Determinantal hypersurface from a geometric perspective

Thesis by

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In Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

2013

(Defended April 29, 2013)

Acknowledgements

I would like to thank my advisor Eric M. Rains for guiding me through my graduate career, Tom Graber for helpful conversations regarding my research, and Babak Hassibi for partly suggesting the project in the first place.

I would also like to thank my wonderful friends at Caltech. I will not have made it this far without any of you, and I will always look back to my graduate student days fondly because of you.

Last but not least, I thank my parents and extended family for their love and support throughout my ten plus years abroad.

Abstract

In this paper, we give a geometric interpretation of determinantal forms, both in the case of general matrices and symmetric matrices. We will prove irreducibility of the determinantal singular loci and state its dimension. We also provide detailed description of the singular locus for small dimensions.

Contents

Chapter 1

Introduction

Given n a positive integer, for an $(n+m)\times(n+m)$ matrix M over $\mathbb{C}$, let m_I denote the determinant of the minor formed by taking the rows and columns indexed by $I\cup\{n+1,\ldots,n+m\}$, where $I\subseteq\{1,2,..,n\}$. The corresponding multilinear form is obtained from taking

$$det(M+K_{n,m})=\sum_I m_I\cdot x_J, \tag{1.0.1}$$

where J is the complement of I in $\{1,2,..,n\}$, and K_{n+m} a diagonal $(n+m)\times(n+m)$ consisting x_i for the first n rows, and 0 for the rest. We assign homogeneous coordinates $[x_{i1}:x_{i2}]\in\mathbb{P}^1$ for each x_i, and call all the homogeneous form of the associated multilinear form $h_M(x_{11},x_{12},x_{21},x_{22},\ldots,x_{n1},x_{n2})$. Additionally, a point $m\in\mathbb{P}^{2^n-1}$ is called a determinantal point if $m=(m_I)$ for some $(n+m)\times(n+m)$ matrix M.

Determinantal points appear in the context of determinantal point process, when the correlation functions of a random point processes are determinantal points. Determinantal points coming from symmetric matrices are of interest in the area of information theory; namely, entropy vectors of n scalar jointly Gaussian random variables are symmetric determinantal.

There are previous results regarding the variety corresponding to symmetric determinantal points given in[HS07], using hyperdeterminantal relations. The explicit expressions that generate the hyperdeterminantal relations become very involved even for small matrices.

In this thesis, we will attempt to give a geometric description by studying the singular loci of the determinantal multilinear forms and the associated matrices. Since scaling preserves the projective multilinear form, matrices of the form

$$\begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix}, \tag{1.0.2}$$

correspond to the same multilinear form as M. Furthermore, when $m > 0$, adding a multiple of one of the last m rows/columns to the first n rows/columns leaves the multilinear form unchanged, as well as an arbitrary change of basis for the last m rows/columns. It is sufficient to consider $m \leq n$, and M of the form

$$\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}, \tag{1.0.3}$$

where A is $n \times n$. Note that $det(M + K_{n+m})$ has degree n. Furthermore, a generic point satisfies $m_\emptyset \neq 0$, and we can take the matrix to be $n \times n$. In the symmetric case, the same types of operations will preserve the symmetry, so $B = C^T$.

The variety of determinantal points is of dimension $n^2 - n + 1$[BR05]. In the symmetric case, note that m_i determines the diagonal entries, and m_I with $|I| = 2$ determines the off-diagonal terms up to signs: therefore, the same multilinear form can correspond to at most a finite number of matrices, whose off-diagonal entries can differ by a multiple of -1. Therefore, the symmetric determinantal points have the same dimension as the symmetric matrices, which is $\frac{n(n+1)}{2}$.

It has been shown that the algebraic group $G_n = GL_2(\mathbb{C})^{\otimes n}$ acts on the determinantal multilinear forms via

$$(x_{i1}, x_{i2}) \mapsto (G_n^{(i)}(x_{i1}, x_{i2})^T),$$

where $G_n^{(i)}$ is the $i-$th copy of $GL_2(\mathbb{C})$ in G_n. To see that this action also preserves symmetric matrices, it is enough to check

$$\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ a_0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{1.0.4}$$

of $GL_2(\mathbb{C})$. The first one corresponds to scaling the i-th row and column both by $(\alpha)^{\frac{1}{2}}$. The second one represents the addition of a_0 to the i-th diagonal entry. Assuming the third one is acting on the first set of coordinates, then it takes a matrix of the form

$$\begin{pmatrix} a & B & C \\ d & E & F \\ g & H & J \end{pmatrix} \tag{1.0.5}$$

to

$$\begin{pmatrix} 0 & 0 & 0 & \sqrt{-1} \\ 0 & E & F & d \\ 0 & H & J & g \\ \sqrt{-1} & b & c & a \end{pmatrix}. \tag{1.0.6}$$

This group action is especially useful for studying symmetric determinantal forms when n is small, when we try to compute the singular loci directly. We will then rely on those results in $n = 3$ and 4 to describe the generic symmetric loci for all n with the following result.

Theorem 1.0.1. *The singular locus of a generic symmetric determinantal multilinear form is irreducible of dimension $n - 3$.*

By generic, we mean the statement holds on a dense open subset of the symmetric determinantal multilinear forms. Since we are thinking of determinantal hypersurfaces in a geometric way, it would be useful to have a more geometric interpretation of the multilinear forms. It turns out that such an interpretation is given by line configurations. Not all determinantal forms have such an interpretation, namely, the ones who have at least one linear factor, but we will show that the line configurations correspond to a dense open set of determinantal forms.

In the general case, we would want to follow a similar approach as in the symmetric case, but direct computation for small dimension cases gets more difficult. We will therefore rely more on the correspondence between determinantal hypersurface and line configurations. The result in the case of general matrices takes the familiar form.

Theorem 1.0.2. *The singular locus of a generic determinantal multilinear form is irreducible of dimension* $n-4$.

Finally, we will take the geometric approach one step further for small n, through vector bundles, with the following result.

Theorem 1.0.3. *There is a birational correspondence between determinantal multilinear* 5*-forms and the moduli space of genus* 1 *curves with certain vector bundle structures.*

For $n = 6$, we attempt to follow a similar approach as in $n = 5$, but since vector bundles in this case are not as well studied, the result we arrive is not as strong, and there are immediate open questions left to future discussion.

Chapter 2

Geometric Interpretation

2.1 General Determinantal Forms

Given a line in $\mathbb{P}^{n-1}$, if viewed as an element in $G(2, n)$, there is a representation by a $2 \times n$ matrix

$$\begin{pmatrix} a_{i,1} & a_{i,2} & \cdots & a_{i,n} \\ b_{i,1} & b_{i,2} & \cdots & b_{i,n} \end{pmatrix}$$

.

If there are n lines, there are n such $2 \times n$ matrices, say with $i = 1, 2, \ldots, n$. We can further rewrite this by

$$A + XB = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix} + \begin{pmatrix} x_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & x_n \end{pmatrix} \begin{pmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n,1} & b_{n,2} & \cdots & b_{n,n} \end{pmatrix},$$

where $A = (a_{i,j})$, and $B = (b_{i,j})$. Then each row is a parametrization of the line, and x_i a parameter. The hypersurface in $(\mathbb{P}^1)^n$ associated with these n lines is given by the equation $det(A + XB) = 0$, i.e. a set of n points, one from each line, that lie on a common hyperplane.

If B is invertible, then $B^{-1} \in GL_n(\mathbb{C})$, and $det(A+XB) = det(B)det(AB^{-1}+X)$. Therefore, we can find a collection of lines in the direction of coordinate vectors that represent the same hypersurface.

