STUDIES ON SEISMIC WAVES

Thesis

by

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Summary

This thesis embodies three separate studies on seismic waves. The first one is concerned with the origin of the oscillatory nature of earthquake waves. Different modes of generation have been discussed in some detail, particularly the theory of Kawasumi and that of Sharpe. The effect of the possible failure of Hooke's law on the nature of the seismic waves is also explained briefly.

The second is a short note dealing with the physical basis of two kinds of observations: the observation of velocity and the observation of period.

The third part is a study on the theory of FLANK WAVES. The basic difficulties of the old theory are pointed out and discussed. The phenomenon is explained in the light of the wave theory, in considerable detail in the case of P waves and very briefly sketched for the case of SH waves.

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I. ON THE ORIGIN OF OSCILLATORY EARTHQUAKE WAVES

1. Introduction

Concerning the origin of the oscillatory nature of the earthquake waves, current treatises on seismology generally give but very brief accounts, if at all. While sustained oscillations are almost always recorded in the seismograms, the origin of the waves is usually impulsive or assumed so - elastic rebounds in most natural earthquakes and explosions in the artificial ones. Besides, no appreciable dispersion has been observed in the body waves traveling through the earth's crust. The explanation is not at all obvious; thus there seems to be a gap in the tracing out of the life history of the disturbance. However, theories, though scattered in different publications, are not entirely wanting. The present study is intended to give a concise appraisal of some of the more plausible explanations. Certain extensions are also attempted so that a better understanding may be derived.

2. Discussion of Some Modes of Generation

There are at least five ways by which impulsive disturbances may result in the manifestation of a train of oscillations. They are: (i) dispersion, (ii) selective absorption, (iii) multiple reflections, (iv) resonance, and (v) repeated faulting. Each of these, to be sure, supplies a possible mode of generation, but it is quite doubtful whether any one of them is adequate by itself.

The phenomenon of dispersion has been applied with considerable success by Gutenberg and Richter (1) to surface waves for the delineation of the different structural units of the earth's crust. But then the mechanism of dispersion is interpreted in a different manner. In the case of body waves, dispersion has not yet been observed in the seismogram.

A pulse may be conceived as composed of an infinite number of harmonic components of different frequencies. If these components, in passing through an absorptive medium, are absorbed differently for different frequencies, the combined effect no longer gives the impression of a pulse but a train of oscillations. Selective absorption has been suggested in Stokes' theory of internal friction. (2) It also plays an important role in elastic hysteresis. (3) Recently N. Ricker (4) applied this effect in his wavelet theory, but his demonstration does not seem to be free from ambiguity. For the longer waves, the observations of the M- and W- phases showed that the absorption is extremely small. That absorption alone is not sufficient to account for the observed oscillatory nature of the earthquake waves has long since been

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(1) B. Gutenberg and C.F. Richter, Ger. Beit. z. Geophys. 47(1936)73-131.
(2) G.G. Stokes, Trans. Camb. Phil. Soc. 8(1845)287-319.
(3) Lord Kelvin, Papers, 3(1890)27
(4) N. Ricker, Geophys. 5(1940)348, 6(1941)254, B.S.S.A. 33(1943)197-228

indicated by Jeffreys.[1]

Multiple reflection is not likely to be of importance in the present phenomenon. Even granted the most favorable layering of the ground so that multiple reflection will occur, the waves as observed in the seismograms of artificial explosions show sustained oscillations before any reflection is observed.

Resonance will cause free oscilations of a certain part of the earth's crust with its characteristic frequency. Then waves arriving at the same station and traveling a similar path must show this same characteristic frequency. There is no conclusive evidence in the seismograms to confirm that this is the case. This cause, even if it were real, cannot be important in every case.

Repeated faulting is probably the most plausible among the above explanations. In the first place, this mechanism conforms with the jerky wave forms as observed in the initial part of most seismograms. Also from the nature of the movements, it is very likely that the gliding along the fault plane will take place only in discontinuous steps. Unless there is no friction along the fault plane (which seems unlikely) in which case there will be no wave radiated[2] from the fault, a finite train of irregular waves should rather be the expected result.

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(1) H. Jeffreys, Geophys. Suppl. 1(1925)282-292.
(2) K. Sezawa, Bull. Earth. Res. Inst., Tokyo, 14(1943)269-279

Associated with this is the finite speed of fracture. This necessitates a certain time duration of the original disturbance which supplies another cause of the oscillatory nature of the waves. This fact has been used by Gutenberg and Richter[1] in explaining the apparent lag of the origin time of $\bar{P}$ behind that of $\bar{S}$. If faulting always starts from one point, it might seem possible to get some information about the length and orientation of the fault from the duration of the initial disturbance in the seismogram. However, the situation is complicated by the fact that the propagation of fracture is not necessarily uni-directional and that the speed of faulting is not constant, being dependent on the initial conditions.

3. Investigations based on the Theories of Small Displacements.

So far, our discussions are confined to the oscillations that are transformed from single pulses, or due to superposition of several pulses. The process, therefore, is a composite one, and the effect is more prominent at a larger distance from the source. The question may arise as to whether it is possible to generate oscillations from a single process. The answer is affirmative, and the origin is to be sought in the hypocentral region. From this result, it will be seen that the spreading of the pulses should be a property inherent in most earthquakes. For the ease of dis-

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(1) B. Gutenberg and C.F. Richter, B.S.S.A. 33(1943)269-279.

cussion and visualization, we assume that the source gives only longitudinal waves. It is evident that when both the latter and the transverse waves are present, the same result must still hold qualitatively.

As an analogy, let us first consider the motion of a pendulum in a resistive medium. If a blow is given to it, a damped oscillation will result, provided that the damping is below the critical value. The equation of motion of the pendulum is

(1) $$\ddot{\theta} + 2\epsilon\dot{\theta} + n^2\theta = \phi(t)$$

where θ is the displacement, ϵ the damping constant and $n/2\pi$ the natural frequency of the pendulum. ϕ is a function which represents the impulse due to the blow. If we put $\alpha = i\epsilon + \sqrt{n^2-\epsilon^2}$ and $\beta = i\epsilon - \sqrt{n^2-\epsilon^2}$, the solution of (1) is given by

(2) $$\theta = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ipt}\,dp}{(p-\alpha)(p-\beta)}\int_{-\infty}^{\infty}\phi(\omega)e^{-ip\omega}\,d\omega$$

Now in an elastic medium, the damping is not expected to be large; and if we imagine the longitudinal wave as generated by a sudden application of a pressure inside a spherical cavity in a homogeneous, infinite medium, we may have some degree of similarity in the effects. The main difference would be that in the present case, a progressive wave should be dealt with. This is exactly the premise of Kawasumi and Yosiyama.(1) Assuming a pressure function $p = \phi(t)$ such that it starts at the time $t = 0$, lasts for a duration t_1, at a constant value $-P_0$ and is then removed, they are able to

(1) H. Kawasumi and R. Yosiyama, Bull. Earth. Inst. Tokyo, 13(1935)496-503

solve the wave equation for small displacements with the boundary conditions that the radial displacement should be continuous and that the normal stress should be equal to the pressure applied at the surface of the cavity. Spherical symmetry is, of course, implied in the assumption that the medium is infinite and homogenous. Their solution is

$$\phi = \frac{aP_0}{2\pi\rho r}\int_{-\infty}^{\infty}\frac{e^{ipt}\,dp}{(p-\alpha)(p-\beta)}\int_{0}^{t_1} e^{-ip\omega}\,d\omega = \frac{a_0 P_0}{2\pi i\rho r}\int_{-\infty}^{\infty}\frac{e^{ip\tau}-e^{ip\tau_1}}{p(p-\alpha)(p-\beta)}\,dp \tag{3}$$

where ϕ = displacement potential whose derivative with respect to the distance r gives the radial displacement,

ρ = density of the medium, a = radius of the cavity,

τ = retarded time = t - (r - a)/V, $\tau_1 = \tau - t$

$\alpha = i\frac{2v^2}{aV} + \frac{2v}{a}\sqrt{1-\left(\frac{v}{V}\right)^2}$, $\beta = i\frac{2v^2}{aV} - \frac{2v}{a}\sqrt{1-\left(\frac{v}{V}\right)^2}$

v = velocity of transverse wave, V = that of longitudinal wave.

Equation (3) is very similar to (2) except that the time is replaced now by the retarded time. This makes the wave a progressive one.

When $\tau_1 > 0$, the radial displacement u at a distance r from the source can be obtained by evaluating (3) and then differentiating with respect to r. The result is

$$u = \frac{a^3 P_0}{4\mu}\left[\frac{1}{r^2}\left\{1-\sqrt{\frac{\lambda+2\mu}{\lambda+\mu}}\,e^{-\frac{2v^2}{aV}\tau}\sin\left(\frac{2v}{a}\sqrt{1-\left(\frac{v}{V}\right)^2}\tau+\tan^{-1}\sqrt{\frac{\lambda+2\mu}{\lambda+\mu}}\right)\right\}-\frac{2}{a}\sqrt{\frac{\mu}{\lambda+\mu}}\frac{1}{r}e^{-\frac{2v^2}{aV}\tau}\sin\frac{2v}{a}\sqrt{1-\left(\frac{v}{V}\right)^2}\,\tau\right]^{*} \tag{4}$$

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* There is a slight misprint in eq. (19) of the paper cited.

where μ and λ are the Lamé constants. When r is large, the first term may be neglected, and we obtain a simple damped oscillation with period T and damping ratio γ given by

$$T = \pi a / v\sqrt{1-(\frac{v}{V})^2}, \quad \gamma = e^{\pi\sqrt{\frac{\mu}{\lambda+\mu}}}$$

These results were originally worked out by Kawasumi and Yosiyama to substantiate the view that the energy carried by the seismic waves is equal to the released strain energy of the medium. This view has been contested by Sezawa,(1) but his arguments seem to be a little obscure. Nevertheless, he did improve the above calculations in one respect; that is, the rapidity with which the pressure is applied also plays an important rôle in the resulting wave form. Recently, J.A. Sharpe(2) reworked Sezawa's result along a more or less similar line and put the final formula in a more elegant form, whereby he was able to explain some of the observations in applied seismology.

Other conditions being the same, Sharpe solved the problem by first assuming the pressure from an explosive source to be a step function of the form

$$p(t) = p_0 \quad \text{for} \quad t = 0$$
$$= 0 \quad \text{for} \quad t < 0$$

and then generalizing the result by use of Duhamel's integral to a pressure function of the general form. For the former case and with a Poisson's ratio of $\frac{1}{4}$, he obtained the radial

(1) K. Sezawa, ibid. 13(1935)740-748, -- and K. Kanai, ibid. 14(1936)10-16,
K. Sezawa, ibid. 14(1936)149-154

(2) J.A. Sharpe, Geophysics, 7(1942)144-154, 311-321, 9(1944) 131-142.

displacement u at a distance several times the radius of the cavity or larger as given by

$$u = \frac{a^2 P_0}{2\sqrt{2}\,\mu r} e^{-\frac{\omega T}{\sqrt{2}}} \sin \omega T \qquad \text{for } T \geqq 0 \qquad (5)$$
$$= 0 \qquad\qquad " \quad T < 0 \qquad \left.\right\} \omega = \frac{2\sqrt{2}\, v}{3a}$$

For the general pressure function, the integral

$$u(T) = \frac{d}{dT} \int_0^T p(n) u(T-n)\, dn \qquad (6)$$

is used. In particular, if $p(\tau)$ is of the form $p_0 (1 - e^{-qt})$, Sezawa and Kanai have obtained u in its dependence on q which signifies the speed at which the pressure is varied. However, their expression is too long to be included here.

From (5), the following conclusions concerning waves generated in an explosion can be drawn:

(i) Larger wave amplitude is obtained if the charge is buried in a less rigid formation;

(ii) Larger amplitude is obtained from a larger cavity;

(iii) Higher frequency of the wave may result from either a higher v or a smaller a;

(iv) Explosives of higher speed give larger amplitudes (from the solution of Sezawa and Kanai).

It must be pointed out here that all these results, either those of Sharpe or those from the Japanese school, hold strictly only for small wave amplitudes. The assumption that the stress-strain relation is linear and contains only the first derivatives has been tacitly made in all the calculations. This is certainly true when the body wave has traveled some distance from the source, and the displacement has become small. But in the near vicinity of the source, especially when it is of the explosive type, Hooke's law is not expected to be valid. The remarks in the next section are just an attempt to present one of the complications that would thus possibly follow.

