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Projective Dirac Operators, Twisted K-Theory, and Local Index Formula

Thesis by

Dapeng ZHANG

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Abstract

We construct a canonical noncommutative spectral triple for every oriented closed Riemannian manifold, which represents the fundamental class in the twisted K-homology of the manifold. This so-called "projective spectral triple" is Morita equivalent to the well-known commutative spin spectral triple provided that the manifold is spin-c. We give an explicit local formula for the twisted Chern character for K-theories twisted with torsion classes, and with this formula we show that the twisted Chern character of the projective spectral triple is identical to the Poincaré dual of the A-hat genus of the manifold.

Keywords. Twisted K-theory, spectral triple, Chern character.

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Introduction

The notion of spectral triple in Connes' noncommutative geometry arises from extracting essential data from the K-homology part of index theory in differential geometry. The following are basic examples of commutative spectral triples:

i) The *spin spectral triple* for a spin^c manifold M with a spinor bundle S:

$$\varsigma_1 = (C^\infty(M), \Gamma(S), \not{D}, \omega),$$

where ω is the grading operator on S. The identity between the analytic and topological indexes of $\not{D}$ is the Atiyah-Singer index formula for spin^c manifold.

ii) The spectral triple for the signature for a Riemannian manifold M:

$$\varsigma_2 = (C^\infty(M), \Omega(M), d + d^*, *(-1)^{\frac{\deg(\deg -1)}{2} - \frac{\dim M}{4}}).$$

The index formula corresponding to this spectral triple is the Hirzebruch signature formula.

iii) The spectral triple for Euler characteristic for a Riemannian manifold M:

$$\varsigma_3 = (C^\infty(M), \Omega(M), d + d^*, (-1)^{\deg}).$$

The local index formula corresponding to this spectral triple is the Gauss-Bonnet-Chern theorem.

In fact, every special case of Atiyah-Singer Index theorem corresponds to an instance of commutative spectral triple (with additional structures when necessary). These spectral triples, like $\varsigma_1, \varsigma_2, \varsigma_3$, have many nice properties such as "the five conditions" in Connes [8], and conversely, it is proved that [8] any commutative spectral triple $(\mathcal{A}, \mathcal{H}, D, \gamma)$ satisfying those five conditions is equivalent to a spectral triple consisting of the algebra of smooth functions on a Riemannian manifold M, the module of sections of a Clifford bundle over M and a Dirac type operator on it. Furthermore, if $(\mathcal{A}, \mathcal{H}, D, \gamma)$ satisfies an additional important property – the Poincaré duality in K-theory – which means $(\mathcal{A}, \mathcal{H}, D, \gamma)$ represents the fundamental class (i.e., a K-orientation) in $K^0(\mathcal{A})$, then it is equivalent to a spin spectral triple ς_1 for some spin^c manifold. The spectral triple for Hirzebruch signature ς_2 (as well as ς_3) does not have the property of Poincaré duality; however, we show in this paper (Corollary 5.4, Theorem 6.1) that for every closed oriented Riemannian manifold there is a canonical noncommutative spectral triple having the property of Poincaré duality in $K^0(M, W_3(M))$, the twisted K-theory of M with local coefficient $W_3(M)$ - the third integral Stiefel-Whitney class. This canonical spectral triple is called the projective spectral triple on M, and its center is unitarily equivalent to ς_2. The projective spectral triple is Morita equivalent to the spin spectral triple provided the underlying manifold is spin^c. On the other hand, in the paper of Mathai-Melrose-Singer [19], a so-called projective spin Dirac operator was defined for every Riemannian manifold; however, this operator is in a formal sense. It turns out that the projective spectral triple, in which the Dirac operator is really an operator acting on a Hilbert space, just plays the role of the projective spin Dirac operator.