Now suppose B is singular, with no zero rows. With the above trick, we can replace B with a lower triangular matrix such that nonzero rows are coordinate vectors. By relabeling the lines, we will assume B is the form

$$\begin{pmatrix} I & 0 \\ B_1 & 0 \end{pmatrix}.$$

Assume $corank(B) = m < n$, i.e. I is of size $m \times m$. Consider the $(n+m)\times(n+m)$ matrix

$$B' = \begin{pmatrix} B & \tilde{I} \\ 0 & I \end{pmatrix},$$

where the i-th of columns $\tilde{I}$ is the $(m+i)$-th coordinate vector with 1 in the $m+i$ position, and 0 otherwise. Note that if we expand A to A' of size $(n+m)\times(n+m)$ with 0 entries, and X to $\begin{pmatrix} X & 0 \\ 0 & I \end{pmatrix}$, then

$$det(A' + X'B') = det \begin{pmatrix} A + XB & \tilde{I} \\ 0 & I \end{pmatrix} = det(A + XB),$$

and the hypersurface remains unchanged. Now by column operations, there is $E' \in GL_{n+m}(\mathbb{C})$ such that

$$B'E' = \begin{pmatrix} I & 0 & \\ -B_1 & I & 0 \end{pmatrix},$$

with I of size n. The original hypersurface is also given by

$$det(\begin{pmatrix} A'E' & & \\ -B_1 & I & 0 \end{pmatrix} + X') = 0.$$

Since $GL_n(\mathbb{C})$ acts on the space of line configurations, any hypersurface associated to a line configuration can be represented by an $n+m \times n+m$ matrix, with $m \leq n+1$. However, in the case when $m > 0$, the multilinear form is of degree strictly less than

n. Since projective lines are preserved by $PGL_2(\mathbb{C})$, $PGL_2^n(\mathbb{C})$ acts on the set of determinantal forms associated to n lines.

In fact, if any of the n lines degenerate into points, we can still represent each point by two dependent vectors a_i and b_i, and repeat the above process to obtain a diagonal X'. Assume b_i is nonzero, then $a_{ij} + b_{ij}x_i$ is a common factor when taking the determinant.

Another representation of line configurations is given by the $n \times 2n$ matrix $\begin{pmatrix} A^T & B^T \end{pmatrix}$. Note that for $det(A + XB) \neq 0$, $\begin{pmatrix} A & B \end{pmatrix}$ can represent an element in $Gr(n, 2n)$. Therefore, even for the degenerate case, when we have points instead of lines, this $n \times 2n$ representation is still valid. The associated hypersurface is given by $\sum_{i \subseteq [n]} det(A_I^T B_{[n]-I}^T)x_I = 0$, i.e. the coefficients are given by the projection of $Gr(n, 2n)$ onto the principal minors.

Since $Gr(n, 2n) \to \mathbb{P}(\wedge^n \bigoplus_i^n V_i)$ under the Plücker embedding, if we consider $Gr(n, 2n)$ onto the principal minors $\tilde{P}_n$, we get

$$\begin{array}{ccc} Gr(n, 2n) & \longrightarrow & \mathbb{P}(\wedge^n \bigoplus_i^n V_i) \\ \downarrow & & \downarrow \\ \tilde{P} & \longrightarrow & \mathbb{P}(\otimes_i^n V_i) \end{array}.$$

$GL(2n)$ acts on $\mathbb{P}(\otimes_i^n V_i)$, and it has been shown the principal minors are invariant under the action of the subgroup $SL(2)^{\times n} \ltimes S_n$.

It is more clear in the $Gr(n, 2n)$ representation that the closure of the space of n lines contains the degenerate cases when there are k points and $n - k$ lines, for some $0 \leq k \leq n$.

2.2 Symmetric Determinantal Forms

For the symmetric matrices, suppose a symmetric matrix is given by n lines in general position, i.e. the corresponding hypersurface is given by $det(A + X) = 0$ where A is a symmetric $n \times n$ matrix. The lines have Plücker coordinates of the form $e_i \wedge e_i A$, where e_i is the $i-th$ standard basis, and the sum of these coordinates is

$$\sum_i e_i \wedge e_i A = \sum_{i,j} a_{ij}(e_i \wedge e_j) = \sum_{i,j}(a_{ij} - a_{ji})(e_i \wedge e_j) = 0$$

. This can be viewed as an extra condition imposed on the lines compared to the general matrix case.

Now suppose we are given n lines, and under $GL(n)$ action as above their parametrization is given by $A+X$, i.e. the Plücker coordinates are $e_i \wedge f_i$. Further, assume the condition that the $\sum_i^n a_i(e_i \wedge f_i) = 0$ for some a_i. Then $\sum_i^n e_i \wedge (a_i f_i) = 0$. Then the new representation of the n lines under n copies of $\begin{pmatrix} 1 & 0 \\ 0 & a_i \end{pmatrix}$ is a symmetric matrix.

Choosing a different representation of the line, i.e. taking into consideration of the $PGL_2(\mathbb{C})$ action, will scale the Plücker coordinates by a constant. Therefore the orbit of a symmetric determinantal form under $PGL_2(\mathbb{C})^n$ consists of symmetric determinantal forms.

In the conditional case when we have to embed the n lines into a larger space to replace with lines in coordinate directions, we have to revise the condition on Plücker coordinates slightly. If the matrix is of size $(n+m) \times (n+m)$, there needs to exist m lines, and a_i for $i = 1, \ldots, n+m$ such that the sum of the Plücker coordinates of the i-th line scaled by a_i is 0.

Since in the conditional case, the determinantal hypersurface is further preserved by left multiplication of the $(n+m) \times (n+m)$ matrix of the form

$$\begin{pmatrix} I & * \\ 0 & * \end{pmatrix},$$

in addition to right multiplication by $GL_{n+m}(\mathbb{C})$, we can again restrict m to be at most $n-1$ in the symmetric case.

In this case, we would consider the Langrangian Grassmannian of a $2n$ dimension vector space instead. Then the corresponding subgroup of $Sp(2n)$, which preserves the symmetric principal minors, is $SL(2)^{\times n} \ltimes S_n$.

Chapter 3

Generic Irreducibility of the Singular Locus

We will show that a point of the generic determinantal locus can be singular only if the corresponding n points on the lines lie on a codimensional higher than 1 subspace. Otherwise, there will be a contradiction to the assumption that n lines are in general position. The n points on the lines can lie on a codimensional higher than 2 subspace, but as we will see, this case fortunately only happens for large n.

Theorem 3.0.1. *For configuration of lines in general position, every singular point of the corresponding hypersurface in* $(\mathbb{P}^1)^n$ *gives a collection of* n *points on the lines lying on a codimension* 2 *subspace.*

Proof. Assume $(q_1, q_2, \ldots, q_n) \in (\mathbb{P}^1)^n$ is a singular point of the hypersurface. Let $\mathscr{Q}$ be the subspace spanned by the n points $(p_1, p_2, \ldots, p_n) \in (\mathbb{P}^{n-1})^n$ corresponding to the singular point. We are given a line configuration $\{l_k\}_{k=1,2,\ldots,n}$ such that $p_i \in l_k$. The singular point assumption implies there is a hyperplane containing all n points $(p_1, p_2, \ldots, p_n)$, and $\mathscr{Q}$ is at least codimension 1. Suppose $\mathscr{Q}$ is exactly codimension 1.

Since we are examining a singular point of the hypersurface, any infinitesimal deformation of the singular point along the coordinate direction will stay on the hypersurface, i.e. any infinitesimal deformation of any one of q_i will remain coplanar with all other q_j for $i \neq j$.

