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Deviation from Standard Inflationary Cosmology and the Problems in Ekpyrosis

Thesis by

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To my family for unconditional support, and to all my friends.

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Abstract

There are two competing models of our universe right now. One is Big Bang with inflation cosmology. The other is the cyclic model with ekpyrotic phase in each cycle. This paper is divided into two main parts according to these two models. In the first part, we quantify the potentially observable effects of a small violation of translational invariance during inflation, as characterized by the presence of a preferred point, line, or plane. We explore the imprint such a violation would leave on the cosmic microwave background anisotropy, and provide explicit formulas for the expected amplitudes $\langle a_{lm} a^*_{l'm'} \rangle$ of the spherical-harmonic coefficients. We then provide a model and study the two-point correlation of a massless scalar (the inflaton) when the stress tensor contains the energy density from an infinitely long straight cosmic string in addition to a cosmological constant. Finally, we discuss if inflation can reconcile with the Liouville's theorem as far as the fine-tuning problem is concerned. In the second part, we find several problems in the cyclic/ekpyrotic cosmology. First of all, quantum to classical transition would not happen during an ekpyrotic phase even for superhorizon modes, and therefore the fluctuations cannot be interpreted as classical. This implies the prediction of scale-free power spectrum in ekpyrotic/cyclic universe model requires more inspection. Secondly, we find that the usual mechanism to solve fine-tuning problems is not compatible with eternal universe which contains infinitely many cycles in both direction of time. Therefore, all fine-tuning problems including the flatness problem still asks for an explanation in any generic cyclic models.

Contents

Chapter 1

Introduction

The standard cosmological model which describes the early development of the universe is the Big Bang theory. In this model, the universe originated in a hot and dense state and has been expanding and cooling ever since. Although the Big Bang theory is extremely successful and can accurately describe the evolution of the universe after the nucleosynthesis, it causes many cosmological puzzles, such as flatness, homogeneous, monopole problems, etc. Two mechanisms are provided to be possible explanations in the literature. The first is inflation [1, 2], a period of accelerated expansion occurring between the Big Bang and nucleosynthesis. The second is ekpyrosis [88, 89], a period of ultra-slow contraction before Big Bang/Big Crunch to an expanding phase. In both mechanisms, there is one dominant energy component which grows faster than all other contributions in the universe, including spatial curvature and anisotropies, and thereby drives the universe into an exponentially flat and isotropic state [4, 90]. Furthermore, they both have the ability to imprint scale-invariant inhomogeneities on superhorizon scales via a causal mechanism [1, 88, 94, 95, 96, 97, 93]. In this paper, we will examine both mechanisms in greater details and try to generalize the ideas or find out the conceptual problems behind them.

1.1 Deviation from the Standard Picture of Inflation

In cosmology, the standard model is characterized by primordial Gaussian perturbations that are statistically homogeneous and isotropic, with an approximately scale-free spectrum. A number of analyses have suggested evidence that deviation from statistical homogeneity might exist in the real world [44]. These include the "axis of evil" alignment of low multipoles [45, 46, 47, 48, 49, 50, 51, 52, 53], the existence of an anomalous cold spot in the CMB [54, 55, 56], an anomalous dipole power asymmetry [57, 58, 59, 60, 61], a claimed "dark flow" of galaxy clusters measured by the Sunyaev-Zeldovich effect [62], as well as a possible detection of a quadrupole power asymmetry of the type predicted by ACW in the WMAP five-year data [33]. In none of these cases is it beyond a reasonable doubt that the effect is more than a statistical fluctuation, or an unknown systematic effect; nevertheless, the combination of all of them is suggestive [34]. Therefore, we perform a corresponding analysis for a small violation of translational invariance in Chapter 2.

After proposing explicit forms for violations of translational invariance motivated by the symmetries that are left unbroken, we explore the formula of the two-point correlation $\langle\delta(\mathbf{k})\delta(\mathbf{q})\rangle$ if translational invariance is broken by the presence of cosmic string that passes through our horizon volume during inflation in Chapter 3. Our result, Eq. (3.28), can be compared with data on the large-scale structure of the universe and the anisotropy of the microwave background radiation.

It is well known that one of the biggest advantage (or goal) of inflation is to make the evolution of our observable universe seem natural. However, it has been recognized for some time that there is tension between this goal and the underlying structure of classical mechanics. Liouville's theorem states that a distribution function in the phase space remains constant along trajectories; roughly speaking, a certain number of states at one time always evolves into precisely the same number of states at any other time. Therefore, the information is conserved. This is in conflict with the philosophy of inflation. Inflation attempts to account for the apparent fine-tuning of our early universe by offering a mechanism by which a relatively natural early condition will robustly evolve into an apparently fine-tuned later condition. But if that evolution is unitary, it is impossible for any mechanism to evolve a large number of states into a smaller number. All statements above are well known, and certainly true. However, does it mean that no choice of early universe Hamiltonian can make the current universe more or less finely tuned? The answer is not obvious and requires more inspection to reach the conclusion. We will discuss this in Chapter 4.

1.2 Problems in Cyclic/Ekpyrotic Cosmology

Primordial density fluctuations are thought to provide the seeds which later become the temperature anisotropies in the cosmic microwave background and the large-scale structure in the universe. This framework of the cosmological perturbation theory is based on the quantum mechanics of scalar fields, where the relevant observable is the amplitude of the field's Fourier modes [4]. Although they originates as quantum mechanical variables, these amplitudes eventually imprint classical stochastic fluctuations on the density field, characterized by the power spectrum. This interpretation proves to be very accurate in the CMB and large-scale structure analyses.

However, in order to make this stochastic interpretation consistent, the density matrix has to be diagonal in the amplitude basis. This criterion implies that interference terms in the density matrix are highly suppressed and can be neglected [99, 100]. Interference is associated with the coherence of the system, i.e., the coherence in the state between different points of configuration space [101, 102]. One way to realize decoherence is to let the system interact with an environment [101].

In the literature, there are various arguments and calculations suggesting that a form of such environment decoherence can indeed occur for inflationary perturbations [103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113]. The coherence length decreases exponentially for wavelengths greater than Hubble radius. Thus perturbations become classical once their wavelength exceeds the Hubble radius. All of these results lend support to the usual heuristic derivation of the spectrum of density perturbations in inflationary models. In Chapter 5, we use a simple model to study whether decoherence can also occur in the ekpyrotic phase. We find that the coherence lengths continue increasing even for the modes outside the horizon. Finally, we strengthen our conclusion by considering a different kind of mechanism, quantum to semi-classical transition without decoherence[98]. We show that the result is the same. The quantum to classical transition would not happen during ekpyrosis. Therefore, the heuristic argument that the modes become classical when they leave the horizon is invalid in the ekpyrotic phase and requires more careful inspection.

Besides the decoherence problem, cyclic and ekpyrotic cosmology has another difficulty to solve. In the literature, it seems that the ekpyrotic phase is the same as inflation as far as the fine-tuning problem is concerned. However, there is a major difference between these two models. In the usual Big Bang plus inflation paradigm, there is a beginning of time corresponding to the initial singularity, i.e. Big Bang; however, ekpyrotic/cyclic cosmology extends the timeline to the infinite past and future. This property makes the analysis which we only consider what happened in a specific cycle incomplete. In other words, taking the whole history of the universe into consideration could be so important that it might dramatically change the conclusion. We will focus on the relationship between cycles in Chapter 6, where we show that the solution of fine-tuning problems is incompatible with the eternal feature of the cyclic universe, and thereby requires another explanation.

Chapter 2

Translational Invariance and the Anisotropy of the Cosmic Microwave Background

2.1 Introduction

Inflationary cosmology, originally proposed as a solution to the horizon, flatness, and monopole problems [1, 2], provides a very successful mechanism for generating primordial density perturbations. During inflation, quantum vacuum fluctuations in a light scalar field are redshifted far outside the Hubble radius, imprinting an approximately scale-invariant spectrum of classical density perturbations [3, 4]. Models that realize this scenario have been widely discussed [5, 6, 7]. The resulting perturbations give rise to large-scale structure and temperature anisotropies in the cosmic microwave background, in excellent agreement with observation [8, 9, 10, 11, 12, 13, 14, 15, 16].

If density perturbations do arise from inflation, they provide a unique window on physics at otherwise inaccessible energy scales. In a typical inflationary model (although certainly not in all of them), the amplitude of density fluctuations is of order $\delta \sim (E/M_{\rm P})^2$, where E^4 is the energy density during inflation and $M_{\rm P}$ is the (reduced) Planck mass. Since we observe $\delta \sim 10^{-5}$, it is very plausible that inflation occurs near the scale of grand unification, and not too far from scales where quantum gravity is relevant. Since direct experimental probes provide very few constraints on physics at such energies, it makes sense to be open-minded about what might happen during the inflationary era.

In a previous paper [17], henceforth "ACW," the possibility that rotational invariance was violated by a small amount during the inflationary era was explored (see also [18, 19, 20, 21, 22, 23, 24]). ACW suggested a simple, model-independent form for the power spectrum of fluctuations in the presence of a small violation of statistical isotropy, characterized by a preferred direction in space, and computed the imprint such a violation would leave on the anisotropy of the cosmic microwave background radiation. A toy model of a dynamical fixed-norm vector field [25, 26, 27, 28, 29, 30] with a spacelike expectation value was presented, which illustrated the validity of the model-independent arguments. The spacelike vector model is not fully realistic due to the presence of instabilities [31], and furthermore it does not provide a mechanism for turning off the violation of rotational invariance at the end of the inflationary era. Nevertheless, it still provides a useful check of the general argument that the terms which violate rotational invariance should be scale invariant. An inflationary era that violates rotational invariance results in a definite prediction, in terms of a few free parameters, for the deviation of the microwave background anisotropy that can be compared with the data [32, 33, 35].

The results of ACW can be thought of as one step in a systematic exploration of the ways in which inflationary perturbations could deviate by small amounts from the standard picture, analogously to how the STU parameters of particle physics [36] parameterize deviations from the Standard Model, or how the Parameterized Post-Newtonian (PPN) formalism of gravity theory parameterizes deviations from general relativity [37]. In cosmology, the fiducial model is characterized by primordial Gaussian perturbations that are statistically homogeneous and isotropic, with an approximately scale-free spectrum. Even in the absence of an underlying dynamical model, it is useful to quantify how well existing and future experiments constrain departures from this paradigm. Deviations from a scale-free spectrum are quantified by the spectral index n_s and its derivatives; deviations from Gaussianity are quantified by the parameter f_{NL} of the three-point function (and its higher-order generalizations) [38, 39, 40, 41, 42, 43]. The remaining features of the fiducial model, statistical homogeneity and isotropy, are derived from the spatial symmetries of the underlying dynamics.

There is another important motivation for studying deviations from pure statistical isotropy of cosmological perturbations: a number of analyses have suggested evidence that such deviations might exist in the real world [44]. These include the "axis of evil" alignment of low multipoles [45, 46, 47, 48, 49, 50, 51, 52, 53], the existence of an anomalous cold spot in the CMB [54, 55, 56], an anomalous dipole power asymmetry [57, 58, 59, 60, 61], a claimed "dark flow" of galaxy clusters measured by the Sunyaev-Zeldovich effect [62], as well as a possible detection of a quadrupole power asymmetry of the type predicted by ACW in the WMAP five-year data [33]. In none of these cases is it beyond a reasonable doubt that the effect is more than a statistical fluctuation, or an unknown systematic effect; nevertheless, the combination of all of them is suggestive [34]. It is possible that statistical isotropy/homogeneity is violated at very high significance in some specific fashion that does not correspond precisely to any of the particular observational effects that have been searched for, but that would stand out dramatically in a better-targeted analysis.

The isometries of a flat Robertson-Walker cosmology are defined by $E(3)$, the Euclidean group in three dimensions, which is generated by the three translations R^3 and the spatial rotations $O(3)$. Our goal is to break as little of this symmetry as is possible in a consistent framework. A preferred vector, considered by ACW [17], leaves all three translations unbroken, as well as an $O(2)$ representing rotations around the axis defined by the vector. If we break some subgroup of the translations, there are three minimal possibilities, characterized by preferred Euclidean submanifolds in space. A preferred point breaks all of the translations, and preserves the entire rotational $O(3)$. A preferred line leaves one translational generator unbroken, as well as one rotational generator around the axis defined by the line. Finally, a preferred plane leaves the two translations within the plane unbroken, as well as a single rotation around an axis perpendicular to that plane. We will consider each of these possibilities in this paper.

A random variable $\phi(\mathbf{x})$ is statistically homogeneous (or translationally invariant) if all of its correlation functions $\langle\phi(\mathbf{x}_1)\phi(\mathbf{x}_2)\cdots\rangle$ depend only on the differences $\mathbf{x}_i-\mathbf{x}_j$, and is statistically isotropic (or rotationally invariant) about some point $\mathbf{z}_*$ if the correlations depend only on dot products of any of the vectors $(\mathbf{x}_i-\mathbf{z}_*)$ and $(\mathbf{x}_i-\mathbf{x}_j)$. The Fourier transform of the two-point function $\langle\phi(\mathbf{x}_1)\phi(\mathbf{x}_2)\rangle$ depends on two wavevectors $\mathbf{k}$ and $\mathbf{q}$, and will be translationally invariant if it only has support when $\mathbf{k}=-\mathbf{q}$. We will show how to perform a systematic expansion in powers of $\mathbf{p}=\mathbf{k}+\mathbf{q}$. ACW showed how a small violation of rotational invariance during inflation would be manifested in a violation of statistical isotropy of the CMB; here we perform a corresponding analysis for a small violation of translational invariance.

At energies accessible to laboratory experiments, translational invariance plays a pivotal role, since it is responsible for the conservation of momentum. Here we are specifically concerned with the possibility that translational invariance may have been broken during inflation by an effect that disappeared after the inflationary era ended. Such a phenomenon could conceivably arise from the presence of some sort of source that remained in our Hubble patch through inflation, although we do not consider any specific models along those lines.

2.2 Setup For a Special Point

In the standard inflationary cosmology the primordial density perturbations $\delta(\mathbf{x})$ have a Fourier transform $\tilde{\delta}(\mathbf{k})$, defined by

$$\delta(\mathbf{x})=\int d^3k e^{i\mathbf{k}\cdot\mathbf{x}}\tilde{\delta}(\mathbf{k}), \tag{2.1}$$

and the power spectrum $P(k)$ is defined by

$$\langle\tilde{\delta}(\mathbf{k})\tilde{\delta}(\mathbf{q})\rangle=P(k)\delta^3(\mathbf{k}+\mathbf{q}). \tag{2.2}$$

so that

$$\langle\delta(\mathbf{x})\delta(\mathbf{y})\rangle=\int d^3k e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}P(k). \tag{2.3}$$

The Dirac delta function in Eq. (2.2) implies that modes with different wavenumbers are uncoupled. This is a consequence of translational invariance during the inflationary era, while the fact that the power spectrum $P(k)$ only depends on the magnitude of the vector $\mathbf{k}$ is a consequence of rotational invariance.

Suppose that during the inflationary era translational invariance is broken by the presence of a special point with comoving coordinates $\mathbf{z}_*$. This is reflected in the statistical properties of the density perturbation $\delta(\mathbf{x})$. It is possible that the violation of translational invariance impacts the classical background for the inflation field during inflation and this induces a one-point function,

$$\langle \delta(\mathbf{x}) \rangle = G\left[|\mathbf{x} - \mathbf{z}_*|\right]. \tag{2.4}$$

Throughout this paper we will assume that this classical piece is small (consistent with current data) and concentrate on the two-point function, which now takes the form

$$\langle \delta(\mathbf{x})\delta(\mathbf{y}) \rangle = F\left[|\mathbf{x} - \mathbf{y}|, |\mathbf{x} - \mathbf{z}_*|, |\mathbf{y} - \mathbf{z}_*|, (\mathbf{x} - \mathbf{z}_*) \cdot (\mathbf{y} - \mathbf{z}_*)\right], \tag{2.5}$$

where F is symmetric under interchange of $\mathbf{x}$ and $\mathbf{y}$. This is the most general form of the two-point correlation that is invariant under the transformations $\mathbf{x} \to \mathbf{x} + \mathbf{a}$, $\mathbf{y} \to \mathbf{y} + \mathbf{a}$, $\mathbf{z}_* \to \mathbf{z}_* + \mathbf{a}$, and rotational invariance about $\mathbf{z}_*$

It is convenient to work with a form for $\langle \delta(\mathbf{x})\delta(\mathbf{y}) \rangle$ that is analogous to Eq. (2.3). We write,

$$\langle \delta(\mathbf{x})\delta(\mathbf{y}) \rangle = \int d^3k \int d^3q \; e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{z}_*)} e^{i\mathbf{q}\cdot(\mathbf{y}-\mathbf{z}_*)} P_t(k, q, \mathbf{k} \cdot \mathbf{q}), \tag{2.6}$$

where P_t is symmetric under interchange of $\mathbf{k}$ and $\mathbf{q}$. This is equivalent to Eq. (2.5) and is the most general form for the density perturbation's two-point correlation that breaks statistical translational invariance by the presence of a special point $\mathbf{z}_*$, preserving rotational invariance about that point. In the limit where the violations of translational invariance are small and can be neglected, the replacement $P_t(|\mathbf{k}|, |\mathbf{q}|, \mathbf{k} \cdot \mathbf{q}) \to P(k)\delta^3(\mathbf{k} + \mathbf{q})$ is valid.

We assume (as is consistent with the data) that violations of translational invariance are small and hence that P_t is strongly peaked about $\mathbf{k} = -\mathbf{q}$. Hence we introduce the variables $\mathbf{p} = \mathbf{k} + \mathbf{q}$, $\mathbf{l} = (\mathbf{k} - \mathbf{q})/2$ and to expand in $\mathbf{p}$ using, for example,

$$k = |\mathbf{l} + \mathbf{p}/2| = l + \frac{\mathbf{p}\cdot\mathbf{l}}{2l} - \frac{(\mathbf{p}\cdot\mathbf{l})^2}{8l^3} + \frac{p^2}{8l} + ... \tag{2.7}$$

It is convenient to introduce $U_t = \ln P_t$ and expand U_t to quadratic order in $\mathbf{p}$, neglecting the higher-order terms since P_t and hence U_t is dominated by wavevectors $\mathbf{p}$ near $\mathbf{p} = 0$,

$$P_t(|\mathbf{k}|, |\mathbf{q}|, \mathbf{k}\cdot\mathbf{q}) = e^{U_t(l,l,-l^2) - A(l)p^2/2 - B(l)(\mathbf{p}\cdot\mathbf{l})^2/(2l^2) + ...} \simeq P_t(l,l,-l^2) e^{-A(l)p^2/2 - B(l)(\mathbf{p}\cdot\mathbf{l})^2/(2l^2)}. \tag{2.8}$$

Note that there are no terms linear in $\mathbf{p}$ because the symmetry under interchange of $\mathbf{k}$ and $\mathbf{q}$ implies symmetry under $\mathbf{l} \to -\mathbf{l}$ and $\mathbf{p} \to \mathbf{p}$.

Plugging the expansion of P_t in Eq. (2.8) into Eq. (2.6) yields

$$\langle \delta(\mathbf{x})\delta(\mathbf{y})\rangle = \int d^3l\, e^{i\mathbf{l}\cdot(\mathbf{x}-\mathbf{y})} P_t(l,l,-l^2) \int d^3p\, e^{-A(l)p^2/2 - B(l)(\mathbf{p}\cdot\mathbf{l})^2/(2l^2)} e^{i\mathbf{p}\cdot\mathbf{z}}, \tag{2.9}$$

where $\mathbf{z} = (\mathbf{x} + \mathbf{y} - 2\mathbf{z}_*)/2$. The integral over d^3p can be performed by completing the square in the argument of the exponential. Introducing the 3×3 matrix,

$$C_{ij} = A(l)\delta_{ij} + B(l)\frac{l_i l_j}{l^2} \tag{2.10}$$

we find that

$$\int d^3p\, e^{-A(l)p^2/2 - B(l)(\mathbf{p}\cdot\mathbf{l})^2/(2l^2)} e^{i\mathbf{p}\cdot\mathbf{z}} = \sqrt{\frac{(2\pi)^3}{\det C}} e^{-z^T C^{-1} z/2} \simeq \sqrt{\frac{(2\pi)^3}{\det C}} (1 - z^T C^{-1} z/2). \tag{2.11}$$

Using this expression the two-point function can be written as

$$\langle \delta(\mathbf{x})\delta(\mathbf{y})\rangle = \int d^3l\, e^{i\mathbf{l}\cdot(\mathbf{x}-\mathbf{y})} P_t(l,l,-l^2) \sqrt{\frac{(2\pi)^3}{\det C}} \left(1 - \frac{z^T C^{-1} z}{2} + \ldots\right), \tag{2.12}$$

where the ellipses represent terms higher than quadratic order in the components of

$\mathbf{z}$. It is straightforward to solve for C^{-1} and $\det C$ in terms of the functions A and B . We find that $\det C = A^3 + A^2 B$ and

$$C_{ij}^{-1} = \frac{1}{A}\delta_{ij} - \frac{B}{A(A+B)}\frac{l_i l_j}{l^2}. \tag{2.13}$$

The part of the two-point correlation that is rotationally invariant is the usual power spectrum $P(l)$, so

$$P(l) = \sqrt{\frac{(2\pi)^3}{\det C}}P_t(l, l, -l^2). \tag{2.14}$$

Next we construct some mathematical examples that illustrate how the term proportional to z^2 is suppressed when P_t is very strongly peaked at $p = 0$. Without any violation of translational invariance, $P_t(|\mathbf{k}|, |\mathbf{q}|, \mathbf{k}\cdot\mathbf{q}) = P(k)\delta^3(\mathbf{k}+\mathbf{q}) = c/k^3\delta^3(\mathbf{k}+\mathbf{q})$ for a scale-invariant Harrison-Zeldovich power spectrum, where c is some constant. We want to construct a form for P_t that reduces to the standard Harrisson-Zeldovich spectrum with translational and rotational invariance as a parameter $d \to \infty$. The three-dimensional delta function can be written as

$$\delta^3(\mathbf{k}+\mathbf{q}) = \lim_{d\to\infty}\left(\frac{d}{\sqrt{\pi}}\right)^3 e^{-d^2(\mathbf{k}+\mathbf{q})^2} \tag{2.15}$$

Therefore, we might try writing P_t as $c/k^3\left(\frac{d}{\sqrt{\pi}}\right)^3 e^{-d^2(\mathbf{k}+\mathbf{q})^2}$ with d a large number. However, this P_t is not symmetric under the interchange of $\mathbf{k}$ and $\mathbf{q}$ because k^3 is not.