A spectral triple that gives rise to Poincaré duality in KK-theory first appeared in Kasparov [17]. In Kasparov's spectral triple (although there was no such terminology at that time), the algebra is noncommutative and $\mathbb{Z}_2$-graded, but in many cases it would be much easier if the algebra is ungraded, especially when considering its Dixmier-Douady class or passing it to cyclic cohomology class via Connes-Chern character. The projective spectral triple constructed in this paper (Corollary 5.4)

$$(\mathcal{A}_{W_3}, \mathcal{H}_{W_3}, D_{W_3}, \gamma_{W_3})$$

has a noncommutative but ungraded algebra, and it is in fact Morita equivalent to that of Kasparov's. To construct such a spectral triple, we first introduce in chapter 4 the notion of Morita equivalence between spectral triples, then find in chapter 5 that the local spin spectral triples on small open subsets of the manifold can be glued together, via Morita equivalence, to form a globally defined spectral triple.

The noncommutative algebras underlying projective spectral triples are examples of Azumaya algebras. In chapter 1 we review some basic theory on Azumaya algebras, such as the fact that Morita equivalent classes of Azumaya algebras are classified by their Dixmier-Douady classes, and that the K-theory of an Azumaya algebra $\mathbf{A}$ coincides with the twisted K-theory of the manifold with the Dixmier-Douady class of $\mathbf{A}$.

Mathai-Stevenson [20] showed that the K-theory (tensoring with $\mathbb{C}$) of an Azumaya algebra $\mathbf{A}$ is isomorphic to the periodic cyclic homology group of $\mathbf{A}$ via Connes-Chern character, and that the latter is isomorphic to the twisted de-Rham cohomology of the manifold with the Dixmier-Douady class of $\mathbf{A}$ via a generalized Connes-Hochschild-Kostant-Rosenberg (CHKR) map.

$$\begin{tikzcd}
K^0(M,\delta(\mathbf{A}))\otimes\mathbb{C} \arrow[rr, "\mathrm{ch}", "\cong"'] \arrow[dr, "\cong", "\mathrm{ch}_{\delta(\mathbf{A})}"'] & & HP_0(\mathbf{A}) \arrow[dl, "\cong"', "\mathrm{Chkr}"] \\
& H^{\mathrm{ev}}_{\mathrm{dR}}(M,\delta(\mathbf{A})) &
\end{tikzcd}$$

In chapter 2, we find an alternative CHKR map (Theorem 2.5) for the special case that the Dixmier-Douady class of $\mathbf{A}$ is torsion.

For an algebra $\mathcal{A}$, a finite projective $\mathcal{A}$-module $\mathcal{E}$ as a K-cocycle in the K-theory of $\mathcal{A}$ has a Connes-Chern character $\mathrm{ch}([\mathcal{E}])$ as a cyclic homology class, whereas a spectral triple on $\mathcal{A}$ as a K-cycle in the K-homology group of $\mathcal{A}$ has also a Connes-Chern character as a cyclic cohomology class, and the index pairing of a K-cocyle and a K-cycle is identical to the index pairing of their Connes-Chern characters ([6, 7]). The main purpose of this paper is to compute the Connes-Chern character of the projective spectral triple and identify it with the Poincaré dual of the A-hat genus of the manifold.

In chapter 7, with the help of the alternative CHKR map ρ and applying the Poincaré duality, we obtain our main result, a local formula for the Connes-Chern character of the projective spectral triple,

$$\mathrm{ch}(\mathcal{A}_{W_3}, \mathcal{H}_{W_3}, D_{W_3}, \gamma_{W_3})(\cdot) = \sum_m \frac{1}{2^n (2m)!} \int_M \hat{A}(M) \circ \rho_{2m}(\cdot).$$

Some conventions and notations

Throughout this paper, we assume the following:

Unless otherwise stated explicitly, all vector spaces, algebras, differential forms, and vector bundles except cotangent bundles are considered over the field $\mathbb{C}$ of *complex* numbers.

The notation $\Gamma(X, E)$ or $\Gamma(E)$ for a fibre bundle E over X always stands for the space of *smooth* sections of E.