- Case 1: All $n - 1$ subset of points among $p_1, p_2, \ldots, p_n$ are independent, then any one of the subset of $n-1$ points alone is enough to determine a hyperplane. We therefore have $n-1$ hyperplanes, each corresponding to a subset of $n-1$ points. But all $n-1$ hyperplanes must be identical to the initial hyperplane $\mathscr{Q}$ containing the n points. This means when we consider an $n-1$ subset, the nth point must also lie on the same hyperplane as the $n-1$ points in the set, and furthermore, infinitesimal deformation allows us to conclude the line containing the n-th point also lies on the same hyperplane. Therefore we obtain $n > n-1$ lines spanning $n-1$ dimensional subspace.

- Case 2: There is a subset S of $n-1$ points among $(p_1, p_2, \ldots, p_n)$ which is dependent. From the codimension 1 condition of $\mathscr{Q}$, the n points $(p_1, p_2, \ldots, p_n)$ need to span $n-1$ dimensions, so the $n-1$ points in S have to span at least $n-2$ dimensions, i.e. there is a unique linear dependency for the points, say $\sum_{k=1}^{n-1} a_k q_k = 0$. Assume $a_i \neq 0$ for $i \in I$, with I a subset of S. Let H be the unique hyperplane in this case, then just as in the previous case, all $|I|$ lines lie on H.

  For $|I| \leq \frac{n}{2}$, the lines $\{l_i\}_{i \in I}$ span at most $2|I|-1$ dimensions.

  For $|I| > \frac{n}{2}$, the lines $\{l_i\}_{i \in I}$ span at most $n-1$ dimensions.

In either case, we have a contradiction to the fact that k lines in general position span $min(2k, n)$ dimensions.

Therefore, all singular points of the hypersurface correspond to n points lying on a codimension of at least 2 subspace.

□

Remark 3.0.2. The above result actually holds for all line configurations. It can be shown using the correspondence with lines in vector space of larger dimension as described in the previous chapter.

Theorem 3.0.1 does not rule out the possibility where a singular point corresponds to an n-tuple lying on a codimension 3 or higher hypersurface. We will show that this indeed cannot happen generically when n is small.

Theorem 3.0.3. *For $4 \leq n < 9$, there is a dense open subset of line configurations such that every singular point of the corresponding hypersurface in $(\mathbb{P}^1)^n$ gives a collection of n points on the lines spanning a codimension 2 subspace.*

Proof. First consider the subset X_3 inside $(\mathbb{P}^{n-1})^n$, which is the projection of $\tilde{X}_3 = \{(\Delta, p_1, p_2, \ldots, p_n) : p_i \in \Delta, \Delta \in Gr(n-3,n)\} \subseteq (\mathbb{P}^{n-1})^n \times Gr(n-3,n)$ onto the first factor, i.e.

$$\begin{array}{c} \tilde{X}_3 \subseteq (\mathbb{P}^{n-1})^n \times Gr(n-3,n) \\ \downarrow \\ X_3 \subseteq (\mathbb{P}^{n-1})^n \end{array}$$

$\tilde{X}_3$ can be viewed as a fiber bundle over $Gr(n-3,n)$ with irreducible fibers that are products of projective spaces, and therefore irreducible. Under projection, X_3 is also irreducible. If we consider the subset X_3' of X_3 where the n points span a codimension 3 hyperplane, then such points have exactly one fiber under the projection. Therefore, the dimension of X_3 is given by the preimage of X_3', which is $n(n-4) + 3n - 9 = n^2 - n - 9$.

Now consider the following diagram,

$$\begin{array}{ccc} \{\text{Line configurations}\} \times (\mathbb{P}^1)^n & \xrightarrow{f} & (\mathbb{P}^{n-1})^n \ , \\ \pi \downarrow & & \\ \{\text{Line configurations}\} & & \end{array}$$

where $(\mathbb{P}^1)^n$ is a set of parameters for each of the n lines, the horizontal map gives points on the n lines, and π is the projection onto the first factor.

For a line configuration to be in the fiber of a point in $(\mathbb{P}^{n-1})^n$, each line needs to pass through a fixed point. Therefore, the fiber dimension is $n(n-2)$. The pullback of X_3 under f has dimension $\leq n^2 - n - 9 + n(n-2) = 2n^2 - 3n - 9 = 2n(n-2) - (9-n)$. For $4 \leq n < 9$, the codimension of $\pi(f^{-1}(X_3))$ is $9 - n > 0$.

□

Remark 3.0.4. In Theorem 3.0.3, we can replace 9 by any $k^2 > 1$; then for any $4 \leq n < k^2$ there is no singular point in the generic singular locus of a determinantal n-form that corresponds to a codimension k or higher subspace.

In general, given n lines $\{l_1, \ldots, l_n\}$ in $\mathbb{P}^{n-1}$ corresponding to M, for each l_i, let $\phi_i : \mathbb{P}^1 \to l_i$ be the parametrization of l_i. Let

$$Sing(M)^c = \{x \in (\mathbb{P}^1)^n : dim(span\{\phi_1(x_1) \ldots, \phi_n(x_n)\}) = n-2, l_i \not\subset span\{\phi_1(x_1) \ldots, \phi_n(x_n)\}\}$$

be the subset of the corank 2 locus inside $Sing(M)$, and $\overline{Sing(M)^c}$ the closure of $Sing(M)^c$.

Let us also denote $Sing(L)$ the singular locus of the determinantal form associated to L(and M).,i.e.

$$Sing(L) := \{x \in (\mathbb{P}^1)^n : dim(span\{\phi_1(x_1) \ldots, \phi_n(x_n)\}) \leq n-2\}.$$

Lemma 3.0.1. *$\overline{Sing(M)^c}$ is irreducible of dimension $n-4$ for generic M.*

Proof. Let $L = \{l_1, l_2, \ldots, l_n\}$ be n lines in general position corresponding to the matrix M. Consider the set of n points, one from each line, that span a codimension 2 hyperplane. Then there is a well defined map

$$\begin{array}{c} Sing(M)^c \\ \downarrow \\ Gr(n-2,n) \times (\mathbb{P}^1)^n \\ \downarrow \\ Gr(n-2,n), \end{array}$$

with the first map given by the inclusion of $Sing(M)^c$ in $(\mathbb{P}^1)^n$, and the $n-2$ plane containing the corresponding n points on n lines, and the second map the projection onto the first factor.

Let $A_i = \{\Delta \in Gr(n-2,n) : l_i \cap \Delta \neq \emptyset\}$, then $Sing(M)^c$ has image $A = \cap_{i=1}^n A_i$. By the Kleiman's version of Bertini theorem[Kle74], A_i intersects transversally, and A is smooth of dimension $2(n-2) - n = n-4$. The projection map has one point fiber. We will show $Sing(M)^c$ is open. Consider

$$C_i := \{x \in (\mathbb{P}^1)^n : dim(span\{\phi_1(x_1) \ldots, \phi_n(x_n)\} + l_i) \leq n-2\}.$$

Let $\bar{l_j}$ be the image of l_j in V/l_i, where V is the corresponding n-dim vector space, and set $L_i = \{\bar{l_j} : j \neq i\}$. Since L is a set of lines in general position, so is L_i in V/l_i. If $\bar{\phi}_j$ is the parametrization of the i-th line in V/l_i, now

$$\begin{aligned} Sing(L_i) &= \{y \in (\mathbb{P}^1)^{n-1} : dim(span\{\bar{\phi}_j(y_j) : j \neq i\}) \leq n-4\} \\ &= \{y \in (\mathbb{P}^1)^{n-1} : dim(span\{\phi_j(y_j) : j \neq i\} + l_i) \leq n-2\} \\ &= \{y = (y_1, \ldots, \hat{y}_i, \ldots, y_n) \in (\mathbb{P}^1)^{n-1} : (y_1, \ldots, y_i, \ldots, y_n) \in C_i \text{ for all } y_i \in \mathbb{P}^1\}. \end{aligned}$$

Therefore, $C_i \cong Sing(L_i) \times \mathbb{P}^1$.