3. Note on a Possible Failure of Hooke's Law

We first notice that the ordinary wave equation will no longer hold in the present case. Let us take the next simplest relation between stress and strain, that it is still linear but contains also the higher derivatives. Then the kinetic energy T and the potential energy W per unit volume of the medium may be written in the following forms:

(7) $$T = \frac{1}{2}\rho\left(\frac{\partial S}{\partial t}\right)^2, \qquad W = \frac{1}{2}a_1\left(\frac{\partial S}{\partial x}\right)^2 + \frac{1}{2}a_2\left(\frac{\partial^2 S}{\partial x^2}\right)^2 + \frac{1}{2}a_3\left(\frac{\partial^3 S}{\partial x^3}\right)^2 + \cdots$$

where S is the displacement, and the a's are constants (we consider only the one-dimensional case). By Hamilton's principle

$$\delta\int_{t_1}^{t_2} dt \int_{x_1}^{x_2} (T - W)\, dx = 0,$$

the equation of motion can be written as

(8) $$\rho\frac{\partial^2 S}{\partial t^2} = \frac{\partial}{\partial x}\frac{\partial W}{\partial S'} - \frac{\partial^2}{\partial x^2}\frac{\partial W}{\partial S''} + \frac{\partial^3}{\partial x^3}\frac{\partial W}{\partial S'''} - \cdots \qquad S' = \frac{\partial S}{\partial x},\ S'' = \frac{\partial^2 S}{\partial x^2},\ etc.$$

Multiplying both sides by $\frac{\partial S}{\partial t}$ and integrating, we obtain

(9) $$\frac{1}{2}\rho\left(\frac{\partial S}{\partial t}\right)^2 = \int^t \frac{\partial}{\partial x}\left[\frac{\partial W}{\partial S'} - \frac{\partial}{\partial x}\frac{\partial W}{\partial S''} + \frac{\partial^2}{\partial x^2}\frac{\partial W}{\partial S''} - \cdots\right]\frac{\partial S}{\partial t}\, dt$$

If we substitute a wave form $S = A\cos(\omega t - kx)$ in this last expression and take the mean value, we have

(10) $\rho\omega^2 \equiv a_1 k^2 + a_2 k^4 + a_3 k^6 + \quad \ldots$ [1]

When Hooke's law holds, all the a's vanish except a_1. Then $\omega = \sqrt{\frac{a_1}{\rho}}\, k$ and $\frac{\partial\omega}{\partial k} = \frac{\omega}{k}$ = wave velocity. This means that the medium is non-dispersive. But if the other a's do not all vanish, $\frac{\partial\omega}{\partial k} \neq \frac{\omega}{k}$ and any sinusoidal wave will be dispersed in the vicinity of the source. If the potential energy is given by an even more complicated expression than (7), then (10) will be more complicated too and $\partial\omega/\partial k$ will not be constant a fortiori.

With this simple picture, we may imagine that a disturbance, even generated by an impulsive force, must necessarily propagate dispersively in the neighborhood of the source, owing to the fact that the displacement is too large to justify Hooke's law. The pulse will then be spread into a spectrum. When the wave travels some distance away from the source, the displacement becomes small, and the ordinary wave equation holds. The whole train will then travel through the medium without dispersion. What is observed as a non-dispersive wave train may actually be the net result of a dispersive process which has already been completed. If this is the case, a more intensive disturbance and a less elastic medium should be associated with a broader initial wave train. On the other hand, if the disturbance is small or if the medium around the source is very rigid, the pulse may travel a-

(1) T.H. Havelock, Camb. Math. Tract, No. 17, 1914.

way without appreciable dispersion. In that case, a solitary wave may be observed in the seismogram.

These remarks are admittedly only qualitative. They are intended to show that many interesting phenomena may be expected when the restriction to Hooke's law is broadened. On the other hand, when the linear relation is not valid, the principle of superposition cannot hold; consequently, Fourier's integral and Duhamel's formula cannot be used in these simple forms. This has a particular significance when a comparison is made between the theoretical ground movement and observations, because no instrument has a constant magnification over all ranges of frequencies.

So far as is known, the general theory of spherical waves of finite amplitude is not available even for a fluid. The problem is, indeed, a very difficult one. Not only is the mathematical difficulty almost insurmountable at present, but also the physical nature of the process is not well understood. In this connection, attention may be called to a paper by J.J. Unwin (1) in which a step-by-step method has been developed for dealing with solutions of problems connected with the production of waves by spherical concentrations of compressed air. Several interesting results have been brought out which are not expected from the theory for small amplitudes. One of these is that from a single region of condensation, a train of waves is produced instead of a single crest propagated outward. This agrees quite well with our qualitative conjecture.

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(1) J.J. Unwin, Proc. Roy. Soc. Lond. A178(1941)153-170

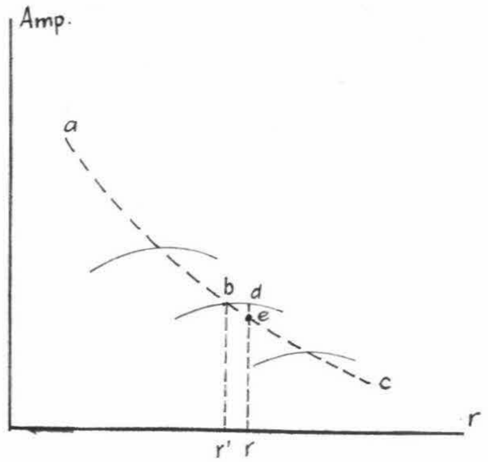
II. ON OBSERVATIONS

1. Observation of Wave Velocity

In finding the velocity of a moving object, we may follow one of three procedures: (i) observing its arrival times at several fixed positions, (ii) observing its positions at certain chosen times, or (iii) observing its change of position with time continuously as in taking a speed-record. This last one is not always possible. The calculation of wave velocity from readings on seismograms at different stations falls into the first category. We have tacitly assumed that the ratio of the distance between two stations to the difference in arrival times would give us the velocity of the wave in the ordinary sense of the word. This is true when the beginnings of the P phase are used. If this is not the case, then there are occasions when the identity of the wave is really ambiguous, because the wave form suffers various changes in the course of its propagation. One of the common practices to identify a wave in applied seismology is to mark the maxima. Then, it appears that the measurements according to the three different procedures listed above do not necessarily give the same velocity. It is, therefore, worth while to make a closer examination of the physical meanings of the quantities defined by the measurements. Indeed, the discrepancy is not expected to be large, and it may even seem to be trivial in some cases. But it is an underlying principle usually taken for granted, which merits some investigation.

If the medium is perfectly elastic and the wave observed is a disturbance strictly obeying the ordinary wave equation, our discussion will be quite superfluous, as will evidently be seen later. But what we have actually observed on the earth's surface is the net result of a composite process owing to several factors. Instead of enquiring about the different mechanisms, we will study the problem just from the kinematic point of view.

Assume that the amplitude of a wave decreases with the distance, according to a certain law (there must be a decrease of amplitude even from purely energy consideration). Since we will not be able to follow the third procedure in seismic observation anyway, we will confine the comparison to the first two. In the diagram, let ac be the locus of the maxima of the amplitude of the wave at different places.



Then at any station whose position is denoted by r, the time corresponding to the maximum amplitude in the seismogram (point d in the diagram) may really refer to the wave whose maximum b corresponds to the position r'. The time corresponding to our real maximum e is not given by the time corresponding to maximum on the seismogram. Since the times thus measured do not correspond to the maximum of the amplitude at the two positions, the ratio of the distance to time will not in general give the true velocity. It will yield the correct result only when the wave is of a certain type.

Probably we can illustrate the idea more clearly by a sample calculation. Let us consider the amplitude factor of the form

$$A \frac{t^m}{x^n} e^{-k\frac{t^2}{x^2}} .$$

This amplitude changes with both time and distance. Keeping the time fixed, we have the maximum given by

$$\frac{d}{dx}\left(x^{-n} e^{-k\frac{t_0^2}{x^2}}\right) = 0$$

$$\therefore \quad x = t_0\sqrt{\frac{2k}{n}} \qquad or \qquad v_t = \sqrt{\frac{2k}{n}} .$$

If the distance is kept constant, then

$$\frac{d}{dt}\left(t^m e^{-k\frac{t^2}{x_0^2}}\right) = 0 , \qquad t = x_0\sqrt{\frac{m}{2k}}$$

and the velocity obtained by the readings at different stations will be given by

$$v_x = \sqrt{\frac{2k}{m}}$$

which is different from the value given above, unless m = n.

In fact, if it is only the maximum of the wave which we are going to use, we can ask the general question: What type of wave functions will give the same velocity by the two methods of observations? Let us assume a wave function of the form $\phi = \phi(x,t)$. Then to find the maximum by keeping the time constant, would amount to putting $\phi_x = 0$. The velocity with which this maximum travels may be found by differentiating this latter equation with respect to time, and we have

$$\phi_{xt} + \phi_{xx}\frac{\partial x}{\partial t} = 0$$

To find the maximum trace in the seismogram, we have $\phi_t = 0$. The velocity with which this trace-maximum travels will be given by

$$\phi_{tt} + \phi_{tx} \frac{\partial x}{\partial t} = 0.$$

In order that these two agree, we should have

$$\begin{vmatrix} \phi_{xt} & \phi_{xx} \\ \phi_{tt} & \phi_{tx} \end{vmatrix} = 0$$

or

$$\phi_{xx} \phi_{tt} - \phi_{xt}^2 = 0$$

In Monge's notation, this is of the type rt - s = 0 whose solution may be readily obtained by use of a Legendre's transformation. Written parametrically, it is as follows:

$$\phi = at - x\,A(a) - B(a)$$

$$0 = t - x\,A'(a) - B'(a)$$

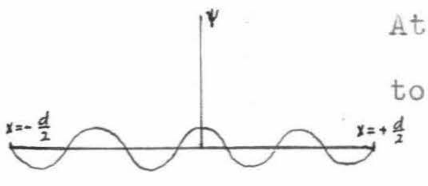
where both A and B are arbitrary functions and a is a parameter.

It is to be pointed out here that each of the solutions of the ordinary wave equation also satisfies the above differential equation. However, being non-linear, it is not satisfied by the sum of different solutions. Physically, this means that the velocity of the trace-maximum will be equal to the wave velocity when we are observing a single wave. When waves of different velocities are observed, the resulting amplitude will not in general satisfy the above condition. In particular, the observation of the group velocity of the dispersed wave train may not be the same quantity as in the theory, and the comparison must be made with care.

2. Observation of the Period

The idea of the period has a precise meaning only when it refers to an infinite train of waves of the same form. When an irregular wave form is considered, a harmonic analysis is usually applied, and the period means any one of its harmonic components. But in the case of the seismograms, the process of harmonic analysis is too tedious to be applicable, and the word period as generally alluded to in the literature has, strictly speaking, only a qualitative meaning. The usual way to determine a period is to find a finite train of waves which are approximately of the same form and then take the average period in that interval. It is usually assumed that if a pure wave form is found in the seismogram, even for a short interval, the reality of that period is vouchsafed; but, actually, this is not exactly the case. The wave function which would give an apparently pure wave form within a finite interval can be found by means of the Fourier's integral. The following analysis is by no means new. It is intended to emphasize the limitation and justification of an experimental procedure which is so often followed in seismometry but seems to have been taken for granted in its theoretical foundation.

Consider the finite wave train within the interval $-d/2 \leq x \leq d/2$. Let the wave velocity be constant and the approximate period of the wave within this interval be T. which corresponds to the wave length λ. Let $k = 2\pi/\lambda$.



