QUANTUM MECHANICS OF THE INTERACTION OF GRAVITY WITH ELECTRONS: THEORY OF A SPIN-TWO FIELD COUPLED TO ENERGY

Thesis by

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ABSTRACT

Two methods of finding the energy tensor from the Lagrangian of a system are those of Belinfante, and Landau and Lifschitz. Neither of these methods are unique; two energy tensors for the same system differ by a term that is symmetric, has zero divergence, and is itself a second derivative. It is shown that such a term in the energy tensor produces physical effects that in one case can be measured experimentally. It is because of this lack of uniqueness of energy tensors that it is not sufficient to consider gravity merely as a spin-two field coupled to energy.

To set up the quantum mechanics of gravity interacting with electrons, the curved space Lagrangian for the Dirac field is expanded in terms of the gravitational fields $h_{\mu\nu}$. It is checked that the expanded Lagrangian has the same transformation properties as the original curved space Lagrangian.

The calculations presented are the gravitational Rutherford scattering of electrons, emission of low energy gravitons by electrons, the scattering of gravitons by electrons, the gravitational self-energy of the electron, and the most divergent part of the vacuum polarization calculation. There is also an investigation of the effects of the spin of the electron by comparison with a spin-zero particle interacting with gravity.

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I. INTRODUCTION

The main purpose of this paper is to consider the interaction of electrons with gravity, where gravity is treated as a quantized spin-two meson field. The quantum mechanics of gravity interacting with a spin-zero particle has been worked out by Feynman, and the author shows that the same general methods can be applied when gravity interacts with a spin-one half field.

On the basis of experiment there is no need to treat gravity as a quantized spin-two field. Due to the weakness of the coupling of gravity to matter all experiments on gravity have used large masses and these experiments have been explained by a classical theory, Einstein's general theory of relativity. Since the classical limit of Feynman's quantized spin-two meson theory of gravity is the general theory of relativity, Feynman's theory is likewise in agreement with experiment, but the quantum nature of gravity is untested.

The spin-two meson theory of gravity differs from general relativity in that the meson theory is consistent with the uncertainty principle. If general relativity were the correct theory then the uncertainty principle would fail for large distances and low velocities; i.e., just in the classical limit. Thus the main purpose of a quantized theory of gravity is that it demonstrates the possibility of maintaining the uncertainty principle even to the classical limit.

The quantization of general relativity has been considered for many years, but appeared to be difficult. Feynman started with the point of view that gravity should be treated from the beginning as another meson field. This point of view is presented in this paper.

There is sufficient experimental evidence to conclude that the source of gravity is energy. This leads directly to the description of the gravitational field as a spin-two meson field. To a very good approximation the total energy of a system may be approximated by the energy of the matter alone, neglecting the energy of the gravitational field. If gravity is coupled only to the energy of matter, then a relatively simple linear theory of gravity results. This linear theory is very accurate but not completely in agreement with experiment. The linear theory is also internally inconsistent.

Feynman corrected the linear theory by demanding that a consistent theory of gravity come from an action principle. This lead in a unique manner to the general theory of relativity. Once the general theory of relativity had been derived from the point of view of meson physics, it was clear how to proceed to the quantum mechanics of gravity.

The author attempted to correct the linear theory of gravity by considering gravity as a spin-two field coupled to energy. The source of the spin-two field in the linear wave equation is the energy tensor of matter. The author added to the source, the energy tensor of the spin-two field itself, to correct the linear theory. A consistent equation for a spin-two field is obtained in this manner, but the equation is not unique because energy tensors are not unique.

One of the possible energy tensors for the spin-two field leads to the same wave equation as general relativity and is therefore in agreement with experiment. Another energy tensor is derived which leads to a different wave equation that is not in agreement with experiment. The conclusion is that gravity is not merely described as a spin-two field coupled to energy, but that an additional restriction is necessary. For Feynman this restriction was that the equations of motion be obtained from an action principle; Einstein required that the gravitational field have a geometrical interpretation. Feynman showed these two restrictions to be equivalent.

This thesis presents the author's work on the theory of a spin-two field coupled to energy as well as the quantum mechanics of the interaction of gravity with electrons.

Much of the author's work on spin-two fields is based on Feynman's description of gravity from the point of view of meson physics. As this is still unpublished, it is described in sections two and three along with the author's description of spin-two fields coupled to energy.

To find the energy momentum tensor of a system, the author uses both the methods of Belinfante (1) and Landau and Lifschitz (2). In part Cl the author extends Belinfante's method to the case where second derivatives are involved in the Lagrangian so that the nonuniqueness of energy tensors may be investigated more completely. The method of Landau and Lifschitz is described in part E and its nonuniqueness is determined by the author in part F. It is in part G that the author gives the condition that selects the correct energy tensor for the gravitational field from the possible energy tensors for a spin-two field.

The quantum mechanics of gravity interacting with electrons is given in sections four, five and six. In section four the author derives the interaction of the spin-one half fields ψ with the spin-two gravitational fields $h_{\mu\nu}$. The expansions involved are new, and were relatively difficult to handle at first. Nor was it clear at first that the expanded Lagrangian possessed the same invariance properties as the curved space Lagrangian, but this is shown to be true in part L.

The curved space Lagrangian for the Dirac field has been derived by several authors, but by relying on ideas not discussed in this paper. Pauli's (3) derivation uses a five dimensional description of space; other derivations which are summarized by Brill and Wheeler (4) depend on spinor analysis. In parts H and J the author presents another derivation which does not require the introduction of spinor analysis or five dimensions.

Section five deals with finite calculations involving gravity and electrons. Part M shows the effects of the spin of the electron by comparing the quantum mechanics of a spin-one half field to a spin-zero field interacting with gravity. These effects are then discussed in more detail in the nonrelativistic limit.

In part N the momentum space representation of the interaction of electrons with gravitons, to second order in $h_{\mu\nu}$, is given explicitly. This interaction is used in part O to calculate the scattering of gravitons by electrons. This calculation is presented as the main test of the gauge invariance of the interaction of electrons with gravity.

In section six, two of the divergent calculations are presented, the gravitational self energy of the electron and the vacuum polarization of the gravitational field. As far as the divergent calculations are carried out, the results appear to be similar to the divergent calculations by Feynman on the spin-zero field. The spin-one half divergent calculations have not been carried as far as for spin-zero because of the increased complexity of the electron-graviton interaction.

II. THEORY OF GRAVITY

Suppose the history of physics were rewritten in the following way. Gravity had not been noticed due to the weakness of its coupling to matter. The theory of quantum electrodynamics, in which electrical forces are explained as the exchange of photons, had been developed and the properties of matter on an atomic scale were understood. Still to be explained were the forces holding nuclei together.

Then heavy mesons and strange particles were discovered. After early difficulties with the theory of heavy mesons, because of the strength of their coupling to nucleons, it was finally shown that the nuclear forces were caused by the exchange of these mesons.

Later, in a famous experiment, it was shown that two large chunks of matter, when separated by distances of the order of a centimeter, attracted each other. Careful checks were made to show that the chunks were electrically neutral and that magnetic forces did not cause the attraction; the remaining known heavy meson forces were of all too short a range to explain the force between the chunks. Thus a new meson called the graviton was invented to explain this force.

In the ensuing experiments to determine the properties of the graviton it was first noted that all materials attracted each other whether the materials were similar or not. More quantitative experiments showed that the graviton was coupled to the mass of the chunk and gave rise to the long range Yukawa potential $-Gm/r$. Careful experiments with moving objects showed that the graviton coupled to the inertial mass, or by the relation $E = mc^2$, to the energy of the object.

From the long range $1/r$ potential it was determined that the graviton had zero rest mass. To determine the spin of the graviton the following points were considered. First, a potential theory does not exist for spin one-half or any half integer mesons. Secondly, mesons of even integer spin give rise to an attractive force while mesons of odd integer spin give rise to a repulsive force between static like objects.* Thus the spin of the graviton was an even integer.

A spin-zero theory of the graviton was eliminated by the experiment that the coupling was proportional to the inertial mass of an object, while a spin-two theory was consistent with this experiment.** In the absence of evidence for a higher spin of the graviton it was decided that the graviton was a spin-two meson.

With this background, let us see how a spin-two theory of gravity could be constructed.

A. GRAVITATIONAL WAVE THEORY

The mechanics of mesons, nucleons, electrons and the basic particles of physics is described by the fields associated with these particles. The equations of motion of the fields allow one to calculate the total amplitude for a given process, the absolute square of this amplitude giving the probability that such a mechanical process should occur. This system of mechanics is consistent with the uncertainty principle. We wish to fit the theory of gravity into this system.

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*In Appendix I these statements are investigated in more detail.

**See Appendix I.

The equations of motion of the fields of particles presumably may be obtained from the principle of least action. It is thus from an action principle that we shall try to determine the equations for a gravitational field. The electromagnetic field being a spin-one, zero rest mass field, it should be closest in form to a spin-two, zero rest mass gravitational field. We shall therefore construct the theory of the gravitational field from an analogy with the theory of the electromagnetic field.

The action for a system may be given by

$$S = \int \mathcal{L}\, d^4x \tag{A-1}$$

where $\mathcal{L}$ is called the Lagrangian density for the system. For electromagnetic fields interacting with matter

$$\mathcal{L} = \mathcal{L}_{em} - j_\mu A_\mu \tag{A-2}$$

where $\mathcal{L}_{em}$ is the Lagrangian density of the free electromagnetic fields and $-j_\mu A_\mu$, the interaction Lagrangian density, is the scalar product of the electromagnetic field A_μ and the electric current density of matter j_μ. $\mathcal{L}_{em}$ is explicitly given by

$$\mathcal{L}_{em} = -\frac{1}{16\pi} F_{\mu\nu} F_{\mu\nu} = -\frac{1}{8\pi}(A_{\mu,\nu}A_{\mu,\nu} - A_{\mu,\nu}A_{\nu,\mu})^* \tag{A-3}$$

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*The notation we shall use in this paper is:

$$A_\mu = (A_4, A_1, A_2, A_3); \quad A_\mu B_\mu = (A_4B_4 - A_1B_1 - A_2B_2 - A_3B_3) \tag{A-4a}$$

$$\nabla_\mu = (\nabla_4, \nabla_1, \nabla_2, \nabla_3) = (\partial/\partial t, -\partial/\partial x, -\partial/\partial y, -\partial/\partial z)$$

We shall define $\nabla_\mu a = a_{,\mu}$

For example

The principle of least action states that $\delta S/\delta\varphi$ is zero for each component φ of the fields included in the action. In terms of the Lagrangian density $\mathcal{L}$ this gives the Euler-Lagrange equations of motion for the field component φ:

$$\frac{\delta\mathcal{L}}{\delta\varphi} = \frac{\partial\mathcal{L}}{\partial\varphi} - \left(\frac{\partial\mathcal{L}}{\partial\varphi_{,\mu}}\right)_{,\mu} = 0 \qquad \text{(A-4)}$$

assuming that $\mathcal{L}$ is a function of the fields and their first derivitives only. It should be noted that if a term of $\mathcal{L}$ is a pure divergence then by equation A-1 the action corresponding to this term may be written as a surface integral evaluated at infinity. If the fields are zero at infinity such a term will not contribute to the action and may be left out of the equations of motion A-4.

By the above procedure the equations of motion of the electromagnetic field become

$$\frac{\delta\mathcal{L}}{\delta A_\mu} = A_{\mu,\nu\nu} - A_{\nu,\mu\nu} = 4\pi j_\mu \qquad \text{(A-5)}$$

which are just Maxwell's equations.

$$A_{\mu,\mu} = (\nabla_4 A_4 - \nabla_1 A_1 - \nabla_2 A_2 - \nabla_3 A_3)$$

$$= \left(\frac{\partial A_4}{\partial t} + \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right)$$

$$A_{\mu,\nu\nu} = \left(\frac{\partial^2 A_4}{\partial t^2} - \frac{\partial^2 A_4}{\partial x^2} - \frac{\partial^2 A_4}{\partial y^2} - \frac{\partial^2 A_4}{\partial z^2},\ \frac{\partial^2 A_1}{\partial t^2} - \frac{\partial^2 A_1}{\partial x^2} - \frac{\partial^2 A_1}{\partial y^2} - \frac{\partial^2 A_1}{\partial z^2},\ \ldots\right)$$

also

$$\vec{\nabla}_i = \left(\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}\right)$$

To construct a gravitational theory in analogy with electromagnetism, the following properties of the electromagnetic theory will be noted. The electric current density j_μ is the source of the electromagnetic field as is seen in equation A-5. The interaction Lagrangian density $j_\mu A_\mu$ leading to this source term in A-5 was the scalar product of the source j_μ and the field A_μ.

The total electric current in a system is conserved; $j_{\mu,\mu} = 0$. The equations of motion A-5 were consistent with that fact;

$$4\pi j_{\mu,\mu} = (A_{\mu,\nu\nu} - A_{\nu,\mu\nu})_{,\mu} = 0 \qquad \text{(A-6)}$$

That is, the Lagrangian for the free electromagnetic fields was designed so that the terms in A_μ in the equation of motion had zero divergence consistent with the conserved source j_μ.

These properties may be carried over to the theory of gravity. As we shall see the source of gravity is the energy of a system. However, energy comes in many forms such as the rest mass, potential energy, and kinetic energy of an object. The description of all of these forms of energy requires the so-called symmetric energy momentum tensor of matter $T^m_{\mu\nu}$, where $T^m_{\mu\nu} = T^m_{\nu\mu}$. The statement of conservation of energy for matter is that

$$T^m_{\mu\nu,\nu} = 0 \qquad \text{(A-7)}$$

By analogy with electromagnetism we shall write the interaction Lagrangian density as the scalar product of the energy tensor $T_{\mu\nu}$ with the gravitational field. To do this the gravitational field itself must be a tensor of the form $h_{\mu\nu}$. The scalar product giving the interaction

Lagrangian density will therefore be $-\frac{K}{2} h_{\mu\nu} T_{\mu\nu}$.* where K is the coupling constant to be determined by experiment. (The choice $KhT_{\sigma\sigma}$ corresponds to a scalar theory of gravity.) We note from the form of the coupling that $h_{\mu\nu}$ may be considered symmetric, for the scalar product of the antisymmetric part of $h_{\mu\nu}$ with the symmetric tensor $T_{\mu\nu}$ is zero, implying that the antisymmetric part of $h_{\mu\nu}$ would not couple to matter and therefore would never be seen.

The Lagrangian density for gravity interacting with matter (in analogy with A-2) is now

$$\mathcal{L} = \mathcal{L}_g - \frac{K}{2} h_{\mu\nu} T^m_{\mu\nu} \tag{A-8}$$

and the Euler-Lagrange equations of motion are now

$$\frac{\delta\mathcal{L}}{\delta h_{\mu\nu}} = \frac{\delta\mathcal{L}_g}{\delta h_{\mu\nu}} - \frac{K}{2} T^m_{\mu\nu} = 0 \tag{A-9}$$

We must now find a Lagrangian density $\mathcal{L}_g$ of the free gravitational fields which has the property that $(\delta\mathcal{L}/\delta h_{\mu\nu})_{,\nu} = 0$ to be consistent with the fact that $T^m_{\mu\nu,\nu} = 0$.

The linear second order differential equation A-5 for the electromagnetic fields was a result of the fact that $\mathcal{L}_{em}$ consisted of terms in which the fields appeared twice and there were two derivatives, e.g.,

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*The factor of -1/2 is a convenient choice for later work.

$A_{\mu,\nu}A_{\mu,\nu}$. By analogy we shall choose for $\mathcal{L}_g$ only those terms that involve the gravitational field twice and have two derivatives. There are only four such terms, (two terms in the Lagrangian density that differ by a pure divergence lead to the same equations of motion and will only be counted once). These terms are

$$h_{\mu\nu,\sigma}h_{\mu\nu,\sigma}$$

$$h_{\mu\mu,\sigma}h_{\nu\nu,\sigma}$$

$$h_{\mu\sigma,\mu}h_{\nu\sigma,\nu}$$

$$h_{\mu\nu,\mu}h_{\sigma\sigma,\nu}$$

(In this paper the notation $h = h_{\sigma\sigma}$ will often be used.) Therefore, the most general Lagrangian density of this form is

$$\mathcal{L}_g = Ah_{\mu\nu,\sigma}h_{\mu\nu,\sigma} + Bh_{,\sigma}h_{,\sigma} + Ch_{\mu\sigma,\mu}h_{\nu\sigma,\nu} + Dh_{\mu\nu,\mu}h_{,\nu}$$

The condition on $\mathcal{L}_g$ is that $(\delta\mathcal{L}_g/\delta h_{\mu\nu})_{,\nu} = 0$. Now

$$\delta\mathcal{L}_g/\delta h_{\mu\nu} = \partial\mathcal{L}_g/\partial h_{\mu\nu} - (\partial\mathcal{L}_g/\partial h_{\mu\nu,\rho})_{,\rho}$$

$$= -2Ah_{\mu\nu,\sigma\sigma} - 2B\delta_{\mu\nu}h_{,\sigma\sigma} - C(h_{\mu\sigma,\nu\sigma} + h_{\nu\sigma,\mu\sigma}) - D(h_{,\mu\nu} + \delta_{\mu\nu}h_{\sigma\rho,\sigma\rho})$$

$$\left(\frac{\delta\mathcal{L}_g}{\delta h_{\mu\nu}}\right)_{,\nu} = -(2A + C)h_{\mu\nu,\nu\sigma\sigma} - (2B + D)h_{,\mu\sigma\sigma} - (C + D)h_{\nu\sigma,\mu\nu\sigma}$$

$$= 0 \qquad \text{(A-10)}$$

In order not to place arbitrary restrictions on the fields themselves we must take the three coefficients each to be zero. This gives

A = arbitrary constant $C = -2A$

$B = -A$ $D = 2A$

However, the coupling constant K has not yet been determined so that if the arbitrary constant is included in K we may take A to be 1/16 (for convenience in later work), giving

$$\mathcal{L}_g = \frac{1}{8}(h_{\mu\nu,\sigma}h_{\mu\nu,\sigma} - h_{,\sigma}h_{,\sigma} - 2h_{\mu\sigma,\mu}h_{\nu\sigma,\nu} + 2h_{\mu\nu,\mu}h_{,\nu}) \qquad \text{(A-11)}$$

The equation of motion A-9 becomes

$$-h_{\mu\nu,\sigma\sigma} - h_{,\mu\nu} + h_{\mu\sigma,\nu\sigma} + h_{\nu\sigma,\mu\sigma} - \delta_{\mu\nu}(h_{\rho\sigma,\rho\sigma} - h_{,\rho\rho}) = 2KT^m_{\mu\nu}$$

We will find it convenient to introduce the notation

$$\bar{A}_{\mu\nu} = A_{\mu\nu} - \frac{1}{2}\delta_{\mu\nu}A_{\sigma\sigma} \qquad \text{(A-12)}$$

It follows immediately that if the operation "bar" defined in A-12 is applied twice we return to the original tensor, i.e.,

$$\bar{\bar{A}}_{\mu\nu} \equiv \bar{A}_{\mu\nu} - \frac{1}{2}\delta_{\mu\nu}\bar{A}_{\sigma\sigma} = A_{\mu\nu} \qquad (\delta_{\mu\nu}\delta_{\mu\nu} = 4)$$

In terms of this notation ($\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\delta_{\mu\nu}h$) the wave equation for the gravitational field becomes

$$-\bar{h}_{\mu\nu,\sigma\sigma} + \bar{h}_{\sigma\nu,\sigma\mu} + \bar{h}_{\sigma\mu,\sigma\nu} - \delta_{\mu\nu}\bar{h}_{\sigma\rho,\sigma\rho} = 2KT^m_{\mu\nu} \qquad \text{(A-13)}$$

We now have a gravitational wave equation with the coupling constant K to be determined experimentally. K is an extremely small number because of the weakness of the coupling of gravity to matter.

Before investigating the properties of the gravitational wave equation A-13 it should be noted that the equation is not quite correct. The derivation of this equation depended on the fact that the total energy, to which gravity coupled, was conserved. This is quite correct. However, in the derivation we assumed that the total energy was given by $T^{m}_{\mu\nu}$, the energy tensor of the matter alone. We then assumed that this energy was conserved, i.e. $T^{m}_{\mu\nu,\nu} = 0$. This is incorrect as can be seen by the following example.

Consider two balls released from rest with a given initial separation. Let there be no external forces on the system. Due to the gravitational attraction between the balls, they will start moving toward each other and each will have a definite velocity just before collision. If we consider the energy $T^{m}_{\mu\nu}$ in the balls, then before release it will just be the sum of the rest masses of the balls; finally it will be the sum of the rest masses plus the sum of the kinetic energy of the balls. Thus, the energy $T^{m}_{\mu\nu}$ is not conserved and we cannot set $T^{m}_{\mu\nu,\nu} = 0$.

The obvious answer is that the total energy is really conserved. We just forgot to include the gravitational potential energy in the above example. That is, we must include the energy in the gravitational field if we want conservation of energy. If we call $T^{g}_{\mu\nu}$ the energy in the gravitational field, then the total energy should be given by $(T^{m}_{\mu\nu} + T^{g}_{\mu\nu})$, where the statement of conservation of the total energy is

$$(T^{m}_{\mu\nu} + T^{g}_{\mu\nu})_{,\nu} = 0$$

Why did we not use this complete energy tensor in the derivation of the gravitational wave equation? The Lagrangian density would be of the form

$$\mathcal{L} = \mathcal{L}_g + \mathcal{L}_m - \frac{K}{2} h_{\mu\nu} [T^{m}_{\mu\nu} + T^{g}_{\mu\nu}(h)]$$

The difficulty is that $T^{g}_{\mu\nu}(h)$ depends explicitly on the fields $h_{\mu\nu}$; thus in deriving the equations of motion

$$\frac{\delta \mathcal{L}}{\delta h_{\mu\nu}} = 0$$

we do not know what $\delta[h_{\mu\nu} T^g_{\mu\nu}(h)]/\delta h_{\mu\nu}$ becomes.

Our approximation has therefore been to neglect the energy of the gravitational field in comparison to the energy of the particles without a gravitational field. In our example this is equivalent to neglecting the kinetic energy of the balls in comparison to the rest energy. For the solar system this corresponds to neglecting the gravitational potential energy in comparison to the rest mass of the planet. The ratio of these energies is in magnitude

$$\frac{GM_s M_p/r}{M_p C^2} = \frac{GM_s}{rC^2}$$

For the earth this ratio is 10^{-8}. Thus the approximation of neglecting the energy in the gravitational field is more than justified for almost any problem.

We have neglected the fact that the energy in the gravitational field is a source of gravity; that is, that gravity itself is a source of gravity. Suppose, for example, we had calculated the gravitational field due to a point mass. We would then have to add to this field the field produced by the energy in the original field. But we would then have to add the field produced by the energy in the field we had just added, and so forth. Because of this non-linear process the fields of two point masses, for example, would not be the superposition of the fields of each of the masses alone.

However, we have seen that the fields produced in practice contain much less energy than the particles had themselves and the corrections to the fields are extremely small. A linear theory of gravity as given by the wave equation A-13 is highly accurate and we shall, in the next few sections, discuss the linear theory of gravity before returning to non-linear corrections.

B. LINEAR THEORY OF GRAVITY

The Lagrangian for the linear theory of gravity is

$$\mathcal{L} = \frac{1}{8}(h_{\mu\nu,\sigma}h_{\mu\nu,\sigma} - h_{,\sigma}h_{,\sigma} - 2h_{\mu\sigma,\mu}h_{\nu\sigma,\nu} + 2h_{\mu\nu,\mu}h_{\nu})$$

$$- \frac{K}{2}h_{\mu\nu}T^{m}_{\mu\nu} \qquad \text{(B-1)}$$

The action corresponding to this Lagrangian density is invariant under the substitution

$$h'_{\mu\nu} = h_{\mu\nu} + \eta_{\mu,\nu} + \eta_{\nu,\mu} \qquad \text{(B-2)}$$

where η_μ is an arbitrary vector. Pure divergences will appear under this substitution as in the following example.

$$\eta_{\mu,\nu}T^{m}_{\mu\nu} = (\eta_\mu T^{m}_{\mu\nu})_{,\nu} - \eta_\mu T^{m}_{\mu\nu,\nu}$$

However, pure divergence in the Lagrangian density does not contribute to the action, and to the accuracy of the linear theory $T^{m}_{\mu\nu,\nu} = 0$; thus for example the term $-\frac{K}{2}h_{\mu\nu}T^{m}_{\mu\nu}$ is invariant under the substitution B-2. We shall call the substitution B-2 a gauge transformation of the gravitational field (in analogy to the gauge transformation of the electromagnetic field $A_\mu = A_\mu + \chi_{,\mu}$) and say that the Lagrangian density

B-1 is gauge invariant.

We also note that the linear gravitational wave equation A-13 is exactly the linear equation given by Einstein's general theory of relativity. See for example Tolman (5). Equation A-13 may be simplified by a particular choice of gauge. If we choose η_μ by the equation

$$\bar{h}'_{\mu\nu,\nu} = 0 = (h_{\mu\nu} + \eta_{\mu,\nu} + \eta_{\nu,\mu})_{,\nu} - \frac{1}{2}\delta_{\mu\nu}(h_{\sigma\sigma} + 2\eta_{\sigma\sigma})_{,\nu}$$

or

$$\eta_{\mu,\nu\nu} + \eta_{\nu,\mu\nu} - \eta_{\sigma,\sigma\mu} = -\bar{h}_{\mu\nu,\nu}$$

Then the wave equation A-13

$$-\bar{h}_{\mu\nu,\sigma\sigma} + \bar{h}_{\sigma\nu,\sigma\mu} + \bar{h}_{\sigma\mu,\sigma\nu} - \delta_{\mu\nu}\bar{h}_{\rho\sigma,\rho\sigma} = 2KT_{\mu\nu}$$

simply becomes

$$-\bar{h}'_{\mu\nu,\sigma\sigma} = 2KT_{\mu\nu} \qquad \text{(B-3)}$$

This is exactly the equation 93.7 given by Tolman (5), derived from a linearized form of general relativity.

B1. Free Fields

Let us consider A-13 in the case there is no source. Let $\bar{h}_{\mu\nu} = \bar{e}_{\mu\nu}e^{-ik\cdot x}$; define $k_\nu\bar{e}_{\mu\nu} = \lambda_\mu$; from A-13 the free equation for $\bar{e}_{\mu\nu}$ is

$$k^2\bar{e}_{\mu\nu} - k_\nu\lambda_\mu - k_\mu\lambda_\nu + \delta_{\mu\nu}(k\cdot\lambda) = 0 \qquad \text{(B1-1)}$$

The change of gauge $h'_{\mu\nu} = h_{\mu\nu} + \eta_{\mu\nu} + \eta_{\nu,\mu}$ becomes $\bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} + \eta_{\mu,\nu} + \eta_{\nu,\mu} - \delta_{\mu\nu}\eta_{\rho,\rho}$ or in momentum representation

$$\bar{e}'_{\mu\nu} = \bar{e}_{\mu\nu} + k_\nu a_\mu + k_\mu a_\nu - \delta_{\mu\nu}(k \cdot a)$$

where

$$\eta_\mu = a_\mu e^{-ik \cdot x}$$

$$\lambda'_\mu = k_\nu \bar{e}'_{\mu\nu} = \lambda_\mu + k^2 a_\mu$$

Let us consider Bl-1 in two cases. First, let us consider that $k^2 \neq 0$. Then by a proper choice of gauge, $a_\mu = -\lambda_\mu / k^2$, we can make $\lambda'_\mu = \lambda_\mu + k^2 a_\mu = 0$. Then the wave equation is $k^2 \bar{e}'_{\mu\nu} = 0$; $\bar{e}'_{\mu\nu} = 0$. Thus for this choice of gauge there is no solution for the gravitational fields and there can be no physical effect. But we must get the same physics for any choice of gauge; therefore, in any gauge for $k^2 \neq 0$ we can have no physical effect.

The second case is where $k^2 = 0$. The free gravitational wave equation is now

$$k_\nu \lambda_\mu + k_\mu \lambda_\nu - \delta_{\mu\nu}(k \cdot \lambda) = 0$$

The solution to these sixteen equations is $\lambda_\mu = 0$; $k_\mu \bar{e}_{\mu\nu} = 0$. Let us choose the case where the plane wave is moving in the x direction

$$k_\mu = (k_4, k_1, 0, 0); \quad k^2 = k_4^2 - k_1^2 = 0; \quad \text{take } k_1 = k_4 = k$$

Now

$$k_\mu \bar{e}_{\mu\nu} = k_4 \bar{e}_{4\nu} - k_1 \bar{e}_{1\nu} = 0; \quad k\bar{e}_{4\nu} = k\bar{e}_{1\nu};$$

we may take

$$\bar{e}_{44} = \bar{e}_{41} = \bar{e}_{11}; \quad \bar{e}_{42} = \bar{e}_{12}; \quad \bar{e}_{43} = \bar{e}_{13}$$

Now choose a special gauge to give us a purely transverse wave.

$$\bar{e}'_{\mu\nu} = \bar{e}_{\mu\nu} + k_\nu a_\mu + k_\mu a_\nu - \delta_{\mu\nu}(k \cdot a)$$

$$\left.\begin{aligned}\bar{e}'_{44} &= \bar{e}_{44} + k(a_1 + a_4)\\ \bar{e}'_{11} &= \bar{e}_{11} + k(a_1 + a_4)\\ \bar{e}'_{41} &= \bar{e}_{41} + k(a_1 + a_4)\end{aligned}\right\} = 0 \text{ by choosing } (a_1 + a_4) = -\bar{e}_{44}/k$$

$$\left.\begin{aligned}\bar{e}'_{43} &= \bar{e}_{43} + ka_3\\ \bar{e}'_{13} &= \bar{e}_{13} + ka_3\end{aligned}\right\} = 0 \text{ by choosing } a_1 = -\bar{e}_{41}/k$$

$$\left.\begin{aligned}\bar{e}'_{42} &= \bar{e}_{42} + ka_2\\ \bar{e}'_{12} &= \bar{e}_{12} + ka_2\end{aligned}\right\} = 0 \text{ by choosing } a_2 = -\bar{e}_{42}/k$$

$$\begin{aligned}\bar{e}'_{33} + \bar{e}'_{22} &= \bar{e}_{33} + \bar{e}_{22} + 2k(a_3 + a_2) + 2k(a_4 - a_1)\\ &= \bar{e}_{33} + \bar{e}_{22} - 2\bar{e}_{43} - 2\bar{e}_{42} + 2k(a_4 - a_1)\end{aligned}$$

$\bar{e}'_{33} + \bar{e}'_{22} = 0$ if we choose $(a_4 - a_1) = (2\bar{e}_{43} + 2\bar{e}_{42} - \bar{e}_{33} - 2\bar{e}_{22})/2k$. We should note that because the trace of $\bar{e}'_{\mu\nu}$ is zero with this choice of gauge, $\bar{e}'_{\mu\nu} = e'_{\mu\nu}$.

$$(e'_{\mu\nu} = \bar{e}'_{\mu\nu} - \tfrac{1}{2}\delta_{\mu\nu}\bar{e}'_{\sigma\sigma};\ \bar{e}'_{\sigma\sigma} = 0)$$

Therefore, by a proper choice of gauge for the case $k^2 = 0$ we are left with only two dependent solutions:

1) First solution

$$e_{22} = -e_{33} = a;\ e^{(1)}_{\mu\nu} = a\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix} = a\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

writing only the y-z part of the tensor.

2) Second solution

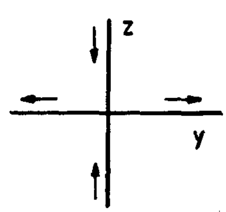
$$e_{32} = e_{23} = b; \quad e^{2}_{\mu\nu} = b \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and the general solution, a linear combination of these two solutions, is

$$e_{\mu\nu} = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}$$

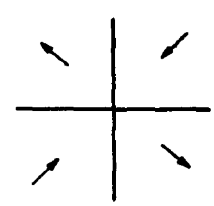
Solution 1) has a positive stress in the y-direction and a negative stress in the z-direction. We shall show that the second solution is the first solution rotated backward through an angle of 45°. Thus, the solutions may be pictured as

1) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$



$e^{(1)}_{\mu\nu}$

2) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$



$e^{(2)}_{\mu\nu}$

To study the properties of these solutions, consider the following vectors in the y-z plane.

$$C_{\mu} = \begin{pmatrix} 1 \\ i \end{pmatrix} ; \quad D_{\mu} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

If we rotate the coordinate system by an angle θ about the x-axis, using $C'_{\mu} = \frac{\partial x_{\alpha}}{\partial x'_{\mu}} C_{\alpha}$, the law of transformation of vectors, we get

$$C'_{\mu} = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{i\theta} ; \quad D'_{\mu} = \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{-i\theta}$$

Now solutions 1) and 2) are outer products of these vectors. Explicitly

1) $$e^{(1)}_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{2}(C_\mu C_\nu + D_\mu D_\nu) = \frac{1}{2}\left[(1, i)\begin{pmatrix} 1 \\ i \end{pmatrix} + (1, -i)\begin{pmatrix} 1 \\ -i \end{pmatrix}\right]$$

$$= \frac{1}{2}\left[\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} + \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}\right]$$

2) $$e^{(2)}_{\mu\nu} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{i}{2}(D_\mu D_\nu - C_\mu C_\nu) = \frac{i}{2}\left[\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} - \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}\right]$$

First let us show that if we rotate solution two by an angle of 45° we get solution one. Consider $e'_{\mu\nu} = e_{\mu\nu}$ rotated by an angle θ.

$$e'^{(2)}_{\mu\nu} = \frac{i}{2}(D'_\mu D'_\nu - C'_\mu C'_\nu) = \frac{i}{2}(D_\mu D_\nu e^{-2i\theta} - C_\mu C_\nu e^{2i\theta})$$

$$= \frac{i}{2}\left[\begin{pmatrix} e^{-2i\theta} & -ie^{-2i\theta} \\ -ie^{-2i\theta} & -e^{-2i\theta} \end{pmatrix} - \begin{pmatrix} e^{2i\theta} & ie^{2i\theta} \\ ie^{2i\theta} & -e^{2i\theta} \end{pmatrix}\right]$$

$$= -\frac{1}{2i}\begin{pmatrix} (e^{-2i\theta} - e^{2i\theta}) & -i(e^{-2i\theta} + e^{2i\theta}) \\ -i(e^{-2i\theta} + e^{2i\theta}) & (e^{2i\theta} - e^{-2i\theta}) \end{pmatrix}$$

$$= \begin{pmatrix} \sin 2\theta & \cos 2\theta \\ \cos 2\theta & -\sin 2\theta \end{pmatrix}$$

For the case $\theta = 45°$

$$e'^{(2)}_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$ which is just solution 1)

Now consider the following linear combinations of our original two solutions

A) $$C_\mu C_\nu = e^{(1)}_{\mu\nu} + ie^{(2)}_{\mu\nu} = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$$

$$\text{B)} \quad D_\mu \overset{D}{C}_\nu = e^{(1)}_{\mu\nu} - i e^{(2)}_{\mu\nu} = \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$$

Under a rotation of the coordinate system by an angle θ about the axis of propagation

A) $C_\mu C_\nu \to C_\mu C_\nu e^{2i\theta}$; the property of a plane wave with plus two units of angular momentum.*

B) $D_\mu \overset{D}{C}_\nu \to D_\mu D_\nu e^{-2i\theta}$; the property of a plane wave with minus two units of angular momentum.

Thus a free gravitation can be represented as a spin-two particle with its polarization directed either with or against its direction of motion. The general solution is a linear combination of these two solutions.

B2. Experimental Tests of the Linear Theory

A fundamental test of the linear theory of gravity would be to test the basic idea that the source of gravity is the total energy of the object. For example, the mass of Pb_{208} is less than the sum of the rest masses of its electrons, neutrons and protons by a factor of 0.825% due mainly to binding of the nucleons in the nucleus; while the mass of a hydrogen atom is reduced only by a factor of the order of 10^{-6}%. Thus the comparison of the weight of a sample of lead with a sample of hydrogen having the same number of nucleons would indicate that the gravitational coupling is reduced by the negative binding energy of the nucleons

*Note that the waves for a spin-zero field φ, a spin-one half field ψ and a spin-one field A_μ transform in the following way under the rotation of the coordinate system by an angle θ.

$$\left.\begin{aligned} &\varphi \to \varphi \\ &\psi \to \psi e^{i\theta/2} \\ &A_\mu \to A_\mu e^{i\theta} \end{aligned}\right\}$$

The rotational properties determined the angular momentum carried in the wave.

in the lead nucleus.

A less direct but far more accurate method of determining that gravity couples to the energy of a particle relies on the assumption that the energy of an object is proportional to the object's inertial mass, i. e. $E = mc^2$. This relation is basic to relativistic mechanics where the relation between the force and the inertial mass is $F = d(mv)/dt$. The inertial mass of a nucleus is directly measured in a mass spectrograph, while the difference in energy between nuclei can be determined from the Q value of nuclear reactions. Thus the equivalence of the energy and the inertial mass may be checked experimentally. The agreement is fairly good, although not completely verified. (See R. H. Dicke (6)).

Assuming the equivalence of energy and inertial mass, then we may interpret the results of the experiment of Eotvos (7) as an accurate test that gravity couples to energy. The idea of Eotvos' experiment is that an object on the surface of the earth is accelerating due to the daily rotation of the earth and the motion of the earth in an orbit about the sun, and this acceleration is produced by gravitational forces. If the gravitational force is exactly proportional to the inertial mass of the object, then all objects, independent of their composition, would have the same acceleration. If not, then objects of different material placed on a tortion balance could produce a torque.

Eotvos' result is that if the gravitational force is proportional to the mass for platinum (he determined the gravitational constant for the case of platinum), then the gravitational force on snakewood is proportional to its inertial mass times a factor $(1 - 0.1 \times 10^{-8} \pm 0.2 \times 10^{-8})$. The snakewood experiment is the most interesting for it compares the light elements of hydrogen and carbon with platinum which has a much greater nuclear binding energy.

We note that a spin-zero theory of gravity would imply that the gravitational force would be proportional to $m(1 - v^2)^{\frac{1}{2}}$,* which for a particle on the earth moving about the sun would be $m(1 - 0.1 \times 10^{-7})$. Thus if an Eotvos experiment showed the proportionality of the inertial mass of an object on the earth to its gravitational attraction to the sun with the same accuracy as Eotvos' original experiment, then a spin-zero theory would be eliminated while the linear theory would still hold.

Most of the accurate experiments on gravity are the result of astronomical observations. Of these all but three may be explained by Newton's original theory of gravity. If we show that in the limit of weak fields and low velocities the linear theory approaches the Newtonian theory, then almost all tests of gravity will be explained.

One observation not explained by a Newtonian theory is the gravitational red shift of light. This is seen in the shift of spectral lines of light emitted from stars, and in the experiment of Pounds (8) where photons were dropped 12.5 meters and their frequency shift measured using the Mossbauer effect. To within the accuracy of the experiments the frequency shift may be explained by assigning an effective mass m_e to a photon, $\hbar\omega = m_e c^2$. The change in kinetic energy of such an object moving from a region of potential φ to a new

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*See Appendix I.

region of potential $\varphi - \Delta\varphi$ would be given by Newtonian physics as $m\Delta\varphi$. The frequency of the photon in this new region by this simple argument is just given by $\hbar\omega' = m_e(1 + \Delta\varphi)c^2$. This simple derivation of the gravitational shift of line spectra is in agreement with the shift predicted by the linear theory of gravity.

An astronomical observation that cannot be explained by an argument even similar to the one used to explain the red shift is the deflection of light passing the sun. If one assumes that a photon should act as a massive particle and that the equation for the trajectory $\vec{z}$ of the particle is given by Newton's laws as $-\vec{\nabla}\varphi = d^2\vec{z}/dt^2$, then the predicted deflection is half that observed. We shall see that the linear theory predicts the full observed deflection.

The final test of gravity not in agreement with the Newtonian theory is the shift in the perihelion of the elliptical orbit of the planet Mercury. In the absence of perturbations the Newtonian theory predicts that the perihelion of a planet in an elliptical orbit should remain fixed. After the perturbations are taken into account the perihelion of Mercury is observed to shift by forty-three seconds of arc per century. The linear theory of gravity does predict a shift in the perihelion of Mercury--two-thirds of that observed. Only in the observation of the shift of the perihelion of Mercury is there an experimental test of the failure of the linear theory.

The inconsistency in, and now the failure of the linear theory is that it neglects the energy in the gravitational field as a source of gravity. We shall show that by correctly including the energy in the fields as a source of gravity, the correct shift in the perihelion of

Mercury is obtained. Thus there is experimental evidence that gravity is itself a source of gravity.

To check the above statements about the linear theory it is only necessary to consider the interaction of gravity with point particles. In the Newtonian limit is is well known that the gravitational field outside of a spherically symmetric object is the same field as that of a point with the same total mass. If we leave to astronomers the calculation of perturbations due to tides, etc. the astronomical theory will be adequately described by point particles.

Let the coordinate for a particle be given by the four vector Z_μ where $Z_\mu = (Z_4, Z_1, Z_2, Z_3)$. The free Lagrangian for such a particle is given by

$$L' = -\frac{m}{2}\dot{Z}_\mu \dot{Z}_\mu \frac{ds}{dZ_4} \quad *$$

where

$$\dot{Z}_\mu = dZ_\mu/ds$$

The energy momentum tensor for such a particle is

$$T^m_{\mu\nu} = m\dot{Z}_\mu \dot{Z}_\nu \frac{ds}{dZ_4} \quad **$$

To the action S_g for the gravitational field must be added S_{mg}, the action of the free particle plus the action of interaction, to obtain the total action for the linear theory of gravity interacting with particles. The Lagrangians corresponding to the terms in S_{mg} are given by

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* See Goldstein (9).

** See Landau (10).

$$L'_m = -\frac{m}{2}\,\dot{Z}_\mu \dot{Z}_\mu \frac{ds}{dZ_4} \qquad \text{(B2-1)}$$

$$L'_{interaction} = -\frac{K}{2} h_{\mu\nu} T^m_{\mu\nu} = -K\frac{m}{2} h_{\mu\nu}(Z)\,\mathring{Z}_\mu \mathring{Z}_\nu \frac{ds}{dZ_4} \qquad \text{(B2-2)}$$

where S_{mg} is given by

$$S_{mg} = \int [\, L'_m + L'_{int}]\, dZ_4$$

We get

$$S_{mg} = -\frac{m}{2}\int [\delta_{\mu\nu} + Kh_{\mu\nu}(Z)]\, \dot{Z}_\mu \dot{Z}_\nu \, ds \qquad \text{(B2-3)}$$

From B2-3 we may define a new Lagrangian $L_{mg}(Z)$ where

$$S_{mg} = \int L_{mg}(Z)\, ds \qquad \text{(B2-4)}$$

$$L_{mg}(Z) = -\frac{m}{2}\,[\, \delta_{\mu\nu} + Kh_{\mu\nu}(Z)]\, \dot{Z}_\mu \dot{Z}_\nu \qquad \text{(B2-5)}$$

In terms of the Lagrangian $L(Z)$ the Euler-Lagrange equations of motion become

$$\frac{\delta L(Z)}{\delta Z_\mu} = \frac{\partial L}{\partial Z_\mu} - \frac{d}{ds}\,\frac{\partial L}{\partial \dot{Z}_\mu} = 0 \qquad \text{(B2-6)}$$

If we wish to describe the particle in the space-time coordinate system $x_\mu = (t, x, y, z)$, the action S_{mg} of B2-3 may be rewritten as

$$S_{mg} = -\frac{m}{2}\iint \delta^4(x-Z)[\, \delta_{\mu\nu} + Kh_{\mu\nu}(x)]\, \dot{Z}_\mu \dot{Z}_\nu \, ds\, d^4x \qquad \text{(B2-6A)}$$

In the previous work on the linear theory, the Lagrangian density $\mathcal{L}_{mg}$ was obtained. The action is given from a Lagrangian density $\mathcal{L}$ by the relation

$$S = \int \mathcal{L} \; d^4x$$

From B2-6A we see that the Lagrangian density $\mathcal{L}_{mg}$ of the free particles plus interaction is given by

$$\mathcal{L}_{mg} = -\frac{m}{2}\int \delta^4(x-Z)[\delta_{\mu\nu} + Kh_{\mu\nu}(x)]\,\dot{Z}_\mu \dot{Z}_\nu \; ds \qquad \text{(B2-7)}$$

The Lagrangian density for a free particle is therefore

$$\mathcal{L}_m = -\frac{m}{2}\int \delta^4(x-Z)\,\dot{Z}_\mu \dot{Z}_\mu \; ds \qquad \text{(B2-8)}$$

and the energy momentum tensor density for a free particle is

$$T^m_{\mu\nu} = m\int \delta^4(x-Z)\dot{Z}_\mu \dot{Z}_\nu \; ds \qquad \text{(B2-9)}$$

In the remainder of this paper we shall use the notation of capital L for the Lagrangian, and script $\mathcal{L}$ for the Lagrangian density, referring to both as the Lagrangian.

With this formalism we may now consider the equation of motion of a particle in a gravitational field. Using the Lagrangian $L_{mg}(Z)$, B2-5, the equation of motion B2-6 becomes

$$[\delta_{\rho\mu} + Kh_{\rho\mu}]\ddot{Z}_\mu + \frac{1}{2}[Kh_{\alpha\rho,\beta} + Kh_{\beta\rho,\alpha} - Kh_{\alpha\beta,\rho}]\dot{Z}_\alpha \dot{Z}_\beta = 0 \qquad \text{(B2-10)}$$

where we have used the symmetry between α and β in $\dot{Z}_\alpha \dot{Z}_\beta$, and noticed that $dh_{\mu\nu}(Z)/ds = h_{\mu\nu,\rho}\dot{Z}_\rho$.

For later work we will want an exact expression for $\ddot{Z}_\mu$. This may be obtained by multiplying B2-10 by $[\delta_{\rho\gamma} + Kh_{\rho\gamma}]^{-1}$, where $[\delta_{\rho\gamma} + Kh_{\rho\gamma}]^{-1}$ is defined by the equation

$$[\delta_{\rho\gamma} + Kh_{\rho\gamma}]^{-1}[\delta_{\rho\mu} + Kh_{\rho\mu}] = \delta_{\gamma\mu} \tag{B2-11}$$

A series expansion of $[\delta_{\rho\gamma} + Kh_{\rho\gamma}]^{-1}$ that obeys B2-11 is given by

$$[\delta_{\rho\gamma} + Kh_{\rho\gamma}]^{-1} = \delta_{\rho\gamma} - Kh_{\rho\gamma} + K^2 h_{\rho\sigma} h_{\sigma\gamma} - K^3 h_{\rho\sigma} h_{\sigma\lambda} h_{\lambda\gamma} + \ldots \tag{B2-12}$$

as may be verified by direct substitution. (B2-12 is essentially the expansion of $(1 + Kh)^{-1}$ with appropriate subscripts.)

Therefore from B2-11 the equation for $\ddot{Z}_\gamma$ is given by

$$\ddot{Z}_\gamma = -\frac{1}{2}[\delta_{\rho\gamma} + Kh_{\rho\gamma}]^{-1}[Kh_{\alpha\rho,\beta} + Kh_{\beta\rho,\alpha} - Kh_{\alpha\beta,\rho}]\dot{Z}_\alpha \dot{Z}_\beta \tag{B2-13}$$

We may abbreviate the notation for the coefficient of $\dot{Z}_\alpha \dot{Z}_\beta$ by

$$\Gamma^\gamma_{\alpha\beta} = \frac{1}{2}[\delta_{\rho\gamma} + Kh_{\rho\gamma}]^{-1}[Kh_{\alpha\rho,\beta} + Kh_{\beta\rho,\alpha} - Kh_{\alpha\beta,\rho}] \tag{B2-14}$$

We are now in a position to demonstrate that the energy momentum tensor is not conserved by explicitly calculating $T^m_{\mu\nu,\nu}$, the divergence of the energy momentum tensor density. By equation B2-9

$$T^m_{\mu\nu} = m\int \delta^4(x-Z(s))\dot{Z}_\mu \dot{Z}_\nu \, ds$$

$$T^m_{\mu\nu,\nu} = m\int [\dot{Z}_\nu \frac{\partial}{\partial x_\nu}\delta^4(x-Z)]\dot{Z}_\mu \, ds = m\int [\dot{Z}_\nu(-)\frac{\partial}{\partial Z_\nu}\delta^4(x-Z)]\dot{Z}_\mu \, ds$$

$$= m\int [-\frac{d}{ds}\delta^4(x-Z)]\dot{Z}_\mu \, ds = m\int \delta^4(x-Z)\ddot{Z}_\mu \, ds$$

where the last expression was obtained by integration by parts and dropping the surface term. Using B2-13 $\ddot{Z}_\mu = -\Gamma^\mu_{\alpha\beta}\dot{Z}_\alpha \dot{Z}_\beta$ we just get

$$T^m_{\mu\nu,\nu} = -\Gamma^\mu_{\alpha\beta} T^m_{\alpha\beta} \tag{B2-15}$$

Now from the linear equation for the gravitational fields A-13 we see that the fields produced by the tensor $T^m_{\mu\nu}$ are of order of magnitude K smaller than $T^m_{\mu\nu}$. But from B2-14 $\Gamma^\mu_{\alpha\beta}$ is of order K smaller than the fields. Thus $T^m_{\mu\nu,\nu}$ is of order K^2 smaller than $T^m_{\mu\nu}$, small to be sure, but not zero.

From the linear theory let us now calculate the gravitational field produced by a particle, e.g., the sun. Using a gauge in which $\bar{h}_{\mu\nu,\nu} = 0$ the wave equation is in the form given by equation B-3

$$-\bar{h}_{\mu\nu,\sigma\sigma} = 2KT^m_{\mu\nu}$$

$T^m_{\mu\nu}$ may be written as:

$$T^m_{\mu\nu} = m\int \delta^4(x-Z)\dot{Z}_\mu \dot{Z}_\nu ds$$

$$= m\int \delta^3(\vec{x} - \vec{Z})\delta(t - Z_4)\dot{Z}_\mu \dot{Z}_\nu \frac{ds}{dZ_4} dZ_4$$

But

$$\dot{Z}_\mu = (1, v_x, v_y, v_z)\frac{dZ_4}{ds} \equiv v_\mu \frac{dZ_4}{ds}$$

$$v_\mu \equiv (1, v_x, v_y, v_z) \qquad \text{(B2-16)}$$

and

$$\frac{dZ_4}{ds} = (1 - v^2)^{-1/2}$$

$$T^m_{\mu\nu} = \frac{m}{(1 - v^2)^{1/2}} \delta^3(\vec{x} - \vec{Z}) v_\mu v_\nu$$

For a particle at rest only T^m_{44} survives

$$T^m_{44} = m\delta^3(\vec{x} - \vec{Z})$$

The field equation B-3 becomes

$$-\bar{h}_{44,\sigma\sigma}(x) = 2Km\delta^3(\vec{x} - \vec{Z})$$

The field produced by a particle at rest will not vary in time ($\bar{h}_{44,t} = 0$) and we may write the equation

$$\frac{\partial^2 \bar{h}_{44}}{\partial x^2} + \frac{\partial^2 \bar{h}_{44}}{\partial y^2} + \frac{\partial^2 \bar{h}_{44}}{\partial z^2} = 2Km\delta^3(\vec{x} - \vec{Z})$$

The solution of this equation known from electrostatics is

$$\bar{h}_{44} = -\frac{Km}{4\pi}\frac{1}{r} \qquad \text{(B2-17)}$$

where

$$r = |\vec{x} - \vec{Z}|$$

The fields $h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\delta_{\mu\nu}\bar{h}_{\sigma\sigma}$. From B2-17 this gives

$$h_{11} = h_{22} = h_{33} = h_{44} = -\frac{Km}{4\pi r} \qquad \text{(B2-18)}$$

These are the linear gravitational potentials produced by a star.

