we define labeling f as:
$f(v_i^{(1)}) = 0;\ i \equiv 1, 4(mod6)$
$= 1;\ i \equiv 0, 5(mod6)$
$= 2;\ i \equiv 2, 3(mod6),\ 1 \leq i \leq n-3$
$= 1;\ i = n-2, n-1$
$= 0;\ i = n$
$f(c_1) = 2;$
$f(v_i^{(2)}) = 0;\ i \equiv 1, 4(mod6)$
$= 1;\ i \equiv 0, 5(mod6)$
$= 2;\ i \equiv 2, 3(mod6),\ 1 \leq i \leq n$
$f(c_2) = 2;\ f(x) = 1.$
Case 6: $n \equiv 5(mod6)$
In this case we define labeling f as:
$f(v_i^{(1)}) = 0;\ i \equiv 1, 4(mod6)$
$= 1;\ i \equiv 2, 3(mod6)$
$= 2;\ i \equiv 0, 5(mod6),\ 1 \leq i \leq n-5$
$= 1;\ i = n-4, n-3$
$= 2;\ i = n-2, n$
$= 0;\ i = n-1$
$f(c_1) = 2;$
$f(v_i^{(2)}) = 0;\ i \equiv 1, 4(mod6)$
$= 1;\ i \equiv 0, 5(mod6)$
$= 2;\ i \equiv 2, 3(mod6),\ 1 \leq i \leq n-5$
$= 0;\ i = n-4, n-1$
$= 1;\ i = n-3, n-2$
$= 2;\ i = n$
$f(c_2) = 0;$
$f(x) = 0;$
The labeling pattern defined above covers all the possible arrangement of vertices and in each case the resulting labeling satisfies the conditions $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $0 \leq i, j \leq 2$ as shown in Table 2. i.e. G admits 3-equitable labeling.
(In Table 2 $n = 6a + b$ and $a \in N \cup \{0\}$)
Let us understand the labeling pattern defined in Theorem 2.5 by means of following Illustration 2.6.

Table 2: Vertex and Edge conditions for f

b	Vertex Condition	Edge Condition
0	$v_f(0)=v_f(1)=v_f(2)$	$e_f(0)=e_f(1)=e_f(2)+1$
1,4	$v_f(0)=v_f(1)+1=v_f(2)$	$e_f(0)=e_f(1)=e_f(2)$
2	$v_f(0)+1=v_f(1)=v_f(2)+1$	$e_f(0)+1=e_f(1)=e_f(2)+1$
3	$v_f(0)=v_f(1)=v_f(2)$	$e_f(0)+1=e_f(1)=e_f(2)$
5	$v_f(0)=v_f(1)+1=v_f(2)+1$	$e_f(0)+1=e_f(1)=e_f(2)+1$

Illustration 2.6 Consider a graph $G = < W_5^{(1)} : W_5^{(2)} >$ Here $n = 5$ i.e $n \equiv 5(mod6)$. The corresponding 3-equitable labeling is shown in Figure 4.

Theorem 2.7 Graph $< W_n^{(1)} : W_n^{(2)} : W_n^{(3)} : \ldots : W_n^{(k)} >$ is 3-equitable.

Proof Let $W_n^{(j)}$ be k copies of wheel W_n, $v_i^{(j)}$ be the rim vertices of $W_n^{(j)}$ where $i = 1, 2, \ldots n$ and $j = 1, 2, \ldots k$. Let c_j be the apex vertex of $W_n^{(j)}$. Consider $G = < W_n^{(1)} : W_n^{(2)} : W_n^{(3)} : \ldots : W_n^{(k)} >$ and vertices $x_1, x_2, \ldots x_{k-1}$ as stated in Theorem 2.3. To define vertex labeling $f : V(G) \to \{0, 1, 2\}$ we consider following cases.