At any time t, we may assume the wave to be represented by

$$\psi \sim e^{ik_0 x} \qquad \text{for} \quad -\tfrac{d}{2} \leq x \leq +\tfrac{d}{2} \ .$$

By Fourier's theorem, we have

$$\psi = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk \int_{-\infty}^{\infty} f(\beta) e^{i(x-\beta)k} d\beta = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx} dk \int_{-d/2}^{d/2} e^{-ik\beta} \cos k_0\beta \, d\beta$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx} dk \int_{-\frac{d}{2}}^{\frac{d}{2}} \frac{1}{2}\left[e^{i(k_0-k)\beta} + e^{-i(k_0+k)\beta}\right] d\beta$$

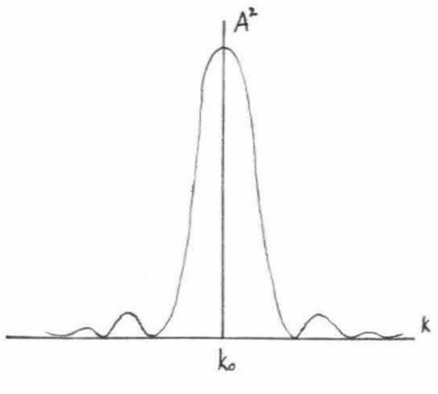
But the second integral is given by

$$\int_{-d/2}^{d/2} = \frac{1}{2}\left[\frac{e^{i(k_0-k)\beta}}{i(k_0-k)} - \frac{e^{-i(k_0+k)\beta}}{i(k_0+k)}\right]_{-d/2}^{d/2} = \frac{\sin(k_0-k)\frac{d}{2}}{k_0-k}$$

where the second term is neglected, being small compared with the first one. This is really the amplitude factor of the harmonic component in the first integral. To see how this amplitude factor varies with k, we may plot its square against k in order to avoid the alternating signs, viz. we will plot the factor

$$A^2 = \frac{\sin^2 \frac{1}{2}(k_0-k)d}{(k_0-k)^2}$$

against k. This is the familiar intensity factor for the diffraction of light by a single slit. The graph is as shown. We have, therefore, several bands of periods, whereas if there is only one period, we would have a single line.



Even if we neglect all the minor maxima, we will still have to consider all the periods embodied in the central region. The width of this region depends upon the value of d in reference to λ. Let there be n whole waves in the interval considered. It is quite easy to determine the inaccuracy in assigning only one period T_0 to the band. The order of magnitude of the error may be measured by the range of period between the central maximum and the first minimum. The latter is given by

$$\sin (k_0 - k)d/2 = \pi$$

Changing into period, we have

$$\frac{2\pi}{T_0 V} - \frac{2\pi}{TV} = \frac{2\pi}{d} - \frac{2\pi}{nT_0 V}$$

or

$$T = T_0 n/(n - 1)$$

The difference of this and the period T_0 which corresponds to the central maximum is

$$\Delta T = T_0 \frac{1}{n-1}$$

It is thus seen that it would be quite impossible to assign a period if there is only one complete pure wave in the interval; but if n is large, the period corresponding to the central maximum which is also the average period in the interval, would give a very fair approximation.

III. THEORY OF THE FLANK WAVE

1. General Introduction

Owing to the ease of visualization and the mathematical simplicity, the method of geometrical optics as applied to the propagation of elastic waves has met with great success in the past and has been responsible for most of the early development in the science of seismology. Even at present, it is still very important in the more complicated problems in that it offers the only feasible way to map out the approximate geometry of the wave paths. Nevertheless, the method has severe limitations especially with regard to the distribution of energy. That it gives only the first approximation of the true picture has also been recognized.

A better approach to the calculation of energy is from the concept of plane waves. The intensity of energy is then derived from the wave amplitude. The calculation is still simple, and the wave paths thus found are precisely the same as those obtained from the ray method, because Snell's law still holds. However, this latter follows not from Fermat's principle, but from the boundary conditions which also determine the partition of the wave amplitudes in reflection and refraction.

In electromagnetic waves which have only transverse components (or in acoustic waves which are longitudinal), the reflected and refracted amplitudes are given by the well known Fresnel's formulae. The energy relation can then be readily derived. In seismic waves, the situation is a little more complicated because of the coexistence of both the longitudinal and the transverse waves. From the continuity of the displacements and the continuity of stresses at the boundary, four linear algebraic equations connecting the different amplitudes may be obtained. The solution of these is thus very simple. All these amplitudes must also satisfy the energy relations which have been derived by Knott and by Blut. The relations given by Knott are expressed in terms of displacement potentials and those of Blut, in terms of the displacements themselves. As an illustration, we will give a simplified derivation of Blut's equations.*

Consider a longitudinal wave incident on a plane boundary. Let A, A_r, B_r, A_f, B_f be the amplitudes of the various waves where A signifies longitudinal, B, transverse, subscript r, the reflected and subscript f the refracted waves. Let the corresponding angles with the normal be denoted by θ, θ_r, ϕ_r, θ_f, ϕ_f (Fig. 1). Now for a plane sinusoidal wave of the form

$$u = Ae^{i(\omega t - \frac{r}{v})}$$

where r is in the direction of propagation, the mean energy per unit volume is given by

$$\frac{1}{2}\rho\overline{\dot{u}^2} = \frac{1}{4}\rho A^2\omega^2$$

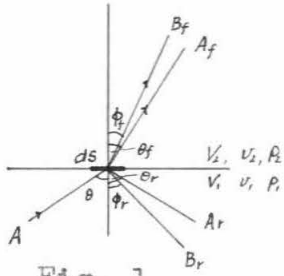


Fig. 1

(1) C.G. Knott, Phil. Mag. (5)48(1899)64-97.
(2) H. Blut, Zeit. f.Geophys. 8(1932) 130-144, 305-322
* In both Blut's papers and in the abstract by Macelwane, the energy relations are obtained by way of the energy integral and the stress-strain relations. However, if the displacements are already known as functions of time, this detour seems to me quite unnecessary, as is indeed shown above.

ρ being the density. The mean energy flux contributed by the incident wave through an element of area ΔS on the boundary is evidently

$$\frac{1}{4}\rho A^2\omega^2 V \Delta S \cos\theta = \frac{1}{8}\rho A^2\omega^2 V \frac{\sin 2\theta}{\sin\theta}\Delta S$$

Equating similar expressions for each of the five waves passing through ΔS, we obtain immediately

$$\frac{\rho_1 A^2 V_1 \sin 2\theta}{\sin\theta} = \frac{\rho_1 A_r^2 V_1 \sin 2\theta_r}{\sin\theta_r} + \frac{\rho_1 B_r^2 v_1 \sin 2\phi_r}{\sin\phi_r} + \frac{\rho_2 A_f^2 V_2 \sin 2\theta_f}{\sin\theta_f} + \frac{\rho_2 B_f^2 v_2 \sin 2\phi_f}{\sin\phi_f}$$

With the aid of Snell's law, this can be written as

$$1 = \frac{A_r^2}{A^2} + \frac{B_r^2}{A^2}\frac{\sin 2\varphi_r}{\sin 2\theta} + \frac{\rho_2}{\rho_1}\frac{A_f^2}{A^2}\frac{\sin 2\theta_f}{\sin 2\theta} + \frac{\rho_2}{\rho_1}\frac{B_f^2}{A^2}\frac{\sin 2\varphi_f}{\sin 2\theta}$$

In exactly the same manner and with similar notations, we have for the case of an incident SH wave,

$$1 = \frac{B_r^2}{B^2} + \frac{\rho_2}{\rho_1}\frac{B_f^2}{B^2}\frac{\sin 2\varphi_f}{\sin 2\varphi}$$

and for the case of an incident SV wave,

$$1 = \frac{A_r^2}{B^2}\frac{\sin 2\theta_r}{\sin 2\varphi} + \frac{B_r^2}{B^2} + \frac{\rho_2}{\rho_1}\frac{A_f^2}{B^2}\frac{\sin 2\theta_f}{\sin 2\varphi} + \frac{\rho_2}{\rho_1}\frac{B_f^2}{B^2}\frac{\sin 2\varphi_f}{\sin 2\varphi}$$

It should be noted here that both the above equations and those of Knott are derived under the assumption that the waves are plane. In case this assumption is not justified, phenomena may exist such that they cannot be explained in the light of the above picture. For the lack of a better name, we shall call these phenomena second order effects. Their existence should not be interpreted as a contradiction of the energy equations, but rather that they demand a modification of the equations themselves. In wave propagation, it is the wave equation and the boundary conditions which are more fundamental. Fresnel's equations in optics and Zoeppritz' equations in seismology, as well as the energy -

equations of Knott or Blut, are all special cases applicable to plane waves only.

Probably the most important as well as the most interesting phenomena which shows the inadequacy of the plane wave assumption, is the refraction of waves in layered media. In both theoretical and applied seismology, the basic formula for the calculation of the thickness of a crustal layer by the travel times of the refracted waves is derived purely from the consideration of geometrical optics. The geometry of the ray path is as shown in the diagram where the ray coming from the source is supposed to be incident on the lower layer at the critical angle, travels along a path parallel to the interface and emerges again at the same angle and is then recorded at the earth's surface. The wave can be observed at different points beyond a certain epicentral region. The path thus constructed agrees with that of the least time. The travel time curve is a straight line, as is confirmed by observations. At first, this appeared to be the correct interpretation. But seismology has been developed to such a stage that one can no longer be content with only a superficial agreement. The above picture, though simple and successful as it is, actually presents grave difficulties. First, if we take both contiguous media to be homogeneous, there is no reason that a ray after entering the lower medium should return to the upper one. Even if it could return, as by assuming an increase of wave velocity with depth

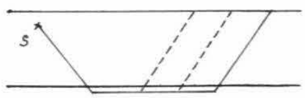


Fig. 2

it should not be observable in a wide region. Second, the grazing rays can come from the source in only one direction. The energy included in a narrow pencil of rays is small. This is also evident from Blut's equations, for if θ_f or ϕ_f is a right angle, sin $2\theta_f$ or sin $2\phi_f$ will vanish. Yet the observed energy of the refracted wave is quite appreciable. These difficulties are due to the fact that when the source is at a finite distance, as is assumed, the wave can no longer be regarded as plane. The interpretation of seismic refraction must therefore be sought directly from the wave equation and the boundary conditions.

The problem of reflection and refraction of seismic waves has ~~actually~~ been studied from the wave theory by Jeffrey's[1] for two particular cases and with the use of the operator method. It was again studied by Muskat[2] for a more general case with the ordinary differential equation analysis. The emphasis of the latter was, however, laid on the justification of the minimal time paths. His final results, though only qualitative, and finally become quite involved, ~~and~~ are really very interesting. So far as mathematical formulism is concerned, the problem may be regarded as solved.

The problem has also been attacked with some success by

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(1) H. Jeffreys, Phil. Mag. 23(1926)472-481; Gerl. Beit. z. Geophys. 30(1931)336-350.

(2) M. Muskat, Physics, 4(1933)14-28

the Japanese authors, among whom are Sezawa, Nakano and Sakai,[1] to mention just a few. The latter made an extensive study of the case of an internal source emitting spherical waves which then are incident on a free boundary. The problem was examined quite exhaustively and may appropriately be regarded as a continuation of Lamb's monumental work.(2) However, since the boundary was assumed free, the question of refraction as discussed above, naturally would not arise.

Yet crucial experiments were performed **recently** by a German scientist, O.V. Schmidt, who showed visually not only what is actually happening at the boundary, but also proved beyond doubt that the wave which we usually attributed to refraction is really a different entity. His experiment will be described in the next section where a mathematical formulation of his theory is also attempted. Since then, theories have been worked out by several authors who linked more closely the propagation of seismic waves with that of electromagnetic radiation. It thus appears that there is still room for more examination. In the later sections, we will approach the problem in a slightly different manner from that of Muskat and thus bring out a few points which were not emphasized in his work. Since this so-called refracted wave is really not what is meant in the ordinary sense of the word, we will follow Ott[3] and call it the "Flank Wave".