Let us now calculate the effect of these potentials on the motion of an object. In dealing with the linear theory we need only keep terms to the lowest order in K in equation B2-13 for $\ddot{Z}$;

$$\ddot{Z}_\gamma = -\frac{K}{2}[h_{\alpha\gamma,\beta} + h_{\beta\gamma,\alpha} - h_{\alpha\beta,\gamma}]\dot{Z}_\alpha\dot{Z}_\beta \qquad \text{(B2-19)}$$

Using the notation of B2-16

$$\dot{Z}_\alpha\dot{Z}_\beta = v_\alpha v_\beta/(1 - v^2)$$

$$v_\alpha v_\alpha = (1 - v_x^2 - v_y^2 - v_z^2) = (1 - v^2)$$

B2-19 now may be rewritten as

$$\ddot{Z}_\gamma(1 - v^2) = - \frac{K}{2}[h_{\alpha\gamma,\beta} + h_{\beta\gamma,\alpha} - h_{\alpha\beta,\gamma}] v_\alpha v_\beta$$

The quantity $\ddot{Z}_\gamma(1 - v^2)$ may be written

$$\ddot{Z}_\gamma(1 - v^2) = (1 - v^2) \frac{d}{ds} \left[\frac{dZ_\gamma}{dt} \frac{dt}{ds}\right]$$

$$= (1 - v^2)^{1/2} \frac{d}{dt}\left[\frac{dZ_\gamma}{dt} (1 - v^2)^{-1/2}\right]$$

$$= \frac{d^2Z_\gamma}{dt^2} + \frac{dZ_\gamma}{dt} (1 - v^2)^{+1/2} \frac{d}{dt}(1 - v^2)^{-1/2}$$

Using $(1 - v^2)^{1/2} = (v_\sigma v_\sigma)^{1/2}$ we get

$$\ddot{Z}_\gamma(1 - v^2) = \frac{d^2Z_\gamma}{dt^2} - \frac{dv_\rho}{dt} \frac{v_\gamma v_\rho}{(1 - v^2)}$$

The equation of motion may be written

$$\frac{d^2Z_\gamma}{dt^2} = - \frac{K}{2} [h_{\alpha\gamma,\beta} + h_{\beta\gamma,\alpha} - h_{\alpha\beta,\gamma}] v_\alpha v_\beta + \frac{dv_\rho}{dt} \frac{v_\gamma v_\rho}{(1-v^2)} \qquad \text{(B2-20)}$$

First let us consider the case of very slow velocities where in equation B2-20 we may take $v_4 = 1$, $v_{1,2,3} = 0$. Using the potentials B2-18 the equation of motion becomes

$$\frac{d^2Z_\gamma}{dt^2} = - \frac{K}{2} [2h_{4\gamma,4} - h_{44,\gamma}]$$

But $h_{4\gamma,4} = 0$ and we are left with

$$\frac{d^2\vec{Z}}{dt^2} = -\vec{\nabla} h_{44} \frac{K}{2} \qquad \text{(B2-21)}$$

Thus the force on the particle is proportional to minus the gradient of the potential h_{44} in agreement with Newtonian physics. The force between two particles of mass M and m separated by a distance R is given by B2-21 and B2-18.

$$F = \frac{K^2 mM}{8\pi R^2}$$

but from classical physics this force is known to be GMm/R^2, thus by going to the Newtonian limit we determine our coupling constant K.

$$K^2 = 8\pi G \qquad \text{(B2-22)}$$

We also see that the relation between the potential h_{44} and the Newtonian potential $\phi = Gm/r$ is

$$Kh_{44} = 2Gm/r = 2\phi \qquad \text{(B2-23)}$$

We finally note that Newtonian physics depends only on the potential h_{44} (see B2-21).

Let us now consider small deflections of high speed particles moving past the sun. Let us calculate the acceleration in the x-direction of a particle moving in the y-direction. From equation B2-20

$$\frac{d^2 Z_x}{dt^2} = -\frac{K}{2}\left[2h_{\alpha x,\beta} - h_{\alpha\beta,x}\right] v_\alpha v_\beta + \frac{dv_\rho}{dt}\frac{v_x v_\rho}{(1-v^2)} \qquad \text{(B2-24)}$$

For small deflections we may calculate the acceleration in the x-direction by assuming the velocity appearing on the right-hand side of B2-24 remains in the y-direction. Thus $v_x = 0$, $h_{\alpha x, \beta}$ does not contribute and we get

$$\frac{d^2 Z_x}{dt^2} = \frac{K}{2}\left[h_{44,x} + v^2 h_{yy,x} \right]; \quad h_{44} = h_{yy} = 2\phi / K$$

$$\frac{d^2 Z_x}{dt^2} = - \frac{\partial \phi}{\partial x}(1 + v^2) \tag{B2-25}$$

We will get a greater deflection by a factor $(1 + v^2)$ than if we had calculated by a Newtonian potential h_{44} alone (corresponding to $d^2 Z_x / dt^2 = - \partial\phi / \partial x$), and in the case of light this deflection will be twice as great. This result as we have mentioned at the beginning of this section is in agreement with experiment.

The calculation of the shift in the perihelion of Mercury is far more involved and we shall not go into it at this point except to note that the answer for the linear theory is two-thirds of the experimental result.

B3. Quantum Mechanics of the Linear Theory of Gravity

The experimental justification for the linear quantum theory of gravity is in a sense stronger than the justification of the linear classical theory. There are no experiments in disagreement with the purely quantum mechanical predictions of the linear theory. This is because there are no tests of the quantum nature of gravity. The quantum effects are all too small.

As with the classical theory the quantum mechanics of the linear theory of gravity will be obtained by an analogy with the theory of electromagnetism. We shall follow the approach given by Feynman (11).

Maxwell's equations for electricity are

$$A_{\mu,\sigma\sigma} - A_{\sigma,\mu\sigma} = 4\pi J_\mu \qquad \text{(B3-1)}$$

In momentum representation, where

$$A_\mu = a_\mu e^{-iq\cdot x} ; \quad J_\mu = j_\mu e^{-iq\cdot x}$$

equation B3-1 becomes

$$-q^2 a_\mu + q_\mu q_\sigma a_\sigma = 4\pi j_\mu \qquad \text{(B3-2)}$$

A solution of this equation, for the case that the current is conserved, is

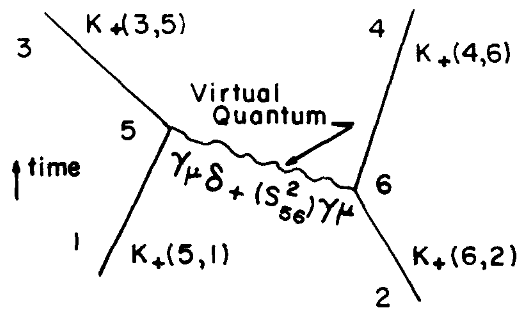
$$a_\mu = -\frac{4\pi}{q^2} j_\mu \qquad \text{(B3-3)}$$

To check this solution, note that the statement of conservation of current, $J_{\mu,\mu} = 0$ becomes in momentum representation $q_\mu j_\mu = 0$. Therefore $q_\mu a_\mu = 0$ and B3-3 is indeed a solution.

The interaction of this field a_μ with a second current j^2_μ is of the form $j^2_\mu a_\mu$. Substituting $a_\mu = -\frac{4\pi}{q^2} j^1_\mu$ we get the interaction of currents in the form

$$-4\pi j^2_\mu \frac{1}{q^2} j^1_\mu \qquad \text{(B3-4)}$$

The interaction of currents in quantum mechanics may be described in the following way. The kernal for the propagation of two electrons exchanging one photon is given by Feynman (12).



$$K'(3,4;1,2) = -ie^2 \int\int K_{+a}(3,5)K_{+b}(4,6)\gamma_{a\mu}\gamma_{b\mu}\, \delta_+(S^2_{56})K_{+a}(5,1)K_{+b}(6,2)d\tau_5 d\tau_6$$

The appearance of $\delta_+(S^2_{56})$ which describes the propagation of the virtual quantum guarantees a delated interaction through only positive energy photons.

If the electrons had wave functions $f_a(\vec{x}_1)$ and $f_b(\vec{x}_2)$ initially, and we wanted the amplitude that they were in the states $g_a(\vec{x}_3)$ and $g_b(\vec{x}_4)$ finally, the matrix element for this process would be

$$-ie^2 \int \bar{g}_a(5)\gamma_\mu f_a(5)\bar{g}_b(6)\gamma_\mu f_b(6)\delta_+(S^2_{56})\, d\tau_5\, d\tau_6$$

In momentum representation, taking

$$f_a(5) = u_1(P_1)e^{-iP_1\cdot x} \qquad \bar{g}_a(5) = \bar{u}_3(P_3)e^{iP_3\cdot x}$$

$$f_b(6) = u_2(P_2)e^{-iP_2\cdot x} \qquad \bar{g}_b(6) = \bar{u}_4(P_4)e^{iP_4\cdot x}$$

where

$$\delta_+(S^2_{56}) = -4\pi \int \frac{e^{-iq\cdot(x_5-x_6)}}{q^2+i\epsilon}\, \frac{d^4q}{(2\pi)^4} \quad *$$

The matrix element in momentum space becomes

$$-i[-4\pi(e\bar{u}_3\gamma_\mu u_1)\frac{1}{q^2+i\epsilon}(e\bar{u}_4\gamma_\mu u_2)]$$

---

* In Feynman's articles replace his d^4K by $4\pi^2 \frac{d^2K}{(2\pi)^4}$.

The momentum space representation for the electron current is $j_\mu = e\bar{u}\gamma_\mu u$. Therefore the matrix element for the electromagnetic interaction of currents is

$$iM_F = -4\pi j^2_\mu \frac{1}{q^2 + i\epsilon} j^1_\mu$$

(If the photon had had a mass μ, the term $(q^2 + i\epsilon)^{-1}$ would have been replaced by $(q^2 - \mu^2 + i\epsilon)^{-1}$. Thus the $+ i\epsilon$, which defines the correct treatment of the pole in quantum mechanics, is obtained by Feynman's rule that all masses are considered to have a negative imaginary part.)

We see from this example that we may obtain $-i$ times the matrix element for the scattering of electrons via one virtual photon by writing a classical formula for the interaction of currents, then adding a negative imaginary part to the mass of the virtual particle. For the case we are considering, the interaction of two currents via one photon, where the momentum q of the virtual particle is known, there is no integration over q^2 and we do not need the $+ i\epsilon$ to tell us how to treat the pole.

Before returning to gravity, let us study the properties of the interaction

$$-4\pi j^2_\mu \frac{1}{q^2} j_\mu = -4\pi[j^2_4 \frac{1}{q^2} j^1_4 - j^2_3 \frac{1}{q^2} j^1_3 - j^2_2 \frac{1}{q^2} j^1_2 - j^2_1 \frac{1}{q^2} j^1_1]$$

Let us assume that the spatial part of the momentum carried by the photon is in the spatial direction 3. The four vector q_μ is then

$$q_\mu = (\omega, 0, 0, Q); \quad q^2 = \omega^2 - Q^2$$

and the directions 1 and 2 are transverse to the direction of motion of the photon.

The fact that current is conserved implies

$$J_{\mu,\mu} = 0$$

or

$$q_\mu j_\mu = 0 = \omega j_4 - Q j_3; \quad j_3 = \frac{\omega}{Q} j_4$$

Therefore the interaction between currents may be written in the form

$$-4\pi j^2_\mu \frac{1}{q^2} j^1_\mu = j^2_4 \frac{4\pi}{Q^2} j^1_4 + 4\pi \sum_{\substack{2 \text{ transverse} \\ \text{directions}}} j^2_{tr} \frac{1}{\omega^2 - Q^2} j^1_{tr}$$

Now $4\pi/Q^2$ is the momentum space representation of the coulomb potential, j_4 is the charge distribution of the current j_μ. Therefore the first term represents an instantaneous coulomb interaction between the currents. The factor $(\omega^2 - Q^2)^{-1} = 1/q^2$ as we have seen represents a delayed interaction through positive energy photons; where in this case the photons have two independent polarizations each transverse to the direction of motion. Thus the electromagnetic interaction of currents is via an instantaneous coulomb interaction plus transverse waves.

With this background let us consider the gravitational interaction of energy. The equation for the gravitational field produced by an energy tensor $T_{\mu\nu}$ is by equation A-13

$$-\bar{h}_{\mu\nu,\sigma\sigma} + \bar{h}_{\sigma\nu,\sigma\mu} + \bar{h}_{\sigma\mu,\sigma\nu} - \delta_{\mu\nu}\bar{h}_{\rho\sigma,\rho\sigma} = 2KT_{\mu\nu}$$

In momentum representation, where

$$h_{\mu\nu} = e_{\mu\nu} e^{-iq\cdot x}; \quad T_{\mu\nu} = t_{\mu\nu} e^{-iq\cdot x}$$

equation A-13 becomes

$$q^2 \bar{e}_{\mu\nu} - q_\mu q_\sigma \bar{e}_{\nu\sigma} - q_\nu q_\sigma \bar{e}_{\mu\sigma} + \delta_{\mu\nu} q_\sigma q_\rho \bar{e}_{\sigma\rho} = 2Kt_{\mu\nu} \qquad \text{(B3-5)}$$

A solution for this equation in the case where energy is conserved ($q_\mu t_{\mu\nu} = 0$), is

$$\bar{e}_{\mu\nu} = 2K \frac{t_{\mu\nu}}{q^2}\ ; \quad \bar{\bar{e}}_{\mu\nu} = e_{\mu\nu} = 2K \frac{\bar{t}_{\mu\nu}}{q^2} \qquad \text{(B3-6)}$$

Again we can check this solution by noting that

$$q_\mu \bar{e}_{\mu\nu} = \frac{2K}{q^2} q_\mu t_{\mu\nu} = 0$$

automatically guaranteeing that the last three terms on the left side of B3-5 are zero for this solution.

The interaction of the field $e_{\mu\nu} = 2K/q^2 \bar{t}_{\mu\nu}$ with a second energy tensor $s_{\mu\nu}$ is of the form $\frac{1}{2} K e_{\mu\nu} s_{\mu\nu}$. (This is the basic interaction we assumed for the derivation of the linear theory of gravity.) Therefore the interaction of energy tensors $s_{\mu\nu}$ and $t_{\mu\nu}$ will be of the form

$$K^2 s_{\mu\nu} \frac{1}{q^2} \bar{t}_{\mu\nu} = -8\pi G(s_{\mu\nu} t_{\mu\nu} - \frac{1}{2} s_{\mu\mu} t_{\nu\nu}) \frac{1}{q^2} \qquad \text{(B3-7)}$$

If, as before, we assume that the spatial part of the momentum carried by the graviton is in the spatial direction 3, then $q_\mu = (0, 0, Q, \omega)$, $q^2 = \omega^2 - Q^2$; and the directions 1 and 2 are transverse. Since both energy tensors are conserved

$$q_\mu t_{\mu\nu} = 0 = \omega t_{4\nu} - Q t_{3\nu}; \quad t_{3\nu} = \frac{\omega}{Q} t_{4\nu}$$

$$q_\mu s_{\mu\nu} = 0 = \omega s_{4\nu} - Q s_{3\nu}; \quad s_{3\nu} = \frac{\omega}{Q} s_{4\nu}$$

We can replace various $t_{3\nu}$ by $t_{4\nu}$, $s_{3\nu}$ by $s_{4\nu}$ and the fundamental interaction B3-7 becomes

$$-4\pi G \frac{s_{44}t_{44}}{Q^2} - \frac{4\pi G}{Q^2} [s_{44}(t_{22}+t_{11}) + t_{44}(s_{22}+s_{11}) - s_{43}t_{43} - 4s_{42}t_{42} - 4s_{41}t_{41}] + \frac{8\pi G}{\omega^2 - Q^2} [\frac{1}{2}(s_{11}-s_{22})(t_{11}-t_{22}) + 2s_{12}t_{12}] \quad \text{(B3-8)}$$

The first two terms represent an instantaneous interaction, the last represents a delayed interaction via positive energy gravitons.

The classical energy momentum tensor density B2-9 may be written

$$T^m_{\mu\nu} = M\delta^3(\vec{x} - \vec{Z}) \dot{Z}_\mu \dot{Z}_\nu (1 - v^2)^{1/2}$$

For a particle at rest the only part that survives is $T_{44} = M\delta^3(\vec{x} - \vec{Z}) = \rho_o$; ρ_o is the mass density. Therefore the first term looks like

$$-4\pi G\rho_{o2} \frac{1}{Q^2} \rho_{o1}$$

Thus the first term corresponds to an interaction energy $-G/r$ between masses; just the Newtonian effect.

For a particle moving in the direction 1 with a velocity v

$$T_{44} = \rho_o(1 - v^2)^{-1/2} = \rho; \quad T_{11} = v^2 T_{44}$$

and the interaction looks like

$$-4\pi G\rho_2 \frac{1}{Q^2} \rho_1(1 + v^2)$$

Thus the attraction exceeds the Newtonian value by a factor $(1 + v^2)$, which gives a factor of 2 for the deflection of light; a fact we saw in the last section.

Terms of the form $s_{42}t_{42}$ represent an instantaneous velocity dependent interaction.

The last term represents a delayed interaction by waves whose source is either $(t_{11} - t_{22})$ or t_{12}. From the wave equation B3-6 we see that the waves generated by these sources are $\bar{e}_{11} = -\bar{e}_{22}$ or $\bar{e}_{12}$. (If the interacting particles are far enough apart so that the only surviving interaction is via these transverse waves, then $\bar{e}_{\sigma\sigma} = e_{11} + e_{22} = 0$ and $e_{ij} = \bar{e}_{ij} - \frac{1}{2}\delta_{ij}\bar{e}_{\sigma\sigma} = \bar{e}_{ij}$ and we can forget the bars.) We have already studied the case of gravitons with the transverse polarizations $e_{11} = -e_{22}$ or e_{12} in part B1, and we see that the last term represents an interaction via spin two-gravitons with two independent transverse polarizations. In this case they are virtual gravitons generated by the transverse components of the density s_{12} or $(s_{11} - s_{22})$.

B4. Example: Gravitational Rutherford Scattering of Electrons

We shall consider the scattering of an electron by a heavy point mass in the Born approximation (i. e. via the exchange of a single virtual graviton). The matrix element for this process is given by equation B3-7

$$iM_F = K^2 s_{\mu\nu} \frac{1}{q^2} \bar{t}_{\mu\nu}$$

where $t_{\mu\nu}$ will be the energy tensor for the heavy particle of mass M,

$s_{\mu\nu}$ for the electron.

Actually much of the work for this problem has been done in the part B2. In that part we found that the gravitational field of a stationary point of mass M is from B2-18 $h_{11} = h_{22} = h_{33} = h_{44} = -2\frac{GM}{Kr}$; $h_{ij} = 0$ $(i \neq j)$. Equation B3-7 was obtained by considering the momentum space representation of the interaction $\frac{K}{2} S_{\mu\nu} h_{\mu\nu}$ where $h_{\mu\nu}$ was the field produced by $T_{\mu\nu}$.

The interaction is therefore

$$\frac{K}{2}[S_{11}h_{11} + S_{22}h_{22} + S_{33}h_{33} + S_{44}h_{44}]$$

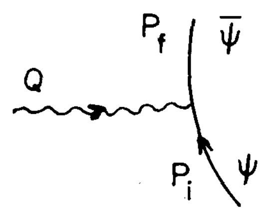
$$= -\frac{GM}{r}[S_{11} + S_{22} + S_{33} + S_{44}] \qquad \text{(B4-1)}$$

In momentum representation this becomes

$$iM_F = -4\pi MG[s_{11} + s_{22} + s_{33} + s_{44}]1/Q^2 \qquad \text{(B4-2)}$$

The symmetric energy momentum tensor for an electron given by Pauli (9) is

$$S_{\mu\nu} = -\frac{1}{4}[-i\bar{\psi}_{,\mu}\gamma_\nu\psi + i\bar{\psi}\gamma_\nu\psi_{,\mu} - i\bar{\psi}_{,\nu}\gamma_\mu\psi + i\bar{\psi}\gamma_\mu\psi_{,\nu}] \qquad \text{(B4-3)}$$



In momentum representation, where

$$\bar{\psi} = \bar{u}_f(p_f)e^{ip_f\cdot x}$$

$$\psi = u_i(p_i)e^{-ip\cdot x}$$

we will get a δ function giving rise to conservation of energy and momentum

$$\vec{P}_f = \vec{P}_i + \vec{Q} \; ; \quad E_f = E_i$$

and the energy tensor $s_{\mu\nu}$ becomes

$$-\frac{1}{4}\left[p_{f\mu}\bar{u}_f\gamma_\nu u_i + p_{i\mu}\bar{u}_f\gamma_\nu u_i + p_{f\nu}\bar{u}_f\gamma_\mu u_i + p_{i\nu}\bar{u}_f\gamma_\mu u_i \right]$$

The matrix element B4-2 becomes

$$iM_f = \frac{2\pi MG}{Q^2} \sum_{n=1}^{4} (p_{fn}\bar{u}_f\gamma_n u_i + p_{in}\bar{u}_f\gamma_n u_i)$$

The free particle Dirac equation may be written in the form

$$\not{p}_i u_i = m u_i \quad \text{and} \quad \bar{u}_f \not{p}_f = m\bar{u}_f$$

where $\not{p} = \gamma_\mu p_\mu$. Using the relation

$$\sum_{n=1}^{4} p_{in}\gamma_n = 2E_i\gamma_t - \not{p}_i$$

the matrix element iM_F becomes

$$iM_F = \frac{2\pi MG}{Q^2}\, \bar{u}_f\left[2(E_i + E_f)\gamma_t - \not{p}_f - \not{p}_i \right] u_i$$

$$M_F = -i\,\frac{4\pi MG}{Q^2}\, \bar{u}_f\left[2E\gamma_t - m\right] u_i \qquad \text{(B4-4)}$$

where $E_i = E_f = E$.

The probability of transition per second P_{fi} is

$$P_{fi} = \frac{2\pi}{(2E_1)(2E_2)} \, |M_F|^2 \rho \qquad (\hbar = c = 1)$$

where ρ = density of final states; E_1 and E_2 are the initial and final energies of the electron, and the normalization of u is $\bar{u}u = 2m$. Now

$$\rho = \frac{d^3p_2}{(2\pi)^3 dE_2} = \frac{p_2^2 \, d\Omega \, dp_2}{(2\pi)^3 dE_2} = \frac{E_2 |p_2| d\Omega}{(2\pi)^3}$$

where $E_2^2 = P_2^2 + m^2$; $E_2 dE_2 = P_2 dP_2$. Also

$$P_{fi} = \sigma v_i = \frac{\sigma |p_1|}{E_1}$$

where σ is the cross section for the reaction.

For our case $E_1 = E_2$, therefore $|p_1| = |p_2|$, and the differential cross section from the matrix element B4-4 becomes

$$\frac{d\sigma}{d\Omega} = \frac{M^2 G^2}{Q^4} \left| 2E\,\bar{u}_f \gamma_t u_i - m^2 \bar{u}_f u_i \right|^2 \qquad \text{(B4-5)}$$

We will evaluate B4-5 for the case where the electron is initially moving in the + x direction with its spin up in the z direction, and the scattering is in the x-y plane. The matrices $\bar{u}_f u_i$ and $\bar{u}_f \gamma_t u_i$ are zero if the spin is flipped and $\bar{u}_f$ represents an electron with its spin down in the z direction.* Thus the total cross section for this case is obtained from the non-spin flip amplitude.

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*For the spin flip case

$$\bar{u}_f = F^{-1/2}(0, F, -P_+, 0)$$

and

$$\bar{u}_f u_i = F^{-1}(0, F, -P_+, 0) \begin{pmatrix} F \\ 0 \\ 0 \\ P_i \end{pmatrix} = 0$$

$$\bar{u}\gamma_t u_i = F^{-1}(0, F, -P_+, 0) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} F \\ 0 \\ 0 \\ P_i \end{pmatrix} = 0$$

See the next paragraph for notation.

The spinor for a free electron moving in the x-y plane with its spin up in the z direction is

$$u = F^{-1/2}\begin{pmatrix} F \\ 0 \\ 0 \\ P_+ \end{pmatrix}; \quad \bar{u} = F^{-1/2}(F, 0, 0, -P_-)$$

where $F = E + m$; $P_+ = P_x + iP_y$; $P_- = P_x - iP_y$; $\bar{u}u = 2m$.

Therefore

$$u_i = F^{-1/2}\begin{pmatrix} F \\ 0 \\ 0 \\ P_i \end{pmatrix}; \quad u_f = F^{-1/2}(F, 0, 0, -P_i e^{-i\theta})$$

where θ is the angle of deflection.

Substituting these spinors into equation B4-5, the differential cross section becomes

$$\frac{d\sigma}{d\Omega} = \frac{4E^4M^2G^2}{Q^4}\left[(1 + v^2)^2 - v^2(v^2 + 3)\sin^2\frac{\theta}{2}\right] \qquad \text{(B4-6)}$$

where

$$Q = 2P \sin\theta/2.$$

The differential cross section if we had used only the Newtonian potential h_{44} would be

$$\frac{d\sigma}{d\Omega} = \frac{4E^2(E^2M^2G^2)}{Q^4}\left[1 - v^2\sin^2\theta/2\right]$$

This corresponds to the electrical Rutherford scattering of an electron with z^2e^4 replaced by $E^2M^2G^2$. This is exactly what we would expect, for the electrical force on the electron is ze^2/r^2, the Newtonian force EMG/r^2. (E approaches m_e, the mass of the electron for slow velocities.)

We see that for small angle deflections at low velocities the cross sections calculated by the Newtonian potential or by the full theory are the same. At high velocities when $v \approx 1$ the full cross section is four times the Newtonian cross section, which is equivalent to a deflection through twice the angle that is predicted by the Newtonian estimate. This is the factor of two we have seen before.

Let us consider the relative size of the gravitational and electrical cross sections. The strength of the electrical interaction is proportional to the dimensionless constant $e^2/\hbar c = 1/137$. The corresponding constant in gravity is

$$\frac{m_e^2 G}{\hbar c} = 1.75 \times 10^{-45}$$

or we may say that for electrons the gravitational force is effectively weaker than the electrical force by a factor 2.4×10^{-43}. Cross sections are smaller by the order of the square of this factor.

We may rewrite the gravitational Rutherford scattering cross section as

$$\frac{d\sigma}{d\Omega} = \frac{M^2 G^2}{4c^4 \sin^4\theta/2} \frac{1}{(v/c)^4} \left[(1 + v^2/c^2)^2 - \frac{v^2}{c^2}\left(\frac{v^2}{c^2} + 3\right)\sin^2\frac{\theta}{2} \right] \quad \text{(B4-7)}$$

where we have restored all factors of $\hbar$ and c. Because of the factor $(v/c)^{-4}$ the largest cross sections come from slow velocity electrons scattering at small angles. However, such a cross section could be obtained by a completely classical calculation using a Newtonian potential. Even so, for a one Kev electron being scattered through an angle of two degrees by a neutron, the cross section is $d\sigma/d\Omega = 2.7 \times 10^{-93}\ \text{cm}^2$.

We can see that quantum mechanical effects, such as the interaction with the spin of the electron, appear in the term $v^2(v^2+3)\sin^2\theta/2$. For example, suppose we had calculated the deflection of the electron neglecting its spin. We could do this by calculating the electrical Rutherford scattering using the Klein-Gordon equation for spinless particles, then replace the electrical force law ze^2/r^2 by $EMG(1 + v^2/c^2)/r^2$. The result is B4-7 without the spin term $v^2(v^2+3)\sin^2\theta/2$.*

For the spin term to be important we should have relativistic velocities and large deflections, just the conditions that make the cross sections even smaller. For relativistic electrons (note that the cross section B4-7 is independent of the energy of the electron once $v \approx 1$) taking $v \approx 1$, we must have a deflection of 21° before the spin term produces a five per cent correction in the cross section. For such an angle $d\sigma/d\Omega = 6.0 \times 10^{-102}$ cm^2. For the five per cent spin correction to be seen, we would have to measure cross sections of the order of 3×10^{-103} cm^2. It is not necessary to comment on the experimental difficulties of such a measurement. In fact it is hard to believe in the validity of an extension of our ideas of quantum mechanics to such a range beyond the limit of present experimental capabilities.

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*The gravitational Rutherford scattering for a spin-zero particle is correctly obtained by using the spin-zero energy tensor $s^o_{\mu\nu}$ in equation B4-2. With

$$S^o_{\mu\nu} = \varphi_{,\mu}\varphi_{,\nu} - \frac{1}{2}\delta_{\mu\nu}(\varphi_{,\rho}\varphi_{,\rho} - m^2\varphi^2)$$
$$s^o_{\mu\nu} = -p^i_\mu p^f_\nu + \frac{1}{2}\delta_{\mu\nu}(p^i \cdot p^f + m^2)$$

The cross section is again B4-7 without the spin term. (For the derivation of $S^o_{\mu\nu}$ see equation C1-42A.)

Consider the validity of the Born approximation (assuming the exchange of only one virtual graviton) for this calculation. For electrical scattering the condition was that $ze^2/\hbar v \ll 1$. For gravitational scattering the condition becomes in the low velocity limit $m_e MG/\hbar v \ll 1$; $v \gg \frac{Mm_e G}{\hbar} \approx 10^{-31}$ cm/sec (B4-8), where M is the mass of the neutron. As we have mentioned, there are corrections to the linear theory of gravity. These corrections become important only when the Born approximation breaks down. If, however, m_e and M were the mass of Mercury and the sun respectively, the velocity of Mercury is not great enough to satisfy condition B4-8 and indeed the corrections to the linear theory can be seen in the shift of the perihelion of Mercury.

In fact we know that the Born approximation breaks down when the particle is in a bound orbit. The smallest orbit that an electron can have around a neutron is the first Bohr orbit. As we can see from B4-8 if we assume only a gravitational force this orbit must be very large to give rise to velocities less than 10^{-31} cm/sec. In fact, the Bohr radius for such an electron-neutron system is 1.1×10^{33} cm. It is only when an electron is in such a large orbit, with such low velocities, that corrections to the linear theory and the Born approximation are necessary.

B5. The Uncertainty Principle for Gravity

We can now see that if gravity were classical, the failure of the uncertainty principle would be on the scale of large distances and slow velocities.

If we think in terms of the classical picture that the electrons in the hydrogen atom are in circular orbits, we would deduce from the uncertainty principle that the smallest orbit has one unit of orbital angular momentum $\hbar$, and a radius given by the Bohr radius. In such an orbit it is known that the product of the momentum of the electron times the lever arm (approximately the distance of the particle from the heavy force center) is exactly $\hbar$. If we do not specify whether the angular momentum comes from a large momentum with a small lever arm, or vice versa, then $\hbar$ essentially measures the product of the uncertainty of the position and momentum of the electron.

From this point of view the first gravitational Bohr orbit of the electron, with a radius of 10^{33} cm and velocities less than 10^{-31} cm/sec, is the smallest orbit consistent with the uncertainty principle. If the uncertainty principle failed and the classical theory were correct, then smaller orbits would be allowed. However, because of the weakness of the known gravitational forces, in comparison to electrical, beta-decay or heavy meson forces gravity may be neglected for smaller orbits, say on the atomic scale. Thus for only slow velocities and large distances would there be an appreciable modification of the uncertainty principle.

C. CORRECTIONS TO THE LINEAR THEORY OF GRAVITY

As we saw at the end of part A, the linear theory of gravity was obtained by neglecting the energy in the gravitational field in comparison to the energy of the matter fields. The linear theory was corrected in a straightforward manner by Feynman, by demanding that a consistent theory be obtained directly from an action principle. This work is described in parts C3 and C4. The author attempted to correct the linear theory by considering gravity as a spin-two field coupled to all forms of energy, including the energy in the field itself. This latter method, described in parts Cl, C2, and the end of part G, was eventually successful, but would probably be found only after the correct answer was known.

The basic problem is that the wave equation A-13 for the linear theory of gravity

$$-\bar{h}_{\mu\nu,\sigma\sigma} + \bar{h}_{\sigma\nu,\sigma\mu} + \bar{h}_{\sigma\mu,\sigma\nu} - \delta_{\mu\nu}\bar{h}_{\rho\sigma,\rho\sigma} = 2KT^{m}_{\mu\nu}$$

is inconsistent. This may be seen if we take the divergence of both sides of A-13.

$$-\bar{h}_{\mu\nu,\sigma\sigma\nu} + \bar{h}_{\sigma\nu,\sigma\mu\nu} + \bar{h}_{\sigma\mu,\sigma\nu\nu} - \bar{h}_{\rho\sigma,\rho\sigma\mu} = 2KT^{m}_{\mu\nu,\nu}$$

The left side is identically zero, while $T^{m}_{\mu\nu}$, the energy tensor of the matter alone, is not conserved in the presence of a gravitational field and thus $T^{m}_{\mu\nu,\nu} \neq 0$. The physically correct idea is to replace $T^{m}_{\mu\nu}$ in A-13 by $T_{\mu\nu}$, the complete symmetric energy tensor of the system, including the energy in the gravitational field as well as the energy of the matter. (We wish $T_{\mu\nu}$ to be symmetric so that the antisymmetric part of $h_{\mu\nu}$ will not be coupled to any form of energy.) As the total energy of the system, including gravitational energy, is conserved, $T_{\mu\nu,\nu} = 0$ and the correct gravitational wave equation will be given by

$$-\bar{h}_{\mu\nu,\sigma\sigma} + \bar{h}_{\sigma\nu,\sigma\mu} + \bar{h}_{\sigma\mu,\sigma\nu} - \delta_{\mu\nu}\bar{h}_{\rho\sigma,\rho\sigma} = 2KT_{\mu\nu} \qquad \text{(C-1)}$$

The problem is now to obtain the correct $T_{\mu\nu}$ to put in equation C-1.

C1. Symmetrized Energy Momentum Tensors

The method of finding conserved quantities such as the energy momentum tensor of a system is to investigate the invariances of the action for that system. For example, if the action is unchanged under the transformation of the time coordinate $t \rightarrow t + a$ where a is an infinitesimal constant, then the total energy of the system is conserved.

More generally if the action for a system is unchanged under the infinitesimal coordinate transformation

$$x_\mu \rightarrow x'_\mu = x_\mu - \eta_\mu \qquad \text{(C1-1)}$$

then depending on the choice of η_μ various quantities will be conserved. If η_μ is a constant infinitesimal vector with four arbitrary components then the total four momentum P_μ of the system is conserved. When η_μ represents an infinitesimal rotation (Lorentz transformation) with six arbitrary constant parameters (angles), it is the angular momentum of the system that is conserved.

The momentum of the system defined in terms of the energy momentum tensor density $T_{\mu\nu}$ is given by

$$P_\mu = \int T_{\mu 4} d^3x \qquad \text{(C1-2)}$$

Thus $T_{\mu 4}$ may be considered the momentum density. The natural choice for the angular momentum density about the origin is

$$x_\mu T_{\nu 4} - x_\nu T_{\mu 4}$$

where the total angular momentum is given by

$$M_{\mu\nu} = \int (x_\mu T_{\nu 4} - x_\nu T_{\mu 4}) d^3x \qquad \text{(C1-3)}$$

The conservation of momentum will follow if $T_{\mu\nu}$ has zero divergence, but $T_{\mu\nu}$ must also be symmetric for conservation of angular momentum. For example,

$$\int T_{\mu\nu,\nu} d^4x = 0 = \frac{\partial}{\partial t}\int T_{\mu 4} d^3x + \sum_{k=1}^{3} \int \frac{\partial}{\partial x_k}(T_{\mu k})\, d^3x$$

Since the fields are zero at infinity we may drop surface integrals, giving

$$\frac{dP_\mu}{dt} = \frac{d}{dt}\int T_{\mu 4}\, d^3x = 0 \qquad \text{(Cl-4)}$$

We also have

$$\frac{dM_{\mu\nu}}{dt} = \frac{d}{dt}\int (x_\mu T_{\nu 4} - x_\nu T_{\mu 4})\, d^3x$$

$$= \int (x_\mu T_{\nu\sigma} - x_\nu T_{\mu\sigma})_{,\sigma}\, d^3x - \sum_{k=1}^{3} \int \frac{\partial}{\partial x_k}(x_\mu T_{\nu k} - x_\nu T_{\mu k}) d^3x$$

Dropping surface terms and using $T_{\mu\sigma,\sigma} = 0$ we get

$$\frac{dM_{\mu\nu}}{dt} = \int (T_{\nu\mu} - T_{\mu\nu}) d^3x \qquad \text{(Cl-5)}$$

From Cl-5 we see that $T_{\mu\nu}$ must be symmetric for conservation of angular momentum, while linear momentum may be defined by an energy tensor whose divergence is zero, but which is not necessarily symmetric.

One procedure for finding an energy momentum tensor of a system is described in the following steps. First investigate the invariance of the action under infinitesimal coordinate translations. We shall see that this leads to a conserved (divergenceless) quantity $\theta_{\mu\nu}$, called the canonical energy momentum tensor. Because the conservation of linear momentum relies only upon the invariance of the action under coordinate translations the total momentum should be correctly given by

$$P_\mu = \int \theta_{\mu 4} d^4x \tag{C1-6}$$

Since the canonical tensor $\theta_{\mu\nu}$ is not obtained by considering the invariance of the action under rotations, there is no guarantee that the angular momentum is correctly given by C1-3 using $\theta_{\mu\nu}$ instead of $T_{\mu\nu}$. In fact the canonical tensor is in general not symmetric thus it does not lead to a conserved angular momentum.

The next step is to investigate the invariance of the action under rotations. This leads to a quantity $S_{\mu\nu}$ which, when added to the canonical tensor $\theta_{\mu\nu}$, gives a symmetric divergenceless energy momentum tensor $T_{\mu\nu}$.*

We shall see that $S_{\mu\nu}$ as well as $\theta_{\mu\nu}$ has zero divergence, thus the divergence of $T_{\mu\nu}$ is zero.

$$T_{\mu\nu,\nu} = (\theta_{\mu\nu} + S_{\mu\nu})_{,\nu} = 0$$

Also $S_{\mu\nu}$ will itself be a pure divergence, therefore its integral over space is zero and the total momentum P_μ is correctly given by

$$P_\mu = \int \theta_{\mu 4} d^3x = \int T_{\mu 4} d^3x \tag{C1-7}$$

*We will use the notation
$\theta_{\mu\nu}$ = canonical energy momentum tensor
$T_{\mu\nu} = \theta_{\mu\nu} + S_{\mu\nu}$ = the symmetric energy momentum tensor.

Let us suppose we desire the energy momentum tensor for a system that involves the scalar, vector and tensor fields φ, A_μ, and $h_{\mu\nu}$ respectively. First consider the action for such a system. The action may be written

$$S = \int \mathcal{L}\,(q^n;q^n_{,\alpha};q^n_{,\alpha\beta})d^4x$$

where the q^n are the components of the fields included in the system. We shall include the possibility that second derivatives of the field components appear in the Lagrangian.

If we write the field components in the form

$$q^n(x,\lambda) = q^n(x) + \lambda\delta q^n(x) \tag{C1-8}$$

then the action may be written in the form

$$S(\lambda) = \int \mathcal{L}\,[\,q^n(x,\lambda);q^n(x,\lambda)_{,\alpha};q^n(x,\lambda)_{,\alpha\beta}]\,d^4x \tag{C1-9}$$

and the condition that the action be an extremum may be written

$$\left.\frac{\partial S}{\partial\lambda}\right|_{\lambda=0} = 0 \tag{C1-10}$$

The quantity $\partial S(\lambda)/\partial\lambda$ may be written

$$\frac{\partial S(\lambda)}{\partial\lambda} = \int\left[\frac{\partial\mathcal{L}}{\partial q^n}\,\frac{\partial q^n(x,\lambda)}{\partial\lambda} + \frac{\partial\mathcal{L}}{\partial q^n_{,\alpha}}\,\frac{\partial q^n(x,\lambda)_{,\alpha}}{\partial\lambda} + \frac{\partial\mathcal{L}}{\partial q^n_{,\alpha\beta}}\,\frac{\partial q^n(x,\lambda)_{,\alpha\beta}}{\partial\lambda}\right]d^4x \tag{C1-11}$$

Using equation C1-8 we get

$$\left.\frac{\partial S(\lambda)}{\partial\lambda}\right|_{\lambda=0} = \int\left[\frac{\partial\mathcal{L}}{\partial q^n}\,\delta q^n + \frac{\partial\mathcal{L}}{\partial q^n_{,\alpha}}\,(\delta q^n)_{,\alpha} + \frac{\partial\mathcal{L}}{\partial q^n_{,\alpha\beta}}\,(\delta q^n)_{,\alpha\beta}\right]d^4x \tag{C1-12}$$

We may integrate by parts, dropping surface terms in C1-12 with the result

$$0 = \int \delta q^n \left[\frac{\partial \mathcal{L}}{\partial q^n} - \left(\frac{\partial \mathcal{L}}{\partial q^n_{,\alpha}} \right)_{,\alpha} + \left(\frac{\partial \mathcal{L}}{\partial q^n_{,\alpha\beta}} \right)_{,\alpha\beta} \right] d^4x$$

Thus the Euler-Lagrange equations of motion are

$$\frac{\partial \mathcal{L}}{\partial q^n} - \left(\frac{\partial \mathcal{L}}{\partial q^n_{,\alpha}} \right)_{,\alpha} + \left(\frac{\partial \mathcal{L}}{\partial q^n_{,\alpha\beta}} \right)_{,\alpha\beta} = 0 \qquad \text{(C1-13)}$$

Let us now consider the consequences when action is invariant under infinitesimal coordinate translations and rotations. This will be the case if the Lagrangian is of the form of a scalar density, which determines the change of the Lagrangian under a coordinate transformation. I.e.,

$$\mathcal{L}(x) = \mathcal{L}'(x'); \quad \mathcal{L}(x) \rightarrow \mathcal{L}'(x) = \mathcal{L}(x) + \delta\mathcal{L} \qquad \text{(C1-14)}$$

Furthermore the Lagrangian must be expressible as a function only of the field components and their derivatives.

$$\mathcal{L} = \mathcal{L}(q^n; q^n_{,\alpha}; q^n_{,\alpha\beta}) \qquad \text{(C1-15)}$$

Under a coordinate transformation the field components will be transformed:

$$q^n \rightarrow q^{n'}(x') = q^n + \delta q^n$$

$$q^n_{,\alpha} \rightarrow q^{n'}_{,\alpha}(x') = q^n_{,\alpha} + \delta(q^n_{,\alpha}) \quad \text{etc.} \qquad \text{(C1-16)}$$

Thus the change in the Lagrangian may also be written

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial q^n}\delta q^n + \frac{\partial\mathcal{L}}{\partial q^n_{,\alpha}}\delta(q^n_{,\alpha}) + \frac{\partial\mathcal{L}}{\partial q^n_{,\alpha\beta}}\delta(q^n_{,\alpha\beta}) \tag{C1-17}$$

The condition that the action is invariant under infinitesimal coordinate translations and rotations is that $\delta\mathcal{L}$ from C1-14 and C1-17 are equal (assuming the coordinate transformations giving C1-14 and C1-16 are infinitesimal translations or rotations).

Let an infinitesimal coordinate transformation be written in the form

$$x_\rho \rightarrow x'_\rho = x_\rho - \eta_\rho(x) \tag{C1-18}$$

If the four components η_ρ are constant, then C1-18 represents a coordinate translation. If $\eta_\rho(x)$ is of the form

$$\eta_\rho(x) = \omega_{\alpha\beta}(x_\alpha\delta_{\rho\beta} - x_\beta\delta_{\rho\alpha}) \tag{C1-19}$$

then C1-18 represents a rotation (Lorentz transformation) about the origin by an arbitrary infinitesimal angle in each of the six planes t-x, t-y, t-z, x-y, x-z, y-z. $\omega_{\alpha\beta} = -\omega_{\beta\alpha}$ has six independent components and $\omega_{\alpha\beta}/2$ is the angle of rotation in the α-β plane.

Under the coordinate transformation C1-18, the field $h_{\mu\nu}$, for example, transforms by the equation

$$h'_{\mu\nu}(x') = h_{\rho\sigma}(x)\frac{\partial x_\rho}{\partial x'_\mu}\frac{\partial x_\sigma}{\partial x'_\nu}$$

(This is the law of transformation of covariant tensors.) For $h_{\mu\nu}$ we get

$$h'_{\mu\nu}(x') = h_{\mu\nu}(x) + h_{\mu\rho}\eta_{\rho,\nu} + h_{\nu\rho}\eta_{\rho,\mu}$$

However we want the non-infinitesimal quantities $h'_{\mu\nu}(x')$ and $h_{\mu\nu}(x)$ expressed at the same point in space, therefore we make a Taylor series expansion of $h'_{\mu\nu}(x'_\rho)$ about the point x_ρ keeping only terms to first order in η.

$$h'_{\mu\nu}(x_\rho - \eta_\rho) = h'_{\mu\nu}(x_\rho) - h'_{\mu\nu,\rho}\eta_\rho \approx h'_{\mu\nu}(x_\rho) - h_{\mu\nu,\rho}\eta_\rho$$

The result is

$$h'_{\mu\nu}(x) = h_{\mu\nu}(x) + h_{\mu\nu,\rho}\eta_\rho + h_{\mu\rho}\eta_{\rho,\nu} + h_{\nu\rho}\eta_{\rho,\mu}$$

Similar arguments give the transformation of scalar and vector fields, leading to the result

$$\varphi'(x) = \varphi(x) + \varphi_{,\rho}\eta_\rho$$

$$A'_\mu(x) = A_\mu(x) + A_{\mu,\rho}\eta_\rho + (A_\rho\eta_{\rho,\mu})$$

$$h'_{\mu\nu}(x) = h_{\mu\nu}(x) + h_{\mu\nu,\rho}\eta_\rho + (h_{\mu\rho}\eta_{\rho,\nu} + h_{\rho\nu}\eta_{\rho,\mu}) \qquad \text{(C1-20)}$$

or

$$q^{n'}(x) = q^n(x) + q^n_{,\rho}\eta_\rho + (\Delta q^n)$$

$$= q^n(x) + \delta q^n \qquad \text{(C1-21)}$$

where all terms of δq^n which involve derivatives of η have been included in Δq^n.

Since $\varphi_{,\alpha}$ is a vector, $A_{\mu,\alpha}$ a tensor, etc. (in Gallelian coordinates)

$$\varphi'_{,\alpha}(x) = \varphi_{,\alpha}(x) + \varphi_{,\alpha\rho}\eta_\rho + \varphi_{,\rho}\eta_{\rho,\alpha}$$

$$A'_{\mu,\alpha}(x) = A_{\mu,\alpha}(x) + A_{\mu,\alpha\rho}\eta_\rho + A_{\rho,\alpha}\eta_{\rho,\mu} + A_{\mu,\rho}\eta_{\rho,\alpha}$$

$$h'_{\mu\nu,\alpha}(x) = h_{\mu\nu,\alpha}(x) + h_{\mu\nu,\alpha\rho}\eta_\rho + h_{\rho\nu,\alpha}\eta_{\rho,\mu}$$

$$+ h_{\mu\rho,\alpha}\eta_{\rho,\nu} + h_{\mu\nu,\rho}\eta_{\rho,\alpha} \qquad \text{(C1-22)}$$

C1-22 may be written in the form

$$q^{n'}_{,\alpha} = q^n_{,\alpha} + \delta(q^n_{,\alpha}) \qquad \text{(C1-23)}$$

Let us now consider the quantity $(\delta A_\mu)_{,\alpha}$ for example.

$$(\delta A_\mu)_{,\alpha} = (A_{\mu,\rho\alpha}\eta_\rho + A_{\mu,\rho}\eta_{\rho,\alpha} + A_{\rho,\alpha}\eta_{\rho,\mu}) + A_\rho\eta_{\rho,\mu\alpha}$$

or

$$(\delta A_\mu)_{,\alpha} = \delta(A_{\mu,\alpha}) + A_\rho\eta_{\rho,\mu\alpha}$$

It is true in general that

$(\delta q^n)_{,\alpha} = \delta(q^n_{,\alpha})$ + terms involving the second derivatives of η_ρ

$(\delta q^n)_{,\alpha\beta} = \delta(q^n_{,\alpha\beta})$ + terms involving the second and third derivatives of η_ρ

However for coordinate translations and rotations, η_ρ has at most a first derivative. Since we are dealing only with such transformations we may take

$$(\delta q^n)_{,\alpha} = \delta(q^n_{,\alpha})$$

$$(\delta q^n)_{,\alpha\beta} = \delta(q^n_{,\alpha\beta}) \qquad \text{(C1-24)}$$

Let us now consider the condition that the two forms of $\delta\mathcal{L}$, from C1-14 and C1-17, are equal. Since $\mathcal{L}$ is a scalar density (so that $S = \int \mathcal{L}\, d^4x$ will be scalar)

$$\mathcal{L}'(x) = \mathcal{L}(x) + \mathcal{L}_{,\rho}\eta_\rho = \mathcal{L}(x) + (\mathcal{L}\eta_\rho)_{,\rho}$$

where the last step follows from the fact that $\eta_{\rho,\rho} = 0$ for translations and rotations. Equating $(\mathcal{L}\eta_\rho)_{,\rho}$ to $\delta\mathcal{L}$ of C1- 17 we get

$$0 = \frac{\partial\mathcal{L}}{\partial q^n}\,\delta q^n + \frac{\partial\mathcal{L}}{\partial q^n_{,\alpha}}\,\delta(q^n_{,\alpha}) + \frac{\partial\mathcal{L}}{\partial q^n_{,\alpha\beta}}\,\delta(q^n_{,\alpha\beta}) - (\mathcal{L}\,\eta_\rho)_{,\rho}$$

Using C1-24 this may be written in the form

$$\delta q^n\left[\frac{\partial\mathcal{L}}{\partial q^n} - \left(\frac{\partial\mathcal{L}}{\partial q^n_{,\alpha}}\right)_{,\alpha} + \left(\frac{\partial\mathcal{L}}{\partial q^n_{,\alpha\beta}}\right)_{,\alpha\beta}\right] - (\mathcal{L}\,\eta_\rho)_{,\rho}$$

$$+\left\{\delta q^n\,\frac{\partial\mathcal{L}}{\partial q^n_{,\alpha}} - \delta q^n\left(\frac{\partial\mathcal{L}}{\partial q^n_{,\alpha\beta}}\right)_{,\beta} + (\delta q^n)_{,\beta}\,\frac{\partial\mathcal{L}}{\partial q^n_{,\alpha\beta}}\right\}_{,\alpha} = 0 \qquad \text{(C1-25)}$$

The terms in the square brackets of C1-25 are zero by the equations of motion C1-13. If we use the notation of C1-21;

$$\delta q^n = q^n_{,\rho}\eta_\rho + \Delta q^n$$

equation C1-25 becomes

$$0 = \left\{\eta_\rho\left[q^n_{,\rho}\,\frac{\partial\mathcal{L}}{\partial q^n_{,\sigma}} - q^n_{,\rho}\left(\frac{\partial\mathcal{L}}{\partial q^n_{,\sigma\lambda}}\right)_{,\lambda} + q^n_{,\rho\lambda}\,\frac{\partial\mathcal{L}}{\partial q^n_{,\sigma\lambda}} - \delta_{\rho\sigma}\mathcal{L}\right]\right\}_{,\sigma} \qquad \text{(a)}$$

$$+\left\{\Delta q^n\,\frac{\partial\mathcal{L}}{\partial q^n_{,\sigma}} - \Delta q^n\left(\frac{\partial\mathcal{L}}{\partial q^n_{,\sigma\lambda}}\right)_{,\lambda} + \Delta q^n_{,\lambda}\,\frac{\partial\mathcal{L}}{\partial q^n_{,\sigma\lambda}} + \eta_{\rho,\lambda}q^n_{,\rho}\,\frac{\partial\mathcal{L}}{\partial q^n_{,\sigma\lambda}}\right\}_{,\sigma} \qquad \text{(b)}$$

(C1-26)

For the fields φ, A_μ, and $h_{\mu\nu}$, the quantities Δq^n are from C1-20

$$\Delta\varphi = 0$$

$$\Delta A_\mu = A_\rho \eta_{\rho,\mu} \tag{C1-27}$$

$$\Delta h_{\mu\nu} = h_{\mu a}\eta_{a,\nu} + h_{\nu a}\eta_{a,\mu}$$

Suppose now that the components of η_ρ are constant, representing a coordinate displacement. Since $\eta_{\rho,\lambda}$ and the Δq^n will be zero, C1-26 reduces to

$$\eta_\rho \left\{ q^n_{,\rho} \frac{\partial \mathcal{L}}{\partial q^n_{,\sigma}} - q^n_{,\rho}\left(\frac{\partial \mathcal{L}}{\partial q^n_{,\sigma\lambda}}\right)_{,\lambda} + q^n_{,\rho\lambda} \frac{\partial \mathcal{L}}{\partial q^n_{,\sigma\lambda}} - \delta_{\rho\sigma}\mathcal{L} \right\}_{,\sigma}$$

$$\equiv \eta_\rho \left\{ \theta_{\rho\sigma} \right\}_{,\sigma} = 0$$

Thus the invariance of the Lagrangian under coordinate displacements leads to the quantity $\theta_{\rho\sigma}$ (given in the curly brackets of equation C1-16) which is conserved. That is, since η_ρ is arbitrary,

$$\theta_{\rho\sigma,\sigma} = 0$$

$\theta_{\rho\sigma}$ is the canonical energy momentum of the system, mentioned at the beginning of this part, and the formula for the total conserved four momentum of the system is

$$P_\mu = \int \theta_{\mu 4} d^4x$$

If η_ρ now represents an infinitesimal rotation about the origin

$$\eta_\rho = \omega_{\alpha\beta}(x_\alpha \delta_{\rho\beta} - x_\beta \delta_{\rho\alpha})$$

the condition on the Lagrangian C1-26 becomes

$$0 = \omega_{\alpha\beta} \left\{ (x_\alpha \delta_{\rho\beta} - x_\beta \delta_{\rho\alpha}) \theta_{\rho\sigma} \right\}_{,\sigma}$$

$$+ \omega_{\alpha\beta} \left\{ (\delta_{\lambda\alpha} \delta_{\rho\beta} - \delta_{\lambda\beta} \delta_{\rho\alpha}) q^n_{,\rho} \frac{\partial \mathcal{L}}{\partial q^n_{,\sigma\lambda}} \right\}_{,\sigma}$$

$$+ \left\{ \Delta q^n \frac{\partial \mathcal{L}}{\partial q^n_{,\sigma}} - \Delta q^n \left(\frac{\partial \mathcal{L}}{\partial q^n_{,\sigma\lambda}} \right)_{,\lambda} + \Delta q^n_{,\lambda} \frac{\partial \mathcal{L}}{\partial q^n_{,\sigma\lambda}} \right\}_{,\sigma} \qquad \text{(C1-29)}$$

where the Δq^n, given in equations C1-27, depend on the spin of the field.