There are many possible ways to resolve this problem. We might imagine replacing k^3 by $k^{3/2}q^{3/2}$, $(k+q)^3/8$, $|\mathbf{k}-\mathbf{q}|^3/8$, $kq(k+q)/2$, $(kq)^{1/2}(k+q)^2/4$, $(\mathbf{k}\cdot\mathbf{q})(k+q)/2$ $\cdots$, or any linear combinations of these. With $\mathbf{p} = \mathbf{k}+\mathbf{q}$, $\mathbf{l} = (\mathbf{k}-\mathbf{q})/2$, we have

$$k^{3/2}q^{3/2} = l^3\left(1 - \frac{3(\mathbf{p}\cdot\mathbf{l})^2}{4l^4} + \frac{3p^2}{8l^2}\right), \frac{1}{8}(k+q)^3 = l^3\left(1 - \frac{3(\mathbf{p}\cdot\mathbf{l})^2}{8l^4} + \frac{3p^2}{8l^2}\right) \quad \cdots, \tag{2.16}$$

to second order in $\mathbf{p}$. Therefore, at quadratic order in $\mathbf{p}$, the most general form of a function which is symmetric under the interchange of $\mathbf{k}$ and $\mathbf{q}$ and reduces to k^3

when $\mathbf{k} = -\mathbf{q}$ is

$$l^3 \left(1 - a\frac{(\mathbf{p}\cdot\mathbf{l})^2}{l^4} - b\frac{p^2}{l^2}\right), \tag{2.17}$$

with two parameters a and b that are independent of l. Hence we arrive at the following form for $P_t(|\mathbf{k}|, |\mathbf{q}|, \mathbf{k}\cdot\mathbf{q})$,

$$P_t(|\mathbf{k}|, |\mathbf{q}|, \mathbf{k}\cdot\mathbf{q}) = \frac{1}{l^3} c \left(1 + a\frac{(\mathbf{p}\cdot\mathbf{l})^2}{l^4} + b\frac{p^2}{l^2}\right)\left(\frac{d}{\sqrt{\pi}}\right)^3 e^{-d^2p^2}, \tag{2.18}$$

which gives the familiar translationally (and rotationally) invariant density perturbations with a Harrison-Zeldovich spectrum as $d \to \infty$. Plugging into Eq. (2.6), the two-point function becomes

$$\langle\delta(\mathbf{x})\delta(\mathbf{y})\rangle = c(1 - \frac{z^2}{4d^2}) \int d^3l\ e^{i\mathbf{l}\cdot(\mathbf{x}-\mathbf{y})} \frac{1}{l^3}\left(1 + \frac{1}{2d^2}\frac{a+3b}{l^2}\right). \tag{2.19}$$

We can construct another example which also gives dependence on $(\mathbf{l}\cdot\mathbf{z})^2$. First notice that the three-dimensional delta function can be written as another form,

$$\delta^3(\mathbf{p}) = \lim_{d\to\infty}\left(\frac{d}{\sqrt{2\pi}}\right)^3 \sqrt{\det U} e^{-\frac{d^2}{2}p^i U_{ij} p^j}, \tag{2.20}$$

where $U_{ij} = 2(\delta_{ij} + f l_i l_j / l^2)$ and f is an arbitrary parameter independent of $\mathbf{l}$. So another possible choice for P_t that has the correct limiting behavior as $d \to \infty$ is

$$P_t(|\mathbf{k}|, |\mathbf{q}|, \mathbf{k}\cdot\mathbf{q}) = \frac{1}{l^3} c(1 + a\frac{(\mathbf{p}\cdot\mathbf{l})^2}{l^4} + b\frac{p^2}{l^2})\left(\frac{d}{\sqrt{2\pi}}\right)^3 \sqrt{\det U} e^{-\frac{d^2}{2}p^i U_{ij} p^j}. \tag{2.21}$$

This gives,

$$\langle\delta(\mathbf{x})\delta(\mathbf{y})\rangle = \int d^3l\ e^{i\mathbf{l}\cdot(\mathbf{x}-\mathbf{y})} \frac{1}{l^3} c\left(1 + \frac{a + (3+2f)b}{2(1+f)d^2 l^2}\right)\left[1 - \frac{z^2}{4d^2} + \frac{f}{4(1+f)d^2}\frac{(\mathbf{l}\cdot\mathbf{z})^2}{l^2}\right]. \tag{2.22}$$

Since observable $|\mathbf{z}|$'s can be as large as our horizon, we need the parameter d to be of that order (or larger) for the leading two terms of the expansion in z to be a good approximation in Eq. (2.19) and (2.22).

The form we have derived in this section is plausible but is not the most general.

For example, it could be that the Fourier transform of the two-point function has the usual form plus a small piece that is proportional to a small parameter ϵ. That is,

$$P_t(|\mathbf{k}|, |\mathbf{q}|, \mathbf{k}\cdot\mathbf{q}) = \frac{c}{k^3}\delta^3(\mathbf{k}+\mathbf{q}) + \epsilon P_t'(|\mathbf{k}|, |\mathbf{q}|, \mathbf{k}\cdot\mathbf{q}) \tag{2.23}$$

If ϵ is small then the effects of the violation of translational invariance in Eq. (2.23) is small even when P_t' is not strongly peaked about $\mathbf{k} = -\mathbf{q}$.

In the next section we discuss how the violation of translational invariance during the inflationary era by the presence of a special point at fixed comoving coordinate impacts the anisotropy of the microwave background. Then in section IV we generalize the results of this section to the possibility that the violation of translation invariance during the inflationary era occurs because of a special line or plane during the inflationary era.

2.3 Microwave Background Anisotropy with a Special Point

We are interested in a quantitative understanding of how the second term in Eq. (2.12) changes the prediction for the microwave background asymmetry from the conventional translationally invariant one. The multipole moments of the microwave background radiation are defined by

$$a_{lm} = \int d\Omega_{\mathbf{e}} Y_l^m(\mathbf{e}) \frac{\Delta T}{T}(\mathbf{e}). \tag{2.24}$$

(Note that our definition differs from the conventional one[1] in which the complex conjugate of Y_l^m appears in the integral.) Since the violation of translational invariance vanishes after the inflationary era ends, the anisotropy of the microwave background temperature T along the direction of the unit vector $\mathbf{e}$ is related to the primordial

[1]To shift our results to what the usual definition gives, $a_{lm} \to a_{lm}^*$.

fluctuations by

$$\frac{\Delta T}{T}(\mathbf{e}) = \int d^3k \sum_l \left(\frac{2l+1}{4\pi}\right) (-i)^l P_l(\hat{\mathbf{k}} \cdot \mathbf{e}) \tilde{\delta}(\mathbf{k}) \Theta_l(k), \tag{2.25}$$

where P_l is the Legendre polynomial of order l and $\Theta_l(k)$ is a known real function of the magnitude of the wave vector $\mathbf{k}$ that includes, for example, the effects of the transfer function.

We are interested in computing $\langle a_{lm} a^*_{l'm'} \rangle$ to first order in the small correction that violates translational invariance. This is related to the two-point function in momentum space via

$$\langle a_{lm} a^*_{l'm'} \rangle = (-i)^{l-l'} \int d^3k d^3q \; Y_l^m(\hat{\mathbf{k}}) Y_{l'}^{m'^*}(\hat{\mathbf{q}}) \Theta_l(k) \Theta_{l'}(q) \langle \tilde{\delta}(\mathbf{k}) \tilde{\delta}^*(\mathbf{q}) \rangle. \tag{2.26}$$

From Eq. (2.12) to Eq.(2.14), we have

$$\begin{aligned} \langle \delta(\mathbf{x}) \delta(\mathbf{y}) \rangle = \int d^3l \; e^{i\mathbf{l}\cdot(\mathbf{x}-\mathbf{y})} P_0(l) + \frac{(\mathbf{x}+\mathbf{y}-2\mathbf{z}_*)^2}{4} \int d^3l \; e^{i\mathbf{l}\cdot(\mathbf{x}-\mathbf{y})} P_1(l) \\ + \int d^3l \; e^{i\mathbf{l}\cdot(\mathbf{x}-\mathbf{y})} P_2(l) \frac{[\mathbf{l}\cdot(\mathbf{x}+\mathbf{y}-\mathbf{2z}_*)]^2}{4l^2} \end{aligned} \tag{2.27}$$

where

$$P_1(l) = -\frac{P_0(l)}{2A(l)} \tag{2.28}$$

$$P_2(l) = \frac{B(l)}{2A(l)\,[A(l)+B(l)]} P_0(l) \tag{2.29}$$

The models in Section II had $P_{1,2}(l)$ proportional to $P_0(l)$. The special point $\mathbf{z}_*$ is characterized by three parameters; the magnitude of its distance from our location and two parameters for its direction (with respect to our location). Hence the corrections to the correlations $\langle a_{lm} a^*_{l'm'} \rangle$ are characterized by just five parameters. The Fourier transform of Eq. (2.27) yields

$$\langle \tilde{\delta}(\mathbf{k}) \tilde{\delta}^*(\mathbf{q}) \rangle = \int \frac{d^3x}{(2\pi)^3} \int \frac{d^3y}{(2\pi)^3} \; e^{-i\mathbf{k}\cdot\mathbf{x}} e^{i\mathbf{q}\cdot\mathbf{y}} \; \langle \delta(\mathbf{x}) \delta(\mathbf{y}) \rangle$$

$$= P_0(k)\delta^3(\mathbf{k}-\mathbf{q}) + \frac{(i\nabla_{\mathbf{k}} - i\nabla_{\mathbf{q}} - 2\mathbf{z}_*)^2}{4} P_1(k)\delta^3(\mathbf{k}-\mathbf{q})$$
$$+ \sum_{i,j=1}^{3} \frac{1}{4}\left(i\frac{\partial}{\partial k_i} - i\frac{\partial}{\partial q_i} - 2z_*^i\right)\left(i\frac{\partial}{\partial k_j} - i\frac{\partial}{\partial q_j} - 2z_*^j\right) \times$$
$$\left[P_2(k)\frac{k_i k_j}{k^2}\delta^3(\mathbf{k}-\mathbf{q})\right]. \qquad (2.30)$$

We therefore define

$$\langle a_{lm} a^*_{l'm'}\rangle = \langle a_{lm} a^*_{l'm'}\rangle_0 + (-i)^{l-l'}\Delta_1(l,m;l',m') + (-i)^{l-l'}\Delta_2(l,m;l',m'), \qquad (2.31)$$

where the subscript 0 denotes the usual translationally invariant piece,

$$\langle a_{lm} a^*_{l'm'}\rangle_0 = \delta_{l,l'}\delta_{m,m'} \int_0^\infty \mathrm{d}k k^2 P_0(k)\Theta_l(k)^2. \qquad (2.32)$$

and the correction coming from $P_1(k)$ is given by

$$\begin{aligned}
\Delta_1(l,m;l',m') \;=\; & \frac{1}{4}\int d^3k P_1(k)\left[-Y_l^m(\hat{\mathbf{k}})\Theta_l(k)\boldsymbol{\nabla}_{\boldsymbol{k}}{}^2\left(Y_{l'}^{m'*}(\hat{\mathbf{k}})\Theta_{l'}(k)\right)\right. \\
& -Y_{l'}^{m'*}(\hat{\mathbf{k}})\Theta_{l'}(k)\boldsymbol{\nabla}_{\boldsymbol{k}}{}^2\left(Y_l^m(\hat{\mathbf{k}})\Theta_l(k)\right) \\
& +2\boldsymbol{\nabla}_{\boldsymbol{k}}\left(Y_l^m(\hat{\mathbf{k}})\Theta_l(k)\right)\cdot\boldsymbol{\nabla}_{\boldsymbol{k}}\left(Y_{l'}^{m'*}(\hat{\mathbf{k}})\Theta_{l'}(k)\right) \\
& +4z_*^2\; Y_l^m(\hat{\mathbf{k}})Y_{l'}^{m'*}(\hat{\mathbf{k}})\Theta_l(k)\Theta_{l'}(k) \\
& +4iY_{l'}^{m'*}(\hat{\mathbf{k}})\Theta_{l'}(k)\;\mathbf{z}_*\cdot\boldsymbol{\nabla}_{\boldsymbol{k}}\left(Y_l^m(\hat{\mathbf{k}})\Theta_l(k)\right) \\
& \left.-4iY_l^m(\hat{\mathbf{k}})\Theta_l(k)\;\mathbf{z}_*\cdot\boldsymbol{\nabla}_{\boldsymbol{k}}\left(Y_{l'}^{m'*}(\hat{\mathbf{k}})\Theta_{l'}(k)\right)\right].
\end{aligned} \qquad (2.33)$$

It is convenient to break up $\Delta_1(l,m;l',m')$ into the parts quadratic in $\mathbf{z}_*$, linear in $\mathbf{z}_*$, and independent of $\mathbf{z}_*$, by writing

$$\Delta_1(l,m;l',m') = \Delta_1^{(2)}(l,m;l',m') + \Delta_1^{(1)}(l,m;l',m') + \Delta_1^{(0)}(l,m;l',m'). \qquad (2.34)$$

The quadratic piece is relatively simple,

$$\Delta_1^{(2)}(l,m;l',m') = \delta_{l,l'}\delta_{m,m'}\; z_*^2 \int_0^\infty \mathrm{d}k k^2 \Theta_l(k)^2 P_1(k). \qquad (2.35)$$

The term linear in $\mathbf{z}_*$ is the most complicated. It can be evaluated using the identity

$$i\boldsymbol{\nabla}_{\boldsymbol{k}}(\Theta_l(k)Y_l^m(\hat{\mathbf{k}})) = i\hat{\mathbf{k}}\left(\frac{\partial\Theta_l(k)}{\partial k}\right)Y_l^m(\hat{\mathbf{k}}) + \frac{1}{k}\hat{\mathbf{k}}\times(\mathbf{L}_{\mathbf{k}}Y_l^m(\hat{\mathbf{k}}))\Theta_l(k), \tag{2.36}$$

where $\mathbf{L}_{\mathbf{k}}$ acts as the angular momentum operator in Fourier space,

$$\mathbf{L}_{\mathbf{k}} = -i\mathbf{k}\times\boldsymbol{\nabla}_{\boldsymbol{k}}. \tag{2.37}$$

It is convenient to divide $\Delta_1^{(1)}(l,m;l',m')$ into a piece coming from the first term in Eq. (2.36) and a term coming from the second term in Eq. (2.36),

$$\Delta_1^{(1)}(l,m;l',m') = \Delta_1^{(1)}(l,m;l',m')_a + \Delta_1^{(1)}(l,m;l',m')_b. \tag{2.38}$$

To evaluate $\Delta_1^{(1)}(l,m;l',m')_{a,b}$, we express the components of $\mathbf{z}_*$ in terms of its "spherical components,"

$$z_+ = -\frac{z_{*1} - iz_{*2}}{\sqrt{2}}, \qquad z_- = \frac{z_{*1} + iz_{*2}}{\sqrt{2}}, \qquad z_0 = z_{*3}, \tag{2.39}$$

and express the components $\hat{\mathbf{k}}$ in terms of the spherical harmonics $Y_1^m(\hat{\mathbf{k}})$. This gives

$$\begin{aligned}\Delta_1^{(1)}(l,m;l',m')_a = i\int_0^\infty \mathrm{d}k\ k^2 P_1(k)\left(\Theta_{l'}(k)\frac{\partial\Theta_l(k)}{\partial k} - \Theta_l(k)\frac{\partial\Theta_{l'}(k)}{\partial k}\right)\times \\ \left(z_+\chi^{(a)+}_{lm;l'm'} + z_-\chi^{(a)-}_{lm;l'm'} + z_0\chi^{(a)0}_{lm;l'm'}\right),\end{aligned} \tag{2.40}$$

where

$$\begin{aligned}\chi^{(a)0}_{l,m;l',m'} = \left[\frac{(l-m+1)(l+m+1)}{(2l+1)(2l+3)}\right]^{1/2}\delta_{l+1,l'}\delta_{m,m'} \\ + \left[\frac{(l-m)(l+m)}{(2l-1)(2l+1)}\right]^{1/2}\delta_{l-1,l'}\delta_{m,m'},\end{aligned} \tag{2.41}$$

$$\chi^{(a)+}_{l,m;l',m'} = \frac{1}{\sqrt{2}}\left[\frac{(l+m+1)(l+m+2)}{(2l+1)(2l+3)}\right]^{1/2}\delta_{l+1,l'}\delta_{m+1,m'}$$
$$-\frac{1}{\sqrt{2}}\left[\frac{(l-m)(l-m-1)}{(2l-1)(2l+1)}\right]^{1/2}\delta_{l-1,l'}\delta_{m+1,m'}, \qquad (2.42)$$

and

$$\chi^{(a)-}_{l,m;l',m'} = \chi^{(a)+}_{l,-m;l',-m'}. \qquad (2.43)$$

For $\Delta_1^{(1)}(l,m;l',m')_b$ we write

$$\Delta_1^{(1)}(l,m;l',m')_b = \Delta_1^{(1)'}(l,m;l',m')_b + \Delta_1^{(1)'}(l',m';l,m)_b{}^*, \qquad (2.44)$$

and find that

$$\Delta_1^{(1)'}(l,m;l',m')_b = -i\int_0^\infty \mathrm{d}k\, kP_1(k)\Theta_l(k)\Theta_{l'}(k)\left(z_+\chi^{(b)+}_{lm;l'm'} + z_-\chi^{(b)-}_{lm;l'm'} + z_0\chi^{(b)0}_{lm;l'm'}\right), \qquad (2.45)$$

where

$$\chi^{(b)0}_{l,m;l',m'} = l\left[\frac{(l-m+1)(l+m+1)}{(2l+1)(2l+3)}\right]^{1/2}\delta_{l+1,l'}\delta_{m,m'}$$
$$-(l+1)\left[\frac{(l-m)(l+m)}{(2l-1)(2l+1)}\right]^{1/2}\delta_{l-1,l'}\delta_{m,m'}, \qquad (2.46)$$

$$\chi^{(b)+}_{l,m;l',m'} = \frac{l}{\sqrt{2}}\left[\frac{(l+m+1)(l+m+2)}{(2l+1)(2l+3)}\right]^{1/2}\delta_{l+1,l'}\delta_{m+1,m'}$$
$$+\frac{l+1}{\sqrt{2}}\left[\frac{(l-m)(l-m-1)}{(2l-1)(2l+1)}\right]^{1/2}\delta_{l-1,l'}\delta_{m+1,m'}, \qquad (2.47)$$

and

$$\chi^{(b)-}_{l,m;l',m'} = \chi^{(b)+}_{l,-m;l',-m'}. \qquad (2.48)$$

Then we evaluate the term independent of $\mathbf{z}_*$ in $\Delta_1(l,m;l',m')$. Using integration by parts, we know

$$
\begin{aligned}
&\int d^3k P_1(k) \boldsymbol{\nabla}_{\boldsymbol{k}} \left(Y_l^m(\hat{\mathbf{k}})\Theta_l(k)\right) \cdot \boldsymbol{\nabla}_{\boldsymbol{k}} \left(Y_{l'}^{m'*}(\hat{\mathbf{k}})\Theta_{l'}(k)\right) \\
= &\int d^3k \left[-Y_{l'}^{m'*}(\hat{\mathbf{k}})\Theta_{l'}(k) \frac{\partial P_1(k)}{\partial k} \hat{\mathbf{k}} \cdot \boldsymbol{\nabla}_{\boldsymbol{k}} \left(Y_l^m(\hat{\mathbf{k}})\Theta_l(k)\right) \right. \\
&\left. \qquad - P_1(k) Y_{l'}^{m'*}(\hat{\mathbf{k}})\Theta_{l'}(k) \boldsymbol{\nabla}_{\boldsymbol{k}}{}^2 \left(Y_l^m(\hat{\mathbf{k}})\Theta_l(k)\right) \right]
\end{aligned}
\tag{2.49}
$$