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(1) T. Sakai, Proc. Phys. Math. Soc. Jap. 15(1933)291; Geophys. Mag. 8(1934)1-71
(2) H. Lamb, Phil. Trans. Lond. (A)203(1904)1-41
(3) H. Ott, Ann. d. Physik, (5) 41(1942)443-466

2. Schmidt's Experiment and the Theory of Characteristics.

In his paper, "Ueber Knallwellenausbreitung in Flüssigkeiten und festen Korpern", published in 1938, O.V. Schmidt[1] showed a series of spark photographs for the propagation of sound waves in two contacting media. The outstanding feature of these photographs is the occurrence of a conical wave front of considerable intensity in the rarer medium (of lower velocity). His results may be graphically represented as follows:

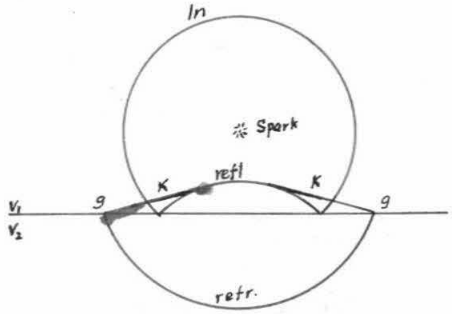

Fig. 3 (liquids)

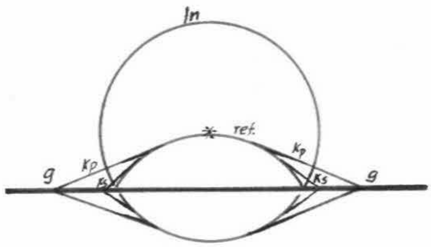

Fig. 4 (solid in water)

The first picture is for two liquids (xylol on NaCl solution) in contact, and the second for solid (aluminum plate or glass) in water. Considering only the first case, we see that the whole wave-pattern consists of four distinct parts: the incident, the reflected, the transmitted and an additional conical wave front. This new wave front is inclined to the interface at exactly that angle as is required by the law of total reflection; namely, if v_1 is

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(1) O.V. Schmidt, Phys. Zeit. 39(1938)868-875

the wave velocity in the upper and rarer medium and v_2 that in the lower medium, then the angle of inclination θ of the conical surface to the plane of contact is given by

$$\sin \theta = v_1 / v_2$$

Since Fresnel's formulae admit of only three of these wave fronts, Schmidt concludes that there is a breakdown of the analogy between geometrical optics and seismology. He interprets the phenomenon by use of an analogy with the shock wave in ballistics. Since the wave trace of the transmitted wave at the boundary travels with a velocity which exceeds the velocity of sound in the upper medium, Schmidt regards this fourth wave front as a "head wave", due to this cause.

In applied seismology, a similar mechanism has been suggested by C.H. Dix.[1] The wave which we have discussed in the previous section and which is supposed to have traveled along the path of Fig. 2, is nothing else than the new wave front in Schmidt's photographs. In the light of the latter, the process should not appropriately be called a refraction.

We will now give a brief mathematical formulation of the head-wave theory. Exactly as in the case of the shock-waves, it is based on the theory of characteristics of a differential equation. Without going into mathematical details, we may say that the characteristic is a surface on which the solution of a differential equation is discontin-

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(1) C.H. Dix, Geophysics, IV(1939)238-241

uous, so that it cannot be uniquely determined from the initial data. This is what we have observed physically as a wave front defined as a traveling surface of discontinuity.

Let us now proceed to find the characteristics, or more exactly the bicharacteristics of the wave equation. Assume that the elastic media are homogeneous and isotropic on both sides of the contact. Then the wave equation is of the form

$$\nabla^2 \phi = \frac{1}{V^2} \frac{\partial^2 \phi}{\partial t^2}$$

where ϕ signifies either the dilatation or the rotation according to whether the wave is longitudinal or transverse. Written in cylindrical coordinates, this becomes

$$\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2} + \frac{\partial^2 \varphi}{\partial z^2} = \frac{1}{V^2}\frac{\partial^2 \varphi}{\partial t^2}$$

Let the source be in the medium 1 in which the wave velocity is v_1. Let the wave velocity in medium 2 be v_2, and $v_1 < v_2$. For an observer moving with the velocity v_2, we can make the substitution $r = \rho - v_2 t$. Then the wave trace would be stationary with respect to this moving observer. We have thus

$$\frac{\partial^2}{\partial r^2} = \frac{\partial^2}{\partial \rho^2}\ , \qquad \frac{\partial^2}{\partial t^2} = V_2^2 \frac{\partial^2}{\partial \rho^2}$$

Since there is an axial symmetry in our problem, the partial derivative with respect to the azimuth θ drops out and we have

$$\left(1-\frac{v_2^2}{v_1^2}\right)\frac{\partial^2\varphi}{\partial\rho^2}+\frac{1}{r}\frac{\partial\varphi}{\partial r}+\frac{\partial^2\varphi}{\partial z^2}=0$$

The differential equation of the characteristics is

$$\left(1-\frac{v_2^2}{v_1^2}\right)\left(\frac{\partial f}{\partial\rho}\right)^2+\left(\frac{\partial f}{\partial z}\right)^2=0$$

The Cauchy's characteristics of this equation are given by

$$d\rho\Big/\left(1-\frac{v_2^2}{v_1^2}\right)f_\rho = dz/f_z$$

Eliminating f_ρ and f_z, we obtain

$$\frac{dz}{d\rho}=\left(\frac{v_2^2}{v_1^2}-1\right)^{-\frac{1}{2}} \qquad \text{or} \qquad \theta=\sin^{-1}\frac{v_1}{v_2}$$

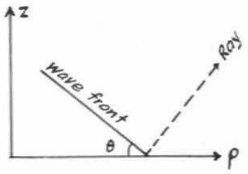

Fig. 5

This is the slope of the wave trace in the p-z plane. Evidently the angle θ is given by $\sin\theta = v_1/v_2$ which is the angle for total reflection. Since it is independent of the azimuth, the wave surface must be a cone. With respect to a stationary observer, this wave will travel forward with a velocity v_2 in the positive ρ-direction. In the direction normal to the front, the velocity of propagation is evidently v_1 .

It should be pointed out here that in either the present treatment or in that of Dix, the intensity of the wave cannot be obtained. The interpretation of Dix is based on the elementary form of the Huygens' principle which has long been known to be inadequate. In the Kirchhoff's formulation, it requires the wave functions on both sides of the boundary to determine the solution uniquely.

The interpretation of Schmidt's experiment from the point of view of the wave theory was first given by Joos and Teltow[1] for the particular case when the source is exactly on the boundary. The result was taken directly from Sommerfeld's classical work[2] on the attenuation of wireless waves. The solution was extended to the case when the source is away from the boundary by Ott[3] with the use of Weyl's[4] method which is a modification of that of Sommerfeld. The detailed paper of Krüger[5] published recently gave the solution even more mathematical rigor and brought out the identity of the two methods of solution.

The results presented in the next sections were actually worked out just after the appearance of the paper of Joos and Teltow and were intended as an extension of their results. The method of solution is the one originally used by Sommerfield in his 1909 paper. When the pole of the integral is very near to its branch point, the method is not very accurate. But the method is much simpler than the others, and for our purpose it seems that the approximation is good enough to bring out the salient points. Besides, it indicates a general character of the problem which has already been exemplified by Muskat's work. We will illustrate this point further by working out the case of the SH wave also. For completeness, we will give a brief account of the setting up of the wave equations and the conditions at the boundary.

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(1) G. Joos and J. Teltow, Phys. Zeit. 40(1939)289-293.
(2) A. Sommerfeld, Ann. d. Phys. 28(1909)665-736
(3) H. Ott, loc.cit.
(4) H. Weyl, Ann. d. Phys. 60(1919)481-500
(5) M. Kruger, Zeit. f. Physik, 121(1943)377-437

3. Equations of Motion and Boundary Conditions.

A. The Equations of Motion. It has been shown by Love that longitudinal and transverse waves cannot be saparated if gravity is taken into account. However, the effect is very small; henceforth in our discussion, the dispersion of elastic waves due to gravity or any other external force will be neglected.

The equations of motion for an isotropic, homogeneous elastic medium are

(1) $$\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z}, \quad etc.,$$

where u, v, w are the components of the displacement, X_x, Y_x, etc. components of the stress and ρ the density of the medium. Substituting from the stress-strain relations

(2) $$X_x = \lambda \delta + 2\mu \frac{\partial u}{\partial x}$$

$$X_y = \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \qquad etc.$$

where $\delta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$ is the dilation and λ, μ are the Lamé constants, we get readily

(3) $$(\lambda + \mu) \nabla \delta + \mu \nabla^2 \bar{S} = \rho \frac{\partial^2 \bar{S}}{\partial t^2}$$

Here, $\bar{S} = u\,\bar{i} + v\,\bar{j} + w\,\bar{k}$ is the displacement vector. By performing the operations of divergence and curl on both sides of (3), we get the usual wave equations for dilation and distortion.

(4) $$(\lambda + 2\mu) \nabla^2 \delta = \rho \frac{\partial^2 \delta}{\partial t^2}$$

(5) $$\mu \nabla^2 \bar{\varpi} = \rho \frac{\partial^2 \bar{\varpi}}{\partial t^2}$$

where $\bar{\varpi} = \frac{1}{2}\nabla \times \bar{s}$ is the rotation.

Wave equations are usually expressed in terms of functions other than δ and $\bar{\varpi}$, because in so doing, the equations might admit more readily of solutions. In this connection, we may apply the well known theorem that any vector may be split into two parts which are respectively lamellar and solenoidal only, viz.,

(6) $$\bar{S} = \nabla\phi + \nabla \times \bar{A}$$

The first part accounts for the dilatation and the second for the distortion. Substituting from (6) the value of $\bar{S}$ in (3), we obtain

$$\nabla[(\lambda+2\mu)\nabla^2\phi - \rho\frac{\partial^2\phi}{\partial t^2}] + \nabla\times[\mu\nabla^2\bar{A} - \rho\frac{\partial^2\bar{A}}{\partial t^2}] = 0$$

Since $\bar{A}$ is arbitrary to the extent of an additional $\nabla\alpha$, α being any scalar, we can choose $\bar{A}$ in such a way that

(7) $$\mu\nabla^2\bar{A} = \rho\frac{\partial^2\bar{A}}{\partial t^2}$$

Then

(8) $$(\lambda+2\mu)\nabla^2\phi - \rho\frac{\partial^2\phi}{\partial t^2} = 0$$

B. The Dilatation and Distortion Functions The quantity $\bar{A}$ is a vector, but from the nature of the problem, we can usually consider only one of its components. This component can be represented by a scalar function such that its derivatives would conform to the condition of a curl and the function itself satisfies (7). To illustrate this, let us consider a plane wave. We can so orient our coordinate axes that one of them lies in the wave front. Let it be z. Then all the derivatives with respect to z will vanish. Writing out the u and v components of (6), we have

$$u = \frac{\partial \phi}{\partial x} + \frac{\partial A_z}{\partial y}$$

$$v = \frac{\partial \phi}{\partial y} - \frac{\partial A_z}{\partial x}$$

It is thus seen that both u and v depend on the derivatives of ϕ and A_z only. On the other hand, we have

$$w = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} + \frac{\partial \phi}{\partial z}$$

which is independent of ~~θ and~~ A_z. We can therefore write A_z as a scalar function ψ and obtain

$$u = \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y} \tag{9}$$

$$v = \frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial x}$$

$$w = w \tag{1}$$

These equations appear to have been first used by Green in his famous paper, "On the Reflexion and Refraction of Light". The functions ϕ and ψ were later applied to elastic displacement and are known as the dilatation and distortion functions. The corresponding wave equations are

$$(\lambda + 2\mu)\nabla^2 \phi = \rho \frac{\partial^2 \phi}{\partial t^2}$$

$$\mu \nabla^2 \psi = \rho \frac{\partial^2 \psi}{\partial t^2} \tag{10}$$

$$\mu \nabla^2 w = \rho \frac{\partial^2 w}{\partial t^2}$$

It must be noted that (9) and (10) are applicable to plane waves only.