For a scalar field where $\Delta\varphi$ is zero, C1-29 becomes

$$0 = \omega_{\alpha\beta}(x_\alpha \theta_{\beta\sigma} - x_\beta \theta_{\alpha\sigma})_{,} + \omega_{\alpha\beta} \left(\varphi_{,\beta} \frac{\partial \mathcal{L}}{\partial \varphi_{,\sigma\alpha}} - \varphi_{,\alpha} \frac{\partial \mathcal{L}}{\partial \varphi_{,\sigma\beta}} \right)_{,\sigma} \qquad \text{(C1-30)}$$

Let us define the quantity $f^o_{\alpha\beta\sigma}$ by

$$f^o_{\alpha\beta\sigma} = \varphi_{,\alpha} \frac{\partial \mathcal{L}}{\partial \varphi_{,\sigma\beta}} - \varphi_{,\beta} \frac{\partial \mathcal{L}}{\partial \varphi_{,\sigma\alpha}}; \quad f^o_{\alpha\beta\sigma} = - f^o_{\beta\alpha\sigma} \qquad \text{(C1-31)}$$

then equation C1-30 becomes

$$0 = \omega_{\alpha\beta} \left\{ (x_\alpha \theta_{\beta\sigma} - x_\beta \theta_{\alpha\sigma})_{,\sigma} + f^o_{\beta\alpha\sigma,\sigma} \right\} \qquad \text{(C1-32)}$$

Now suppose we can write

$$(x_\alpha \theta_{\beta\sigma} - x_\beta \theta_{\alpha\sigma})_{,\sigma} + f^o_{\beta\alpha\sigma,\sigma} = [x_\alpha(\theta_{\beta\sigma} + S^o_{\beta\sigma}) - x_\beta(\theta_{\alpha\sigma} + S^o_{\alpha\sigma})]_{,\sigma} \qquad \text{(C1-33)}$$

Then since $\omega_{\alpha\beta}$ is arbitrary, C1-32 gives

$$[x_\alpha(\theta_{\beta\sigma} + S^o_{\beta\sigma}) - x_\beta(\theta_{\alpha\sigma} + S^o_{\alpha\sigma})]_{,\sigma} = 0 \tag{C1-34}$$

If in addition $S_{\beta\sigma,\sigma} = 0$, then we will get from C1-34

$$(\theta_{\beta\alpha} + S^o_{\beta\alpha}) - (\theta_{\alpha\beta} + S^o_{\alpha\beta}) = 0 \tag{C1-35}$$

Thus if we define

$$T_{\alpha\beta} = \theta_{\alpha\beta} + S^o_{\alpha\beta}$$

then $T_{\alpha\beta}$ will be a symmetric tensor with zero divergence.

Equation C1-33 may be written in the form

$$f^o_{\beta\alpha\sigma,\sigma} = (x_\alpha S^o_{\beta\sigma} - x_\beta S^o_{\alpha\sigma})_{,\sigma} \tag{C1-36}$$

A solution of this equation is given by Belinfante (1)

$$S^o_{\alpha\beta} = \frac{1}{2}(f^o_{\alpha\beta\sigma} + f^o_{\sigma\alpha\beta} + f^o_{\sigma\beta\alpha})_{,\sigma} \tag{C1-37}$$

where

$$S^o_{\alpha\beta,\beta} = \frac{1}{2}(f^o_{\alpha\beta\sigma} + f^o_{\sigma\alpha\beta} + f^o_{\sigma\beta\alpha})_{,\sigma\beta} = 0$$

since $f^o_{ijk} = - f^o_{jik}$.

It is seen in C1-37 that $S_{\alpha\beta}$, aside from having zero divergence, is itself a derivative. If we write

$$S_{\alpha\beta} = \psi_{\alpha\beta\sigma,\sigma} \quad \text{where} \quad \psi_{\alpha\beta\sigma} = - \psi_{\alpha\sigma\beta} \tag{C1-38}$$

then

$$\int S_{\mu 4} d^3x = \int \psi_{\mu t\sigma,\sigma} d^3x$$

$$= \frac{\partial}{\partial t} \psi_{\mu tt} d^3x + \sum_{k=1}^{3} \int \frac{\partial}{\partial x_k}(\psi_{\mu tk}) d^3x$$

Since $\psi_{\mu tt} = -\psi_{\mu tt} = 0$ and the second term is a surface integral,

$$\int S_{\mu 4} d^3x = 0$$

Thus the tensor $T_{\alpha\beta} = \theta_{\alpha\beta} + S_{\alpha\beta}$ is not only symmetric and has zero divergence, but

$$\int T_{\mu 4} d^4x = \int \theta_{\mu 4} d^4x = P_\mu$$

Thus $T_{\alpha\beta}$ is a complete energy tensor from which the total linear and angular momentum may be obtained by equations Cl-2 and Cl-3.

Before the energy tensor for a spin-zero field is considered in more detail, let us note that equation Cl-26, for the choice $\eta_\rho = \omega_{\alpha\beta}(x_\alpha \delta_{\rho\beta} - x_\beta \delta_{\rho\alpha})$, can always be written in the form

$$0 = \omega_{\alpha\beta}(x_\alpha \theta_{\beta\sigma} - x_\beta \theta_{\alpha\sigma})_{,\sigma} + \omega_{\alpha\beta} f_{\beta\alpha\sigma,\sigma} \qquad \text{(Cl-39)}$$

where Cl-26b has been written in the form $\omega_{\alpha\beta} f_{\beta\alpha\sigma,\sigma}$. This can be done because $\omega_{\alpha\beta}$ apears in each term of Cl-26b, and being a constant may be factored outside of the derivatives.

Since Cl-39 is the same form as Cl-32 the same arguments used to find $S_{\alpha\beta}$ in the form of equation Cl-37 may be applied in general. We may summarize the results for finding the energy tensor $T_{\alpha\beta}$ from a Lagrangian involving fields of any integer spin and up to second derivatives of the field components by the following relations. (If the Lagrangian involves higher derivatives than the second, the formulas may clearly be extended.)

$$T_{\alpha\beta} = \theta_{\alpha\beta} + S_{\alpha\beta} \qquad \text{(Cl-40a)}$$

$$\theta_{\alpha\beta} = q^n_{,\alpha} \frac{\partial \mathcal{L}}{\partial q^n_{,\beta}} - q^n_{,\alpha}\left(\frac{\partial \mathcal{L}}{\partial q^n_{,\beta\lambda}}\right)_{,\lambda} + q^n_{,\alpha\lambda} \frac{\partial \mathcal{L}}{\partial q^n_{,\beta\lambda}} - \delta_{\alpha\beta}\mathcal{L} \tag{C1-40b}$$

$$S_{\alpha\beta} = \frac{1}{2}\left(f_{\alpha\beta\sigma} + f_{\sigma\alpha\beta} + f_{\sigma\beta\alpha}\right)_{,\sigma} \tag{C1-40c}$$

where

$$\omega_{\alpha\beta} f_{\beta\alpha\sigma,\sigma} = \omega_{\alpha\beta}\left\{ q^n_{,\beta} \frac{\partial \mathcal{L}}{\partial q^n_{,\sigma\alpha}} - q^n_{,\alpha} \frac{\partial \mathcal{L}}{\partial q^n_{,\sigma\beta}} \right\}_{,\sigma}$$

$$+ \left\{ \Delta q^n \frac{\partial \mathcal{L}}{\partial q^n_{,\sigma}} - \Delta q^n \left(\frac{\partial \mathcal{L}}{\partial q^n_{,\sigma\lambda}}\right)_{,\lambda} + \Delta q^n_{,\lambda} \frac{\partial \mathcal{L}}{\partial q^n_{,\sigma\lambda}} \right\}_{,\sigma} \tag{C1-40d}$$

The Δq^n are given by the formula

$$\Delta q^n = \delta q^n - q^n_{,\rho}\eta_\rho \tag{C1-40e}$$

where δq^n is the change of q^n under the coordinate transformation

$$x_\rho \rightarrow x'_\rho = x_\rho - \eta_\rho(x) \tag{C1-40f}$$

and η_ρ for our case is given by

$$\eta_\rho = \omega_{\alpha\beta}(x_\alpha \delta_{\rho\beta} - x_\beta \delta_{\rho\alpha}) \tag{C1-40g}$$

For a Lagrangian involving a spin-zero field φ, a vector field A_μ, and a symmetric tensor field $h_{\mu\nu}$ (including first and second derivatives of these fields), the formula for $f_{\alpha\beta\sigma}$ is given by

$$f_{\alpha\beta\sigma} = \varphi_{,\alpha} \frac{\partial \mathcal{L}}{\partial \varphi_{,\beta\sigma}} + A_\alpha \frac{\partial \mathcal{L}}{\partial A_{\beta,\sigma}} + A_{\alpha,\lambda} \frac{\partial \mathcal{L}}{\partial A_{\beta,\sigma\lambda}} + A_{\lambda,\alpha} \frac{\partial \mathcal{L}}{\partial A_{\lambda,\sigma\beta}}$$

$$- A_\alpha \left(\frac{\partial \mathcal{L}}{\partial A_{\beta,\sigma\lambda}}\right)_{,\lambda} + 2\,\delta_{\gamma\beta} h_{\delta\alpha} \frac{\partial \mathcal{L}}{\partial h_{\delta\gamma,\sigma}} + 2\delta_{\gamma\beta} h_{\delta\alpha,\lambda} \frac{\partial \mathcal{L}}{\partial h_{\delta\gamma,\sigma\lambda}}$$

$$+ 2\delta_{\gamma\lambda} h_{\delta\lambda,\alpha} \frac{\partial \mathcal{L}}{\partial h_{\delta\gamma,\sigma\beta}} - 2\delta_{\beta\gamma} h_{\delta\alpha} \left(\frac{\partial \mathcal{L}}{\partial h_{\delta\gamma,\sigma\lambda}} \right)_{,\lambda}$$

- the same terms with α and β reversed.* (Cl-40h)

In using Cl-40h for the fields $h_{\mu\nu}$, care must be taken taken that the variation of $\mathcal{L}$ with respect to $h_{\gamma\delta}$ be kept symmetric in γ and δ.

The method described by equations Cl-40 for finding a symmetric energy momentum of a system is essentially an extension of the method of Belinfante (1) to the case where second derivatives are included in the Lagrangian.

There are several interesting points to be noted about the tensors $\theta_{\alpha\beta}$ and $S_{\alpha\beta}$ given in equations Cl-40. First the tensor $T_{\alpha\beta} = \theta_{\alpha\beta} + S_{\alpha\beta}$ has zero divergence and is symmetric only if the equations of motion are used. For example, since the divergence of $S_{\alpha\beta}$ is identically zero,

$$T_{\alpha\beta,\beta} = \theta_{\alpha\beta,\beta} = - q^n_{,\alpha} \left[\frac{\partial \mathcal{L}}{\partial q^n} - \left(\frac{\partial \mathcal{L}}{\partial \beta^n_{,\beta}} \right)_{,\beta} + \left(\frac{\partial \mathcal{L}}{\partial q^n_{,\beta\lambda}} \right)_{,\beta\lambda} \right]$$

which is only zero by the equations of motion Cl-13.

Second, Belinfante** was lead to consider $S_{\mu\nu}$ as the "spin momentum density," and the quantity

$$x_\lambda S_{\mu\nu} - x_\mu S_{\lambda\nu}$$

*The term $\partial \mathcal{L}/\partial A_{\beta,\sigma\lambda}$, for example, must be kept symmetric in σ and λ since the order of partial differential is reversable.

**Belinfante (1).

as the "spin angular momentum density" of the field. However Belinfante considered that the Lagrangian was a function only of the field components and their first derivatives. In this case a scalar field does not give rise to a term $S_{\mu\nu}$, and thus has no "spin momentum density." Belinfante's interpretation cannot be extended to the case where second derivatives are included in the Lagrangian, for then a scalar, or spin-zero field has a "spin momentum density" $S_{\mu\nu}$.

Consider for example the Lagrangian for a spin-zero field of rest mass μ.

$$\mathcal{L}^a = \frac{1}{2}(\varphi_{,\rho}\varphi_{,\rho} - \mu^2\varphi^2) \tag{C1-41}$$

If a pure divergence $\frac{1}{2}(\varphi_{,\rho}\varphi)_{,\rho}$ is added to $\mathcal{L}^a$ we get

$$\mathcal{L}^b = \varphi_{,\rho}\varphi_{,\rho} + \frac{1}{2}\varphi\varphi_{,\rho\rho} - \frac{1}{2}\mu^2\varphi^2 \tag{C1-42}$$

The tensors $\theta_{\alpha\beta}$ and $S_{\alpha\beta}$ corresponding to $\mathcal{L}^a$ and $\mathcal{L}^b$ are

$$\theta^a_{\alpha\beta} = \varphi_{,\alpha}\varphi_{,\beta} - \frac{1}{2}\delta_{\alpha\beta}(\varphi_{,\rho}\varphi_{,\rho} - \mu^2\varphi^2)$$

$$S^a_{\alpha\beta} = 0$$

$$\theta^b_{\alpha\beta} = \frac{3}{2}\varphi_{,\alpha}\varphi_{,\beta} + \frac{1}{2}\varphi_{,\alpha\beta}\varphi - \frac{1}{2}\delta_{\alpha\beta}(2\varphi_{,\rho}\varphi_{,\rho} + \varphi_{,\rho\rho}\varphi - \mu^2\varphi^2)$$

$$S^b_{\alpha\beta} = -\frac{1}{2}\varphi_{,\alpha}\varphi_{,\beta} - \frac{1}{2}\varphi_{,\alpha\beta}\varphi - \frac{1}{2}\delta_{\alpha\beta}(-\varphi_{,\rho}\varphi_{,\rho} - \varphi_{,\rho\rho}\varphi)$$

also

$$T^b_{\alpha\beta} = \theta^b_{\alpha\beta} + S^b_{\alpha\beta} = \varphi_{,\alpha}\varphi_{,\beta} - \frac{1}{2}\delta_{\alpha\beta}(\varphi_{,\rho}\varphi_{,\rho} - \mu^2\varphi^2) \tag{C1-42A}$$

We see that the two complete energy tensors $T^a_{\alpha\beta}$ and $T^b_{\alpha\beta}$ are the same, but there is a so-called "spin momentum density" $S^{b}_{\alpha\beta}$ for a spin-zero field when derived from Lagrangian $\mathcal{L}^{b}$.

It is interesting to note that $\theta^{b}_{\alpha\beta}$ is already symmetric and therefore might be considered a possible energy tensor for a spin-zero field. In fact

$$\theta^{b}_{\alpha\beta} - \theta^{a}_{\alpha\beta} = \frac{1}{4}\left[(\delta_{\rho\alpha}\delta_{\beta\sigma} - \delta_{\alpha\beta}\delta_{\rho\sigma})\varphi\varphi\right]_{,\rho\sigma} \qquad \text{(C1-43)}$$

Thus the two energy tensors differ by a term that is symmetric, is a pure divergence, and, as may be easily checked, has zero divergence. Although the prescription of equations C1-40 does not lead to $\theta^{b}_{\alpha\beta}$ as a complete energy tensor since $S^{b}_{\alpha\beta}$ has not been added, we shall later discuss another method of obtaining energy tensors that can lead directly to $\theta^{b}_{\alpha\beta}$.

Finally, energy tensors resulting from equations C1-40 are not unique. Only for spin-zero fields do equations C1-40 lead to the same tensor starting from equivalent Lagrangians (i.e., starting from Lagrangians differing only by a pure divergence). We have not considered adding to the Lagrangian a term such as $(\varphi_{,\rho\sigma}\varphi_{,\sigma})_{,\rho}$ since this term has different dimensions than the rest of the terms in the Lagrangian. The only term that can be added to a spin-zero field with the same number of field components and of the same dimensions is $(\varphi_{,\rho}\varphi)_{,\rho}$ and we saw that this did not change the energy tensor.

Let us however consider the following Lagrangian for a vector field.

$$4\pi \mathcal{L}^{c} = \frac{1}{2}\left(A_{\mu,\nu}A_{\mu,\nu} - A_{\mu,\nu}A_{\nu,\mu}\right) \tag{C1-44}$$

This is the usual Lagrangian for the electromagnetic field, and by equations C1-40 gives the energy tensor

$$T^{c}_{\mu\nu} = -\frac{1}{4\pi}\left\{F_{\mu\alpha}F_{\alpha\nu} - \frac{\delta_{\mu\nu}}{4}F_{\alpha\beta}F_{\alpha\beta}\right\}^{*} \tag{C1-44A}$$

where

$$F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu}$$

$T^{c}_{\mu\nu}$ is just the energy momentum tensor of the electromagnetic field, as given in several texts.**

We might also have started with the Lagrangian

$$4\pi \mathcal{L}^{d} = \frac{1}{2}\left(A_{\mu,\nu}A_{\mu,\nu} - A_{\mu,\mu}A_{\nu,\nu}\right) \tag{C1-45}$$

because $\mathcal{L}^{d}$ and $\mathcal{L}^{c}$ differ by a pure divergence.

$$4\pi\mathcal{L}^{d} - 4\pi\mathcal{L}^{c} = \frac{1}{2}\left[A_{\nu,\nu}A_{\mu} - A_{\mu,\nu}A_{\nu}\right]_{,\mu} \tag{C1-46}$$

However the energy tensor that results from $\mathcal{L}^{d}$ is

$$T^{d}_{\mu\nu} = T^{c}_{\mu\nu} + W_{\mu\nu}$$

where

$$W_{\mu\nu} = \frac{1}{8\pi}\left[\delta_{\mu\nu}A_{\rho}A_{\sigma} + \delta_{\rho\sigma}A_{\mu}A_{\nu} - \delta_{\mu\rho}A_{\nu}A_{\sigma} - \delta_{\rho\nu}A_{\mu}A_{\sigma}\right]_{,\rho\sigma} \tag{C1-47}$$

Since the expression for $W_{\mu\nu}$ is anti-symmetric between ν and ρ,

$$W_{\mu\nu,\nu} = 0$$

---

*In this equation and in the rest of this part we will neglect terms that are zero by the equations of motion.

**See Landau (12), Tolman (13), Eddington (14).

We have thus generated, by equations C1-40, two energy momentum tensors that differ by a term $W_{\mu\nu}$ that is symmetric, has zero divergence, and is itself a second derivative. We should note that the tensors $T^{c}_{\mu\nu}$ and $T^{d}_{\mu\nu}$ were obtained from Lagrangians that involved only the first derivatives of the field components. Thus the ambiguity is not a result of the fact that we have allowed the Lagrangian to involve second derivatives of the field components.

In fact, from working several examples, the author believes that the inclusion of second derivatives does not contribute any further ambiguity in the energy tensors calculated by equations C1-40. In the case of spin-zero fields we have seen that the inclusion in the Lagrangian of the term $(\varphi_{,\rho}\varphi)_{,\rho}$ does not lead to a different energy tensor $T_{\mu\nu}$.

For vector fields, when the terms

$$\frac{1}{8\pi}(A_{\nu,\nu}A_{\mu})_{,\mu} - \frac{1}{8\pi}(A_{\mu,\nu}A_{\nu})_{,\nu}$$

$$= \frac{1}{8\pi}(A_{\nu,\nu}A_{\mu,\mu} - A_{\mu,\nu}A_{\nu,\mu}) \qquad \text{(C1-48)}$$

were added to the Lagrangian, the additional term $W_{\mu\nu}$ of equation C1-47 appeared in the energy tensor. When we add just one of the terms of C1-48, namely

$$\frac{1}{8\pi}(A_{\nu,\nu}A_{\mu})_{,\mu} = \frac{1}{8\pi}(A_{\nu,\nu\mu}A_{\mu} + A_{\nu,\nu}A_{\mu,\mu})$$

the Lagrangian will involve second derivatives, but the additional term in the energy tensor is just $W_{\mu\nu}/2$. The term

$$-\frac{1}{8\pi}(A_{\mu,\nu}A_{\nu})_{,\mu}$$

likewise adds $W_{\mu\nu}/2$ to the energy tensor, which agrees with the fact that the sum of the two terms C1-48 just adds $W_{\mu\nu}$. We might consider the possibility of adding

$$(A_{\mu,\nu}A_{\mu})_{,\nu} \tag{C1-49}$$

but this term does not lead to any change in the energy tensor.

Thus for vector fields the only ambiguity in the energy tensor calculated by C1-40 is the term $W_{\mu\nu}$ of C1-47, which may be obtained from Lagrangians involving first derivatives only. For tensor fields there are many more forms of the Lagrangian that differ by a pure divergence and still involve only first derivatives. Due to the algebra involved we have not investigated whether new forms of the energy tensor may be obtained by equations C1-40 if second derivatives are allowed in the Lagrangian for the tensor field.

The author does not guarantee that equations C1-40 are unique, for we shall later, by another method, derive symmetric energy tensors which cannot be obtained by equations C1-40. An example of such a tensor will be $\Theta^{b}_{\mu\nu}$ of equation C1-43.

The difference between energy tensors of a given system, whether obtained by equations C1-40 or any other method used by the author, is always of the form $W_{\mu\nu}$--that is the difference is symmetric, has zero divergence, and is itself a second derivative. We will show later in part G why this form always appears. We might argue that a term of this form in the energy tensor has no physical meaning, for $W_{\mu\nu}$ will not contribute to either the total momentum or the total angular momentum by equations C1-2 and C1-3. However the energy momentum tensor appears as the source of the gravitational waves in equation A-13. We will see in the next part that a term of the form $W_{\mu\nu}$ does have physical meaning in connection with gravity, and that a theory using the wrong choice of $W_{\mu\nu}$ leads to the wrong experimental result.

C2. Theory of a Spin-Two Field Coupled to Energy

At the beginning of part C, we said that the correct gravitational wave equation would result if we used as the source of gravity the complete energy tensor of the system $T_{\mu\nu}$, rather than $T^m_{\mu\nu}$, the energy tensor of the matter alone. $T_{\mu\nu}$ would be obtained from the complete Lagrangian of the system whereas $T^m_{\mu\nu}$ would come from a Lagrangian in which the gravitational fields have been neglected.

In this part we shall use equations C1-40 to obtain a complete energy tensor for a system of point particles interacting with gravity. It will then be checked whether this energy tensor, used as a source of gravity, gives the correct shift in the perihelion of Mercury. Because energy tensors obtained by equations C1-40 are not unique, the first energy tensor chosen might not work.

In using equations C1-40, we do not need to deal with the complete Lagrangian all at one time. If the Lagrangian can be written in the form

$$\mathcal{L} = \mathcal{L}^1 + \mathcal{L}^2 + \mathcal{L}^3 + \dots$$

where each part of the Lagrangian, $\mathcal{L}_i$, is Lorentz invariant, then the energy tensor may be broken up into corresponding parts

$$T_{\mu\nu} = T^1_{\mu\nu} + T^2_{\mu\nu} + T^3_{\mu\nu} + \dots$$

The tensor $T^i_{\mu\nu}$ is obtained by applying equations C1-40 to the part of the Lagrangian $\mathcal{L}^i$.

For the case of gravity interacting with matter, we will find it convenient to break the Lagrangian up in the form

$$\mathcal{L} = \mathcal{L}_g + \mathcal{L}_m + \mathcal{L}_{int} \tag{C2-1}$$

where $\mathcal{L}_g$ is the Lagrangian for the free gravitational fields (given by equation A-11), $\mathcal{L}_m$ is the Lagrangian for the matter without the presence of gravity, and $\mathcal{L}_{int}$ represents the interaction between gravity and all forms of energy. The corresponding energy tensors are given by

$$T_{\mu\nu} = T^g_{\mu\nu} + T^m_{\mu\nu} + T^{int}_{\mu\nu} \tag{C2-2}$$

We see that the linear theory of gravity was obtained by approximating $-\frac{K}{2} h_{\mu\nu} T_{\mu\nu}$ by $-\frac{K}{2} h_{\mu\nu} T^m_{\mu\nu}$ in the interaction Lagrangian. It is consistent in lowest order to neglect $T^g_{\mu\nu}$ and $T^{int}_{\mu\nu}$ for they each contain at least one factor of $h_{\mu\nu}$ which, by the linear wave equations, is smaller than the source of $h_{\mu\nu}$ by a factor K. We shall see in fact that both $T^g_{\mu\nu}$ and $T^{int}_{\mu\nu}$ are smaller than $T^m_{\mu\nu}$ by a factor K^2.

To get $T^{int}_{\mu\nu}$ we must apply equations C1-40 to $\mathcal{L}_{int}$. However $\mathcal{L}_{int}$ should be of the form

$$\mathcal{L}_{int} = -\frac{K}{2} h_{\mu\nu} T_{\mu\nu} = -\frac{K}{2} h_{\mu\nu} T^m_{\mu\nu} - \frac{K}{2} h_{\mu\nu} T^g_{\mu\nu} - \frac{K}{2} h_{\mu\nu} T^{int}_{\mu\nu} \tag{C2-3}$$

$\mathcal{L}_{int}$ involves $T^{int}_{\mu\nu}$ which we are trying to calculate; thus we cannot solve the problem exactly in one step. We will therefore consider only the lowest order corrections to the linear theory in this part.

For example to lowest order $\mathcal{L}_{int}$ is given by

$$\mathcal{L}_{int} = -\frac{K}{2} h_{\mu\nu} T^{m}_{\mu\nu} \tag{C2-4}$$

Since $\mathcal{L}_{int}$ involves one factor of K and one factor of $h_{\mu\nu}$, $T^{int}_{\mu\nu}$ derived from $\mathcal{L}_{int}$ will be of order K^2 smaller than $T^{m}_{\mu\nu}$. As $T^{g}_{\mu\nu}$ will involve two factors of $h_{\mu\nu}$ it will also be of order K^2 smaller than $T^{m}_{\mu\nu}$.

Equations C1-40 applied to $\mathcal{L}_g$ of equation A-11 give the result

$$T^{g}_{\mu\nu} = U_{\mu\nu} + \frac{1}{2} h_{\mu\alpha}(-\bar{h}_{\alpha\nu,\sigma\sigma} + \bar{h}_{\sigma\nu,\sigma\alpha} + \bar{h}_{\sigma\alpha,\sigma\nu} - \delta_{\alpha\nu}\bar{h}_{\rho\sigma,\rho\sigma}) \tag{C2-5}$$

where $U_{\mu\nu}$ is the symmetric part of

$$\begin{aligned} \frac{1}{2}\{ & h_{\gamma\mu,\gamma} h_{,\nu} + h_{\gamma\rho,\gamma} h_{\rho\mu,\nu} - h_{\rho\mu,\gamma} h_{\rho\gamma,\nu} + h_{\mu\nu,\gamma} h_{\lambda\gamma,\lambda} + h_{\mu\gamma,\gamma} h_{\lambda\nu,\lambda} \\ & - \frac{1}{2} h_{\mu\nu,\gamma} h_{,\gamma} + \frac{1}{2} h_{\mu\nu} h_{,\sigma\sigma} - h_{\mu\rho} h_{\rho\nu,\sigma\sigma} + h_{\rho\nu} h_{\rho\lambda,\lambda\mu} + h_{\rho\gamma} h_{\rho\mu,\gamma\nu} \\ & - h_{\rho\nu} h_{,\rho\mu} + \frac{1}{2} h_{\rho\sigma,\mu} h_{\rho\sigma,\nu} - \frac{1}{2} h_{,\mu} h_{,\nu} + \delta_{\mu\nu} [\frac{1}{4} h_{,\rho} h_{,\rho} + \frac{1}{2} h_{,\rho\sigma} h_{\rho\sigma} \\ & - h_{\rho\gamma} h_{\lambda\rho,\gamma\lambda} - \frac{1}{2} h_{\rho\gamma,\gamma} h_{\rho\lambda,\lambda} - \frac{1}{4} h_{\rho\sigma,\lambda} h_{\rho\sigma,\lambda}] \} \end{aligned} \tag{C2-6}$$

We note that if the complete system were described by $\mathcal{L}_g$ alone, then the wave equation gives

$$(-\bar{h}_{\alpha\nu,\sigma\sigma} + \bar{h}_{\sigma\nu,\sigma\alpha} + \bar{h}_{\sigma\alpha,\sigma\nu} - \delta_{\alpha\nu}\bar{h}_{\rho\sigma,\rho\sigma}) = 0$$

and $T^g_{\mu\nu}$ of equation C1-6 would be symmetric by the use of the equations of motion. If the exact equations of motion are used, then only the complete tensor $T_{\mu\nu} = T^g_{\mu\nu} + T^m_{\mu\nu} + T^{int}_{\mu\nu}$ is in general symmetric.

Let us first suppose that gravity is interacting with matter fields rather than point particles. For example a scalar field whose Lagrangian involves only first derivatives has an energy tensor given by

$$T^m_{\mu\nu} = \varphi_{,\mu} \frac{\partial \mathcal{L}}{\partial \varphi_{,\mu}} - \delta_{\mu\nu} \mathcal{L}$$

(Such a Lagrangian is given in equation C1-41.) The interaction Lagrangian to lowest order K is given by

$$\mathcal{L}_{int} = -\frac{K}{2} h_{\mu\nu} T^m_{\mu\nu}$$

By equations C1-40 the interaction energy tensor is just given by

$$T^{int}_{\mu\nu} = -\frac{K}{2} h_{\rho\sigma} \left[\varphi_{,\mu} \frac{\partial T^m_{\rho\sigma}}{\partial \varphi_{,\nu}} - \delta_{\mu\nu} T^m_{\rho\sigma} \right] \qquad \text{(C2-6A)}$$

(In both $T^m_{\mu\nu}$ and $T^{int}_{\mu\nu}$ no terms $S_{\mu\nu}$ appear in this example.) More algebra is involved when we consider matter fields of a higher spin, or if second derivatives are involved in the Lagrangians, but the calculation of $T^m_{\mu\nu}$ and $T^{int}_{\mu\nu}$ is still straightforward.

A problem arises however when we consider gravity interacting with point particles. Equations C1-40 apply to a Lagrangian involving only fields, but the author has been unable to apply equations C1-40 directly to a Lagrangian involving point particles. This would lead to no problem if the particles did not interact with gravity, for $T^g_{\mu\nu}$ would be obtained by equations C1-40, and we already know $T^m_{\mu\nu}$ from equation B2-9. However $\mathcal{L}_{int}$ involves both particles and fields and $T^{int}_{\mu\nu}$ could not be obtained directly.

The following method was used by the author to obtain $T^{int}_{\mu\nu}$ when gravity interacts with particles. The complete energy tensor must have zero divergence, and any approximation to the complete tensor which is to be an improvement over $T^{m}_{\mu\nu}$ must have a divergence that is smaller than $T^{m}_{\mu\nu,\,\nu}$ by at least a factor of K.

We claim that the improved tensor is to be of the form

$$T_{\mu\nu} = T^{g}_{\mu\nu} + T^{m}_{\mu\nu} + T^{int}_{\mu\nu}$$

where $T^{g}_{\mu\nu}$ is given in equation C2-5 and $T^{m}_{\mu\nu}$ is from equation B2-9. $T^{int}_{\mu\nu}$ is to be determined by demanding that $T_{\mu\nu,\,\nu}$ be less than $T^{m}_{\mu\nu,\,\nu}$ by at least a factor of K. We know from equation B2-15 that

$$T^{m}_{\mu\nu,\,\nu} = -\Gamma^{\mu}_{\rho\nu}\,T^{m}_{\rho\nu} \approx -\frac{1}{2}\left[2h_{\mu\rho,\,\nu} - h_{\rho\nu,\,\mu}\right] K T^{m}_{\rho\nu} \qquad \text{(C2-6B)}$$

Thus $T^{m}_{\mu\nu,\,\nu}$ is already of order K^2 smaller than $T^{m}_{\mu\nu}$.

By differentiating $T^{g}_{\mu\nu}$ we get (From C2-5)

$$T^{g}_{\mu\nu,\,\nu} = \frac{1}{4}\left(\left[2h_{\mu\rho,\,\nu} - h_{\rho\nu,\,\mu}\right] + 2h_{\mu\rho,\,\nu}\right)\left(-\bar{h}_{\rho\nu,\,\alpha\alpha} + \bar{h}_{\rho\alpha,\,\nu\alpha} + \bar{h}_{\nu\alpha,\,\rho\alpha} + \delta_{\rho\nu}\bar{h}_{\alpha\beta,\,\alpha\beta}\right)$$

$$\text{(C2-7)}$$

which checks that $T^{g}_{\mu\nu,\,\nu}$ would be zero if we used the equations of motion for the free gravitational fields (obtained from $\mathcal{L}_{g}$ alone).

We now see that if

$$T^{int}_{\mu\nu} = -\frac{1}{2}h_{\mu\rho}\left(-\bar{h}_{\rho\nu,\,\alpha\alpha} + \bar{h}_{\rho\alpha,\,\nu\alpha} + \bar{h}_{\nu\alpha,\,\rho\alpha} - \delta_{\rho\nu}\bar{h}_{\alpha\beta,\,\alpha\beta}\right) \qquad \text{(C2-8)}$$

then

$$T^{int}_{\mu\nu,\nu} = -\frac{1}{2} h_{\mu\rho,\nu}(-h_{\rho\nu,\alpha\alpha} + \bar{h}_{\rho\alpha,\nu\alpha} + \bar{h}_{\nu\alpha,\rho\alpha} - \delta_{\rho\nu}\bar{h}_{\alpha\beta,\alpha\beta})$$

and the divergence of the complete energy tensor is given by

$$T_{\mu\nu,\nu} = (T^{g}_{\mu\nu} + T^{int}_{\mu\nu} + T^{m}_{\mu\nu})_{,\nu}$$

$$= \frac{1}{4}(2h_{\mu\rho,\nu} - h_{\rho\nu,\mu})[-\bar{h}_{\rho\nu,\alpha\alpha} + \bar{h}_{\rho\alpha,\nu\alpha} + \bar{h}_{\nu\alpha,\rho\alpha}$$

$$- \delta_{\rho\nu}\bar{h}_{\alpha\beta,\alpha\beta} - 2KT^{m}_{\rho\nu}]$$

The term in the square brackets is of order K^3 (zero in the linear theory) thus $T_{\mu\nu,\nu}$ is of order K^2 smaller than $T^{m}_{\mu\nu,\nu}$. To lowest order $T^{int}_{\mu\nu}$ of equation C2-8 may be written

$$T^{int}_{\mu\nu} = -Kh_{\mu\rho}T^{m}_{\rho\nu} \qquad \text{(C2-9)}$$

which checks that $T^{int}_{\mu\nu}$ is the tensor involving both the fields and particles that should be added to $T^{g}_{\mu\nu} + T^{m}_{\mu\nu}$.*

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* The form $T^{int}_{\mu\nu} = -Kh_{\mu\rho}T^{m}_{\rho\nu}$ is particularly simple, and to lowest order may be expressed entirely in terms of the fields $h_{\mu\nu}$ as in equation C2-8. One might wonder if this is the general form of $T^{int}_{\mu\nu}$ even for the case of gravity interacting with fields. This is not so as may be seen in the case of spin-zero fields.

The derivation of $T^{int}_{\mu\nu} = -Kh_{\mu\rho}T^{m}_{\rho\nu}$ for gravity interacting with particles relies on the relation $T^{m}_{\mu\nu,\nu} = -\Gamma^{\mu}_{\rho\nu}T^{m}_{\rho\nu}$ of equation C2-6P. For spin-zero fields where the energy tensor $T^{mo}_{\mu\nu}$ is from the Lagrangian C1-41,

$$T^{mo}_{\mu\nu,\nu} = \varphi_{,\mu}[\varphi_{,\rho\rho} + \mu^2\varphi^2]$$

$$\approx Kh_{\rho\sigma}\varphi_{,\mu}\varphi_{,\rho\sigma} + Kh_{\rho\sigma,\sigma}\varphi_{,\mu}\varphi_{,\rho} - \frac{K}{2}h_{,\rho}\varphi_{,\mu}\varphi_{,\rho}$$

where we used the equation of motion (to lowest order in $h_{\mu\nu}$).

$$\varphi_{,\rho\rho} + \mu^2\varphi^2 = Kh_{\rho\sigma}\varphi_{,\rho\sigma} + Kh_{\rho\sigma,\sigma}\varphi_{,\rho} - \frac{K}{2}h_{,\rho}\varphi_{,\rho}$$

It is clear that $T^{mo}_{\mu\nu,\nu}$ is not of the form $-\Gamma^{\mu}_{\rho\nu}T^{mo}_{\rho\nu}$.

What happens in the case of a spin-zero field interacting with

It is interesting that $T^{int}_{\mu\nu}$ just cancels the part of $T^{g}_{\mu\nu}$ that was not symmetric (see equation C2-5), and we get

$$T^{g}_{\mu\nu} + T^{int}_{\mu\nu} = U_{\mu\nu}$$

where $U_{\mu\nu}$ is the symmetric tensor given in equation C2-6. The corrected wave equation to be tested on the orbit of Mercury is now given by

$$-\bar{h}_{\mu\nu,\sigma\sigma} + \bar{h}_{\sigma\nu,\sigma\mu} + \bar{h}_{\sigma\mu,\sigma\nu} - \delta_{\mu\nu}\bar{h}_{\rho\sigma,\rho\sigma} = 2K(T^{m}_{\mu\nu} + U_{\mu\nu}) \quad \text{(C2-10)}$$

where the inconsistency in C2-10 is down by a factor of K^2 from the linear theory.

To solve C2-10 to check the perihelion shift of Mercury, the fields produced by a point particle were first obtained by neglecting $U_{\mu\nu}$. Next these fields were used in formula C2-6 to give $U_{\mu\nu}$. Then the wave equation including $U_{\mu\nu}$ was solved giving the corrected fields $h'_{\mu\nu}$ due to a point mass. To actually check the effect on the shift of the perihelion, the work of Eddington (15) was followed closely, replacing the fields given in general relatively by $h'_{\mu\nu}$. The result: the term $2KU_{\mu\nu}$ produced a correction to the fields, but no correction to the perihelion shift, and the same wrong answer as given by the linear theory was obtained.

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gravity is that the correct $T^{int}_{\mu\nu}$ given by equation C2-6A may be written in the form

$$T^{int}_{\mu\nu} = -Kh_{\mu\rho}T^{mo}_{\rho\nu} + B_{\mu\nu}$$

It is the quantity $B_{\mu\nu} + T^{mo}_{\mu\nu}$ that satisfies the relation

$$(B_{\mu\nu} + T^{mo}_{\mu a})_{,\nu} = -\Gamma^{\mu}_{\rho\nu}(B_{\rho\nu} + T^{mo}_{\rho\nu}).$$

As we have seen, energy tensors for a given system are not unique. We should ask if there is another energy tensor for the system of gravity interacting with particles that, when used as the source of gravity in equation C-1, does give the correct shift in the perihelion of Mercury. There is, and this energy tensor is given by

$$T^F_{\mu\nu} = U_{\mu\nu} + W_{\mu\nu} + T^m_{\mu\nu} \qquad \text{(C2-11)}$$

where $W_{\mu\nu}$ is given by

$$\frac{\partial^2}{\partial x_\rho \partial x_\sigma}\Big\{\frac{1}{2} h_{\mu\sigma} h_{\nu\rho} + \frac{1}{4}\delta_{\mu\sigma} h_{\nu\alpha} h_{\alpha\rho} + \frac{1}{4}\delta_{\nu\sigma} h_{\mu\alpha} h_{\alpha\rho} - \frac{1}{2} h_{\mu\nu} h_{\rho\sigma}$$

$$- \frac{1}{4}\delta_{\sigma\rho} h_{\mu\alpha} h_{\nu\alpha} - \frac{1}{4}\delta_{\nu\rho}\delta_{\mu\sigma} h_{\alpha\lambda} h_{\alpha\lambda} + \frac{1}{4}\delta_{\rho\sigma} h_{\mu\nu} h - \frac{1}{4}\delta_{\nu\sigma} h_{\mu\rho} h$$

$$- \frac{1}{4}\delta_{\mu\sigma} h_{\nu\rho} h + \frac{1}{8}\delta_{\nu\rho}\delta_{\mu\sigma} hh + \delta_{\mu\nu}\Big[\frac{1}{4} h_{\rho\sigma} h - \frac{1}{4} h_{\sigma\lambda} h_{\rho\lambda}$$

$$+ \frac{1}{4}\delta_{\rho\sigma} h_{\alpha\lambda} h_{\alpha\lambda} - \frac{1}{8}\delta_{\rho\sigma} hh\Big]\Big\} \qquad \text{(C2-12)}$$

We shall see at the end of part G that $T^F_{\mu\nu}$ may be directly obtained as an energy tensor from the Lagrangian of gravity interacting with particles. $T^F_{\mu\nu}$ was obtained in a unique manner by Feynman using the methods to be described in part C3.

$T^F_{\mu\nu}$ and the tensor we obtained by equations C1-40 differ by the quantity $W_{\mu\nu}$. $W_{\mu\nu}$, being symmetric, divergenceless, and itself a second derivative, is just the form of the difference between tensors obtained by equations C1-40. (See equation C1-47 which gives the difference between two tensors for vector fields.) We will see at the end of part G that $T^F_{\mu\nu}$ is not likely to be obtained directly from equation C1-40, but will be obtained as an energy tensor by the methods used in part G. Thus we will verify the idea that gravity is a spin-two field coupled to all forms of energy.

The interesting result of this part is that two energy tensors which differ by a term $W_{\mu\nu}$ that is symmetric, divergenceless, and is itself a divergence, are not physically equivalent tensors. Both tensors would lead to the same total momentum and angular momentum for the system by equations C1-2 and C1-3, but this is not sufficient to determine the energy tensor. Gravity interacts with the local distribution of energy; a term that disturbs the local distribution of energy, even though it does not affect the total energy of the system, changes the gravitational fields.

This change in the gravitational field is tested experimentally, for the fields arising from $T^F_{\mu\nu}$ gave the correct shift in the perihelion of Mercury, while $T_{\mu\nu} = U_{\mu\nu} + T^m_{\mu\nu}$ lead to the wrong shift. Since $T^F_{\mu\nu} = T_{\mu\nu} + W_{\mu\nu}$, $W_{\mu\nu}$ has a physical significance.

C3. Feynman's Derivation of a Theory of Gravity

The linear wave equation A-13 written in the form $\delta \mathcal{L}_g / \delta h_{\mu\nu} = - \delta \mathcal{L}_m / \delta h_{\mu\nu}$ becomes

$$\frac{1}{4}\left(-\bar{h}_{\mu\nu,\sigma\sigma} + \bar{h}_{\sigma\nu,\sigma\mu} + \bar{h}_{\sigma\mu,\sigma\nu} - \delta_{\mu\nu}\bar{h}_{\sigma\rho,\sigma\rho}\right) = \frac{K}{2}\, T^m_{\mu\nu}$$

The left-hand side has zero divergence, while the divergence of $T^m_{\mu\nu}$ from C2-6B is

$$T^m_{\mu\nu,\nu} = -\frac{1}{2}[2h_{\mu\rho,\sigma} - h_{\rho\sigma,\mu}]KT^m_{\rho\sigma}$$

To the lowest order in K the quantity $KT^m_{\mu\nu}/2$ may be replaced by fields using A-13. We get

$$\frac{K}{2}T^m_{\mu\nu,\nu} = -\frac{K}{8}[2h_{\mu\rho,\sigma} - h_{\rho\sigma,\mu}][-\bar{h}_{\rho\sigma,\alpha\alpha} + \bar{h}_{\rho\alpha,\sigma\alpha}$$

$$+ \bar{h}_{\sigma\alpha,\rho\alpha} - \delta_{\rho\sigma}\bar{h}_{\alpha\beta,\alpha\beta}] \qquad \text{(C3-1)}$$

The problem is now to correct the left-hand side of A-13 so that it will have a divergence given by C3-1.

Suppose, for example, a term of the form

$$Kf^3 = Kh_{\alpha\beta,\sigma}h_{\alpha\beta,\sigma}h_{\rho\rho} \qquad \text{(C3-2)}$$

were added to the Lagrangian of the free gravitational fields. This would add to the left side of the equation of motion A-13

$$K\frac{\delta f^3}{\delta h_{\mu\nu}} = \delta_{\mu\nu}kh_{\alpha\beta,\sigma}h_{\alpha\beta,\sigma} - 2Kh_{\mu\nu,\sigma\sigma}h_{\rho\rho} - 2Kh_{\mu\nu,\sigma}h_{\rho\rho,\sigma} \qquad \text{(C3-3)}$$

The divergence on the left-hand side of the equation of motion comes solely from C3-3 and is now

$$2Kh_{\alpha\beta,\sigma\mu}h_{\alpha\beta,\sigma} - 2Kh_{\mu\nu,\sigma\sigma\nu}h_{\rho\rho} - 2Kh_{\mu\nu,\sigma\sigma}h_{\rho\rho,\nu}$$

$$-2Kh_{\mu\nu,\sigma\nu}h_{\rho\rho,\sigma} - 2Kh_{\mu\nu,\sigma}h_{\rho\rho,\sigma\nu} \qquad \text{(C3-4)}$$

These terms are not equal to $KT_{\mu\nu,\nu}/2$ given by C3-1, but they are of the correct form and suggest the following procedure used by Feynman for obtaining the correct wave equation.

Write all independent terms (i.e., terms that differ by more than a pure divergence) of the form C3-2, that is, with three h's and two derivatives. Assign an arbitrary coefficient A → N to each of the fourteen terms obtained this way, calling the sum of these terms F^3. To the equation of motion will be added

$$K\delta F^3/\delta h_{\mu\nu}$$

Then the relation that must be satisfied for a consistent wave equation (to a higher order in K) is

$$K\left(\frac{\delta F^3}{\delta h_{\mu\nu}}\right)_{,\nu} = \frac{K}{2} T^m_{\mu\nu,\nu} \qquad \text{(C3-5)}$$

where $KT^m_{\mu\nu}/2$ is given by C3-1.

C3-5 may be rearranged so that it is in a form that closely resembles equation A-10; that is, an equation in which the sum of a series of terms of the form $(-1 + A - 2D \ldots)Kh_{\rho\sigma,\mu}h_{\rho\sigma,\alpha\alpha}$ is set equal to zero. As before, in order not to put arbitrary restrictions on the fields, each coefficient is set equal to zero. This gives a large number of equations for the arbitrary coefficients A, B, C, ... which can be solved to determine the coefficients uniquely. The resulting F^3 given by Feynman is

$$KF^3 = \frac{K}{8}\left(h_{\alpha\beta}\bar{h}_{\gamma\delta}\bar{h}_{\alpha\beta,\gamma\delta} + h_{\gamma\beta}h_{\gamma\alpha}\bar{h}_{\alpha\beta,\delta\delta} - 2h_{\alpha\gamma}h_{\beta\delta}\bar{h}_{\alpha\beta,\gamma\delta}\right.$$
$$\left. + 2\bar{h}_{\alpha\beta}\bar{h}_{\alpha\gamma,\gamma}\bar{h}_{\beta\delta,\delta} + \frac{1}{2}h_{\alpha\beta}h_{\alpha\beta}\bar{h}_{\gamma\delta,\gamma\delta} + \frac{1}{4}h_{\alpha\alpha}h_{\beta\beta}\bar{h}_{\gamma\delta,\gamma\delta}\right) \qquad \text{(C3-6)}$$

This expression has been considerably simplified by judicious use of the notation $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\delta_{\mu\nu}h_{\sigma\sigma}$. ~~The~~ Most derivatives have been put on the last h by appropriate integration by parts. The simple form C3-6 will be useful in later calculations.

The Lagrangian for the free gravitational fields consists of terms involving two h's (equation A-11) which leads to the linear wave equation A-13, and now terms involving three h's. For higher order corrections to the theory we would add terms involving four h's, five h's, etc. Let us label this series by

$$\mathcal{L}_g = F = F^2 + KF^3 + K^2F^4 + K^3F^5 + \ldots$$

where F^2 includes the terms with two h's, F^3 the terms with three h's, etc. The exact condition that must be satisfied by the series F is

$$\frac{\delta F}{\delta h_{\mu\nu}} = \frac{K}{2} T^{\mu}_{\mu\nu}$$

or

$$\left(\frac{\delta F}{\delta h_{\mu\nu}}\right)_{,\nu} = \frac{K}{2} T_{\mu\nu,\nu} = -\Gamma^{\mu}_{\alpha\beta}\frac{K}{2} T_{\alpha\beta} \qquad \text{(C3-7)}$$

where $T_{\mu\nu,\nu} = -\Gamma^{\mu}_{\alpha\beta}T_{\alpha\beta}$ is the exact relation B2-15, and $\Gamma^{\mu}_{\alpha\beta}$ which is defined in B2-14 may be expressed in an infinite series in powers of K. We will show in the next part the method used by Feynman to sum the series in F, consistent with C3-7.

Let us first check that the addition of F^3 to $\mathcal{L}_g$ gives a theory which correctly predicts the shift in the perihelion of Mercury. We have

$$\mathcal{L} = F^2 + KF^3 - \frac{1}{2}(\delta_{\mu\nu} + Kh_{\mu\nu})T^m_{\mu\nu}$$

The gravitational wave equation may be written

$$\frac{\delta F^2}{\delta h_{\mu\nu}} = \frac{K}{2}\left(T^m_{\mu\nu} - 2\,\frac{\delta F^3}{\delta h_{\mu\nu}}\right) \qquad \text{(C3-8)}$$

Thus we can interpret the term $-2\delta F^3/\delta h_{\mu\nu}$ as the energy in the gravitational field, where $-2\delta F^3/\delta h_{\mu\nu}$ is given by

$$\begin{aligned} -2\,\frac{\delta F^3}{\delta h_{\mu\nu}} = T^F_{\mu\nu} = \frac{1}{2}\{ & h_{\alpha\mu,\sigma} h_{\sigma\nu,\alpha} - h_{\alpha\nu,\sigma} h_{\mu\alpha,\sigma} - h_{,\alpha} h_{\alpha\mu,\nu} \\ & + \frac{1}{2} h_{,\alpha} h_{\mu\nu,\alpha} - \frac{1}{2} h_{\sigma\lambda,\nu} h_{\sigma\lambda,\mu} - h_{\sigma\lambda} h_{\sigma\lambda,\mu\nu} + 2h_{\sigma\lambda,\sigma} h_{\lambda\mu,\nu} \\ & + 2h_{\sigma\lambda} h_{\mu\lambda,\sigma\nu} - h_{\sigma\lambda,\sigma} h_{\mu\nu,\lambda} - h_{\sigma\lambda} h_{\mu\nu,\sigma\lambda} - 2h_{\mu\alpha} h_{,\nu\alpha} \\ & - 2h_{\mu\alpha} h_{\nu\alpha,\sigma\sigma} + 2h_{\mu\alpha} h_{\nu\sigma,\alpha\sigma} + 2h_{\mu\alpha} h_{\sigma\alpha,\sigma\nu} + h_{\mu\nu} h_{,\sigma\sigma} \\ & - h_{\mu\nu} h_{\sigma\lambda,\sigma\lambda} + \frac{1}{2} h h_{\mu\nu,\sigma\sigma} + \frac{1}{2} h h_{,\mu\nu} - h h_{\sigma\nu,\sigma\mu} + \delta_{\mu\nu}\left(\frac{3}{4} h_{\alpha\lambda,\sigma} h_{\alpha\lambda,\sigma}\right. \\ & - \frac{1}{4} h_{,\alpha} h_{,\alpha} - \frac{1}{2} h_{\alpha\lambda,\sigma} h_{\sigma\lambda,\alpha} + h_{,\alpha} h_{\alpha\lambda,\lambda} + h_{\sigma\lambda} h_{\sigma\lambda,\rho\rho} \\ & - h_{\sigma\lambda,\sigma} h_{\alpha\lambda,\alpha} - \frac{1}{2} h h_{,\sigma\sigma} - 2h_{\sigma\lambda} h_{\alpha\lambda,\sigma\alpha} + h_{\sigma\lambda} h_{,\sigma\lambda} \\ & \left. + \frac{1}{2} h h_{\sigma\lambda,\sigma\lambda}\right) \} \end{aligned} \qquad \text{(C3-9)}$$

where

$$h = h_{\sigma\sigma}.$$

We have already mentioned in part C2 that $T^F_{\mu\nu}$ from C3-9 is related to the tensor $U_{\mu\nu}$ by the equation

$$T^F_{\mu\nu} = U_{\mu\nu} + W_{\mu\nu}$$

where $U_{\mu\nu}$ was obtained by using equations C1-40, and $W_{\mu\nu}$ is a term that alters only the local distribution of energy and momentum. It is $T^F_{\mu\nu}$ and not $U_{\mu\nu}$ that gives the correct shift in the perihelion of the planet Mercury.