Another familiar result of spherical harmonics is

$$
-\nabla_{\mathbf{k}}^2 Y_l^m(\hat{\mathbf{k}})\Theta_l(k) = \left[-\frac{1}{k^2}\frac{\partial}{\partial k}\left(k^2 \frac{\partial \Theta_l(k)}{\partial k}\right) + \frac{l(l+1)}{k^2}\Theta_l(k)\right] Y_l^m(\hat{\mathbf{k}}). \tag{2.50}
$$

Combining Eq. (2.36), (2.49), and (2.50) implies that,

$$
\begin{aligned}
\Delta_1^{(0)}(l,m;l',m') = \delta_{l,l'}\delta_{m,m'} \int_0^\infty \mathrm{d}k &\left[-P_1(k)\Theta_l(k) \frac{\partial}{\partial k}\left(k^2 \frac{\partial \Theta_l(k)}{\partial k}\right) \right. \\
&\left. + l(l+1) P_1(k)\Theta_l(k)^2 - \frac{1}{2} k^2 \frac{\partial P_1(k)}{\partial k}\frac{\partial \Theta_l(k)}{\partial k}\Theta_l(k) \right]
\end{aligned}
\tag{2.51}
$$

The next step is to calculate the correction coming from $P_2(k)$.

$$
\begin{aligned}
\Delta_2(l,m;l',m') = \frac{1}{4}\int d^3k P_2(k) &\left[4(\hat{\mathbf{k}}\cdot\mathbf{z}_*)^2 \; Y_l^m(\hat{\mathbf{k}}) Y_{l'}^{m'*}(\hat{\mathbf{k}})\Theta_l(k)\Theta_{l'}(k) \right. \\
&+ 4i\left(\hat{\mathbf{k}}\cdot\mathbf{z}_*\right)\left(Y_{l'}^{m'*}(\hat{\mathbf{k}})\Theta_{l'}(k) \; \hat{\mathbf{k}}\cdot\boldsymbol{\nabla}_{\boldsymbol{k}}\left(Y_l^m(\hat{\mathbf{k}})\Theta_l(k)\right) - Y_l^m(\hat{\mathbf{k}})\Theta_l(k) \; \hat{\mathbf{k}}\cdot\boldsymbol{\nabla}_{\boldsymbol{k}}\left(Y_{l'}^{m'*}(\hat{\mathbf{k}})\Theta_{l'}(k)\right)\right) \\
&- \sum_{i,j=1}^{3} \frac{k_i k_j}{k^2}\left(Y_l^m(\hat{\mathbf{k}})\Theta_l(k) \frac{\partial}{\partial k_i}\frac{\partial}{\partial k_j}\left(Y_{l'}^{m'*}(\hat{\mathbf{k}})\Theta_{l'}(k)\right) + Y_{l'}^{m'*}(\hat{\mathbf{k}})\Theta_{l'}(k)\frac{\partial}{\partial k_i}\frac{\partial}{\partial k_j}\left(Y_l^m(\hat{\mathbf{k}})\Theta_l(k)\right)\right) \\
&\left. + 2\; \left(\hat{\mathbf{k}}\cdot\boldsymbol{\nabla}_{\boldsymbol{k}}\left(Y_l^m(\hat{\mathbf{k}})\Theta_l(k)\right)\right)\left(\hat{\mathbf{k}}\cdot\boldsymbol{\nabla}_{\boldsymbol{k}}\left(Y_{l'}^{m'*}(\hat{\mathbf{k}})\Theta_{l'}(k)\right)\right) \right].
\end{aligned}
\tag{2.52}
$$

We also break $\Delta_2(l,m;l',m')$ into terms quadratic in $\mathbf{z}_*$, linear and containing no factors of $\mathbf{z}_*$.

$$
\Delta_2(l,m;l',m') = \Delta_2^{(2)}(l,m;l',m') + \Delta_2^{(1)}(l,m;l',m') + \Delta_2^{(0)}(l,m;l',m') \tag{2.53}
$$

The term quadratic in $\mathbf{z}_*$ can be written as

$$\Delta_2^{(2)}(l,m;l',m') = \xi_{lm;l'm'} \int_0^\infty \mathrm{d}k k^2 P_2(k)\Theta_l(k)\Theta_{l'}(k) \tag{2.54}$$

where

$$\xi_{lm;l'm'} = \int \mathrm{d}\Omega_{\mathbf{k}} (\hat{\mathbf{k}} \cdot \mathbf{z}_*)^2 Y_l^m(\hat{\mathbf{k}}) Y_{l'}^{m'*}(\hat{\mathbf{k}}), \tag{2.55}$$

For the computation of $\xi_{l,m;l'm'}$, we use the "spherical" components of $\mathbf{z}_*$ in Eq. (2.39). $\xi_{lm;l'm'}$ was calculated in [17] where violation of rotational invariance was considered. It is convenient to decompose $\xi_{lm;l'm'}$ into coefficients of the quadratic quantities $z_i z_j$, via

$$\xi_{lm;l'm'} = z_+^2 \xi^{++}_{lm;l'm'} + z_-^2 \xi^{--}_{lm;l'm'} + 2z_+z_- \xi^{+-}_{lm;l'm'} + 2z_+z_0 \xi^{+0}_{lm;l'm'} + 2z_-z_0 \xi^{-0}_{lm;l'm'} + z_0^2 \xi^{00}_{lm;l'm'}. \tag{2.56}$$

ACW [17] found that

$$\begin{aligned}
\xi^{++}_{lm;l'm'} &= -\delta_{m',m+2}\Bigg[\delta_{l',l}\frac{\sqrt{(l^2-(m+1)^2)(l+m+2)(l-m)}}{(2l+3)(2l-1)} \\
&\quad -\frac{1}{2}\delta_{l',l+2}\sqrt{\frac{(l+m+1)(l+m+2)(l+m+3)(l+m+4)}{(2l+1)(2l+3)^2(2l+5)}} \\
&\quad -\frac{1}{2}\delta_{l',l-2}\sqrt{\frac{(l-m)(l-m-1)(l-m-2)(l-m-3)}{(2l+1)(2l-1)^2(2l-3)}}\Bigg], \\
\xi^{--}_{lm;l'm'} &= \xi^{++}_{l'm';lm}, \\
\xi^{+-}_{lm;l'm'} &= \frac{1}{2}\delta_{m',m}\Bigg[-2\,\delta_{l',l}\frac{(-1+l+l^2+m^2)}{(2l-1)(2l+3)} + \delta_{l',l+2}\sqrt{\frac{((l+1)^2-m^2)((l+2)^2-m^2)}{(2l+1)(2l+3)^2(2l+5)}} \\
&\quad + \delta_{l',l-2}\sqrt{\frac{(l^2-m^2)((l-1)^2-m^2)}{(2l-3)(2l-1)^2(2l+1)}}\Bigg], \\
\xi^{+0}_{lm;l'm'} &= \frac{\delta_{m',m+1}}{\sqrt{2}}\Bigg[\delta_{l',l}\frac{(2m+1)\sqrt{(l+m+1)(l-m)}}{(2l-1)(2l+3)} \\
&\quad + \delta_{l',l+2}\sqrt{\frac{((l+1)^2-m^2)(l+m+2)(l+m+3)}{(2l+1)(2l+3)^2(2l+5)}}
\end{aligned}$$

$$
\begin{aligned}
& \qquad -\delta_{l',l-2}\sqrt{\frac{(l^2-m^2)(l-m-1)(l-m-2)}{(2l-3)(2l-1)^2(2l+1)}}\Bigg], \\
\xi^{-0}_{lm;l'm'} &= -\xi^{+0}_{l'm';lm}, \\
\xi^{00}_{lm;l'm'} &= \delta_{m,m'}\left[\delta_{l,l'}\frac{(2l^2+2l-2m^2-1)}{(2l-1)(2l+3)}+\delta_{l',l+2}\sqrt{\frac{((l+1)^2-m^2)((l+2)^2-m^2)}{(2l+1)(2l+3)^2(2l+5)}}\right. \\
& \qquad \left.+\delta_{l',l-2}\sqrt{\frac{(l^2-m^2)((l-1)^2-m^2)}{(2l-3)(2l-1)^2(2l+1))}}\right]. \qquad (2.57)
\end{aligned}
$$

The term linear in $\mathbf{z}_*$ has already been evaluated before.

$$
\Delta_2^{(1)}(l,m;l',m') = i\int_0^\infty \mathrm{d}k\ k^2 P_2(k)\left(\Theta_{l'}(k)\frac{\partial\Theta_l(k)}{\partial k}-\Theta_l(k)\frac{\partial\Theta_{l'}(k)}{\partial k}\right)\times \\
\left(z_+\chi^{(a)+}_{lm;l'm'}+z_-\chi^{(a)-}_{lm;l'm'}+z_0\chi^{(a)0}_{lm;l'm'}\right) \qquad (2.58)
$$

where all $\chi^{(a)}$'s are given from Eq. (2.41) to (2.43).

The term independent of $\mathbf{z}_*$ can be evaluated using the identity

$$
\begin{aligned}
& \sum_{i,j=1}^{3}\frac{k_ik_j}{k^2}Y_l^m(\hat{\mathbf{k}})\Theta_l(k)\frac{\partial}{\partial k_i}\frac{\partial}{\partial k_j}\left(Y_{l'}^{m'*}(\hat{\mathbf{k}})\Theta_{l'}(k)\right) \\
&= \sum_{i,j=1}^{3}\frac{k_i}{k}Y_l^m(\hat{\mathbf{k}})\Theta_l(k)\frac{\partial}{\partial k_i}\left(\frac{k_j}{k}\frac{\partial}{\partial k_j}\left(Y_{l'}^{m'*}(\hat{\mathbf{k}})\Theta_{l'}(k)\right)\right) \\
&= Y_l^m(\hat{\mathbf{k}})\Theta_l(k)\hat{\mathbf{k}}\cdot\boldsymbol{\nabla}_{\boldsymbol{k}}\left[\hat{\mathbf{k}}\cdot\boldsymbol{\nabla}_{\boldsymbol{k}}\left(Y_{l'}^{m'*}(\hat{\mathbf{k}})\Theta_{l'}(k)\right)\right] \qquad (2.59)
\end{aligned}
$$

From Eq. (2.36), we know that

$$
\hat{\mathbf{k}}\cdot\boldsymbol{\nabla}_{\boldsymbol{k}}\left(Y_{l'}^{m'*}(\hat{\mathbf{k}})\Theta_{l'}(k)\right) = \frac{\partial\Theta_{l'}(k)}{\partial k}Y_{l'}^{m'*}(\hat{\mathbf{k}}) \qquad (2.60)
$$

and

$$
\hat{\mathbf{k}}\cdot\boldsymbol{\nabla}_{\boldsymbol{k}}\left[\hat{\mathbf{k}}\cdot\boldsymbol{\nabla}_{\boldsymbol{k}}\left(Y_{l'}^{m'*}(\hat{\mathbf{k}})\Theta_{l'}(k)\right)\right] = \frac{\partial^2\Theta_{l'}(k)}{\partial k^2}Y_{l'}^{m'*}(\hat{\mathbf{k}}) \qquad (2.61)
$$

These give

$$
\Delta_2^{(0)}(l,m;l',m') = \frac{1}{2}\delta_{l,l'}\delta_{m,m'}\int_0^\infty \mathrm{d}k\ k^2P_2(k)\left[\left(\frac{\partial\Theta_l(k)}{\partial k}\right)^2-\Theta_l(k)\frac{\partial^2\Theta_l(k)}{\partial k^2}\right] \qquad (2.62)
$$

To recap: the modification of the correlations $\langle a_{lm}a^*_{l'm'}\rangle$ caused by the violation of translational invariance is defined by Eq. (2.31). It can be decomposed into two pieces, $\Delta_1(l,m,l',m')$ and $\Delta_2(l,m,l',m')$, and each can be expressed as three components depending on their dependence on $\mathbf{z}_*$ in Eq. (2.34) and (2.53). The quadratic piece in $\Delta_1(l,m,l',m')$ is given by (2.35), the $\mathbf{z}_*$-independent piece by (2.51), and the linear piece by (2.38), whose terms are given by (2.40-2.48). Meanwhile, the quadratic piece in $\Delta_2(l,m,l',m')$ is given by (2.54), the linear piece by (2.58), and the $\mathbf{z}_*$-independent piece by (2.62).

While these expressions appear formidable, the good news is that coefficients at multipole l are only correlated with those at $l-2 \leq l' \leq l+2$. The correlation matrix is sparse, making the analysis of CMB data computationally tractable [33].

2.4 Set up for A Special Line or Plane

In this section we extend the results obtained for the case of a preferred point in space to the cases where translational invariance is broken by a special line or point. Since many of the steps are similar to the special point case we will be brief.

To specify the location of a preferred line in space requires a point $\mathbf{z}_*$ and a unit tangent vector $\mathbf{n}$. (Note that we place Earth at the center of our coordinate system, so that the specification of any point defines a vector pointing from us to the point.) Since any point on the line will do, without loss of generality we can take $\mathbf{z}_*$ to be the point closest to us, implying the constraint $\mathbf{n}\cdot\mathbf{z}_* = 0$. This is illustrated by the diagram on the left in Figure 2.1.

In order to simplify the calculation, we first align the preferred direction with the z axis. In that case, the rotational invariance about the z axis and the translational invariance along this preferred direction are left unbroken. These symmetries imply that the most general form of the two-point correlation of energy density correlations is

$$\langle\tilde{\delta}(\mathbf{k})\tilde{\delta}(\mathbf{q})\rangle = \delta(k_z+q_z)e^{-i(\mathbf{k}_\perp+\mathbf{q}_\perp)\cdot\mathbf{z}_*}P_t(k_\perp,q_\perp,k_z,\mathbf{k}_\perp\cdot\mathbf{q}_\perp), \tag{2.63}$$

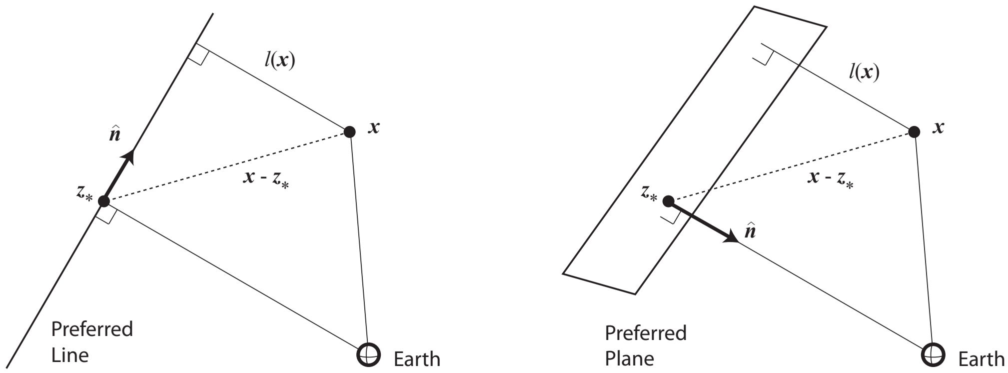

Figure 2.1: A preferred line in space can be specified by its closest point, $\mathbf{z}_*$, and a unit tangent vector $\hat{\mathbf{n}}$; a preferred plane can be specified by its closest point and a unit normal vector. The distance $l(\mathbf{x})$ to any point $\mathbf{x}$ in space is measured perpendicularly to the line or plane.

so that

$$\begin{aligned} \langle\delta(\mathbf{x})\delta(\mathbf{y})\rangle &= \int d^3k \int d^3q \; e^{i\mathbf{k}\cdot\mathbf{x}} e^{i\mathbf{q}\cdot\mathbf{y}} \langle\tilde{\delta}(\mathbf{k})\tilde{\delta}(\mathbf{q})\rangle \\ &= \int dk_z \int d^2k_\perp \int d^2q_\perp \; e^{ik_z(x_z-y_z)} e^{i\mathbf{k}_\perp\cdot(\mathbf{x}_\perp-\mathbf{z}_{*\perp})} e^{i\mathbf{q}_\perp\cdot(\mathbf{y}_\perp-\mathbf{z}_{*\perp})} \times \\ & \quad P_t(k_\perp, q_\perp, k_z, \mathbf{k}_\perp\cdot\mathbf{q}_\perp) \end{aligned} \tag{2.64}$$

with P_t symmetric under the interchange of $\mathbf{k}_\perp$ and $\mathbf{q}_\perp$. Here we have decomposed the position and wave vectors along the z axis and the two-dimensional subspace perpendicular to that which is denoted by a subscript $\perp$. In the limit that there is no violations of translational (and rotational) invariance, $P_t(k_\perp, q_\perp, k_z, \mathbf{k}_\perp\cdot\mathbf{q})$ reduces to $P(k)\delta^2(\mathbf{k}_\perp + \mathbf{q}_\perp)$, where $k = \sqrt{{\mathbf{k}_\perp}^2 + k_z^2}$. We now assume the violations of translational (and rotational) invariance are small and hence that P_t is strongly peaked about $\mathbf{k}_\perp = -\mathbf{q}_\perp$. We introduce the variables $\mathbf{p}_\perp = \mathbf{k}_\perp + \mathbf{q}_\perp$, $\mathbf{l}_\perp = (\mathbf{k}_\perp - \mathbf{q}_\perp)/2$ and follow the same steps in the point case. Then,

$$\begin{aligned} \langle\delta(\mathbf{x})\delta(\mathbf{y})\rangle = \int dk_z \int d^2l_\perp \; e^{ik_z(x_z-y_z)} e^{i\mathbf{l}_\perp\cdot(\mathbf{x}_\perp-\mathbf{y}_\perp)} P_t(l_\perp, l_\perp, k_z, -l_\perp^2) \cdot \\ \int d^2p_\perp \; e^{-A(l_\perp,k_z)p_\perp^2/2 - B(l_\perp,k_z)(\mathbf{p}_\perp\cdot\mathbf{l}_\perp)^2/(2l_\perp^2)} e^{i\mathbf{p}_\perp\cdot\mathbf{z}_\perp} \end{aligned} \tag{2.65}$$

where $\mathbf{z}_\perp = (\mathbf{x}_\perp + \mathbf{y}_\perp - 2\mathbf{z}_{*\perp})/2$. Performing the integral over $d^2p_\perp$, we find that,

$$\langle\delta(\mathbf{x})\delta(\mathbf{y})\rangle = \int dk_z \int d^2l_\perp\ e^{ik_z(x_z-y_z)} e^{i\mathbf{l}_\perp\cdot(\mathbf{x}_\perp-\mathbf{y}_\perp)} P_t(l_\perp, l_\perp, k_z, -l_\perp^2)\sqrt{\frac{(2\pi)^2}{\det C}}\times \left(1 - \frac{z_\perp^T C^{-1} z_\perp}{2} + \dots\right) \tag{2.66}$$

where $C_{ij} = A(l_\perp, k_z)\delta_{ij} + B(l_\perp, k_z)\frac{l_{\perp i} l_{\perp j}}{l_\perp^2}$ is a 2×2 matrix, $\det C = A^2 + AB$, and

$$C_{ij}^{-1} = \frac{1}{A}\delta_{ij} - \frac{B}{A(A+B)}\frac{l_{\perp i} l_{\perp j}}{l_\perp^2} \tag{2.67}$$

We can define

$$P(l_\perp, k_z) = \sqrt{\frac{(2\pi)^2}{\det C}} P_t(l_\perp, l_\perp, k_z - l_\perp^2) \tag{2.68}$$

and plug in the expression of C_{ij}^{-1} in terms of $A(l_\perp, k_z)$ and $B(l_\perp, k_z)$. This gives after relabeling, $\mathbf{l}_\perp \to \mathbf{k}_\perp$

$$\langle\delta(\mathbf{x})\delta(\mathbf{y})\rangle = \int d^3k\ e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} P(k_\perp, k_z)\left[1 - \frac{z_\perp^2}{2A} + \frac{B}{2A(A+B)}\frac{(\mathbf{k}_\perp\cdot\mathbf{z}_\perp)^2}{k_\perp^2}\right] \tag{2.69}$$

Note that we want the leading term in the expansion in z to correspond to the standard cosmology and hence $P(k_\perp, k_z) = P(k)$, where $k = \sqrt{k_\perp^2 + k_z^2}$. Finally, to make the preferred direction arbitrary, we replace all position vectors a_z with $\mathbf{n}\cdot\mathbf{a}$ and also replacing $\mathbf{a}_\perp$ with $\mathbf{a} - \mathbf{n}(\mathbf{n}\cdot\mathbf{a})$ in Eq. (2.69).