Since we are more interested in waves which have symmetry about an axis, we will next find the appropriate dilatation and distortion functions for this case. For this purpose, we may express the vector $\bar{A}$ as the curl of another vector $\bar{B}$, Then

$$\nabla \times \bar{A} = \nabla \times \nabla \times \bar{B} = \nabla \nabla \cdot \bar{B} - \nabla^2 \bar{B}$$

If we assume that $\bar{B}$ has only the z-component, then we can write

$$\bar{S} = \nabla \varphi + \nabla \frac{\partial B_z}{\partial z} - \bar{k} \nabla^2 B_z$$

It is also evident that if B_z satisfies (7), so will $\bar{A}$. If we use cylindrical coordinates and write ψ for B_z, we may express the components of the displacement as follows:

$$u = \frac{\partial \phi}{\partial r} + \frac{\partial^2 \psi}{\partial r \partial z}$$

(11)

$$w = \frac{\partial \phi}{\partial z} + \frac{\partial^2 \psi}{\partial z^2} - \frac{\rho}{\mu} \frac{\partial^2 \psi}{\partial t^2}$$

The corresponding wave equations are:

$$(\lambda + 2\mu) \nabla^2 \phi = \rho \frac{\partial^2 \phi}{\partial t^2} \tag{12}$$

$$\mu \nabla^2 \psi = \rho \frac{\partial^2 \psi}{\partial t^2} \tag{13}$$

It should be pointed out here that in writing $\bar{A}$ as the curl of another vector which has only the z-component, we would be dealing with displacements in the plane of incidence only. This is suitable for the case of P and SV waves. Since the displacement in SH wave is perpendicular to the plane of incidence, we may treat it separately. For this case, we may assume that the vector $\bar{A}$ itself has only the z-component. On account of the axial symmetry, both the r- and z- components of the displacement vanish. Denoting the ϕ-component of the displacement by v and A_z by $-\chi$, we have

$$u = 0, \quad v = \frac{\partial \chi}{\partial r}, \quad w = 0 \tag{14}$$

and the wave equation

(15) $$\mu \nabla^2 \chi = \rho \frac{\partial^2 \chi}{\partial t^2}$$

C. Boundary Conditions. For a slipless contact of two media, the fundamental conditions at the boundary are the continuity of displacements and the continuity of stresses. For the case of plane waves, the conditions have been discussed in detail in Macelwane's book. We will discuss here the more general cases:

(i) When there are only longitudinal waves. We have then

$$\bar{s} = \nabla \varphi$$

The continuity of normal displacement gives

(16) $$\frac{\partial \phi_1}{\partial z} = \frac{\partial \phi_2}{\partial z} \qquad \text{at } z = 0.$$

The normal stress is

$$T_{zz} = \lambda \delta = \lambda \nabla^2 \varphi = \rho \frac{\partial^2 \phi}{\partial t^2}$$

The continuity of this gives

(17) $$\rho_1 \varphi_1 = \rho_2 \varphi_2 \qquad \text{at } z = 0,$$

since this condition should hold at all times.

(ii) When there are only SH waves. Here, the continuity of the ϕ-component of the displacement gives

$$\frac{\partial \chi_1}{\partial r} = \frac{\partial \chi_2}{\partial r} \qquad \text{at } z = 0.$$

But since this holds for all r's, we have

(18) $$\chi_1 = \chi_2 \qquad \text{at } z = 0.$$

The continuity of the tangential stress gives

(19) $$\mu_1 \frac{\partial \chi_1}{\partial z} = \mu_2 \frac{\partial \chi_2}{\partial z} \qquad \text{at } z = 0.$$

(iii) When both P and SV are present. For the present case, we have to use both the tangential and the normal components of the displacement and the stress.

From (11), the continuity of u gives

$$\frac{\partial \varphi_1}{\partial r} + \frac{\partial^2 \psi_1}{\partial r \partial z} = \frac{\partial \varphi_2}{\partial r} + \frac{\partial^2 \psi_2}{\partial r \partial z} \quad \text{at } z = 0. \tag{20}$$

The continuity of w gives

$$\frac{\partial \varphi_1}{\partial z} + \frac{\partial^2 \psi_1}{\partial z^2} - \frac{\rho_1}{\mu_1} \frac{\partial^2 \psi_1}{\partial t^2} = \frac{\partial \varphi_2}{\partial z} + \frac{\partial^2 \psi_2}{\partial z^2} - \frac{\rho_2}{\mu_2} \frac{\partial^2 \psi_2}{\partial t^2} \quad \text{at } z = 0. \tag{21}$$

The tangential stress is given by

$$T_{rz} = \mu \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right)$$

By equation (11), the continuity of this results in

$$\mu_1 \frac{\partial}{\partial r}\left[2 \frac{\partial \varphi_1}{\partial z} + 2 \frac{\partial^2 \psi_1}{\partial z^2} - \frac{\rho_1}{\mu_1} \frac{\partial^2 \psi_1}{\partial t^2} \right] = \mu_2 \frac{\partial}{\partial r}\left[2 \frac{\partial \varphi_2}{\partial z} + 2 \frac{\partial^2 \psi_2}{\partial z^2} - \frac{\rho_2}{\mu_2} \frac{\partial^2 \psi_2}{\partial t^2} \right] \text{at } z = 0. \tag{22}$$

The normal stress is given by

$$T_{zz} = \lambda \delta + 2\mu \frac{\partial w}{\partial z}.$$

The continuity of this gives

$$\mu_1 \left(\frac{k_1^2}{h^2} - 2 \right) \nabla^2 \varphi_1 + 2\mu_1 \left(\frac{\partial^2 \phi_1}{\partial z^2} + \frac{\partial^3 \psi_1}{\partial z^3} - \frac{\rho_1}{\mu_1} \frac{\partial^3 \psi_1}{\partial z \partial t^2} \right) = \mu_2 \left(\frac{k_2^2}{h_2^2} - 2 \right) \nabla^2 \phi_2 + 2\mu_2 \left(\frac{\partial^2 \phi_2}{\partial z^2} + \frac{\partial^3 \psi_2}{\partial z^3} - \frac{\rho_2}{\mu_2} \frac{\partial^3 \psi_2}{\partial z \partial t^2} \right) \tag{23}$$

at z = 0, *where* $k_1^2 = \omega^2 \rho / \mu$, $h^2 = \omega^2 \rho / (\lambda + 2\mu)$.

4. On Longitudinal Waves in Two Media in Contact

A. <u>The Problem</u>. Let us first consider the case which has been briefly described by Joos and Teltow. The source of disturbance is supposed to be exactly on the boundary between two media, and the transverse waves are absent. Admittedly, this situation is too much idealized, but the salient features of the method of solution appear to be more easily brought out by this simplification.

As shown in Fig 6, we take the source point as the origin of a cylindrical coordinate system and the surface of contact between the two media as the plane z = 0. If the media are assumed homogeneous, we should have a symmetrical wave pattern about the z-axis. Let the densities of the media be denoted by ρ_1 and ρ_2, the elastic constants by λ_1 and λ_2, and the wave velocities in the media by v_1 and v_2. The rigidities of both media are assumed zero, and the velocities refer to the longitudinal waves. Let a harmonic spherical wave train be generated at the source, and we will study the stationary state only. For this case, we have the radial and the vertical components of the displacement given by

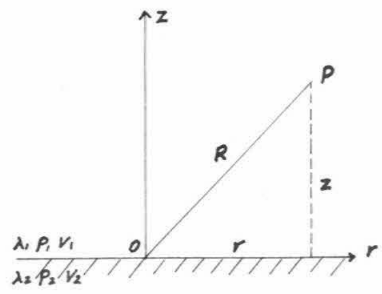

fig. 6.

(24) $$u = \frac{\partial\varphi}{\partial r}, \qquad w = \frac{\partial\varphi}{\partial z}$$

The wave equation for the stationary stste is

(25) $$\nabla^2\varphi + h^2\varphi = 0$$

where $h^2 = \omega^2/v^2$, the time factor $e^{-i\omega t}$ being omitted from all the calculations since it does not affect the results and can be taken into account at any time. The boundary conditions are

(26) $$\frac{\partial\varphi_1}{\partial z} = \frac{\partial\varphi_2}{\partial z}$$

(27) $$\rho_1\varphi_1 = \rho_2\varphi_2$$

at z = 0. These have been derived fully in the previous section. The present problem is to solve (25) under the conditions (26) and (27). In order to determine the solution uniquely, we must also impose the conditions that there are only divergent waves and that the wave functions must vanish at infinity. If there is absorption in the medium, viz., if h is complex, Sommerfeld [(1)] has proved that the solution of this problem is unique. But if neither of our media is absorptive, as we would assume that both h_1 and h_2 are real, the solution will not be unique in the sense that we can superpose a free oscillation on the solution. Since we are not interested in the free oscillations, we will assume that they are absent and consider at first that both h_1 and h_2 are complex and then make their imaginary parts approach zero.

B. Method of Solution. By the method of separation of variables, it is readily seen that the particular solution of (25) is of the form

$$C\, J_0(\xi r)\, e^{\pm\sqrt{\xi^2-h^2}\, z}$$

where J_0 is the usual notation for Bessel function of the first kind and zero order, ξ a parameter and C an arbitrary constant. To make the solution finite when z approaches infinity, we may write

$$C_1 J_0(\xi r)\, e^{-\sqrt{\xi^2-h^2}\, z} \qquad \text{for } z>0$$

$$C_2 J_0(\xi r)\, e^{+\sqrt{\xi^2-h^2}\, z} \qquad \text{for } z<0$$

the real part of the square root being always taken to be positive when ξ is real and large.

Since the wave equation is linear, we can write C_1 and C_2 as arbitrary functions of ξ and then integrate with respect to ξ from 0 to ∞ . The resulting integrals will still be solutions. Let the disturbance be a spherical wave of

(1) A. Sommerfeld, loc. cit.

the form e^{ihR}/R. Then we may write the expression of ϕ_1 and ϕ_2 as follows:

(28) $$\varphi_1 = \frac{e^{ih_1R}}{\rho_1 R} + \int_0^\infty f_1(\xi) J_0(\xi r) e^{-\sqrt{\xi^2-h_1^2}\, z} d\xi \qquad z > 0$$

(29) $$\varphi_2 = \frac{e^{ih_2R}}{\rho_2 R} + \int_0^\infty f_2(\xi) J_0(\xi r) e^{\sqrt{\xi^2-h_2^2}\, z} d\xi \qquad z < 0$$

where $R^2 = r^2 + z^2$, f_1 and f_2 are two arbitrary functions to be determined from the boundary conditions. The amplitudes of the waves in the two media are so chosen that they will satisfy the condition (27) when R is very small.

To determine f_1 and f_2, we will make use of the well known formulae

(30) $$\begin{aligned} \frac{e^{ihR}}{R} &= \int_0^\infty \frac{\xi}{\sqrt{\xi^2-h^2}} J_0(\xi r) e^{-\sqrt{\xi^2-h^2}\, z} d\xi \qquad z > 0 \\ &= \int_0^\infty \frac{\xi}{\sqrt{\xi^2-h^2}} J_0(\xi r) e^{\sqrt{\xi^2-h^2}\, z} d\xi \qquad z < 0 \end{aligned}$$

in the theory of Bessel functions which were due to Lamb[1] and Sommerfeld[2] independently. Substituting from (28), (29) and (30) in (27), we find

$$\int_0^\infty \left[\frac{\xi}{\sqrt{\xi^2-h_1^2}} + \rho_1 f_1(\xi)\right] J_0(\xi r) d\xi = \int_0^\infty \left[\frac{\xi}{\sqrt{\xi^2-h_2^2}} + \rho_2 f_2(\xi)\right] J_0(\xi r) d\xi .$$

By means of the Fourier-Bessel theorem, we can equate the integrands and obtain

(31) $$\frac{\xi}{\sqrt{\xi^2-h_1^2}} - \frac{\xi}{\sqrt{\xi^2-h_2^2}} = \rho_2 f_2(\xi) - \rho_1 f_1(\xi).$$

To satisfy (27), we note that $\frac{\partial}{\partial z}\frac{e^{ihR}}{R} = 0$ when $z = 0$. Differentiating under the integral signs ~~of~~ the second parts of ϕ_1 and ϕ_2 and then ~~integ~~equating the integrands, we have

(32) $$-\sqrt{\xi^2-h_1^2}\; f_1(\xi) = \sqrt{\xi^2-h_2^2}\; f_2(\xi).$$

(31) and (32) are two linear algebraic equations for f_1 and

(1) H. Lamb, loc. cit. (2) A. Sommerfeld, loc. cit.

f_2, and the solutions are readily obtained as

$$(33)\qquad \begin{aligned} f_1(\xi) &= \frac{\xi}{\sqrt{\xi^2-h_1^2}}\;\frac{\sqrt{\xi^2-h_1^2}-\sqrt{\xi^2-h_2^2}}{P_1\sqrt{\xi^2-h_2^2}+P_2\sqrt{\xi^2-h_1^2}} \\ f_2(\xi) &= \frac{\xi}{\sqrt{\xi^2-h_2^2}}\;\frac{\sqrt{\xi^2-h_2^2}-\sqrt{\xi^2-h_1^2}}{P_1\sqrt{\xi^2-h_2^2}+P_2\sqrt{\xi^2-h_1^2}} \end{aligned}$$

Substituting (33) in (28) and (29) and simplifying, we get the final integral expressions for ϕ_1 and ϕ_2 as follows:

$$(34)\qquad \varphi_1 = \int_0^\infty \frac{P_1+P_2}{P_1 N}\, J_0(\xi r)\, e^{-\sqrt{\xi^2-h_1^2}\,z}\,\xi\, d\xi \qquad z>0,$$

$$(35)\qquad \varphi_2 = \int_0^\infty \frac{P_1+P_2}{P_2 N}\, J_0(\xi r)\, e^{\sqrt{\xi^2-h_2^2}\,z}\,\xi\, d\xi \qquad z<0.$$

where

$$N = P_1\sqrt{\xi^2-h_2^2} + P_2\sqrt{\xi^2-h_1^2}$$

In these integrals, the paths of integration as well as the constants h_1 and h_2 are real. However, in order to use the method of contour integrals, we consider h_1 and h_2 as complex numbers and then assume their imaginary parts approaching zero. The integrations will then be carried out in a complex ξ-plane.