Since $Sing(L_i)$ is closed, so is C_i. Now $Sing(M)^c$ is the complement of $\cup_i C_i$ in $Sing(M)$ and therefore is open. Hence $\overline{Sing(M)^c}$ is irreducible of dimension $n-4$.

□

We will see for some small n, $Sing(M) = Sing(M)^c$ for a generic M.

Proposition 3.0.1. *There is a morphism from the generic singular locus into $Gr(n-2, n)$ for $4 \leq n < 9$, and for $4 \leq n < 7$ the map is injective.*

Proof. The statement regarding $4 \leq n < 9$ is a rephrasing of Theorem 3.0.3.

The ambiguity of injection comes from the fact that we could have $n-2$ out of the n points spanning an $n-2$ plane which contains one line in our configuration, but this line does not contain any of the previous $n-2$ points. However, if this is the case, we could consider the quotient space of the $n-2$ plane by the line contained in it. In this regard, we obtain an $n-4$ plane that intersects $n-1$ lines in general position. By Kleinman's Bertini Theorem, the set of $Gr(n-4, n-2)$ we are considering has dimension $dim(Gr(n-4, n-2)) - (n-1) = n-7$.

For $4 \leq n < 7$, there cannot be such an $n-2$ plane containing a line generically. Therefore, all points in the generic singular locus correspond to corank 2 matrices which intersect each line at exactly one point. □

Since for $4 \leq n < 7$, $Sing(M) = Sing(M)^c$ for a generic line configuration, and $Sing(M)$ is irreducible of dimension $n-4$, we will show that in general this statement also holds for higher n, except $Sing(M) = \overline{Sing(M)^c}$.

Theorem 3.0.5. *The generic singular locus is irreducible of dimension $n-4$.*

Proof. Let $L = \{l_1, \ldots, l_n\}$ be a set of n lines in general position in $\mathbb{P}^{n-1}$. For each l_i, let $\phi_i : \mathbb{P}^1 \to l_i$ be the parametrization of l_i, and $\phi = (\phi_i)$.

We want to show that $Sing(L)$ is irreducible.

For $4 < n < 9$, we already know that $Sing(L)$ is irreducible for generic L. We proceed by induction and assume $Sing(L_i)$ is irreducible for all $1 \leq i \leq n$, which implies $C_i \cong Sing(L_i) \times \mathbb{P}^1$ is also irreducible for $1 \leq i \leq n$.

Suppose there is a component C in $Sing(L)$ such that $C \neq Sing(L)^c$. Then since

$$Sing(L) = Sing(L)^c \cup \cup_{i=1}^n C_i,$$

if $C \cup Sing(L)^c \neq \emptyset$,

$$C = (C \cap Sing(L)^c) \cup (C \cap \cup_{i=1}^n C_i)$$

would contradict C being irreducible.

So we will assume $C \cap Sing(L)^c = \emptyset$, i.e. $C \subset \cup_{i=1}^n C_i$. Note that for any $x \in C_i \cap C$,

$$dim(span\{\phi_1(x_1) \ldots, \phi_n(x_n)\} + l_i) \leq n-2$$

and

$$dim(span\{\phi_1(x_1) \ldots, \phi_n(x_n)\}) < n-2,$$

which implies

$$dim(span\{\phi_1(x_1) \ldots, \phi_n(x_n)\} + l_j) \leq n-2.$$

So $x \in C_j$, and $C \cap C_i = C \cap C_j$, i.e. $C = C \cap C_i \subset C_i$. Since C_i is irreducible, $C = C_i$. Therefore C_is are all identical. However, from the definition of C_i, C is then the whole $(\mathbb{P}^1)^n$, which contradicts the codimension 2 condition of $Sing(L)$.

We can now conclude that $Sing(L)^c$ is the only component of $Sing(L)$, and $dim(Sing(L)) = dim(Sing(L)^c) = n-4$. ∎

Chapter 4

Description of the Generic Singular Locus for $n = 5$ and 6

4.1 n=5

Theorem 4.1.1. *The generic singular locus for $n = 5$ is a genus 1 curve.*

Proof. We already know that the singular locus is of dimension 1, from 3.0.5. Using previous notation, we'll calculate the canonical bundle of A.

Since A_i is of codimension 1, viewing as a divisor, we have $K_{A_i} = (K_{G(3,5)} + A_i)|_{A_i} = (-c_1(G(3,5)) + A_i)|_{A_i}$. Iterating over the five linearly equivalent divisors, we have $K_A = (-c_1(G(3,5)) + A_1 + A_2 + A_3 + A_4 + A_5)|_A = (-c_1(G(3,5)) + 5A_i)|_A$.

Consider the tautological exact sequence over $G(3,5)$

$$0 \to S \to \mathbb{C}^5 \to V \to 0,$$

then the tangent bundle is given by

$$TG = Hom(S, Q) = S^\vee \otimes Q.$$

Since the TG is generated by its sections[Ful98],

$$c_1(S^\vee \otimes Q) = 5[A_i]$$

So K_A is trivial. □

As usual, we restrict our attention to the line configurations in a sufficiently small dense open subset. By doing so, there is an injective morphism from the singular locus to $G(3,5)$. Since non-vanishing of the determinant imposes open conditions, for every singular point p, where rows r_1, r_2, r_3 of the corresponding matrix are independent, for $1 \leq r_i \leq 5$, it is possible to find an open set around p where r_1, r_2, r_3 stay independent. Let W be the pullback of the universal bundle on $G(3,5)$. Then W is naturally embedded in $\mathscr{O}^5_{\mathscr{C}}$. The quotient V is then the pullback of the universal quotient bundle of $G(3,5)$, then V a vector bundle of rank 2 with five global sections. We denote the fiber of W over p by $\sigma(p)$.

In addition, we can construct five line bundles $\mathscr{L}_i \sim \mathscr{O}$ to $\mathscr{C}$, which correspond to the five lines l_i whose parametrizations are given by line configurations via $l_i/\sigma(p) \cap l_i$. $\mathscr{L}_i$ is well-defined, since local trivialization is given by W. Furthermore, $\Gamma(\mathscr{L}_i)$ is given by l_i, and $\mathscr{L}_i$ is a line bundle of degree 2.

Theorem 4.1.2. *V is an indecomposable rank 2 vector bundle of degree 5.*

Proof. Note that $\Gamma(V)$ is generated by five global sections, each corresponding to a row in the matrix M. Furthermore, since the corresponding lines have general configuration, $\Gamma(V) = 5$. Suppose V is decomposable, then either $V = \mathscr{O}_{\mathbb{C}} \oplus \mathscr{L}'_4$ or $\mathscr{L}'_2 \oplus \mathscr{L}'_3$, where the subscript denotes the degree of the line bundle, and $H^0(V)$ is the direct sum of the global sections of its line bundle decompositions.

In the first case, V splits into a short exact sequence

$$0 \to \mathscr{O}_{\mathbb{C}} \to V \to \mathscr{L}'_4 \to 0. \tag{4.1.1}$$

On the other hand, each $\mathscr{L}_i$ maps into V. Since $\mathscr{L}_i$ are degree 2, it has nontrvial image in $\mathscr{L}'_4$. Taking global sections, we have $l_1, l_2, \ldots, l_5$ which are contained in $\Gamma(\mathscr{L}'_4)$, which is a 4-dimensional vector space. But this contradicts the generic line configuration assumption.