Experimentally equation C3-8 is a completely satisfactory equation for the gravitational fields, there are no known tests more accurate than the shift of the perihelion of Mercury. The work of the next part, to sum the series $F^2 + KF^3 + K^2F^4 + \ldots$, will be to merely formulate a consistent theory of gravity.

C4. A Consistent Theory of Gravity

For this part we shall adopt the following notation

$$g_{\mu\nu} \equiv (\delta_{\mu\nu} + Kh_{\mu\nu}) \qquad \text{(C4-1)}$$

$$g^{\mu\nu} \equiv g^{-1}_{\mu\nu} \;;\; \text{that is} \quad g^{\mu\alpha}g_{\alpha\nu} = \delta^{\mu}_{\nu} \qquad \text{(C4-2)}$$

$$\Gamma_{\mu,\alpha\beta} \equiv \frac{K}{2}\left[h_{\mu\alpha,\beta} + h_{\mu\beta,\alpha} - h_{\alpha\beta,\mu}\right]$$

$$= \frac{1}{2}\left[g_{\mu\alpha,\beta} + g_{\mu\beta,\alpha} - g_{\alpha\beta,\mu}\right] \qquad \text{(C4-3)}$$

Then $\Gamma^{\nu}_{\alpha\beta} = g^{\nu\mu}\Gamma_{\mu,\alpha\beta}$ as given by equation B2-14. From our point of view this notation is strictly a matter of convenience. The $g^{\mu\nu} = g^{-1}_{\mu\nu}$ can be expressed in terms of the fields as is seen in B2-12

$$g^{\mu\nu} = \delta_{\mu\nu} - Kh_{\mu\nu} + K^2h_{\mu\alpha}h_{\alpha\nu} - K^3h_{\mu\alpha}h_{\alpha\beta}h_{\beta\nu} + \ldots$$

Finally, for purposes of notation the coordinate of a particle Z^μ will be written with the index as a superscript, although there is no change in meaning. Feynman's summation convention A-4A will still apply to repeated indices.

In the above notation we have for the case of gravity acting on a point particle

$$\mathcal{L} = F - \frac{1}{2} g_{\mu\nu} T_m^{\mu\nu} \qquad \text{(C4-4)}$$

where

$$\frac{\delta F}{\delta h_{\mu\nu}} = \frac{K}{2}\, T_m^{\mu\nu} \qquad \text{(C4-5)}$$

and the condition that F must satisfy is given by C3-7

$$\left(\frac{\delta F}{\delta h_{\mu\nu}} \right)_{,\,\nu} = - \Gamma^{\mu}_{\alpha\beta} \frac{K}{2}\, T^{\alpha\beta}$$

To determine F we shall show that equation C3-7 implies that $\int F\, d^4x$ has a certain invariance property. We will then look for the quantities of the form $\int F\, d^4x$ which have this invariance property, and then decide which of these quantities we have been generating by the series $F^2 + KF^3 + \ldots$

As an example of this technique suppose we had the function $G(\varphi)$ which had the property that

$$\int G(\varphi)\, d^4x \qquad \text{(C4-6)}$$

was unchanged under the substitution $\varphi \rightarrow \varphi + \lambda$, where λ is an infinitesimal quantity. Under this substitution ~~C3-4~~ C4-6 becomes

$$\int G(\varphi+\lambda)d^4x = \int G(\varphi)\,d^4x + \int \frac{\delta G(\varphi)}{\delta\varphi}\,\lambda\, d^4x + \text{terms of order } \lambda^2$$

The quantity $\int \frac{\delta G(\varphi)}{\delta\varphi}\,\lambda\, d^4x$ must be zero if C4-6 is to be unchanged under the substitution $\varphi \rightarrow \varphi + \lambda$.

To find the invariance properties of F we shall therefore look for an equation of the form

$$\int \frac{\delta F}{\delta h_{\mu\nu}}\,\lambda_{\mu\nu}\, d^4x = 0$$

Equation C3-7 is

$$\left(\frac{\delta F}{\delta h_{\mu\nu}}\right)_{,\,\nu} + \Gamma^{\mu}_{\alpha\beta}\,\frac{K}{2}\,T^{\alpha\beta}_{m} = 0$$

Multiplying through by $g_{\mu\lambda}$, noting C4-3, we get

$$g_{\mu\lambda}\left(\frac{\delta F}{\delta h_{\mu\nu}}\right)_{,\,\nu} + \Gamma_{\lambda,\,\mu\nu}\,\frac{K}{2}\,T^{\mu\nu}_{m} = 0 \qquad \text{(C4-7)}$$

Multiply equation C4-7 by an arbitrary vector quantity A^{λ} and integrate the result over all space.

$$\int \left[g_{\mu\lambda}\left(\frac{\delta F}{\delta h_{\mu\nu}}\right)_{,\,\nu} + \Gamma_{\lambda,\,\mu\nu}\,\frac{K}{2}\,T^{\mu\nu}_{m}\right] A^{\lambda} d^4x = 0$$

Assume $A^{\lambda} = A^{\lambda}(x)$ goes to zero at infinity. Integrate by parts the term involving $\delta F/\delta h_{\mu\nu}$ throwing out the surface terms and there results:

$$\int \frac{\delta F}{\delta h_{\mu\nu}}\left[-g_{\mu\lambda,\,\nu}A^{\lambda} - g_{\mu\lambda}A^{\lambda}_{\ ,\,\nu} + \Gamma_{\lambda,\,\mu\nu}A^{\lambda}\right] d^4x = 0$$

where we have used $\frac{K}{2}\,T^{\mu\nu}_{m} = \delta F/\delta h_{\mu\nu}$. This last equation may be rewritten substituting for $\Gamma_{\lambda,\mu\nu}$ and noting that $\delta F/\delta h_{\mu\nu}$ is symmetric in μ and ν:

$$\frac{1}{2}\int \frac{\delta F}{\delta h_{\mu\nu}}[g_{\mu\lambda}A^{\lambda}_{,\nu} + g_{\nu\lambda}A^{\lambda}_{,\mu} + g_{\mu\nu,\lambda}A^{\lambda}]\, d^4x = 0 \tag{C4-8}$$

This is the equation that we wanted, and we see that the quantity $\int F(h_{\mu\nu})d^4x$, which is the action for the gravitational field, remains unchanged if the fields $h_{\mu\nu}$ are replaced by the fields $h^*_{\mu\nu}$ given by

$$h^*_{\mu\nu} = h_{\mu\nu} + \epsilon[g_{\mu\lambda}A^{\lambda}_{,\nu} + g_{\nu\lambda}A^{\lambda}_{,\mu} + g_{\mu\nu,\lambda}A^{\lambda}] \tag{C4-9}$$

The ϵ is an infinitesimal constant to guarantee that the substitution $h_{\mu\nu} \to h^*_{\mu\nu}$ corresponds to an infinitesimal change of the fields. (Remember that $A^{\lambda}(x)$ is an arbitrary vector.)

The problem is to find the quantities $\int F(h_{\mu\nu})\, d^4x$ which are invariant under the transformation $h_{\mu\nu} \to h^*_{\mu\nu}$. To do this the author has found it convenient to study the transformation C4-9 itself.

By adding $\delta_{\mu\nu}$ to both sides of C4-9, expanding $g_{\mu\alpha} = \delta_{\mu\alpha} + h_{\mu\alpha}$ and taking $\epsilon A^{\lambda} = (1/K)\eta^{\lambda}$, we get

$$\delta_{\mu\nu} + Kh^*_{\mu\nu} = \delta_{\mu\nu} + \delta_{\mu\alpha}\eta^{\alpha}_{,\nu} + \delta_{\nu\alpha}\eta^{\alpha}_{,\nu} + \delta_{\nu\alpha}\eta^{\alpha}_{,\mu} + \delta_{\mu\nu,\alpha}\eta^{\alpha}$$
$$+ K(h_{\mu\nu} + h_{\mu\alpha}\eta^{\alpha}_{,\nu} + h_{\nu\alpha}\eta^{\alpha}_{,\mu} + h_{\mu\nu,\alpha}\eta^{\alpha}) \tag{C4-10}$$

Now suppose we were originally working in Cartesian coordinates and make an arbitrary infinitesimal coordinate transformation

$$x^{\mu} \to x'^{\mu} = x^{\mu} - \eta^{\mu}$$

From equations C1-22 we see that the tensors $\delta_{\mu\nu}$ and $Kh_{\mu\nu}$ will be transformed into

$$\delta_{\mu\nu} \to \delta'_{\mu\nu} = \delta_{\mu\nu} + \delta_{\mu\alpha}\eta^{\alpha}_{,\nu} + \delta_{\nu\alpha}\eta^{\alpha}_{,\mu} + \delta_{\mu\nu,\alpha}\eta^{\alpha}$$

$$Kh_{\mu\nu} \to Kh'_{\mu\nu} = Kh_{\mu\nu} + Kh_{\mu\alpha}\eta^{\alpha}_{,\nu} + Kh_{\nu\alpha}\eta^{\alpha}_{,\mu} + Kh_{\mu\nu,\alpha}\eta^{\alpha} \quad \text{(C4-11)}$$

It is now seen that the sum of $\delta'_{\mu\nu}$ plus $Kh'_{\mu\nu}$ just gives the right-hand side of equation C4-10. Thus we have the relation

$$\delta_{\mu\nu} + Kh^{*}_{\mu\nu} = \delta'_{\mu\nu} + Kh'_{\mu\nu} \quad \text{(C4-12)}$$

For an interpretation of this equation let us consider the action for particles interacting with gravity:

$$S = \int \mathcal{L}\, d^4x = \int F\, d^4x - \frac{1}{2}\int (\delta_{\mu\nu} + Kh_{\mu\nu}(x)\,)\, T^{\mu\nu}(x)\, d^4x$$

Because the Lagrangian is a scalar quantity the action is invariant under a coordinate transformation of the Lagrangian, with the result

$$\int [\,\delta_{\mu\nu}(x) + Kh_{\mu\nu}(x)]\, T^{\mu\nu}(x)\, d^4x$$
$$= \int [\,\delta'_{\mu\nu}(x) + Kh'_{\mu\nu}(x)]\, T'^{\mu\nu}(x)\, d^4x$$

But by equation C4-11

$$\int [\,\delta'_{\mu\nu}(x) + Kh'_{\mu\nu}(x)]\, T'^{\mu\nu}(x)\, d^4x$$
$$= \int [\,\delta_{\mu\nu}(x) + Kh^{*}_{\mu\nu}(x)]\, T'^{\mu\nu}(x)\, d^4x \quad \text{(C4-13)}$$

Suppose we were originally working in the x' coordinate system with the metric tensor $\delta'_{\mu\nu}$ and the fields $h'_{\mu\nu}(x)$ were present. The equation C3-12 states that we must get the same physics (since $\int F(h_{\mu\nu})d^4x = \int F(h^*_{\mu\nu})\,d^4x$) if we use the new fields $h^*_{\mu\nu}$, but use a different metric tensor $\delta_{\mu\nu}(x)$.

That is, if we make the transformation of the gravitational fields $h_{\mu\nu} \rightarrow h^*_{\mu\nu}$ we must change the geometry of our system in order to obtain the same physics. It is therefore suggested that we look to the theory of geometry for the quantities $\int F(h_{\mu\nu})\,d^4x$ that are invariant under the transformations C4-9.

Such functions are known from geometry to be

$$\int (-g)^{1/2} d^4x \,, \int R(-g)^{1/2} d^4x \,, \int R^{\mu\nu} R_{\mu\nu} (-g)^{1/2} d^4x \qquad \text{(C4-14)}$$

and other functions constructed from the curvature tensor $R_{\alpha\mu\beta\nu}$ *, where

g = determinant of $g_{\mu\nu}$

$$R_{\mu\nu} = g^{\alpha\beta} R_{\alpha\mu\nu\beta}$$

$$R = g^{\mu\nu} R_{\mu\nu}$$

It is now a matter of testing which of the functions ~~C3-13~~ C4-14 is being generated by the series $\int (F^2 + KF^3 + \ldots)d^4x$. The answer is

$$\int F(h_{\mu\nu})d^4x = \frac{1}{2K^2} \int R(-g)^{1/2}\, d^4x$$

The action for the problem of point particles interacting with gravity

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*See Landau (16). We shall use the opposite sign for the curvature tensor than that used by Landau (16). Our choice of sign corresponds to that of Tolman (17).

is now given by

$$S = \frac{1}{2K^2}\int R(-g)^{1/2} d^4x - \frac{1}{2}\int g_{\mu\nu}\dot{Z}^\mu \dot{Z}^\nu\, ds \qquad \text{(C4-15)}$$

It is consistent with the action C4-15 to interpret $g_{\mu\nu} = \delta_{\mu\nu} + Kh_{\mu\nu}$ as the metric of space-time assuming that there are no gravitational fields present, giving gravity a completely geometrical description. This was the starting point of Einstein when he formulated the general theory of relativity. The general theory of relativity may be formulated from an action principle, where the action for gravity interacting with point particles is just given by C4-15.

It is thus not unlikely that a meson physicist, first noticing gravity only when he built too large a chunk of matter, would eventually arrive at the geometrical description of gravity postulated by Einstein.

The meson physicist, being rather excited by the geometrical description of the gravitational force, would probably set out to find a geometrical description of the other known meson forces such as electro-magnetic, β-decay, and nuclear forces. Our history does not tell us whether he succeeded, but if he did he would have ample reason to believe in the significance of a geometrical description of the world. Failing to include the other forces in a geometrical description he would have to content himself with the fact that the geometrical description of gravity is a great convenience in solving gravitational problems, such as the formulation of the spin-two meson theory of gravity.

III. USE OF THE GEOMETRICAL FORMULATION OF GRAVITY

In the general theory of relativity the effects of a gravitational field are correctly described if one works in a curved space of metric $g_{\mu\nu} = \delta_{\mu\nu} + Kh_{\mu\nu}$. This implies that the correct method for obtaining the interaction of a system with gravity is to write the action for that system in a curved space of metric $g_{\mu\nu}$. If then the explicit dependence of the action on the fields $h_{\mu\nu}$ is desired, the functions of the metric tensor that appear in the curved space action may be expanded in terms of $h_{\mu\nu}$.

For example, the action for a system of non-interacting particles is from part B2

$$S_m = -\frac{m}{2}\int \delta_{\mu\nu}\dot{z}^{\mu}\dot{z}^{\nu}ds$$

To write this action in a curved space we replace the metric $\delta_{\mu\nu}$ by $g_{\mu\nu} = \delta_{\mu\nu} + Kh_{\mu\nu}$, giving

$$S_m = -\frac{m}{2}\int [\delta_{\mu\nu} + Kh_{\mu\nu}]\dot{z}^{\mu}\dot{z}^{\nu}ds$$

which is exactly the result we had in equation B2-5.

The correct prescription for writing an action in curved space is that the action be a scalar quantity in that curved space. Since the definition of a scalar quantity is a quantity that is invariant under a transformation of the coordinate system, this prescription implies that the action for a system interacting with gravity shall be invariant under a coordinate transformation.

We shall show that this condition on the action does in fact lead to a consistent theory of gravity, and gives a method of calculating a conserved symmetric energy momentum tensor for the complete system. The prescription of general relativity does not, as we shall see, lead to a unique theory of gravity, nor to a unique energy tensor for the system even in the absence of gravity.

D. A CONSISTENT THEORY OF GRAVITY INTERACTING WITH FIELDS

For the case of gravity interacting with point particles, the equation of motion for the gravitational fields was

$$\frac{\delta F}{\delta h_{\mu\nu}} = \frac{K}{2} T_m^{\mu\nu} \tag{C4-5}$$

where F was the Lagrangian of the free gravitational fields and $T_m^{\mu\nu}$, the energy tensor of the particles, satisfied the relation

$$T^{\mu\nu}_{m,\nu} = - \Gamma^{\mu}_{\alpha\beta} T^{\alpha\beta}_m \tag{B2-15}$$

Thus the condition on F that lead to a consistent theory of gravity was

$$\left.\frac{\delta F}{\delta h_{\mu\nu}}\right._{,\nu} = - \Gamma^{\mu}_{\alpha\beta} T^{\alpha\beta}_m = - \frac{2}{K} \Gamma^{\mu}_{\alpha\beta} \frac{\delta F}{\delta h_{\alpha\beta}} \tag{C3-7}$$

Suppose now that the Lagrangian for gravity interacting with matter fields is written in the form

$$\mathcal{L} = F + \mathcal{L}_{mg} \tag{D-1}$$

where L_{mg} includes the Lagrangian of the free matter fields plus the Lagrangian representing the interaction between these fields and gravity. The equations of motion for the gravitational field from D-1 is

$$\frac{\delta F}{\delta h_{\mu\nu}} = - \frac{\delta \mathcal{L}_{mg}}{\delta h_{\mu\nu}} \tag{D-2}$$

If we define

$$T^{\mu\nu}_{mg} = - \frac{2}{K} \frac{\delta \mathcal{L}_{mg}}{\delta h_{\mu\nu}} = - 2 \frac{\delta \mathcal{L}}{\delta K h_{\mu\nu}} \tag{D-3}$$

then the equation of motion D-2 becomes

$$\frac{\delta F}{\delta h_{\mu\nu}} = \frac{K}{2} T^{\mu\nu}_{mg} \tag{D-4}$$

We see that if

$$T^{\mu\nu}_{mg,\nu} = - \Gamma^{\mu}_{\alpha\beta} T^{\alpha\beta}_{mg} \tag{D-5a}$$

or

$$\left(\frac{\delta \mathcal{L}_{mg}}{\delta K h_{\mu\nu}} \right)_{,\nu} = - \Gamma^{\mu}_{\alpha\beta} \frac{\delta \mathcal{L}}{\delta K h_{\alpha\beta}} \tag{D-5b}$$

then we get the same condition on F as given in C3-7. Since F was determined by condition C3-7, the theory of gravity interacting with fields will be consistent as long as $\mathcal{L}_{mg}$ satisfies the relation D-5.

We will now show that if $\mathcal{L}_{mg}$ is obtained by rewriting $\mathcal{L}_m$ in the curved space of metric $g_{\mu\nu} = \delta_{\mu\nu} + Kh_{\mu\nu}$, in such a way that $\int \mathcal{L}_{mg}\, d^4x$ is a scalar quantity in that space, then $\mathcal{L}_{mg}$ satisfies condition D-5.

Let the components of the matter fields be given by q^n. Let us assume for simplicity that $\mathcal{L}_{mg}$, which is a function of the q^n and of the metric tensor $g_{\mu\nu}$, is written in a form that involves q^n, $g_{\mu\nu}$ and only the first derivatives of q^n and $g_{\mu\nu}$. I.e.,

$$\mathcal{L}_{mg} = \mathcal{L}_{mg}(g_{\mu\nu}; g_{\mu\nu,\alpha}; q^n, q^n_{,\alpha}) \qquad \text{(D-}\ \text{)}$$

If the action $S_{mg} = \int \mathcal{L}_{mg} d^4x$ is to be a scalar quantity it must be unchanged under the coordinate transformation

$$x^\mu \rightarrow x^{\mu'} = x^\mu - \eta^\mu(x) \qquad \text{(D-7)}$$

where $\eta^\mu(x)$ is an arbitrary infinitesimal vector. $\mathcal{L}_{mg}$, considered as a function of $g_{\mu\nu}; g_{\mu\ ,\alpha}; q^n; q^n_{,\alpha}$ will however be subject to change. If under the coordinate transformation D-7

$$g_{\mu\nu} \rightarrow g_{\mu\nu} + \delta g_{\mu\nu}$$

$$q^n \rightarrow q^n + \delta q^n \qquad \text{(D-8)}$$

then

$$g_{\mu\nu,\alpha} \rightarrow g_{\mu\nu,\alpha} + (\delta g_{\mu\nu})_{,\alpha}$$

$$q^n_{,\alpha} \rightarrow q^n_{,\alpha} + (\delta q^n)_{,\alpha} \ {}^* \qquad \text{(D-9)}$$

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*For example for a vector field A_μ under the transformation $x^\mu \rightarrow x^{'\mu} = x^\mu - \eta^\mu$

$$A'_\mu(x') = A_\rho(x)\frac{\partial x^\rho}{\partial x'^\mu}$$

$$\frac{\partial A'_\mu(x')}{\partial x'^\alpha} = A_\rho(x)\frac{\partial^2 x^\rho}{\partial x'^\alpha \partial x'^\mu} + \frac{\partial A_\rho(x)}{\partial x'^\alpha}\ \frac{\partial x^\rho}{\partial x'^\mu}$$

We may write

$$\frac{\partial A'_\mu(x')}{\partial x'^\alpha} = \frac{\partial A'_\mu(x-\eta)}{\partial x^\sigma}\ \frac{\partial x^\sigma}{\partial x'^\alpha} = \frac{\partial A'_\mu(x)}{\partial x^\alpha} - A_{\mu,\rho\alpha}\eta^\rho$$

and

$$\frac{\partial A_\rho(x)}{\partial x'^\alpha}\ \frac{\partial x^\rho}{\partial x'^\mu} = \frac{\partial A_\rho}{\partial x^\sigma}\ \frac{\partial x^\sigma}{\partial x'^\alpha}\ \frac{\partial x^\rho}{\partial x'^\mu} = \frac{\partial A_\mu(x)}{\partial x^\alpha} + A_{\mu,\rho}\eta^\rho_{,\alpha} + A_{\rho,\alpha}\eta^\rho_{,\mu}$$

Putting these equations together, we get and the change in S_{mg} may be written in the form

$$\delta S_{mg} = \int \left[\frac{\partial \mathcal{L}_{mg}}{\partial g_{\mu\nu}} - \left(\frac{\partial \mathcal{L}}{\partial g_{\mu\nu,\alpha}} \right)_{,\alpha} \right] \delta g_{\mu\nu} d^4x \qquad \text{(a)}$$

$$+ \int \sum_n \left[\frac{\partial \mathcal{L}_{mg}}{\partial q^n} - \left(\frac{\partial \mathcal{L}_{mg}}{\partial q^n_{,\alpha}} \right)_{,\alpha} \right] \delta q^n \, d^4x \qquad \text{(b)}$$

(D-10)

where we have dropped the surface terms that usually appear in such a calculation. The quantity D-10b is zero by the equations of motion for the field components q^n,* thus the first integral D-10a must be zero if S_{mg} is to be unchanged by a coordinate transformation.

Writing

$$\frac{\partial \mathcal{L}_{mg}}{\partial g_{\mu\nu}} - \left(\frac{\partial \mathcal{L}_{mg}}{\partial g_{\mu\nu,\alpha}} \right)_{,\alpha} = \frac{\partial \mathcal{L}_{mg}}{\partial g_{\mu\nu}} = \frac{\partial \mathcal{L}_{mg}}{\partial Kh_{\mu\nu}} = -\frac{1}{2} T^{\mu\nu}_{mg} \qquad \text{(D-11)}$$

we therefore get the condition

$$0 = -\frac{1}{2} \int T^{\mu\nu}_{mg} \, \delta g^{\mu\nu} \, d^4x \qquad \text{(D-12)}$$

However under the coordinate transformation D-7 the quantity $\delta g_{\mu\nu}$ is given by

$$\delta g_{\mu\nu} = g_{\mu\nu,\alpha}\eta^\alpha + g_{\mu\alpha}\eta^\alpha_{,\nu} + g_{\alpha\nu}\eta^\alpha_{,\mu\alpha} \qquad \text{(see equations C1-20)}$$

$$\frac{\partial A'_\mu(x)}{\partial x^\alpha} = A_{\mu,\rho\alpha}\eta^\rho + A_{\mu,\rho}\eta^\rho_{,\alpha} + A_{\rho,\alpha}\eta^\rho_{,\mu} + A_\rho \eta^\rho_{,\mu\alpha}$$

$$= (A_{\mu,\rho}\eta^\rho + A_\rho\eta^\rho_{,\mu})_{,\alpha} = (\delta A_\mu)_{,\alpha}$$

The proof of D-9 for $g_{\mu\nu}$ or fields of various spin is similar to this proof for vector fields.

*The quantity $[(\partial \mathcal{L}_{mg}/\partial g_{\mu\nu}) - (\partial \mathcal{L}_{mg}/\partial g_{\mu\nu,\alpha})_{,\alpha}]$ is not zero by the equations of motion for the gravitational field because the Lagrangian of the free gravitational fields has not been included.

Thus the condition D-12 becomes (noticing $T^{\mu\nu}_{mg} = T^{\nu\mu}_{mg}$)

$$0 = -\frac{1}{2}\int [\, 2T^{\mu\nu}_{mg} g_{\mu\alpha}\eta^{\alpha}_{,\nu} + T^{\mu\nu}_{mg} g_{\mu\nu,\alpha}\eta^{\alpha}]\, d^4x$$

Integrating by parts and writing $\eta^{\alpha} = g^{\alpha\beta}\eta_{\beta}$ we get

$$0 = \int \eta_{\beta}[\, g^{\beta\alpha} g_{\alpha\mu} T^{\mu\nu}_{mg,\nu} + T^{\mu\nu}_{mg}\, g^{\alpha\beta}(g_{\mu\alpha,\nu} - \frac{1}{2} g_{\mu\nu,\alpha})]\, d^4x \qquad \text{(D-13)}$$

Equation D-13 may be written in the form

$$\int \eta_{\beta}(x)[\, T^{\beta\nu}_{mg,\nu} + \Gamma^{\beta}_{\mu\nu} T^{\mu\nu}_{mg}]\, d^4x = 0$$

Since $\eta_{\beta}(x)$ is an arbitrary vector, we must have the relation

$$T^{\mu\nu}_{mg,\nu} = -\Gamma^{\mu}_{\alpha\beta} T^{\alpha\beta}_{mg}$$

which is just equation D-5a which is what we wished to prove.

E. CONSERVED SYMMETRIC ENERGY TENSORS

We are now in a position to write the general formula for a symmetric conserved energy tensor for a system interacting with gravity. To do this let us write the Lagrangian for the system in the form

$$\mathcal{L} = F^2 + (F - F^2) + \mathcal{L}_{mg} \qquad \text{(E-1)}$$

In E-1, $F = R(-g)^{1/2}/2K^2$. F^2 includes those terms in the expansion of F that involve the fields $h_{\mu\nu}$ twice and is given by equation A-11.* $\mathcal{L}_{mg}$ is the Lagrangian of the matter fields rewritten in the curved

*$R(-g)^{1/2}$ also involves terms that are first order in $h_{\mu\nu}$, but these form a pure divergence and have been dropped from F.

space of metric $g_{\mu\nu} = (\delta_{\mu\nu} + Kh_{\mu\nu})$.

The wave equation for the gravitational fields may now be rewritten in the form

$$\frac{\delta F^2}{\delta h_{\mu\nu}} = \frac{K}{2}\left[- 2\,\frac{\delta \mathcal{L}_{mg}}{\delta K h_{\mu\nu}} - 2\,\frac{\delta(F - F_2)}{\delta K h_{\mu\nu}} \right] \qquad \text{(E-2)}$$

We have seen however that for a consistent theory of gravity the term in the brackets must be $T^{\mu\nu}$, the complete, symmetric, conserved energy momentum tensor of the system. (See for example equation C-1. Notice that C-1 has been multiplied through by a factor of four.)

If we use the notation of equation D-11

$$\frac{1}{2}\,T^{\mu\nu}_{mg} = \frac{\partial \mathcal{L}_{mg}}{\delta g_{\mu\nu}} = \frac{\delta \mathcal{L}_{mg}}{\delta K h_{\mu\nu}}$$

we get

$$T^{\mu\nu} = T^{\mu\nu}_{mg} - 2\,\frac{\delta(F - F^2)}{\delta K h_{\mu\nu}} \qquad \text{(E-3)}$$

From the derivation we know that $T^{\mu\nu}$ is symmetric for it was obtained by variation with respect to the symmetric quantity $h_{\mu\nu}$, and from the fact that

$$\left(\frac{\delta F^2}{\delta h_{\mu\nu}}\right)_{,\nu} = 0$$

we must have $T^{\mu\nu}_{\ ,\nu} = 0$ for we have shown that equation E-2 is consistent.

Suppose we wish the symmetric energy tensor of a system in the absence of gravity. The formalism of writing the canonical energy tensor and then symmetrizing involves a rather cumbersome calculation.

We now have a direct method of obtaining a symmetric energy tensor by using E-3 and then taking the limit as $Kh_{\mu\nu} \rightarrow 0$. Since $(F - F^2)$ will not contribute in this limit, the symmetric energy tensor of matter is given by

$$-\frac{1}{2} T_m^{\mu\nu} = \left. \frac{\delta \mathcal{L}_{mg}}{\delta K h_{\mu\nu}} \right|_{Kh_{\mu\nu} = 0} \tag{E-4}$$

Equation E-4 does not actually rely on the existence of gravity or curved space. Let the flat space action for a system be given by

$$S_m = \int \mathcal{L}_m \, d^4x \tag{E-5}$$

Now suppose that the action S_m is invariant under the coordinate transformation

$$x^\mu \rightarrow x'^\mu = x^\mu - \eta^\mu(x) \tag{E-6}$$

where $\eta^\mu(x)$ is an arbitrary infinitesimal vector. Since $\eta^\mu(x)$ can take on an arbitrary infinitesimal value at each point in space, the invariance of the action under the transformation E-6 will mean that there exists a quantity that is conserved at each local region in the space. This conserved quantity will be identified as the energy momentum tensor of the system.

This method of finding conserved energy momentum tensors differs from the method described in part C1. In part C1 we considered the invariance of the action only under coordinate translations and Lorentz transformations. Under these transformations the Gallelian metric tensor $\delta_{\mu\nu}$,

$$\delta_{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

is unchanged. Thus if the action was originally written in a Gallelian metric, then it will remain in a Gallelian metric under the transformations used in part C1.

Suppose that the action is originally written in a Gallelian metric and we use the coordinate transformation E-6. The coordinate system x'^{μ} would then have a metric $g_{\mu\nu}$ defined by the relation

$$s^2 = \delta_{\mu\nu} x^{\mu} x^{\nu} = g_{\mu\nu} x'^{\mu} x'^{\nu}$$

If we write $g_{\mu\nu}$ in the form

$$g_{\mu\nu} = \delta_{\mu\nu} + k_{\mu\nu}$$

then

$$k_{\mu\nu} = \delta_{\mu a} \eta^{a}_{,\nu} + \delta_{\nu a} \eta^{a}_{,\mu} + \text{terms of order } \eta^2$$

as may be checked by direct substitution. Thus the transformation E-6 leaves us with an action that is no longer in a Gallelian metric, but which must be described by curvilinear coordinates.

We would therefore be led to write the action in curvilinear coordinates. This would give us the same form for the action as $\mathcal{L}_{mg}$, except the metric would be

$$g_{\mu\nu} = \delta_{\mu\nu} + k_{\mu\nu}$$

a flat space metric. Designating the action written in curvilinear coordinates by S_{mk}, the condition that the action be unchanged under the coordinate transformation E-6 is

$$\int \left(\frac{\delta \mathcal{L}_{mk}}{\delta k_{\mu\nu}} \right) \delta k_{\mu\nu} = 0$$

in analogy with equation D-12. The resulting conserved energy momentum tensor is given by

$$-\frac{1}{2} T_m^{\mu\nu} = \frac{\delta \mathcal{L}_{mk}}{\delta k_{\mu\nu}} \Bigg|_{k_{\mu\nu}=0}^{*} \qquad \text{(E-7)}$$

for all the steps are similar to those used for the case that gravity was present.

The only difference between equations E-7 and E-4 is that E-7 does not rely on the use of a curved space for its derivation.

F. NONUNIQUENESS OF THE INTERACTION OF GRAVITY WITH FIELDS

In the previous parts of this section we have given the prescription for finding a consistent theory for the interaction of gravity with fields of matter, and for finding the total conserved energy momentum tensor for that system. The prescription relied on writing the action for the system as a scalar quantity in the space of metric $g_{\mu\nu} = \delta_{\mu\nu} + Kh_{\mu\nu}$. Since there is no unique method for rewriting, in a space of metric $g_{\mu\nu}$, an action known only in a flat space of Gallelian metric $\delta_{\mu\nu}$, the above prescriptions are not unique.

Consider for example the action for the electromagnetic field. Two flat space Lagrangians, already discussed in part C1, that lead to

* This is the method used by Landau and Lifschitz (2) for obtaining symmetric energy momentum tensors.

the same action are

$$4\pi\mathcal{L}_m^c = \frac{1}{2}(A_{\mu,\nu}A_{\mu,\nu} - A_{\mu,\nu}A_{\nu,\mu}) \tag{C1-44}$$

$$4\pi\mathcal{L}_m^d = \frac{1}{2}(A_{\mu,\nu}A_{\mu,\nu} - A_{\mu,\mu}A_{\nu,\nu}) \tag{C1-45}$$

$4\pi\mathcal{L}_m^d$ is obtained by adding to $4\pi\mathcal{L}_m^c$ the quantity

$$-\frac{1}{2}(A_{\nu,\nu}A_{\mu})_{,\mu} + \frac{1}{2}(A_{\nu,\mu}A_{\mu})_{,\nu} \tag{F-1}$$

$$= -\frac{1}{2}(A_{\nu,\nu}A_{\mu,\mu} - A_{\nu,\mu}A_{\mu,\nu} + [A_{\nu,\nu\mu} - A_{\nu,\mu\nu}]A_{\mu}) \tag{F-2}$$

Since the order of partial differentiation may be reversed the terms in the square brackets cancel, and F-2 added to $4\pi\mathcal{L}_m^c$ gives $4\pi\mathcal{L}_m^d$.

To write $\mathcal{L}_m^c$ and $\mathcal{L}_m^d$ in the space of metric $g_{\mu\nu}$, we shall introduce the metric tensor for each summed index, replace partial derivatives by covariant derivatives,* and replace the volume element d^4x in the action by $(-g)^{1/2}\,d^4x$. $\mathcal{L}_m^c$ and $\mathcal{L}_m^d$ become

$$\mathcal{L}_{mg}^c = -\frac{1}{8\pi}g^{\alpha\beta}g^{\rho\sigma}(A_{\alpha;\rho}A_{\beta;\sigma} - A_{\alpha;\rho}A_{\sigma;\beta})(-g)^{1/2} \tag{F-3}$$

$$\mathcal{L}_{mg}^d = -\frac{1}{8\pi}g^{\alpha\beta}g^{\rho\sigma}(A_{\alpha;\rho}A_{\beta;\sigma} - A_{\alpha;\beta}A_{\rho;\sigma})(-g)^{1/2} \tag{F-4}$$

Although in flat space

$$\int \mathcal{L}_m^c\, d^4x = \int \mathcal{L}_m^d\, d^4x$$

the corresponding actions in curved space are related by the equation

*We will use the notation $;\mu$ for a covariant derivative. For example

$$\varphi_{;\mu} = \varphi_{,\mu}$$
$$A_{\nu;\mu} = A_{\nu,\mu} - \Gamma^{\rho}_{\mu\nu}A_{\rho}$$
$$T^{\alpha\beta}_{;\mu} = T^{\alpha\beta}_{,\mu} + \Gamma^{\alpha}_{\mu\rho}T^{\rho\beta} + \Gamma^{\beta}_{\mu\rho}T^{\alpha\rho}$$

etc.

$$\int \mathcal{L}^{d}_{mg}\, d^4x = \int \mathcal{L}^{c}_{mg}\, d^4x - \frac{1}{8\pi}\int R^{\mu\nu} A_\mu A_\nu (-g)^{1/2} d^4x \qquad \text{(F-5)}$$

where $R^{\mu\nu}$ is the curvature tensor. Since the action of the free gravitational fields is given by

$$S_g = \frac{1}{2K^2}\int g_{\mu\nu} R^{\mu\nu} (-g)^{1/2}\, d^4x$$

only in the absence of gravity can we set the curvature tensor $R^{\mu\nu}$ equal to zero. Thus we have generated two nonequivalent, but consistent, theories for the interaction of gravity with electromagnetic fields.

The reason that the two actions $\int \mathcal{L}^{d}_{mg}\, d^4x$ and $\int \mathcal{L}^{c}_{mg}\, d^4x$ differ by the term $-\frac{1}{8\pi}\int R^{\mu\nu} A_\mu A_\nu (-g)^{1/2} d^4x$ may be seen as follows. If either of the terms of F-1 are separately written in curved space they remain a pure divergence; i.e.,

$$-\frac{1}{2} g^{\alpha\beta} g^{\rho\sigma} [A_{\alpha;\beta;\rho} A_\sigma + A_{\alpha;\beta} A_{\sigma;\rho}] (-g)^{1/2} = -\frac{1}{2} [A^{\alpha}_{;\alpha} A^{\sigma} (-g)^{1/2}]_{,\sigma}{}^{*} \qquad \text{(F-6)}$$

and

$$+\frac{1}{2} g^{\alpha\beta} g^{\rho\sigma} [A_{\alpha;\rho;\beta} A_\sigma + A_{\alpha;\rho} A_{\sigma;\beta}] (-g)^{1/2} = \frac{1}{2} [A^{\alpha}_{;\sigma} A^{\sigma} (-g)^{1/2}]_{,\alpha} \qquad \text{(F-7)}$$

However to obtain $\mathcal{L}^{d}_{mg}$ we did not add the term

$$\frac{1}{2} g^{\alpha\beta} g^{\rho\sigma} [(A_{\alpha;\beta;\rho} - A_{\alpha;\rho;\beta}) A_\sigma] (-g)^{1/2} \times \frac{1}{4\pi} \qquad \text{(F-8)}$$

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*For example F-6 may be written (noticing $A^{\alpha}_{;\alpha}$ is scalar)

$$[(A^{\alpha}_{;\alpha})_{,\sigma} A^{\sigma} + A^{\alpha}_{;\alpha} A^{\sigma}_{;\sigma}] (-g)^{1/2} = (A^{\alpha}_{;\alpha})_{,\sigma} A^{\sigma} (-g)^{1/2} + A^{\alpha}_{;\alpha} A^{\sigma}_{,\sigma} (-g)^{1/2} + A^{\alpha}_{;\alpha} A^{\sigma} \Gamma^{\delta}_{\sigma\delta} (-g)^{1/2} = [A^{\alpha}_{;\alpha} A^{\sigma} (-g)^{1/2}]_{,\sigma}$$

where $(-g)^{1/2}{}_{,\sigma} = (-g)^{1/2} \Gamma^{\delta}_{\sigma\delta}$.

because $(A_{\nu,\nu\mu} - A_{\nu,\mu\nu})A_{\mu}$ was cancelled in flat space before the Lagrangians were written in curved space.

Since

$$A_{\alpha;\beta;\rho} - A_{\alpha;\rho;\beta} = A_{\lambda}R^{\lambda}_{\alpha\rho\beta} \quad *$$

the term F-8 is just

$$\frac{1}{8\pi} R^{\mu\nu} A_{\mu} A_{\nu} (-g)^{1/2} \qquad \text{(F-9)}$$

and this is just the term missing in $\mathcal{L}^{d}_{mg}$.

Let us assume that the only restrictions placed on a curved space action are the following. First, the action shall be a scalar quantity in the space of metric $g_{\mu\nu}$. Second, in the limit of flat space we return to the original flat space action. Third, the same number of field components q^n shall appear in each term of a given curved space Lagrangian. Fourth, each term of the curved space Lagrangian shall be of the same dimensions without the introduction of new constants. (This limits the number of derivatives appearing in a given term in the curved space Lagrangian.) Let these restrictions be labeled by F-10.

Let us denote by $\mathcal{L}^{So}_{mg}$, $\mathcal{L}^{Vo}_{mg}$, and $\mathcal{L}^{To}_{mg}$ a particular choice of curved space Lagrangians for the scalar, vector, and tensor fields φ, A_{μ}, and $h_{\mu\nu}$. The most general curved space Lagrangians consistent with the restrictions F-10 are

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*See for example Landau (18). Note again that we are using the opposite sign than that used by Landau for the curvature tensor.

$$\mathcal{L}^{S}_{mg} = \mathcal{L}^{So}_{mg} + fR\varphi\varphi(-g)^{1/2} \qquad \text{(F-11a)}$$

$$\mathcal{L}^{V}_{mg} = \mathcal{L}^{Vo}_{mg} + mR^{\alpha\beta}A_{\alpha}A_{\beta}(-g)^{1/2} + nRg^{\alpha\beta}A_{\alpha}A_{\beta}(-g)^{1/2} \qquad \text{(F-11b)}$$

$$\mathcal{L}^{T}_{mg} = \mathcal{L}^{To}_{mg} + aR^{\rho\alpha\beta\sigma}h_{\rho\sigma}h_{\alpha\beta}(-g)^{1/2} + bR^{\alpha\beta}g^{\rho\sigma}h_{\alpha\rho}h_{\sigma\beta}(-g)^{1/2}$$

$$+ cR^{\alpha\beta}g^{\rho\sigma}h_{\alpha\beta}h_{\rho\sigma}(-g)^{1/2} + dRg^{\alpha\beta}g^{\rho\sigma}h_{\alpha\rho}h_{\beta\sigma}(-g)^{1/2}$$

$$+ eRg^{\alpha\beta}g^{\rho\sigma}h_{\alpha\beta}h_{\rho\sigma}(-g)^{1/2} \qquad \text{(F-11c)}$$

where the constants a, b, c, d, e, f, m, n are arbitrary.

We might include the further restriction that we only consider curved space Lagrangians that are obtained from a flat space Lagrangian "directly," that is, by introducing the metric tensor for each summed index, replacing partial derivatives by covariant derivatives and introducing the $(-g)^{1/2}$ for the volume element. (Restriction F-12.)

If the flat space Lagrangian involves only first derivatives of the field components, and the $\mathcal{L}^{o}_{mg}$ are obtained under restriction F-12, then the most general curved space Lagrangians are

$$\mathcal{L}^{S}_{mg} = \mathcal{L}^{So}_{mg} \qquad \text{(F-13a)}$$

$$\mathcal{L}^{V}_{mg} = \mathcal{L}^{Vo}_{mg} + mR^{\alpha\beta}A_{\alpha}B_{\beta}(-g)^{1/2} \qquad \text{(F-13b)}$$

$$\mathcal{L}^{T}_{mg} = \mathcal{L}^{To}_{mg} + bR^{\alpha\beta}g^{\rho\sigma}h_{\alpha\rho}h_{\sigma\beta}(-g)^{1/2} \qquad \text{(F-13c)}$$

The terms with arbitrary constants m and b only appear due to the possibility of interchanging the order of covariant differentiation. Since the fields $h_{\mu\nu}$ are symmetric, we get the same number of arbitrary terms for the tensor field as for the vector field.

If the Lagrangian involves second derivatives, and a second derivative is written in curved space in the following manner (using A_μ as an example)

$$A_{\mu,\alpha\beta} \rightarrow A_{\mu;\alpha;\beta} \text{ or } A_{\mu;\beta;\alpha} \qquad \text{(F-14)}$$

then the most general curved space Lagrangians are still given by equations F-13. (We may also add pure divergences to the Lagrangians in F-11 and F-13, but such terms will not lead to different actions.)

We will see in the next part that the restriction F-12 is not the correct restriction for all fields.

G. NONUNIQUENESS OF ENERGY MOMENTUM TENSORS

At the end of part E we described a method of obtaining energy momentum tensors that relied on writing the Lagrangian for a system in flat space curvilinear coordinates. The procedure for writing a Lagrangian in flat space curvilinear coordinates is similar to the procedure for writing the Lagrangian in curved space as was described in part F. The only difference is that in flat space the metric tensor $g_{\mu\nu} = \delta_{\mu\nu} + k_{\mu\nu}(x)$ may be obtained from the Gallelian metric $\delta_{\mu\nu}$ by a coordinate transformation, while the curved space metric $g_{\mu\nu} = \delta_{\mu\nu} + Kh_{\mu\nu}$ cannot be so obtained. Thus the formulas F-11 and F-13 may be considered as formulas for Lagrangians written in flat space curvilinear coordinates, as long as we use the metric $g_{\mu\nu} = \delta_{\mu\nu} + k_{\mu\nu}$.

In equations F-11 and F-13 the Lagrangians differed by terms involving the curvature tensor. In flat space the curvature tensor is zero, thus the possible Lagrangians for a given system are equivalent. (They may differ by a pure divergence.) The question is therefore, will we get the same energy tensor for a system by equation E-7

$$-\frac{1}{2}T^m_{\mu\nu} = \left.\frac{\delta \mathcal{L}_{mk}}{\delta k_{\mu\nu}}\right|_{k_{\mu\nu}=0} \tag{G-1}$$

if we use equivalent flat space Lagrangians?

The answer is that two Lagrangians that differ by a term involving the curvature tensor lead to different energy tensors. We may investigate the difference between energy tensors for a given system by writing all the possible terms involving the curvature tensor and applying equation E-7.

Let the Lagrangian for a given system be written in the form

$$\mathcal{L}_{mk} = \mathcal{L}^o_{mk} + f(R) \tag{G-2}$$

where $f(R)$ is a function of the curvature tensor, and $\mathcal{L}^o_{mk}$ is one particular choice of the curvilinear Lagrangian. The energy tensor from $\mathcal{L}_{mk}$ is

$$T^m_{\mu\nu} = -\left.\frac{2\delta\mathcal{L}^o_{mk}}{\delta k_{\mu\nu}}\right|_{k_{\mu\nu}=0} - \left.\frac{2\delta f(R)}{\delta k_{\mu\nu}}\right|_{k_{\mu\nu}=0} \tag{G-3a}$$

$$T^m_{\mu\nu} = T^o_{\mu\nu} + W_{\mu\nu} \tag{G-3b}$$

We shall call $W_{\mu\nu}$, obtained from $f(R)$, the difference tensor.

To find $W_{\mu\nu}$ (or $T^o_{\mu\nu}$) by equation E-7, we need to keep only the terms in the curvilinear Lagrangian that are first order in $k_{\mu\nu}$. To lowest order in $k_{\mu\nu}$ the various forms of the curvature tensor are

$$2R^{\rho\alpha\beta\sigma} = g_{\rho\sigma,\alpha\beta} + g_{\alpha\beta,\rho\sigma} - g_{\rho\beta,\alpha\sigma} - g_{\sigma\alpha,\rho\beta}$$

$$2R^{\alpha\beta} = g_{\sigma\sigma,\alpha\beta} + g_{\alpha\beta,\sigma\sigma} - g_{\sigma\beta,\sigma\alpha} - g_{\sigma\alpha,\sigma\beta}$$

$$2R = 2g_{\sigma\sigma,\rho\rho} - 2g_{\sigma\rho,\sigma\rho} \qquad \text{(G-4)}$$

When $g_{\mu\nu} = \delta_{\mu\nu} + k_{\mu\nu}$ has been obtained from $\delta_{\mu\nu}$ by a coordinate transformation, i.e.

$$k_{\mu\nu} = \eta_{\mu,\nu} + \eta_{\nu,\mu}$$

then the terms in G-4 will cancel. However, if we apply equation E-7 before canceling the terms in the curvature tensor, we get a non-zero contribution.

The most general curvilinear Lagrangians of the form G-2 for scalar, vector and tensor fields are given by equations F-11. (We are assuming the restrictions F-10 and that we are dealing with linear theories.) The energy tensors corresponding to the Lagrangians $\mathcal{L}^{S}_{mk}$, $\mathcal{L}^{V}_{mk}$, and $\mathcal{L}^{T}_{mk}$ of equations F-11 are given by

$$\mathcal{L}^{S}_{mk} = \mathcal{L}^{So}_{mk} + fR\varphi^2(-g)^{1/2}$$

$$T^{S}_{\mu\nu} = T^{So}_{mk} - 2f\{(\delta_{\mu\nu}\delta_{\rho\sigma} - \delta_{\mu\rho}\delta_{\nu\sigma})\varphi^2\}_{,\rho\sigma} \qquad \text{(G-5a)}$$

$$\mathcal{L}^{V}_{mk} = \mathcal{L}^{Vo}_{mk} + mR^{\alpha\beta}A_{\alpha}A_{\beta}(-g)^{1/2} + nRg^{\alpha\beta}A_{\alpha}A_{\beta}(-g)^{1/2}$$

$$T^{V}_{\mu\nu} = T^{Vo}_{\mu\nu} - 2n\{(\delta_{\mu\nu}\delta_{\rho\sigma} - \delta_{\mu\rho}\delta_{\nu\sigma})A_{\alpha}A_{\alpha}\}_{,\rho\sigma}$$

$$- m\{\delta_{\mu\nu}A_{\rho}A_{\sigma} + \delta_{\rho\sigma}A_{\mu}A_{\nu} - \delta_{\mu\sigma}A_{\rho}A_{\nu} - \delta_{\nu\sigma}A_{\rho}A_{\mu}\}_{,\rho\sigma} \qquad \text{(G-5b)}$$

and for $\mathcal{L}^T_{mg}$ given in F-11c, the energy tensor $T^T_{\mu\nu}$ is

$$T^T_{\mu\nu} = T^{To}_{\mu\nu} + W^T_{\mu\nu}$$

where $W^T_{\mu\nu}$ is the symmetric part of

$$- \{2ah_{\rho\sigma}h_{\mu\nu} - 2ah_{\nu\sigma}h_{\mu\rho} + b\delta_{\mu\nu}h_{\rho\alpha}h_{\sigma\alpha} + b\delta_{\rho\sigma}h_{\mu\alpha}h_{\alpha\nu}$$

$$- 2b\delta_{\mu\rho}h_{\nu\alpha}h_{\sigma\alpha} + c\delta_{\mu\nu}hh_{\rho\sigma} + c\delta_{\rho\sigma}hh_{\mu\nu} - 2c\delta_{\mu\rho}hh_{\nu\sigma}$$

$$+ 2d(\delta_{\mu\nu}\delta_{\rho\sigma} - \delta_{\mu\rho}\delta_{\nu\sigma})h_{\alpha\beta}h_{\alpha\beta} + 2e(\delta_{\mu\nu}\delta_{\rho\sigma} - \delta_{\mu\rho}\delta_{\nu\sigma})hh\}_{,\rho\sigma} \quad \text{(G-5c)}$$

When the a, b, c, d, e, f, m, n are arbitrary constants equations G-5 give the most general energy tensors obtainable from equation E-7. (Again assuming a linear theory and restrictions F-10.)

We note that all difference tensors $W_{\mu\nu}$ are symmetric, have zero divergence, and are a second derivative. This is the same form as the difference tensor as obtained by equations C1-40.

If we add the restriction that the curvilinear Lagrangian is to be written "directly" from the Gallelian Lagrangian (analogous to restriction F-12) and that the Gallelian Lagrangian involve only first derivatives of the field components, then the only arbitrary constants left in G-5 are m and b. This means that for a scalar field the energy tensor is unique, and that for a vector field $W^V_{\mu\nu}$ must be of the form

$$W^V_{\mu\nu} = \{\delta_{\mu\alpha}A_\rho A_\sigma + \delta_{\rho\sigma}A_\mu A_\nu - \delta_{\mu\sigma}A_\rho A_\nu - \delta_{\nu\sigma}A_\rho A_\mu\}_{,\rho\sigma}$$

But this is exactly the situation we had when we used equations C1-40.

(See equations C1-47 through C1-49.)

Possible Lagrangians for scalar, vector and tensor fields that involve only first derivatives are

$\mathcal{L}^a$ of equation C1-41 for scalar fields

$\mathcal{L}^c$ of equation C1-44 for vector fields

$\mathcal{L}_g$ of equation A-11 for tensor fields

When these Lagrangians are written "directly" in curvilinear coordinates and equation E-7 is used to find the energy tensor, the results are the same as obtained by equations C1-40.*

It therefore seems likely that equations C1-40 are equivalent to E-7 when E-7 is applied only to a curvilinear Lagrangian written "directly" from a Gallelian Lagrangian involving only first derivatives of the field components.

We should mention that we obtained a difference tensor

$$W^S_{\mu\nu} = \{(\delta_{\mu\nu}\delta_{\rho\sigma} - \delta_{\mu\rho}\delta_{\nu\sigma})\varphi^2\}_{,\rho\sigma}$$

for scalar fields by comparing the symmetric cannonical energy tensor $\theta^b_{\mu\nu}$ with the complete energy tensor. (See equation C1-43.) In this sense equations C1-40 gave the same general results as in G-5a for scalar fields. This indicates that some modification of equations C1-40 might lead the full range of energy tensors in G-5.

With the spectrum of energy tensors presented in G-5, and with

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*C1-40 also gave rise to terms that destroyed the symmetry of the energy tensor, but these terms were zero by the equations of motion. We are neglecting such terms.

the evidence from part C2 that there is physical meaning to a particular choice of the difference tensors $W_{\mu\nu}$, is there any generally valid rule for selecting a particular energy tensor for a given system? The author has found none. For scalar and vector fields there exists only one energy tensor in G-5 that involves only first derivatives of the field components,* but no such energy tensor exists for spin-two fields. Thus the requirement that energy tensors have only first derivatives is not general. In the one case where we have experimental evidence (the energy tensor of gravity), second derivatives are involved.