Since $J_0(\xi r)$ is infinite when $\xi \to \pm i\infty$, we will split it into the form

$$2J_0(\xi r) = H_1(\xi r) + H_2(\xi r)$$

where H_1 and H_2 are the two Hankel functions whose asymptotic expressions are

$$(36)\qquad \begin{aligned} H_1(x) &\sim \sqrt{\frac{2}{\pi x}}\, e^{i(x-\frac{\pi}{4})} \\ H_2(x) &\sim \sqrt{\frac{2}{\pi x}}\, e^{-i(x-\frac{\pi}{4})}. \end{aligned}$$

H_1 vanishes in the first and H_2, in the fourth quadrant when $x \to \infty$. Hence (34) and (35) may be rewritten in the forms

(37) $$\phi_1 = \int_0^\infty \frac{\rho_1+\rho_2}{2\rho_1 N} H_1(\xi r) e^{-\sqrt{\xi^2-h_1^2}\, z} \xi d\xi + \int_0^\infty \frac{\rho_1+\rho_2}{2\rho_1 N} H_2(\xi r) e^{-\sqrt{\xi^2-h_1^2}\, z} \xi d\xi , \qquad z>0,$$

(38) $$\varphi_2 = \int_0^\infty \frac{\rho_1+\rho_2}{2\rho_2 N} H_1(\xi r) e^{+\sqrt{\xi^2-h_2^2}\, z} \xi d\xi + \int_0^\infty \frac{\rho_1+\rho_2}{2\rho_2 N} H_2(\xi r)^{+\sqrt{\xi^2-h_2^2}\, z} \xi d\xi \qquad z<0,$$

where the first integrals are carried out in the first quadrant and the second integrals in the fourth.

The singular points of the integrands are the two branch points h and h and the pole given by N = o. We will assume that in our problem, the upper medium is rarer so that $\rho_2 > \rho_1$ and $h_1 > h_2$, ($v_2 > v_1$). In Sommerfeld's solution of the electromagnetic problem, the integrals around the poles will give rise to terms which are inversely proportional to $\sqrt{r}$. These terms were identified by Sommerfeld as the surface waves. It is on the realization of these surface waves that controversies have arisen, and a large number of papers have been published all centering on the rigorous evaluations of Sommerfeld's integrals. But in our case where both h_1 and h_2 are real, and $h_1 > h_2$, and also $\rho_2 > \rho_1$, this question of surface waves will not arise because the pole of the integral lies in an inaccessible region of the complex plane. To show this, we have, when N = 0,

$$\rho_1^2 (\xi^2 - h_2^2) = \rho_2^2 (\xi^2 - h_1^2).$$

Hence

$$\xi^2 = \frac{\rho_1^2 h_2^2 - \rho_2 h_1^2}{\rho_1^2 - \rho_2^2}.$$

By our assumed inequalities, both the numerator and the denominator are negative. It follows that ξ^2 is positive and ξ is real. Now

$$-\frac{\rho_1}{\rho_2} = \frac{\sqrt{\xi^2-h_1^2}}{\sqrt{\xi^2-h_2^2}} \qquad (N=0).$$

The left hand side is negative and real. We have only to vary ξ along the real axis to see whether this is possible.. When ξ is greater than both h_1 and h_2, both radicals should be taken positive according to our previous conventions which are required for the convergence of the integrals. Hence the ratio is positive. When ξ is greater than h_2, but smaller than h_1, then $\sqrt{\xi^2 - h_1^2}$ is purely imaginary, and the above ratio is not real. When ξ is smaller than both h_1 and h_2, then the arguments of both radicals change by $-\pi$ (we will draw our path of integration below the two branch points, because we will assume that they approach the real axis from the first quadrant). Their ratio is, therefore, again positive. Thus, there is no point in the sheet of Riemann's surface chosen that will make N vanish. We may, therefore, disregard the surface wave entirely in the present problem.

The integral along the real axis in the first quadrant may be replaced by the sum of the integrals along OB, along a curve at infinity and around the branch cuts drawn from h_1 and h_2 to infinity (Fig. 7). In the fourth quadrant, the integral along OA may be replaced by the sum of integrals along OB$'$ and along a curve at infinity. The two integrals along paths at infinity vanish on account of the asymptotic properties of H_1 and H_2. The integrals along OB and OB$'$ cancel each other,

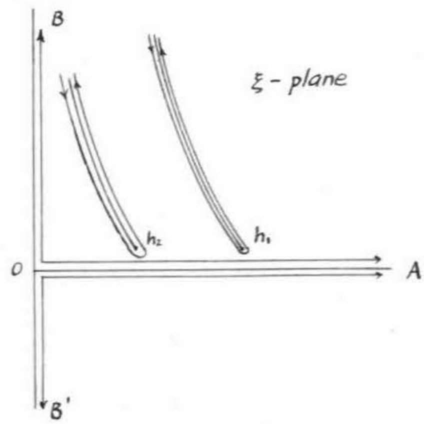

Fig 7

because

$$H_2(\xi r e^{-i\pi}) = -H_1(\xi r).$$

Hence we have

$$\int_0^{i\infty} H_2(\xi r e^{-i\pi}) \frac{\rho_1+\rho_2}{2\rho_1 N} e^{-\sqrt{\xi^2-h_1^2}\,z}\xi d\xi = -\int_0^{i\infty} H_1(\xi r) \text{ ----} d\xi$$

The right hand side is $-I_{OB}$. On the left hand side, let $\xi e^{-i\pi} = \xi'$. We obtain at once

$$I_{OB} + I_{OB'} = 0$$

Thus, our integrals are reduced to those components along the branch cuts only. Let us denote the integral for ϕ_1 by R and that for ϕ_2 by R'.

Since we take the real parts of $\sqrt{\xi^2-h_1^2}$ and $\sqrt{\xi^2-h_2^2}$ to be positive so long as we do not cross the cuts, we will draw the latter in such a way that along them, the real parts of the radicals vanish. Then

$$R_1 = \frac{\rho_1+\rho_2}{2\rho_1}\int \frac{\xi d\xi}{N} H_1(\xi r) e^{-\sqrt{\xi^2-h_1^2}\,z} \tag{39}$$

where R_1 is the integral around the branch point h . Let $\sqrt{\xi^2-h_1^2} = \pm i\beta$ along two sides of the cut and substitute in the above equation. We have then

$$R_1 = \frac{\rho_1+\rho_2}{2\rho_1}\int_{\beta=-\infty}^{\beta=+\infty} \frac{H_1(\xi r)e^{-i\beta z}\xi d\xi}{\rho_2 i\beta + \rho_1\sqrt{-\beta^2+h_1^2-h_2^2}} \tag{40}$$

This integral has been evaluated by Sommerfeld and is equal to

$$R_1 = -\frac{ih_1(\rho_1+\rho_2)}{\rho_1}\left[\frac{\rho_2}{\rho_1^2(h_1^2-h_2^2)} + \frac{z}{\rho_1\sqrt{h_1^2-h_2^2}}\right]\frac{e^{ih_1 r}}{r^2} \qquad z>0, \tag{41}$$

provided that r is very large and z is small compared with r, because the asymptotic expansion of H_1 has been used. Replacing $-\sqrt{\xi^2-h_1^2}\,z$ by $\sqrt{\xi^2-h_2^2}\,z$ in (39) and $-i\beta z$ by $\sqrt{-\beta^2+h_1^2-h_2^2}$ in (40) we can eveluate the integral in a similar manner and obtain

$$R_1' = -\frac{ih_1(\rho_1+\rho_2)}{\rho_1^2(h_1^2-h_2^2)}\frac{1}{r^2}e^{ih_1 r+\sqrt{h_1^2-h_2^2}\,z} \qquad z<0, \tag{42}$$

for the second medium. In exactly the same manner, we can evaluate the integrals around the other branch point h_2 and obtain

$$R_2 = -\frac{ih_2(\rho_1+\rho_2)}{\rho_2^2(h_2^2-h_1^2)}\frac{1}{r^2}e^{ih_2 r-\sqrt{h_2^2-h_1^2}\,z} \qquad z>0 \tag{43}$$

$$R_2' = -\frac{ih_2(\rho_1+\rho_2)}{\rho_2}\left[\frac{\rho_1}{\rho_2^2(h_2^2-h_1^2)} + \frac{z}{\rho_2\sqrt{h_2^2-h_1^2}}\right]\frac{e^{ih_2 r}}{r^2} \qquad z<0 \tag{44}$$

We recall here that R_1 and R_2 are the two space waves in medium 1 and R_1' and R_2' are those in medium 2. We shall discuss some of the consequences of these equations in the following section.

C. Discussion of Results. In solving the problem, we have tacitly assumed the following:

1. Only longitudinal waves exist;
2. The disturbance is represented by a harmonic wave of angular frequency ω;
3. For the media, $\rho_2 > \rho_1$, $v_2 (=\sqrt{\lambda_2/\rho_2}) > v_1 (=\sqrt{\lambda_1/\rho_1})$ and therefore $h_1 > h_2$;
4. A stationary state has been reached;
5. In evaluating the integrals, asymptotic expansions of H_1 and H_2 are used so that r is fairly large and $z \ll r$. However, it is known that the asymptotic expansions of the Hankel functions give very good approximations even when

their arguments are as small as 3.

6. The source of disturbance is exactly on the plane of the contact. This condition, however, can be generalized and we will treat the case when the source is at a small distance away from the plane of contact in a later section.

Equation (41) represents a space wave in medium 1 whose amplitude decreases as $1/r^2$ and increases linearly with z. But this increase cannot be indefinite, because z must be small compared wih r.

Equation (44) represents the corresponding wave in medium 2; but here, since $\sqrt{h_2^2 - h_1^2}$ is imaginary, the wave really consists of two parts whose phases differ by 90°. Both this wave and the above one are propagated in the direction of r and are, therefore, at grazing incidence to the plane of contact. The phase velocity of the wave in medium 1 is given by $v_1 = \omega/h_1$ and that in medium 2 given by $v_2 = \omega/h_2$, in agreement with our initial assumptions.

Since $\sqrt{h_1^2 - h_2^2}$ is real, so in (42), $\exp(h_1^2 - h_2^2)^{\frac{1}{2}} z$ decreases very rapidly with z $(z<0)$ and we have a wave propagating in the direction of r with a velocity v_1, but confined within a very thin layer below the interface.

Equation (43) is the most intetersing. Rewriting it as follows

(43a) $$R = A \frac{1}{r^2} e^{i(h_2 r - \sqrt{h_1^2 - h_2^2}\, z)}$$

where A is a constant amplitude factor, we see at once that this represents a conical wave propagating in the medium 1 with a phase velocity given by

(45) $$\omega/\sqrt{h_2^2+(h_1^2-h_2^2)} = \omega/h_1 = v_1 .$$

In any vertical plane passing through the source, the trace of the wave front is given by

(46) $$h_2 r - \sqrt{h_1^2-h_2^2}\ z = \text{constant}.$$

This is inclined to the interface at an angle given by

(47) $\theta = \text{arc tan } h_2/\sqrt{h_1^2-h_2^2} = \text{arc sin } h_2/h_1 = \text{arc sin } v_1/v_2$ which is exactly the angle required in total reflection.