In the case where $V = \mathscr{L}'_2 \oplus \mathscr{L}'_3$, since all $\mathscr{L}_i$ are non-isomorphic, there is at most one $\mathscr{L}_i = \mathscr{L}'_2$. Then at least four $\mathscr{L}_i$ all have a nontrivial image in $\mathscr{L}'_4$, and four of $l_1, l_2, \ldots, l_5$ are contained in a 3-dimensional vector space, which again is a contradiction to the generality assumption.

Therefore V is indecomposable. By the classification of indecomposable vector bundles over an elliptic curve, we know $degV = 5$ [Ati57]. □

Suppose we are now given a genus 1 curve with a rank 2 indecomposable vector bundle V of degree 5, and five non-isomorphic line bundles L_i of degree 2 that map into V, there is a corresponding line configuration.

Consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S & \longrightarrow & \Gamma(V) & \xrightarrow{j} & V & \longrightarrow & 0 \\ & & & & \uparrow & & \psi\uparrow & & \\ & & & & \Gamma(\mathscr{L}_i) & \longrightarrow & \mathscr{L}_i & \longrightarrow & 0 \end{array}$$

where the morphism from global sections are defined by the the base-point free linearly independent global sections, and S is the kernel of j, which has a codimension 2 fiber over each point.

Since $\mathscr{L}_i$ injects into V, the images of $\Gamma(\mathscr{L}_i)$ in V are proper. Therefore the image of $\Gamma(\mathscr{L}_i)$ meets S in $\Gamma(V)$. Therefore, the line configuration obtained from the geometric data given above gives rise to a singular locus containing a genus 1 curve.

In fact, by counting dimensions, we can conclude that the singular locus we constructed cannot be any bigger. The moduli space $\mathscr{M}(\mathscr{E}, \{\mathscr{L}_i\}_{i=1}^5, V)$ in question consists of an elliptic curve $\mathscr{E}$, sets of line bundles $\{\mathscr{L}_i\}_{i=1}^5$ of degree 2 belonging to different isomorphism classes, and a rank 2 vector bundle V of degree 5, along with maps $\mathscr{L}_i \to V$ for $i = 1, 2, \ldots, 5$. Consider the projection

$$\mathscr{M}(\mathscr{E}, \{\mathscr{L}_i\}_{i=1}^5, V) \to \mathscr{M}(\mathscr{E}, \mathscr{L}_1, V) \to \mathscr{M}(\mathscr{E}, \mathscr{L}_1, \mathscr{K} = W/\mathscr{L}_1),$$

where $\mathscr{M}(\mathscr{E}, \mathscr{L}_1, V)$ is the moduli space of $\mathscr{E}$ with V and just one of the above mentioned $\mathscr{L}_1$ along with $\mathscr{L}_i \to V$, and $\mathscr{M}(\mathscr{E}, \mathscr{L}_1, K = W/L_1)$ the moduli space of $\mathscr{E}$ with $\mathscr{L}_1$ and a degree 3 line bundle $\mathscr{K}$, which has dimension 2. [HM98]

Now $Ext^1(\mathscr{K}, \mathscr{L}_1) = H^1(L \otimes Q^v)$ which has dimension $-(deg(\mathscr{L}_1) - deg(\mathscr{K})) = 1$, and $Ext^1(\mathscr{K}, \mathscr{L}_1)$ is entirely given by scaling.

Since we have a short exact sequence,

$$0 \to \mathscr{L}_1 \to W \to \mathscr{K} \to 0,$$

we can consider the following long exact sequence

$$Hom(\mathscr{L}_i, \mathscr{L}_1) \to Hom(\mathscr{L}_i, W) \to Hom(\mathscr{L}_i, \mathscr{K}) \to Ext^1(\mathscr{L}_i, \mathscr{L}_1).$$

From the assumption that $\mathscr{L}_i$ and $\mathscr{L}_1$ are not isomorphic for $i = 2, 3, 4, 5$, we have $Hom(\mathscr{L}_i, W) \cong Hom(\mathscr{L}_i, \mathscr{K})$, and the first projection of moduli space has fiber of dimension 4. Therefore $\mathscr{M}(\mathscr{E}, \{\mathscr{L}_i\}_{i=1}^5, V)$ is 6-dimensional.

On the other hand, if we consider configurations of five lines in $\mathbb{P}^4$, it is of dimension $5 \cdot dim(G(2,5)) - (dim(GL_5) - 1) = 6$, where -1 accounts for the scalar matrices. Therefore, given an elliptic curve with a rank 2 indecomposable vector bundle V of degree 5, and five non-isomorphic line bundles L_i of degree 2 that map into V, we indeed obtain a line configuration that corresponds to an elliptic curve. We have proven the following statement:

Theorem 4.1.3. *There is a natural correspondence between the set of line configurations and the moduli space of genus* 1 *curves with six vector bundles whose properties are stated as above.*

4.2 n=6

Similar to the discussion for $n = 5$, the generic singular locus here can be viewed as the intersection of $G(4,6)$ with six lines in $\mathbb{P}^5$ in general position, and we obtain a dimension 2 variety with trivial canonical class, which is a $K3$ surface.

In fact, for higher n, we can consider the corank 2 locus X, i.e. the image of $G(n-2, n)$ inside the singular locus. Let H be a hyperplane section of X, i.e. $X = Y \cap H$ for some subvariety Y. Since the canonical divisor of $G(n-2, n)$ is given by $-nH$[Con82], $K_Y = -H$, by Kodaira's theorem, $H^i(\mathscr{O}_Y) = H^i(\mathscr{O}_Y(K_Y + (-K_Y))) = 0$ for $i > 0$. Furthermore, the dual form of Kodaira's theorem gives $H^i(\mathscr{O}_Y(-H)) = 0$ for $0 < i < dim(X) + 1$. If we consider the long exact exact sequence in cohomology associated to

$$0 \to \mathscr{O}_Y(-H) \to \mathscr{O}_Y \to \mathscr{O}_X \to 0,$$

we obtain $H^i(\mathscr{O}_X) = 0$ for all $0 < i < dim(X)$. Therefore, X is a Calabi-Yau manifold.

For $n = 6$, we have the following sturcture on a $K3$ surface. There is a rank 2 vector bundle V such that $h^0(V) = 6$, and six line bundles $\mathscr{L}_i$ for $i = 1, 2, \ldots, 6$, with each $h^0(\mathscr{L}_i) = 2$, and $\mathscr{L}_i$ injects into V . Furthermore,

$$\mathscr{L}_i{\cdot}\mathscr{L}_j = |\{\Delta \in G(4,6) : \Delta \text{ contains 2 distinct points and intersects 4 lines in general position}\}| =$$

$$\mathscr{L}_i^2 = \{\Delta \in G(4,6) : \Delta \text{ contains 1 line and intersects 5 lines in general position } \} = 0$$

$$\mathscr{L}_i \cdot V = \{\Delta \in G(4,6) : \Delta \text{ intersects 8 lines in general position } \} = 5$$

$$V^2 = \{\Delta \in G(4,6) : \Delta \text{ contains 1 point and intersects 7 lines in general position } \} = 14,$$

since the degree of the Grassmannian of lines embedded via the Plücker embedding are given by Catalan numbers [Muk93]. We then compute the intersection matrix to be nonsingular.

If we consider the moduli space of $K3$ surfaces and vector bundles in the above assumption, the dimension would be at most $20 - 7 = 13$. Note that in this case, the configuration of six lines in $\mathbb{P}^5$ gives a dimenson count of $6 \cdot dim(G(2,6)) - dim(GL_6) - 1 = 13$.