Case 1: For $n \equiv 0(mod6)$.
In this case we define labeling function f as follows
Subcase 1: For $k \equiv 0(mod3)$.
For $j \equiv 1, 2(mod3)$
$f(v_i^{(j)}) = 0;$ if $i \equiv 1, 4(mod6)$.
$= 1;$ if $i \equiv 0, 5(mod6)$.
$= 2;$ if $i \equiv 2, 3(mod6),\ i \leq n-3$.
$f(v_i^{(j)}) = 1;$ if $i \geq n-2$.
$f(c_j) = 0.$
$f(x_j) = 2;$ if $j \equiv 1(mod3)$.
$= 0;$ if $j \equiv 2(mod3)$.
For $j \equiv 0(mod3)$
$f(v_i^{(j)}) = 0;$ if $i \equiv 1, 4(mod6)$.
$= 1;$ if $i \equiv 0, 5(mod6)$.
$= 2;$ if $i \equiv 2, 3(mod6)$.
$f(c_j) = 2.$
$f(x_j) = 0,\ j \neq k.$
Subcase 2: For $k \equiv 1(mod3)$.
$f(v_i^{(1)}) = 0;$ if $i \equiv 1, 4(mod6)$.
$= 1;$ if $i \equiv 0, 5(mod6)$.
$= 2;$ if $i \equiv 2, 3(mod6)$.
$f(c_1) = 2.$
$f(x_1) = 0.$
For remaining vertices take $j = k-1$ and label them as in subcase 1.
Subcase 3: For $k \equiv 2(mod3)$.
$f(v_i^{(1)}) = 0;$ if $i \equiv 1, 4(mod6)$.
$= 1;$ if $i \equiv 0, 5(mod6)$.
$= 2;$ if $i \equiv 2, 3(mod6)$.
$f(c_1) = 0.$
$f(x_1) = 2.$

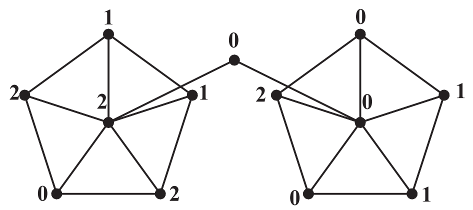

Figure 4: 3-equitable labeling of graph G.