To survey the possible energy tensors for spin-two fields, choose the Lagrangian A-11 written "directly" in curvilinear coordinates as $\mathcal{L}^{To}_{mk}$. The energy tensor we get from $\mathcal{L}^{To}_{mk}$ is

$$-2 \left. \frac{\partial \mathcal{L}^{To}_{mk}}{\partial k_{\mu\nu}} \right|_{k_{\mu\nu}=0} = U_{\mu\nu} \qquad \text{(G-6)}$$

where $U_{\mu\nu}$ is just the energy tensor of equation C2-6. $U_{\mu\nu}$ was obtained from A-11 by equations C1-40 (within terms that are zero by the equation of motion of the free fields) which checks the equivalence of the two methods of finding energy tensors.

We can now see that there is no energy tensor for the spin-two fields that involves only first derivatives of the field components. Using equation G-5c, the possible terms in the energy tensor that have a factor $\delta_{\mu\nu}$ are

*Namely the tensors $T^{b}_{\alpha\beta}$ and $T^{c}_{\alpha\beta}$ of equations C1-42a and C1-44a. Any $W_{\mu\nu}$ added to these tensors involves second derivatives of the field components.

$$\delta_{\mu\nu}\{\frac{1}{4}h_{,\rho\sigma}h_{\rho\sigma}-\frac{1}{2}h_{\rho\alpha}h_{\sigma\alpha,\rho\sigma}\} \qquad \text{(from } U_{\mu\nu})$$

$$+\delta_{\mu\nu}\{-b(h_{\rho\alpha,\rho\sigma}h_{\sigma\alpha}+h_{\rho\alpha}h_{\sigma\alpha,\rho\sigma})-4e(hh_{,\sigma\sigma})$$

$$-c(h h_{\rho\sigma,\rho\sigma}+h_{,\rho\sigma}h_{\rho\sigma})-4d(h_{\alpha\beta,\sigma\sigma}h_{\alpha\beta})\} \qquad \text{(G-7)}$$

By no choice of the constants b, c, d, e, can we eliminate the terms involving second derivatives.

There is, however, at least one well defined choice of a curvilinear Lagrangian for spin-two fields (leading to an equally well defined energy tensor). The Lagrangian

$$\mathcal{L}_{gk}=(-g)^{1/2}\{\frac{1}{8}g^{\alpha\beta}g^{\rho\sigma}g^{\gamma\delta}(h_{\alpha\rho;\gamma}h_{\beta\sigma;\delta}-h_{\alpha\beta;\gamma}h_{\rho\sigma;\delta}$$

$$-2h_{\alpha\gamma;\beta}h_{\rho\delta;\sigma}+2h_{\alpha\gamma;\beta}h_{\rho\sigma;\delta})+\frac{1}{4}R^{\rho\alpha\beta\sigma}h_{\rho\sigma}h_{\alpha\beta}$$

$$+\frac{1}{4}R^{\alpha\beta}g^{\rho\sigma}(h_{\alpha\rho}h_{\sigma\beta}-h_{\alpha\beta}h_{\rho\sigma})$$

$$-\frac{1}{8}Rg^{\alpha\beta}g^{\rho\sigma}(h_{\alpha\rho}h_{\beta\sigma}-\frac{1}{2}h_{\alpha\beta}h_{\rho\sigma})\} \qquad \text{(G-8)}$$

which corresponds in F-llc to the choice of the constants

$$a=b=-c=\frac{1}{4},\; d=-\frac{1}{8},\; e=\frac{1}{16} \qquad \text{(G-9)}$$

is symmetric between the fields $h_{\mu\nu}$ and the geometrical quantities $k_{\mu\nu}$ (if we keep the terms in $\mathcal{L}_{mk}$ that are first order in $k_{\mu\nu}$). By this symmetry, we mean that if a term, for example of the form $k_{\mu\nu}h_{\mu\nu,\sigma}h_{,\sigma}$ appears, then the terms $h_{\mu\nu}k_{\mu\nu,\sigma}h_{,\sigma}$ and $h_{\mu\nu}h_{\mu\nu,\sigma}k_{,\sigma}$

also appear with equal coefficients.

The energy tensor from $\mathcal{L}_{gk}$ is

$$- 2 \left. \frac{\partial \mathcal{L}_{gk}}{\partial k_{\mu\nu}} \right|_{k_{\mu\nu}=0} = U_{\mu\nu} + W_{\mu\nu} = T^F_{\mu\nu} \tag{G-10}$$

where $W_{\mu\nu}$ is the difference tensor given in equation C2-12, and $T^F_{\mu\nu}$, the tensor obtained by Feynman, is given in equation C3-9.

Because this derivation of the tensor $U_{\mu\nu} + W_{\mu\nu}$ required the use of all five arbitrary constants, it is probably not derivable by equations C1-40 as they now stand. However, $U_{\mu\nu} + W_{\mu\nu}$ was derived as an energy tensor (without the use of curved space). Since $U_{\mu\nu} + W_{\mu\nu}$, the same tensor as obtained by Feynman and Einstein, is the energy tensor that gives the correct shift in the perihelion of Mercury, we may consider gravity as a spin-two field coupled to energy. The difficulty is that we need an extra condition to define the energy.

Thus the connection between gravity and geometry, originally stated by Einstein in the general theory of relativity, is also seen when gravity is treated as a spin-two field coupled to energy.

IV. INTERACTION OF GRAVITY WITH ELECTRONS

In this section we shall derive the interaction of gravity with electrons in a manner that treats gravity as a spin-two field. The method will be to rewrite the action for the free electron fields in a curved space of metric $g_{\mu\nu} = \delta_{\mu\nu} + Kh_{\mu\nu}$. We have seen from part D that this method leads to a consistent theory.

Our first problem is to find the correct flat space action that is to be rewritten in curved space.

H. FLAT SPACE LAGRANGIAN FOR THE ELECTRON FIELDS

The flat space Lagrangian for the two independent electron fields ψ and $\overline{\psi}$ is usually given by the formula

$$\mathcal{L}_e = i\overline{\psi}\overline{\gamma}_\mu\psi_{,\mu} - m\overline{\psi}\psi \tag{H-1}$$

where the Dirac γ matrices are defined by the commutation relation

$$\overline{\gamma}_\mu\overline{\gamma}_\nu + \overline{\gamma}_\nu\overline{\gamma}_\mu = 2\delta_{\mu\nu}\ {}^{*} \tag{H-2}$$

The equations of motion for the fields $\overline{\psi}$ and ψ are given by

$$\frac{\delta\mathcal{L}_e}{\delta\psi} = 0 \qquad \text{and} \qquad \frac{\delta\mathcal{L}_e}{\delta\overline{\psi}} = 0$$

For example, the equation for ψ is given from $\delta\mathcal{L}_e/\delta\overline{\psi}$ as

$$i\overline{\gamma}_\mu\psi_{,\mu} - m\psi = 0$$

---

* As we will later need to distinguish between a curved space and a flat space γ matrix, the notation $\overline{\gamma}_\mu$ will be used for flat space γ matrices.

which is just the Dirac equation for a free electron.

The electron Lagrangian H-1 is not the only form for a flat space Lagrangian for the electron fields. First of all any Lagrangian differing by a pure divergence gives an equivalent action for an electron. Secondly, the equation H-2 does not uniquely define the flat space γ matrices; any set of γ matrices satisfying H-2 will lead to the same physics of the electron provided the correct wave function is used.

To see the possible choices for a flat space Lagrangian suppose that the particular set of γ matrices $\bar{\gamma}'_\mu$ had been chosen. Let the wave functions associated with this choice of γ matrices be given by ψ' and $\bar{\psi}'$. If one choice of the Lagrangian for these fields is given by

$$\mathcal{L}_e^1 = i\bar{\psi}'\bar{\gamma}'_\mu \psi'_{,\mu} - m\bar{\psi}'\psi' \tag{H-3}$$

then another choice is

$$\mathcal{L}_e^2 = -i\bar{\psi}'_{,\mu}\bar{\gamma}'_\mu \psi' - m\bar{\psi}'\psi' \tag{H-4}$$

for $\mathcal{L}_e^1$ and $\mathcal{L}_e^2$ differ by a pure divergence. The commutation relation satisfied by the γ matrices $\bar{\gamma}'_\mu$ is

$$\bar{\gamma}'_\mu\bar{\gamma}'_\nu + \bar{\gamma}'_\nu\bar{\gamma}'_\mu = 2\delta_{\mu\nu} \tag{H-5}$$

Now let us assume that we wished to use a different choice (or representation) of the γ matrices, namely $\bar{\gamma}_\mu$, related to $\bar{\gamma}'_\mu$ by the equation

$$\bar{\gamma}'_\mu = S^{-1}\bar{\gamma}_\mu S \tag{H-6}$$

where the operation $S^{-1}\bar{\gamma}_\mu S$ represents a unitary transformation on the 4 x 4 Dirac γ matrices; $S^{-1}S = 1$.

Equation H-5 becomes

$$S^{-1}\bar{\gamma}_\mu S S^{-1}\gamma_\nu S + S^{-1}\bar{\gamma}_\nu S S^{-1}\bar{\gamma}_\mu S = 2\delta_{\mu\nu}$$

or

$$\bar{\gamma}_\mu\bar{\gamma}_\nu + \bar{\gamma}_\nu\bar{\gamma}_\mu = 2S\delta_{\mu\nu}S^{-1} = 2\delta_{\mu\nu} \qquad \text{(H-7)}$$

We get the last step for S must commute with $\delta_{\mu\nu}$; thus matrices $\bar{\gamma}_\mu$ satisfy the correct commutation relation and are a possible representation of the γ matrices.

The Lagrangians $\mathcal{L}_e^1$ and $\mathcal{L}_e^2$ expressed in terms of the Dirac matrices $\bar{\gamma}_\mu$ now become:

$$\mathcal{L}_e^1 = i\bar{\Psi}'S^{-1}\bar{\gamma}_\mu S\psi'_{,\mu} - m\,\bar{\Psi}'\psi'; \qquad S\psi'_{,\mu} = (S\psi')_{,\mu} - S_{,\mu}\psi'$$

$$\mathcal{L}_e^1 = i(\bar{\Psi}'S^{-1})\bar{\gamma}_\mu[(S\psi')_{,\mu} - S_{,\mu}S^{-1}(S\psi')] - m(\bar{\Psi}'S^{-1})(S\psi')$$

If we define

$$\bar{\Psi}'S^{-1} = \bar{\Psi}$$

$$S\psi' = \psi$$

$$\bar{\Gamma}_\mu = S_{,\mu}S^{-1} \,{}^*$$

we get

$$\mathcal{L}_e^1 = i\bar{\Psi}\gamma_\mu[\psi_{,\mu} - \bar{\Gamma}_\mu\psi] - m\bar{\Psi}\psi \qquad \text{(H-8)}$$

We will also get

$$\mathcal{L}_e^2 = -i[\bar{\Psi}_{,\mu} + \bar{\Psi}\bar{\Gamma}_\mu]\bar{\gamma}_\mu\psi - m\bar{\Psi}\psi \qquad \text{(H-9)}$$

---

*Note that $\bar{\Gamma}_\mu$ is a spin matrix as well as a spatial vector.

Any linear combination of these two Lagrangians is the most general Lagrangian we can write for an electron in flat space.

We note that in flat space the quantities $\bar{\Gamma}_\mu$ can always be eliminated by an appropriate choice of the representation of the γ matrices. For the so-called standard representations where the γ matrices are not a function of position the $\bar{\Gamma}_\mu$ will be zero.

J. CURVED SPACE LAGRANGIAN FOR THE ELECTRON FIELDS

A prescription for writing a curved space Lagrangian is to introduce the metric tensor for each summed index, replace partial derivatives by covariant derivatives, and replace the volume element d^4x in the action by $(-g)^{1/2}\, d^4x$.

It was noted in section III that in writing the curved space action, terms involving the curvature tensor might also be added, for in the limit of flat space the curvature tensor goes to zero and such terms would not contribute to the flat space Lagrangian. These terms do contribute to the energy tensor of the matter fields however, changing the coupling of gravity to matter even in the limit of the linear theory of gravity.

In the case of electrons we have a unique prescription for identifying and excluding such terms. We will see that the energy momentum tensor for the free electron field involves only terms that have a single derivative, whereas terms derived from an expression including the curvature tensor each have at least two derivatives.*
We shall therefore assume that the curved space action for the

*See part G.

electron field does not involve the curvature tensor.

To write the electron Lagrangian in curved space, we will adopt the convention that the fields ψ and $\bar{\psi}$ are scalar quantities, while the γ matrices transform as vector quantities.* As the partial derivative of a scalar quantity is the covariant derivative, no change in the quantities $\psi_{,\mu}$ and $\bar{\psi}_{,\mu}$ need be made.

The flat space γ matrices satisfied the commutation relation

$$\bar{\gamma}_\mu \bar{\gamma}_\nu + \bar{\gamma}_\nu \bar{\gamma}_\mu = 2\delta_{\mu\nu}$$

We will assume that the curved space γ matrices are obtained by merely replacing the metric tensor $\delta_{\mu\nu}$ by $g_{\mu\nu}$ giving

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2g_{\mu\nu} \qquad \text{(J-1)}$$

where γ_μ is now our curved space γ matrix, satisfying the commutation relation J-1.

Finally, we will assume that the quantity $\bar{\Gamma}_\mu$ appearing in the flat space Lagrangians $\mathcal{L}_e^1$ and $\mathcal{L}_e^2$ becomes Γ_μ in curved space. The curved space Lagrangians corresponding to $\mathcal{L}_e^1$ and $\mathcal{L}_e^2$ now become

$$\mathcal{L}_{eg}^1 = (-g)^{1/2} \{ i\bar{\psi} \gamma^\mu [\psi_{,\mu} - \Gamma_\mu \psi] - m\bar{\psi}\psi \} \qquad \text{(J-2)}$$

$$\mathcal{L}_{eg}^2 = (-g)^{1/2} \{ - i [\bar{\psi}_{,\mu} + \bar{\psi}\Gamma_\mu] \gamma^\mu \psi - m\bar{\psi}\psi \} \qquad \text{(J-3)}$$

where we have used the notation

$$\gamma^\mu = g^{\mu\nu}\gamma_\nu$$

*An equivalent choice is to assume that the γ matrices do not transform under coordinate transformations, and that the fields ψ and $\bar{\psi}$ transform as spinors. This leads to the subject of spinor analysis (see W. L. Bade (19)) which is somewhat cumbersome.

It follows that γ^μ satisfies the relation

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} \tag{J-4}$$

The question is, which Lagrangian, $\mathcal{L}^1_{eg}$ or $\mathcal{L}^2_{eg}$, is the correct curved space Lagrangian. In general they will not lead to the same physics for they do not differ by a pure divergence. It was true that the flat space Lagrangians $\mathcal{L}^1_e$ and $\mathcal{L}^2_e$ differed by a pure divergence, but this property was lost in the generalization to curved space. If for example we have $\Gamma_\mu = 0$, then

$$\mathcal{L}^2_{eg} = \mathcal{L}^1_{eg} - ((-g)^{1/2} i \bar{\psi} \gamma^\mu \psi)_{,\mu} + i(-g)^{1/2} \bar{\psi} \gamma^\mu_{,\mu} \psi + i(-g)^{1/2}_{,\mu} \bar{\psi} \gamma^\mu \psi$$

We can see that the last two terms, which destroy the equivalence of the two Lagrangians would be zero in flat space. (In flat space we could choose constant γ matrices and use a coordinate system where $(-g)^{1/2} = 1$.)

The fact that $\mathcal{L}^1_{eg}$ and $\mathcal{L}^2_{eg}$ are not equivalent leaves us with an infinite number of nonequivalent theories for the interaction of electrons with gravity, depending on the linear combination of $\mathcal{L}^1_{eg}$ and $\mathcal{L}^2_{eg}$ that is chosen for the Lagrangian of the system. One way out of this difficulty is to find a value of the quantity Γ_μ which makes $\mathcal{L}^1_{eg}$ and $\mathcal{L}^2_{eg}$ equivalent.

If we drop the restriction $\Gamma_\mu = 0$, and note that

$$(-g)^{1/2}_{,\mu} = (-g)^{1/2} \Gamma^\sigma_{\mu\sigma}$$

we get

$$\mathcal{L}^2_{eg} = \mathcal{L}^1_{eg} + (-g)^{1/2}\, i\bar{\psi}\,[\,\gamma^{\mu}_{,\mu} + \gamma^{\mu}\Gamma^{\sigma}_{\mu\sigma} - \Gamma_{\mu}\gamma^{\mu} + \gamma^{\mu}\Gamma_{\mu}\,]\,\psi$$

$$+ \text{ a pure divergence} \qquad \text{(J-5)}$$

For $\mathcal{L}^2_{eg}$ to be equivalent to $\mathcal{L}^1_{eg}$ we must find a value of Γ_μ for which the term of J-5 in the square brackets vanishes. That is, we must satisfy the relation

$$-\gamma^{\mu}\Gamma_{\mu} + \Gamma_{\mu}\gamma^{\mu} = \gamma^{\mu}_{,\mu} + \gamma^{\mu}\Gamma^{\sigma}_{\mu\sigma} \equiv \gamma^{\mu}_{;\mu} \qquad \text{(J-6)}$$

where the quantity $\gamma^{\mu}_{;\nu} \equiv \gamma^{\mu}_{,\nu} + \gamma^{\rho}\Gamma^{\mu}_{\rho\nu}$ is just the covariant derivative of the contravariant vector γ^{μ}, and in J-6 we have summed over the indices μ and ν.*

The equation actually solved was not J-6 but the more general form

$$-\gamma^{\nu}\Gamma_{\mu} + \Gamma_{\mu}\gamma^{\nu} = \gamma^{\nu}_{;\mu} \qquad \text{(J-7)}$$

with the result**

*This is different than the notation in the literature. The literature is concerned with the ideas of spinor analysis in which the transformation properties are put into the fields $\bar{\psi}$ and ψ, and the γ's are considered a metric tensor in spin space. It is desired that the covariant derivative of the spin metric be zero (in analogy with the choice for tensor analysis), therefore the covariant derivative of a γ matrix is taken as

$$\gamma^{\mu}_{:\nu} = \gamma^{\mu}_{,\nu} + \gamma^{\rho}\Gamma^{\mu}_{\rho\nu} - \Gamma_{\nu}\gamma^{\mu} + \gamma^{\mu}\Gamma_{\nu} = 0$$

consistent with our condition J-6.

ψ and $\bar{\psi}$ are no longer scalar quantities, but spinors, and as such have a covariant derivative given by

$$\psi_{:\mu} = \psi_{,\mu} - \Gamma_{\mu}\psi\,; \qquad \bar{\psi}_{:\mu} = \bar{\psi}_{,\mu} + \bar{\psi}\Gamma_{\mu}$$

See Wheeler (4).

**See Appendix III for the solution of J-7.

$$\Gamma_\mu = \frac{1}{4}\gamma_{\alpha;\mu}\gamma^\alpha \tag{J-8}$$

where $\gamma_{\alpha;\mu} = \gamma_{\alpha,\mu} - \gamma_\rho \Gamma^\rho_{\alpha\mu}$ is the covariant derivative of the covariant vector γ. We note that a solution of J-7 is also a solution of J-6.

With the choice J-8 for Γ_μ, $\mathcal{L}^1_{eg}$ and $\mathcal{L}^2_{eg}$ differ only by a pure divergence and are thus equivalent. It is convenient for later calculations to choose a symmetric combination of the two Lagrangians $\mathcal{L}_{eg} = \frac{1}{2}(\mathcal{L}^1_{eg} + \mathcal{L}^2_{eg})$.*

$$\mathcal{L}_{eg} = (-g)^{1/2}\{ \tfrac{i}{2}\bar\psi\gamma^\mu\psi_{,\mu} - \tfrac{i}{2}\bar\psi_{,\mu}\gamma^\mu\psi - m\bar\psi\psi - i\bar\psi\frac{[\gamma^\mu\Gamma_\mu + \Gamma_\mu\gamma^\mu]}{2}\psi\} \tag{J-9}$$

where

$$\frac{\gamma^\mu\Gamma_\mu + \Gamma_\mu\gamma^\mu}{2} = \frac{1}{8}[\gamma^\mu\gamma_{\alpha;\mu}\gamma^\alpha + \gamma_{\alpha;\mu}\gamma^\alpha\gamma^\mu] \tag{J-10}$$

K. EXPANSION OF THE ELECTRON LAGRANGIAN IN TERMS OF THE FIELDS $h_{\mu\nu}$

To find the dependence of the curved space Lagrangian J-9 on the gravitational fields $h_{\mu\nu}$ we must first find an expansion of the curved space matrices that satisfy the commutation relation

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$$

To do this we shall look for a relation between the vector quantity γ^μ

*$\mathcal{L}^1_{eg}$ alone was originally used by the author, but the test of the gauge invariance of certain calculations became too difficult to carry out. The symmetric form of the interaction leads to a far simpler calculation of problems such as the scattering of gravitons by electrons.

in curved space, and the vectors $\bar{\gamma}^{a}$ in flat space.

A theorem of curved spaces is that at each point x in space time it is possible to transform from general coordinates x^i to a system $\bar{x}^i$ whose metric is flat at that point, i.e.,

$$dx^{\mu} = a^{\mu}_{a} d\bar{x}^{a} \; ; \; d\bar{x}^{\beta} = b^{\beta}_{\nu} dx^{\nu} \tag{K-1}$$

We must use a different transformation at different points in space, therefore

$$a^{\mu}_{a} = a^{\mu}_{a}(x) \quad ; \quad b^{\beta}_{\nu} = b^{\beta}_{\nu}(x)$$

At a particular point in space time we have the relations

$$ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu} = g_{\mu\nu} a^{\mu}_{a} a^{\nu}_{\beta} d\bar{x}^{a} d\bar{x}^{\beta} = \delta_{a\beta} d\bar{x}^{a} d\bar{x}^{\beta}$$

$$g_{\mu\nu} a^{\mu}_{a} a^{\nu}_{\beta} = \delta_{a\beta} \; ; \quad a^{\mu}_{a} a^{\nu}_{\beta} = \delta_{a\beta} g^{\mu\nu}$$

For this point in space the relation

$$\gamma^{\mu} = a^{\mu}_{a} \bar{\gamma}^{a}$$

implies

$$2g^{\mu\nu} = \gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = a^{\mu}_{a} a^{\nu}_{\beta} (\bar{\gamma}^{a}\bar{\gamma}^{\beta} + \bar{\gamma}^{\beta}\bar{\gamma}^{a})$$

$$= g^{\mu\nu} \delta_{a\beta} (\bar{\gamma}^{a}\bar{\gamma}^{\beta} + \bar{\gamma}^{\beta}\bar{\gamma}^{a})$$

or

$$g^{\mu\nu} (\bar{\gamma}^{a}\bar{\gamma}^{\beta} + \bar{\gamma}^{\beta}\bar{\gamma}^{a}) = 2\delta^{a\beta} g^{\mu\nu} \tag{K-2}$$

For an arbitrary metric $g_{\mu\nu}$ the solution of K-2 is

$$\bar{\gamma}^{\alpha}\bar{\gamma}^{\beta} + \bar{\gamma}^{\beta}\bar{\gamma}^{\alpha} = 2\delta^{\alpha\beta}$$

which is the correct commutation relation for the flat space γ matrices. Thus at any point in space time we may take as the relation between the curved space γ matrix γ^{μ} and the flat space γ matrices $\bar{\gamma}^{\alpha}$

$$\gamma^{\mu} = a^{\mu}_{\alpha}(x)\bar{\gamma}^{\alpha}$$

The following relations between the various γ matrices and transformation matrices may easily be verified:

$$\gamma^{\mu} = a^{\mu}_{\alpha}(x)\bar{\gamma}^{\alpha} \qquad \bar{\gamma}^{\alpha} = b^{\alpha}_{\mu}(x)\gamma^{\mu}$$

$$\gamma_{\mu} = b^{\alpha}_{\mu}(x)\bar{\gamma}_{\alpha} \qquad \bar{\gamma}_{\alpha} = a^{\mu}_{\alpha}(x)\gamma_{\mu}$$

$$a^{\mu}_{\alpha}(x)\, b^{\alpha}_{\nu}(x) = \delta^{\mu}_{\nu}$$

$$\delta^{\alpha\beta}a^{\mu}_{\alpha}(x)a^{\nu}_{\beta}(x) = g^{\mu\nu}$$

$$\delta_{\alpha\beta}b^{\alpha}_{\mu}(x)\, b^{\beta}_{\nu}(x) = g_{\mu\nu} \tag{K-3}$$

We now wish to use the point of view that a curved space may be replaced by a flat space plus gravitational fields in order to find the dependence of the quantities $a^{\mu}_{\alpha}(x)$ and $b^{\mu}_{\alpha}(x)$ on the fields $h_{\mu\nu}$. For example, the relation $\delta_{\alpha\beta}b^{\alpha}_{\mu}b^{\beta}_{\nu} = g_{\mu\nu}$ becomes

$$b_{\mu\alpha}b_{\nu\alpha} = \delta_{\mu\nu} + Kh_{\mu\nu} \tag{K-4}$$

To obtain a series expansion for $b_{\mu\nu}$ that satisfies K-4 consider the expansion

$$b = (1 + Kh)^{1/2} = 1 + \frac{K}{2}h - \frac{K^2}{8}h^2 + \frac{3K^3}{48}h^3 + \ldots \tag{K-5}$$

If indices are now put in K-5 we might believe that the correct expansion for $b_{\mu\alpha}$ is given by

$$b_{\mu\alpha} = \delta_{\mu\alpha} + \frac{K}{2} h_{\mu\alpha} - \frac{K^2}{8} h_{\mu\rho}h_{\rho\alpha} + \ldots$$

That this is indeed the correct expansion may be checked by substituting into K-4.

From the relation $a^{\mu}_{\alpha} b^{\alpha}_{\nu} = \delta^{\mu}_{\nu}$ we would expect that $a_{\mu\alpha}$ would be the reciprocal of $b_{\mu\alpha}$, or that the numerical coefficients for the expansion of $a_{\mu\alpha}$ are given by the series

$$a = (1 + Kh)^{-1/2} = 1 - \frac{K}{2}h + \frac{3K^2}{8}h^2 - \frac{15K^3}{48}h^3 + \ldots$$

This is correct and the series expansions for $a_{\mu\alpha}$ and $b_{\mu\alpha}$ are given by

$$b_{\mu\alpha} = \delta_{\mu\alpha} + \frac{K}{2} h_{\mu\alpha} - \frac{K^2}{8} h_{\mu\rho}h_{\rho\alpha} + \frac{3K^2}{48} h_{\mu\rho}h_{\rho\sigma}h_{\sigma\alpha} + \ldots$$

$$a_{\mu\alpha} = \delta_{\mu\nu} - \frac{K}{2} h_{\mu\nu} + \frac{3K^2}{8} h_{\mu\rho}h_{\rho\alpha} - \frac{15K^2}{48} h_{\mu\rho}h_{\rho\sigma}h_{\sigma\alpha} + \ldots$$

(K-6)

We note that the choice K-6 implies that the quantities $b_{\mu\alpha}$ and $a_{\mu\alpha}$ are symmetric in the indices μ and α. This is because we demanded that $b_{\mu\alpha}$ and $a_{\mu\alpha}$ be expressed as a series involving only the symmetric fields in a Lorentz invariant manner. It was thus impossible to write an antisymmetric part for $b_{\mu\alpha}$ and $a_{\mu\alpha}$.

We shall now write the curved space Lagrangian J-9 exactly in terms of the quantities $a_{\mu\alpha}$, $b_{\mu\alpha}$, and $(-g)^{1/2}$. As the expansion for these quantities in terms of $h_{\mu\nu}$ are now known, it will be merely a matter of using these expansions to the desired accuracy to obtain the Lagrangian for electrons interacting with the gravitational fields $h_{\mu\nu}$.

$\mathcal{L}_{eg}$ is given by

$$\mathcal{L}_{eg} = (-g)^{1/2} \{\tfrac{i}{2}\bar{\psi}\gamma^{\mu}\psi_{,\mu} - \tfrac{i}{2}\bar{\psi}_{,\mu}\gamma^{\mu}\psi - \tfrac{1}{8}\bar{\psi}(\gamma^{\mu}\gamma_{\alpha;\mu}\gamma^{\alpha} + \gamma_{\alpha;\mu}\gamma^{\alpha}\gamma^{\mu})\psi\}$$

In Appendix III it is shown that

$$\tfrac{1}{8}(\gamma^{\mu}\gamma_{\alpha;\mu}\gamma^{\alpha} + \gamma_{\alpha;\mu}\gamma^{\alpha}\gamma^{\mu}) = \tfrac{1}{4}\,\underline{\bar{\gamma}_{\mu}\bar{\gamma}_{\nu}\bar{\gamma}_{\rho}}\, a_{\rho\beta}a_{\alpha\nu}b_{\mu\alpha,\beta} \qquad \text{(K-7)}$$

where $\underline{\bar{\gamma}_{\mu}\bar{\gamma}_{\nu}\bar{\gamma}_{\rho}}$ is the completely antisymmetric combination of the three γ matrices $\bar{\gamma}_{\mu}$, $\bar{\gamma}_{\nu}$, and $\bar{\gamma}_{\rho}$. That is

$$\underline{\bar{\gamma}_{\mu}\bar{\gamma}_{\nu}\bar{\gamma}_{\rho}} = \tfrac{1}{6}[\bar{\gamma}_{\mu}\bar{\gamma}_{\nu}\bar{\gamma}_{\rho} - \bar{\gamma}_{\nu}\bar{\gamma}_{\mu}\bar{\gamma}_{\rho} + \bar{\gamma}_{\nu}\bar{\gamma}_{\rho}\bar{\gamma}_{\mu} - \bar{\gamma}_{\rho}\bar{\gamma}_{\nu}\bar{\gamma}_{\mu}$$

$$+ \bar{\gamma}_{\rho}\bar{\gamma}_{\mu}\bar{\gamma}_{\nu} - \bar{\gamma}_{\mu}\bar{\gamma}_{\rho}\bar{\gamma}_{\nu}] \qquad \text{(K-8)}$$

$\underline{\bar{\gamma}_{\mu}\bar{\gamma}_{\nu}\bar{\gamma}_{\rho}}$ may also be written in the form

$$\underline{\bar{\gamma}_{\mu}\bar{\gamma}_{\nu}\bar{\gamma}_{\rho}} = \epsilon_{\mu\nu\rho\sigma}\bar{\gamma}_5\bar{\gamma}_{\sigma}$$

where $\epsilon_{\mu\nu\rho\sigma}$ is the completely antisymmetric unit tensor, defined so that its components are zero unless $\mu \neq \nu \neq \rho \neq \sigma$, and equal to ± 1 according to whether $\mu\nu\rho\sigma$ is an even or odd permutation of $xyzt$. The matrix $\bar{\gamma}_5$ is defined by

$$\bar{\gamma}_5 = \bar{\gamma}_x\bar{\gamma}_y\bar{\gamma}_z\bar{\gamma}_t$$

The Lagrangian $\mathcal{L}_{eg}$ using K-7 now becomes

$$\mathcal{L}_{eg} = (-g)^{1/2}\{\frac{i}{2}(\bar{\psi}\bar{\gamma}_\alpha \bar{\psi}_{,\beta} - \bar{\psi}_{,\beta}\bar{\gamma}_\alpha \psi) a_{\alpha\beta} - m\bar{\psi}\psi$$

$$- \frac{i}{4}\bar{\psi}\,\underline{\bar{\gamma}_\mu\bar{\gamma}_\nu\bar{\gamma}_\rho}\,\psi a_{\rho\beta} a_{\alpha\nu} b_{\mu\alpha,\beta}\} \qquad \text{(K-9)}$$

where

$$(-g)^{1/2} = 1 + \frac{K}{2}h + \frac{K^2}{8}h^2 - \frac{K^2}{4}h_{\alpha\beta}h_{\alpha\beta} + \dots$$

$$a_{\alpha\beta} = \delta_{\alpha\beta} - \frac{K}{2}h_{\alpha\beta} + \frac{3K}{8}h_{\alpha\alpha}h_{\alpha\beta} + \dots$$

$$b_{\alpha\beta} = \delta_{\alpha\beta} + \frac{K}{2}h_{\alpha\beta} - \frac{K}{8}h_{\alpha\rho}h_{\rho\beta} + \dots$$

L. INVARIANCE OF THE ELECTRON LAGRANGIAN UNDER CO-ORDINATE TRANSFORMATIONS

To find the interaction of a system with gravity, the action for that system is to be rewritten as a scalar quantity in the space of metric $g_{\mu\nu}$. This prescription of general relativity is in agreement with experiment wherever it is tested, and leads to a consistent theory of gravity.

For the action to be scalar, it must be unchanged under the coordinate transformation $x^\mu \to x^{\mu'} = x^\mu - \eta^\mu$. This will be the case if the Lagrangian changes by a pure divergence under the coordinate transformation, for the pure divergence will not contribute to the action. For electrons interacting with gravity, the Lagrangian

$$\mathcal{L} = (-g)^{1/2}\{\frac{i}{2}\bar{\psi}\gamma^\mu\psi_{,\mu} - \frac{i}{2}\bar{\psi}_{,\mu}\gamma^\mu\psi - m\bar{\psi}\psi - i\bar{\psi}\,\frac{\gamma^\mu\Gamma_\mu + \Gamma_\mu\gamma^\mu}{2}\,\psi\} \qquad \text{(J-9)}$$

does change by a pure divergence when the following transformation properties are assumed.

$$\psi(x) \rightarrow \psi'(x) = \psi(x) + \psi_{,\alpha}\eta^{\alpha} \quad \text{(a)}$$

$$\bar{\psi}(x) \rightarrow \bar{\psi}'(x) = \bar{\psi}(x) + \bar{\psi}_{,\alpha}\eta^{\alpha} \quad \text{(b)}$$

$$\psi_{,\mu}(x) \rightarrow \psi'_{,\mu}(x) = [\psi'(x)]_{,\mu} \quad \text{(c)}$$

$$\bar{\psi}_{,\mu}(x) \rightarrow \bar{\psi}'_{,\mu}(x) = [\bar{\psi}'(x)]_{,\mu} \quad \text{(d)}$$

$$\gamma^{\mu}(x) \rightarrow \gamma^{\mu\prime}(x) = \gamma^{\mu}(x) + \gamma^{\mu}_{,\rho}\eta^{\rho} - \gamma^{\rho}\eta^{\mu}_{,\rho} \quad \text{(e)}$$

$$\gamma_{\mu}(x) \rightarrow \gamma'_{\mu}(x) = \gamma_{\mu}(x) + \gamma_{\mu,\rho}\eta^{\rho} + \gamma_{\rho}\eta^{\mu}_{,\rho} \quad \text{(f)}$$

$$\Gamma_{\mu}(x) \rightarrow \Gamma'_{\mu}(x) = \Gamma_{\mu}(x) + \Gamma_{\mu,\rho}\eta^{\rho} + \Gamma_{\rho}\eta^{\rho}_{,\mu} \quad \text{(g)}$$

$$g_{\mu\nu}(x) \rightarrow g'_{\mu\nu}(x) = g_{\mu\nu}(x) + g_{\mu\nu,\rho}\eta^{\rho} + g_{\rho\nu}\eta^{\rho}_{,\mu} + g_{\mu\rho}\eta^{\rho}_{,\nu} \quad \text{(h)}$$

$$(-g)^{1/2} \rightarrow (-g)^{1/2\prime} = (-g)^{1/2} + (-g)^{1/2}_{,\rho}\eta^{\rho} + (-g)^{1/2}\eta^{\rho}_{,\rho} \quad \text{(i)}$$

(L-1)

Under the transformations L-1 we get

$$\mathcal{L}(x) \rightarrow \mathcal{L}'(x) = \mathcal{L}(x) + [\eta^{\rho}\mathcal{L}(x)]_{,\rho} \quad \text{(L-2)}$$

Thus the change in the action δS is

$$\delta S = \int [\eta^{\rho}\mathcal{L}(x)]_{,\rho}\, d^4x = 0$$

and the action is invariant under the transformations L-1.

The transformations L-1 follow from the assumption, originally made in deriving the electron Lagrangian, that ψ and $\overline{\psi}$ transform as scalars, while the γ matrices transform as vectors. Since $\Gamma_\mu = \frac{1}{4}\gamma_{a;\mu}\gamma^a$, Γ_μ should likewise transform as a vector. The transformation of scalars, vectors and tensors is given in equation C-1 and the transformation of the derivatives of these quentities is given in equation D-9. The transformation of $(-g)^{1/2}$ follows from the transformation of $g_{\mu\nu}$ as may be seen in the following derivation. Landau (20) shows that the change in g is related to the change in $g_{\mu\nu}$ by the equation

$$\delta g = g g^{\mu\nu}\delta g_{\mu\nu} \tag{L-3}$$

From L-3 we get the relations

$$\delta(-g)^{1/2} = \frac{1}{2}(-g)^{1/2} g^{\mu\nu}\delta g_{\mu\nu} \tag{L-4}$$

and

$$(-g)^{1/2}_{,a} = \frac{1}{2}(-g)^{1/2} g^{\mu\nu} g_{\mu\nu,a} \tag{L-5}$$

Substituting into L-4 the quantity $\delta g_{\mu\nu}$ from L-1h we get

$$\delta(-g)^{1/2} = \frac{1}{2}(-g)^{1/2} g^{\mu\nu} g_{\mu\nu,\rho}\eta^\rho + \frac{1}{2}(-g)^{1/2} g^{\mu\nu}(g_{\mu\rho}\eta^\rho_{,\nu} + g_{\nu\rho}\eta^\rho_{,\mu})$$

$$\delta(-g)^{1/2} = (-g)^{1/2}_{,\rho}\eta^\rho + (-g)^{1/2}\eta^\rho_{,\rho}$$

Let us now consider the curved space electron Lagrangian expanded in terms of the fields $h_{\mu\nu}$. To do this we may replace the metric tensor $g_{\mu\nu}$ by $\delta_{\mu\nu} + h_{\mu\nu}$, where $\delta_{\mu\nu}$ is the Gallelian metric of flat space. We now have a flat space Lagrangian representing the interaction of electrons with the gravitational field $h_{\mu\nu}$, plus the Lagrangian of the free electron fields. When we add to the Lagrangian the quantity

$$F = \frac{1}{2K^2} R(-g)^{1/2}$$

expanded in terms of $h_{\mu\nu}$, we have the complete flat space Lagrangian for gravity interacting with electrons.

In part C4 we saw that the action for gravity in the absence of matter

$$S = \int F \, d^4x = \frac{1}{2K^2} \int R(-g)^{1/2} \, d^4x$$

was invariant under the substitution

$$Kh_{\mu\nu} \to Kh^*_{\mu\nu} = Kh_{\mu\nu} + \eta_{\mu,\nu} + \eta_{\nu,\mu} + Kh_{\mu\nu,\rho}\eta_\rho$$

$$+ Kh_{\rho\nu}\eta_{\rho,\mu} + Kh_{\mu\rho}\eta_{\rho,\nu} \qquad \text{(L-6)}$$

If we add $\delta_{\mu\nu}$ to both sides of L-6 we get

$$g_{\mu\nu}(h) \to g_{\mu\nu}(h^*) = g_{\mu\nu} + g_{\mu\nu,\rho}\eta_\rho + g_{\mu\rho}\eta_{\rho,\nu} + g_{\nu\rho}\eta_{\rho,\mu}$$

which is equation L-1h. Thus the invariance of the gravitational field under coordinate transformations in curved space corresponds to the invariance under the substitution $h_{\mu\nu} \to h^*_{\mu\nu}$ in flat space.

We now wish to find the substitutions that leave unchanged the complete flat space action for electrons interacting with the gravitational fields $h_{\mu\nu}$. The substitutions will correspond to a kind of gauge transformation of the fields $h_{\mu\nu}$ and ψ under which the theory is invariant. It is clear that if the gauge substitutions correspond to a coordinate transformation of the curved space action, as they do for the gravitational fields alone, then the flat space action will be gauge invariant.*

One possible transformation of the flat space Lagrangian would be the simultaneous substitutions

$$\psi(x) \rightarrow \psi'(x)$$

$$\bar{\psi}(x) \rightarrow \bar{\psi}'(x)$$

$$h_{\mu\nu} \rightarrow h^{*}_{\mu\nu} \qquad \text{(L-7)}$$

where $\psi'(x)$ and $\bar{\psi}'(x)$ have the same values as given in equation L-1. The substitutions L-7 automatically correspond to the transformations L-1 (a, b, c, d, h, i). If we show that the transformation of the γ matrices by equations L-7 give the same results as L-1 (e, f, g) then we will know that the action is invariant under the substitutions L-7.

All we have to show is that equation L-7 leads to the transformation

$$\gamma_{\mu}(x) \rightarrow \gamma_{\mu}(x) + \gamma_{\mu,\rho}\eta^{\rho} + \gamma_{\rho}\eta^{\rho}_{,\mu} \qquad \text{(L-1f)}$$

for then the quantities

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*It should be noted that the name "gauge transformation" has already been used to describe the substitution given by equation B-2. We shall retain the name "gauge transformation" for equation B-2 and no longer refer to $h_{\mu\nu} \rightarrow h^{*}_{\mu\nu}$ as a gauge transformation.

$$\gamma^{\mu} = g^{\mu\nu}\gamma_{\nu}$$

$$\Gamma_{\mu} = g^{\beta\gamma}{}_{;\mu}\gamma_{\beta}$$

will automatically transform by equations L-1e and L-1g. This may be checked in detail, or seen from the fact that $g_{\mu\nu}$ transforms in the same way by L-7 as by L-1.

In flat space the relation L-1f may be written

$$\gamma_{\mu}(x) = b_{\mu\alpha}\bar{\gamma}_{\alpha} \rightarrow (b_{\mu\alpha} + b_{\mu\alpha,\rho}\eta_{\rho} + b_{\rho\alpha}\eta_{\rho,\mu})\bar{\gamma}_{\alpha} \qquad \text{(L-8)}$$

where

$$b_{\mu\alpha} = \delta_{\mu\alpha} + \frac{K}{2} h_{\mu\alpha} - \frac{K^2}{8} h_{\mu\beta}h_{\beta} + \ldots$$

We have used the relation between curved and flat space γ matrices given by equations K-3 and K-6. If equation L-8 can be shown to follow from the substitution $h_{\mu\nu} \rightarrow h^{*}_{\mu\nu}$, then the substitutions L-7 will leave the Lagrangian invariant.

Under the substitution $h_{\mu\nu} \rightarrow h^{*}_{\mu\nu}$ we may directly calculate the change in $b_{\mu\alpha}$. The result to first order in K is

$$b_{\mu\alpha}(h^{*}) = b_{\mu\alpha}(h) + \frac{1}{2}\eta_{\mu,\alpha} + \frac{1}{2}\eta_{\alpha,\mu} + \frac{3K}{8} h_{\mu\rho}\eta_{\rho,\alpha} + \frac{3K}{8} h_{\alpha\rho}\eta_{\rho,\mu}$$

$$- \frac{K}{8} h_{\mu\rho}\eta_{\alpha,\rho} - \frac{K}{8} h_{\rho\alpha}\eta_{\mu,\rho} + \frac{K}{2} h_{\mu\alpha,\rho}\eta_{\rho} \qquad \text{(L-9)}$$

This expansion is rather a mess which does not directly lead to equation L-8. At this point we can say that the substitutions L-7 are not equivalent to a coordinate transformation, and, as may be checked to zero order in K, do not leave the action invariant.

We need not give up, for the following relation does hold. Defining

$$b'_{\mu\alpha} = b_{\mu\alpha} + b_{\mu\alpha,\rho}\eta_\rho + b_{\rho\alpha}\eta_{\rho,\mu} \qquad \text{(L-10)}$$

we get

$$b'_{\mu\alpha} = b_{\mu\beta}(h^*)[\delta_{\beta\alpha} + D_{\beta\alpha}] \qquad \text{(L-11)}$$

where $D_{\mu\nu}$ is an infinitesimal antisymmetric tensor given to order K^2 by

$$\begin{aligned} D_{\mu\nu} = {} & \frac{1}{4}\eta_{\nu,\alpha}[\delta_{\mu\alpha} - \frac{K}{2}h_{\mu\alpha} + \frac{K^2}{4}h_{\mu\beta}h_{\beta\alpha} + \ldots] \\ & - \frac{1}{4}\eta_{\mu,\alpha}[\delta_{\nu\alpha} - \frac{K}{2}h_{\nu\alpha} + \frac{K^2}{4}h_{\nu\beta}h_{\beta\alpha} + \ldots] \\ & + \frac{1}{4}\eta_{\alpha,\mu}[\delta_{\nu\alpha} + \frac{K}{2}h_{\nu\alpha} - \frac{K^2}{4}h_{\nu\beta}h_{\beta\alpha} + \ldots] \\ & - \frac{1}{4}\eta_{\alpha,\nu}[\delta_{\mu\alpha} + \frac{K}{2}h_{\mu\alpha} - \frac{K^2}{4}h_{\mu\beta}h_{\beta\alpha} + \ldots] \end{aligned} \qquad \text{(L-12)}$$

Suppose at the same time we make the substitution $h_{\mu\nu} \to h^*_{\mu\nu}$ we also change the representation of the flat space γ matrices, $\bar{\gamma}_\alpha \to S^{-1}\bar{\gamma}_\alpha S$. Under these simultaneous substitutions we get

$$b_{\mu\alpha}\bar{\gamma}_\alpha \to b_{\mu\alpha}(h^*)S^{-1}\bar{\gamma}_\alpha S \qquad \text{(L-13)}$$

Now suppose we can solve the relation

$$[\delta_{\beta\alpha} + D_{\beta\alpha}]\bar{\gamma}_\beta = S^{-1}\bar{\gamma}_\alpha S \qquad \text{(L-14)}$$

then equation L-13 becomes

$$b_{\mu\alpha}\bar{\gamma}_\alpha \to b_{\mu\alpha}(h^*)[\delta_{\beta\alpha} + D_{\beta\alpha}]\bar{\gamma}_\alpha \qquad \text{(L-15)}$$

$$b_{\mu\alpha}\bar{\gamma}_\alpha \rightarrow b'_{\mu\alpha}\bar{\gamma}_\alpha = (b_{\mu\alpha} + b_{\mu\alpha,\rho}\eta_\rho + b_{\rho\alpha}\eta_{\rho,\mu})\bar{\gamma}_\alpha \qquad \text{(L-16)}$$

where we have used L-11 to go from L-15 to L-16.

Equation L-16 is just the transformation property we needed for the γ matrices to maintain the invariance of the action.

The solution of L-14 for S is

$$S = [1 + \tfrac{1}{4} D_{\rho\sigma}\underline{\bar{\gamma}_\rho\bar{\gamma}_\sigma}] \qquad \text{(L-17a)}$$

where $\underline{\bar{\gamma}_\rho\bar{\gamma}_\sigma} = \frac{1}{2}(\bar{\gamma}_\rho\bar{\gamma}_\sigma - \bar{\gamma}_\sigma\bar{\gamma}_\rho)$ is the antisymmetric combination of the two γ matrices. Since $D_{\rho\sigma}$ is infinitesimal

$$S^{-1} = [1 - \tfrac{1}{4} D_{\rho\sigma}\underline{\gamma_\rho\gamma_\sigma}] \qquad \text{(L-17b)}$$

To check that S is the correct solution for L-14 we have

$$[1 - \tfrac{1}{4} D_{\rho\sigma}\underline{\bar{\gamma}_\rho\bar{\gamma}_\sigma}]\bar{\gamma}_\alpha[1 + \tfrac{1}{4} D_{\rho\sigma}\underline{\bar{\gamma}_\rho\bar{\gamma}_\sigma}]$$

$$= \bar{\gamma}_\alpha + \tfrac{1}{4} D_{\rho\sigma}(\bar{\gamma}_\alpha\underline{\bar{\gamma}_\rho\bar{\gamma}_\sigma} - \underline{\bar{\gamma}_\rho\bar{\gamma}_\sigma}\bar{\gamma}_\alpha)$$

Using the commutation relation, which may easily be checked,

$$\underline{\bar{\gamma}_\rho\bar{\gamma}_\sigma}\bar{\gamma}_\alpha = \bar{\gamma}_\alpha\underline{\bar{\gamma}_\rho\bar{\gamma}_\sigma} + 2\delta_{\alpha\sigma}\bar{\gamma}_\rho - 2\delta_{\rho\alpha}\bar{\gamma}_\sigma \qquad \text{(L-18)}$$

we get L-14.

We now have the situation where the flat space action is unchanged if we simultaneously make the substitutions

$$\psi \rightarrow \psi' \qquad h_{\mu\nu} \rightarrow h^*_{\mu\nu}$$

$$\bar{\psi} \rightarrow \bar{\psi}' \qquad \bar{\gamma}_\alpha \rightarrow S^{-1}\bar{\gamma}_\alpha S \qquad \text{(L-19)}$$

The difficulty with the substitutions L-19 is that it is convenient to work with a given simple set of γ matrices. We would rather have a set of substitutions on the field components alone that leave the action unchanged. As we shall see this may be done by redefining the transformed fields ψ' and $\overline{\psi}'$.

Let us start with the curved space Lagrangian

$$\mathcal{L} = \left(\frac{i}{2}\overline{\psi}\gamma_\rho\psi_{,\sigma} - \frac{i}{2}\overline{\psi}_{,\sigma}\gamma_\rho\psi - m\overline{\psi}\psi - i\overline{\psi}\frac{(\gamma_\rho\Gamma_\sigma + \Gamma_\sigma\gamma_\rho)}{2}\psi\right)g^{\rho\sigma}(-g)^{1/2} \tag{J-9}$$

Under a coordinate transformation this becomes

$$\mathcal{L}' = \left(\frac{i}{2}\overline{\psi}'\gamma_\rho'\psi_{,\sigma}' - \frac{i}{2}\overline{\psi}_{,\sigma}'\gamma_\rho'\psi' - m\overline{\psi}'\psi' - i\overline{\psi}'\frac{(\gamma_\rho'\Gamma_\sigma' + \Gamma_\sigma'\gamma_\rho')}{2}\psi'\right)g^{\rho\sigma'}(-g)^{1/2'} \tag{L-20}$$

Since $\mathcal{L}$ and $\mathcal{L}'$ differ by a pure divergence, they lead to the same action.

We have shown that

$$g^{\rho\sigma'} = g^{\rho\sigma}(h^*) \tag{L-21a}$$

$$(-g)^{1/2'} = (-g)^{1/2}(h^*) \tag{L-21b}$$

$$\gamma_\rho' = b_{\rho\sigma}(h^*)S^{-1}\overline{\gamma}_\sigma S \tag{L-21c}$$

We also need the relation

$$\Gamma_\sigma' = S^{-1}[\Gamma_\sigma(h^*) - S_{,\sigma}S^{-1}]S \tag{L-21d}$$

which is proved in appendix IV.

If we define

$$\psi^* = S\psi'$$
$$\overline{\psi}^* = \overline{\psi}'S^{-1} \tag{L-22}$$

then we have the relations

$$\psi' = S^{-1}\psi^* \qquad \psi'_{,\sigma} = S^{-1}(-S_{,\sigma}S^{-1})\psi^* + S^{-1}\psi^*_{,\sigma}$$

$$\overline{\psi}' = \overline{\psi}^* S \qquad \overline{\psi}'_{,\sigma} = \overline{\psi}^*(S_{,\sigma}S^{-1})S + \overline{\psi}^*_{,\sigma}S \qquad \text{(L-23)}$$

where $SS^{-1}_{,\sigma} = (-S_{,\sigma}S^{-1})$.

If we now substitute the relations L-21 and L-23 into the equation L-20 for $\mathcal{L}'$ we get

$$\mathcal{L}' = (\tfrac{i}{2}\psi^* b_{\rho a}(h^*)\overline{\gamma}_a \psi^*_{,\sigma} - \tfrac{i}{2}\psi^*_{,\sigma} b_{\rho a}(h^*)\overline{\gamma}_a\psi^* - m\overline{\psi}^*\psi$$

$$- i\overline{\psi}^* b_{\rho a}(h^*)\,\frac{[\overline{\gamma}_a\Gamma_\sigma(h^*) + \Gamma_\sigma(h^*)\overline{\gamma}_a]}{2}\,\psi^*) \qquad \text{(L-24)}$$

However equation L-24 for $\mathcal{L}'$ is exactly what we would get if we made the substitutions $\psi \to \psi^*$, $\overline{\psi} \to \overline{\psi}^*$, $h_{\mu\nu} \to h^*_{\mu\nu}$ directly in the original Lagrangian.

To summarize the preceding work, we have the following results. Under the coordinate transformation $x^\mu \to x^{\mu'} = x^\mu - \eta^\mu$ the curved space Lagrangian $\mathcal{L}$, representing the interaction of electrons with gravity, transformed into $\mathcal{L}'$ where

$$\mathcal{L}'(x) = \mathcal{L}(x) + [\eta^\rho \mathcal{L}(x)]_{,\rho}$$

If both $\mathcal{L}(x)$ and $\mathcal{L}'(x)$ are expanded in terms of the gravitational fields $h_{\mu\nu}$ we obtain $\mathcal{L}'$ from $\mathcal{L}$ by substituting $h^*_{\mu\nu}$, ψ^*, and $\overline{\psi}^*$ for $h_{\mu\nu}$, ψ and $\overline{\psi}$ in $\mathcal{L}$. That is

$$\mathcal{L}(h^*_{\mu\nu}, \psi^*, \overline{\psi}^*) = \mathcal{L}'(h_{\mu\nu}, \psi, \overline{\psi}) \qquad \text{(L-25)}$$

Since $\mathcal{L}$ and $\mathcal{L}'$ differ by a pure divergence, the action remains unchanged under the above substitution for the fields $h_{\mu\nu}$, ψ and $\overline{\psi}$.