The above discussion leads to a question we can ask. Do all $K3$ surfaces in $(\mathbb{P}^1)^6$ with the above vector bundle structures come from our determinantal hypersurface construction? If not, what would be a natural way to realize the vector bundles?

If the $K3$ surface in $(\mathbb{P}^1)^6$ is the singular locus of a hypersurface, and furthermore, when we project onto any copy $\mathbb{P}^1$, it is a genus 1 curve most of the time, i.e. determinantal in the case of $n = 5$, is this $K3$ surface associated to a determinantal hypersurface?

Chapter 5

Description of Generic Symmetric Singular Loci for $n = 3, 4, 5$

For symmetric matrices, since there is an extra condition on the Plücker coordinates of line configurations, we expect the dimension of a singular locus would go up.

In this chapter, we will explicitly find a dense open set inside symmetric determinantal multilinear n-forms and describe the singular locus of a multilinear form inside this open set in the case of small n. The singular loci are obtained from solving systems of equations for $n = 3$ and 4. For $n = 4$, direct computation can in fact give us all possible singular loci, but we will not state it here. For $n = 5$, we have to simplify the set of equations with its Gröbner Basis (in our case, we utilize the software Magma).

Note, It is easy to see that in the case where $n = 2$, we obtain all possible bilinear forms.

5.1 n=3

Take=ing a Zariski open subset inside symmetric determinantal trilinear forms corresponding to $m_\emptyset \neq 0$, we can then consider 3×3 matrices $\{a_{ij}\}$. The determinant of minors have relations

$$(m_{123}-m_{12}m_3-m_{13}m_2-m_{23}m_1+2m_1m_2m_3)^2 = 4(m_1m_2-m_{12})(m_2m_3-m_{23})(m_1m_3-m_{13}), \tag{5.1.1}$$

The singular point is computed to be

$$[\frac{a_{12}a_{13}-a_{11}a_{23}}{a_{23}}, \frac{a_{12}a_{23}-a_{13}a_{22}}{a_{13}}, \frac{a_{13}a_{23}-a_{12}a_{33}}{a_{12}}] \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$$

when at most one $a_{ij} = 0$, for $i \neq j$, i.e. no lines degenerate into points. Call this type of singular locus, consisting of just one point, T_1.

Furthermore, we can describe all possible singular loci. With $m_\emptyset = 1$, a point outside the dense open set would have at least two $a_{ij} = 0$. The associated multilinear forms satisfies $m_I/m_{I\cup\{k\}}$ is a constant for all $I \subseteq [3]$ with $|I| \leq 2$, and a fixed k not in I, i.e. the multilinear forms have at least a linear factor. We then have a case in which either

- Exactly two a_{ij} vanish, i.e. two lines and one point, in which case the singular locus is just $\mathbb{P}^1$(Type T_2).
- All off-diagonal entries are zero, i.e. three points in general position, then

$$Det(M + K_{3,0}) = (x_1 - m_1)(x_2 - m_2)(x_3 - m_3),$$

  and the singular locus is three copies of $\mathbb{P}^1$ intersecting at a point (Type T_3).

The three types of singular loci are shown in the figure below.

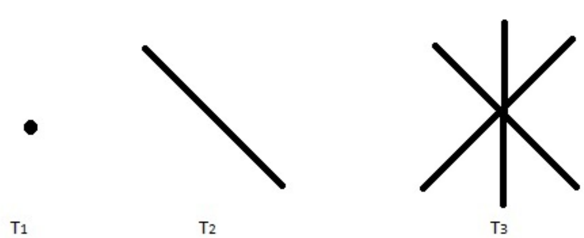

Figure 5.1: Singular loci for determinantal points of 3×3 matrices

5.2 n=4

Theorem 5.2.1. *There is a dense open set of symmetric determinantal quadrilinear forms whose singular locus under $PGL_2^{\otimes 4}$ action is given by the diagonal embedding from $\mathbb{P}^1$ to $(\mathbb{P}^1)^4$, given by $k \mapsto (k, k, k, k)$.*

Proof. Consider the multilinear form

$$a(x_1x_2 + x_3x_4) + b(x_1x_3 + x_2x_4) - (a + b)(x_2x_3 + x_1x_4) \tag{5.2.1}$$

whose singular locus is given by the diagonal embedding. Furthermore, it is realizable by a symmetric matrix M given by

$$\begin{pmatrix} 0 & 0 & 0 & -a^{\frac{1}{4}} & \frac{\sqrt{b}}{a^{\frac{1}{4}}} & \frac{\sqrt{-a-b}}{a^{\frac{1}{4}}} \\ 0 & 0 & a^{\frac{1}{4}} & -\frac{a^{\frac{1}{4}}\sqrt{-a-b}}{\sqrt{b}} & 0 & -\frac{a^{\frac{3}{4}}}{\sqrt{b}} \\ 0 & a^{\frac{1}{4}} & 0 & -\frac{\sqrt{a}d_{14}}{\sqrt{-a-b}} + \frac{\sqrt{a}d_{24}}{\sqrt{b}} & \frac{\sqrt{b}d_{14}}{\sqrt{-a-b}} - \frac{\sqrt{b}d_{34}}{\sqrt{-a-b}} & d_{14} \\ -a^{\frac{1}{4}} & -\frac{a^{\frac{1}{4}}\sqrt{-a-b}}{\sqrt{b}} & -\frac{\sqrt{a}d_{14}}{\sqrt{-a-b}} + \frac{\sqrt{a}d_{24}}{\sqrt{b}} & 0 & -\frac{\sqrt{-a-b}d_{24}}{\sqrt{b}} + \frac{\sqrt{-a-b}d_{34}}{\sqrt{a}} & d_{24} \\ \frac{\sqrt{b}}{a^{\frac{1}{4}}} & 0 & \frac{\sqrt{b}d_{14}}{\sqrt{-a-b}} - \frac{\sqrt{b}d_{34}}{\sqrt{-a-b}} & -\frac{\sqrt{-a-b}d_{24}}{\sqrt{b}} + \frac{\sqrt{-a-b}d_{34}}{\sqrt{a}} & 0 & d_{34} \\ \frac{\sqrt{-a-b}}{a^{\frac{1}{4}}} & -\frac{a^{\frac{3}{4}}}{\sqrt{b}} & d_{14} & d_{24} & d_{34} & 0 \end{pmatrix},$$

where d_{ij} are arbitrary constants.

Note that $dim(PGL_2(\mathbb{C})^{\otimes 4}) = 12$, and symmetric determinantal points for $n = 4$ are of dimension 10, so a generic symmetric determinantal quadrilinear form should have stabilizers of dimension 2. Indeed , in this case, it is given by four copies of $\begin{pmatrix} \frac{\alpha}{-\beta^2+\alpha^2} & b \\ \frac{\beta}{-\beta^2+\alpha^2} & \alpha \end{pmatrix}$, where $\alpha, \beta \in \mathbb{C}$.

Now the orbit of $det(M + K_{3,0})$ has dimension $dim(PGL_2(\mathbb{C})^{\otimes 4}) - dim(Stab) = 12 - 2 = 10$, so this orbit is a dense open subset in the set of all symmetric determinantal quadrilinear forms where the singular locus is given by the diagonal embedding up to PGL_2 action. □

Remark 5.2.2. Theorem 5.2.1 implies that every symmetric determinantal quadrilinear form is in the closure of the orbit of $det(M + K_{3,0})$ under the $PGL_2^{\otimes 4}$ action.

The symmetric determinantal quadrilinear forms are the closure of one orbit.