$f(v_i^{(2)}) = 0$; if $i \equiv 1, 4(mod6)$.
$= 1$; if $i \equiv 0, 5(mod6)$.
$= 2$; if $i \equiv 2, 3(mod6)$, $i \leq n-3$.
$f(v_i^{(2)}) = 1$; if $i \geq n-2$.
$f(c_2) = 0$.
$f(x_2) = 0$.
For remaining vertices take $j = k-2$ and label them as in subcase 1.
Case 2: For $n \equiv 1(mod6)$.
In this case we define labeling function f as follows
Subcase 1: For $k \equiv 0(mod3)$.
$f(v_i^{(j)}) = 0$; if $i \equiv 1, 4(mod6)$.
$= 1$; if $i \equiv 0, 5(mod6)$.
$= 2$; if $i \equiv 2, 3(mod6)$, $i \leq n-1$.
$f(v_n^{(j)}) = 0$; if $j \equiv 1(mod3)$.
$f(v_n^{(j)}) = 1$; if $j \equiv 0, 2(mod3)$.
$f(c_j) = 2$; if $j \equiv 1(mod3)$.
$f(c_j) = 0$; if $j \equiv 0, 2(mod3)$.
$f(x_j) = 1$; if $j \equiv 1(mod3)$.
$= 2$; if $j \equiv 0, 2(mod3)$, $j \neq k$.
Subcase 2: For $k \equiv 1(mod3)$.
$f(v_i^{(1)}) = 0$; if $i \equiv 1, 4(mod6)$.
$= 1$; if $i \equiv 0, 5(mod6)$.
$= 2$; if $i \equiv 2, 3(mod6)$, $i \leq n-1$.
$f(v_n^{(1)}) = 1$;
$f(c_1) = 2$.
$f(x_1) = 0$.
For remaining vertices take $j = k-1$ and label them as in subcase 1.
Subcase 3: For $k \equiv 2(mod3)$.
For $j = 1, 2$
$f(v_i^{(j)}) = 0$; if $i \equiv 1, 4(mod6)$.
$= 1$; if $i \equiv 0, 5(mod6)$.
$= 2$; if $i \equiv 2, 3(mod6)$, $i \leq n-1$.
$f(v_n^{(j)}) = 1$;
$f(c_1) = 0$.
$f(c_2) = 2$.
$f(x_1) = 2$.
$f(x_2) = 0$.
For remaining vertices take $j = k-2$ and label them as in subcase 1.
Case 3: For $n \equiv 2(mod6)$.
In this case we define labeling function f as follows
Subcase 1: For $k \equiv 0(mod3)$.
For $j \equiv 1, 2(mod3)$
$f(v_i^{(j)}) = 0$; if $i \equiv 1, 4(mod6)$.
$= 1$; if $i \equiv 0, 5(mod6)$.
$= 2$; if $i \equiv 2, 3(mod6)$, $i \leq n-4$.
$f(v_{n-3}^{(j)}) = 2$.
$f(v_i^{(j)}) = 1$; if $i \geq n-2$.
$f(c_j) = 0$; if $j \equiv 1(mod3)$.
$f(c_j) = 2$; if $j \equiv 2(mod3)$.
$f(x_j) = 0$.
For $j \equiv 0(mod3)$
$f(v_i^{(j)}) = 0$; if $i \equiv 1, 4(mod6)$.
$= 1$; if $i \equiv 2, 3(mod6)$.
$= 2$; if $i \equiv 0, 5(mod6)$, $i \leq n-2$.
$f(v_i^{(j)}) = 1$; if $i \geq n-1$.
$f(c_j) = 2$.
$f(x_j) = 0$, $j \neq k$.
Subcase 2: For $k \equiv 1(mod3)$.
$f(v_i^{(1)}) = 0$; if $i \equiv 1, 4(mod6)$.
$= 1$; if $i \equiv 0, 5(mod6)$.
$= 2$; if $i \equiv 2, 3(mod6)$, $i \leq n-2$.
$f(v_{n-1}^{(1)}) = 2$.
$f(v_n^{(1)}) = 0$.
$f(c_1) = 0$.
$f(x_1) = 1$.
For remaining vertices take $j = k-1$ and label them as in subcase 1.
Subcase 3: For $k \equiv 2(mod3)$.
For $j = 1, 2$
$f(v_i^{(j)}) = 0$; if $i \equiv 1, 4(mod6)$.
$= 1$; if $i \equiv 0, 5(mod6)$.
$= 2$; if $i \equiv 2, 3(mod6)$, $i \leq n-4$.
$f(v_{n-3}^{(j)}) = 2$;
$f(v_i^{(j)}) = 1$; if $i \geq n-2$.
$f(c_j) = 0$.
$f(x_1) = 1$.
$f(x_2) = 0$.
For remaining vertices take $j = k-2$ and label them as in subcase 1.
Case 4: For $n \equiv 3(mod6)$.
In this case we define labeling function f as follows
Subcase 1: For $k \equiv 0(mod3)$.
$f(v_i^{(j)}) = 0$; if $i \equiv 1, 4(mod6)$.
$= 1$; if $i \equiv 0, 5(mod6)$.
$= 2$; if $i \equiv 2, 3(mod6)$, $i \leq n-3$.
If $j \equiv 1(mod3)$
$f(v_i^{(j)}) = 1$; if $i \geq n-2$.
$f(c_j) = 0$.
$f(x_j) = 1$.
If $j \equiv 2(mod3)$
$f(v_{n-2}^{(j)}) = 0$.
$f(v_{n-1}^{(j)}) = 2$.
$f(v_n^{(j)}) = 1$.
$f(c_j) = 0$.
$f(x_j) = 2$.
If $j \equiv 0(mod3)$
$f(v_i^{(j)}) = 0$; $if j = n-1, n-2$.
$f(v_n^{(j)}) = 2$.
$f(c_j) = 2$.
$f(x_j) = 2$, $j \neq k$.
Subcase 2: For $k \equiv 1(mod3)$.
$f(v_i^{(1)}) = 0$; if $i \equiv 1, 4(mod6)$.
$= 1$; if $i \equiv 2, 3(mod6)$.
$= 2$; if $i \equiv 0, 5(mod6)$, $i \leq n-3$.
$f(v_i^{(1)}) = 2$; if $i \geq n-2$.
$f(c_1) = 0$.