As in the special point case we note that another way to get a small violation of translational is if there is a small parameter ϵ and P_t takes the form,

$$P_t(k_\perp, q_\perp, k_z, \mathbf{k}_\perp\cdot\mathbf{q}_\perp) = \frac{c}{k^3}\delta(\mathbf{k}+\mathbf{q}) + \epsilon P_t'(k_\perp, q_\perp, k_z, \mathbf{k}_\perp\cdot\mathbf{q}_\perp) \tag{2.70}$$

where P_t' cannot be expanded in any simple way. This is what happened in Ref. ([63]).

A preferred plane can be specified by a point $\mathbf{z}_*$ and a unit normal vector $\mathbf{n}$. We can again choose $\mathbf{z}_*$ to be the point on the plane closest to us, implying a constraint $\mathbf{n}\times\mathbf{z}_* = 0$, as shown on the right-hand side of Figure 2.1. Notice that the rotational invariance about the **n** axis and the translational invariance along the **n** direction are unbroken. These symmetries imply

$$\langle\tilde{\delta}(\mathbf{k})\tilde{\delta}(\mathbf{q})\rangle = \delta^2(\mathbf{k}_\parallel + \mathbf{q}_\parallel)e^{-i(k_n+q_n)z_{*n}}P_t(k_\parallel, k_n, q_n) \tag{2.71}$$

so that

$$\begin{aligned}\langle\delta(\mathbf{x})\delta(\mathbf{y})\rangle &= \int d^3k \int d^3q \; e^{i\mathbf{k}\cdot\mathbf{x}}e^{i\mathbf{q}\cdot\mathbf{y}}\langle\tilde{\delta}(\mathbf{k})\tilde{\delta}(\mathbf{q})\rangle \\ &= \int d^2k_\parallel \int dk_n \int dq_n \; e^{i\mathbf{k}_\parallel\cdot(\mathbf{x}_\parallel - \mathbf{y}_\parallel)}e^{ik_n(x_n - z_{*n})}e^{iq_n(y_n - z_{*n})} \times \\ &\quad P_t(k_\parallel, k_n, q_n)\end{aligned} \tag{2.72}$$

Here we have decomposed the position and wave vectors along the normal vector **n** and the two-dimensional subspace parallel to the plane which is denoted by a subscript $\|$. Then we change variables $p_n = k_n + q_n$, $l_n = (k_n - q_n)/2$ and perform the integral over dp_n to get

$$\langle\delta(\mathbf{x})\delta(\mathbf{y})\rangle = \int d^2k_\parallel \int dl_n \; e^{i\mathbf{k}_\parallel\cdot(\mathbf{x}_\parallel - \mathbf{y}_\parallel)}e^{il_n(x_n - y_n)}P_t(k_\parallel, l_n, l_n)\sqrt{\frac{2\pi}{A}}\left(1 - \frac{z_n^2}{2A} + \dots\right) \tag{2.73}$$

After relabeling $l_n \to k_n$ and defining

$$P(k_\parallel, l_n) = \sqrt{\frac{2\pi}{A}}P_t(k_\parallel, l_n, l_n) \tag{2.74}$$

we have

$$\langle\delta(\mathbf{x})\delta(\mathbf{y})\rangle = \int d^3k \; e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}P(k_\parallel, k_n)\left[1 - \frac{z_n^2}{2A}\right] \tag{2.75}$$

Finally, for the reason that we want the leading-order term to correspond to the standard cosmology, we replace $P(k_\parallel, k_n)$ with $P(k)$, where $k = \sqrt{k_\parallel^2 + k_n^2}$.

2.5 Conclusion

We have investigated the observational consequences of a small violation of translational invariance on the temperature anisotropies in the cosmic microwave background. Three cases were investigated, based on the assumption of a preferred point, line, or plane in space, and a quadratic dependence on the distance to the preferred locus of points. Explicit formula were presented for the correlations $\langle a_{lm} a^*_{l'm'} \rangle$ between spherical harmonic coefficients of the CMB temperature field in the case of a special point. The expressions we have derived may be used to directly compare CMB observations against the hypothesis of perfect translational invariance during the inflationary era, as part of a systematic framework for constraining deviations from the standard paradigm of primordial perturbations. Explicit expressions for the correlations $\langle a_{lm} a^*_{l'm'} \rangle$ can also be derived for the special line and plane cases.

One can also test the hypothesis of perfect translational invariance during the inflationary era using data on the large-scale distribution of galaxies and clusters of galaxies, using, in the special point case,

$$\langle \delta(\mathbf{x})\delta(\mathbf{y}) \rangle = \int \frac{d^3l}{(2\pi)^3} e^{i\mathbf{l}\cdot(\mathbf{x}-\mathbf{y})} P_0(l) + \frac{(\mathbf{x}+\mathbf{y}-2\mathbf{z}_*)^2}{4} \int \frac{d^3l}{(2\pi)^3} e^{i\mathbf{l}\cdot(\mathbf{x}-\mathbf{y})} P_1(l) + \int \frac{d^3l}{(2\pi)^3} e^{i\mathbf{l}\cdot(\mathbf{x}-\mathbf{y})} P_2(l) \frac{[\mathbf{l}\cdot(\mathbf{x}+\mathbf{y}-\mathbf{2z}_*)]^2}{4l^2}. \quad (2.76)$$

The work in Section II suggests that $P_1(k)$ and $P_2(k)$ are proportional to $P_0(k)$ and so the corrections to the microwave background anisotropy and the large-scale distribution of galaxies are characterized by five parameters, two are these constants of proportionality and three are the parameters to specify the special point including the direction and the magnitude of $\mathbf{z}_*$.

Chapter 3

Inflaton Two-Point Correlation in the Presence of a Cosmic String

3.1 Introduction

The inflationary cosmology is the standard paradigm for explaining the horizon problem [1, 2]. In its simplest form inflation predicts an almost scale-invariant spectrum of approximately Gaussian density perturbations [3, 4]. Rotational and translational invariance dictate that the two-point correlation of the Fourier transform of the primordial density perturbations $\delta(\mathbf{k})$ has the form,

$$\langle \delta(\mathbf{k})\delta(\mathbf{q})\rangle = P(k)(2\pi)^3\delta(\mathbf{k}+\mathbf{q}), \tag{3.1}$$

where $k = |\mathbf{k}|$ and P is called the power spectrum. In Eq. (3.1) the fact that P only depends on the magnitude of the wave-vector $\mathbf{k}$ is a consequence of rotational invariance and the delta function of $\mathbf{k}+\mathbf{q}$ arises from translational invariance. Let $\chi(\mathbf{k})$ be the Fourier transform of a massless scalar field with canonical normalization. Its two-point correlation in de-Sitter space is

$$\langle \chi(\mathbf{k})\chi(\mathbf{q})\rangle = P_\chi(k)(2\pi)^3\delta(\mathbf{k}+\mathbf{q}), \tag{3.2}$$

where H is the Hubble constant during inflation and

$$P_\chi(k) = \frac{H^2}{2k^3}. \tag{3.3}$$

In the inflationary cosmology the almost scale-invariant density perturbations that are probed by the microwave background anisotropy and the large-scale structure of our observed universe have a power spectrum that differs from $P_\chi(k)$ normalization factor that has weak k dependence[1] and a transfer function that arises from the growth of fluctuations at late times after they reenter the horizon [8, 9, 10, 11, 12, 13, 14, 15, 16].

Inflation occurs at an early time when the energy density of the universe is large compared to energy scales that can be probed by laboratory experiments. It is possible that there are paradigm shifts in our understanding of the laws of nature, as radical as the shift from classical physics to quantum physics, that are needed to understand physics at the energy scale associated with the inflationary era. Motivated by the lack of direct probes of physics at the inflationary scale Ackerman et. al. wrote down the general form that Eq. (3.1) would take [17] if rotational invariance was broken by a small amount during the inflationary era (but not today) by a preferred direction and computed its impact on the microwave background anisotropy (see also [18, 19, 20, 21, 23, 24, 64]). They also wrote down a simple field theory model that realizes this form for the density perturbations where the preferred direction is associated with spontaneous breaking of rotational invariance by the expectation value of a vector field. This model serves as a nice pedagogical example, however, it cannot be realistic because of instabilities [31]. Evidence in the WMAP data for the violation of rotational evidence was found in Ref. [32, 33, 35]. Another anomaly in the data on the anisotropy of the microwave background data is the "hemisphere effect" [57, 59]. This cannot be explained by the model of Ackerman et. al. Erickeck et. al. proposed an explanation based on the presence of a very long wavelength (superhorizon) perturbation [61]. This long wavelength mode picks out a preferred wave-number and can give rise to a hemisphere effect. It violates translational invariance and there

[1] We will treat this factor as a constant and denote it by κ^2.

are very strong constraints from the observed large-scale structure of the universe on this [65, 66, 67]. The generation of large-scale temperature fluctuations in the microwave background temperature by superhorizon perturbations is known as the Grishchuk-Zel'dovich effect [68].

Carroll et. al. proposed explicit forms for violations of translational invariance [69], in the energy density perturbation two-point correlation, motivated by: the symmetries that are left unbroken, the desire to have a prediction for the two-point correlation of multipole moments of the microwave background anisotropy $\langle a_{lm} a^*_{l'm'} \rangle$ that is non-zero for at most a few l's that are different from l', and the desire to introduce at most a few new parameters. To get a feeling for what can happen in general consider a case where there is a special point $\mathbf{x}_0$ during inflation. Its presence violates translational invariance, however translational invariance is restored if in addition to translating the spatial coordinates we also translate $\mathbf{x}_0$. So in coordinate space $\langle \delta(\mathbf{x})\delta(\mathbf{y}) \rangle$ must be a function of $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{x}_0$ that is invariant under translations $\mathbf{x} \to \mathbf{x} + \mathbf{a}$, $\mathbf{y} \to \mathbf{y} + \mathbf{a}$, $\mathbf{x}_0 \to \mathbf{x}_0 + \mathbf{a}$ and rotations $\mathbf{x} \to R\mathbf{x}$, $\mathbf{y} \to R\mathbf{y}$, $\mathbf{x}_0 \to R\mathbf{x}_0$. Furthermore it must be symmetric under interchange of $\mathbf{x}$ and $\mathbf{y}$. Ref. [69] assumed $\langle \delta(\mathbf{x})\delta(\mathbf{y}) \rangle$ only depends on the two variables, $(\mathbf{x} - \mathbf{x}_0)^2 + (\mathbf{y} - \mathbf{x}_0)^2$and $|\mathbf{x} - \mathbf{y}|$, and expanded in the dependence the first of these. However in the general case of a special point $\mathbf{x}_0$ Eq. (3.1) becomes

$$\langle \delta(\mathbf{k})\delta(\mathbf{q}) \rangle = e^{i(\mathbf{k}+\mathbf{q})\cdot\mathbf{x}_0} P(k, q, \mathbf{k} \cdot \mathbf{q}), \tag{3.4}$$

where P is symmetric under interchange of $\mathbf{k}$ and $\mathbf{q}$. Without further simplifying assumptions about the form of P and the value of $\mathbf{x}_0$ this will result in a very complicated matrix[2]$\langle a_{lm} a^*_{l'm'} \rangle$.

In this chapter we explore the form of the two-point correlation $\langle \delta(\mathbf{k})\delta(\mathbf{q}) \rangle$ if translational invariance is broken by the presence of cosmic string that passes through our horizon volume during inflation. We will assume that the string becomes unstable and disappears near the end of inflation and approximate the string as infinitely long

[2] l, m label the rows and l', m' the columns.

and having infinitesimal thickness. In that case rotational invariance about the string axis and translational invariance along the string direction are left unbroken. Aligning the preferred direction with the z axis these symmetries imply that the two-point correlation of energy density correlations takes the form,

$$\begin{aligned}\langle\delta(\mathbf{k})\delta(\mathbf{q})\rangle =&(2\pi)\delta(k_z+q_z)e^{i(\mathbf{k}_\perp+\mathbf{q}_\perp)\cdot\mathbf{x}_0}\\ &P(k_\perp,q_\perp,k_z,\mathbf{k}_\perp\cdot\mathbf{q}_\perp),\end{aligned}\tag{3.5}$$

with P symmetric under interchange of $\mathbf{k}_\perp$ and $\mathbf{q}_\perp$. Here we have decomposed the wave vectors along the z axis and the two-dimensional subspace perpendicular to that is denoted by a subscript $\perp$. $\mathbf{x}_0$ is a point on the string. If the preferred direction is along an arbitrary direction $\hat{\mathbf{n}} = R\hat{\mathbf{z}}$, where R is a rotation that leaves the point $\mathbf{x}_0$ fixed, then on the right-hand side of Eq. (3.5) the wave vectors are replaced by the rotated ones; $\mathbf{k} \to R\mathbf{k}$ and $\mathbf{q} \to R\mathbf{q}$. The goal of this paper is to derive an explicit expression for the function $P(k_\perp, q_\perp, k_z, \mathbf{k}_\perp \cdot \mathbf{q}_\perp)$.

Using cylindrical spatial coordinates the metric for the inflationary spacetime with an infinitely long infinitesimally thin straight string directed along z direction and passing through the origin is [70]

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2\left[\mathrm{d}\rho^2 + \rho^2(1-4G\mu)^2\mathrm{d}\theta^2 + \mathrm{d}z^2\right],\tag{3.6}$$

where $a(t) = e^{Ht}$ is just the ordinary inflationary scale factor and μ is the tension along the string. We compute the Fourier transform of the two-point correlation of χ in this space-time. This is a simplified model for inflation where χ plays the role of the inflaton and $\delta(\mathbf{k}) \propto \chi(\mathbf{k})$. We focus on the cosmic string case because of the simplicity of the metric and not because of a strong physical motivation. Unless there is " just enough inflation" it is very unlikely that there would be a cosmic string in our horizon volume during inflation. If there was just enough inflation [71, 72, 73] there could be other sources of violations of translational and rotational invariance [74, 75, 76, 77, 78, 79]. However, the cosmic string case does provides a simple physical model where the form of the violation of translational and rotational invariance can be explicitly calculated and it depends on only the parameter $G\mu$ and four other parameters that specify the location and alignment of the string. Real cosmic strings have a thickness of order $1/\sqrt{\mu}$ and so for it to be treated as thin we need $H^2 << \mu$ which implies that the dimensionless quantity $\epsilon = G\mu$ is much greater than, GH^2.

It is also possible to violate translational invariance by a point defect that existed during the inflationary era. In the conclusions we briefly discuss how the cosmic string case differs from the case of a black hole located in our horizon volume during the inflationary era [80].

3.2 The Two-Point Correlation Function of a Massless Scalar

The metric for an inflationary spacetime with an infinitely long string along z direction and through the origin is taken of the form [70]

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2 \left[\mathrm{d}\rho^2 + \rho^2(1-4G\mu)^2\mathrm{d}\theta^2 + \mathrm{d}z^2\right], \tag{3.7}$$

where $a(t) = e^{Ht}$ is just the ordinary inflationary scale factor. We let $\alpha = 1 - 4G\mu$. In these coordinates the Lagrangian density for a massless scalar field is

$$\begin{aligned} \mathcal{L}_\chi &= -\frac{\sqrt{-g}}{2} g^{\mu\nu}\partial_\mu\chi\partial_\nu\chi \\ &= \frac{a(t)^3}{2}\rho\alpha\left(\frac{\partial\chi}{\partial t}\right)^2 - \frac{a(t)}{2}\rho\alpha\left(\frac{\partial\chi}{\partial z}\right)^2 - \frac{a(t)}{2}\rho\alpha\left(\frac{\partial\chi}{\partial \rho}\right)^2 - \frac{a(t)}{2\rho\alpha}\left(\frac{\partial\chi}{\partial \theta}\right)^2 . \end{aligned} \tag{3.8}$$

The Hamiltonian[3] is,

$$H = \int \mathrm{d}^3x(\pi\dot{\chi} - \mathcal{L})$$

[3]The same symbol is used for the Hamiltonian and Hubble constant during inflation, however the meaning of the symbol should be clear from the context.

$$= \int \rho \mathrm{d}\rho \mathrm{d}\theta \mathrm{d}z \, \frac{\alpha}{2} \left[a(t)^3 \left(\frac{\partial \chi}{\partial t} \right)^2 + a(t) \left(\frac{\partial \chi}{\partial z} \right)^2 \right.$$
$$\left. + a(t) \left(\frac{\partial \chi}{\partial \rho} \right)^2 + \frac{a(t)}{\rho^2 \alpha^2} \left(\frac{\partial \chi}{\partial \theta} \right)^2 \right]. \tag{3.9}$$

It is convenient to introduce the conformal time,

$$\tau = -\frac{1}{H} e^{-Ht} \tag{3.10}$$

and as t goes from $-\infty$ to ∞ the conformal time τ goes from $-\infty$ to 0. Since the metric only differs from de Sitter space by the presence of a conical singularity at $\rho = 0$ the (equal time) two-point correlation of χ can easily be shown to be,

$$\langle \chi(\rho, \theta, z, \tau) \chi(\rho', \theta', z', \tau) \rangle = \int_0^\infty \frac{\mathrm{d}k_\perp}{2\pi} k_\perp \int_{-\infty}^{\infty} \frac{\mathrm{d}k_z}{2\pi} e^{ik_z(z-z')} \times$$
$$\sum_{m=-\infty}^{\infty} \frac{e^{im(\theta-\theta')}}{2\pi} J_{|m/\alpha|}(k_\perp \rho) J_{|m/\alpha|}(k_\perp \rho') \frac{|\chi_k(\tau)|^2}{\alpha}. \tag{3.11}$$

Here $\chi_k(\tau)$ are the usual mode functions in de Sitter space,

$$\chi_k(\tau) = \frac{H}{\sqrt{2k}} e^{-ik\tau} \left(\tau - \frac{i}{k} \right). \tag{3.12}$$

We are interested in the late time, $k\tau \to 0$ behavior. Using the explicit form of $\chi_k(\tau)$ above,

$$\langle \chi(\rho, \theta, z, 0) \chi(\rho', \theta', z', 0) \rangle = \frac{H^2}{2\alpha} \int_0^\infty \frac{\mathrm{d}k_\perp}{2\pi} k_\perp \int_{-\infty}^{\infty} \frac{\mathrm{d}k_z}{2\pi} e^{ik_z(z-z')} \times$$
$$\sum_{m=-\infty}^{\infty} \frac{e^{im(\theta-\theta')}}{2\pi} \frac{J_{|m/\alpha|}(k_\perp \rho) J_{|m/\alpha|}(k_\perp \rho')}{(k_\perp^2 + k_z^2)^{3/2}}. \tag{3.13}$$

The observed universe is consistent with the standard predictions of the inflationary cosmology. Therefore the violation of translational invariance due to the string is a small perturbation, and is parametrized by the small quantity $\epsilon = 4G\mu$. There are two approaches to calculate the Fourier transform of the two-point correlation of χ.

One is to expand the Bessel functions in Eq. (3.13) about $\epsilon = 0$ and then change to Cartesian coordinates. Another approach, which is the one we take, is to abandon the exact result in Eq. (3.13) and just do quantum mechanical perturbation theory about the unperturbed, $\epsilon = 0$ background, *i.e.*, de Sitter space.