Just to give an idea of the order of magnitude, we will estimate the ratio of the r-component of the displacement in this flank wave to that of the source. It is easy to see that this ratio $|Q_r|$ is given by

(48) $$|Q_r| \cong \left| \frac{h_2^2 \rho_1(\rho_1+\rho_2)}{r h_1 (h_1^2-h_2^2)} \right|$$

to the order of $1/r$. With period = 5 sec., v_1 = 3 km/sec, v_2 = 5 km/sed, = 2.5, ρ = 2.7 and r = 100 km, we have

$$|Q_r| = 2.3\ \%.$$

Since the present case is too particular, we will give a more detailed calculation for the case in which the source is not on the contact.

D. <u>Longitudinal Waves when the Source is not on the Boundary</u>. We will generalize the present problem by assuming that the source is at a distance "d" from the boundary. As before, we assume the solution to be of the form

(49) $$\phi_1 = \frac{e^{ih_1 R}}{\rho_1 R} + \int_0^\infty g_1(\xi) J_0(\xi r) e^{-\sqrt{\xi^2-h_1^2}\,z}\, d\xi \qquad z>0,$$

(50) $$\phi_2 = \int_0^\infty g_2(\xi) e^{\sqrt{\xi^2-h_2^2}\, z} J_0(\xi r)\, d\xi, \qquad z<0,$$

and proceed to determine $g_1(\xi)$ and $g_2(\xi)$ from the boundary conditions (26) and (27). There is only one term in ϕ_2, because the source is in the first medium. It should be noted that here $R^2 = r^2 + (z-d)^2$

In exactly the same manner as before, we obtain

(51) $$g_1(\xi) = \frac{\xi e^{-\sqrt{\xi^2-h_1^2}\,d}}{\rho_1\sqrt{\xi^2-h_1^2}} \; \frac{\rho_2\sqrt{\xi^2-h_1^2} - \rho_1\sqrt{\xi^2-h_2^2}}{N}$$

$$g_2(\xi) = \frac{2\xi e^{-\sqrt{\xi^2-h_1^2}\,d}}{\rho_1 N}$$

Since we are interested only in the waves in medium 1, we will investigate ϕ_1 only. We have then

(52) $$\phi_1 = \frac{e^{ih_1R}}{\rho_1 R} + \int_0^\infty \frac{\xi}{\rho_1} \frac{e^{-\sqrt{\xi^2-h_1^2}(d+z)}}{\sqrt{\xi^2-h_1^2}} \frac{\rho_2\sqrt{\xi^2-h_1^2} - \rho_1\sqrt{\xi^2-h_2^2}}{N} J_0\, d\xi, \quad z > 0$$

The integral represents the effect given rise by the discontinuity in the medium. We may rewrite it in the form

(52a) $$\phi_1 = \frac{e^{ih_1R}}{\rho_1 R} + \int_0^\infty \frac{\rho_2}{\rho_1}\frac{\xi}{N} e^{-\sqrt{\xi^2-h_1^2}(z+d)} J_0(\xi r)\, d\xi - \int_0^\infty \frac{\xi}{N} \frac{\sqrt{\xi^2-h_2^2}}{\sqrt{\xi^2-h_1^2}} e^{-\sqrt{\xi^2-h_1^2}(d+z)} J_0(\xi r)\, d\xi$$

The first integral is of the same form as (34) and presents no additional feature. Let us denote the second integral by I. Splitting J_0 as before and using the same kind of contours, we may write

(53) $$I = -\frac{1}{2}\int \frac{\xi}{N} \frac{\sqrt{\xi^2-h_2^2}}{\sqrt{\xi^2-h_1^2}} e^{-\sqrt{\xi^2-h_1^2}(z+d)} H_1(\xi r)\, d\xi$$

The branch points are still h_1 and h_2. Denoting $\xi^2 - h_2^2 = -\beta^2$, we have

$$I_2 = -\frac{1}{2}\int_{\beta=-\infty}^{\beta=+\infty} \frac{i\beta\,\xi\,H_1(\xi r)\,e^{-\sqrt{-\beta^2+h_2^2-h_1^2}\,(z+d)}}{\sqrt{-\beta^2+h_2^2-h_1^2}\,[i\rho_1\beta+\rho_2\sqrt{-\beta^2+h_2^2-h_1^2}]}\,d\xi$$

Integrating by parts and using the asymptotic expression

$$H_1'(\xi r) \sim \sqrt{\frac{2}{\pi\xi r}}\; e^{i(\xi r+\frac{\pi}{4})}$$

we obtain

$$I_2 = \frac{e^{i\frac{\pi}{4}}}{r\sqrt{2\pi r}}\int_{\beta=-\infty}^{\beta=+\infty} \frac{\xi\sqrt{\xi}\,e^{i\xi r}}{\beta}\frac{d}{d\beta}\,\frac{i\beta\,e^{-\sqrt{-\beta^2+h_2^2-h_1^2}\,(z+d)}}{\sqrt{-\beta^2+h_2^2-h_1^2}\,[i\rho_1\beta+\rho_2\sqrt{\beta^2+h_2^2-h_1^2}]}\,d\xi$$

By use of the formula

$$\lim_{r\to\infty}\int_0^{i\infty} e^{itr} f(t)\frac{dt}{\sqrt{t}} \to f(0)\sqrt{\frac{\pi}{r}}\,e^{i\frac{\pi}{4}},$$

the integral can be evaluated and we find

$$(54) \qquad I_2 = -\frac{i h_2}{\rho_2(h_2^2-h_1^2)}\,\frac{1}{r^2}\,e^{+i[h_2 r-\sqrt{h_1^2-h_2^2}\,(z+d)]}$$

Here, we get a similar equation to (43). The wave front is given by

$$(55) \qquad h_2 r - \sqrt{h_1^2-h_2^2}\,(z+d) = \text{constant},$$

or

$$h_2 r - \sqrt{h_1^2-h_2^2}\,z = \text{constant}$$

It is, therefore, a conical wave. The velocity of propagation is again

$$(56) \qquad \omega/\sqrt{h_2^2+h_1^2-h_2^2} = V_1$$

and the angle of inclination of the front to the plane of contact,

$$(57) \qquad \theta = \tan^{-1} h_2/\sqrt{h_1^2-h_2^2} = \sin^{-1} V_1/V_2$$

which are the same as we obtained before. The integral around h_1 will again give an inhomogeneous wave which is not of particular interest here. To calculate the ratios of the amplitudes, we have to evaluate the first integral of (52a). Comparing with (43), we see at once that it is equal to

$$-ih_2 \frac{\rho_2}{\rho_1} \frac{\rho_1}{\rho_2^2(h_1^2-h_2^2)} \frac{1}{r^2} e^{i[h_2 r - \sqrt{h_1^2-h_2^2}(z+d)]}$$

Combining this with (54), we get finally the expression for the flank wave

(58) $$R_2 = \frac{2ih_2}{\rho_2(h_1^2-h_2^2)} \frac{1}{r^2} e^{i[h_2 r - \sqrt{h_1^2-h_2^2}(z+d)]}$$

using the same notations as before. Equations (43) and (58) represent nothing else than the fourth wave fronts in v. Schmidt's photographs. The present solution is an extension of the result of Joos and Teltow, who treated only the case when the source is exactly on the boundary. It should be pointed out here that this method of evaluating the integrals is valid only when the velocity contrast is large; otherwise, the two branch points would be too near, and the present separation would break down. In addition, the formulae hold only when z is small and r large.

By equations (24), it is quite easy to calculate the ratios of the radial and vertical components of this flank wave to those of the source wave at the same point.
We get

(59) $$\left|\frac{u}{u_0}\right| = \frac{2h_2^2}{h_1(h_1^2-h_2^2)} \frac{\rho_1}{\rho_2} \frac{R^2}{r^3}$$

(60) $$\left|\frac{w}{w_0}\right| = \frac{2h_2}{h_1 d} \frac{1}{\sqrt{h_1^2-h_2^2}} \frac{\rho_1}{\rho_2} \frac{R^2}{r^2}.$$

Table 1.* Ratio of the Horizontal Displacements in the Flank Waves to those in the Source-Waves at the same point, with the following data: v_1 = velocity of waves in the rarer medium = 6 km/sec., γ = velocity ratio = v_2/v_1 = 1.2, α = density ratio = ρ_2/ρ_1 = 1.11, d = distance of the source drom the interface = 15 km., r = horizontal distance from the source, T = period of the waves.

T \ r (sec.)	100	200	400	600	1000	2000 kms
0.1	0.0339	0.0196	0.0098	0.0065	0.0039	0.0020
0.2	0.0798	0.0392	0.0196	0.0130	0.0078	0.0040
0.5	0.1994	0.0983	0.0489	0.0326	0.0196	0.0098
1.0	0.3988	0.1965	0.0978	0.0651	0.0391	0.0195

Table 2.* Ratios of the Vertical Displacements for the same Data as above

T \ r (sec.)	100	200	400	600	1000	2000 kms.
0.1	0.176	0.175	0.174	0.174	0.174	0.174
0.5	0.882	0.873	0.870	0.870	0.870	0.870

* Equations (59a) and (60a) hold only for wave lengths small compared with r. Hence T cannot be too large.

From these, we notice that the ratio of the horizontal components varies approximately as the inverse epicentral distance when the latter is large, while the ratio of the vertical components is almost constant. Contrary to the first order effect, these ratios depend not only on the density and velocity contrasts, but also on the velocities themselves. This is to be expected, because the amplitude of this flank wave varies as the inverse square of the distance.

For convenience of tabulation, let $\gamma = v_2/v_1$ be the velocity contrast, $\alpha = \rho_2/\rho_1$ the density contrast and T the period of the wave. Then (59) and (60) may also be written in the following forms:

$$\left|\frac{u}{u_0}\right| = \frac{T}{\pi}\,\frac{V_1}{\alpha}\,\frac{1}{\gamma^2-1}\,\frac{R^2}{r^3} \tag{59a}$$

$$\left|\frac{w}{w_0}\right| = \frac{T}{\pi}\,\frac{V_1}{\alpha}\,\frac{1}{\sqrt{\gamma^2-1}}\,\frac{R^2}{r^2 d} \tag{60a}$$

The tables are calculated for different values of periods and epicentral distances (valid only for small T, because of the approximation we have used).

To integrate equation (52a) around the branch point h_1, we see that the first integral is of the form (39), and it will give rise to a wave of the form (41) which is inhomogeneous, and its amplitude decreases as $1/r^2$. The second integral has a pole at h_1 coinciding with the branch point. However, the integral is convergent and can be integrated in a slightly different way. Rewrite it in the form

$$I_2 = \frac{1}{2}\int_{i\infty}^{(0-i0)} \frac{\xi}{N}\,\frac{\sqrt{\xi^2-h_2^2}}{\sqrt{\xi^2-h_1^2}}\, e^{-\sqrt{\xi^2-h_1^2}\,(z+d)}\, H_1(\xi r)\,d\xi .$$

Let us make a vertical cut passing through h_1 as shown in Fig. 8. Let $t = -h_1$. Then the limit of t will be from $i\infty$ to 0 and from 0 to $i\infty$. Making the substitution, we have

$$I_2 = \frac{1}{2}\left(\int_{i\infty}^{0}+\int_{0}^{i\infty}\right)\frac{\xi}{N}\,\frac{\sqrt{\xi^2-h_2^2}}{\sqrt{t(t+2h_1)}}\, e^{-\sqrt{t(t+2h_1)}\,(z+d)}\sqrt{\frac{2}{\pi\xi r}}\, e^{i(\xi r-\frac{\pi}{4})}\,dt$$

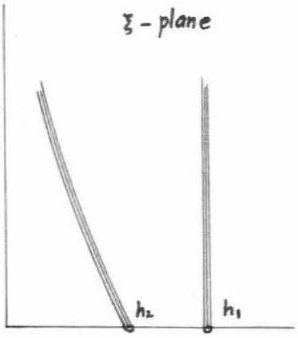

Fig. 8.