$f(x_1) = 1$.
For remaining vertices take $j = k - 1$ and label them as in subcase 1.
Subcase 3: For $k \equiv 2(mod3)$.
For $j = 1, 2$
$f(v_i^{(j)}) = 0$; if $i \equiv 1, 4(mod6)$.
$= 1$; if $i \equiv 0, 5(mod6)$.
$= 2$; if $i \equiv 2, 3(mod6)$, $i \leq n - 3$.
$f(v_i^{(1)}) = 1$; if $i = n - 1, n - 2$.
$f(v_n^{(1)}) = 0$.
$f(v_i^{(2)}) = 2$; if $i \geq n - 2$.
$f(c_j) = 0$.
$f(x_1) = 1$.
$f(x_2) = 2$.
For $n = 3$ label rim vertices of $W_n^{(1)}$ by $0, 1, 0$ and apex vertex by 1.
For remaining vertices take $j = k - 2$ and label them as in subcase 1.
Case 5: For $n \equiv 4(mod6)$.
In this case we define labeling function f as follows
Subcase 1: For $k \equiv 0(mod3)$.
For $j \equiv 0, 1, 2(mod3)$
$f(v_i^{(j)}) = 0$; if $i \equiv 1, 4(mod6)$.
$= 1$; if $i \equiv 0, 5(mod6)$.
$= 2$; if $i \equiv 2, 3(mod6)$, $i \leq n - 4$.
$f(v_{n-3}^{(j)}) = 0$; if $j \equiv 0, 1(mod3)$.
$f(v_{n-3}^{(j)}) = 2$; if $j \equiv 2(mod3)$.
$f(v_i^{(j)}) = 1$; if $j \equiv 1, 2(mod3)$, $i \geq n - 2$.
$f(v_i^{(j)}) = 2$; if $j \equiv 0(mod3)$, $i \geq n - 2$.
$f(c_j) = 2$, $j \equiv 1, 2(mod3)$.
$f(c_j) = 0$, $j \equiv 0(mod3)$.
$f(x_j) = 0$, $j \neq k$.
Subcase 2: For $k \equiv 1(mod3)$.
$f(v_i^{(1)}) = 0$; if $i \equiv 1, 4(mod6)$.
$= 1$; if $i \equiv 0, 5(mod6)$.
$= 2$; if $i \equiv 2, 3(mod6)$.
$f(c_1) = 0$.
$f(x_1) = 1$.
For remaining vertices take $j = k - 1$ and label them as in subcase 1.
Subcase 3: For $k \equiv 2(mod3)$.
$f(v_i^{(1)}) = 0$; if $i \equiv 1, 4(mod6)$.
$= 1$; if $i \equiv 2, 3(mod6)$.
$= 2$; if $i \equiv 0, 5(mod6)$.
$f(v_i^{(2)}) = 0$; if $i \equiv 1, 4(mod6)$.
$= 1$; if $i \equiv 0, 5(mod6)$.
$= 2$; if $i \equiv 2, 3(mod6)$.
$f(c_1) = 2$.
$f(c_2) = 0$.
$f(x_1) = 1$.
$f(x_2) = 2$.
For remaining vertices take $j = k - 2$ and label them as in subcase 1.

Case 6: For $n \equiv 5(mod6)$.
In this case we define labeling function f as follows
Subcase 1: For $k \equiv 0(mod3)$.
For $j \equiv 1, 2(mod3)$
$f(v_i^{(j)}) = 0$; if $i \equiv 1, 4(mod6)$.
$= 1$; if $i \equiv 2, 3(mod6)$.
$= 2$; if $i \equiv 0, 5(mod6)$, $i \leq n - 2$.
$f(v_{n-1}^{(j)}) = 1$.
$f(v_n^{(j)}) = 2$; if $j \equiv 1(mod3)$.
$f(v_n^{(j)}) = 0$; if $j \equiv 2(mod3)$.
$f(c_j) = 2$; if $j \equiv 1(mod3)$.
$f(c_j) = 0$; if $j \equiv 2(mod3)$.
$f(x_j) = 1$; if $j \equiv 1(mod3)$.
$f(x_j) = 2$; if $j \equiv 2(mod3)$.
For $j \equiv 0(mod3)$
$f(v_i^{(j)}) = 0$; if $i \equiv 1, 4(mod6)$.
$= 1$; if $i \equiv 0, 5(mod6)$.
$= 2$; if $i \equiv 2, 3(mod6)$ $i \leq n - 1$.
$f(v_n^{(j)}) = 2$.
$f(c_j) = 0$.
$f(x_j) = 2$, $j \neq k$.
Subcase 2: For $k \equiv 1(mod3)$.
$f(v_i^{(1)}) = 0$; if $i \equiv 1, 4(mod6)$.
$= 1$; if $i \equiv 0, 5(mod6)$.
$= 2$; if $i \equiv 2, 3(mod6)$, $i \leq n - 2$.
$f(v_i^{(1)}) = 1$; if $i \geq n - 1$.
$f(c_1) = 0$.
$f(x_1) = 2$.
For remaining vertices take $j = k - 1$ and label them as in subcase 1.
Subcase 3: For $k \equiv 2(mod3)$.
For $j = 1, 2$
$f(v_i^{(j)}) = 0$; if $i \equiv 1, 4(mod6)$.
$= 1$; if $i \equiv 0, 5(mod6)$.
$= 2$; if $i \equiv 2, 3(mod6)$, $i \leq n - 2$.
$f(v_i^{(j)}) = 1$, $i \geq n - 1$.
$f(c_1) = 0$.
$f(c_2) = 2$.
$f(x_j) = 0$.
For remaining vertices take $j = k - 2$ and label them as in subcase 1.
The labeling pattern defined above covers all possible arrangement of vertices. In each case, the graph G under consideration satisfies the conditions $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $0 \leq i, j \leq 2$ as shown in Table 3. i.e. G admits 3-equitable labeling.
(In Table 3 $n = 6a + b$ and $k = 3c + d$ where $a \in N \cup \{0\}$,$c \in N$)

The labeling pattern defined above is demonstrated by means of following Illustration 2.8.