5.3 n=5

In the $n = 5$ case, one point to note is that both symmetric determinantal multilinear 5-forms and $PGL_2^{\otimes 5}$ have dimension 15: if we want to follow what happens in $n = 4$, we would hope to find determinantal points with finite stabilizers under the action of $PGL_2^{\otimes 5}$. In fact, this is the case, and additionally, we can describe the generic singular locus as a quintic del Pezzo surface.

In our case, we will consider the del Pezzo surface dP_5 as obtained from blowing up $\mathbb{P}^1 \times \mathbb{P}^1$ at three different points, with five maps to $\mathbb{P}^1$, which gives a canonical embedding of dP_5 into $(\mathbb{P}^1)^5$ given by $(a, b) \mapsto (a, b, \frac{1+b}{a}, \frac{a+b-1}{ab}, \frac{1-a}{b})$, up to PGL_2 actions.

Proposition 5.3.1. *Matrices whose singular loci are dP_5 embedded as described above are dense in the set of symmetric matrices for $n = 5$.*

Proof. First we need to show that there exists a matrix whose singular locus is the quintic del Pezzo surface. We consider

$$\begin{pmatrix} z & x & y & y & x \\ x & z & x & y & y \\ y & x & z & x & y \\ y & y & x & z & x \\ x & y & y & x & z \end{pmatrix}.$$

We take M to be the matrix when $x = 0, y = 1, z = 0$, and its associated multilinear form ϕ is the homogeneous form of

$$2 + x_1 + x_2 + x_3 + x_4 + x_5 - x_1x_2x_4 - x_1x_3x_4 - x_1x_3x_5 - x_2x_3x_5 - x_2x_4x_5 + x_1x_2x_3x_4x_5. \tag{5.3.1}$$

The singular locus W of ϕ has a Gröbner basis consisting of 10 cubics,

$$
\begin{aligned}
&x_{11}x_{21}x_{41} - x_{11}x_{22}x_{42} - x_{12}x_{21}x_{42} - x_{12}x_{22}x_{42},\\
&x_{11}x_{31}x_{41} - x_{12}x_{31}x_{42} - x_{12}x_{32}x_{41} - x_{12}x_{32}x_{42},\\
&x_{11}x_{31}x_{51} - x_{11}x_{32}x_{52} - x_{12}x_{32}x_{51} - x_{11}x_{31}x_{51},\\
&x_{11}x_{22}x_{31} - x_{12}x_{21}x_{32} - x_{12}x_{22}x_{32},\\
&x_{11}x_{41}x_{52} - x_{12}x_{42}x_{51} - x_{12}x_{42}x_{52},\\
&x_{11}x_{22}x_{52} - x_{12}x_{21}x_{51} - x_{12}x_{22}x_{52},\\
&x_{21}x_{31}x_{51} - x_{21}x_{32}x_{52} - x_{22}x_{31}x_{52} - x_{22}x_{32}x_{52},\\
&x_{21}x_{41}x_{51} - x_{22}x_{41}x_{52} - x_{22}x_{42}x_{51} - x_{22}x_{42}x_{52},\\
&x_{21}x_{32}x_{41} - x_{22}x_{31}x_{42} - x_{22}x_{32}x_{42},\\
&x_{31}x_{42}x_{51} - x_{32}x_{41}x_{52} - x_{32}x_{42}x_{52}.
\end{aligned}
$$

The projection map onto the first two factors gives a birational equivalence $W \mapsto \mathbb{P}^1 \times \mathbb{P}^1$ via the open set that is the preimage of the complement of the points $\{(0,-1),(-1,0),(\infty,\infty)\}$.

Every stabilizer of the multilinear form induces an automorphism of W, whose automorphism group is S_5. Therefore the stabilizer of the multilinear form is trivial under $PGL_2(\mathbb{C})^5$, and the orbit of this multilinear form is of dimension $dim(PGL_2(\mathbb{C})^{\otimes 5}) = 15 = dim(C_5)$. Hence, this orbit gives a dense open subset of symmetric determinantal multilinear 5-forms where the singular locus is a del Pezzo surface blowing up at three points with embedding given to $(\mathbb{P}^1)^5$ as above. □

Note that for the small n we have just discussed, the generic type of singular locus is irreducible, and the dimension of the singular locus goes up by one each time. This will serve as a motivation for higher n.

Chapter 6

Generic Irreducibility of the Symmetric Singular Locus

We will again look for a dense open set in $n \times n$ symmetric matrices and attempt to describe the properties of the associated singular loci. Similar to the general case, we will show that a generic singular locus is irreducible by first showing that there exists an irreducible component of the appropriate dimension, and that this is the unique irreducible component in this singular locus.

We start with two results concerning intersection numbers.

Definition 6.0.1. For a subvariety X of dimension k in $(\mathbb{P}^1)^n$, X represents a class $[X]$ in the group $A^{n-k}((\mathbb{P}^1)^n)$ inside the Chow ring of $(\mathbb{P}^1)^n$. Let $H_{j,n} \in A((\mathbb{P}^1)^n)$ denote the generator of the j-th copy of $A(\mathbb{P}^1)$ under the projection map, and $H_{j_1 \cdots j_k,n} = H_{j_1,n} \cdots H_{j_k,n} \in A^k((\mathbb{P}^1)^n)$, where $j_i \in \{1, \dots, n\}$ are distinct. Then we define the intersection number $[X].[H_{j_1 \cdots j_k,n}]$ of X with respect to the subset $\{j_1, \dots, j_k\}$ as the image of $[X]$ and $[H_{j_1 \cdots j_k,n}]$ under the natural pairing $A^{n-k}((\mathbb{P}^1)^n) \times A^k((\mathbb{P}^1)^n) \to \mathbb{Z}$.

Lemma 6.0.1. *For a subvariety $X \subseteq (\mathbb{P}^1)^n$ such that $dim X = k$, there exists a subset of $(\mathbb{P}^1)^n$ of size k such that the intersection number with X is nonzero.*

Proof. We induct on n. For $n = 1$, $A^1(\mathbb{P})^1$ is generated by any point, in particular, X if $k = 0$. For $k = n = 1$, $X = (\mathbb{P}^1)^n$, the statement also holds.

Assume the statement is true for $n - 1$. Let $\iota_\alpha : (\mathbb{P}^1)^{n-1} \to (\mathbb{P}^1)^n$ via $x \mapsto (x, \alpha)$, then the preimage X' of X is either X itself, or a $k - 1$ dimensional subvariety of $(\mathbb{P}^1)^{n-1}$.

Suppose $[X][H_{j_1\cdots j_k,n}] = 0$ for some $j_1, \ldots, j_k$, and $X' = X$. Let $p : (\mathbb{P}^1)^n \to (\mathbb{P}^1)^{n-1}$ be the projection map. Since $(\iota_\alpha)_*([X'][H_{j_1\cdots j_k,n-1}]) = [X][H_{j_1\cdots j_k,n}] = 0$, and $p_*(\iota_\alpha)_* = id$, we have $([X'][H_{j_1\cdots j_k,n-1}]) = (p_*(\iota_\alpha)_*)((([X'][H_{j_1\cdots j_k,n-1}])) = 0$. This is a contradiction.

Suppose X' is a $k-1$ dimensional variety inside $\mathbb{P}^{n-1}$. Then under pullback $p^*([X'][H_{j_1\cdots j_{k-1},n-1}]) = [X][H_{j_1\cdots j_{k-1},n}]$. Now $[X][H_{j_1\cdots j_{k-1}j',n}] = 0$ if $j' = j_l$ for some l, or otherwise by induction hypothesis. Since p^* is a split monomorphism, we have $[X'][H_{j_1\cdots j_{k-1},n-1}] = 0$.