Explicitly, the substitution that leaves the action invariant is given by

$$h_{\mu\nu} \rightarrow h_{\mu\nu}(x) + \eta_{\mu,\nu} + \eta_{\nu,\mu} + h_{\mu\nu,\rho}\eta_\rho + h_{\mu\rho}\eta_{\rho,\nu} + h_{\nu\rho}\eta_{\rho,\mu} \qquad \text{(a)}$$

$$\psi(x) \rightarrow [1 + \tfrac{1}{4} D_{\rho\sigma}\underline{\overline{\gamma}_\rho\overline{\gamma}_\sigma}][\psi(x) + \psi_{,\alpha}\eta_\alpha] \qquad \text{(b)}$$

$$\overline{\psi}(x) \rightarrow [\overline{\psi}(x) + \overline{\psi}_{,\alpha}\eta_\alpha][1 - \tfrac{1}{4} D_{\rho\sigma}\underline{\overline{\gamma}_\rho\overline{\gamma}_\sigma}] \qquad \text{(c)}$$

(L-26)

where $D_{\rho\sigma}$ is an infinitesimal antisymmetric matrix, given as a series in the coupling constant K to order K^2 by equation L-12.

The work in this part to find the substitutions L-26 was carried out because the author originally had considerable difficulty in checking the calculation of the scattering of gravitons by electrons. It was noted that the action was invariant under the substitutions L-1 in which

$$\gamma_\mu \rightarrow \gamma_\mu + \gamma_{\mu,\rho}\eta^\rho + \gamma_\rho\eta^\rho_{,\mu} \qquad \text{(L-1f)}$$

But L-1f seemed to imply that the transformation of $b_{\mu\alpha}$ should be given by

$$b_{\mu\alpha} \rightarrow b'_{\mu\alpha} = b_{\mu\alpha} + b_{\mu\alpha,\rho}\eta^\rho + b_{\rho\alpha}\eta_{\rho,\mu} \qquad \text{(L-27)}$$

The difficulty with L-27 is that it does not treat the subscripts μ and α symmetrically, while the author's expansion for $b_{\mu\alpha}$ in terms of the fields $h_{\mu\nu}$ makes $b_{\mu\alpha}$ symmetric. This suggested that the difficulty in checking calculations based on the expansions K-6 was due to an error in the expansions themselves.

It was then that the author found the substitutions L-26 that leave the action invariant when the expansions K-6 are used. By a rather lengthy calculation it was checked to order $Kh\eta$ that the substitutions L-26 did in fact leave the action unchanged. The difficulty of checking calculations was later solved by following a suggestion of Dr. Feynman's that a symmetric form for the interaction of electrons with gravity be used.

From the existence of the substitution L-26 and from the check of later calculations, the author suggests that there is good evidence for the validity of the electron-graviton Lagrangian K-9. The only difficulties are that no Lagrangian involving gravity is unique as has been discussed in part F, and there are no experiments to check the theory.

V. QUANTUM MECHANICS OF THE INTERACTION OF GRAVITY WITH ELECTRONS

In this section we shall deal with the quantum mechanics of gravity interacting with electrons. The first part will be an investigation of the gravitational Dirac equation to find the effects of the spin of the electron. This will be done by comparing the Dirac equation with the equation for a spin-zero particle interacting with gravity.

In the next part, part N, we will show how to write Feynman diagrams for the case of gravity interacting with electrons. The emmision of low frequency gravitons will be discussed as a simple example of the diagrams.

In the final part we give the calculation of the scattering of gravitons by electrons. This calculation to lowest order includes the nonlinear effects of the gravitational field. The check of this calculation for gauge invariance offers the best proof of the correctness of the electron Lagrangian in the expanded form given by the author.

We will not discuss radiative corrections to these calculations until the next section, and even there the discussion of radiative correction will not be complete.

M. GRAVITATIONAL DIRAC EQUATION - EFFECTS OF THE SPIN OF THE ELECTRON

The gravitational Dirac equation may be obtained most easily from the Lagrangian $\mathcal{L}_{eg}$ of equation J-2.

$$\mathcal{L}_{eg} = (-g)^{1/2} \{ i\bar{\psi}\gamma^{\mu}[\psi_{,\mu} - \Gamma_{\mu}\psi] - m\bar{\psi}\psi \}$$

The equation of motion is

$$\gamma^{\mu}(i\nabla_{\mu} - i\Gamma_{\mu})\psi = m\psi \qquad \text{(M-1)}$$

To find the effect of the spin of the electron we wish to compare M-1 with the equation of motion for a spin-zero particle interacting with gravity. The Lagrangian for a spin-zero particle of mass m may be written

$$\mathcal{L}_m = \frac{1}{2}(\varphi_{,\mu}\varphi_{,\mu} - m^2\varphi^2) \qquad \text{(M-2)}$$

A general curved space Lagrangian that reduces to M-2 in the limit $g_{\mu\nu} \rightarrow \delta_{\mu\nu}$ is

$$\mathcal{L}_{mg} = \frac{1}{2}(-g)^{1/2}[g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu} - (m^2 + \alpha R)\varphi^2] \qquad \text{(M-3)}$$

where α is an arbitrary constant.

The gravitational Klein-Gordon equation corresponding to M-3 is

$$-g^{\mu\nu}\varphi_{,\mu;\nu} - \alpha R\varphi = m^2\varphi$$

or

$$[-g^{\mu\nu}(\nabla_{\mu}\nabla_{\nu} - \Gamma^{\rho}_{\mu\nu}\nabla_{\rho}) - \alpha R]\varphi = m^2\varphi \qquad \text{(M-4)}$$

We could find the effects of the spin of the electron, at least in the nonrelativistic linear limit by directly expanding equations M-1 and M-4, adjusting α so that the equations are as similar as possible. The extra terms that appear in the expansion of M-1 will then be due to the spin of the electron. The author has attempted this but the calculation is extremely cumbersome.

For a more direct method of finding the spin of the electron, let us first consider the interaction of electromagnetic fields with electrons. The Dirac equation is

$$\underline{\gamma}_\mu(i\nabla_\mu - eA_\mu)\psi = m\psi \tag{M-5}$$

while the spin-zero Klein-Gordon equation is

$$(i\nabla_\mu - eA_\mu)(i\nabla_\mu - eA_\mu)\varphi = m^2\varphi \tag{M-6}$$

If we operate with $\overline{\gamma}_\mu(i\nabla_\mu - eA_\mu)$ on both sides of M-5 we get

$$[\overline{\gamma}_\mu(i\nabla_\mu - eA_\mu)\overline{\gamma}_\nu(i\nabla_\nu - eA_\nu)] = m\gamma_\mu(i\nabla_\mu - eA_\mu)\psi = m^2\psi \tag{M-7}$$

Equation M-7 reduces to

$$(i\nabla_\mu - eA_\mu)(i\nabla_\mu - eA_\mu)\psi - \frac{ie}{2}\,\underline{\overline{\gamma}_\mu\overline{\gamma}_\nu}F_{\mu\nu}\psi = m^2\psi \tag{M-8}$$

where

$$\underline{\overline{\gamma}_\mu\overline{\gamma}_\nu} = \frac{1}{2}(\overline{\gamma}_\mu\overline{\gamma}_\nu - \overline{\gamma}_\nu\overline{\gamma}_\mu)$$

Comparing M-6 with M-8 we see that the effects of the spin, such as the Dirac magnetic moment and spin orbit coupling, arise from the term

$$-\frac{ie}{2}\,\underline{\overline{\gamma}_\mu\overline{\gamma}_\nu}\,F_{\mu\nu}\psi \tag{M-8a}$$

Let us use the same technique on the gravitational Dirac equation to find the part of the interaction that is due to the spin of the electron. We have

$$i(\gamma^\beta\nabla_\beta - \gamma^\beta\Gamma_\beta)\psi = m\psi$$

Operating again with the quantity $i(\gamma^\alpha\nabla_\alpha - \gamma^\alpha\Gamma_\alpha)$ we get

$$(i\gamma^\alpha\nabla_\alpha - i\gamma^\alpha\Gamma_\alpha)(i\gamma^\beta\nabla_\beta - i\gamma^\beta\Gamma_\beta)\psi = m^2\psi \qquad \text{(M-9)}$$

M-9 becomes

$$(-\gamma^\alpha\nabla_\alpha\gamma^\beta\nabla_\beta + \gamma^\alpha\nabla_\alpha\gamma^\beta\Gamma_\beta + \gamma^\alpha\Gamma_\alpha\gamma^\beta\nabla_\beta - \gamma^\alpha\Gamma_\alpha\gamma^\beta\Gamma_\beta)\psi$$
$$= m^2\psi \qquad \text{(M-10)}$$

Now

$$-\gamma^\alpha\nabla_\alpha\gamma^\beta\nabla_\beta = -\gamma^\alpha\gamma^\beta\nabla_\alpha\nabla_\beta - \gamma^\alpha\gamma^\beta_{,\alpha}\nabla_\beta$$

But

$$\gamma^\beta_{,\alpha} = \gamma^\beta_{;\alpha} - \Gamma^\beta_{\alpha\rho}\gamma^\rho = \Gamma_\alpha\gamma^\beta - \gamma^\beta\Gamma_\alpha - \Gamma^\beta_{\alpha\rho}\gamma^\rho$$

where by equation J-7

$$\gamma^\beta_{;\alpha} = \Gamma_\alpha\gamma^\beta - \gamma^\beta\Gamma_\alpha$$

Also

$$\gamma^\alpha\nabla_\alpha\gamma^\beta\Gamma_\beta = \gamma^\alpha\gamma^\beta\Gamma_\beta\Gamma_\alpha + \gamma^\alpha(\gamma^\beta\Gamma_\beta)_{,\alpha}$$

Since $\gamma^\beta\Gamma_\beta$ is a scalar quantity

$$\gamma^\alpha(\gamma^\beta\Gamma_\beta)_{,\alpha} = \gamma^\alpha(\gamma^\beta_{;\alpha}\Gamma_\beta + \gamma^\beta\Gamma_{\beta;\alpha})$$
$$= \gamma^\alpha\Gamma_\alpha\gamma^\beta\Gamma_\beta - \gamma^\alpha\gamma^\beta\Gamma_\alpha\Gamma_\beta + \gamma^\alpha\gamma^\beta\Gamma_{\beta;\alpha}$$

where we again used J-7 for $\gamma^\beta_{;\alpha}$.

Putting these relations together equation M-10 becomes

$$\{-g^{\alpha\beta}[\nabla_\alpha\nabla_\beta - \Gamma^\rho_{\alpha\beta}\nabla_\rho] + g^{\alpha\beta}[2\Gamma_\beta\nabla_\alpha + \Gamma_{\alpha;\beta} - \Gamma_\alpha\Gamma_\beta]$$
$$- \underline{\gamma^\alpha\gamma^\beta}(\Gamma_{\alpha;\beta} + \Gamma_\alpha\Gamma_\beta)\}\psi = m^2\psi \qquad \text{(M-11)}$$

where we have split $\gamma^{a}\gamma^{\beta}$ into symmetric and antisymmetric parts

$$\gamma^{a}\gamma^{\beta} = g^{a\beta} + \underline{\gamma^{a}\gamma^{\beta}}$$

In appendix V we show that

$$-\underline{\gamma^{a}\gamma^{\beta}}(\Gamma_{a;\beta} + \Gamma_{a}\Gamma_{\beta}) = -\frac{1}{4}R \qquad \text{(M-12)}$$

The gravitational Dirac equation M-11 becomes

$$\{-g^{a\beta}[\nabla_{a}\nabla_{\beta} - \Gamma^{\rho}_{a\beta}\nabla_{\beta}] - \frac{1}{4}R\}\psi + g^{a\beta}[2\Gamma_{a}\nabla_{\beta} + \Gamma_{a;\beta} - \Gamma_{a}\Gamma_{\beta}]\psi$$
$$= m^{2}\psi$$

For the choice $a = \frac{1}{4}$ the spin-zero equation M-4 becomes

$$\{-g^{a\beta}[\nabla_{a}\nabla_{\beta} - \Gamma^{\rho}_{a\beta}] - \frac{1}{4}R\}\varphi = m^{2}\varphi \qquad \text{(M-14)}$$

By comparing equations M-13 and M-14 we see that we are left with the additional term in the Dirac equation

$$g^{a\beta}[2\Gamma_{a}\nabla_{\beta} + \Gamma_{a;\beta}]\psi \qquad \text{(M-15a)}$$

$$-g^{a\beta}\Gamma_{a}\Gamma_{\beta}\psi \qquad \text{(M-15b)}$$

To study the Dirac terms M-15, we note that the matrix Γ_{a} is given by appendix III, equations 13 and 14;

$$\Gamma_{a} = \frac{1}{4}(b_{\nu\lambda,a}b_{\lambda\mu} + \Gamma_{\nu,a\mu})\underline{\gamma^{\mu}\gamma^{\nu}} \qquad \text{(M-16a)}$$

or

$$\Gamma_{a} = \frac{1}{4}a_{\lambda\nu}(b_{\lambda\mu,a} + \Gamma_{\lambda,a\delta}a_{\delta\mu})\underline{\overline{\gamma}_{\mu}\overline{\gamma}_{\nu}} \qquad \text{(M-16b)}$$

Using equation M-16b we see that the Dirac term M-15a is purely a spin term.

$$g^{\alpha\beta}[\,2\Gamma_{\alpha}\nabla_{\beta} + \Gamma_{\alpha;\beta}] = A_{\mu\nu}\,\underline{\overline{\gamma}_{\mu}\overline{\gamma}_{\nu}} \qquad \text{(M-17)}$$

where

$$A_{\mu\nu} = \frac{1}{4}\, g^{\alpha\beta}\,\{2a_{\lambda\nu}(b_{\lambda\mu,\alpha} + \Gamma_{\lambda,\,\alpha\delta}a_{\delta\mu})\nabla_{\beta}$$

$$+ [\,a_{\lambda\nu}(b_{\lambda\mu,\,\alpha} + \Gamma_{\lambda,\,\alpha\delta}a_{\delta\mu})]_{,\,\beta}$$

$$- \Gamma^{\rho}_{\alpha\beta}a_{\lambda\nu}(b_{\lambda\rho,\,\alpha} + \Gamma_{\lambda,\,\alpha\delta}a_{\delta\mu})\,\} \qquad \text{(M-17a)}$$

To lowest order in K the Dirac term M-15a reduces to

$$g^{\alpha\beta}[\,2\Gamma_{\alpha}\nabla_{\beta} + \Gamma_{\alpha;\beta}]\psi$$

$$= \frac{K}{4}\,[\,\nabla_{\alpha}h_{\nu\alpha,\,\mu} + h_{\nu\alpha,\,\mu}\nabla_{\alpha}]\,\underline{\overline{\gamma}_{\mu}\overline{\gamma}_{\nu}}\,\psi \qquad \text{(M-18)}$$

which is in close analogy to the electromagnetic term M-8a.

The Dirac term M-15b is more difficult to handle and harder to interpret. Using equation M-16a for Γ_{α} we get

$$-g^{\alpha\beta}\Gamma_{\alpha}\Gamma_{\beta}\psi = -\frac{1}{2}g^{\alpha\beta}(\Gamma_{\alpha}\Gamma_{\beta} - \Gamma_{\beta}\Gamma_{\alpha})\psi$$

$$= -\frac{1}{16}\,g^{\alpha\beta}(b_{\lambda\nu,\,\alpha}b_{\lambda\mu} + \Gamma_{\nu,\,\alpha\mu})(b_{\sigma\delta,\,\beta}b_{\delta\rho} + \Gamma_{\sigma,\,\beta\rho})$$

$$\times\ \frac{1}{2}\,[\,\underline{\gamma^{\mu}\gamma^{\nu}}\,\underline{\gamma^{\rho}\gamma^{\sigma}} + \underline{\gamma^{\rho}\gamma^{\sigma}}\,\underline{\gamma^{\mu}\gamma^{\nu}}\,]$$

From appendix II, equation 14, we have

$$\frac{1}{2}[\underline{\gamma^\mu\gamma^\nu}\,\underline{\gamma^\rho\gamma^\sigma} + \underline{\gamma^\rho\gamma^\sigma}\,\underline{\gamma^\mu\gamma^\nu}]$$

$$= \underline{\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma} + g^{\mu\sigma}g^{\nu\rho} - g^{\mu\rho}g^{\nu\sigma}$$

Writing

$$\underline{\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma} = a_{\mu\xi}a_{\nu\eta}a_{\rho\gamma}a_{\sigma\delta}\,\underline{\overline{\gamma}_\xi\overline{\gamma}_\eta\overline{\gamma}_\gamma\overline{\gamma}_\delta}$$

and noting that

$$\underline{\overline{\gamma}_\xi\overline{\gamma}_\eta\overline{\gamma}_\gamma\overline{\gamma}_\delta} = \epsilon_{\xi\eta\gamma\delta}\overline{\gamma}_5$$

we get for the term M-15b,

$$-g^{\alpha\beta}\Gamma_\alpha\Gamma_\beta\psi = B\,\overline{\gamma}_5\psi + C\psi \tag{M-19}$$

where

$$B = -\frac{1}{16}g^{\alpha\beta}(b_{\nu\lambda,\,\alpha}b_{\lambda\mu} + \Gamma_{\nu,\,\alpha\mu})(b_{\sigma\delta,\beta}b_{\delta\rho} + \Gamma_{\sigma,\,\beta\rho})$$

$$\times\, a_{\mu\xi}a_{\nu\eta}a_{\rho\gamma}a_{\sigma\delta}\epsilon_{\xi\eta\lambda\delta} \tag{M-19a}$$

$$C = \frac{1}{4}(g^{\alpha\beta}g^{\mu\rho}g^{\nu\sigma} - g^{\alpha\beta}g^{\mu\sigma}g^{\nu\rho})$$

$$\times\,(b_{\nu\lambda,\,\alpha}b_{\lambda\mu} + \Gamma_{\nu,\,\alpha\mu})(b_{\sigma\delta,\,\beta}b_{\delta\rho} + \Gamma_{\sigma,\,\beta\rho}) \tag{M-19b}$$

To lowest order in K M-15b becomes

$$-g^{\alpha\beta}\Gamma_\alpha\Gamma_\beta = -\frac{K^2}{16}h_{\nu\alpha,\,\mu}h_{\sigma\alpha,\,\rho}\epsilon_{\mu\nu\rho\sigma}\overline{\gamma}_5\psi$$

$$-\frac{K^2}{2}(h_{\nu\alpha,\,\mu}h_{\nu\alpha,\,\mu} - h_{\nu\alpha,\,\mu}h_{\mu\alpha,\,\nu})\psi \tag{M-20}$$

At first it appears that the quantity $C\psi$ may not be a spin term for it involves no γ matrices. We might try to include $C\psi$ in the spin-zero part of the equation by adding the term

$$(-g)^{1/2}C\varphi^2 \tag{M-21}$$

to the Lagrangian M-3 for the spin-zero field. This would just lead to the equation

$$[-g^{\mu\nu}(\nabla_\mu\nabla_\nu - \Gamma^\rho_{\mu\nu}\nabla_\rho) - \tfrac{1}{4}R + C]\varphi = m^2\varphi \tag{M-22}$$

for a spin-zero field, and $C\psi$ would not be considered as arising due to the spin of the electron.

This is not the correct interpretation of the term $C\psi$ as may be seen by investigating the transformation properties of the term $(-g)^{1/2}C\varphi^2$ which we were supposed to add to the spin-zero Lagrangian. If we suppose that under a coordinate transformation

$$C \rightarrow C + \delta C$$

then from the transformation of $(-g)^{1/2}$ and the scalar field φ given in equations L-1

$$(-g)^{1/2} \rightarrow (-g)^{1/2} + (-g)^{1/2}_{,\rho}\eta^\rho + (-g)^{1/2}\eta^\rho_{,\rho}$$

$$\varphi \rightarrow \varphi + \varphi_{,\rho}\eta^\rho$$

$$\varphi^2 \rightarrow \varphi^2 + 2\varphi\varphi_{,\rho}\eta^\rho = \varphi^2 + \varphi^2_{,\rho}\eta^\rho$$

the transformation of $(-g)^{1/2}C\varphi^2$ is

$$(-g)^{1/2}C\varphi^2 \rightarrow (-g)^{1/2}C\varphi^2 + [\eta^\rho(-g)^{1/2}C\varphi^2]_{,\rho}$$

$$+ (-g)^{1/2}\varphi^2(\delta C - C_{,\rho}\eta^\rho) \tag{M-23}$$

Since the change in $(-g)^{1/2}C\varphi^2$ must be a pure divergence in order that the action be unchanged under coordinate transformations, we must have

$$\delta C = C_{,\rho}\eta^{\rho} \qquad \text{(M-24)}$$

To lowest order in K, C is given from equation M-20

$$C = -\frac{K^2}{2}(h_{\nu\alpha,\mu}h_{\nu\alpha,\mu} - h_{\nu\alpha,\mu}h_{\mu\alpha,\nu})$$

To lowest order in K the transformation of the fields $h_{\mu\nu}$ is given by equation L-6

$$Kh_{\mu\nu} \rightarrow Kh_{\mu\nu} + \eta_{\mu,\nu} + \eta_{\nu,\mu} \qquad \text{(M-25)}$$

Using M-25 we get the change in C, to lowest order in K, to be

$$\delta C = 2Kh_{\nu\alpha,\mu}(\eta_{\nu,\mu} - \eta_{\mu,\nu})_{,\alpha} \qquad \text{(M-26)}$$

δC is an order of K larger than $C_{,\rho}\eta^{\rho}$ and does not satisfy equation M-26. Thus if the quantity $(-g)^{1/2}C\varphi^2$ were added to the curved space Lagrangian for a spin-zero field, the resulting action would no longer be invariant under coordinate transformations. We therefore conclude that the appearance of the term $C\psi$ in the Dirac equation is due exclusively to the properties of the spin of the electron.

Collecting the results of the last few pages, we can write the gravitational Dirac equation in the form

$$\{-g^{\alpha\beta}[\nabla_\alpha\nabla_\beta - \Gamma^{\rho}_{\alpha\beta}\nabla_\rho] - \frac{1}{4}R\}\psi$$
$$+ \quad A_{\mu\nu}\underline{\overline{\gamma}_\mu\overline{\gamma}_\nu} + B\overline{\gamma}_5 + C\}\psi = m^2\psi \qquad \text{(M-27)}$$

where $A_{\mu\nu}$, B and C are given in equations M-17a, M-19a, and M-19b respectively. Comparing with a possible equation for a spin-zero field

$$\{-g^{\alpha\beta}[\nabla_\alpha\nabla_\beta - \Gamma^\rho_{\alpha\beta}\nabla_\rho] - \frac{1}{4}R\}\varphi = m^2\varphi \qquad \text{(M-14)}$$

we conclude that the terms

$$\{A_{\mu\nu}\overline{\gamma}_\mu\overline{\gamma}_\nu + B\overline{\gamma}_5 + C\}\psi \qquad \text{(M-27a)}$$

arise due to the effects of the spin of the electron.

To order K, or in the linear approximation, only the term $A_{\mu\nu}$ remains in M-27a and the spin term is given by

$$\frac{K}{4}[\nabla_\alpha h_{\nu\alpha,\mu} + h_{\nu\alpha,\mu}\nabla_\alpha]\ \underline{\overline{\gamma}_\mu\overline{\gamma}_\nu}\ \psi \qquad \text{(M-18)}$$

The terms $\{B\overline{\gamma}_5 + C\}\psi$ are of order K^2, and to that order are given by

$$-K^2[\frac{1}{16}(h_{\alpha\nu,\mu}h_{\sigma\alpha,\rho}\epsilon_{\mu\nu\rho\sigma})\overline{\gamma}_5 - \frac{1}{2}(h_{\nu\alpha,\mu}h_{\nu\alpha,\mu} - h_{\nu\alpha,\mu}h_{\mu\alpha,\nu})]\psi$$

Finally, the linear approximation to equation M-27 is given by

$$(-\nabla^2 + Kh_{\mu\nu}\nabla_\mu\nabla_\nu + Kh_{\mu\nu,\mu}\nabla_\nu - \frac{K}{2}h_{,\mu}\nabla_\mu$$

$$+ \frac{K}{4}h_{\rho\sigma,\rho\sigma} - \frac{K}{4}h_{,\sigma\sigma})\psi$$

$$+ \frac{K}{4}(2h_{\alpha\nu,\mu}\nabla_\alpha + h_{\alpha\nu,\alpha\mu})\underline{\overline{\gamma}_\mu\overline{\gamma}_\nu}\psi = m^2\psi \qquad \text{(M-28)}$$

M1. Dirac Equation for an Electron Moving in the Gravitational Field of a Point Mass - Nonrelativistic Limit

For a more detailed investigation of the effects of the spin of the electron, it is instructive to consider the nonrelativistic limit of the linear gravitational Dirac equation. The fact that we are dealing with the linear equation and can neglect terms that involve the product of two gravitational field components allows us to proceed to the non-relativistic limit in about the same way as for the electromagnetic Dirac equation. The only difficulty is that the gravitational field has so many components that the result is very long and would require an elaborate investigation of the physical effects.

Rather than consider the electron moving in an arbitrary gravitational field, the results are far simpler if the electron moves in the field of a stationary point mass. Such fields are experimentally observed and there are still effects due to the spin of the electron.

We may use either the first order form of the Dirac equation M-1, or the second order equation M-28. It turns out to be slightly easier to handle the first order form. The linear expansion of M-1 is

$$\left[i\gamma_\mu \nabla_\mu - \frac{i}{2} h_{\mu\nu}\gamma_\nu\nabla_\mu - \frac{i}{4} h_{\mu\nu,\mu}\gamma_\nu + \frac{i}{4} h_{,\mu}\gamma_\mu \right] \psi = m\psi \qquad \text{(M1-1)}$$

The gravitational field arising from a stationary point mass is given by equation B2-18

$$Kh_{11} = Kh_{22} = Kh_{33} = Kh_{44} = 2\phi = -2\frac{GM}{r}$$

where ϕ is the Newtonian potential. Using these fields the three dimensional form of equation M1-1 becomes

$$[i\gamma_i\nabla_i + i\gamma_t\nabla_t + i\phi\gamma_i\nabla_i - i\phi\gamma_t\nabla_t - \frac{i}{2}\phi_{,i}\gamma_i]\psi = m\psi \qquad \text{(M1-2)}$$

where the latin letters stand for three dimensional quantities. For example

$$\nabla_i = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$$

$$\nabla_i\nabla_i = (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})$$

$$\nabla_t = \frac{\partial}{\partial t} \;;\; \phi_{,i} = \nabla_i\phi \qquad \text{(M-3)}$$

Multiplying M1-2 through by γ_t we get

$$[(1-\phi)E - (1+\phi)\alpha_i p_i - \frac{i}{2}\phi_{,i}\alpha_i - \gamma_t m]\psi = 0 \qquad \text{(M1-4)}$$

where

$$\alpha_i = \gamma_t\gamma_i$$

and we have defined the operators p_i and E by

$$p_i = -i\nabla_i$$

$$E = i\nabla_t \qquad \text{(M1-5)}$$

If ψ is a stationary state, E is no longer an operator, and

$$E = M + V \qquad \text{(M1-6)}$$

where V is the kinetic energy of the electron. (Since $\nabla_t\phi = 0$, E will not operate on ϕ.)

In terms of the two component electron wave functions ψ_a and ψ_b and the Pauli spinors σ,

$$\psi = \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix}; \quad \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$$

equation M1-4 becomes the two equations

$$(1-\phi)E\psi_a - m\psi_a - \sigma \cdot \pi\psi_b = 0 \qquad \text{(M1-6a)}$$

$$(1-\phi)E\psi_b + m\psi_b - \sigma \cdot \pi\psi_a = 0 \qquad \text{(M1-6b)}$$

where

$$\pi_i = (1 + \phi)p_i + \frac{i}{2}\phi_{,i} \qquad \text{(M1-7)}$$

Using $E = M + W$ equations M1-6 become

$$(W - E\phi)\psi_a = \sigma \cdot \pi\psi_b \qquad \text{(M1-8a)}$$

$$\psi_b = \frac{1}{W - E\phi + 2m}\sigma \cdot \pi\psi_a \qquad \text{(M1-8b)}$$

Substituting for ψ_b in equation M1-8a we get

$$W\psi_a = \sigma \cdot \pi \frac{1}{W - E\phi + 2m} \sigma \cdot \pi\psi_a + E\phi\psi_a \qquad \text{(M1-9)}$$

This is the same form as for electromagnetism, with the electric potential V replaced by $E\phi$, and the momentum operator

$$\pi_i = (p_i - eA_i)$$

replaced by

$$\pi_i = [(1 + \phi)p_i + \frac{i}{2}\phi_{,i}]$$

Due to the similarity of the electromagnetic and gravitational equations, we expect similar effects such as spin orbit coupling.

We shall expand equation M1-9 to order v^4. Assuming that the kinetic and potential energy of the electron are of the same order of magnitude as in the gravitational hydrogen atom, we have

$$m\phi \sim \frac{p^2}{2m} \sim v^2$$

or we shall take ϕ to be of the order of v^2. Expanding the fraction in M1-8b we get

$$\frac{1}{W - E\phi + 2m} \approx \frac{1}{2m} - \frac{W - E\phi}{4m^2} + \ldots$$

Equation M1-9 becomes to order v^4

$$W\psi_a = \{M\phi + W\phi + \frac{(\sigma\cdot\pi)^2}{2m} - \frac{1}{4m}\sigma\cdot\pi(W - E\phi)\sigma\cdot\pi\}\psi_a \qquad \text{(M1-10)}$$

Using the relation

$$(\sigma\cdot A)(\sigma\cdot B) = A\cdot B + i\sigma\cdot(A \times B) \qquad \text{(M1-11)}$$

we get

$$(\sigma\cdot\pi)(\sigma\cdot\pi) = \pi\cdot\pi + i\sigma\cdot(\pi \times \pi)$$

Since ϕ is a $1/r$ potential $\phi_{,ii} = 0$ and

$$i\sigma\cdot(\pi \times \pi)\psi_a = 0$$

$$\pi\cdot\pi = (1 + 2\phi)p^2$$

or

$$(\sigma\cdot\pi)^2 = (1 + 2\phi)p^2 \qquad \text{(M1-12)}$$

To lowest order equation M1-10 is

$$W\psi_a = [\frac{p^2}{2m} + m\phi]\psi_a \qquad \text{(M-13)}$$

Using M-13 to replace W on the right side of M1-10, taking care that W is treated as a number, we get to order v^4

$$W\psi_a = \{M\phi + \phi\frac{p^2}{2m} + (1 + 2\phi)\frac{p^2}{2m} - (\frac{p^2}{8m^2} + \frac{\phi p^2}{4m})$$

$$+ \frac{1}{4m}(\sigma \cdot p)\phi(\sigma \cdot p)\}\psi_a \qquad \text{(M1-14)}$$

Now the correct normalization of the wave functions is

$$\int \psi^* \psi \, d^3x = 1$$

or

$$\int (|\psi_a|^2 + |\psi_b|^2) d^3x = 1$$

From equation M1-8b to lowest order in v we have

$$\psi_b = \frac{1}{2m}\sigma \cdot p\psi_a \; ; \qquad \psi_b^2 = \frac{1}{4m^2}(\sigma \cdot p)^2\psi_a^2$$

or

$$\psi_b^2 = \frac{p^2}{4m^2}\psi_a^2 + \text{terms of order } v^4$$

The normalizing integral is now

$$\int \psi_a^* [1 + \frac{p^2}{4m^2}]\psi_a \, d^3x = 1$$

By the substitution

$$\chi = [1 + \frac{p^2}{8m^2}]\psi_a \quad \text{or} \quad \psi_a = [1 - \frac{p^2}{8m^2}]\chi \qquad \text{(M1-15)}$$

the normalization integral is simply

$$\int \chi^* \chi \, d^3x = 1 \qquad . \qquad \text{(M1-16)}$$

Substituting $[1 - p^2/8m^2]\chi$ for ψ_a in equation M1-14, and noting that to lowest order

$$W \frac{p^2}{8m^2}\chi = -\frac{p^4}{8m^2}\chi - m\phi\frac{p^2}{8m^2}\chi$$

we get no change in the form of M1-14. Thus we may consider ψ_a normalized to one, or replace ψ_a by χ in M1-14.

The term $1/4m(\sigma \cdot p)\phi(\sigma \cdot p)\chi$ may be written

$$\frac{1}{4m}\phi(\sigma \cdot p)^2\chi - \frac{i}{4m}(\sigma \cdot (\vec{\nabla}\phi)\sigma \cdot p)\chi$$

By equation M1-11 this becomes

$$\{\frac{\phi}{4m}p^2 - \frac{i}{4m}(\vec{\nabla}\phi) \cdot p + \frac{1}{4m}\sigma \cdot [(\vec{\nabla}\phi) \times p]\}\chi$$

Substituting this into equation M1-14 we get

$$W\chi = (\frac{p^2}{2m} - \frac{p^4}{8m^3})\chi \qquad \text{(a)}$$

$$+ \phi(m + \frac{3p^2}{2m})\chi \qquad \text{(b)}$$

$$+ \frac{1}{4m}\sigma \cdot [(\vec{\nabla}\phi) \times p]\chi \qquad \text{(c)}$$

$$- \frac{i}{4m}(\vec{\nabla}\phi) \cdot p\chi \qquad \text{(d)}$$

(M1-17)

where

$$\int \chi^* \chi \, d^3x = 1$$

The interpretation of the terms in M1-17 is as follows. The terms (a) and (b) are independent of the spin of the particle. In fact if we dropped (c) and (d) we would just have the equation for a spin-zero particle as may be checked from equation M-4.

(a) is just the kinetic energy of the particle, which is $p^2/2m$ plus the relativistic correction $-p^4/8m^3$. The first part of (b), namely $m\phi$ is just the Newtonian potential energy. To interpret all of the term (b) we note that to this order the inertial mass m_i is just $m(1+v^2/2)$. In terms of the inertial mass (b) may be written, to order v^4

$$\phi m_i(1 + v^2)$$

Thus we see the extra factor of $(1+v^2)$ which appeared in the classical mechanics of the linear theory (see equation B2-25). It is this factor that accounts for the deflection of light moving past the sun.

We should note that we obtained M1-17 from a linear approximation to the gravitational Dirac equation, and therefore did not keep any factors of ϕ^2 in going from M1-2 to M1-17. However in approximating M1-9 we assumed that the kinetic energy and gravitational potential energy were of the same order of magnitude, or that ϕ was of order v^2. Thus in dropping terms with a factor ϕ^2 we have dropped a term of order v^4. But to keep such a term would be inconsistent with the linear approximation to the Dirac equation with which we started.

If terms with a factor ϕ^2 are kept, we will have the following additional terms on the right-hand side of M-17 (to order v^4),

$$m\phi^2\chi - \frac{1}{8m}\phi_{,i}\phi_{,i}\chi$$

To this order there are no new terms in which a σ matrix appears. Thus the only terms of order v^4 that may be taken seriously in (b), (c) and (d) of Ml-17 is the spin term (c).

(c) is similar to the spin-orbit coupling term that appears for the electromagnetic Dirac equation. The electromagnetic spin-orbit coupling term that is proportional to σ is

$$\frac{e}{4m^2}\sigma\cdot[(\vec{p} - e\vec{A})\times E]\chi \qquad \text{(Ml-18a)}$$

For the coulomb field of a proton this is just

$$\frac{e^2}{4m^2}\sigma\cdot(\vec{p}\times\vec{r})/r^3 \qquad \text{(Ml-18b)}$$

For the case that ϕ is the field of a point of mass M, (c) is just

$$\frac{GMm}{4m^2}\sigma\cdot(\vec{p}\times\vec{r})/r^3 \qquad \text{(Ml-19)}$$

Thus we have the same term if we replace the electrical force e^2/r^2 by the gravitational force GMm/r^2.

Since $\vec{p}\times\vec{r}$ may be interpreted as the angular momentum $\vec{L}$, Ml-19 may be written

$$\frac{GMm}{4m^2}(\vec{\sigma}\cdot\vec{L})/r^3 \qquad \text{(Ml-19a)}$$

It is thus clear that (c) is a spin-orbit coupling term.

N. FEYNMAN DIAGRAMS: EMISSION OF LOW ENERGY GRAVITONS

The Feynman diagrams for gravity interacting with electrons are obtained by analogy with quantum-electrodynamics using the methods presented in Feynman's articles, "Theory of Positrons" and "Quantum Electrodynamics" (7) .

Let the Lagrangian for an electron interacting with an external field be given by

$$\mathcal{L} = i\bar{\psi}\not{\nabla}\psi - m\bar{\psi}\psi + \mathcal{L}_c{}^{*}$$

where $\mathcal{L}_c$ is the interaction part of the Lagrangian. The equation of motion for ψ is

$$(i\not{\nabla} - m)\psi = -\frac{\delta\mathcal{L}_c}{\delta\bar{\psi}} = O_p\psi \qquad \text{(N-1)}$$

The operator O_p is a quantity involving the external field components. For example, for the electromagnetic field

$$\mathcal{L}_c = -j_\mu A_\mu = -e\bar{\psi}\gamma_\mu\psi A_\mu$$

$$-\frac{\delta\mathcal{L}_c}{\delta\bar{\psi}} = e\not{A}\psi = O_p\psi$$

or

$$O_p = e\not{A}$$

---

*By definition $\not{\nabla} = \bar{\gamma}_\mu A_\mu$. Since we shall only deal with flat space γ matrices in the rest of this paper, we shall represent them by γ_μ rather than $\bar{\gamma}_\mu$.

The amplitude for the transition from the state ψ_i to the state ψ_f when the electron interacts with an external potential is given by the perturbation series

$$P_{fi} = -i \int \bar{\psi}_f(1) O_p(1) \psi_i(1)\, d^4x_1 \qquad \text{(a)}$$

$$- \int\int \bar{\psi}_f(2) O_p(2) K_+(2,1) O_p(1) \psi_i(1) d^4x_1 d^4x_2 \qquad \text{(b)}$$

$$+ \ldots \qquad \text{(N-2)}$$

where $K_+(2,1)$ is the free electron kernal to go from 1 to 2. For electromagnetism equation N-2 just becomes

$$P_{fi} = -ie \int \bar{\psi}_f(1) A\!\!\!/(1) \psi(1)\, d^4x_1 + \ldots \qquad \text{(N-3)}$$

N-3 is just the transition amplitude given by Feynman (21).

Rather than differentiating $\mathcal{L}_c$ to obtain O_p and then substituting in N-2a, the first order transition amplitude, N-2a, may be directly obtained from the action. Suppose we rewrote the interaction part of the electron action so that it is a function of $\bar{\psi}_f$ and ψ_i rather than just arbitrary fields $\bar{\psi}$ and ψ.

$$S_c \rightarrow \int \mathcal{L}_c(\bar{\psi}_f, \psi_i)\, d^4x$$

Since $\mathcal{L}_c(\bar{\psi}_f, \psi_i)$ is a linear function of $\bar{\psi}_f$ and its first derivative,

$$\mathcal{L}_c(\overline{\psi}_f, \psi_i) = \overline{\psi}_f \frac{\partial \mathcal{L}_c}{\partial \overline{\psi}_f} + \overline{\psi}_{f,\mu} \frac{\partial \mathcal{L}_c}{\partial \overline{\psi}_{f,\mu}}$$

$$= \overline{\psi}_f \left[\frac{\partial \mathcal{L}_c}{\partial \overline{\psi}_f} - \left(\frac{\partial \mathcal{L}_c}{\partial \overline{\psi}_{f,\mu}} \right)_{,\mu} \right] + \left(\overline{\psi}_f \frac{\partial \mathcal{L}_c}{\partial \overline{\psi}_{f,\mu}} \right)_{,\mu}$$

Dropping the surface term

$$S_c(\overline{\psi}_f, \psi_i) = \int \overline{\psi}_f \frac{\delta \mathcal{L}_c}{\delta \overline{\psi}_f} \, d^4x$$

Writing

$$O_p \psi_i = \delta \mathcal{L}_c / \delta \overline{\psi}_f$$

we get

$$S_c(\overline{\psi}_f, \psi_i) = \int \overline{\psi}_f(1) O_p(1) \psi_i(1) \, d^4x_1 \qquad \text{(N-4)}$$

which is $-i$ times the first order transition amplitude.

Thus we have the simple rule that we get $-i$ times the first order transition amplitude by simply writing the interaction part of the action and replacing $\overline{\psi}$ and ψ by $\overline{\psi}_f$ and ψ_i.

The transition amplitude N-2 is for an electron moving in an external potential. The transition ampliutde for an electron emitting or absorbing quantized particles, photons or gravitons, is obtained by expanding the external field in plane waves. Each wave acts only once, the waves proportional to $e^{+iq \cdot x}$ acting when the particle is absorbed, $e^{-iq \cdot x}$ when emitted.

For electrons interacting with gravity, the first order transition amplitude, obtained by replacing $\overline{\psi}$ and ψ by $\overline{\psi}_f$, ψ_i in $iS_c(\overline{\psi}, \psi)$ is

$$P_{fi} = -i\frac{K}{2}\int \{ \frac{h_{\alpha\beta}}{2}[\,\overline{\psi}_f\gamma_\alpha(i\nabla_\beta\psi_i) + (-i\nabla_\beta\overline{\psi}_f)\gamma_\alpha\psi_i\,]$$

$$- \frac{h}{2}[\overline{\psi}_f\gamma_\alpha(i\nabla_\alpha\psi_i) + (-i\nabla_\alpha\overline{\psi}_f)\gamma_\alpha\psi_i - 2m\overline{\psi}_f\psi_i\,]\}\,d^4x$$

$$+ \text{ terms of order } K^2 \qquad \text{(N-5)}$$

To get N-17 we used the elecgron Lagrangian K-9 expanded to first order in K.

Consider the case that the electron is initially in a plane wave state of momentum p^i and we wish the amplitude that after the emission or absorption of a single graviton the electron is in a state of momentum p^f. We have

$$\psi_i = u(p^i)e^{-p^i\cdot x} \qquad i\nabla_\alpha\psi_i = p^i_\alpha\psi_i$$

$$\overline{\psi}_f = \overline{u}(p^f)e^{+ip^f\cdot x} \qquad -i\nabla_\alpha\overline{\psi}_f = p^f_\alpha\overline{\psi}_f \qquad \text{(N-6)}$$

For absorption of a single graviton the field $h_{\alpha\beta}$ is replaced by

$$h_{\alpha\beta} \rightarrow e_{\alpha\beta}e^{-iq\cdot x} \qquad \text{(N-7a)}$$

and for emission

$$h_{\alpha\beta} \rightarrow e_{\alpha\beta}e^{+iq\cdot x} \qquad \text{(N-7b)}$$

These gravitational potentials are normalized to 2ω gravitons per cubic centimeter, the same normalization used by Feynman for electromagnetic potentials.

The first order transition amplitude for this case becomes

$$P_{fi} = -\frac{K}{2}\bar{u}(p^f)\{\frac{p^f_\beta + p^i_\beta}{2}\gamma_\alpha e_{\alpha\beta} - [\frac{\not{p}^f + \not{p}^i}{2} - m]\frac{e_{\alpha\alpha}}{2}\}u(p^i)$$

$$\times \int e^{-i(p^i - p^f \pm q)\cdot x_3}\, d^4x_3 \qquad \text{(N-8)}$$

where the choice of $+q$ or $-q$ in the exponent depends on whether a graviton was emitted or absorbed. In either case the integral corresponds to $(2\pi)^4$ times a δ function representing conservation of momentum at the point where the graviton was absorbed or emitted.

For the emission of a graviton, N-8 may be written

$$P_{fi} = -i\bar{u}(p^f)\frac{K}{2}\{\hat{p}_\beta\gamma_\alpha e_{\alpha\beta} + (\hat{\not{p}} - m)(-\frac{e_{\alpha\alpha}}{2})\}u(p^i)$$

$$\times (2\pi)^4\delta^4(p^f - p^i - q) \qquad \text{(N-9)}$$

where

$$\hat{p}_\beta = \frac{p^f_\beta + p^i_\beta}{2}$$ = the average of the electron momentum just before and after the emission of the graviton (N-10)

Now the first order transition amplitude for the emission of a photon by an electron is

$$P_{fi} = -i\bar{u}(p^f)\ \{(4\pi e^2)^{1/2}\not{e}\}u(p^i)$$

$$\times (2\pi)^4\delta^4(p^f - p^i - q) \qquad \text{(N-11)}$$

Comparing N-9 with N-11, we get the amplitude for the emission of a graviton (in analogy with the amplitude $(4\pi e^2)^{1/2}$ for a photon) as

$$\frac{K}{2}\{\hat{p}_\beta\gamma_\alpha e_{\alpha\beta} + (\hat{\not{p}} - m)(-\frac{e_{\alpha\alpha}}{2})\} \qquad \text{(N-12)}$$

The corresponding diagram is

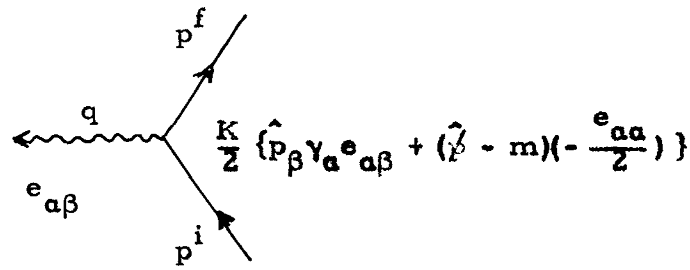


Figure 1

Iterations of the first order matrix element corresponding to the second order transition amplitude of equation N-2b follow in a manner similar to electrodynamics. The diagram for N-2b to order K^2 is

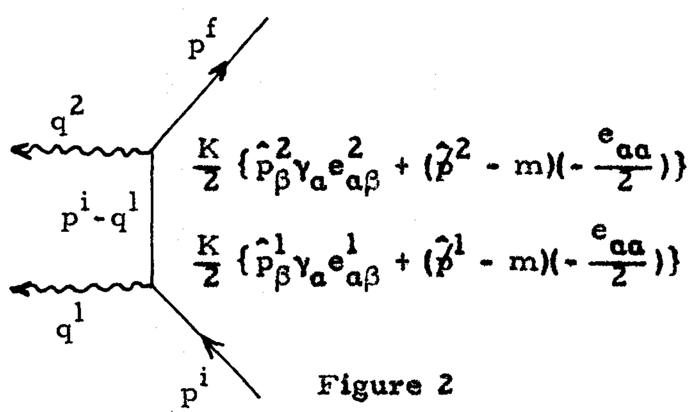


Figure 2

where

$$\hat{p}^1_\beta = \frac{p^i_\beta + (p^i_\beta + q^1_\beta)}{2} \; ; \quad \hat{p}^2_\beta = \frac{(p^i_\beta - q^1_\beta) + p^f_\beta}{2} \qquad \text{(N-13)}$$

One difference between gravity and electrons is that for gravity we have higher powers of the gravitational field in the interaction Lagrangian. This means we will have the possibility of emitting or absorbing two or more gravitons at one point.

Let us consider the absorption of two gravitons at a single point. We obtain the amplitude for this process by expanding the interaction part of the action to second order in K, and proceed as we did for one graviton. One term in the action that offers some new problems is the last term in K-9

$$-\frac{i}{4}\int(-g)^{1/2}\bar{\psi}_f\underset{\sim\sim\sim}{\gamma_\mu\gamma_\nu\gamma_\rho}\,\psi_i a_{\rho\beta}a_{\alpha\nu}b_{\mu\alpha,\beta}\,d^4x$$

Expanded in terms of $h_{\mu\nu}$ we get to lowest order

$$\frac{K^2}{16}\,i\int\bar{\psi}_f\,\underset{\sim\sim\sim}{\gamma_\mu\gamma_\nu\gamma_\rho}\,\psi_i\,h_{\mu\alpha,\rho}h_{\alpha\nu}d^4x \qquad \text{(N-14)}$$

The diagram for the absorption of two gravitons is

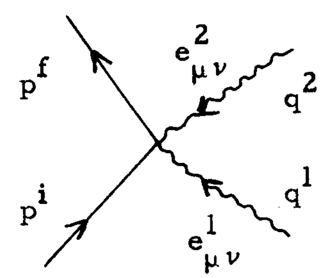


Figure 3

To obtain the transition amplitude we expand the external fields $h_{\mu\alpha}$ and $h_{\nu\alpha}$ in plane waves, each of which acts only once.

$$h_{\mu\alpha} = e_{\mu\alpha}e^{-iq^1\cdot x} + e_{\mu\alpha}e^{-iq^2\cdot x} + \ldots$$

$$h_{\nu\alpha} = e_{\nu\alpha}e^{-iq^1\cdot x} + e_{\nu\alpha}e^{-iq^2\cdot x} + \ldots$$

For the case shown in fig. 3 where one graviton carries in a momentum q^1 and the other q^2,

$$h_{\mu\alpha,\rho}h_{\alpha\nu} \to -iq^1_\rho e^1_{\mu\alpha} e^2_{\alpha\nu} e^{-i(q^1+q^2)\cdot x}$$

$$-iq^2_\rho e^2_{\mu\alpha} e^1_{\alpha\nu} e^{-i(q^1+q^2)\cdot x} \qquad \text{(N-15)}$$

The transition amplitude becomes

$$i\frac{K^2}{16}\,\bar{u}(p^f)\,\underline{\gamma_\mu\gamma_\nu\gamma_\rho}\;u(p^i)[\,q^1_\rho e^1_{\mu\alpha} e^2_{\alpha\nu} + q^2_\rho e^2_{\mu\alpha} e^1_{\alpha\nu}]$$

$$\times\,(2\pi)^4\delta^4(p^f - p^i - q^1 - q^2) \qquad \text{(N-16)}$$

Since $\underline{\gamma_\mu\gamma_\nu\gamma_\rho}$ is antisymmetric in μ, ν and ρ, N-16 may be written

$$-i\frac{K^2}{16}\,\bar{u}\,(p^f)\,\underline{\gamma_\mu(\not{q}^1 - \not{q}^2)\gamma_\nu}\;e^1_{\alpha\mu} e^2_{\nu\alpha}\,\bar{u}\,(p^f)$$

$$\times\,(2\pi)^4\delta^4(p^f - p^i - q^1 - q^2) \qquad \text{(N-17)}$$

Comparing this with equation N-9 we see that the amplitude for the absorption of two gravitons corresponding to the term N-14 is

$$\frac{K^2}{16}\;e^1_{\alpha\mu}\;\underline{\gamma_\mu(\not{q}^1 - \not{q}_2^2)\gamma_\nu} e^2_{\alpha\nu} \qquad \text{(N-18)}$$

By a similar calculation we can expand the rest of the action to order K^2 to find the complete amplitude for absorption of two gravitons at a point. The resulting amplitude for the absorption of one or two gravitons at a point is given by

$$\frac{1}{2}\{p_\beta\gamma_\alpha[Ke_{\alpha\beta} - \frac{3K^2}{4}e_{\alpha\rho}e_{\rho\beta} + \frac{K^2}{2}e_{\rho\rho}e_{\alpha\beta}]$$

$$+ (\hat{\not p} - m)[-Ke_{\alpha\alpha} - \frac{K^2}{4}e_{\alpha\alpha}e_{\beta\beta} + \frac{K^2}{2}e_{\alpha\beta}e_{\alpha\beta}]\}$$

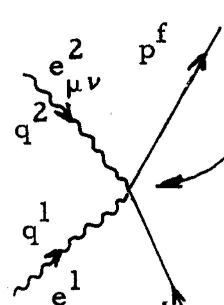
$$+ \frac{1}{16}e^1_{\alpha\mu}\,\underline{\gamma_\mu(\not A^1 - \not A^2)\gamma_\nu}\,e^2_{\nu\alpha} \qquad \text{(N-19)}$$

where $\hat{p}$ always stands for the average of the electron momentum just before and after the point of absorption.