Illustration 2.8 Consider a graph $G =< W_6^{(1)} : W_6^{(2)} : W_6^{(3)} : W_6^{(4)} >$. Here $n = 6$ and $k = 4$. The corresponding 3-equitable labeling is as shown in Figure 5.

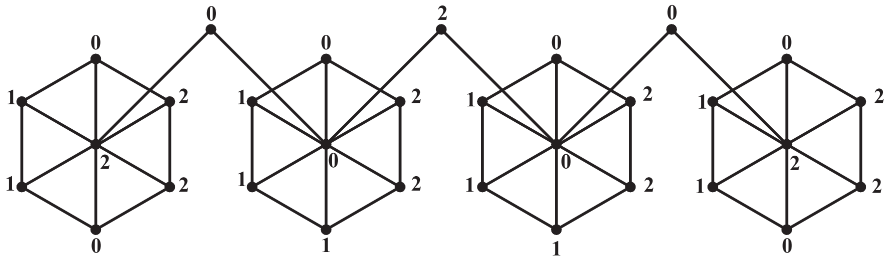

Figure 5: 3-equitable labeling of graph G.

Table 3: Vertex and Edge conditions for f

b	d	Vertex Condition	Edge Condition
0	0	$v_f(0)+1=v_f(1)=v_f(2)$	$e_f(0)+1=e_f(1)=e_f(2)+1$
	1	$v_f(0)+1=v_f(1)+1=v_f(2)$	$e_f(0)=e_f(1)=e_f(2)$
	2	$v_f(0)=v_f(1)=v_f(2)$	$e_f(0)=e_f(1)=e_f(2)+1$
1	0	$v_f(0)=v_f(1)=v_f(2)+1$	$e_f(0)+1=e_f(1)=e_f(2)+1$
	1	$v_f(0)=v_f(1)=v_f(2)+1$	$e_f(0)+1=e_f(1)=e_f(2)$
	2	$v_f(0)=v_f(1)=v_f(2)+1$	$e_f(0)=e_f(1)=e_f(2)$
2	0	$v_f(0)+1=v_f(1)=v_f(2)$	$e_f(0)+1=e_f(1)=e_f(2)+1$
	1	$v_f(0)=v_f(1)=v_f(2)$	$e_f(0)+1=e_f(1)=e_f(2)+1$
	2	$v_f(0)+1=v_f(1)=v_f(2)+1$	$e_f(0)+1=e_f(1)=e_f(2)+1$
3	0	$v_f(0)=v_f(1)=v_f(2)+1$	$e_f(0)+1=e_f(1)=e_f(2)+1$
	1	$v_f(0)+1=v_f(1)+1=v_f(2)$	$e_f(0)=e_f(1)=e_f(2)$
	2	$v_f(0)=v_f(1)=v_f(2)$	$e_f(0)=e_f(1)+1=e_f(2)$
4	0	$v_f(0)+1=v_f(1)=v_f(2)$	$e_f(0)+1=e_f(1)=e_f(2)+1$
	1	$v_f(0)=v_f(1)+1=v_f(2)$	$e_f(0)=e_f(1)+1=e_f(2)$
	2	$v_f(0)=v_f(1)+1=v_f(2)$	$e_f(0)=e_f(1)=e_f(2)$
5	0	$v_f(0)=v_f(1)=v_f(2)+1$	$e_f(0)+1=e_f(1)=e_f(2)+1$
	1	$v_f(0)=v_f(1)=v_f(2)$	$e_f(0)+1=e_f(1)=e_f(2)+1$
	2	$v_f(0)=v_f(1)+1=v_f(2)+1$	$e_f(0)+1=e_f(1)=e_f(2)+1$

3 Concluding Remarks

Cordial and 3-equitable labeling of some star and shell related graphs are reported in Vaidya et al.[14], [15] while the present work corresponds to cordial and 3-equitable labeling of some wheel related graphs. Here we provide cordial and 3-equitable labeling for the larger graphs constructed from the standard graph.

Further scope of research

Similar investigations can be carried out in the context of different graph labeling techniques and for various standard graphs.

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