In the standard inflationary cosmology with one field, the inflaton, perturbations in the gauge invariant quantity, that reduces to the density perturbations for modes with wavelengths well within the horizon, are calculated from the two-point function of a massless scalar field. Precisely how this field is related to the gravitational and scalar degrees of freedom depends on the choice of gauge. One can work in a gauge where the scalar field has no fluctuations and then the massless field lives in the gravitational degrees of freedom. (See [82] for a calculation in this gauge.). For a more conventional approach where fluctuations in the inflaton field itself are computed, see for example, [83]. We assume a similar calculation holds in the case we are discussing so approach the problem by computing the fluctuations in a massless scalar field χ and take, $\delta = \kappa\chi$. We need to compute the two-point correlation function $\langle\chi(\mathbf{x}, t)\chi(\mathbf{y}, t)\rangle$. Treating ϵ as a small perturbation and using the "in-in" formalism, to first order of ϵ, (see Ref. [84]).

$$\langle\chi(\mathbf{x}, t)\chi(\mathbf{y}, t)\rangle \simeq \langle\chi_I(\mathbf{x}, t)\chi_I(\mathbf{y}, t)\rangle + i\int_{-\infty}^{t} \mathrm{d}t' e^{-\epsilon'|t'|} \langle[H_I(t'), \chi_I(\mathbf{x}, t)\chi_I(\mathbf{y}, t)]\rangle \,, \tag{3.14}$$

where ϵ' is an infinitesimal parameter that cuts off the early time part of the integration. In this case the interaction-picture Hamiltonian $H_I(t)$ is given by

$$\begin{aligned} H_I &= \int \rho \mathrm{d}\rho \mathrm{d}\theta \mathrm{d}z \left(-\frac{\epsilon}{2}\right) \left[a^3 \left(\frac{\partial\chi_I}{\partial t}\right)^2 + a\left(\frac{\partial\chi_I}{\partial z}\right)^2 + a\left(\frac{\partial\chi_I}{\partial\rho}\right)^2 - \frac{a}{\rho^2}\left(\frac{\partial\chi_I}{\partial\theta}\right)^2\right] \\ &= \int \rho \mathrm{d}\rho \mathrm{d}\theta \mathrm{d}z \left(-\frac{\epsilon}{2}\right) \left[a^3 \left(\frac{\partial\chi_I}{\partial t}\right)^2 + a\left(\frac{\partial\chi_I}{\partial z}\right)^2 + a\left(\frac{\partial\chi_I}{\partial\rho}\right)^2 + \frac{a}{\rho^2}\left(\frac{\partial\chi_I}{\partial\theta}\right)^2 - \frac{2a}{\rho^2}\left(\frac{\partial\chi_I}{\partial\theta}\right)^2\right] \\ &= -\epsilon H_0 + \epsilon\int \rho \mathrm{d}\rho \mathrm{d}\theta dz \frac{a}{\rho^2}\left(\frac{\partial\chi_I}{\partial\theta}\right)^2 , \end{aligned} \tag{3.15}$$

where the interaction picture field χ_I has its time evolution governed by the unperturbed Hamiltonian,

$$H_0 = \int \rho \mathrm{d}\rho \mathrm{d}\theta \mathrm{d}z \frac{1}{2}\left[a^3\left(\frac{\partial \chi_I}{\partial t}\right)^2 + a\left(\frac{\partial \chi_I}{\partial z}\right)^2 + a\left(\frac{\partial \chi_I}{\partial \rho}\right)^2 + \frac{a}{\rho^2}\left(\frac{\partial \chi_I}{\partial \theta}\right)^2\right]. \tag{3.16}$$

Because we are interested in the effects that violate rotational and/or translational invariance in, $\Delta\langle\chi(\mathbf{x},t)\chi(\mathbf{y},t)\rangle = \langle\chi(\mathbf{x},t)\chi(\mathbf{y},t)\rangle - \langle\chi_I(\mathbf{x},t)\chi_I(\mathbf{y},t)\rangle$, we will drop the term proportional to H_0 in the interaction Hamiltonian leaving us with,

$$H_I = \epsilon \int \rho \mathrm{d}\rho \mathrm{d}\theta \mathrm{d}z \frac{a}{\rho^2}\left(\frac{\partial \chi_I}{\partial \theta}\right)^2, \tag{3.17}$$

to first order in ϵ. The free field obeys the unperturbed equation of motion,

$$\frac{d^2\chi_I}{\mathrm{d}t^2} + 3H\frac{\mathrm{d}\chi_I}{\mathrm{d}t} - \frac{1}{a(t)^2}\frac{\mathrm{d}^2\chi_I}{\mathrm{d}\mathbf{x}^2} = 0. \tag{3.18}$$

Upon quantization, χ_I becomes a quantum operator

$$\begin{aligned}\chi_I(\mathbf{x},\tau) &= \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}}\left[\chi_k(\tau)\beta(\mathbf{k}) + \chi_k^*(\tau)\beta^\dagger(-\mathbf{k})\right] \\ &= \int \frac{d^3k}{(2\pi)^3} e^{ik_z z} e^{ik_\perp \rho\cos(\theta-\theta_k)}\left[\chi_k(\tau)\beta(\mathbf{k}) + \chi_k^*(\tau)\beta^\dagger(-\mathbf{k})\right]\end{aligned} \tag{3.19}$$

where $\chi_k(\tau)$ is given by Eq. (3.12). Note that we have converted to the conformal time $\tau = -e^{-Ht}/H$ and used cylindrical coordinates for $\mathbf{k}$ and $\mathbf{x}$ in the exponential. $\beta(\mathbf{k})$ annihilates the vacuum state and satisfies the usual commutation relations, $[\beta(\mathbf{k}), \beta^\dagger(\mathbf{q})] = (2\pi)^3\delta(\mathbf{k}-\mathbf{q})$. Combining these definitions that interaction Hamiltonian can be written in terms of creation and annihilation operators as,

$$\begin{aligned}H_I(\tau') = \epsilon\left(\frac{1}{H\tau'}\right)\int \frac{d^3k}{(2\pi)^3}\int \frac{d^3q}{(2\pi)^3}\int d^3x'\ e^{i\mathbf{k}\cdot\mathbf{x}'+i\mathbf{q}\cdot\mathbf{x}'}\frac{(y'k_x - x'k_y)(y'q_x - x'q_y)}{x'^2+y'^2} \\ \times\left[\chi_k(\tau')\beta(\mathbf{k}) + \chi_k^*(\tau')\beta^\dagger(-\mathbf{k})\right]\left[\chi_q(\tau')\beta(\mathbf{q}) + \chi_q^*(\tau')\beta^\dagger(-\mathbf{q})\right]\end{aligned} \tag{3.20}$$

Next we use the above results to compute the needed commutator,

$$\langle[H_I(\tau'),\chi_I(\mathbf{x},\tau)\chi_I(\mathbf{y},\tau)]\rangle = \langle[H_I(\tau'),\chi_I(\mathbf{x},\tau)]\,\chi_I(\mathbf{y},\tau)\rangle+\langle\chi_I(\mathbf{x},\tau)\,[H_I(\tau'),\chi_I(\mathbf{y},\tau)]\rangle\,. \tag{3.21}$$

This gives,

$$\begin{aligned}\langle[H_I(\tau'),\chi_I(\mathbf{x},\tau)\chi_I(\mathbf{y},\tau)]\rangle &= \frac{H^3\epsilon}{2\tau'}\int\frac{d^3k}{(2\pi)^3}\int\frac{d^3q}{(2\pi)^3}\times\\ &\int d^3x'\ e^{i\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x})+i\mathbf{q}\cdot(\mathbf{x}'-\mathbf{y})}\frac{(y'k_x-x'k_y)(y'q_x-x'q_y)}{(x'^2+y'^2)}\times\\ &\frac{1}{kq}\left[e^{-i(k+q)(\tau'-\tau)}\left(\tau'-\frac{i}{k}\right)\left(\tau'-\frac{i}{q}\right)\left(\tau+\frac{i}{k}\right)\left(\tau+\frac{i}{q}\right)-\text{h.c.}\right].\end{aligned} \tag{3.22}$$

Converting the integration over t' in Eq. (3.14) to the integration over the conformal time τ' $\left(\mathrm{d}t'=-\dfrac{\mathrm{d}\tau'}{H\tau'}\right)$, using Eq. (3.22), and noting that cutoff involving ϵ' removes the influence at the very early time, we integrate over τ' to get

$$\begin{aligned}\Delta\langle\chi(\mathbf{x},\tau)\chi(\mathbf{y},\tau)\rangle &= -H^2\epsilon\int\frac{d^3k}{(2\pi)^3}\int\frac{d^3q}{(2\pi)^3}\int d^3x'\ e^{i\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x})+i\mathbf{q}\cdot(\mathbf{x}'-\mathbf{y})}\times\\ &\qquad\frac{(y'k_x-x'k_y)(y'q_x-x'q_y)}{x'^2+y'^2}\left[\frac{kq+q^2+k^2+k^2q^2\tau^2}{k^3q^3(k+q)}\right]\end{aligned} \tag{3.23}$$

Using,

$$\frac{(y'k_x-x'k_y)(y'q_x-x'q_y)}{x'^2+y'^2}=\mathbf{k}_\perp\cdot\mathbf{q}_\perp-\frac{(\mathbf{x}'_\perp\cdot\mathbf{k}_\perp)(\mathbf{x}'_\perp\cdot\mathbf{q}_\perp)}{{x'_\perp}^2}, \tag{3.24}$$

gives

$$\begin{aligned}\Delta\langle\chi(\mathbf{x},\tau)\chi(\mathbf{y},\tau)\rangle &= -H^2\epsilon\int\frac{d^3k}{(2\pi)^3}\int\frac{d^3q}{(2\pi)^3}e^{-i\mathbf{k}\cdot\mathbf{x}-i\mathbf{q}\cdot\mathbf{y}}2\pi\delta(k_z+q_z)\times\\ &\left[\frac{kq+q^2+k^2+k^2q^2\tau^2}{k^3q^3(k+q)}\right]\times\\ &\left[\int d^2x'_\perp\ e^{i(\mathbf{k}_\perp+\mathbf{q}_\perp)\cdot\mathbf{x}'_\perp}\left(\mathbf{k}_\perp\cdot\mathbf{q}_\perp-\frac{(\mathbf{x}'_\perp\cdot\mathbf{k}_\perp)(\mathbf{x}'_\perp\cdot\mathbf{q}_\perp)}{{x'_\perp}^2}\right)\right],\end{aligned} \tag{3.25}$$

where $x'_\perp=|\mathbf{x}'_\perp|$. It remains to perform the integration over x'. We find that,

$$\int\mathrm{d}^2x'_\perp e^{i\mathbf{p}_\perp\cdot\mathbf{x}'_\perp}\frac{x'_{\perp i}x'_{\perp j}}{{x'_\perp}^2}=(2\pi)^2\delta(\mathbf{p}_\perp)\frac{\delta_{ij}}{2}+\frac{4\pi}{{p_\perp}^2}\left(\frac{\delta_{ij}}{2}-\frac{p_{\perp i}p_{\perp j}}{{p_\perp}^2}\right), \tag{3.26}$$

and so

$$\Delta\langle\chi(\mathbf{x},\tau)\chi(\mathbf{y},\tau)\rangle = -H^2\epsilon\int\frac{d^3k}{(2\pi)^3}\int\frac{d^3q}{(2\pi)^3}e^{-i\mathbf{k}\cdot\mathbf{x}-i\mathbf{q}\cdot\mathbf{y}}2\pi\delta(k_z+q_z)\frac{kq+q^2+k^2+k^2q^2\tau^2}{k^3q^3(k+q)}$$
$$\left[\frac{\mathbf{k}_\perp\cdot\mathbf{q}_\perp}{2}(2\pi)^2\delta(\mathbf{k}_\perp+\mathbf{q}_\perp)-\frac{4\pi}{(\mathbf{k}_\perp+\mathbf{q}_\perp)^2}\left(\frac{\mathbf{k}_\perp\cdot\mathbf{q}_\perp}{2}-\frac{\mathbf{k}_\perp\cdot(\mathbf{k}_\perp+\mathbf{q}_\perp)\mathbf{q}_\perp\cdot(\mathbf{k}_\perp+\mathbf{q}_\perp)}{(\mathbf{k}_\perp+\mathbf{q}_\perp)^2}\right)\right] \tag{3.27}$$

The second term in the large square brackets of Eq. (3.27) appears naively to give rise to a logarithmic divergence in the integrations over q and k near $\mathbf{p}_\perp = \mathbf{k}_\perp + \mathbf{q}_\perp = 0$. However after doing the angular integration over the direction of $\mathbf{p}_\perp$ this potentially divergent term vanishes.

Writing the density perturbations as $\delta = \kappa\chi$ we arrive at the following expression for the part of $P(k_\perp, q_\perp, k_z, \mathbf{k}_\perp\cdot\mathbf{q}_\perp)$ in Eq. (3.5) that violates rotational and/or translational invariance,

$$\Delta P(k_\perp, q_\perp, k_z, \mathbf{k}_\perp\cdot\mathbf{q}_\perp) = -\kappa^2H^2\epsilon\left(\frac{kq+q^2+k^2}{k^3q^3(k+q)}\right)\left[\frac{\mathbf{k}_\perp\cdot\mathbf{q}_\perp}{2}(2\pi)^2\delta(\mathbf{k}_\perp+\mathbf{q}_\perp)\right.$$
$$-\frac{4\pi}{(\mathbf{k}_\perp+\mathbf{q}_\perp)^2}\left(\frac{\mathbf{k}_\perp\cdot\mathbf{q}_\perp}{2}\right.$$
$$\left.\left.-\frac{\mathbf{k}_\perp\cdot(\mathbf{k}_\perp+\mathbf{q}_\perp)\mathbf{q}_\perp\cdot(\mathbf{k}_\perp+\mathbf{q}_\perp)}{(\mathbf{k}_\perp+\mathbf{q}_\perp)^2}\right)\right]. \tag{3.28}$$

In Eq. (3.28) $k = \sqrt{k_z^2+k_\perp^2}$ and $q = \sqrt{k_z^2+q_\perp^2}$. Eq. (3.28) is the main result of this paper. Notice that the dependence on the wave-vectors is scale invariant, which is mainly due to the assumption of massless inflaton. However, since the actual inflaton potential $V(\chi)$ steepens towards the end of inflation, there will be a scale-dependent spectral tilt on cosmologically observable scales. Furthermore, the scale invariance is also broken by the dependence on $\mathbf{x}_0$ that arises when one considers a string that does not pass through the origin. The first term in the large square brackets of Eq. (3.28) violates rotational invariance but not translational invariance. It is consistent with the form proposed by Ackerman et. al. [17]. The second term in the large square brackets of Eq. (3.28) violates translational invariance.

Recall that in the model we have adopted the unperturbed density perturbations have a power spectrum $P_0(k) = \kappa^2 H^2/(2k^3)$ and so ϵ characterizes the overall strength of the violations of rotational and translational invariance. For $(\mathbf{k}_\perp + \mathbf{q}_\perp) \cdot \mathbf{x}_0 \gg 1$ the exponential dependence on this quantity in Eq. (3.5) oscillates rapidly and this suppresses the impact of ΔP on observable quantities which depend on integrals of $\langle \delta(\mathbf{k}) \delta(\mathbf{q}) \rangle$ over the components of $\mathbf{q}$ and $\mathbf{k}$. Over some range of k these oscillations may be observable. For a discussion of density perturbations that are modulated by an oscillating term see [85].

3.3 Conclusion

We have computed (with some simplifying assumptions) the impact that an infinitely long and infinitesimally thin straight string that exists during inflation and passes through our horizon volume has on the perturbations of the energy density of the universe. We have assumed that the string disappears towards the end of the inflationary era. It may be possible to remove some of these assumptions or provide dynamics that realizes them. However, even without that, Eq. (3.28) provides a simple and physically motivated functional form (after modifying it so the string can have an arbitrary location and orientation) for the part of the density perturbations two-point correlation function that violates translational and rotational invariance. It can be compared with data on the large-scale structure of the universe and the anisotropy of the microwave background radiation.

Computations analogous to those performed in this paper can be done for a point defect (located at the origin $x = 0$) in de Sitter space using the metric [86, 87],

$$\mathrm{d}s^2 = -\frac{\left(1 - \frac{r_0}{a(t)x}\right)^2}{\left(1 + \frac{r_0}{a(t)x}\right)^2}\mathrm{d}t^2 + a(t)^2 \left(1 + \frac{r_0}{a(t)x}\right)^4 (\mathrm{d}x^2 + x^2 \mathrm{d}\Omega_2^2), \tag{3.29}$$

where $a(t) = e^{Ht}$. In this case, perturbing about de Sitter space, yields the following interaction Hamiltonian for a massless scalar field χ,

$$H_I = 4r_0 \int \mathrm{d}^3x \left(\frac{1}{x}\right) a(t)^2 \left(\frac{\mathrm{d}\chi_I}{\mathrm{d}t}\right)^2 = 4r_0 \int \mathrm{d}^3x \left(\frac{1}{x}\right) \left(\frac{\mathrm{d}\chi_I}{\mathrm{d}\tau}\right)^2 . \tag{3.30}$$

For the point mass case the perturbative calculation of $\langle\chi(\mathbf{x}, t)\chi(\mathbf{y}, t)\rangle$ using Eq. (3.14) has greater sensitivity to earlier times t'. For example, the factor of $e^{-\epsilon'|t'|}$ does not regulate the t' integration. An exponential regulator in conformal time would work but then the Fourier transform of the part of this two-point function that violates translation invariance vanishes as $q\tau$ and $k\tau$ go to zero. This case was considered in Ref. [80].

Chapter 4

Dynamical Fine-tuning in Inflation

4.1 Introduction

Inflation cosmology has come to play a central role in our modern understanding of the universe[1]. One of the biggest advantage (or goal) of inflation is to make the evolution of our observable universe seem natural. However, it has been recognized for some time that there is tension between this goal and the underlying structure of classical mechanics[81]. Liouville's theorem states that a distribution function in the phase space remains constant along trajectories; roughly speaking, a certain number of states at one time always evolves into precisely the same number of states at any other time. Therefore, the information is conserved. This is in conflict with the philosophy of inflation. Inflation attempts to account for the apparent fine-tuning of our early universe by offering a mechanism by which a relatively natural early condition will robustly evolve into an apparently fine-tuned later condition. But if that evolution is unitary, it is impossible for any mechanism to evolve a large number of states into a smaller number. All statements above are well known, and certainly true. However, does it mean that no choice of early universe Hamiltonian can make the current universe more or less finely tuned? The answer is not obvious and requires more inspection to reach the conclusion. Let's start with a simple example. Assuming the Hamiltonian is $H = -p\,q$, we can easily get the equation of motion,

$$\dot{q} = -q, \qquad \dot{p} = p \tag{4.1}$$

This means that if we start with an arbitrary normalizable probability distribution and evolution under this Hamiltonian will cause the probability that $|q| < \epsilon$, for any $\epsilon > 0$, to approach 1 at late times. (This is also equivalent to saying that the shape of the distribution function becomes extremely skinny at late times.) By shifting the Hamiltonian, we could also cause q to approach any other number for sure. By making a canonical transformation on the Hamiltonian, we could also arrange the evolution to fix the final value of p instead of q, or if we prefer, $x + p$, etc. Therefore, we have a system obeying Liouville's theorem and it can cause an arbitrary normalizable initial state to evolve so that one of its two canonical coordinates becomes fine-tuned to arbitrary precision, with probability approaching 1 at late times, to a value that is determined entirely by the choice of the Hamiltonian. With this example, it seems fine-tuning can happen even though Liouville's theorem is satisfied. This is far from true because of the following statements.

- For any non-periodic system, we can always find a canonical transformation such that one of the new canonical coordinates approaches to zero with probability 1 as time goes to infinity.
- For any non-periodic system, we can always find a canonical transformation such that the shape of the distribution function in the new coordinate system remains fixed as time goes by.
- There is no coordinate-independent quantity which can change with time.

The first statement tells us the phenomenon that one of the canonical coordinates approaches to zero at late time cannot be called fine-tuning. It is just a fact of any Hamiltonian system which is not periodic, so this phenomenon is more like a choice of coordinates than fine-tuning. The idea of this is simple. For any system, we should not be surprised that we can find a combination of p and q such that it approaches to zero as time goes to infinity. All we have to do is to find this combination, and prove that it is canonical transformation. We will do this in section 4.2. The second statement strengthens our argument by saying that the shape of the distribution function cannot be used to describe any fine-tuning since it is only a consequence of the coordinate choice. The reason why the last statement is important is because if fine-tuning is a real physical quantity, we prefer it not depending on what coordinates we choose. In other words, we should be able to find a coordinate-independent quantity to describe it.

However, there are some special coordinates in physics, called observables and there is no reason that fine-tuning phenomenon should be independent of coordinates. In this case, we have already shown that inflation can provide solutions to fine-tuning problem without violating Liouville's theorem.

4.2 Proof of statements

Notice that we only consider non-periodic systems. It is obvious that fine-tuning cannot happen in any periodic system because all physical quantities would go back to their original values after one cycle which contradicts with the idea of dynamical fine-tuning. In the phase space diagram, non-periodicity also means that the path in phase space is not closed. We also know that the paths in phase space cannot cross, since if they did, a given point could evolve in more than one direction, violating the deterministic nature of classical mechanics. Therefore, any path in phase space can only be parameterized by one parameter t given an initial condition (q^*, p^*), and any points (q, p) on the path cannot have two different values of t. This is equivalent to saying that (q, p) is only a function of time t and this function is invertible. We can write it as $t = t(q, p)$. Our first goal is to prove that given any Hamiltonian $H(q, p)$[1], if this system is not periodic, we can find a canonical transformation such that new canonical coordinate Q approaches to zero as time goes to infinity. Let

$$Q = e^{-t(q,p)}, \qquad P = H(q,p)e^{t(q,p)} \tag{4.2}$$

[1]We assume the Hamiltonian does not depend on time explicitly.

It is not difficult to get

$$\{Q, P\}_{q,p} = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} = \frac{\partial H}{\partial p}\frac{\partial t}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial t}{\partial p} \tag{4.3}$$

We also know that

$$1 = \frac{dt(p(t), q(t))}{dt} = \frac{\partial t}{\partial q}\dot{q} + \frac{\partial t}{\partial p}\dot{p} = \frac{\partial t}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial t}{\partial p}\frac{\partial H}{\partial q} \tag{4.4}$$

The last equality comes from the Hamiltonian equation of motion. Combining Eq. (4.3) and Eq. (4.4), (Q, P) is a set of canonical coordinates and Eq. (4.2) is a canonical transformation. The Hamiltonian can be written as $H = -QP$, and therefore Q has the desired property. This completes the proof of the first statement.