To use formula N-19, we must symmetrically treat two gravitons arriving at a point. For example, the term $\frac{K^2}{2}e_{\rho\rho}e_{\alpha\beta}$ is to be written $\frac{K^2}{2}e^1_{\rho\rho}e^2_{\alpha\beta} + \frac{K^2}{2}e^2_{\rho\rho}e^1_{\alpha\beta}$. If we wish one of the gravitons to be emitted, we change the sign of the momentum of that graviton but maintain conservation of momentum. We also note that if gravitons are converging on a single point, and the electrons entering and leaving that point are free, then $(\hat{\not p} - m) = 0$. That is because

$$\bar{u}(p^f)\hat{\not p}u(p^i) = \bar{u}(p^f)\left(\frac{\not p^f + \not p^i}{2}\right)u(p^i) = \bar{u}mu$$

Explicitly the diagram for two gravitons being absorbed at a point is given by equation N-19



$$\frac{K^2}{4}[\hat{p}_\beta\gamma_\alpha(-\frac{3}{2}e^1_{\alpha\rho}e^2_{\rho\beta} - \frac{3}{2}e^2_{\alpha\rho}e^1_{\rho\beta} + e^1_{\rho\rho}e^2_{\alpha\beta}$$

$$+ e^2_{\rho\rho}e^1_{\alpha\beta}) + (\hat{\not p} - m)(-e^1_{\alpha\alpha}e^2_{\beta\beta} + 2e^1_{\alpha\beta}e^2_{\alpha\beta})$$

$$+ \frac{1}{4}e^1_{\alpha\mu}\,\underline{\gamma_\mu(\not A^1 - \not A^2)\gamma_\nu e^2_{\nu\alpha}}]$$

Figure 4

Suppose one of the particles is a positron rather than an electron. For example, one of the diagrams for two-graviton pair annihilation is

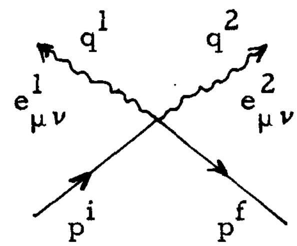


Figure 5

The matrix element for fig. 5 is exactly the same as for fig. 4 except that the signs of q^1 and q^2 are changed since the gravitons are emitted, and p^f is the negative of the four momentum of the positron. For example

$$\not{p}^i = (E_- \gamma_t - \vec{p}_- \cdot \vec{\gamma})$$

$$\not{p}^f = -(E_+ \gamma_t - \vec{p}_+ \cdot \vec{\gamma}) = -\not{p}_+ \tag{N-20}$$

The conservation of momentum

$$p^i_\mu - p^f_\mu = q^1_\mu + q^2_\mu$$

now becomes

$$p^i_\mu + p^f_\mu = q^1_\mu + q^2_\mu$$

The quantity $\hat{p}_\beta$ appearing in equation N-19 is now

$$\hat{p}_\beta = \frac{p^i_\beta + p^f_\beta}{2} = \frac{p^i_\beta - p^+_\beta}{2} \tag{N- 21}$$

To order K^2 there is another possible type of diagram representing the interaction of two external gravitons with an electron. This diagram represents the possibility of gravity interacting with itself, and is of the form

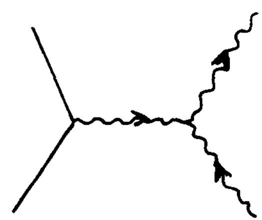


Figure 6

This diagram will be described in the next part when we consider the scattering of gravitons by electrons. No such diagrams appear to first order in K, and we may correctly consider the emission of a single graviton by an electron using formula N-19.

Let us consider low energy gravitational bremstahlung when an electron is scattered by some potential V. The Feynman diagrams for the processes are

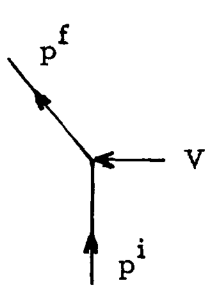


Figure 7

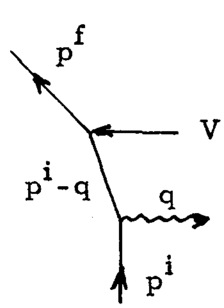


Figure 8

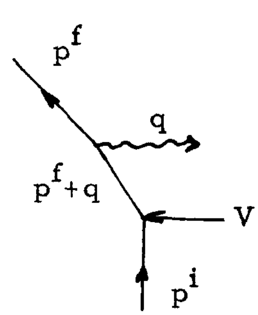


Figure 9

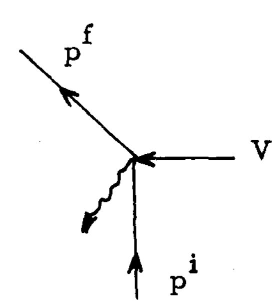


Figure 10

Figure 7 is the diagram for the process without the emission of gravitons. Since there is an energy of interaction between the potential V and the electron, the diagram of fig. 10 represents the possibility of graviton emission from this energy. Gravitons may also be emitted from the potential V, but these are being neglected now.

The amplitude for the diagram of fig. 8 is proportional to

$$M = - \bar{u}(p^f)V \frac{1}{\not p^i - \not q - m} \frac{K}{2} [\gamma_\alpha \hat{p}_\beta e_{\alpha\beta} - (\hat{\not p} - m)e_{\alpha\alpha}] u(p_i) \quad \text{(N-22)}$$

Since $p^2 = m^2$ and $q^2 = 0$, the propagator becomes

$$\frac{1}{\not p^i - \not q - m} = \frac{\not p^i - \not q + m}{-2 p^i \cdot q}$$

which in the limit of low energy gravitons goes as $1/\omega$. To calculate the amplitude for emission low energy gravitons, we may neglect q in comparison to p in the numerator of N-22 with the result

$$M = \frac{K}{4p^i \cdot q} \bar{u}(p^f)Ve_{\alpha\beta}[\not p^i \gamma_\alpha \hat{p}_\beta + \gamma_\alpha m \hat{p}_\beta] u(p^i) \quad \text{(N-23)}$$

The term $(\hat{\not p} - m)e_{\alpha\alpha}u(p^i)$ does not contribute in this limit since

$$\hat{\not p} u(p^i) \approx \not p^i u(p^i) = mu(p^i)$$

The term in the square brackets of N-23 may be written

$$[\not p^i \gamma_\alpha p^i_\beta + \gamma_\alpha \not p^i p^i_\beta] u(p^i) \quad \text{(N-24)}$$

where we have replaced $mu(p^i)$ by $\not p^i u(p^i)$, and approximated $\hat{p}_\beta$ by

p^i_β. N-24 may be written

$$[\gamma_\rho\gamma_\alpha + \gamma_\alpha\gamma_\rho] p^i_\rho p^i_\beta u(p^i) = 2p^i_\alpha p^i_\beta u(p^i)$$

Thus the amplitude N-23 becomes

$$\frac{K}{2p^i \cdot q} \bar{u}(p^f) V (e_{\alpha\beta} p^i_\alpha p^i_\beta) u(p^i) \tag{N-25}$$

Since $e_{\alpha\beta}p^i_\alpha p^i_\beta$ does not involve γ matrices N-25 becomes

$$\frac{K}{2p^i \cdot q} e_{\alpha\beta} p^i_\alpha p^i_\beta [\bar{u}(p^f) V u(p^i)] \tag{N-26}$$

The term in the square brackets is just the amplitude for the scattering of the electron by the potential V without the emission of gravitons, corresponding to the diagram of fig. 7. If we call this amplitude A, then the amplitude for fig. 8 in the low frequency limit is

$$\frac{K}{2\omega} A a_7 \tag{N-27}$$

where

$$a_7 = \omega e_{\alpha\beta} \frac{p^i_\alpha p^i_\beta}{p^i \cdot q} \tag{N-27a}$$

The amplitude for the diagram of fig. 9 is similar to that of fig. 8, except that p^i is replaced by p^f, and the overall sign is changed since the propagator is now for the virtual momentum $p^f + q$ rather than $p^i - q$. The amplitude for the two diagrams of fig. 8 and fig. 9 is

$$M = \frac{K}{2\omega} Aa \tag{N-28}$$

where

$$a = \omega\left(\frac{e_{\alpha\beta}p^{i}_{\alpha}p^{i}_{\beta}}{p^{i}\cdot q} - \frac{e_{\alpha\beta}p^{f}_{\alpha}p^{f}_{\beta}}{p^{f}\cdot q}\right) \tag{N-28a}$$

Since $(p^{i}\cdot q)^{-1}$ and $(p^{f}\cdot q)^{-1}$ go as $1/\omega$ as $\omega \rightarrow 0$, the quantity a of N-28a is finite in the limit of $\omega \rightarrow 0$. Thus the amplitude N-28 diverges as $1/\omega$ and we get an infra-red bremstrahlumg from the diagrams. From the diagram of fig. 10 we have no electron propagator, thus no factor of $(p \cdot q)^{-1}$. As long as the momentum transferred to the electron by the potential is much larger than ω, we have no infra-red divergence for this diagram and it does not contribute in the low frequency limit.

By similar arguments Feynman showed that the amplitude for low energy graviton emission from a spin-zero particle is given by

$$M = \frac{K}{2\omega} Aa \tag{N-29}$$

where

A = the amplitude for scattering without graviton emission

$$a = \omega \sum_{i} \frac{p^{i}_{\mu}p^{i}_{\nu}}{p^{i}\cdot q}\, e_{\mu\nu}$$

where the sum over i is the sum over all incoming particles, minus the sum over all outgoing particles.

It is easily checked that the amplitude of emission of a low frequency graviton from a photon is of the same form as N-28. Thus the formula for low energy graviton emission will be independent of the type of particle or the kind of force scattering the particle.*

In terms of the amplitude for a given process, M, the transition probability/second is given by

$$P_{fi} = \frac{2\pi}{\Pi(N)} |M|^2 \rho$$

where ρ is the density of final states and $\Pi(N)$ is the normalization factor. For example we have normalized $\bar{u}u$ to $2M$, or the quantity

$$u^* u = \bar{u}\gamma_t u = 2E$$

Thus we have normalized the electrons to $2E$ per cubic centimeter rather than one per cubic centimeter, and this normalization factor $2E$ is divided out in $\Pi(N)$. As we mentioned after equation N-20, the gravitons are normalized to 2ω gravitons per cubic centimeter and this 2ω is likewise divided out in $\Pi(N)$. Thus $\Pi(N)$ is given by the product of twice the energy of all the external particles.

Now the transition probability for the scattering of an electron without the emission of gravitons, corresponding to the diagram of fig. 7, is

$$P_{fi} = \frac{2\pi}{\Pi(N)} |A|^2 \rho_o$$

Granting that such a scattering has occurred, the mean number of gravitons of polarization $e_{\alpha\beta}$, momentum q, emitted per scattering

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*The author has not explicitly checked this statement for the case that a graviton itself is the source of the infra-red gravitons.

is, for small ω

$$\frac{1}{2\omega}\left[\, |a|^2 \frac{K^2}{4\omega^2} \right] \rho_g \, d\omega$$

where ρ_g, the density of final states of the graviton

$$\rho_g = \frac{E_2 |p_2|}{(2\pi)^3} \, d\Omega \approx 4\omega^2 \frac{d\Omega}{4\pi} \times \frac{1}{8\pi^2}$$

The number of gravitons emitted per collision is therefore given by

$$N = |a|^2 \times \frac{d\Omega}{4\pi} \times \frac{d\omega}{\omega} \times \frac{K^2}{16\pi^2} \qquad \text{(N-30)}$$

where in general

$$a = \omega \sum_i \left(\frac{p^i_\alpha p^i_\beta e_{\alpha\beta}}{p^i \cdot q} \right) \qquad \text{(N-30a)}$$

The sum over i is the sum over all incoming particles minus the sum over all outgoing particles. As we have seen equation N-30 applies not only for incoming and outgoing electrons, but all spin-zero, spin-one half and spin-one particles.

The radiation formula N-30 is discussed in considerable detail by Feynman in a letter to Weisskopf (21), including comparison to classical calculations of gravitational radiation.

O. SCATTERING OF GRAVITONS BY ELECTRONS: GAUGE INVARIANCE

In this part we shall describe the scattering of gravitons by electrons. We will not be as interested in the resulting cross section as we will be in using the calculation as a check of the rules for diagrams given in the last part. In fact we shall only give the amplitude for the scattering of gravitons by electrons since this amplitude is rather complicated.

The check of a calculation with gravitons involved is similar to the check of a calculation in quantum electrodynamics for gauge invariance. For electrodynamics the interaction part of the action

$$S_c = -\int J_\mu A_\mu d^4x$$

is unchanged under the substitution

$$A_\mu \rightarrow A_\mu + \chi_{,\mu}$$

because the current J_μ is conserved. I.e.,

$$-\int J_\mu (A_\mu + \chi_{,\mu}) d^4x = -\int J_\mu A_\mu d^4x - \int J_\mu \chi_{,\mu} d^4x$$

where

$$\int J_\mu \chi_{,\mu} d^4x = \int (J_\mu \chi)_{,\mu} d^4x = 0$$

since

$$J_{\mu,\mu} = 0$$

In quantum mechanics this means that whenever we have a photon emitted by a conserved source, the cross section for the emission of that photon will be unchanged if the photon's potential A_μ is replaced by $A_\mu + \chi_{,\mu}$. In terms of Feynman diagrams the cross section for the process

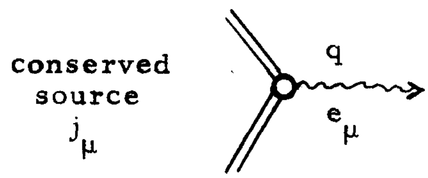

Fig. 11

is unchanged if we make the gauge transformation

$$e_\mu \to e_\mu + q_\mu \xi \tag{O-1}$$

in the matrix element of the diagram. The quantity ξ in O-1 is the momentum space representation of χ.

We may use the same procedure to define gauge invariance when gravity interacts with matter. From part E we have the Lagrangian for gravity interacting with matter,

$$\mathcal{L} = F^2 + (F - F^2) + \mathcal{L}_{mg} \tag{E-1}$$

where $F = R(-g)^{1/2}/2K^2$, and F^2 is the Lagrangian for the linear gravitational fields. The gravitational wave equation from E-1 is

$$\frac{\delta F^2}{\delta h_{\mu\nu}} = \frac{K}{2}\left[-2\frac{\delta \mathcal{L}_{mg}}{\delta K h_{\mu\nu}} - 2\,\frac{\delta(F - F^2)}{\delta K h_{\mu\nu}}\right] = \frac{K}{2}\,T_{\mu\nu}$$

In terms of $h_{\mu\nu}$ this is

$$\frac{1}{4}[-\bar{h}_{\mu\nu,\sigma\sigma} + (\bar{h}_{\sigma\mu,\sigma\nu} + \bar{h}_{\sigma\nu,\sigma\mu} - \delta_{\mu\nu}\bar{h}_{\rho\sigma,\rho\sigma})] = \frac{K}{2} T_{\mu\nu} \qquad \text{(O-2)}$$

Equation O-2 is consistent only if the source term $T_{\mu\nu}$ is conserved. The consistency of the gravitational theory has been demonstrated in part D, thus

$$T_{\mu\nu,\nu} = \left[\frac{-2\delta(\mathcal{L}_{mg} + F - F^2)}{\delta K h_{\mu\nu}} \right]_{,\nu} = 0$$

Consider the following action

$$S = \int [F_2(h'_{\mu\nu}) + \frac{K}{2} h'_{\mu\nu} T_{\mu\nu}(h_{\mu\nu})] d^4x \qquad \text{(O-3)}$$

in which we consider the gravitational fields $h'_{\mu\nu}$ in $F^2(h'_{\mu\nu})$ separately from the gravitational fields $h_{\mu\nu}$ involved in the source term $T_{\mu\nu}(h_{\mu\nu})$. Variation of O-3 with respect to $h'_{\mu\nu}$ just gives

$$\frac{1}{4}[\bar{h}'_{\mu\nu,\sigma\sigma} + (\bar{h}'_{\mu\sigma,\nu\sigma} + \bar{h}'_{\sigma\nu,\sigma\mu} - \delta_{\mu\nu}\bar{h}'_{\rho\sigma,\rho\sigma})] = \frac{K}{2} T_{\mu\nu}(h_{\mu\nu}) \qquad \text{(O-4)}$$

which is just the correct gravitational wave equation if we drop the primes.

The action O-3 is now invariant under the gauge transformation

$$h'_{\mu\nu} \rightarrow h'_{\mu\nu} + \eta_{\mu,\nu} + \eta_{\nu,\mu} \qquad \text{(O-5a)}$$

$$h_{\mu\nu} \rightarrow h_{\mu\nu} \qquad \text{(O-5b)}$$

O-5 is similar in form to the gauge invariance of the linear theory given in equation B-2. From the action O-3 we see that it is the gravitational field $h'_{\mu\nu}$ that is emitted or absorbed by the conserved source $T_{\mu\nu}$.

The gauge invariance of a process in which a graviton is emitted or absorbed will be tested by replacing the field $h'_{\mu\nu}$ of this graviton by $h'_{\mu\nu} + \eta_{\mu,\nu} + \eta_{\nu,\mu}$. The fields in the source are not to be changed.

We shall see that a source $T_{\mu\nu}$ emits the gravitational wave $\bar{h}_{\mu\nu}$. Thus the diagram for the emission of a graviton from a conserved source $t_{\mu\nu}$ is

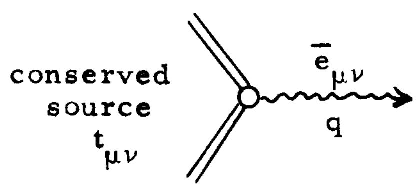

Fig. 12

To check this diagram for gauge invariance, we replace the field $e_{\mu\nu}$ of the emitted graviton;

$$e_{\mu\nu} \rightarrow e_{\mu\nu} + \xi_\mu q_\nu + \xi_\nu q_\mu \tag{O-6a}$$

or

$$\bar{e}_{\mu\nu} \rightarrow \bar{e}_{\mu\nu} + \xi_\nu q_\mu + \xi_\mu q_\nu - \delta_{\mu\nu} q \cdot \xi \tag{O-6b}$$

This substitution is made only on the external graviton, and the result should be that the cross section for the process is unchanged by the substitution O-6.

Before calculating the amplitude for the scattering of gravitons by electrons, let us return to the exact wave equation O-2.

$$\frac{1}{4}\left[-\bar{h}_{\mu\nu,\sigma\sigma} + (\bar{h}_{\sigma\mu,\sigma\nu} + \bar{h}_{\sigma\nu,\sigma\mu} - \delta_{\mu\nu}\bar{h}_{\rho\sigma,\rho\sigma})\right] = \frac{K}{2}\left[T^{g}_{\mu\nu} + T^{mg}_{\mu\nu}\right]$$

where

$$T^{g}_{\mu\nu} = -2\delta(F - F^2)/\delta K h_{\mu\nu} \qquad \text{(O-7a)}$$

$$T^{mg}_{\mu\nu} = -2\delta \mathcal{L}_{mg}/\delta K h_{\mu\nu} \qquad \text{(O-7b)}$$

The momentum space representation of O-2 is

$$q^2 \bar{e}_{\mu\nu} + (q_\sigma q_\nu \bar{e}_{\sigma\mu} + q_\sigma q_\mu \bar{e}_{\sigma\nu} - \delta_{\mu\nu} q_\rho q_\sigma \bar{e}_{\sigma\nu}) = 2K[t^{g}_{\mu\nu} + t^{mg}_{\mu\nu}] \qquad \text{(O-8)}$$

where $t^{g}_{\mu\nu}$ and $t^{mg}_{\mu\nu}$ are the momentum space representations of $T^{g}_{\mu\nu}$ and $T^{mg}_{\mu\nu}$. Since the source $[t^{g}_{\mu\nu} + t^{mg}_{\mu\nu}]$ is conserved,

$$q_\mu [t^{g}_{\mu\nu} + t^{mg}_{\mu\nu}] = 0$$

A solution of O-8 is

$$\bar{e}_{\mu\nu} = \frac{2}{q^2} K[t^{g}_{\mu\nu} + t^{mg}_{\mu\nu}] \; * \qquad \text{(O-9)}$$

The solution O-9 is of the same form as the solution for $\bar{e}_{\mu\nu}$ given by equation B3-5 for the linear theory. The only difference is that we now have an exactly conserved source, while in the linear theory we made the approximation that the incomplete source was conserved. In the linear theory the matrix element for the interaction of two energy tensors $t_{\mu\nu}$ and $s_{\mu\nu}$ was given by

$$M = K^2 s_{\mu\nu} \frac{1}{q^2} \bar{t}_{\mu\nu} \qquad \text{(B3-7)}$$

*The steps here are entirely similar to those for the linear theory. For a more complete description see part B3, equations B3-5 and B3-6.

Replacing the linear approximations to the energy in B3-7 by the complete tensors $(t^{g}_{\mu\nu} + t^{mg}_{\mu\nu})$ and $(s^{g}_{\mu\nu} + s^{mg}_{\mu\nu})$, we get the matrix element for the interaction,

$$M = K^2 (s^{g}_{\mu\nu} + s^{mg}_{\mu\nu}) \frac{1}{q^2} (\bar{t}^{g}_{\mu\nu} + \bar{t}^{mg}_{\mu\nu}) \qquad \text{(O-10)}$$

From equation O-10 we see there are new types of interaction not considered in the linear theory. Consider the term in O-10

$$K^2 s^{mg}_{\mu\nu} \frac{1}{q^2} \bar{t}^{g}_{\mu\nu} \qquad \text{(O-11)}$$

Suppose that $s^{mg}_{\mu\nu}$ is the energy tensor of an electron, and $t^{g}_{\mu\nu}$ is the energy tensor of an external graviton passing by the electron. The term O-11 represents the possibility of the exchange of a virtual graviton between the electron and the external graviton.

To obtain the diagram for this process, we expand the tensors $s^{mg}_{\mu\nu}$ and $\bar{t}^{g}_{\mu\nu}$ in orders of K. To lowest order in K, $t^{g}_{\mu\nu}$ is just the energy tensor of a single graviton, and $s^{mg}_{\mu\nu}$ is $s^{g}_{\mu\nu}$, the energy tensor of a single electron. The diagram is therefore

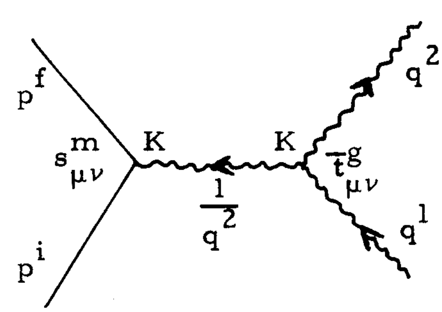

Fig. 13

We see that this diagram corresponds to the scattering of a graviton by an electron. This type of diagram does not have an analogy in electromagnetism and was not included in the last part. This is the only diagram for the gravitational Compton effect that has a factor of $1/q^2$. For small momentum transfers of this diagram will be dominant, and at zero momentum transfer will lead to an infinite cross section.

Let us now obtain the quantity $t^g_{\mu\nu}$ (to lowest order in K) that appears in the diagram of fig. 13. $t^g_{\mu\nu}$ is the momentum space representation of $T^g_{\mu\nu}$ given by

$$T^g_{\mu\nu} = -2\delta(F - F^2)/\delta K h_{\mu\nu} \qquad \text{(C-7a)}$$

The gravitational Lagrangian $F = R(-g)^{1/2}/2K^2$ may be expanded in the series

$$F = F^2 + KF^3 + K^2F^4 + \ldots$$

where KF^3 is of order K smaller than F^2 etc. This series is described in considerable detail in parts C3 and C4. To lowest order in K the quantity $(F - F^2)$ is just KF^3, given in equation C3-6. The source term O-7a to the lowest order in K is now

$$T^g_{\mu\nu} = -2\delta F^3/\delta h_{\mu\nu} \qquad \text{(O-13)}$$

where $-\delta F^3/\delta h_{\mu\nu}$ is given explicitly in equation C3-9.

A typical term of F^3 is

$$f^3 = -\frac{1}{8} h_{\alpha\beta,\sigma} h_{\alpha\beta,\sigma} h_{\rho\rho} \qquad \text{(O-14)}$$

$T'^g_{\mu\nu}$ for this term is

$$T'^{g}_{\mu\nu} = -2\delta f^3/\delta h_{\mu\nu} = +\frac{1}{4}\delta_{\mu\nu}h_{\alpha\beta,\gamma}h_{\alpha\beta,\gamma} - \frac{1}{2}h_{\mu\nu,\gamma\gamma}h_{\delta\delta}$$

$$-\frac{1}{2}h_{\mu\nu,\gamma}h_{\delta\delta,\gamma} \quad \text{(O-14a)}$$

If we label the gravitons on the right-hand side of fig. 13 by $e^1_{\mu\nu}$, $e^2_{\mu\nu}$, and $e^3_{\mu\nu}$, and in analogy to electromagnetism assign a factor $e^{+iq\cdot x}$ for emitted gravitons, $e^{-iq\cdot x}$ for those absorbed, we have for this part of the diagram

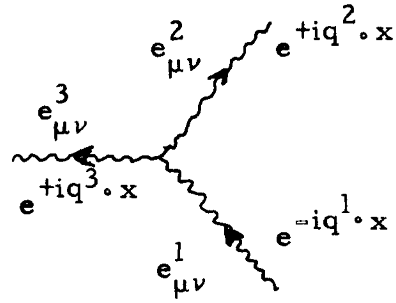

Fig. 14

To obtain the momentum space representation of the source $T'^{g}_{\mu\nu}$ we expand the classical fields $h_{\mu\nu}$ appearing in O-13a in plane waves. Using the rule that each of these waves acts only once, we get for $t'^{g}_{\mu\nu}$ corresponding to the diagram of fig. 14

$$t'^{g}_{\mu\nu} = (\frac{1}{2}\delta_{\mu\nu}q^1\cdot q^2 e^1_{\alpha\beta}e^2_{\alpha\beta} - \frac{1}{2}q^1\cdot q^1 e^1_{\mu\nu}e^2_{\delta\delta} - \frac{1}{2}q^2\cdot q^2 e^1_{\delta\delta}e^2_{\mu\nu}$$

$$-\frac{1}{2}q^1\cdot q^2 e^1_{\mu\nu}e^2_{\delta\delta} - \frac{1}{2}q^1\cdot q^2 e^1_{\delta\delta}e^2_{\mu\nu} \quad \text{(O-14b)}$$

There is a more direct method of obtaining $t'^{g}_{\mu\nu}$ than the one outlined above. The action corresponding to our typical term f^3 is

$$-\frac{K}{8}\int h_{\alpha\beta,\sigma}h_{\alpha\beta,\sigma}h_{\rho\rho}\, d^4x \quad \text{(O-14c)}$$

Suppose the action O-14c is directly expanded in plane waves corresponding to the diagram of fig. 14. We get

$$-\frac{K}{8}\left[q^1\cdot q^2 e^1_{\alpha\beta}e^2_{\alpha\beta}e^3_{\rho\rho} + q^1\cdot q^2 e^1_{\alpha\beta}e^3_{\alpha\beta}e^2_{\rho\rho} + q^1\cdot q^2 e^2_{\alpha\beta}e^1_{\alpha\beta}e^3_{\rho\rho}\right.$$

$$\left. - q^2\cdot q^3 e^2_{\alpha\beta}e^3_{\alpha\beta}e^1_{\rho\rho} + q^1\cdot q^3 e^3_{\alpha\beta}e^1_{\alpha\beta}e^2_{\rho\rho} - q^3\cdot q^2 e^3_{\alpha\beta}e^2_{\alpha\beta}e^1_{\rho\rho}\right]$$

$$\times (2\pi)^4\delta^4(q^3 - q^1 + q^2) \qquad \text{(O-14d)}$$

Eliminating q^3 by the δ function O-14d may be written

$$e^3_{\mu\nu}\left[-\frac{K}{2}\left(\frac{1}{2}\delta_{\mu\nu}q^1\cdot q^2 e^1_{\alpha\beta}e^2_{\alpha\beta} - \frac{1}{2}q^1\cdot q^2 e^1_{\mu\nu}e^2_{\delta\delta} - \frac{1}{2}q^1\cdot q^1 e^1_{\mu\nu}e^2_{\delta\delta}\right.\right.$$

$$\left.\left. + \frac{1}{2}q^1\cdot q^2 e^2_{\mu\nu}e^1_{\delta\delta} + \frac{1}{2}q^2\cdot q^2 e^2_{\mu\nu}e^1_{\delta\delta}\right)\right] \times (2\pi)^4 \qquad \text{(O-14e)}$$

The $(2\pi)^4$ is taken care of in the cross section. Neglecting the factor $(2\pi)^4$, $-\frac{K}{2}t'^{g}_{\mu\nu}$ is obtained directly from the action as the coefficient of the polarization vector of the emitted graviton. Thus we may write the momentum space representation of the action O-14c as

$$-\frac{K}{2}e^3_{\mu\nu}\,t'^{g}_{\mu\nu} \qquad \text{(O-14d)}$$

The momentum space representation of the complete action corresponding to fig. 14 will be

$$\int F^3 d^4x \rightarrow -\frac{K}{2}t^g_{\mu\nu}e^3_{\mu\nu}$$

In a similar manner the energy tensor of the electron $s_{\mu\nu}$ is obtained directly from the electron action $\int \mathcal{L}_{mg} d^4x$ by the formula

$$\int \mathcal{L}_{mg} d^4x \rightarrow -\frac{K}{2} s^m_{\alpha\beta} e^3_{\alpha\beta}$$

The expansion of the action $\int \mathcal{L}_{mg} d^4x$ or equivalently $\int \mathcal{L}_c d^4x$ is given in the last part by equation N-9. The amplitude for the emission or absorption of a graviton is given by N-11 and is proportional to the coefficient of $e^3_{\alpha\beta}$ in the expansion of the action. Thus we have taken the amplitude for the absorption of a graviton from an electron to be $-\frac{K}{2} s^m_{\alpha\beta}$, corresponding to the diagram

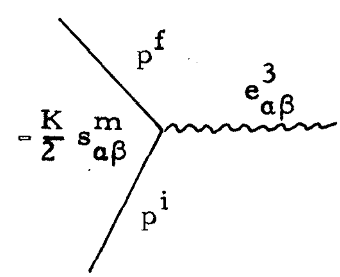

Fig. 15

In a similar manner we take the amplitude for the emission of a graviton from a graviton to be the coefficient of $e^3_{\mu\nu}$ in $\int F^3 d^4x$, or as $-\frac{K}{2} t^g_{\mu\nu}$

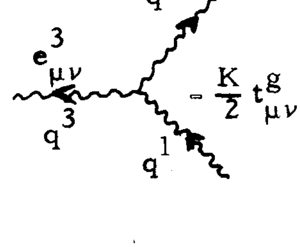

Fig. 16

From equation O-11 the amplitude for the interaction of a graviton with an electron via the exchange of a graviton is given by

$$K^2 s^m_{\mu\nu} \frac{1}{q^2} \overline{t}^g_{\mu\nu}$$

$$= \left(-\frac{K}{2}\right) s^m_{\mu\nu} \quad \frac{2(\delta_{\mu\alpha}\delta_{\nu\beta} + \delta_{\nu\alpha}\delta_{\mu\beta} - \delta_{\mu\nu}\delta_{\alpha\beta})}{q^2} \quad \left(-\frac{K}{2}\right) t^g_{\alpha\beta} \tag{O-15}$$

If we take

$$2(\delta_{\mu\alpha}\delta_{\nu\beta} + \delta_{\nu\alpha}\delta_{\mu\beta} - \delta_{\mu\nu}\delta_{\alpha\beta})/q^2 \tag{O-16}$$

as the propagator of the graviton then we may put the diagrams of fig. 15 and fig. 16 together to get the diagram of fig. 13.

We should note that the above analysis which leads to O-16 as the propagator for a graviton is based on the solution to the wave equation given in equation O-8. This solution relied on the assumption that the source of the emitted or absorbed graviton is conserved. If the source or absorber of a particular virtual graviton is not conserved we must deal with the full wave equation O-8. For the diagram of fig. 13 both the source and absorber $t^g_{\mu\nu}$ and $s^m_{\mu\nu}$ of the virtual graviton have a divergence which is higher order in K. We are calculating the diagram to lowest order in K and may assume that these sources are conserved.

We are now in a position to calculate the gravitational Compton effect. The diagrams for the process are

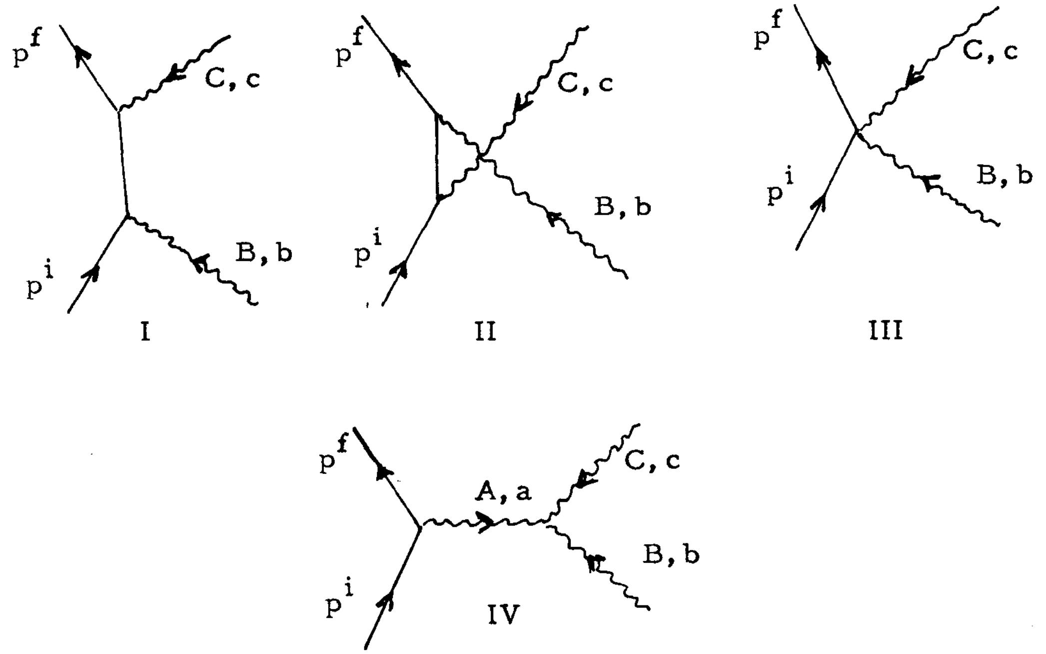

Fig. 17

where we have chosen the sign of the graviton momenta to represent the absorption of these gravitons.

We will use the following notation. B and C are the polarization tensors of the two external gravitons and b and c are their associated momenta.

$$B = e^1_{\mu\nu} \qquad b = q^1$$
$$C = e^2_{\mu\nu} \qquad c = q^2 \tag{O-17a}$$

The variable p will be the average electron momentum

$$p = \frac{p^i + p^f}{2} \tag{O-17b}$$

To eliminate the complication of subscripts in the calculation we shall use a form of matrix multiplication, explained by the following examples.

$$\gamma_\alpha e^1_{\alpha\beta} p_\beta = \gamma B p$$

$$\gamma_\alpha e^1_{\alpha\rho} e^2_{\rho\beta} p_\beta = \gamma B C p$$

$$e^2_{\rho\rho} \gamma_\alpha e^1_{\alpha\beta} p_\beta = (C) \gamma B p$$

$$\not p e^1_{\alpha\beta} e^2_{\alpha\beta} = \not p (BC)$$

$$e^1_{\rho\alpha} \underline{\gamma_\alpha (\not q^1 - \not q^2) \gamma_\beta} e^2_{\beta\rho} = (B \underline{\gamma(\not p - \not q) \gamma} C) \qquad \text{(O-17c)}$$

In this notation capital letters with the exception of K represent tensors; small letters, vectors. A capital letter surrounded by two small letters represents the dot product of that tensor into the two vectors. A parenthesis around capital letters indicates that the first subscript of first tensor in the parenthesis is dotted into the last subscript of the last tensor.

The amplitude for absorption of gravitons by an electron is given by equation N-31

$$\frac{1}{2} \{ \hat{p}_\beta \gamma_\alpha [K e_{\alpha\beta} - \frac{3K^2}{4} e_{\alpha\rho} e_{\rho\beta} + \frac{K^2}{2} e_{\rho\rho} e_{\alpha\beta}]$$

$$+ (\hat{\not p} - m) [-K e_{\alpha\alpha} + \frac{K^2}{4} e_{\alpha\alpha} e_{\beta\beta} + \frac{K^2}{2} e_{\alpha\beta} e_{\alpha\beta}] \}$$

$$+ \frac{K^2}{16} e'_{\rho\alpha} \underline{\gamma_\alpha (\not q^1 - \not q^2) \gamma_\beta} e^2_{\rho\beta} \qquad \text{(N-31)}$$

where $\hat{p}_\beta$ is the average of the electron momenta just before and after the point of emission of a graviton.

In the notation described above, the amplitude for the diagrams I, II, and III of fig. 17 are

$$(I) = \frac{K^2}{4}[(p + \frac{b}{c})C\gamma + \frac{\not{c}}{2}(C)] \frac{1}{\not{p}^2 + \not{b} - m} [\gamma B(p - \frac{c}{2}) - \frac{\not{b}}{2}(B)] \qquad (O\text{-}18)$$

$$(II) = \frac{K^2}{4}[(p + \frac{c}{2})B\gamma + \frac{\not{b}}{2}(B)] \frac{1}{\not{p}^2 + \not{c} - m} [\gamma C(p - \frac{b}{2}) - \frac{\not{c}}{2}(C)] \qquad (O\text{-}19)$$

$$(III) = \frac{K^2}{4}[-\frac{3}{2}\gamma BCp - \frac{3}{2}\gamma CBp + (B)\gamma Cp + (C)(\gamma Bp)$$

$$- \frac{1}{4}(C\,\underline{\gamma\not{b}\gamma}\,B) - \frac{1}{4}(B\,\underline{\gamma\not{c}\gamma}\,C)] \qquad (O\text{-}20)$$

After some standard algebra the sum of these three diagrams is given by

$$(I) + (II) + (III) = \frac{K^2}{4}[-\frac{3}{2}\gamma BCp - \frac{1}{4}C\,\underline{\gamma\not{b}\gamma}\,B$$

$$+ \frac{1}{2b(p - \frac{c}{2})}(p + \frac{b}{2})C(2\gamma p + b\gamma - \gamma c - \not{b} + \underline{\gamma\not{b}\gamma})B(p - \frac{c}{2})]$$

+ the same terms with B exchanged for C, b for c

(O-21)

For additional examples of notation, we have

$$pC\gamma pBp = p_\mu e^2_{\mu\nu}\gamma_\nu p_\rho e^1_{\rho\sigma}p_\sigma$$

$$pC\not{p}Bp = \not{p}pCBp$$

Suppose we try to check these terms alone for gauge invariance.

We do this by replacing the polarization tensor $e_{\mu\nu}$ for <u>one</u> of the external gravitons by

$$e_{\mu\nu} \rightarrow e_{\mu\nu} + q_\mu \xi_\nu + q_\nu \xi_\mu \tag{O-6a}$$

Under this substitution the cross section should be unchanged provided we have all the diagrams for a real physical process. Let us make this substitution on the graviton B,

$$B \rightarrow B + b\xi + \xi b$$

or if we replace B by $b\xi + \xi b$ we should get zero.

Making the substitution in O-21,

$$B \rightarrow b\xi + \xi b \tag{O-22}$$

we do not get zero. In fact O-21 becomes under this substitution

$$B \rightarrow b\xi + \xi b$$

$$(I) + (II) + (III) \rightarrow \frac{K^2}{4}[-2(pC\xi)\not{b} - 2(\xi C\gamma)bp - 2(pC\gamma)c\xi] \tag{O-21a}$$

Thus the diagrams (I) + (II) + (III) alone are not gauge invariant and therefore do not completely describe the Compton effect.

To get a gauge invariant amplitude we must include diagram IV. From equation O-14 we see that we get

$$-\frac{K}{2}\, t^g_{\mu\nu} e^3_{\mu\nu}$$

by expanding $\int KF^3 d^4x$ in plane waves. Let F^3 be represented by the diagram

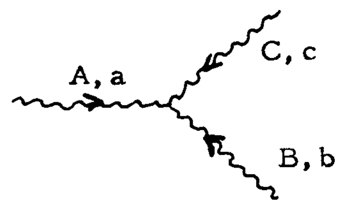

Fig. 16

where A is $e^3_{\mu\nu}$.

In terms of the fields $h_{\mu\nu}$, F^3 is given by equation C3-6 as

$$F^3 = \frac{1}{8}\left[h_{\alpha\beta}\bar{h}_{\gamma\delta}\bar{h}_{\alpha\beta,\gamma\delta} + h_{\gamma\beta}h_{\gamma\alpha}\bar{h}_{\alpha\beta,\delta\delta} - 2h_{\alpha\gamma}h_{\beta\delta}\bar{h}_{\alpha\beta,\gamma\delta} \right.$$

$$\left. + 2\bar{h}_{\alpha\beta}\bar{h}_{\alpha\gamma,\gamma}\bar{h}_{\beta\delta,\delta} + \frac{1}{2} h_{\alpha\beta}h_{\alpha\beta}\bar{h}_{\gamma\delta,\gamma\delta} + \frac{1}{4} h_{\alpha\alpha}h_{\beta\beta}\bar{h}_{\gamma\delta,\gamma\delta}\right]$$

To expand F^3 in plane waves we take all possible combinations of each term. (For reference, see the example given in equations O-14c and O-14d.) Using the notation described in O-17, the momentum space representation of F^3 is

$$KF^3 = \frac{K}{8}\left[(A\bar{C})c\bar{B}c + (A\bar{B})b\bar{C}b + (B\bar{C})c\bar{A}c + (C\bar{B})b\bar{A}b + (B\bar{A})a\bar{C}a \right.$$

$$+ (C\bar{A})a\bar{B}a + 2(AB\bar{C})cc + 2(AC\bar{B})bb + 2(\bar{A}BC)aa - 4cA\bar{C}Bc$$

$$- 4bA\bar{B}Cb - 4aB\bar{A}Ca + 4c\bar{C}\bar{A}\bar{B}b + 4a\bar{A}\bar{B}\bar{C}c + 4a\bar{A}\bar{C}\bar{B}b$$

$$+ a\bar{A}a(BC) + b\bar{B}b(AC) + c\bar{C}c(AB) + \frac{1}{2}a\bar{A}a(B)(C)$$

$$\left. + \frac{1}{2}b\bar{B}b(A)(C) + \frac{1}{2}c\bar{C}c(A)(B)\right] \qquad \text{(O-23)}$$

In equation O-23, the coefficient of e^3, or A is $-\frac{K}{2} t^g_{\mu\nu}$. More conveniently, the coefficient of $\bar{A}$ is $-\frac{K}{2}\bar{t}^g_{\mu\nu}$. Since the electron in diagram IV is free, the matrix element for diagram IV is from equation O-16

$$M_{IV} = K^2 s^m_{\mu\nu} \frac{1}{(q^3)^2} \bar{t}^g_{\mu\nu} = K(\gamma_\mu p_\nu + p_\nu \gamma_\mu) \frac{1}{(q^3)^2} (-\frac{K}{2} \bar{t}^g_{\mu\nu}) \qquad \text{(O-24)}$$

This matrix element may be obtained simply by replacing $\bar{A}$ or $\bar{e}^3_{\mu\nu}$ in O-23 by

$$\bar{e}^3_{\mu\nu} \rightarrow \frac{K}{(q^3)^2} (\gamma_\mu p_\nu - \gamma_\nu p_\mu) \qquad \text{(O-25a)}$$

$$\bar{A} \rightarrow \frac{K}{a^2} [\gamma p + p \gamma] \qquad \text{(O-25b)}$$

As an example of the substitution O-25b we have

$$\frac{K}{2} aB\bar{A}Ca \rightarrow \frac{K^2}{2a^2} [(aB\gamma)(pCa) + (aBp)(\gamma Ca)]$$

Before making the substitution O-25b in O-23, we note that O-23 may be simplified. Consider the factor

$$a\bar{A} = q^3_\mu e^3_{\mu\nu}$$

Under the substitution O-25a this becomes

$$\frac{K}{(q^3)^2} q^3_\mu (\gamma_\mu p_\nu + \gamma_\nu p_\mu) = \frac{K}{(q^3)^2} [\not{q}^3 p_\nu + \gamma_\nu p \cdot q^3] = 0 \qquad \text{(O-26)}$$

This expression is zero because

$$\not{q}^3 = \not{p}^f - \not{p}^i$$

$$\bar{u}(p^f)\not{q}^3 u(p^i) = \bar{u}(p^f)(\not{p}^f - \not{p}^i)u(p^i) = \bar{u}(m - m)u = 0$$

and

$$2p \cdot q = (p^i + p^f) \cdot (p^i - p^f) = (p^i)^2 - (p^f)^2 = m^2 - m^2 = 0$$

Thus any expression in O-23 containing a factor $a\bar{A}$ is zero.

We also have conservation of momentum which gives us the relation

$$q_\mu^3 = - q_\mu^2 - q_\mu^1$$

$$a = - b - c \tag{O-27}$$

This relation can be used to eliminate a in O-23. There are various manipulations possible in O-23. For example, we get

$$(A\bar{C}) = (\bar{A}C)$$

$$(\bar{B}C) = (BC) - \frac{1}{2}(B)(C)$$

by manipulation of the operation "bar". Using conservation of momentum and the fact that $a\bar{A}$ is zero, we can get

$$b\bar{A} = - c\bar{A} \qquad \text{etc.}$$

A convenient form for O-22, using $a\bar{A} = 0$ is

$$KF^3 = \frac{K}{8}[\, 2(\bar{A}C)cBc + (BC)b\bar{A}b + 2(\bar{A}BC)bc$$

$$- 4c\bar{A}CBc - 2cB\bar{A}Cb] \qquad \text{(a)}$$

+ the same terms with B exchanged for C, b for c

$$+ \frac{K}{8}[\, 4b^2(\bar{A}CB) - b^2(C)(\bar{A}B) - b^2(\bar{A})(CB) - b^2(B)(\bar{A}C)$$

$$+ \frac{1}{2}b^2(\bar{A})(B)(C) + 2b\bar{B}b(\bar{A}C) + 2b\bar{B}c(\bar{A}C) - b\bar{B}b(\bar{A})(C)$$

$$- 4b\bar{B}\bar{A}Cb + 2b\bar{B}Cb(\bar{A}) + 2b\bar{B}\bar{A}b(C)] \qquad \text{(b)}$$

+ the same terms with B exchanged for C, b for c

(O-28)

It is more convenient to test for gauge invariance by making the substitution O-22,

$$B \to b\xi + \xi b \qquad \text{(O-22)}$$

before making the substitution O-25b for $\bar{A}$. In using O-22 we will have for example

$$(\bar{A}BC) \to \xi\bar{A}Cb + b\bar{A}C\xi$$

$$(B) \to 2b \cdot \xi$$

$$b\bar{B}b \to b \cdot \xi$$

$$b\bar{B}\bar{A}b \to \xi\bar{A}b \qquad \text{etc.}$$

Making this substitution O-22 in part a of O-28 (including the terms with B and C, b and c exchanged) we simply get

$$KF^3_{(a)} \rightarrow \frac{K}{4} 2bc[(\overline{A}C)c \cdot \xi + 2b\overline{A}C\xi] \qquad \text{(O-29a)}$$

The substitution $B \rightarrow b\xi + \xi b$ in part b of O-28 (including the exchanged terms) does not give a simple result. We only get a simple result if we assume that the graviton $e^2_{\mu\nu}$ is free, or that $(q^2)^2 = 0$. If this graviton is free it satisfies the wave equation for free fields, equation B1-1. This equation automatically implies that $q^2_\mu \overline{e}^2_{\mu\nu} = 0$ (see part B1). Thus if $e^2_{\mu\nu}$ or C is free,

$$c^2 = 0$$

$$c\overline{C} = 0$$

and the only part of $KF^3_{(b)}$ that remains is that shown explicitly in O-28b without the exchange of B for C, b for c. Under this condition KF^3 becomes under the substitution $B \rightarrow b\xi + \xi b$

$$KF^3_{(b)} \rightarrow \frac{K}{4} b^2[(\overline{A}C)\xi \cdot c + 2b\overline{A}C\xi] \qquad \text{(O-30)}$$

If we now make the substitution

$$A \rightarrow \frac{K}{a^2}[\gamma p + p\gamma]$$

we get

$$M_{IV}(\xi) = \frac{K^2}{4} \frac{(b^2 + 2bc)}{a^2}[2(pC\xi)\not{b} + 2(\xi C\gamma)bp + 2(pC\gamma)c\xi] \qquad \text{(O-31)}$$

However $a^2 = b^2 + 2bc + c^2 = b^2 + 2bc$ for $c^2 = 0$, and O-31 just cancels O-21a. Thus the amplitude for the scattering of gravitons by electrons is gauge invariant, provided we include all four diagrams, and provided that the external graviton which is not tested for gauge invariance (namely C) is free.

The reason that C must be free for a test of gauge invariance is as follows. If C is not free, it was recently emitted from another source. Including this source the four diagrams of fig. 17 become

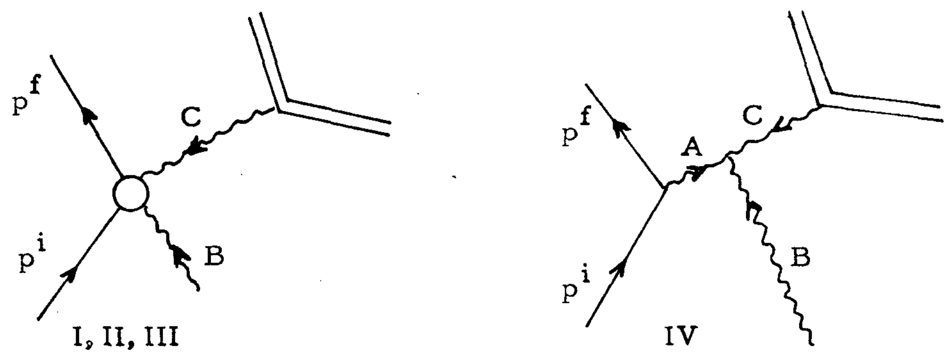

Fig. 18

There is however another physical process that can occur, namely

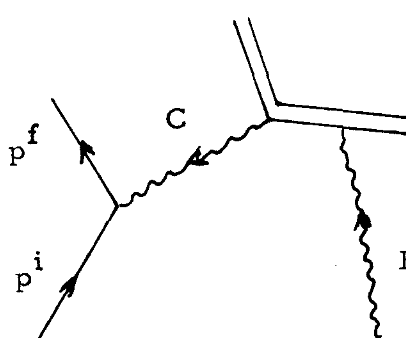

Fig. 19 V

If the graviton B is to interact with a complete conserved system, we must include the possibility of diagram V. We pointed out in deriving the gauge test O-6 that the external graviton must interact with a conserved source for the gauge test to work. For the case C is not free, we must include the interaction of B with the source of C, as shown in diagram V.

Only if C is free, or nearly free, is its source so far away that we do not need to include the possibility of B interacting with that source. It is then that we are able to successfully test B for gauge invariance.

Finally, we give the complete amplitude for the scattering of gravitons by electrons.

$$M = \frac{K^2}{4}\Big[-\frac{3}{2}\gamma BCp - \frac{1}{4} C\underline{\gamma b \gamma} B$$

$$+ \frac{1}{2b(p-\frac{c}{2})}(p+\frac{b}{2})C(2\gamma p + b\gamma - \gamma c - \not{b} + \underline{\gamma\not{b}\gamma})B(p-\frac{c}{2})$$

$$+ \frac{1}{b^2 + 2bc + c^2}\{2\gamma CpcBc - cBp\gamma Cb - cB\gamma pCb + bc\gamma BCp$$

$$+ bcpBC\gamma - 2cp\gamma CBc - 2\not{c}pCBc + bp\not{b}(BC) + 2b^2pCB\gamma$$

$$+ 2b^2\gamma CBp - b^2\not{p}(CB) - b^2\gamma Bp(C) - b^2\gamma Cp(B) + \frac{1}{2}b^2\not{p}(B)(C)$$

$$+ 2b\overline{B}bpC\gamma + 2b\overline{B}c\gamma Cp - b\overline{B}b\not{p}(C) - 2b\overline{B}p\gamma Cb - 2b\overline{B}\gamma pCb$$

$$+ 2b\overline{B}Cb\not{p} + b\overline{B}\gamma pb(C) + b\overline{B}p\not{b}(C)\}\Big]$$

+ the same terms with B exchanged for C, b for c.

VI. DIVERGENT CALCULATIONS

The subject of divergences in the quantum theory of gravity is quite complicated. For the case of gravity interacting with spin-zero particles Feynman has worked out the lowest order divergent diagrams for such processes as the gravitational self-energy of the particle, vacuum polarization, and the corrections to the scattering in an external potential. At present there are still some problems with this last calculation.

Because of the added complication of the graviton-electron interaction, the corresponding divergent calculations are even longer when gravity interacts with electrons. For this reason we shall present only the calculation of gravitational self-energy of the electron, and the most divergent part of the vacuum polarization. The rest of the vacuum polarization calculation involves considerable algebra which the author has not yet had time to check.

Harold Yura has been applying dispersion theory techniques to the problem of divergences in the quantum theory of gravity. This leads to a slightly different emphasis of what is to be calculated, for it is assumed that one is already working with a gauge invariant renormalized theory. Thus certain renormalization constants are not calculated. However the calculation of the remaining quantities appears to be far simpler. It therefore seems reasonable that any further divergent calculations involving electrons, with the more complicated electron graviton interaction, should be done using the dispersion theory technique.