□

Corollary 6.0.1. *For a subvariety $X \subseteq (\mathbb{P}^1)^n$ such that the intersection number of X with any subsets of $(\mathbb{P}^1)^n$ of size k is 0, then $dimX < k$.*

Proof. For any $k' \geq k$, we can find a subset of k' where the intersection number with X is nonzero. Applying Lemma 6.0.1, we have $dimX \neq k'$. □

For $n = 3$ or 4, we say a determinantal multilinear form is generically singular if its singular locus is a point (when $n = 3$), or given by the diagonal embedding as described in the previous chapter up to PGL_2 action (when $n = 4$).

Suppose $n > 4$. Take a singular locus $Sing(f)$ which corresponds to a determinantal multilinear n-form f, and consider projection maps $p_I : Sing(f) \to (\mathbb{P}^1)^{|I|}$, where $I \subset [n]$, and $|I| = n-3$ or $n-4$. Denote $Z_I(f) = \{z \in Sing : f|_{p_I(z)}$ is generically singular$\}$. For an $n \times n$ matrix M, denote $Z_I(M) = Z_I(det(M + K_{n,0}))$

Proposition 6.0.2. *For $|I| = n-3$, there is a dense open set of $n \times n$ symmetric matrices where $Z_I(M)$ is an irreducible subvariety of dimension $n-3$.*

Proof. Let $z \in Sing(M)$; then it corresponds to one point on each line $\tilde{Z} = \{\tilde{z}_1, \ldots, \tilde{z}_n\}$. Let V_{n-3} be the space spanned by $n-3$ points in $\tilde{Z}$ indexed by I. A necessary condition for $z \in Z_I$ is that the projection of three lines not indexed by I onto the quotient by V_{n-3} corresponds to three lines in general position.

It suffices to consider configurations of n lines in $\mathbb{P}^{n-1}$ that satisfy the Plücker coordinate condition as described in Chapter 1. There is a rational projection

$$\begin{array}{c} Gr(2,n) \times (\mathbb{P}^{n-1})^{n-3} \, , \\ \Big\downarrow \\ Gr(2,3) \end{array}$$

given by projection onto the complementary space spanned by the $n-3$ points in $\mathbb{P}^{n-1}$. If the $n-3$ points are independent, and the line is not in the span, the projection is well-defined. Furthermore, it is surjective, since we can split into n-dim vector space as $\mathbb{C}^n = \mathbb{C}^3 \oplus \mathbb{C}^{n-3}$, and can therefore embed an element in $Gr(2,3)$ into the first $\mathbb{C}^3$.

Let $\mathscr{F}_n$ be the $(\mathbb{P}^1)^{n-3}$ bundle over $G(2,n)^n$, and $\mathscr{F}'_{n-3}$ be the natural $(\mathbb{P}^1)^{n-3}$ bundle over $G(2,n)^{n-3}$. Then we have

$$\begin{array}{c} \mathscr{F}_n = Gr(2,n)^3 \times \mathscr{F}'_{n-3} \\ \Big\downarrow \\ Gr(2,n)^3 \times (\mathbb{P}^{n-1})^{n-3} \\ \Big\downarrow \\ G(2,3)^3, \end{array}$$

where the first arrow is a morphism given by points in $(\mathbb{P}^{n-1})^{n-3}$ on the n lines, and the second arrow three copies of the rational projection mentioned above, which is also surjective. Therefore, given an open set $G(2,3)^3$ that corresponds to the generic singular locus in $(\mathbb{P}^1)^3$, i.e. three lines in general position, it pulls back to an open set U in $\mathscr{F}_n$. In other words, there is a open subset of configurations of n lines in $\mathbb{P}^{n-1}$, along with an open subset of each fiber, that corresponds to three lines in general position under the projection map. Under the quotient correspondence, the fiber would lie on a $1 + (n-3) = n-2$ plane and indeed belongs to the singular locus.

The Plücker coordinate condition is preserved under push forward. Furthermore, given an element $G(2,3)^3$, if we represent it by a 3×3 matrix R, then the $n \times n$ matrix

$$\begin{pmatrix} R & 0 \\ 0 & I' \end{pmatrix},$$

where I' consists of 1 on the diagonal, superdiagonal and subdiagonal, and 0 elsewhere, represents n lines in $\mathbb{P}^{n-1}$ that satisfy the Plücker coordinate condition and are in the preimage. Therefore, U is non-empty.

Z_I has irreducible image under p_I, with irreducible fiber of constant dimension (a single point), therefore Z_I is irreducible of dimension $dim((\mathbb{P}^1)^{n-3}) = n-3$.

□

Although the above result can possibly be proved with matrix representation, the advantage of adapting the line configuration is that we would not accidentally miss the points at infinity. Since we only care about the generic situation, it is enough to consider line configurations and ignore the degenerate cases when points replace lines.

It may seem like we get many Z_I as we vary I, but they turn out to be an identical component inside the singular locus.

Lemma 6.0.2. *$Z_I(M)$ is open in $Sing(M)$ for a generic M.*

Proof. For $|I| = n-3$, refer to Theorem 6.0.1. In general, as we have seen above, Z_I is open if the set of generic multilinear $n-|I|$ forms is open. For $|I| = n-4$, Z_I is the preimage of an orbit under PGL_2 action of maximal dimension. □

Let $I_k = I - \{k\}$ for some $k \in I$, with $|I| = n-3$. We can regard p_{I_k} as a composition $\phi_k \cdot p_I$, where $\phi_k : (\mathbb{P}^1)^I \to (\mathbb{P}^1)^{I_k}$ deletes the k-th coordinate. Then $Z_I \cap Z_{I_k} \neq \emptyset$, and since they are both irreducible open sets, $\bar{Z}_I = \bar{Z}_{I_k}$ for all $k \in I$. For $k \neq k'$, $\bar{Z}_{I_k} = \bar{Z}_{I'_k}$, and $Z_{I_k} \cap Z_{I'_k} \neq \emptyset$. We will denote Z_{n-3} the nontrivial intersection of all Z_I and Z_{I_k} varying over all I, i.e. $Z_{n-3} = \cap_{\{I:|I|=n-3\}} \cap_{\{I_k:k\in I\}} Z_{I_k}$.

Finally, we will show that Z_{n-3} is the unique irreducible component inside the singular locus, and therefore it is oftentimes enough to use Z_{n-3} to describe the singular locus.

Theorem 6.0.1. *$Sing(M)$ is irreducible of dimension $n-3$ for a generic M.*

Proof. Suppose there is an irreducible component C disjoint from $\bar{Z}$, then $p_I(C)$ is not dominant for $|I| = n-3$. Therefore, there exists an open subset $U \subset (\mathbb{P}^1)^{n-3}$ such that $p_I^{-1}(U) \cap C = \emptyset$, which means that for $[p^{-1}(U)] \in A^{n-3}(\mathbb{P}^1)^n$, $[C][p^{-1}(U)] = 0$. Applying Lemma 6.0.1, $dimC < n-3$.

Since at each singular point the matrix determinant and its minors vanish, together with the symmetric condition, we associate matrices of co-rank 2 generically. Given a generic M, there is a well-defined morphism v from the $Sing(M)$ to the corank 2 subvariety, $M_{2,n}$, and $im(v)$ has a covering defined by hypersurfaces everywhere. Since an open set $U \in im(v)$ is of $dim(M_{2,n})$ and v is injective, $v^{-1}(U)$ is at most co-dimension 3. Therefore, such C cannot exist, and Z_{n-3} is the only irreducible component in $Sing(M)$.

Since Z_{n-3} is open, $dim(Sing(M)) = dim(Z_{n-3}) = n-3$. □

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