The proof of the second statement is very similar to the first one, so we will be brief. Let

$$Q = t(q, p), \qquad P = H(q, p) \tag{4.5}$$

we have

$$\{Q, P\}_{q,p} = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} = \frac{\partial H}{\partial p}\frac{\partial t}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial t}{\partial p} = 1 \tag{4.6}$$

Therefore, Eq. (4.5) is a canonical transformation, and the Hamiltonian can be written as $H = P$. The equation of motions in terms of new canonical coordinates are

$$\dot{Q} = 1, \qquad \dot{P} = 0 \tag{4.7}$$

which means the distribution function in phase space only makes translations during evolution so the shape of the distribution function remains unchanged.

In order to prove the last statement, we first review what we have from the geometry of the phase space. The state of a classical system is described by a point in a phase space with canonical coordinates (q, p). The classical equation of motion are

Hamilton's equations,

$$\dot{q} = \frac{\partial H}{\partial p}, \qquad \dot{p} = -\frac{\partial H}{\partial q} \tag{4.8}$$

Equivalently, evolution is governed by a Hamiltonian phase flow with tangent vector

$$\vec{v} = \frac{\partial H}{\partial p}\hat{e}_q - \frac{\partial H}{\partial q}\hat{e}_p \tag{4.9}$$

Phase space is a symplectic manifold, which means it naturally comes equipped with a symplectic form, which is a closed 2-form:

$$\omega = dp \wedge dq, \qquad d\omega = 0 \tag{4.10}$$

The existence of the symplectic form provides us with a naturally-defined measure on phase space, so there is a nature way to define integration over regions of phase space. Given any distribution function $f(q,p,t)$, there are only two scalars we can form on this manifold. One is $\vec{v}\cdot\vec{\nabla}f$, and the other is f itself. From Eq. (4.9),

$$\vec{v}\cdot\vec{\nabla}f = \frac{\partial H}{\partial p}\frac{\partial f}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial f}{\partial p} = \{f,H\}_{q,p} \tag{4.11}$$

Also from Liouville's theorem, we have

$$0 = \frac{df}{dt} = \frac{\partial f}{\partial t} + \{f,H\}_{q,p} \tag{4.12}$$

which gives

$$\vec{v}\cdot\vec{\nabla}f = -\frac{\partial f}{\partial t} \tag{4.13}$$

Another trivial result is

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{df}{dt}\right) &= \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial t} + \frac{\partial H}{\partial p}\frac{\partial f}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial f}{\partial p}\right)\\ &= \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial t}\right) + \frac{\partial}{\partial q}\left(\frac{\partial f}{\partial t}\right)\frac{\partial H}{\partial p} - \frac{\partial}{\partial p}\left(\frac{\partial f}{\partial t}\right)\frac{\partial H}{\partial q}\end{aligned}$$

$$= \frac{d}{dt}\left(\frac{\partial f}{\partial t}\right) \tag{4.14}$$

if the Hamiltonian is not dependent on time explicitly. With these, it is straightforward to get $\vec{v} \cdot \vec{\nabla} f$ is also independent of time. If we claim that there is a real fine-tuning in this system, we should be able to find a quantity $T(t)$ independent of the choice of coordinates such that $T(t) \to 0$ as $t \to \infty$. The most general form of $T(t)$ can be written as

$$T(t) = \iint K\left(f(q,p,t),\ \vec{v} \cdot \vec{\nabla} f(q,p,t)\right) dqdp \tag{4.15}$$

where $K(x, y)$ is an arbitrary analytical function of two variables. Then the hand-waving argument gives us

$$\frac{dT}{dt} = \iint \left[\frac{\partial K}{\partial x}\frac{df}{dt} + \frac{\partial K}{\partial y}\frac{d\left(\vec{v} \cdot \nabla f\right)}{dt}\right] dqdp = 0 \tag{4.16}$$

4.3 Conclusion

We investigate the implication of Liouville's theorem more carefully and find that no early universe Hamiltonian can make the current universe more or less finely tuned under the assumption that all fine-tuning can be described in a coordinate-independent way. However, there are still some preferred coordinates in physics, e.g., observables. If the description of fine-tuning phenomenon does not have to be independent of coordinates, inflation can provide solutions and reconcile with Liouville's theorem at the same time.

Chapter 5

Decoherence Problem in Ekpyrotic Phase

5.1 Introduction

From cosmological observations we know that the current universe is to a good approximation flat, homogeneous and isotropic on large scales [8, 16]. It is well known that in standard Big Bang cosmology this requires an enormous amount of fine-tuning on the initial conditions. Two mechanisms are provided to be possible explanations. The first is inflation [1, 2], a period of accelerated expansion occurring between the Big Bang and nucleosynthesis. The second is ekpyrosis [88, 89, 90, 91, 92], a period of ultra-slow contraction before Big Bang/Big Crunch to an expanding phase. Both mechanisms not only manage to address the standard cosmological puzzles but also have the ability to imprint scale-invariant inhomogeneities on superhorizon scales via a causal mechanism [1, 88, 94, 95, 96, 97, 93]. These inhomogeneities are thought to provide the seeds which later become the temperature anisotropies in the cosmic microwave background and the large-scale structure in the universe. This framework of the cosmological perturbation theory is based on the quantum mechanics of scalar fields, where the relevant observable is the amplitude of the field's Fourier modes [4]. Although they originates as quantum mechanical variables, these amplitudes eventually imprint classical stochastic fluctuations on the density field, characterized by the power spectrum. This interpretation proves to be very accurate in the CMB and large-cale structure analyses.

However, in order to make this stochastic interpretation consistent, the density matrix has to be diagonal in the amplitude basis. This criterion implies that interference terms in the density matrix are highly suppressed and can be neglected [99, 100]. Interference is associated with the coherence of the system, i.e., the coherence in the state between different points of configuration space [101, 102]. A measure of this is the coherence length which gives the configuration distance over which off-diagonal terms are correlated [103].

An isolated system described by the Schrödinger equation cannot lose its coherence; a pure state always remains pure. However, if it is coarse grained, it may evolve from a pure to a mixed state. One way to realize coarse graining is to let the system interact with an environment [101]. The environment consists of all fields whose evolution we are not interested in. The state of the system is obtained by tracing over all possible states of the environment. Now, even if the state describing system plus environment is pure, the state of the system alone will in general be mixed.

In the literature, there are various arguments and calculations suggesting that a form of such environment decoherence can indeed occur for inflationary perturbations [103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113]. The coherence length decreases exponentially for wavelengths greater than Hubble radius. Thus perturbations become classical once their wavelength exceeds the Hubble radius. All of these results lend support to the usual heuristic derivation of the spectrum of density perturbations in inflationary models. In this chapter, we use a simple model to study whether decoherence can also occur in the ekpyrotic phase. We find that the coherence lengths continue increasing even for the modes outside the horizon. Finally, we strengthen our conclusion by considering a different kind of mechanism, quantum to semi-classical transition without decoherence[98]. We show that the result is the same. The quantum to classical transition would not happen during ekpyrosis. Therefore, the heuristic argument that the modes become classical when they leave the horizon is invalid in the ekpyrotic phase and requires more careful inspection.

5.2 The model

A crucial question is how to model the environment. Any realistic model will be very complicated and hard to analyze. However, the basic physics should emerge from the simplest models. Hence, we choose a model [103] which can be solved exactly: the system is a real massless scalar field ϕ_1, and the environment is taken to be a second massless real scalar field ϕ_2 interacting with ϕ_1 through their gradients.

The action of system and environment is

$$S = \int d^4x \mathcal{L} = \int d^4x \sqrt{-g}\, \frac{1}{2} \left(-\partial_\mu \phi_1 \partial^\mu \phi_1 - \partial_\mu \phi_2 \partial^\mu \phi_2 - 2c \partial_\mu \phi_1 \partial^\mu \phi_2\right) \tag{5.1}$$

where g is the determinant of the background metric which is given by

$$ds^2 = a^2(\eta)(-d\eta^2 + d\mathbf{x}^2) \tag{5.2}$$

and $c \ll 1$ is the coupling constant describing the interaction between two fields. Note that this Lagrangian is quadratic in the derivative of the fields and can hence be diagonalized for which the interaction term disappears and the whole Lagrangian becomes a free field theory. If there is no other field or interaction in our universe, this argument is true. However, we suppose there is a hidden interaction such that we can only obeserve the first field ϕ_1 but not the environment ϕ_2. In other words, we assume the environment and the observed system do not form the diagonal basis. This assumption is reasonable since any observed scalar fields (whose reduced density matrix we want) will interact with gravitational perturbations (which is a part of the environment).

Then, the canonical momenta π_i conjugate to the fields ϕ_i, $i = 1, 2$ are

$$\pi_1 = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_1} = a^2 \left(\dot{\phi}_1 + c\dot{\phi}_2\right) \tag{5.3}$$

$$\pi_2 = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_2} = a^2 \left(\dot{\phi}_2 + c\dot{\phi}_1\right) \tag{5.4}$$

where "·" denotes derivative with respect to η. This allows us to write the Hamiltonian

H as

$$
\begin{aligned}
H &= \int d^3x(\pi_i\dot{\phi}_i - \mathcal{L}) \\
&= \int d^3x \left\{ \frac{1}{2a^2(1-c^2)} \left(\pi_1^2 + \pi_2^2 - 2c\pi_1\pi_2\right) + \right. \\
&\left. \frac{a^2}{2}\left[(\nabla\phi_1)^2 + (\nabla\phi_2)^2 + 2c(\nabla\phi_1)\cdot(\nabla\phi_2)\right] \right\}
\end{aligned}
\tag{5.5}
$$

To study decoherence, it is more convenient to use the functional Schrödinger picture[114]. The commutation relation $[\phi_i(\mathbf{x}), \pi_j(\mathbf{y})] = i\delta_{ij}\delta^3(\mathbf{x}-\mathbf{y})$ is equivalent to making the replacement $\pi_i(\mathbf{x}) \to -i\frac{\delta}{\delta\phi_i(\mathbf{x})}$. The wave functional $\Psi[\phi_1, \phi_2]$ obeys the Schrödinger equation

$$
i\frac{\partial}{\partial\eta}\Psi = \hat{H}\Psi \tag{5.6}
$$

We make a Gaussian ansatz for Ψ to be able to find the vacuum or ground state solution:

$$
\begin{aligned}
\Psi[\phi_1, \phi_2] = \mathcal{N}\exp\left[-\frac{1}{2}\int d^3x d^3y \left(\phi_1(\mathbf{x})\phi_1(\mathbf{y}) + \phi_2(\mathbf{x})\phi_2(\mathbf{y})\right) A(\mathbf{x},\mathbf{y},\eta) + \right. \\
\left. 2\phi_1(\mathbf{x})\phi_2(\mathbf{y})B(\mathbf{x},\mathbf{y},\eta)\right]
\end{aligned}
\tag{5.7}
$$

Note that we have already used the $\phi_1 \leftrightarrow \phi_2$ symmetry of the Lagrangian. Furthermore, because of the $x \leftrightarrow y$ symmetry of the above integration, we have to require

$$
A(\mathbf{x},\mathbf{y},\eta) = A(\mathbf{y},\mathbf{x},\eta) \tag{5.8}
$$

$$
B(\mathbf{x},\mathbf{y},\eta) = B(\mathbf{y},\mathbf{x},\eta) \tag{5.9}
$$

Plug Eq. (5.7) into Schrödinger equation (5.6), it is not difficult to get

$$
\begin{aligned}
\frac{i}{2}\frac{\partial A(\mathbf{x},\mathbf{y},\eta)}{\partial\eta} = \int d^3z \frac{1}{2a^2(1-c^2)}\left[A(\mathbf{x},\mathbf{z},\eta)A(\mathbf{y},\mathbf{z},\eta) + B(\mathbf{x},\mathbf{z},\eta)B(\mathbf{y},\mathbf{z},\eta)\right. \\
\left. -2cA(\mathbf{x},\mathbf{z},\eta)B(\mathbf{y},\mathbf{z},\eta)\right] + \frac{a^2}{2}\nabla_y^2\delta^3(\mathbf{x}-\mathbf{y})
\end{aligned}
\tag{5.10}
$$

$$\frac{i}{2}\frac{\partial A(\mathbf{x},\mathbf{y},\eta)}{\partial\eta} = \int d^3z \ \frac{1}{2a^2(1-c^2)} \left[B(\mathbf{x},\mathbf{z},\eta)B(\mathbf{y},\mathbf{z},\eta) + A(\mathbf{x},\mathbf{z},\eta)A(\mathbf{y},\mathbf{z},\eta) \right.$$
$$\left. -2cB(\mathbf{x},\mathbf{z},\eta)A(\mathbf{y},\mathbf{z},\eta)\right] + \frac{a^2}{2}\nabla_y^2\delta^3(\mathbf{x}-\mathbf{y}) \quad (5.11)$$

$$\frac{i}{2}\frac{\partial B(\mathbf{x},\mathbf{y},\eta)}{\partial\eta} = \int d^3z \ \frac{1}{2a^2(1-c^2)}\left[2A(\mathbf{x},\mathbf{z},\eta)B(\mathbf{y},\mathbf{z},\eta) + 2B(\mathbf{x},\mathbf{z},\eta)A(\mathbf{y},\mathbf{z},\eta) \right.$$
$$\left. -2cA(\mathbf{x},\mathbf{z},\eta)A(\mathbf{y},\mathbf{z},\eta) - 2cB(\mathbf{x},\mathbf{z},\eta)B(\mathbf{y},\mathbf{z},\eta)\right]$$
$$+\frac{a^2}{2}\cdot 2c\nabla_y^2\delta^3(\mathbf{x}-\mathbf{y}) \quad (5.12)$$

$$i\frac{\partial \ln\mathcal{N}}{\partial\eta} = \frac{1}{2a^2(1-c^2)}\int d^3z \left[2A(\mathbf{z},\mathbf{z},\eta) - 2B(\mathbf{z},\mathbf{z},\eta)\right] \quad (5.13)$$

All the above equations come from the comparison of the coefficients in front of $\phi_i(\mathbf{x})\phi_j(\mathbf{y})$. It is easy to see that Eq. (5.11) and Eq. (5.10) are equivalent, which is just the result of the symmetry of ϕ_1 and ϕ_2. In order to satisfy Eq. (5.10) - (5.12), we have to require $B(\mathbf{x},\mathbf{y},\eta) = cA(\mathbf{x},\mathbf{y},\eta)$, which gives

$$\Psi[\phi_1,\phi_2] = \mathcal{N}\exp\left\{-\frac{1}{2}\int d^3x d^3y \left[\phi_1(\mathbf{x})\phi_1(\mathbf{y}) + \phi_2(\mathbf{x})\phi_2(\mathbf{y}) + 2c\phi_1(\mathbf{x})\phi_2(\mathbf{y})\right] A(\mathbf{x},\mathbf{y},\eta)\right\} \quad (5.14)$$

$$i\frac{\partial \ln\mathcal{N}}{\partial\eta} = \frac{1}{a^2}\int d^3z A(\mathbf{z},\mathbf{z},\eta) \quad (5.15)$$
$$i\frac{\partial A(\mathbf{x},\mathbf{y},\eta)}{\partial\eta} = \frac{1}{a^2}\int d^3z A(\mathbf{x},\mathbf{z},\eta)A(\mathbf{y},\mathbf{z},\eta) + a^2\nabla_y^2\delta^3(\mathbf{x}-\mathbf{y}) \quad (5.16)$$

It is more convenient to solve Eq. (5.16) in momentum space. Upon writing

$$\phi_i(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3}\phi_i(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} \quad (5.17)$$
$$A(\mathbf{x},\mathbf{y},\eta) = \int \frac{d^3k}{(2\pi)^3}A(\mathbf{k},\eta)e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \quad (5.18)$$

we get

$$i\frac{\partial A(\mathbf{k},\eta)}{\partial\eta} = \frac{1}{a^2}A^2(\mathbf{k},\eta) - a^2k^2 \quad (5.19)$$

Here we have already used the relation $A(-\mathbf{k},\eta) = A(\mathbf{k},\eta)$ coming from Eq. (5.8). Note that $A(\mathbf{k},\eta)$ is only a function of $|\mathbf{k}|$, so we will write it as $A_k(\eta)$ from now on. This differential equation can be easily solved by assuming

$$A_k(\eta) = -ia^2(\eta)\left[\frac{\dot{u}_k(\eta)}{u_k(\eta)} - \frac{\dot{a}(\eta)}{a(\eta)}\right] \tag{5.20}$$

Then Eq. (5.19) becomes

$$\ddot{u}_k + \left(k^2 - \frac{\ddot{a}}{a}\right) u_k = 0 \tag{5.21}$$

The wave functional can also be expressed in momentum space,

$$\begin{aligned}\Psi[\phi_1,\phi_2] &= \mathcal{N}\exp\left\{-\frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\left[\phi_1^*(\mathbf{k})\phi_1(\mathbf{k}) + \phi_2^*(\mathbf{k})\phi_2(\mathbf{k})\right.\right.\\ &\qquad\left.\left. + c\phi_1^*(\mathbf{k})\phi_2(\mathbf{k}) + c\phi_2^*(\mathbf{k})\phi_1(\mathbf{k})\right] A_k(\eta)\right\}\\ &\equiv \prod_k \Psi_k \end{aligned} \tag{5.22}$$

where

$$\Psi_k = \mathcal{N}_k \exp\left\{-\frac{1}{2}\left[\phi_1^*(\mathbf{k})\phi_1(\mathbf{k}) + \phi_2^*(\mathbf{k})\phi_2(\mathbf{k}) + c\phi_1^*(\mathbf{k})\phi_2(\mathbf{k}) + c\phi_2^*(\mathbf{k})\phi_1(\mathbf{k})\right] A_k(\eta)\right\} \tag{5.23}$$

and $\phi_i(-\mathbf{k}) = \phi_i^*(\mathbf{k})$ for the real scalar field. Because there is no coupling between modes with different $\mathbf{k}$, we will only consider a single wavelength and drop the index $\mathbf{k}$ for convenience from now on.

5.3 The density matrix and the coherence length

We now have the wave functional for all modes with single wavelength $\mathbf{k}$. The next step is to calculate the reduced density matrix for ϕ_1 by tracing out ϕ_2.

$$\begin{aligned}\rho(\phi_1,\bar{\phi}_1;\eta) &= \int d\phi_2 d\phi_2^*\ \Psi_k^*(\phi_1,\phi_2,\eta)\Psi_k(\bar{\phi}_1,\phi_2,\eta) \qquad (5.24)\\ &= |\mathcal{N}_k|^2\int d\phi_2 d\phi_2^* \exp\left[-\frac{1}{2}(\phi_1\phi_1^* + \phi_2\phi_2^* + c\phi_1\phi_2^* + c\phi_2\phi_1^*)A^*\right.\end{aligned}$$

$$-\frac{1}{2}(\bar{\phi}_1\bar{\phi}_1^* + \phi_2\phi_2^* + c\bar{\phi}_1\phi_2^* + c\phi_2\bar{\phi}_1^*)A\Big] \tag{5.25}$$

This can be computed from the Gaussian integral:

$$\rho(\phi_1, \bar{\phi}_1; \eta) = \frac{4\pi}{A + A^*}|\mathcal{N}_k|^2 \exp\left(R + iI\right), \tag{5.26}$$

where

$$\begin{aligned} R = -\frac{A + A^*}{4}\left(|\phi_1|^2 + |\bar{\phi}_1|^2\right) + \frac{c^2}{8(A + A^*)}&[(A + A^*)^2[\left(|\phi_1|^2 + |\bar{\phi}_1|^2 + \phi_1^*\bar{\phi}_1 + \phi_1\bar{\phi}_1^*\right) \\ &+(A^* - A)^2\left(|\phi_1|^2 + |\bar{\phi}_1|^2 - \phi_1^*\bar{\phi}_1 - \phi_1\bar{\phi}_1^*\right)] \end{aligned} \tag{5.27}$$

$$iI = -(1 - c^2)\frac{A^* - A}{4}\left(|\phi_1|^2 - |\bar{\phi}_1|^2\right) \tag{5.28}$$

To determine the coherence length of the reduced density matrix, it is convenient to introduce the new variables:

$$\chi \equiv \frac{1}{2}(\phi_1 + \bar{\phi}_1) \tag{5.29}$$

$$\Delta \equiv \frac{1}{2}(\phi_1 - \bar{\phi}_1) \tag{5.30}$$

In terms of these variables, the reduced density matrix (5.26) becomes

$$\rho(\phi_1, \bar{\phi}_1; \eta) = \frac{4\pi}{A + A^*}|\mathcal{N}_k|^2 \exp\left[-\left(\frac{|\chi|^2}{\sigma^2} + \frac{|\Delta|^2}{l_c^2} + \beta(\chi\Delta^* + \chi^*\Delta)\right)\right] \tag{5.31}$$

Because $\beta = \frac{1-c^2}{2}(A^* - A)$ is purely imaginary, the third term in the exponential just gives a complex phase. The first term gives the dispersion of the system, the dispersion coefficient σ being

$$\sigma = \sqrt{\frac{2}{(1 - c^2)(A + A^*)}} \tag{5.32}$$

The second term describes how fast the density matrix decays when considering the off-diagonal terms. Hence, l_c is called the coherence length and is given by

$$l_c = \sqrt{\frac{2}{(A+A^*)\left[1-c^2\left(\frac{A^*-A}{A+A^*}\right)^2\right]}} \tag{5.33}$$

5.4 Decoherence in the usual inflation model

For usual inflation, $a(t) = e^{Ht}$ which is equivalent to $a(\eta) = -\dfrac{1}{H\eta}$. Here, H is the Hubble constant. Eq. (5.21) then tells us

$$u_k(\eta) = c_1 \frac{e^{-ik\eta}}{\sqrt{2k}}\left(1 - \frac{i}{k\eta}\right) + c_2 \frac{e^{ik\eta}}{\sqrt{2k}}\left(1 + \frac{i}{k\eta}\right) \tag{5.34}$$

Considering the wave functional (5.23), we have to require a positive real part of A for obvious reasons. Therefore, we choose $c_1 = 0$ and

$$A_k(\eta) = \frac{k}{H^2\eta^2}\frac{1}{1+\frac{i}{k\eta}} \tag{5.35}$$

Then, Eq. (5.33) gives us the coherence length [1]:

$$l_c = \frac{H(1+k^2\eta^2)^{1/2}}{k^{3/2}\left(1+\frac{c^2}{k^2\eta^2}\right)^{1/2}} \tag{5.36}$$

We see that if no interaction is present ($c = 0$), the coherence length approaches a constant value. Adding even a small interaction will reduce it to zero (See Fig. 5.1). Besides, the coherence length starts to decrease exponentially when the wavelength crosses the Hubble radius, which justifies our heuristic derivation in cosmological perturbation theory.