For each function $f : \mathbb{N} \to \mathbb{C}$, we define the operator $f(\deg) : \Omega(X) \to \Omega(X)$ acting on the space of differential forms on X to be the linear map given by

$$f(\deg)\,\omega = f(k)\,\omega, \quad \forall \omega \in \Omega^k(X).$$

Chapter 1

Azumaya Bundles, Twisted K-theory, and Twisted Cohomology

Suppose X is a closed oriented manifold. We use the notations

$$\mathrm{M}_n = \begin{cases} \mathrm{M}_n(\mathbb{C}), & n = 1, 2, \dots \\ \mathrm{K}(H), & n = \infty \end{cases}, \quad \mathrm{U}_n = \begin{cases} \mathrm{U}(n), & n = 1, 2, \dots \\ \mathrm{U}(H), & n = \infty \end{cases},$$

where n could either be a positive integer or infinity, H is an infinite dimensional separable Hilbert space, $\mathrm{K}(H)$ is the C^*-algebra of compact operators on H, and $\mathrm{U}(H)$ is the topological group of unitary operators with the operator norm topology. Kuiper's theorem states that $\mathrm{U}(H)$ is contractible. Let $\mathrm{PU}_n = \mathrm{U}_n/\mathrm{U}(1)$ be the projective unitary groups. In particular $\mathrm{PU}(H) = \mathrm{PU}_\infty$ is endowed with the topology induced from the norm topology of $\mathrm{U}(H)$.

Let $\mathrm{Aut}(\mathrm{M}_n)$ be the group of automorphisms the C^*-algebra M_n.

Fact 1. *For every element $g \in \mathrm{Aut}(\mathrm{M}_n)$, there exists $\tilde{g} \in \mathrm{U}_n$, such that $g = \mathrm{Ad}\tilde{g}$. For every $u \in \mathrm{U}_n$, $\mathrm{Ad}\ u = 1$ if and only if u is scalar. In other words, as groups $\mathrm{PU}_n \cong \mathrm{Aut}(\mathrm{M}_n)$.*

Fact 2. *If n is finite, $\mathrm{U}_n/\mathrm{U}(1) \cong \mathrm{SU}(n)/\{z \in \mathbb{C} \mid z^n = 1\}$.*

Definition 1.1. An *Azumaya bundle* over X of rank n (possibly $n = \infty$) is a vector bundle over X with fibre M_n and structure group PU_n.

Every Azumaya bundle of rank n is associated with a principal PU_n-bundle and vice versa.

Definition 1.2. The space $\mathbf{A} = \Gamma^0(A)$ of continuous sections of an Azumaya bundle A over X forms a C^*-algebra called an *Azumaya algebra* over X.

The following are examples of Azumaya algebras over X:

i) the algebra of complex valued continuous functions $C^0(X)$;

ii) $C^0(X) \otimes \mathrm{M}_n$;

iii) if E is a finite rank vector bundle over X, the algebra of continuous sections of $\mathrm{End}(E)$, $\Gamma^0(\mathrm{End}(E))$;

iv) if X is an even dimensional Riemannian manifold, the algebra of continuous sections of the Clifford bundle $\mathbb{C}l(T^*X)$, $\Gamma^0(\mathbb{C}l(T^*X))$;

v) if E is a real vector bundle over X of even rank with a fiberwise inner product, the algebra of continuous sections of the Clifford bundle $\mathbb{C}l(E)$, $\Gamma^0(\mathbb{C}l(E))$.

Note that examples i), ii), iii) are Morita equivalent (in the category of C*-algebras, i.e., strongly Morita equivalent) to $C^0(X)$, while example iv) or v) is Morita equivalent to $C^0(X)$ if and only if X or E is spin^c respectively.