P. GRAVITATIONAL SELF-ENERGY OF THE ELECTRON

The diagrams for the self-energy of the electron are the following

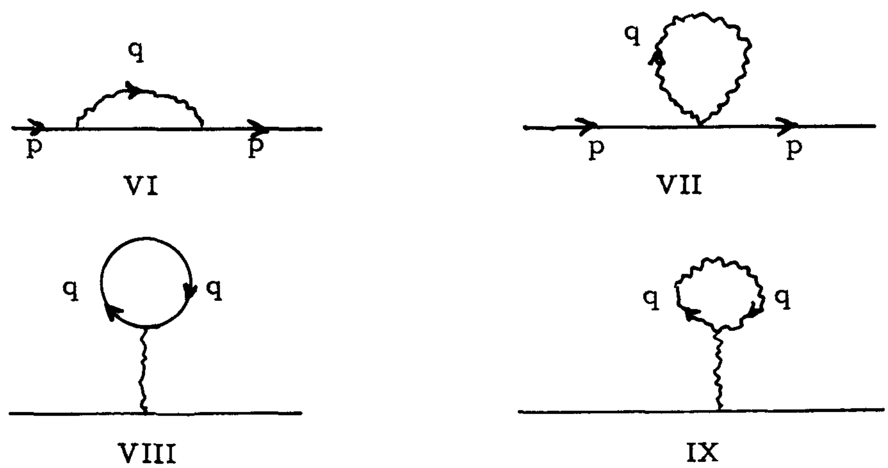

Fig. 20

To calculate diagrams VI and VII we will use the graviton-electron coupling N-19 and the graviton propagator O-16. The matrix element for diagram VI becomes

$$VI = \frac{K^2}{2}[\gamma_\rho(p - \frac{q}{2})_\sigma + \delta_{\rho\sigma}\frac{\not{q}}{2}]\frac{1}{\not{p} - \not{q} - m}$$

$$\times[\gamma_\mu(p - \frac{q}{2})_\nu + \frac{\not{q}}{2}\delta_{\mu\nu}] \times [\delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\nu\rho}\delta_{\mu\sigma} - \delta_{\mu\nu}\delta_{\rho\sigma}]/q^2 \quad \text{(P-1)}$$

Using $\not{p} = m$; $p^2 = m^2$ since the external electron lines are free this matrix element becomes

$$\frac{K^2}{2}[\not{p}(-4p\cdot q + \frac{3}{2}q^2) - 4mp\cdot q + \frac{5}{2}m^2q^2 + 2m^3]$$

$$\times [q^2(q^2 - 2p\cdot q)]^{-1} \qquad \text{(P-2)}$$

The matrix element for diagram VII is

$$\text{VII} = \frac{K^2}{2}\ [\gamma_\alpha p_\sigma(-\frac{3}{2}\delta_{\mu\alpha}\delta_{\nu\rho} + \delta_{\mu\nu}\delta_{\alpha\rho})$$

$$\times (\delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho} - \delta_{\mu\nu}\delta_{\rho\sigma})/q^2] \qquad \text{(P-3)}$$

Due to the symmetry of the propagator, the term in the interaction that is proportional to the antisymmetric combination of three γ matrices gives zero. The matrix element P-3 reduces to

$$\text{VII} = \frac{K^2}{2}[-\frac{8m}{q^2}] \qquad \text{(P-4)}$$

It is interesting to think of these two diagrams, VI and VII, as being obtained by connecting the external gravitons in the Compton effect. The relevant Compton effect diagrams are given in fig. 21.

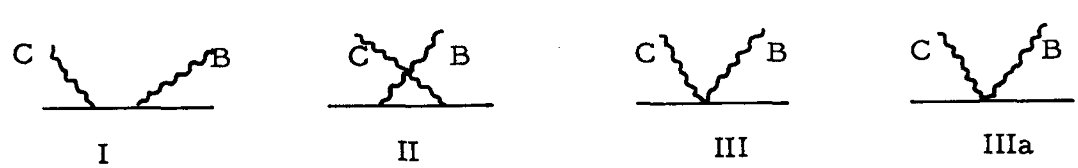

Fig. 21

If we connect the external gravitons in these diagrams by the propagator O-16, we will get just twice the self-energy diagrams VI and VII.

We get the factor two because connecting the gravitons of diagram I gives the same result as for diagram II, and each will give diagram VI. Similarly diagrams III and IIIa will each give diagram VII when the external diagrams are connected.

For the complete self-energy of the electron we should include diagrams VIII and IX. However, the calculation of these diagrams is not straightforward since the graviton that connects the electron with the external loop carries zero momentum. The propagator for that graviton, being inversely proportional to the square of the graviton's momentum, is therefore $1/0$.

Diagrams VI and VII lead to an infinite contribution to the self-energy of the electron, but only in the limit that q, the momentum of the virtual graviton, goes to infinity. If we cut off the momentum of the virtual graviton to a finite value, say λ, then the diagrams give only a finite contribution to the electron self-energy. However, no such cutoff can be used for diagrams VIII and IX since the connecting graviton has identically zero momentum and therefore a propagator that is always proportional to $1/0$.

Despite this factor $1/0$ in diagrams VIII and IX, we will show at the end of the next part that these diagrams give no contribution to the self-energy of the electron, thus the total contribution to the gravitational self-energy is from diagrams VI and VII.

The complete matrix element is obtained from P-2 and P-4 by integrating over all possible momenta q of the virtual graviton. The result is

$$M = \frac{K^2}{2}\int_{-\infty}^{\infty}\left\{\frac{-4\not{A}\,p\cdot q}{q^2(q^2-2p\cdot q)} + \frac{\frac{3}{2}\not{A} + \frac{5}{2}m}{(q^2-2p\cdot q)} - \frac{2\not{A}\,p\cdot q}{q^4} - \frac{6m}{q^2}\right\}\frac{d^4q}{(2\pi)^4} \quad \text{(a)}$$

$$+ \frac{K^2}{2}\int_{-\infty}^{\infty}\frac{(-4m\,p\cdot q + 2m^2)}{q^2(q^2-2p\cdot q)}\frac{d^4q}{(2\pi)^4} \quad \text{(b)}$$

(P-5)

In terms of this matrix element M the correction to the mass is given by

$$\Delta m\bar{u}u = \frac{1}{i}\bar{u}\,Mu^*$$

We shall calculate only the most divergent part of the self-energy. This means that the integral P-5b, which is not as divergent as P-5a, will not contribute. Using Feynman's (23) technique for calculating integrals, P-5a becomes

$$M = \frac{23}{128}\,\frac{K^2\lambda^2}{\pi^2 i}\,m \qquad \text{(P-6)}$$

where we used the cutoff $\lambda^4/(q^2-\lambda^2)$ in calculating the integrals.

The value of K^2 is given in equation B2-22 as

$$K^2 = 8\pi G$$

thus we get for Δm

$$\frac{\Delta m}{m} = -\frac{23}{16\pi}G\lambda^2 \qquad \text{(P-7)}$$

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*See Feynman (22).

Let us compare this value of $\Delta m/m$ with the value of $\Delta m_o/m_o$ for a spin-zero particle. Feynman gives the result that gravitational self-energy of a spin-zero particle is

$$\frac{\Delta m_o}{m_o} = -\frac{2}{\pi} G\lambda^2 + \text{finite terms}$$

which is not the same correction as for an electron.

It would be interesting if all gravitational mass corrections were the same. We can take as a basic unit of length the Compton wave length of any of the fundamental particles. These units of length are proportional to the mass of the particles. Thus if the mass of these particles were all renormalized by the same factor, each unit of length would be changed by the same factor and the gravitational mass renormalization could be interpreted as merely a uniform change in the scale of lengths. Since the electron mass renormalization is not the same as for a spin-zero particle, this interpretation is not correct.

O. VACUUM POLARIZATION

The diagrams for vacuum polarization are the following

Fig. 22

The matrix elements for these diagrams are

$$X = K^2 \int \frac{tr}{4} \left\{ \frac{1}{\not p - \frac{\not q}{2} - m} [\ \gamma Bp - (B)(\not p - m)]\ \frac{1}{\not p + \frac{\not q}{2} - m} \right.$$

$$\left. \times [\ \gamma Ap - (A)(\not p - m)] \right\} d^4p \qquad \text{(Q-1)}^*$$

$$XI = K^2 \int \frac{tr}{4} \left\{ \frac{1}{\not p - m} [(A)\gamma Bp + (B)\gamma Ap - \frac{3}{2}\gamma ABp - \frac{3}{2}\gamma BAp] \right.$$

$$\left. + \frac{(\not p - m)}{(\not p - m)} [\ 2(AB) - (A)(B)\ ] \right\} d^4p \qquad \text{(Q-2)}$$

where we are using the notation given in equations O-17. This notation is very convenient for taking traces. For example

$$\frac{tr}{4}[\ \not a \not b \not c \not d] = [\ (ab)(cd) + (ad)(cb) - (ac)(bd)]$$

Therefore the trace of a quantity such as $\not p \gamma Bp \not q \gamma Ap$ may be done by inspection:

$$\frac{tr}{4}[\not p \gamma Bp \not q \gamma Ap] = pBpqAp + qBppAp - p \cdot qpBAp$$

The matrix elements X and XI may be evaluated by techniques entirely similar to those used by Feynman (24) for the problem of vacuum polarization in quantum electrodynamics. The traces and integrals involved have been done by the author, but are not yet checked. There are no inherent difficulties or complicated integrals involved in

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*The part of the second order interaction proportional to $\underline{\gamma \not a \gamma}$ does not not contribute in the trace since $\underline{\gamma_\mu \gamma_\nu \gamma_\rho} = \epsilon_{\mu\nu\rho\sigma}\gamma_\sigma\gamma_5$.

calculating the divergent parts of X and XI; but quite a few pages of algebra are involved. The finite parts of X and XI involve one final integral that is rather difficult, but for physical interpretation there is apparently little gained by performing this integral.

It is far easier to calculate the most divergent part of X and XI. The most divergent part of these amplitudes is independent of the momentum q of the external particles and may be calculated for q = 0. In this case the amplitude X reduces to

$$X(q=0) = \int \frac{tr}{4} \left\{ \frac{\not{p}\gamma Ap\not{p}\gamma A\not{p} + m^2 \gamma Ap\gamma Ap}{(p^2 - m^2)^2} \right.$$

$$\left. + \frac{-\not{p}\gamma Ap(B) - \not{p}\gamma Bp(A)}{p^2 - m^2} + (A)(B) \right\} d^4p \qquad (Q\text{-}3)$$

There is no change in the form of XI for z = 0.

Taking the traces in Q-2 and Q-3 the amplitudes X and XI may be written

$$X(q=0) = \int \left[\frac{2pAppBp}{(p^2 - m^2)^2} - \frac{(pABp + pAp(B) + pBp(A)}{p^2 - m^2} \right] d^4p \qquad (Q\text{-}4)$$

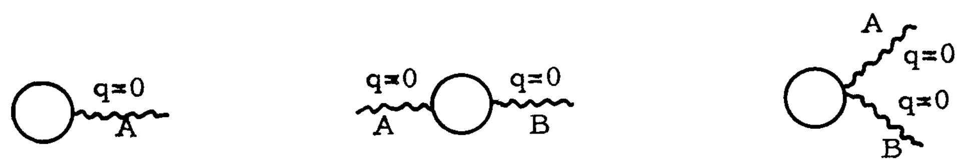
$$XI(q=0) = \int \left[\frac{-3pABp + (A)pBp + (B)pAp}{p^2 - m^2} \right] d^4p \qquad (Q\text{-}5)$$

where we have left out terms of the form

$$\int (AB) d^4p \qquad (Q\text{-}6)$$

Feynman's technique for evaluating integrals appearing in vacuum polarization problems involves calculating the integral for two different masses of the electron, namely the normal mass and a very large mass, and taking the difference. Therefore any integral such as Q-6 which does not involve the electron mass will not contribute.

Before evaluating Q-4 and Q-5 let us include the amplitude for the emission of a single graviton from a closed electron loop. This process only occurs for zero momentum of the graviton. We are therefore calculating the amplitude of the following diagrams.



XII Xa XIa

Fig. 23

The amplitude for XII is

$$XII = \int \frac{tr}{4} \left\{ \frac{1}{\not{p} - m} \left[2\gamma Ap - 2(\not{p} - m)(A) \right] \right\} d^4p$$

$$= \int \frac{2pAp}{p^2 - m^2} \, d^4p$$

The total amplitude for diagrams Xa, XIa and XII for $q = 0$ may be written in the following form

$$Xa + XIa + XII = 2A_{\mu\nu}B_{\rho\sigma} \int \frac{p_\mu p_\nu p_\rho p_\sigma}{(p^2 - m^2)} \, d^4p$$

$$+ \left[2A_{\mu\nu} - 4({}_\mu AB_\nu) \right] \int \frac{p_\mu p_\nu}{(p^2 - m^2)} \, d^4p \qquad \text{(Q-7)}$$

Using Feynman's methods for evaluating vacuum polarization integrals we get

$$\int \frac{p_\mu p_\nu p_\rho p_\sigma}{(p^2-m^2)^2}\ d^4p$$

$$= \frac{1}{16\pi^2 i} \times \frac{1}{4}(\delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\nu\rho}\delta_{\mu\sigma} + \delta_{\mu\nu}\delta_{\rho\sigma})[\rho^2(\ln\rho - \frac{3}{2})]_{m^2}^{m^2+\lambda^2} \qquad \text{(Q-8)}$$

$$\int \frac{p_\mu p_\nu}{(p^2-m^2)}\ d^4p = \frac{\delta_{\mu\nu}}{16\pi^2 i} \times \frac{1}{4}[\rho^2(\ln\rho - \frac{3}{2}]_{m^2}^{m^2+\lambda^2} \qquad \text{(Q-9)}$$

The complete contribution of these diagrams for $q = 0$ is

$$\frac{1}{16\pi^2 i}[\rho^2(\ln\rho - \frac{3}{2})]_{m^2}^{m^2+\lambda^2} \times [\frac{(A)}{2} - \frac{1}{2}(AB) + \frac{1}{4}(A)(B)] \qquad \text{(Q-10)}$$

The divergent part of $[\rho^2(\ln\rho - \frac{3}{2})]_{m^2}^{m^2+\lambda^2}$ may be obtained in the following way. Using

$$\ln(m^2+\lambda^2) \approx \ln[\frac{\lambda^2}{m^2}(1+\frac{m^2}{\lambda^2})] = \ln\frac{\lambda^2}{m^2} + \frac{m^2}{\lambda^2} - \frac{m^4}{2\lambda^4} + \ldots$$

we get

$$[\rho^2(\ln\rho - \frac{3}{2})] \approx -\lambda^4 \ln\frac{\lambda^2}{m^2} - 2m^2\lambda^2 \ln\frac{\lambda^2}{m^2} - m^4 \ln\frac{\lambda^2}{m^2} \qquad \text{(Q-11)}$$

where we have dropped all terms that are finite or contain only a factor of λ^2 or λ^4. (See Feynman (24).)

From Q-10 and Q-11 we get the most divergent part of the vacuum polarization diagrams X and XI:

$$X + XI = -\frac{1}{128\,\pi^2 i}\,\lambda^4 \ln \frac{\lambda^2}{m^2}\,[-2(AB) + (A)(B)]$$

$$+ \text{ less divergent terms} \qquad \text{(Q-12)}$$

This is the same degree of divergence for the vacuum polarization as for the case of gravity interacting with spin-zero particles.

In the case of electromagnetism the amplitude for vacuum polarization is less divergent for electrons than for spin-zero particles, namely $\ln \lambda^2/m^2$ compared to $\lambda^2 \ln(\lambda^2/m^2)$. This mildness of the divergence for electrons is not repeated in the case of gravity, and both spin-zero particles and electrons give rise to the higher divergence $\lambda^4 \ln (\lambda^2/m^2)$.

Finally let us return to the amplitude Q-10 representing the diagrams of fig. 23. Q-10 may be written

$$N(\lambda)[\tfrac{1}{2}(A) - \tfrac{1}{2}(AB) + \tfrac{1}{4}(A)(B)] \qquad \text{(Q-10a)}$$

where $N(\lambda)$ is a divergent constant. Suppose we added to the action the quantity

$$-\int N(\lambda)(-g)^{1/2} d^4x \qquad \text{(Q-13)}$$

The term Q-13 is a scalar quantity in the space of metric $g_{\mu\nu}$ and thus preserves the invariance properties of the action.

Expanding $(-g)^{1/2}$, we have added to the action

$$\int -N(\lambda)[1 + \frac{h}{2} + \frac{h^2}{8} - \frac{1}{4} h_{\mu\nu}h_{\mu\nu} + \ldots]\, d^4x \qquad \text{(Q-14)}$$

This term in the action gives rise to the diagrams

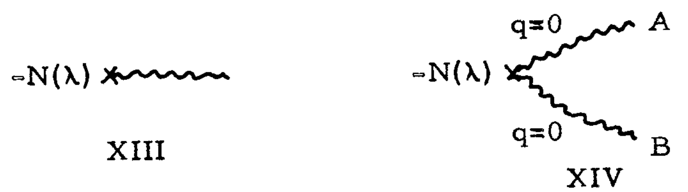


Fig. 24

The amplitude for these diagrams (treating A and B symmetrically in XIV) is

$$-N(\lambda)\left[\frac{1}{2}(A) - \frac{1}{2}(AB) + \frac{1}{4}(A)(B)\right]$$

which exactly cancels Q-10. Thus Q-13 may be considered the counter term in the action that removes the effect of the diagrams in fig. 23.

The physical interpretation of the diagrams in fig. 23 is as follows. The vacuum state in field theory is not represented by a real vacuum, but by the lowest state of the oscillators of the fields, in this case the electron field. The energy of the lowest state of a quantum oscillator is not zero but $\hbar\omega/2$, thus the vacuum state has an energy equal to $\hbar\omega/2$ for each oscillator of the field, or an infinite energy. To get the correct vacuum state this energy should have been subtracted. This is not usually done for the zero of energy has no meaning in most problems.

However gravity couples to all forms of energy including the energy of the so-called vacuum state, thus we must be careful to subtract this energy. In terms of diagrams the energy of the vacuum appears to lowest order in the form of an unconnected closed loop. That this closed loop is a source of gravity is seen in the diagrams of fig. 23. Thus when we add the counter term Q-13 to the action, we are subtracting off the energy of the vacuum state as well as the gravitational fields produced by that energy.

We can now return to the diagrams VIII and IX for the self-energy of the electron.

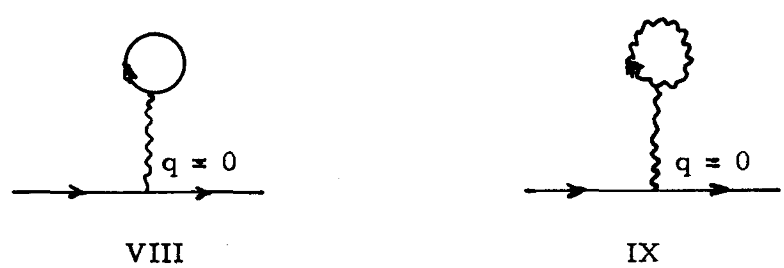


Fig. 25

Diagram VIII represents the energy of the vacuum producing a graviton of zero momentum which later interacts with the electron. Since the vacuum state of the gravitational field also has an energy, diagram IX has the same interpretation as VIII.

We have already introduced a counter term in the action so that the amplitude for the emission of a single graviton from a closed electron loop is zero. A similar counter term should be introduced so that the amplitude for a closed graviton loop to emit a single graviton is zero. With these counter terms the amplitude for diagrams VIII and IX will be proportional to zero times the propagator of the zero momentum graviton, or 0/0. Thus the magnitude of the contribution from these diagrams is undefined.

These diagrams will still have no physical effect for the following reason. All particles are affected by the gravitational field $h'_{\mu\nu}$ from the closed loops, thus these fields may be replaced by a space of metric $g'_{\mu\nu} = \delta_{\mu\nu} + Kh'_{\mu\nu}$ in which all particles move. Furthermore these fields are constant (there is an equal amplitude for the graviton to arrive at any point in space) and the metric $g'_{\mu\nu} = \delta_{\mu\nu} + Kh'_{\mu\nu}$ is a flat space metric. Thus by a suitable choice of the scales of length and time the effects of these fields will not be noticed.

This argument, invented by Feynman, shows that although the amplitude for diagrams VIII and IX may not be zero (they are proportional to 0/0), their physical effects are inobservable and should not be included in the calculation of the self-energy of the electron.

APPENDIX I

At the beginning of section II we pointed out that a potential theory could not be set up for a half integer-spin meson. The reason for this is simple. Assume an arrangement of the sources of this meson that gives rise to a static potential. If one of these sources is moved, the potential is changed. As with electric potentials, the change in potential must be brought about by the radiation of an infinite number of very low energy mesons. (This is the source of the infra-red divergence in quantum electrodynamics.)

Let us assume that a proton is the source of the mesons. To conserve angular momentum the proton could emit an integer-spin meson into a state of the opposite angular momentum and not change its own state. However, there are no angular distributions of radiation that correspond to half a unit of angular momentum, thus the proton must change its own state when emitting a half integer-spin meson. But this cannot happen in the limit that the proton emits an infinite number of such mesons corresponding to a change in potential. Thus a potential theory does not exist for a half integer-spin meson.

A potential theory may be constructed for integer spin mesons, but for even integer spins the force is attractive while for odd integer spins the force is repulsive between static like objects. This is a consequence of the fact that the energy in radiation fields must be positive and therefore those components of a meson field that contribute to the radiation field must have positive energy.

For the example of a single component spin-zero meson field the energy in the field must be positive. This immediately leads to an attractive force between like objects exchanging spin-zero mesons. Consider the case of two like parallel plates. The field between them will be uniform as long as the spin-zero meson has zero rest mass. (The $1/r$ Yukawa potential gives a force field of the same form as electrostatics.) As the plates are brought together the region of the fields, and thus the energy in the fields decreases. Thus the force must have been attractive. This result is also true for non-zero rest mass spin-zero mesons as is seen in the binding of nuclear matter by π mesons.

For a spin-one or vector meson the vector field consists of a time component, a longitudinal and two transverse spatial components. By a proper choice of gauge the longitudinal component may be eliminated (Coulomb gauge). The radiation fields are made up of the transverse components, thus these components must carry positive energy. Thus for electromagnetism like objects interacting via the transverse components of the field, such as parallel currents, attract each other.

To determine the sign of the energy in the time component of the field, one may consider the Hamiltonian density of the field. (See for example Schweber (25).) The over-all sign of the Hamiltonian density is determined by the fact that the transverse components of the field carry positive energy. The result, for the static vector field which we are considering, is that the energy in the time component is negative. Since the interaction of static particles is only through the time component (Coulomb field) this implies that like objects (charges) repel each other when interacting via a spin-one meson.

For the spin-two field like objects interact only through the time-time component of the tensor that describes the spin-two field. Here the sign of the energy is reversed again and static like objects attract. The general rule as stated earlier is that static like objects attract for the exchange of mesons of even integer spins, and repel for mesons of odd integer spins.

To distinguish the physical effects of a spin-zero and a spin-two gravitational theory, we can look at the form of the coupling for the two fields. Consider the case of gravity interacting with point particles. For static particles gravity must couple to the mass of the particle, and the only corrections to the coupling can be due to the velocity of the particle.

For a spin-zero gravitational field φ we might write the interaction part of the action as

$$S_{int} = \int \varphi(\vec{z})\, m\, dt \tag{1}$$

where $\vec{z}$ is the coordinate of the particle.

However, this action is not a scalar quantity under Lorentz transformations, and in the relativistic limit would not lead to conservation of energy and momentum. This may be corrected by replacing dt in 1 by

$$ds = (1 - v^2)^{1/2} dt$$

with the result

$$S_{int} = \int \varphi[\, m(1 - v^2)^{1/2}]\, dt\,.$$

Thus for a spin-zero theory of gravity the coupling would be proportional to $m(1 - v^2)^{1/2}$, or would be reduced for a moving particle.

For a spin-two gravitational field $h_{\mu\nu}$, in order that the action be scalar, gravity must be coupled to the energy tensor of the particle. This is because the energy tensor is the only tensor constructed from the mass and velocity of the particle that reduces to the mass when the velocity goes to zero. Thus for a moving particle which has more energy than when it is at rest, the coupling to gravity would be increased.

Finally we note that a theory of a spin-zero, zero rest mass meson field does not exist at present. If we started with a zero rest mass spin-zero particle, it would shortly create its own rest mass via virtual interactions with itself. There are no such self energy corrections for a photon, because of the gauge invariance properties of the electromagnetic field. We may also prevent self energy corrections of a zero rest mass spin-two meson by gauge invariance, but no gauge invariance exists for spin-zero.

APPENDIX II. RELATIONS AMONG γ MATRICES

The curved space γ matrices are defined by the relation

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu} \tag{1}$$

We define the antisymmetric combination of γ matrices by

$$\underline{\gamma_\mu\gamma_\nu} = \frac{1}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu) \tag{2}$$

$$\underline{\gamma_\mu\gamma_\nu\gamma_\rho} = \frac{1}{3}(\gamma_\mu\underline{\gamma_\nu\gamma_\rho} - \gamma_\nu\underline{\gamma_\mu\gamma_\rho} + \gamma_\rho\underline{\gamma_\mu\gamma_\nu}) \tag{3}$$

$$\underline{\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma} = \frac{1}{4}(\gamma_\mu\underline{\gamma_\nu\gamma_\rho\gamma_\sigma} - \gamma_\rho\underline{\gamma_\mu\gamma_\nu\gamma_\sigma}$$

$$+ \gamma_\rho\underline{\gamma_\mu\gamma_\nu\gamma_\sigma} - \gamma_\sigma\underline{\gamma_\mu\gamma_\nu\gamma_\rho}) \tag{4}$$

The quantities $\underline{\gamma_\mu\gamma_\nu}$, $\underline{\gamma_\mu\gamma_\nu\gamma_\rho}$ and $\underline{\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma}$ are all antisymmetric in their indices μ, ν, ρ, σ, and normalized to one. That is if $A_{\mu\nu}$, $A_{\mu\nu\rho}$, and $A_{\mu\nu\rho\sigma}$ are antisymmetric tensors

$$\gamma_\mu\gamma_\nu A_{\mu\nu} = \underline{\gamma_\mu\gamma_\nu} A_{\mu\nu}$$

$$\gamma_\mu\gamma_\nu\gamma_\rho A_{\mu\nu\rho} = \underline{\gamma_\mu\gamma_\nu\gamma_\rho} A_{\mu\nu\rho}$$

$$\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma A_{\mu\nu\rho\sigma} = \underline{\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma} A_{\mu\nu\rho\sigma} .$$

We can express the quantities $\underline{\gamma_\mu\gamma_\nu\gamma_\rho}$ and $\underline{\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma}$ in terms of the γ matrix $\gamma_5 = \gamma_x\gamma_y\gamma_z\gamma_t$ by the equations

$$\underline{\gamma_\mu\gamma_\nu\gamma_\rho} = \epsilon_{\mu\nu\rho\sigma}\gamma_5\gamma_\sigma \tag{5}$$

$$\underline{\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma} = \epsilon_{\mu\nu\rho\sigma}\gamma_5 \tag{6}$$

where $\epsilon_{\mu\nu\rho\sigma}$ is the antisymmetric unit tensor which is zero unless $\mu \neq \nu \neq \rho \neq \sigma$ and equal to + 1 or - 1 according to whether μ, ν, ρ, σ is an even or odd permutation of x, y, z, t.

The following are a set of relations among the matrices.

$$\gamma_\rho\underline{\gamma_\mu\gamma_\nu} - \underline{\gamma_\mu\gamma_\nu}\gamma_\rho = 2g_{\rho\mu}\gamma_\nu - 2g_{\rho\nu}\gamma_\mu \tag{7}$$

$$\gamma_\rho\underline{\gamma_\mu\gamma_\nu} = \underline{\gamma_\mu\gamma_\nu\gamma_\rho} + g_{\rho\mu}\gamma_\nu - g_{\rho\nu}\gamma_\mu \tag{8}$$

$$\underline{\gamma_\mu\gamma_\nu}\gamma_\rho = \underline{\gamma_\mu\gamma_\nu\gamma_\rho} - g_{\rho\mu}\gamma_\nu + g_{\rho\nu}\gamma_\mu \tag{9}$$

$$\underline{\gamma_\mu\gamma_\nu\gamma_\rho} = \gamma_\mu\gamma_\nu\gamma_\rho - g_{\mu\nu}\gamma_\rho + g_{\mu\rho}\gamma_\nu - g_{\nu\rho}\gamma_\mu \tag{10}$$

$$\underline{\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma} = \frac{1}{6}\Big[\underline{\gamma_\mu\gamma_\nu}\,\underline{\gamma_\rho\gamma_\sigma} + \underline{\gamma_\rho\gamma_\sigma}\,\underline{\gamma_\mu\gamma_\nu} + \underline{\gamma_\mu\gamma_\sigma}\,\underline{\gamma_\nu\gamma_\rho}$$

$$+ \underline{\gamma_\nu\gamma_\rho}\,\underline{\gamma_\mu\gamma_\sigma} + \underline{\gamma_\rho\gamma_\mu}\,\underline{\gamma_\nu\gamma_\sigma} + \underline{\gamma_\sigma\gamma_\nu}\,\underline{\gamma_\mu\gamma_\rho}\Big] \tag{11}$$

$$\frac{1}{2}\Big[\underline{\gamma_\mu\gamma_\nu}\,\underline{\gamma_\rho\gamma_\sigma} + \underline{\gamma_\rho\gamma_\sigma}\,\underline{\gamma_\mu\gamma_\nu}\Big] = \frac{1}{2}\Big[\underline{\gamma_\mu\gamma_\sigma}\,\underline{\gamma_\nu\gamma_\rho} + \underline{\gamma_\nu\gamma_\rho}\,\underline{\gamma_\mu\gamma_\sigma}\Big]$$

$$+ g_{\mu\nu}g_{\rho\sigma} + g_{\rho\nu}g_{\mu\sigma} - 2g_{\mu\rho}g_{\nu\sigma} \tag{12}$$

$$\frac{1}{2}\Big[\underline{\gamma_\mu\gamma_\nu}\,\underline{\gamma_\rho\gamma_\sigma} + \underline{\gamma_\rho\gamma_\sigma}\,\underline{\gamma_\mu\gamma_\nu}\Big] = \frac{1}{2}\Big[\underline{\gamma_\rho\gamma_\mu}\,\underline{\gamma_\nu\gamma_\sigma} + \underline{\gamma_\nu\gamma_\sigma}\,\underline{\gamma_\rho\gamma_\mu}\Big]$$

$$- g_{\mu\nu}g_{\rho\sigma} - g_{\mu\rho}g_{\nu\sigma} + 2g_{\mu\sigma}g_{\rho\nu} \tag{13}$$

From equation 11, 12, and 13 we get

$$\frac{1}{2}\left[\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma + \gamma_\rho\gamma_\sigma\gamma_\mu\gamma_\nu\right] = \gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma$$

$$+ g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma} \tag{14}$$

We also have

$$\frac{1}{2}\left[\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma - \gamma_\rho\gamma_\sigma\gamma_\mu\gamma_\nu\right] = 2\left[g_{\rho\mu}\gamma_\sigma\gamma_\nu + g_{\rho\nu}\gamma_\mu\gamma_\sigma\right.$$

$$\left. + g_{\sigma\mu}\gamma_\nu\gamma_\rho + g_{\sigma\nu}\gamma_\rho\gamma_\mu\right] \tag{15}$$

APPENDIX III. THE MATRIX Γ_μ

We wish to solve equation J-7

$$\gamma^{\nu}_{;\mu} = \Gamma_\mu \gamma^\nu - \gamma^\nu \Gamma_\mu$$

Multiplying through by $g_{\rho\nu}$ and noting that the covariant derivative of $g_{\rho\nu}$ is zero, we get the equivalent equation

$$\gamma_{\rho;\mu} = \Gamma_\mu \gamma_\rho - \gamma_\rho \Gamma_\mu \tag{1}$$

Now the commutation relation satisfied by γ_ρ is

$$\gamma_\alpha \gamma_\rho + \gamma_\rho \gamma_\alpha = 2 g_{\alpha\rho} \tag{2}$$

Taking the covariant derivative of both sides of 2 we get

$$\gamma_{\alpha;\mu}\gamma_\rho + \gamma_\alpha \gamma_{\rho;\mu} + \gamma_{\rho;\mu}\gamma_\alpha + \gamma_\rho \gamma_{\alpha;\mu} = 0 \tag{3}$$

Multiplying 3 on the left by γ^α, noting $\gamma^\alpha \gamma_\alpha = 4$, we get

$$\gamma^\alpha \gamma_{\alpha;\mu}\gamma_\rho + 4\gamma_{\rho;\mu} + \gamma^\alpha \gamma_{\rho;\mu}\gamma_\alpha + \gamma^\alpha \gamma_\rho \gamma_{\alpha;\mu} = 0 \tag{4}$$

But

$$\gamma^\alpha \gamma_\rho \gamma_{\alpha;\mu} = - \gamma_\rho \gamma^\alpha \gamma_{\alpha;\mu} + 2\gamma_{\rho;\mu} \tag{5}$$

Equation 4 becomes

$$6\gamma_{\rho;\mu} + \gamma^\alpha \gamma_{\rho;\mu}\gamma_\alpha = \gamma_\rho(\gamma^\alpha \gamma_{\alpha;\mu}) - (\gamma^\alpha \gamma_{\alpha;\mu})\gamma_\rho \tag{6}$$

At this point the author has been unable to proceed without expanding γ_ρ in terms of the flat space matrices by the relations K-3

$$\gamma_\rho = b_{\rho\delta}\bar{\gamma}_\delta \quad ; \quad \bar{\gamma}_\delta = b_{\sigma\delta}\gamma^\sigma$$

From this we get

$$\gamma_{\rho;\mu} = \gamma_{\rho,\mu} - \Gamma_{\sigma,\rho\mu}\gamma^\sigma = (b_{\rho\delta,\mu}b_{\sigma\delta} - \Gamma_{\sigma,\rho\mu})\gamma^\sigma \qquad (7)$$

where we have used $(b_{\rho\delta}\bar{\gamma}_\delta)_{,\mu} = b_{\rho\delta,\mu}\bar{\gamma}_\delta$ since we are assuming a representation of the flat space γ matrices where

$$\bar{\gamma}_{\delta,\mu} = 0$$

Using equation 7 we get

$$\gamma^a\gamma_{\rho;\mu}\gamma_a = (b_{\rho\delta,\mu}b_{\sigma\delta} - \Gamma_{\sigma,\rho\mu})\gamma^a\gamma^\sigma\gamma_a \qquad (8)$$

Now

$$\gamma^a\gamma^\sigma\gamma_a = -\gamma^a\gamma_a\gamma^\sigma + 2\gamma^a\delta^\sigma_a = -2\gamma^\sigma$$

Therefore

$$\gamma^a\gamma_{\rho;\mu}\gamma_a = -2(b_{\rho\delta,\mu}b_{\sigma\delta} - \Gamma_{\sigma,\mu})\gamma^\sigma = -2\gamma_{\rho;\mu} \qquad (9)$$

The author has been unable to obtain equation 9 directly from the commutation relations of γ_ρ. It is not known whether this is a failure on the part of the author, or whether the relation K-3 puts an added restriction on the curved space γ matrices that allows us to obtain the relation 9. This question is not important for the quantum mechanics of gravity interacting with electrons, since the quantum mechanics is based on the expansion K-3.

Substituting equation 9 into 6 we get

$$\gamma_{\rho;\mu} = \gamma_\rho\left(\frac{1}{4}\gamma^\alpha\gamma_{\alpha;\mu}\right) - \left(\frac{1}{4}\gamma^\alpha\gamma_{\alpha;\mu}\right)\gamma_\rho \tag{10}$$

Comparing with equation 1

$$\gamma_{\rho;\mu} = \gamma_\rho(-\Gamma_\mu) - (-\Gamma_\mu)\gamma_\rho$$

we get

$$\Gamma_\mu = -\frac{1}{4}\gamma^\alpha\gamma_{\alpha;\mu} \tag{11}$$

Since

$$(\gamma^\alpha\gamma_\alpha)_{;\mu} = 0 = \gamma^\alpha{}_{;\mu}\gamma_\alpha + \gamma^\alpha\gamma_{\alpha;\mu} = \gamma_{\alpha;\mu}\gamma^\alpha + \gamma^\alpha\gamma_{\alpha;\mu}$$

we also get

$$\Gamma_\mu = \frac{1}{4}\gamma_{\alpha;\mu}\gamma^\alpha \tag{12}$$

which is the solution given in equation J-8.

Alternate forms of Γ_μ in terms of the quantities a and b may be obtained in the following way.

$$\Gamma_\mu = \frac{1}{4}\gamma_{\alpha;\mu}\gamma^\alpha = -\frac{1}{4}\gamma^\alpha\gamma_{\alpha;\mu} = \frac{1}{8}(\gamma_{\alpha;\mu}\gamma^\alpha - \gamma^\alpha\gamma_{\alpha;\mu}) \tag{13}$$

Using equation 7 we get

$$\Gamma_\mu = \frac{1}{4}(b_{\beta\lambda,\mu}b_{\lambda\alpha} + \Gamma_{\beta,\mu\alpha})\underline{\gamma^\alpha\gamma^\beta} \tag{14}$$

In terms of the flat space γ matrices $\overline{\gamma}_\alpha$ this becomes

$$\Gamma_\mu = \frac{1}{4}(b_{\lambda\alpha,\mu}a_{\lambda\beta} + \Gamma_{\lambda,\mu\delta}a_{\delta\alpha}a_{\lambda\beta})\underline{\overline{\gamma}_\alpha\overline{\gamma}_\beta} \tag{15}$$

To lowest order in $h_{\mu\nu}$ we get

$$\Gamma_\mu = \frac{1}{4} h_{\beta\mu,\alpha}\underline{\bar{\gamma}_\alpha \bar{\gamma}_\beta} \tag{16}$$

In equation K-7 we desired the quantity

$$\frac{1}{2}[\Gamma_\mu \gamma^\mu + \gamma^\mu \Gamma_\mu] \tag{17}$$

From equation 14 we get

$$\frac{1}{2}(\Gamma_\mu \gamma^\mu + \gamma^\mu \Gamma_\mu) = \frac{1}{8}(b_{\beta\delta,\mu} b_{\delta\alpha} + \Gamma_{\beta,\mu\alpha})(\underline{\gamma^\alpha \gamma^\beta}\gamma^\mu + \gamma^\mu \underline{\gamma^\alpha \gamma^\beta}) \tag{18}$$

From equations 8 and 9 in appendix II we get

$$(\underline{\gamma^\alpha \gamma^\beta}\gamma^\mu + \gamma^\mu \underline{\gamma^\alpha \gamma^\beta}) = 2\underline{\gamma^\alpha \gamma^\beta \gamma^\mu} \tag{19}$$

Since $\Gamma_{\beta,\mu\alpha}$ is symmetric in μ and α, it gives zero when multiplied by $\underline{\gamma^\alpha \gamma^\beta \gamma^\mu}$. Expressing the remainder of equation 17 in terms of flat space matrices we get K-7

$$\frac{1}{2}[\Gamma_\mu \gamma^\mu + \gamma^\mu \Gamma_\mu] = \frac{1}{4} b_{\alpha\mu,\beta} a_{\alpha\nu} a_{\beta\rho} \underline{\bar{\gamma}_\mu \bar{\gamma}_\nu \bar{\gamma}_\rho} \tag{20}$$

APPENDIX IV

To prove equation L-20 we use the relation

$$\Gamma'_\kappa = \frac{1}{4}(\gamma_{\alpha;\kappa}\gamma^\alpha)' = \frac{1}{4}(\gamma_{\alpha,\kappa}\gamma_\beta)' g^{\alpha\beta'} - \frac{1}{4}\Gamma^{\rho'}_{\alpha\kappa} g^{\alpha\beta'}\gamma'_\rho\gamma'_\beta$$

Substituting $\gamma'_\alpha = b_{\alpha\sigma}(h^*)S^{-1}\overline{\gamma}_\sigma S$ and noting that

$$\Gamma^{\rho'}_{\alpha\kappa} = \Gamma^\rho_{\alpha\kappa}(h^*)$$

we get

$$\begin{aligned}\Gamma'_\kappa = \frac{1}{4}S^{-1}\{g^{\alpha\beta}(h^*)[\, & b_{\alpha\rho,\kappa}(h^*)b_{\beta\sigma}(h^*)\overline{\gamma}_\rho\overline{\gamma}_\sigma \\ & - \Gamma^\rho_{\alpha\kappa}(h^*)b_{\alpha\rho}(h^*)b_{\beta\sigma}(h^*)\overline{\gamma}_\rho\overline{\gamma}_\sigma \\ & + b_{\alpha\rho}(h^*)b_{\beta\sigma}(h^*)(\overline{\gamma}_\rho S_{,\kappa}S^{-1}\overline{\gamma}_\sigma - S_{,\kappa}S^{-1}\overline{\gamma}_\rho\overline{\gamma}_\sigma)]\}\, S\end{aligned}$$

where we have used

$$\gamma_{\alpha,\kappa} = [\, b_{\alpha\rho}S^{-1}\overline{\gamma}_\rho S]_{,\kappa} = S^{-1}\{S[b_{\alpha\rho}S^{-1}\overline{\gamma}_\rho S]_{,\kappa}S^{-1}\}S$$

and

$$SS^{-1}_{,\kappa} = -S_{,\kappa}S^{-1}$$

If we note that

$$\begin{aligned}\Gamma_\kappa(h^*) = \frac{1}{4}g^{\alpha\beta}(h^*)[\, & b_{\alpha\rho}(h^*)b_{\beta\sigma}(h^*)\overline{\gamma}_\rho\overline{\gamma}_\sigma \\ & - \Gamma^\rho_{\alpha\kappa}(h^*)b_{\alpha\rho}(h^*)b_{\beta\sigma}(h^*)\overline{\gamma}_\rho\overline{\gamma}_\sigma]\end{aligned}$$

and that from equation K-3

$$g^{\alpha\beta}b_{\alpha\rho}b_{\beta\sigma} = g^{\alpha\beta}(h^*)b_{\alpha\rho}(h^*)b_{\beta\sigma}(h^*)$$

we get

$$\Gamma'_{\kappa} = S^{-1}[\,\Gamma_{\kappa}(h^{*}) + \frac{1}{4}\,\overline{\gamma}_{\rho} S_{,\kappa} S^{-1} \overline{\gamma}_{\rho} - S_{,\kappa} S^{-1}]\,S$$

However, from equations L-17 we get

$$\overline{\gamma}_{\rho} S_{,\kappa} S^{-1} \overline{\gamma}_{\rho} = \frac{1}{4} D_{\mu\nu,k} \overline{\gamma}_{\rho} \underline{\overline{\gamma}_{\mu} \overline{\gamma}_{\nu}} \overline{\gamma}_{\rho} = 0$$

since $\overline{\gamma}_{\rho} \underline{\overline{\gamma}_{\mu} \overline{\gamma}_{\nu}} \overline{\gamma}_{\rho} = 0$. Thus we have proved equation L-20.

$$\Gamma'_{\kappa} = S^{-1}[\,\Gamma_{\kappa}(h^{*}) - S_{,\kappa} S^{-1}]\,S$$

APPENDIX V

We wish to investigate the quantity

$$-\underline{\gamma^{\alpha}\gamma^{\beta}}(\Gamma_{\alpha;\beta}+\Gamma_{\alpha}\Gamma_{\beta}) = -\frac{1}{2}\gamma^{\alpha}\gamma^{\beta}(\Gamma_{\alpha,\beta}-\Gamma_{\beta,\alpha}+\Gamma_{\alpha}\Gamma_{\beta}-\Gamma_{\beta}\Gamma_{\alpha}) \qquad (1)$$

where we used the relation

$$\Gamma_{\alpha;\beta}-\Gamma_{\beta;\alpha} = \Gamma_{\alpha,\beta}-\Gamma^{\rho}_{\alpha\beta}\Gamma_{\rho}-\Gamma_{\beta,\alpha}+\Gamma^{\rho}_{\beta\alpha}\Gamma_{\rho} = \Gamma_{\alpha,\beta}-\Gamma_{\beta,\alpha}$$

We can see the meaning of the terms in 1 by the following investigation.

The order of partial differentiation is interchangeable, thus

$$\gamma_{\rho,\mu\nu}-\gamma_{\rho,\nu\mu} = 0 \qquad (2)$$

Now

$$\gamma_{\rho,\mu} = \gamma_{\rho;\mu}+\Gamma^{\sigma}_{\rho\mu}\gamma_{\sigma} = \Gamma_{\mu}\gamma_{\rho}-\gamma_{\rho}\Gamma_{\mu}+\Gamma^{\sigma}_{\rho\mu}\gamma_{\sigma}$$

$$\begin{aligned}\gamma_{\rho,\mu\nu} = {} & \Gamma_{\mu,\nu}\gamma_{\rho}-\gamma_{\rho}\Gamma_{\mu,\nu}+\Gamma_{\mu}\Gamma_{\nu}\gamma_{\rho}-\Gamma_{\mu}\gamma_{\rho}\Gamma_{\nu}+\Gamma^{\sigma}_{\rho\nu}\Gamma_{\mu}\gamma_{\sigma} \\ & -\Gamma_{\nu}\gamma_{\rho}\Gamma_{\mu}+\gamma_{\rho}\Gamma_{\nu}\Gamma_{\mu}-\Gamma^{\sigma}_{\rho\nu}\gamma_{\sigma}\Gamma_{\mu}+\Gamma^{\sigma}_{\rho\mu,\nu}\gamma_{\sigma} \\ & +\Gamma^{\sigma}_{\rho\mu}\Gamma_{\nu}\gamma_{\sigma}-\Gamma^{\sigma}_{\rho\mu}\gamma_{\sigma}\Gamma_{\nu}+\Gamma^{\delta}_{\rho\mu}\Gamma^{\sigma}_{\delta\nu}\gamma_{\sigma}\end{aligned} \qquad (3)$$

And in a similar manner

$$\begin{aligned}-\gamma_{\rho,\nu\mu} = {} & \gamma_{\rho}\Gamma_{\nu,\mu}-\Gamma_{\nu,\mu}\gamma_{\rho}+\Gamma_{\mu}\gamma_{\rho}\Gamma_{\nu}-\gamma_{\rho}\Gamma_{\mu}\Gamma_{\nu}+\Gamma^{\sigma}_{\rho\mu}\gamma_{\sigma}\Gamma_{\nu} \\ & -\Gamma_{\nu}\Gamma_{\mu}\gamma_{\rho}+\Gamma_{\nu}\gamma_{\rho}\Gamma_{\mu}-\Gamma^{\sigma}_{\rho\mu}\Gamma_{\nu}\gamma_{\sigma}-\Gamma^{\sigma}_{\rho\nu,\mu}\gamma_{\sigma} \\ & +\Gamma^{\sigma}_{\rho\nu}\gamma_{\sigma}\Gamma_{\mu}-\Gamma^{\sigma}_{\rho\nu}\Gamma_{\mu}\gamma_{\sigma}-\Gamma^{\delta}_{\rho\nu}\Gamma^{\sigma}_{\delta\mu}\gamma_{\sigma}\end{aligned} \qquad (4)$$

Substituting 3 and 4 in 2 we get

$$(\Gamma_{\mu,\nu} - \Gamma_{\nu,\mu} + \Gamma_\mu\Gamma_\nu - \Gamma_\nu\Gamma_\mu)\gamma_\rho - \gamma_\rho(\Gamma_{\mu,\nu} - \Gamma_{\nu,\mu} + \Gamma_\mu\Gamma_\nu - \Gamma_\nu\Gamma_\mu)$$

$$= +\gamma_\sigma[-\Gamma^\sigma_{\rho\mu,\nu} + \Gamma^\sigma_{\rho\nu,\mu} - \Gamma^\delta_{\rho\mu}\Gamma^\sigma_{\delta\nu} + \Gamma^\delta_{\rho\nu}\Gamma^\sigma_{\delta\mu}] \qquad (5)$$

The terms in the square brackets are just $R^\sigma_{\rho\mu\nu}$. If we define

$$\Gamma_{\mu,\nu} - \Gamma_{\nu,\mu} + \Gamma_\mu\Gamma_\nu - \Gamma_\nu\Gamma_\mu = C_{\mu\nu} \qquad (6)$$

we get

$$\gamma_\rho C_{\mu\nu} - C_{\mu\nu}\gamma_\rho = -\gamma_\sigma R^\sigma_{\rho\mu\nu} = +\gamma^\sigma R_{\rho\sigma\mu\nu} \qquad (7)$$

Try as a solution of equation 7

$$C_{\mu\nu} = +\frac{1}{4} R_{\alpha\beta\mu\nu}\underline{\gamma^\alpha\gamma^\beta} \qquad (8)$$

We get

$$\gamma_\rho C_{\mu\nu} - C_{\mu\nu}\gamma_\rho = +\frac{1}{4} R_{\alpha\beta\mu\nu}(\gamma_\rho\underline{\gamma^\alpha\gamma^\beta} - \underline{\gamma^\alpha\gamma^\beta}\gamma_\rho)$$

$$= +\frac{1}{2} R_{\alpha\beta\mu\nu}(\delta^\alpha_\rho\gamma^\beta - \delta^\beta_\rho\gamma^\alpha)$$

$$= +\gamma^\sigma R_{\rho\sigma\mu\nu}$$

which checks equation 8.

Equation 8 may be written

$$(\Gamma_{\alpha,\beta} - \Gamma_{\beta,\alpha} + \Gamma_\alpha\Gamma_\beta - \Gamma_\beta\Gamma_\alpha) = +\frac{1}{4}\underline{\gamma^\mu\gamma^\nu}R_{\mu\nu\alpha\beta}$$

$$= +\frac{1}{4}\gamma^\mu\gamma^\nu R_{\mu\nu\alpha\beta}$$

since $R_{\mu\nu\alpha\beta} = - R_{\nu\mu\alpha\beta}$. We now get from equation 1

$$-\gamma^{\alpha}\gamma^{\beta}(\Gamma_{\alpha;\beta} + \Gamma_{\alpha}\Gamma_{\beta}) = -\frac{1}{8}\gamma^{\alpha}\gamma^{\beta}\gamma^{\mu}\gamma^{\nu}R_{\mu\nu\alpha\beta} \tag{9}$$

The quantity $-\gamma^{\alpha}\gamma^{\beta}\gamma^{\mu}\gamma^{\nu}R_{\mu\nu\alpha\beta}$ may be written

$$-\gamma^{\alpha}\gamma^{\beta}\gamma^{\mu}\gamma^{\nu}R_{\alpha\beta\mu\nu} = -\frac{1}{3}(\gamma^{\mu}\gamma^{\nu}\gamma^{\alpha}\gamma^{\beta}R_{\mu\nu\alpha\beta} + \gamma^{\mu}\gamma^{\beta}\gamma^{\nu}\gamma^{\alpha}R_{\mu\beta\nu\alpha}$$
$$+ \gamma^{\mu}\gamma^{\alpha}\gamma^{\beta}\gamma^{\nu}R_{\mu\alpha\beta\nu}) \tag{10}$$

We have

$$\gamma^{\mu}\gamma^{\beta}\gamma^{\nu}\gamma^{\alpha} = \gamma^{\mu}\gamma^{\nu}\gamma^{\alpha}\gamma^{\beta} + 2g^{\beta\nu}\gamma^{\mu}\gamma^{\alpha} - 2g^{\alpha\beta}\gamma^{\mu}\gamma^{\nu}$$

$$\gamma^{\mu}\gamma^{\alpha}\gamma^{\beta}\gamma^{\nu} = \gamma^{\mu}\gamma^{\nu}\gamma^{\alpha}\gamma^{\beta} + 2g^{\beta\nu}\gamma^{\mu}\gamma^{\alpha} - 2g^{\alpha\nu}\gamma^{\mu}\gamma^{\beta}$$

Therefore

$$-\gamma^{\alpha}\gamma^{\beta}\gamma^{\mu}\gamma^{\nu}R_{\alpha\beta\mu\nu} = -\frac{1}{3}[\gamma^{\mu}\gamma^{\nu}\gamma^{\alpha}\gamma^{\beta}(R_{\mu\nu\alpha\beta} + R_{\mu\beta\nu\alpha} + R_{\mu\alpha\beta\nu})$$
$$+ 2\gamma^{\mu}\gamma^{\alpha}R_{\mu\alpha} + 2\gamma^{\mu}\gamma^{\nu}R_{\mu\nu} + 2\gamma^{\mu}\gamma^{\beta}R_{\mu\beta}] \tag{11}$$

Using the identity

$$R_{\mu\nu\alpha\beta} + R_{\mu\beta\nu\alpha} + R_{\mu\alpha\beta\nu} = 0$$

equation 11 becomes

$$-\gamma^{\alpha}\gamma^{\beta}\gamma^{\mu}\gamma^{\nu}R_{\alpha\beta\mu\nu} = -2R$$

Substitution of 12 into 9 gives

$$-\gamma^{\alpha}\gamma^{\beta}(\Gamma_{\beta;\alpha} + \Gamma_{\beta}\Gamma_{\alpha}) = -\frac{1}{4}R$$

which is relation M-12 which we set out to prove.

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