Since we have

(61) $$\lim_{r\to\infty}\int_{+i\infty}^{(0-i0)} e^{itr} f(t)\frac{dt}{\sqrt{t}} \to 2f(0)\sqrt{\frac{\pi}{r}}\, e^{i\frac{\pi}{4}}$$

we can also write the above equation as

(62) $$I_2 = \int_{0}^{i\infty}\frac{\xi}{N}\,\frac{\sqrt{\xi^2-h_2^2}}{\sqrt{t(t+2h_1)}}\, e^{-\sqrt{t(t+2h_1)}\,(z+d)}\sqrt{\frac{2}{\pi\xi r}}\, e^{i(\xi r-\frac{\pi}{4})}\,dt ,$$

whence we obtain

(63) $$I_2 = \frac{h_1}{\rho_1\sqrt{h_1^2-h_2^2}}\,\frac{\sqrt{h_1^2-h_2^2}}{\sqrt{2h_1}}\sqrt{\frac{2}{h_1}}\,\frac{1}{r}\,e^{ih_1 r} = \frac{e^{ih_1 r}}{\rho_1 r}$$

This is a cylindrical wave whose amplitude decreases only as the inverse first power of r. Hence at a large distance, this should dominate all the other waves we obtained before.

The above evaluation is made under the assumptions that r is large and z small compared with r. This means that the observation should be made very near to the boundary surface. When z is comparable with r, we can get a better approximation by taking out the factor $e^{-\sqrt{t(t+2h_1)}\,(z+d)}$ from f(t) and evaluating the integral

$$\int_0^{i\infty} e^{itr}\, e^{-\sqrt{t(t+2h_1)}\,(z+d)}\,\frac{dt}{\sqrt{t}}$$

instead of using (61). Let $t = s\, e^{i\frac{\pi}{2}}$. The above integral is then changed into

$$e^{i\frac{\pi}{4}}\int_0^{i\infty} e^{-sr}e^{-e^{i\frac{\pi}{4}}(z+d)\sqrt{s}\sqrt{2h_1+se^{i\frac{\pi}{2}}}}\ \frac{ds}{\sqrt{s}}$$

Making first the substitution $s = u^2$ and then the substitution $v = \sqrt{r}\,u$, we reduce the integral further into

$$\frac{2e^{-i\frac{\pi}{4}}}{\sqrt{r}}\int_0^{\infty} e^{-v^2-\frac{z+d}{\sqrt{r}}ve^{i\frac{\pi}{4}}\sqrt{2h_1+\frac{v^2}{r}e^{i\pi/2}}}\ dv$$

The second term under the radical sign may be neglected in comparison with the first. Completing the square in the exponent, the integral can readily be integrated, and we finally obtain

(61a) $$\lim_{r\to\infty}\int_{i\infty}^{(0-i0)} e^{itr}e^{-\sqrt{t(t+2h_1)}(z+d)}f(t)\frac{dt}{\sqrt{t}} \to 2f(0)\sqrt{\frac{\pi}{r}}\,e^{i\frac{\pi}{4}}e^{\frac{ih_1(z+d)^2}{2r}}$$

By use of this formula, we can get a second approximation of (62) which runs as:

(63a) $$I_2 = \frac{1}{P_1 r}e^{ih_1\left[r+\frac{1}{2}\frac{(z+d)^2}{r}\right]}$$

The wave front is, therefore, given by

(64) $$2r^2+(z+d)^2 = \text{const. } r$$

which is the equation of an ellipse with center off the origin. By rotating this around the z-axis, we will get a toroidal surface which seems to be strange. This is due to the various approximations we have made. We notice that the expression within the parenthesis in the exponent of (63a) is really the first two terms of the expansion of

$$R' = [r^2+(z+d)^2]^{\frac{1}{2}} = r+\frac{1}{2}\frac{(z+d)^2}{r}+\cdots$$

and $1/r$ is the first term of the expansion of $1/R'$, both being to the first power of $1/r$. Substituting from these in (63a), we get

$$I_2 = \frac{1}{P_1 R'} e^{ih_1 R'} \tag{63b}$$

which is simply a spherical wave from the image point of the source. This is to be expected when the velocity contrast is fairly large. Our calculations, therefore, indicate very clearly the trend of approximations to the true picture.

As a check of our calculations, we may point out that equations (58) and (63a) are exactly the same as the equations (18) and (20) in Muskat's[(1)] paper, which are obtained in a different way (there is a difference in the factor $1/P_1$ which we have assumed for the source, but this is quite arbitrary).

From these calculations, it seems that the method may also be applied to other problems of wave propagation in layered media. If the wave function in the rth medium is given by

$$\phi_r = \int_0^\infty F_r(\xi, h_1, h_2 \cdots h_r, \cdots h_n) J_0(\xi r)\, d\xi$$

with branch points at h_1, h_2, ..., we can obtain the different kinds of waves by evaluating the integrals around each of these. In particular, if a contiguous layer has a constant h_{r+1}, which is smaller than h_r, the integrals around this may give rise to a flank wave (the refracted wave from this layer in the customary usage of the term) due to this layer. To illustrate this point, we will apply the same

(1) M. Muskat, loc. cit.

method to the SH wave in a two-layer crust bounded by a free surface.

5. On SH Waves in a Two Layered Earth's Crust with a Free Boundary

A. The Problem. This problem has been examined in one aspect by Jeffreys[1] for the interpretation of the origin of Love waves. The present study differs from his both in the method of solution and in the point of view.

As shown in Fig. 9, we have a layer of thickness H resting on a mass of infinite depth. On top of this is free air. A source of spherical SH wave is situated at a distance d from the free boundary z = 0. According to section 3, let the distortion function be denoted by ψ. Using cylindrical coordinates, we have the components of displacement given by $(0, \frac{\partial \psi}{\partial r}, 0)$. Let the various constants be as shown in the figure. For the stationary state and for a source of the form

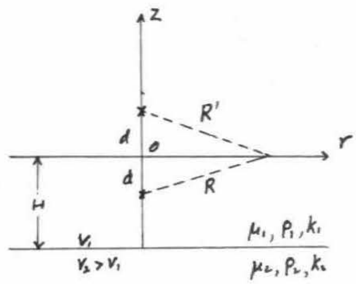

Fig. 9.

$$\psi_0 = \frac{e^{ik_1 R}}{R}$$

where $k_1 = \omega/v_1 = 2\pi/\lambda_1$ the wave equation is

$$\nabla^2 \psi + k^2 \psi = 0$$

and the boundary conditions are

(65) $$\frac{\partial \psi_1}{\partial z} = 0 \quad at \quad z = 0,$$

(1) H. Jeffreys, loc. cit.

(66) $$\psi_1 = \psi_2, \quad \mu_1\left(\frac{\partial \psi_1}{\partial z}\right) = \mu_2\left(\frac{\partial \psi_2}{\partial z}\right) \qquad at \quad z = -H.$$

B. The Formal Solution. For convenience, let us assume an image source of the same strength ~~put~~ at the image point. This is permissible because it is a solution of the wave equation. As in the case of longitudinal waves, designating ~~by~~ the wave function within the layer by ψ_1 and that below by ψ_2, we have evidently,

(67) $$\psi_1 = \frac{e^{ik_1R}}{R} + \frac{e^{ik_1R'}}{R'} + \frac{1}{2}\int_0^\infty A(\xi)e^{-\sqrt{\xi^2-k_1^2}\,z}J_0(\xi r)\,d\xi + \frac{1}{2}\int_0^\infty B(\xi)e^{\sqrt{\xi^2-k_1^2}\,z}J_0(\xi r)\,d\xi$$

(68) $$\psi_2 = \int_0^\infty C(\xi)e^{\sqrt{\xi^2-k_2^2}\,z}J_0(\xi r)\,d\xi$$

By (65), we obtain at once $A(\xi) = B(\xi)$. Thus ψ_1 may be written as

(67a) $$\psi_1 = \frac{e^{ik_1R}}{R} + \frac{e^{ik_1R'}}{R'} + \int_0^\infty A(\xi)\,ch\sqrt{\xi^2-k_1^2}\,z\cdot J_0(\xi r)\,d\xi$$

Let $$\gamma_1 = \sqrt{\xi^2-k_1^2}, \qquad \gamma_2 = \sqrt{\xi^2-k_2^2}.$$

Substituting into (66) with the aid of (30), we find

$$\frac{\xi}{\gamma_1}e^{\gamma_1(d-H)} + \frac{\xi}{\gamma_1}e^{-\gamma_1(d+H)} + A(\xi)\,ch\,\gamma_1 H = C(\xi)e^{-\gamma_2 H}$$

$$\mu_1\xi e^{\gamma_1(d-H)} + \mu_1\xi e^{-\gamma_1(d+H)} + \mu_1\gamma_1 A(\xi)\,sh\,\gamma_1(-H) = \mu_2\gamma_2 C(\xi)e^{-\gamma_2 H},$$

whence,

(69) $$A(\xi) = \frac{2\xi}{\gamma_1}\,\frac{e^{-\gamma_1 H}ch\,\gamma_1 d(\mu_1\gamma_1-\mu_2\gamma_2)}{\mu_1\gamma_1\,sh\,\gamma_1 H + \mu_2\gamma_2\,ch\,\gamma_1 H}$$

(70) $$C(\xi) = \frac{2\xi}{e^{-\gamma_2 H}}\,\frac{\mu_1 ch\,\gamma_1 d}{\mu_1\gamma_1\,sh\,\gamma_1 H + \mu_2\gamma_2\,ch\,\gamma_1 H}$$

With these values of A and C, equations (67) and (68) may be written as

(71) $$\psi_1 = \frac{e^{ik_1R}}{R} + \frac{e^{ik_1R'}}{R'} + \int_0^\infty \frac{\xi}{\gamma_1}\,\frac{ch\,\gamma_1 d(\mu_1\gamma_1-\mu_2\gamma_2)}{\mu_1\gamma_1\,sh\,\gamma_1 H + \mu_2\gamma_2\,ch\,\gamma_1 H}\left[e^{-\gamma_1(z+H)} + e^{\gamma_1(z-H)}\right]J_0(\xi r)\,d\xi$$

(72) $$\psi_2 = \int_0^\infty \frac{2\xi}{\gamma_2} e^{\gamma_2(z+H)} \frac{\mu_1 \gamma_2 \, ch\gamma_1 d}{\mu_1\gamma_1 \, sh\gamma_1 H + \mu_2\gamma_2 \, ch\gamma_1 H} J_0(\xi r)\, d\xi$$

The sign conventions of the square roots are the same as in the previous section, in order to insure the convergence of the integrals. Here, we notice that there are an infinite number of poles of the integrals. These have been studied by Jeffreys in connection with the surface waves. We are concerned in this section only with the flank wave , which again may be obtained by the integration around the branch point k_2, as will be shown below:

We will consider ψ_1, because this alone will give rise to waves propagating in the upper layer. The first two terms give the source and the wave reflected from the free surface. Of the two parts of the integral, the first one will account for the effect of the surface z = -H, and the second will account for the effect of the free surface. Since this latter will not yield the flank wave, we will consider the first part. We will try the following ~~integral~~ expression which is only a part of the integral.

(73) $$I = -\frac{1}{2}\int \frac{\xi\, d\xi}{\gamma_1} e^{-\gamma_1(z+H-d)} \frac{\mu_2\gamma_2}{\mu_1\gamma_1 \, sh\gamma_1 H + \mu_2\gamma_2 \, ch\gamma_1 H} H_1(\xi r)\, d\xi$$

Make a cut through the branch point k_2 in such a way that the real part of γ_2 is zero, and then the substitution $\gamma_2^2 = -\beta^2$. By the same formula (61) we get

(74) $$I = -\frac{1}{4} \frac{\mu_2}{\mu_1} \frac{k_1}{k_2} \frac{1}{\sqrt{k_1^2-k_2^2}} \frac{1}{\sin\sqrt{k_1^2-k_2^2}\, H} e^{i[k_2 r - \sqrt{k_1^2-k_2^2}\,(z+H-d)]}$$

which is seen of the same forn as (54) and the wave front is again given by

$$k_2 r - \sqrt{k_1^2 - k_2^2}\; z = \text{constant}$$

There are still other terms of the integral which will contribute similar results as (74), but we need not go further than this illustration. The method gives only a rough estimate and may not be reliable unless all the singularities have been examined.