Fact 3. *The center of a finite Azumaya algebra over X is $C^0(X)$.*

Fact 4. *An Azumaya algebra $\mathbf{A}$ over X is locally Morita equivalent to $C^0(X)$.*

The obstruction to an Azumaya algebra being (globally) Morita equivalent to its "center" is characterized by its Dixmier-Douady class:

Definition 1.3. Every Azumaya bundle $\pi : A \to X$ of rank n is associated with a cohomology class $\delta(A)$ in $H^3(X, \mathbb{Z})$, called the *Dixmier-Douady class* of A, constructed as follows:

Let $\{U_i\}_{i\in I}$ be a good covering of X, and write $U_{i_1\cdots i_n}$ for the intersection of $U_{i_1}, U_{i_2}, \cdots, U_{i_n}$. Suppose

$$\psi_i : U_i \times \mathrm{M}_n \xrightarrow{\simeq} \pi^{-1}(U_i), \quad \forall i \in I,$$

provide a local trivialization of A. Then $\psi_i^{-1}\psi_j : U_{ij} \times \mathrm{M}_n \to U_{ij} \times \mathrm{M}_n$ give rise to the transition functions $g_{ij} \in C^0(U_{ij}, \mathrm{Aut}(\mathrm{M}_n))$. Pick $\tilde{g}_{ij} \in C^0(U_{ij}, \mathrm{U}_n)$ such that $\mathrm{Ad}\tilde{g}_{ij} = g_{ij}$ and $\tilde{g}_{ij} = \tilde{g}_{ji}^{-1}$. Thus $\mathrm{Ad}(\tilde{g}_{ij}\tilde{g}_{jk}\tilde{g}_{ki}) = g_{ij}g_{jk}g_{ki} = 1$, which implies

$$\mu_{ijk} := \tilde{g}_{ij}\tilde{g}_{jk}\tilde{g}_{ki} \in C^0(U_{ijk}, \mathrm{U}(1)).$$

Therefore μ is a Čech 2-cocycle with coefficient sheaf $\mathscr{U}(1) : U \mapsto C^0(U, \mathrm{U}(1))$, since $(\partial\mu)_{ijkl} = \mu_{jkl}\mu_{ikl}^{-1}\mu_{ijl}\mu_{ijk}^{-1} = 1$. The Čech 2-cocycle μ is also called the *bundle gerbe* structure of A. The short exact sequence of sheaves

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathscr{R} \xrightarrow{\exp 2\pi i \cdot} \mathscr{U}(1) \longrightarrow 0,$$

where $\mathscr{R}$ is the sheaf $U \mapsto C^0(U, \mathbb{R})$, induces an isomorphism of Čech cohomology groups

$$\partial : \check{H}^2(X, \mathscr{U}(1)) \xrightarrow{\cong} \check{H}^3(X, \mathbb{Z}).$$

Define the Dixmier-Douady class by $\delta(A) := \partial[\mu]$. More explicitly, pick $\nu_{ijk} \in C^0(U_{ijk}, \mathbb{R})$ such that

$$\exp 2\pi i \nu_{ijk} = \mu_{ijk}.$$

Then $\exp 2\pi i(\partial\nu)_{ijkl} = (\partial\mu)_{ijkl} = 1$, which implies $(\partial\nu)_{ijkl} = \nu_{jkl} - \nu_{ikl} + \nu_{ijl} - \nu_{ijk} \in \mathbb{Z}$ are locally constant integers on U_{ijkl}. In fact, $\delta(A) = [\partial\nu] \in \check{H}^3(X, \mathbb{Z})$.

Definition 1.4. Suppose that A is an Azumaya bundle, that $\mathbf{A}$ is the Azumaya algebra corresponding to A, and that P is the principal PU-bundle associated to A. We say $\delta(\mathbf{A}) = \delta(P) = \delta(A)$ are the Dixmier-Douady class of $\mathbf{A}$ and P respectively.

As a consequence of Kuiper's theorem,

Proposition 1.5. *For every cohomology class δ in $H^3(X, \mathbb{Z})$, there is a unique (up to isomorphism) infinite rank Azumaya bundle (or algebra) with Dixmier-Douady class δ.*

Proposition 1.6. *Let A be an Azumaya bundle. If $\delta(A) = 0$, then one can choose $\tilde{g}_{ij}$ so that $\tilde{g}_{ij}$*

are the transition functions of a certain Hermitian bundle E over X, and A is isomorphic to $\mathrm{K}(E)$, *the bundle over X with fibres* $\mathrm{K}(E_x)$.