[1] We recover the results in [103] after accounting for some typos in that paper.

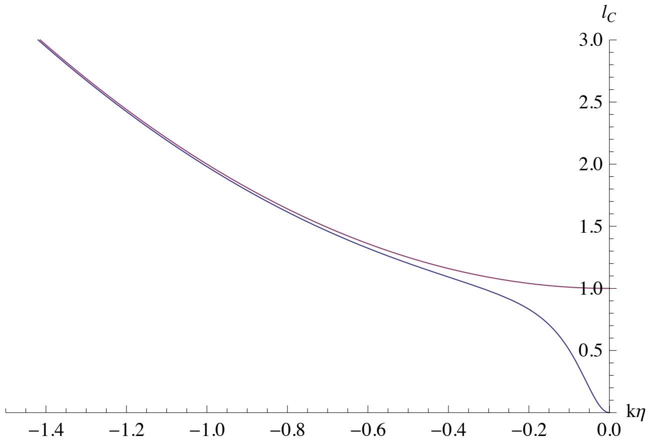

Figure 5.1: The relation of coherence length and the conformal time for usual inflation. The horizontal axis is $k\eta$ and the vertical axis is normalized coherence length. The upper (red) line corresponds to no interaction, and the lower (blue) line corresponds to $c = 0.15$. If there is an interaction, the coherence length starts decreasing and eventually becomes zero for the superhorizon modes.

Table 5.1: Table (comparing power law inflation and ekpyrosis)

	power law inflation	ekpyrotic phase
range of t	$0 \le t \le \infty$	$-\infty \le t \le 0$
$a(t)$	t^p	$(-t)^p$
p	$p \gg 1$	$p \ll 1$
range of η	$-\infty \le \eta \le 0$	$-\infty \le \eta \le 0$
$a(\eta)$	$[(1-p)\eta]^{p/(1-p)}$	$[-(1-p)\eta]^{p/(1-p)}$
$\frac{\dot{a}}{a}$	$\frac{p}{(1-p)}\frac{1}{\eta}$	$\frac{p}{(1-p)}\frac{1}{\eta}$
$\frac{\ddot{a}}{a}$	$\frac{p(2p-1)}{(1-p)^2}\frac{1}{\eta^2}$	$\frac{p(2p-1)}{(1-p)^2}\frac{1}{\eta^2}$

5.5 Decoherence in power law inflation and ekpyrotic phase

The scale factor behaviors of power law inflation and ekpyrosis are very similar so we consider them at the same time. We list some properties of their scale factors in the Table 5.1.

Because both of the power law inflation and ekpyrosis have the same $\dfrac{\ddot{a}}{a}$, they share the same solution of u_k. The differential equation of (5.21) can be solved exactly by

$$u_k = \sqrt{-k\eta}\left[c_1 H_\alpha^{(1)}(-k\eta) + c_2 H_\alpha^{(2)}(-k\eta)\right] \tag{5.37}$$

where $H_\alpha^{(1,2)}$ are Hankel functions, and we have defined

$$\alpha \equiv \sqrt{\frac{\ddot{a}}{a}\eta^2 + \frac{1}{4}} = \left|\frac{1-3p}{2(1-p)}\right| \tag{5.38}$$

As before, we want $A_k(\eta)$ to have a positive real part, so we take $c_1 = 0$, and Eq. (5.20) tells us

$$A_k(\eta) = -ia^2(\eta)\left[\frac{1-3p}{2(1-p)}\frac{1}{\eta} - \frac{k}{2}\frac{H^{(2)}_{\alpha-1}(-k\eta) - H^{(2)}_{\alpha+1}(-k\eta)}{H^{(2)}_\alpha(-k\eta)}\right] \tag{5.39}$$

Notice that they are the same for both power law inflation and ekpyrotic phase except $p \gg 1$ for the former and $p \ll 1$ for the latter. We can then use Eq. (5.33) to calculate the coherence length for both cases. The numerical solutions are plotted in the Fig. 5.2 and Fig. 5.3.

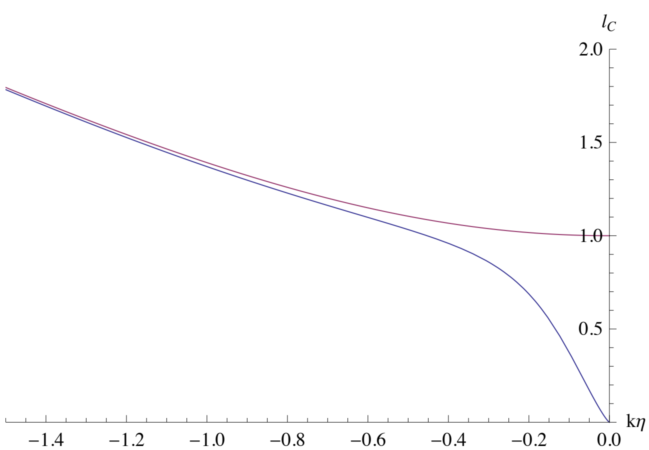

Figure 5.2: The relation of coherence length and the conformal time for power law inflation. We choose $p = 10$ in this plot. The upper (red) line corresponds to no interaction, and the lower (blue) line corresponds to $c = 0.15$.

In order to get the behavior of the coherence length l_c when the modes are well outside the Hubble radius, we need the asymptotic form of the Hankel function as

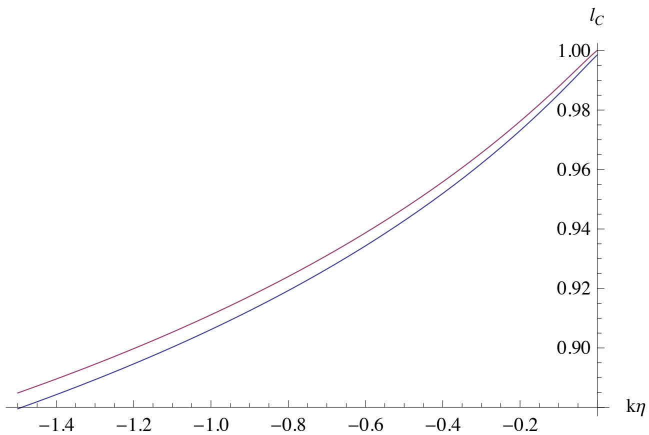

Figure 5.3: The relation of coherence length and the conformal time for ekpyrosis with $p = 0.1$. The upper (red) line corresponds to no interaction, and the lower (blue) line corresponds to $c = 0.15$. It is clear that even the modes go outside the horizon, the coherence length continues growing and approaches to a nonzero constant in the end.

$x \to 0$:

$$H_\alpha^{(2)}(x) \to \left[\frac{1}{\Gamma(\alpha+1)}\left(\frac{x}{2}\right)^{\alpha} - \frac{1}{\Gamma(\alpha+2)}\left(\frac{x}{2}\right)^{\alpha+2}\right] + i\left[\frac{\Gamma(\alpha)}{\pi}\left(\frac{x}{2}\right)^{-\alpha} + \frac{\Gamma(\alpha-1)}{\pi}\left(\frac{x}{2}\right)^{2-\alpha}\right] \tag{5.40}$$

where $\alpha > 0$ and $\Gamma(\alpha)$ is the Euler gamma function. After some manipulation of algebra, we have

$$A_k(\eta) \approx \begin{cases} 2^{1-2\alpha}|1-p|^{1-2\alpha}k^{2\alpha}\left[\dfrac{\pi}{\Gamma(\alpha)^2} + i\dfrac{1}{\alpha-1}\left(\dfrac{-k\eta}{2}\right)^{2-2\alpha}\right] & \text{, if } \alpha > \frac{1}{2} \\ 2^{1-2\alpha}|1-p|^{1-2\alpha}k^{2\alpha}\left[\dfrac{\pi}{\Gamma(\alpha)^2} + i\dfrac{\pi^2}{2\alpha\Gamma(\alpha)^4}\left(\dfrac{-k\eta}{2}\right)^{2\alpha}\right] & \text{, if } \alpha < \frac{1}{2} \end{cases} \tag{5.41}$$

as $-k\eta \ll 1$.

For power law inflation, $p \gg 1$, we have $\alpha = \frac{3}{2} + \frac{1}{p-1} = \frac{3}{2} + \epsilon$, $0 < \epsilon \ll 1$. Therefore,

$$l_c \approx l_0 \left[\frac{1}{1 + c^2\dfrac{\Gamma(\alpha)^4}{(\alpha-1)^2\pi^2}\left(\dfrac{-k\eta}{2}\right)^{-2-4\epsilon}}\right]^{\frac{1}{2}} \tag{5.42}$$

where

$$l_0^2 = |2(1-p)|^{2+2\epsilon} k^{-3-2\epsilon} \frac{\Gamma(\alpha)^2}{\pi} \tag{5.43}$$

From Eq. (5.42), it is obvious that if no interaction is present, the coherence length approaches a constant value l_0. However, even a small interaction will reduce the coherence length to zero just like what happened in the usual inflationary case.

As for the ekpyrotic phase, $p \ll 1$, and $\alpha = \frac{1}{2} - \frac{p}{1-p} = \frac{1}{2} - \epsilon$, $0 < \epsilon \ll 1$. Use Eq. (5.41), it is not difficult to get

$$l_c \approx l_0 \left[\frac{1}{1 + c^2 \frac{\pi^2}{4\alpha^2 \Gamma(\alpha)^4} \left(\frac{-k\eta}{2} \right)^{2-4\epsilon}} \right]^{\frac{1}{2}} \tag{5.44}$$

This means the coherence length approaches a nonzero constant value no matter whether the interaction is present or not, in agreement with our numerical results in Fig. 5.3.

5.6 Quantum to Semi-classical Transition without Decoherence

Even though we showed that the decoherence phenomenon would not happen during ekpyrotic phase, it is still possible that the prediction of observation remains unchanged. In [98], D. Polarski and A. A. Starobinsky prove that the quantum perturbations are indistinguishable from the perturbations of a classical stochastic system if the quantum state is extremely squeezed, namely the squeezing parameter $|\gamma_k| \gg 1$. Note that this mechanism is not the same as the usual decoherence because this kind of quantum to classical transition has nothing to do with possible interactions with environment: it is only an effect of the spacetime dynamics. In the following, we would like to show whether this kind of quantum to semi-classical transition can happen during ekpyrosis. Let us consider a real massless scalar field ϕ

$$S = \int d^4x \mathcal{L} = -\frac{1}{2}\int d^4x\sqrt{-g}\ \partial_\mu\phi\partial^\mu\phi \tag{5.45}$$

with the background metric

$$ds^2 = a^2(\eta)(-d\eta^2 + d\mathbf{x}^2) \tag{5.46}$$

We can then write down the classical Hamiltonian H in terms of the field $y \equiv a\phi$,

$$H = \frac{1}{2}\int d^3\mathbf{k}\left[p(\mathbf{k})p^*(\mathbf{k}) + k^2 y(\mathbf{k})y^*(\mathbf{k}) + \frac{\dot{a}}{a}\left(y(\mathbf{k})p^*(\mathbf{k}) + p(\mathbf{k})y^*(\mathbf{k})\right)\right] \tag{5.47}$$

where

$$p \equiv \frac{\partial\mathcal{L}(y,\dot{y})}{\partial\dot{y}} = \dot{y} - \frac{\dot{a}}{a}y \tag{5.48}$$

and "$\cdot$" stands for derivative with respect to the conformal time. From [98], we know a classical stochastic system can be described by an equation of motion and an initial distribution of probability in phase space. That is,

$$\begin{aligned} y(\mathbf{k}) &= \sqrt{2k}f_{k_1}(\eta)y(\mathbf{k},\eta_0) - \sqrt{\frac{2}{k}}f_{k_2}(\eta)p(\mathbf{k},\eta_0) \\ p(\mathbf{k}) &= \sqrt{\frac{2}{k}}g_{k_1}(\eta)p(\mathbf{k},\eta_0) + \sqrt{2k}g_{k_2}(\eta)y(\mathbf{k},\eta_0) \end{aligned} \tag{5.49}$$

where

$$\begin{aligned} \ddot{f}_k(\eta) + \left(k^2 - \frac{\ddot{a}}{a}\right)f_k(\eta) &= 0 \\ \ddot{g}_k(\eta) + \left(k^2 - \frac{\ddot{\left(\frac{1}{a}\right)}}{\left(\frac{1}{a}\right)}\right)g_k(\eta) &= 0 \end{aligned} \tag{5.50}$$

with $f_{k_1} = \mathrm{Re}(f_k)$, $f_{k_2} = \mathrm{Im}(f_k)$, $g_{k_1} = \mathrm{Re}(g_k)$, and $g_{k_2} = \mathrm{Im}(g_k)$. On scales much smaller than the horizon, the curvature of the spacetime is negligible so we can impose the boundary conditions corresponding to the Minkowski vacuum:

$$\begin{aligned} f_k(\eta) &\rightarrow \frac{1}{\sqrt{2k}} e^{-ik\eta} \\ g_k(\eta) &\rightarrow \sqrt{\frac{k}{2}} e^{-ik\eta} \end{aligned} \tag{5.51}$$

as $k\eta \rightarrow -\infty$. We see from [98] that semi-classicality is implied if the following condition is satisfied

$$|F(k)| \equiv |\mathrm{Im}(f_k^* g_k)| \gg 1 \tag{5.52}$$

It is clear that this requires the quantum state to be extremely squeezed, namely $|\gamma_k| \gg 1$, where

$$\gamma_k = \frac{1}{2|f_k|^2} - i\frac{F(k)}{|f_k|^2} \tag{5.53}$$

For usual inflation, $a(\eta) = -\frac{1}{H\eta}$, Eq. (5.50) and (5.51) imply

$$\begin{aligned} f_k(\eta) &= \frac{1}{\sqrt{2k}} e^{-ik\eta} \left(1 - \frac{i}{k\eta}\right) \\ g_k(\eta) &= \sqrt{\frac{k}{2}} e^{-ik\eta} \end{aligned} \tag{5.54}$$

, so the semi-classicality condition is satisfied at late times. This means the mode is in a squeezed state and this system is asymptotically indistinguishable from the classical one. Next, we consider the power law inflation and ekpyrotic phase. From Table 5.1, the field modes satisfy

$$\begin{aligned} \ddot{f}_k(\eta) + \left[k^2 - \frac{p(2p-1)}{(1-p)^2}\frac{1}{\eta^2}\right] f_k(\eta) &= 0 \\ \ddot{g}_k(\eta) + \left[k^2 - \frac{p}{(1-p)^2}\frac{1}{\eta^2}\right] g_k(\eta) &= 0 \end{aligned} \tag{5.55}$$

Plugging the boundary conditions (5.51), it is not difficult to get

$$f_k(\eta) = \frac{1}{2}\sqrt{\frac{\pi}{k}} e^{i(\frac{\alpha}{2}+\frac{1}{4})\pi} \sqrt{-k\eta} H_\alpha^{(1)}(-k\eta)$$

$$g_k(\eta) = \frac{1}{2}\sqrt{\pi k}e^{i(\frac{\beta}{2}+\frac{1}{4})\pi}\sqrt{-k\eta}H_\beta^{(1)}(-k\eta) \tag{5.56}$$

, where

$$\begin{aligned}\alpha &= \left|\frac{1-3p}{2(1-p)}\right| \\ \beta &= \left|\frac{1+p}{2(1-p)}\right|\end{aligned} \tag{5.57}$$

and $H_{\alpha,\beta}^{(1)}$ are Hankel functions of the first kind. The semi-classicality testing function (5.52) can then be expressed as

$$F(k) = \mathrm{Im}(f_k^* g_k) = \mathrm{Im}\left[\frac{\pi}{4}e^{i\frac{\pi}{2}(\beta-\alpha)}(-k\eta)H_\beta^{(1)}(-k\eta)H_\alpha^{*(1)}(-k\eta)\right] \tag{5.58}$$

For power law inflation, $p \gg 1$, we have $\beta - \alpha = -2$. Together with the asymptotic form of Hankel function as $x \to 0$,

$$H_\alpha^{(1)}(x) \to \frac{1}{\Gamma(1+\alpha)}\left(\frac{x}{2}\right)^\alpha - i\frac{\Gamma(\alpha)}{\pi}\left(\frac{x}{2}\right)^{-\alpha} \tag{5.59}$$

, we can show that

$$|F(k)| \to \frac{\pi}{4}(-k\eta)^{\beta-\alpha+1} \gg 1, \quad \text{as } -k\eta \to 0 \tag{5.60}$$

By the same token, we can examine this phenomenon in ekpyrosis, where $p \ll 1$. After some manipulation of algebra, it is not difficult to get

$$F(k) = \frac{1}{2\pi}\Gamma(\alpha)\Gamma(\beta)\sin\left[\frac{\pi}{2}(\beta-\alpha)\right] + O(-k\eta) \tag{5.61}$$

, where $\beta - \alpha = \dfrac{2p}{1-p} \ll 1$. Therefore, the semi-classicality condition is satisfied at late times in power law inflation but not in ekpyrotic phase. In other words, this kind of quantum to semi-classical transition would also occur during power law inflation but not ekpyrosis. This result strengthens our conclusion from previous sections.

5.7 Conclusion

We have studied a simple model with two free scalar fields interacting via a gradient coupling term in three different background spacetime: the usual inflation, the power law inflation, and the ekpyrosis. We also calculate the reduced density matrix and the corresponding coherence length by summing over one of the fields in all three cases.

Our results are that if no interaction is present, the coherence length approaches a constant value. Adding even a small interaction will reduce it to zero in either usual inflation or power law inflation case. Since this decoherence starts at Hubble crossing, the quantum fluctuations evaluated at $k\eta = -1$ give the classical initial density perturbations which become the seeds of inhomogenities of our universe later on. However, this argument does not work for ekpyrosis whose coherence length never hits zero. This means the quantum coherence would not disappear even when the modes leave the horizon. Therefore, the heuristic argument that the quantum fluctuation can become classical for superhorizon modes is not valid for ekpyrotic phase. The implication of our result is that the power spectrum of CMB fluctuations is not directly related to the ekpyrotic phase. Even though at the end of ekpyrosis the scalar field has a scale-invariant power spectrum, it is hard to say anything about what we observe right now, since that depends on the "classical" initial density perturbations. This puts some doubts on the analyses of the cosmological perturbations in the cyclic/ekpyrotic universe.

However, even though we show the decoherence would not happen during ekpyrosis, it is still possible that the prediction of observation remains unchanged [98]. We also examine this possibility and find out that this kind of quantum to semi-classical transition without decoherence still cannot happen during ekpyrotic phase. This result strengthens our conclusion that the analyses of the cosmological perturbations in cyclic/ekpyrotic universe require more inspection.

We derived our results using a very simple model. In principle, if we would like to claim the decoherence phenomenon cannot occur in ekpyrosis, we have to consider all kinds of interactions between systems and environment which is almost impossible to do. However, we believe the basic physics should emerge from simple models. We can easily generalize our analyses to a massive scalar field, and the results wouldn't change too much. We could also consider different kinds of interactions, but we will leave it to the future work.