Corollary 1.7. *An Azumaya algebra $\mathbf{A}$ over X is Morita equivalent to $C^0(X)$ if and only if $\delta(\mathbf{A}) = 0$.*

Corollary 1.8. *Two Azumaya algebras $\mathbf{A}_1$, $\mathbf{A}_2$ over X are Morita equivalent if and only if $\delta(\mathbf{A}_1) = \delta(\mathbf{A}_2)$. Namely, Morita equivalence classes of Azumaya algebras are parameterized by $H^3(X, \mathbb{Z})$.*

As a consequence of **Fact 2**,

Proposition 1.9. *If A is an Azumaya bundle of finite rank n, then $n\delta(A) = 0$.*

For example, suppose X is a $2m$-dimensional smooth manifold. The Clifford bundle $\mathbb{C}l(T^*X)$ is an Azumaya bundle of rank 2^m. Its Dixmier-Douady class $\delta(\mathbb{C}l(T^*X)) = W_3(X)$ is the third integral Stiefel-Whitney class of X, and $2W_3(X) = 0$.

Proposition 1.10. *If A_1, A_2 are two Azumaya bundles over X, then*

$$\delta(A_1 \otimes A_2) = \delta(A_1) + \delta(A_2).$$

Proposition 1.11. *If $\mathbf{A}$ is an Azumaya algebra, then its opposite algebra $\mathbf{A}^{\mathrm{op}}$ is also an Azumaya algebra and*

$$\delta(\mathbf{A}^{\mathrm{op}}) = -\delta(\mathbf{A}).$$

Let δ be a cohomology class in $H^3(X, \mathbb{Z})$. Recall that (Rosenberg [23], Atiyah-Segal [1]) the *twisted K-theory* $K^0(X, \delta)$ can be defined by

$$K^0(X, \delta) = [P \to \mathrm{Fred}(H)]_{\mathrm{PU}(H)},$$

the abelian group of homotopy classes of maps $P \to \mathrm{Fred}(H)$ that are equivariant under the natural action of $\mathrm{PU}(H)$, where P is a principal $\mathrm{PU}(H)$-bundle over X with Dixmier-Douady class $\delta(P) = \delta$; and where $\mathrm{Fred}(H)$ is the space of Fredholm operators on H. Twisted K-theory can also be defined

with K-theory of a C*-algebra:

$$K^0(X,\delta) = K_0(\mathbf{A}),$$

where $\mathbf{A}$ is an (infinite rank) Azumaya algebra over X with Dixmier-Douady class $\delta(\mathbf{A}) = \delta$. One can also define the twisted K^1-group by $K^1(X,\delta) = K_1(\mathbf{A})$. The above two definitions of twisted K-theory are equivalent (Rosenberg [23]). We will always use the second definition in this paper.

Proposition 1.12. *The direct sum of twisted K-groups of X*

$$\bigoplus_{\delta \in H^3(X,\mathbb{Z})} K^\bullet(X,\delta)$$

forms a $\mathbb{Z}_2 \times H^3(X,\mathbb{Z})$-bigraded ring. The product $K^i(X,\delta_1) \times K^j(X,\delta_2) \to K^{i+j}(X,\delta_1+\delta_2)$ is naturally defined.

Definition 1.13. Let $c \in \Omega^3(X)$ be a closed 3-form, the *twisted de Rham complex* is the following periodic sequence

$$\xrightarrow{d_c} \Omega^{\mathrm{ev}}(X) \xrightarrow{d_c} \Omega^{\mathrm{odd}}(X) \xrightarrow{d_c},$$

where $d_c\omega = d\omega + c \wedge \omega$. The *twisted de Rham cohomology* is $H^*_{\mathrm{dR}}(X,c) = H^*(\Omega^*(X), d_c)$.

Proposition 1.14. *If c is a closed 3-form, then $H^*_{\mathrm{dR}}(X,c) \cong H^*_{\mathrm{dR}}(X,zc)$ as isomorphic vector spaces for all nonzero $z \in \mathbb{C}$.*

In particular, $H^*_{\mathrm{dR}}(X,c) \cong H^*_{\mathrm{dR}}(X,-c)$ as vector spaces. In fact, in some literatures such as [20], the twisted coboundary $d_c\omega$ of ω is defined by $d\omega - c \wedge \omega$.