Finally, we model the environment with a scalar field, which is convincing but might be an oversimplified assumption. The environment can also be taken to consist of the short wavelength modes which are coupled to the long wavelength modes via non-linear couplings [104, 105, 106, 107, 108, 109, 110]. Hence, this might be another possible way to generate decoherence during ekpyrosis.

Chapter 6

Fine-tuning Problem in Cyclic Cosmology

6.1 Introduction

In both inflation and ekpyrosis, there is one dominant energy component which grows faster than all other contributions in the universe, including spatial curvature and anisotropies, and thereby drives the universe into an exponentially flat and isotropic state [4, 90]. This is basically how people resolve the homogeneous and flatness problems in the literature. The purpose of this chapter is to understand whether this kind of arguments can be applied to the cyclic cosmology.

Before we start, we have to precisely define what we mean by resolving the fine-tuning problems in cosmology: ***"Given any generic initial conditions, we can find a mechanism such that the current observation is insensitive to the initial conditions."***

From the above definition, it is clear that a purely periodic universe does not belong to this category. For a periodic universe, all quantities go back to their original values in the previous cycle, so does the curvature energy density. Therefore, no mechanism can help to solve the flatness problem so we have to put this condition into periodic models by hand. However, ekpyrotic and cyclic model [91] is not purely periodic. In that model, the universe undergoes a large expansion during each cycle.

The approximate number of e-folds by which the scale factor grows per cycle is [93]

$$N_{DE} + N_{rad} + \frac{2\gamma_{ke}}{3} \tag{6.1}$$

where N_{DE} is the number of e-folds of dark energy domination, N_{rad} is approximately the number of e-folds of matter and radiation domination and the last term quantifies the expansion during kinetic energy domination phases. During ekpyrotic phase the scale factor decreases by a very small amount, so that its contribution to the overall change of scale factor can be neglected.

While the scale factor grows with each new cycle, locally measurable quantities like Hubble parameter and the density of matter, radiation, entropy, etc., undergo periodic evolution and return to their original values after each cycle [93]. The Friedmann equation can be written as

$$\Omega_m + \Omega_r + \cdots + \Omega_k = 1 \tag{6.2}$$

,where $\Omega_k = \dfrac{-k}{(aH)^2}$, $\Omega_m = \dfrac{\rho_m}{3H^2}$, $\Omega_r = \dfrac{\rho_r}{3H^2}$, and so on. Because H and all the densities undergo periodic evolution, we can get the implicit assumption in the cyclic model that Ω_k also returns to its original value after each cycle. We regard this assumption problematic in two respects. Firstly, we know that although all the matters and radiation in the previous cycle would be diluted out at the end of ekpyrosis, the new matter and radiation can be produced when the branes collide to each other, i.e., during Big Bang [88]. Furthermore, the dark energy and ekpyrotic phase come from the small attractive force between the two branes and should evolve periodically. However, there is no mechanism to curve the space such that the curvature energy density goes back to its originally value after each cycle. Secondly, if there is really a hidden mechanism to make Ω_k periodic, the ekpyrotic and cyclic model would fall into the same category as the purely periodic model and thereby, the ekpyrotic phase is no longer a solution to the flatness problem. In other words, Ω_k at present could, in principle, be any value and we have to make Ω_k small by assumption.

In this chapter, when we discuss the flatness problem of ekpyrotic and cyclic cosmology, we will assume all the densities, except for the curvature energy density, go back to their original value after each cycle while Ω_k, and of course H, do not have to. We will show that even with this assumption, Ω_k at present still has to be put in by hand and therefore, flatness problem in ekpyrotic and cyclic universe remains unsolved.

Similar arguments can also be applied to the classical perturbation. We will also extend our discussion to all cyclic-like universe models.

This chapter is organized as follows. In section 6.2, we review some results of ekpyrotic and cyclic cosmology which we will use later. Section 6.3 reviews how the flatness problem is solved in the literature for both inflation and cyclic models. Section 6.4 describes the reasons why ekpyrotic/cyclic model cannot resolve these fine-tuning problems and is the main topic of this paper. We extend this argument to any cyclic model with infinitely many cycles in the past and in the future in section 6.5. Finally, section 6.6 provides a brief conclusion.

6.2 Review of the Ekpyrotic and Cyclic Cosmology

The ekpyrotic and cyclic model is an ambitious attempt at providing a complete history of the universe. Here, we briefly review the timeline of the cyclic universe and all the behavior of key quantities such as scale factors and Hubble parameters. We will closely follow the presentation of [93, 89] in the following.

The cyclic model is based on the braneworld picture of the universe, in which spacetime is effectively 5-dimension with the extra dimension does not extend to infinity but have a finite size. Away from a bounce, the universe can be treated using the four-dimensional effective theory. The main imprint of the higher-dimensional theory on the effective picture is through the addition of one scalar fields ϕ with a potential $V(\phi)$ [1].

We are now at the start of a dark energy phase in which our universe expands exponentially. At some point in the future, our universe reverts from expansion to contraction and enters an ekpyrotic phase, which locally flattens and homogenizes the universe. After a brief phase of kinetic energy domination, we enter the phase of Big Bang during which matter and radiation are produced. After the bang, our universe experiences another short period of kinetic energy domination phase and soon undergoes the usual period of radiation and matter domination. Eventually, the dark energy comes to dominate the energy density of the universe and the whole cycle starts again.

During the dark energy domination, the scale factor a grows by an amount $e^{N_{DE}}$, where by definition, N_{DE} is the number of e-folds of dark energy domination. During this phase, the Hubble parameter remains almost constant. Then during ekpyrotic phase, the scale factor contracts slowly while the Hubble parameter grows immensely, by an amount $e^{N_{ek}}$, where N_{ek} denotes the number of e-folds of ekpyrosis. As the brane collide, radiation and matter is produced on the brane with finite temperature [92]. There are kinetic energy phases just before the brane collision and after the bang. It is proven that all kinetic energy dominated phases can be combined to a single kinetic phase with scale factor growing by $e^{2\gamma_{ke}/3}$ and Hubble parameter shrinking by $e^{-2\gamma_{ke}}$. Finally, during radiation and matter domination, the universe grows by a factor $e^{N_{rad}}$ and Hubble parameter shrinks by $e^{-2N_{rad}}$. Therefore, the scale factor grows by a total of

$$N_{DE} + \frac{2\gamma_{ke}}{3} + N_{rad} \tag{6.3}$$

e-folds over the course of a single cycle. Furthermore, we can also prove the Hubble parameter actually returns to its original value after one cycle [93] under certain assumptions. We notice that one of the assumptions is that the curvature density parameter Ω_k is always negligible during the whole cycle, and we cannot make this assumption when we would like to deal with the flatness problem. Therefore, we will

[1] The general shape of $V(\phi)$ can be found in [93].

assume H does not have to be periodic in the following.

6.3 Solution of Flatness Problem in the Literature

First of all, let us briefly quantify the flatness problem. Consider a Friedmann-Robertson-Walker(FRW) metric

$$ds^2 = -dt^2 + a(t)^2 \left(\frac{dr^2}{1-kr^2} + r^2 d\Omega_2^2 \right) \tag{6.4}$$

where $a(t)$ denotes the scale factor of the universe and $k = -1, 0, 1$ for an open, flat, or closed universe, respectively. If we consider the universe to be filled with a perfect fluid, then the Einstein equations reduced to the so-called Friedmann equations

$$H^2 = \frac{1}{3}\rho - \frac{k}{a^2} \tag{6.5}$$

$$\frac{\ddot{a}}{a} = -\frac{1}{6}\left(\rho + 3P\right), \tag{6.6}$$

where the Hubble parameter is defined by $H \equiv \frac{\dot{a}}{a}$ and a dot denotes a derivative with respect to time t. For a fluid with a constant equation of state $w \equiv \frac{P}{\rho}$, we have

$$\rho \propto a^{-3(1+w)}. \tag{6.7}$$

Now we define the quantity $\Omega(t) \equiv \frac{\rho(t)}{\rho_{crit}(t)}$ with $\rho_{crit} = 3H^2$ being the critical density. Then the first Friedmann equation (6.5) can be written as

$$\Omega - 1 = \frac{k}{(aH)^2} \tag{6.8}$$

and it expresses how close the universe is to flatness. At the present time, observations show that [16]

$$|\Omega - 1|_0 \lesssim 10^{-2}. \tag{6.9}$$

Extrapolating back in time, this means that at the Planck time

$$\frac{|\Omega - 1|_{Pl}}{|\Omega - 1|_0} = \frac{(aH)_0^2}{(aH)_{Pl}^2}. \tag{6.10}$$

If we assume a radiation-dominated universe ($w = 1/3$), which is a good approximation for the present calculation, then

$$|\Omega - 1|_{Pl} \lesssim \frac{t_P}{t_0} 10^{-2} \sim 10^{-62}. \tag{6.11}$$

Even though extrapolating all the way back to the Planck time is probably exaggerated, this simple estimate shows that the universe must have been extremely flat at early times. Clearly, this peculiar observation asks for an explanation.

Inflation postulates a period of rapid expansion immediately following the big bang, during which the equation of states $w = \frac{P}{\rho}$ of the universe is close to -1 [1, 2]. One way to model a matter component with the required equation of state is to have a scalar field ϕ with canonical kinetic energy and with a very flat potential $V(\phi)$, i.e., we add the following terms to the Lagrangian

$$\sqrt{-g} \left(-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right). \tag{6.12}$$

Then a quick calculation shows that the equation of state is given by

$$w_\phi = \frac{P_\phi}{\rho_\phi} = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)}, \tag{6.13}$$

which is close to -1 if the potential is sufficiently flat so that the fields are rolling very slowly. In the presence of different matter types, represented here by their energy densities ρ, the Friedmann equation (6.5) generalizes to

$$H^2 = \frac{1}{3} \left(\frac{-3k}{a^2} + \frac{\rho_m}{a^3} + \frac{\rho_r}{a^4} + \frac{\sigma^2}{a^6} + \ldots + \frac{\rho_\phi}{a^{3(1+w_\phi)}} \right) \tag{6.14}$$

In an expanding universe, as the scale factor a grows, matter components with a slower fall-off of their energy density come to dominate. Eventually, the inflaton, whose energy density is roughly constant, dominates the cosmic evolution and determines the (roughly constant) Hubble parameter while causing the scale factor to grow exponentially. During inflation, the relative energy density in the curvature $\Omega_k \equiv \frac{-k}{(aH)^2}$ falls off quickly, and the universe is rendered exponentially flat; according to (6.11) the flatness puzzle is then resolved as long as the scale factor grows by at least 60 e-folds.

Similar argument is used in the literature [89] to solve the flatness problem by having a slowly contracting phase before the standard expanding phase of the universe. From (6.14), if there is a matter component with $w > 1$, it will come to dominate the cosmic evolution in a contracting universe in the same way as the inflaton comes to dominate in an expanding universe. A concrete example to model this weird matter component is to have a scalar field with a negative exponential potential

$$V(\phi) = -V_0 e^{-c\phi}, \tag{6.15}$$

where $V_0 > 0$ and $c \gg 1$ are constants. In this example, it is not difficult to get the equation of state

$$w_\phi = \frac{c^2}{3} - 1 \gg 1. \tag{6.16}$$

With this extra term in the Friedmann equation (6.14), the fractional energy density Ω_k quickly decays and therefore, neglecting quantum effects, the universe is left exponentially flat as it approaches the big crunch.

6.4 Fine-Tuning Problems in Ekpyrotic and Cyclic Cosmology

From the previous section, it seems that the ekpyrotic phase is the same as inflation as far as the fine-tuning problem is concerned. However, there is a major difference between these two models. In the usual Big Bang plus inflation paradigm, there is a beginning of time corresponding to the initial singularity, i.e., Big Bang; however, ekpyrotic/cyclic cosmology extends the timeline to the infinite past and future. This property makes the analysis in section 6.3 incomplete where we only consider what happened in a specific cycle. In other words, taking the whole history of the universe into consideration could be so important that it might dramatically change the conclusion. We will focus on the relationship between cycles in the following.

We can keep track of what happens to Ω_k using the results of section 6.2. In each cycle, if we know the curvature energy density parameter Ω_k at one point, we can calculate Ω_k for the whole cycle. Therefore, we only have to pick out one moment to represent the whole cycle. Without loss of generality, we can use the beginning of dark energy phase (which is today in the present cycle) as the representative. The Friedmann equation is

$$H^2 + \frac{k}{a^2} = \frac{1}{3}\rho \tag{6.17}$$

in the reduced Planck units ($8\pi G = 1$). From the definition of Ω_k,

$$\Omega_k \equiv \frac{-k}{(aH)^2} = \frac{1}{1 - \frac{a^2}{3k}\rho}. \tag{6.18}$$

Notice that the scale factor undergoes a large expansion while all energy densities (except for the curvature) return to their original values after each cycle, so the $a^2\rho/3k$ term in Ω_k will eventually dominate, and $|\Omega_k|$ would be suppressed exponentially by an amount

$$|\Omega_k| \propto \frac{1}{a^2} = e^{-2(N_{DE}+N_{rad}+\frac{2\gamma_{ke}}{3})} \tag{6.19}$$

during each cycle. This seems a nice result at first sight for the reason that no matter what the initial Ω_k is, it would be suppressed rapidly and become almost zero in finite cycles. However, if we read this result more closely, we will find that it suggests the opposite. The reason is as follows.

We know that one of the advantages for ekpyrotic and cyclic cosmology is that there is no mysterious moment of creation. For inflation, the Big Bang corresponds to the beginning of time and therefore the inflation occurs only once. For ekpyrotic and cyclic model, the cosmological history is continued to the infinite past so that there are infinitely many number of cycles in the past and in the future. Because of this, we should be able to extrapolate backward in time and Eq. (6.18) suggests that our universe was curvature dominated finite cycles ago. More specifically, if we do have endless cycles in the past and in the future, $|\Omega_k|$ at present can range from 0 to 1 in the whole history of universe. There is no reason that we should be in the cycle with small curvature, and therefore the flatness problem remains. This problem might be relieved in two ways. Firstly, we can argue that the universe we observe today came from a tiny patch of empty space from the cycle before. Similarly, each such patch of sufficiently empty space in the observable universe today will evolve into a new region like ours a cycle from now. In other words, we assume that only regions which is flat enough can evolve to the next cycles. This argument can only be justified by using anthropic principle or putting it by hand. No matter what, we have to put in new initial conditions for each cycle, which corresponds to infinitely fine-tuning in our opinion. Furthermore, this argument is also equivalent to saying that no flatness puzzle needs to be considered. It is just an assumption in this model. Secondly, we can relieve this problem by assuming finite cycles in the past. In this case, we cannot extrapolate backwards all the way to the infinite past. However, if one of the main purpose of introducing the cyclic model is to avoid having a "beginning", we should not make this kind of assumption.

In fact, the argument above is very general and does not rely on any special properties of curvature. Any quantity which seems fine-tuned without inflation also needs to be considered carefully in the ekpyrotic/cyclic model. Let's take the classical perturbation as another example. For a cosmological model to succeed, it not only has to address the standard cosmological puzzles but also needs the ability to imprint nearly scale-invariant inhomogeneities on superhorizon scales. These inhomogeneities are thought to provide the seeds which later become the temperature anisotropies in the cosmic microwave background and the large-scale structure in the universe. This framework of the cosmological perturbation theory is based on the quantum mechanics of scalar fields, and thereby requires the amplitude of classical perturbation being much smaller than that of quantum fluctuation near the end of ekpyrosis. This seems not a problem at all because the ekpyrosis, as it is designed for, can dilute out all pre-ekpyrotic inhomogeneities and anisotropies. However, this argument is true only in a single cycle. More sophisticatedly, we need to consider the whole history of the universe at once.

In principle, we should be able to know how the classical perturbation evolves in a single cycle. More specifically, we know the classical perturbation would grow in the radiation/matter dominated era and decay during dark energy domination and ekpyrosis. We also assume that it would not change too much during kinetic phase and Big Bang/Big Crunch. Of course without complete knowledge of quantum gravity, we are not able to describe the crunch with certainty, but it is just a technical issue. Theoretically, once we know the information of classical perturbation at any specific time, we should be able to get all information in the whole cycle. Therefore, we can pick any moment to represent the whole cycle. Let's choose the end of the ekpyrotic phase as our representative in this case.

Let the amplitude of the classical perturbation at the end of ekpyrosis be X. Then X can either go up or go down or remain the same after one cycle. We take all of them into consideration, respectively, as follows. If X returns to its original value, the argument that ekpyrosis can be used to dilute the classical inhomogeneities is not valid. That is, even though there are one or more phases during which the classical perturbation is suppressed, there are also other phases to compete this effect in the same cycle. Another case also needs to be considered. As we mentioned before, we can argue that only patches which give negligible X can evolve to the next cycle. In both cases, the condition that X is small compared to quantum effects can only be a boundary condition and have to be put in by hand. If X goes up from cycle to cycle, it will either diverge or reach an upper bound ϵ in the future. Because there is no reason to prefer us to be present in one cycle to the others, we cannot let X blow up in the future and the upper bound ϵ has to be very small to make the cosmological perturbation theory works. This requirement is equivalent to assuming X is small in the entire history of universe so the homogeneous problem remains unsolved. Finally, we consider the case where X decreases after each cycle. This is exactly the time reversal version of above and can be deduced easily.

To sum up, the fine-tuning problems can not be resolved using similar arguments to inflation in the ekpyrotic/cycle model. We need another fundamental reasons to explain the closeness of critical density and the smallness of the classical homogeneities in order to make ekpyrotic/cyclic model to work.

6.5 Fine-Tuning Problems in all cyclic models

We notice that our argument is so general that it can be used not only in ekpyrotic/cyclic cosmology but also all cyclic-like models attempting to describe our universe. First of all, we have to define what we mean by cyclic models:

1. A cyclic model is any of cosmological models in which the universe follows infinite, self-sustaining cycles. There is no beginning of time and no end either.

2. All cycles do not have to be exactly the same; however, all densities, except for curvature density, return to their original values after each cycle.

3. We can only assign the initial condition once and all the dynamics of our universe can be obtained by evolving this initial condition forwards and backwards in time.

The first definition is crucial in that we can extrapolate in both directions of time for infinitely many cycles. If there is a beginning or an end, this cyclic model is nearly as good as inflation as far as the fine-tuning problem is concerned. The second definition is two-folded. We make this model cyclic-like by requiring almost all physical quantities are periodic. However, if we also make curvature energy density periodic, the flatness problem can only be solved by assuming Ω_k negligible for the whole history of the universe. The third definition is based on the unitarity of the physics. Furthermore, we avoid the situation of infinite fine-tuning by assigning initial conditions on every cycles.

Under these assumptions, we can easily argue that many fine-tuning problems are unsolved in this cyclic model. Let's take the flatness problem as an example to demonstrate the idea. From definition 2 and 3, $|\Omega_k|$ can either go up or go down or remain the same after one cycle. If $|\Omega_k|$ returns to its original value, this value has to be extremely small in order to fit the observation. There are only two possibilities: either our universe is purely periodic or only flat enough patches can evolve to the next cycle. The second possibility violates definition 3 in that it requires resetting the initial conditions for each cycle. Moreover, purely periodicity means $|\Omega_k|$ at present could be any value and have to be put in by hand. If $|\Omega_k|$ goes up from cycle to cycle, it will eventually reach an upper bound ϵ between 0 and 1 in the future. The observed value of $|\Omega_k|$ at present could be any number between 0 and ϵ, and therefore ϵ has to be very small. This requirement is equivalent to assuming $|\Omega_k|$ is small in the entire history of universe so the flatness problem remains unsolved. Finally, we consider the case where $|\Omega_k|$ decreases after each cycle. This is exactly the time reversal version of above and can be deduced easily.

From the above discussion, we found that the usual solutions of fine-tuning problems are incompatible with the intrinsic nature of cyclic models. Without further reasons, we claim that the fine-tuning problems, including the flatness problem, remain unsolved in all cyclic models.

6.6 Conclusion

Under the assumption that all densities, except for the curvature, return to their original values, we have computed how Ω_k evolves in the whole history of ekpyrotic/cyclic scenario, not only in one cycle. The curvature density parameter Ω_k is suppressed exponentially from one cycle to another. However, if we go backward in time, $|\Omega_k|$ in the previous cycle would be larger than that at present, which means our universe was curvature dominated sometime in the past. Because $|\Omega_k|$ can be any value between 0 and 1, the flatness problem does not have a convincing solution in the ekpyrotic/cyclic model. If we assume there are only finite cycles in the past, then our universe would eventually become extremely flat independent of the initial conditions because it is an attractor solution. However, the cyclic universe will lose one of his main attractions which is avoiding the origin of time.

The similar arguments can also be extended to the more general case. Basically, the mechanism which inflation uses to solve fine-tuning problems is not compatible with eternal universe which contains infinitely many cycles in both direction of time. Therefore, flatness problem still asks for an explanation in any generic cyclic models.

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