Proposition 1.15. *If a closed 3-form $c_1 = c_2 + d\beta$ for some $\beta \in \Omega^2(X)$, then*

$$\begin{tikzcd}
\arrow[r] & \Omega^{\mathrm{ev}}(X) \arrow[r, "d_{c_1}"] \arrow[d, "\wedge \exp\beta"] & \Omega^{\mathrm{odd}}(X) \arrow[r] \arrow[d, "\wedge \exp\beta"] & {} \\
\arrow[r] & \Omega^{\mathrm{ev}}(X) \arrow[r, "d_{c_2}"] & \Omega^{\mathrm{odd}}(X) \arrow[r] & {}
\end{tikzcd}$$

is a chain isomorphism. Therefore $H^*_{\mathrm{dR}}(X,c_1) \cong H^*_{\mathrm{dR}}(X,c_2)$ as vector spaces.

Chapter 2

Generalized Connes-Hochschild-Kostant-Rosenberg Theorem

In this section, we assume that M is a smooth oriented closed manifold, and that A is an Azumaya bundle over M with a smooth structure in the sense that the transition functions for the vector bundle A are smooth functions valued in the (Banach) Lie group PU_n. Let $\mathcal{A}$ be the space of trace class smooth sections of A, then $\mathcal{A}$ is a Fréchet pre-C*-algebra densely embedded in $\mathbf{A} = \Gamma^0(A)$. In particular, if the rank n of A is finite, then $\mathcal{A} = \Gamma(A)$.

Given a PU_n-connection $\nabla : \Omega^k(M, A) \to \Omega^{k+1}(M, A)$ on A, the image of the Dixmier-Douady class $\delta(A)$ in $H^3_{\mathrm{dR}}(M, \mathbb{R})$ can be represented by a differential 3-form in terms of the connection and curvature (e.g., Freed [13]) as follows:

Let $\{U_i\}$ be a good covering of M, and $\psi_i : U_i \times \mathrm{M}_n \to A|_{U_i}$ be a local trivialization compatible with the smooth structure on A. Denote by $g_{ji} \in C^\infty(U_{ij}, \mathrm{PU}_n)$ the transition function corresponding to $\psi_j^{-1}\psi_i$. Pick $\tilde{g}_{ji} \in C^\infty(U_{ij}, \mathrm{U}_n)$ so that $\mathrm{Ad}\tilde{g}_{ji} = g_{ji}$. Let θ_i be the local connection forms of ∇ on U_i,

$$\nabla(\psi_i(O)) = \psi_i(dO + \theta_i(O)), \forall O \in C^\infty(U_i, \mathrm{M}_n).$$

Then $\theta_i = g_{ji}^{-1}\theta_j g_{ji} + g_{ji}^{-1} dg_{ji}$. Pick $\tilde{\theta}_i \in \Omega^1(U_i, \mathrm{M}_n)$ if $n \neq \infty$, or pick $\tilde{\theta}_i \in \Omega^1(U_i, \mathrm{B}(H))$ if $n = \infty$, so that $\theta_i = \mathrm{ad}\tilde{\theta}_i$. Thus

$$\tilde{\theta}_i = \tilde{g}_{ji}^{-1}\tilde{\theta}_j\tilde{g}_{ji} + \tilde{g}_{ji}^{-1} d\tilde{g}_{ji} + \alpha_{ij},$$

for some scalar valued 1-form $\alpha_{ij} \in \Omega^1(U_{ij})$. Let ω_i be the local curvature forms of $\Omega = \nabla^2 : \Gamma(A) \to \Omega^2(X, A)$ on U_i,

$$\Omega(\psi_i(O)) = \psi_i(\omega_i(O)), \forall O \in C^\infty(U_i, \mathrm{M}_n).$$

So $\omega_i = d\theta_i + \theta_i \wedge \theta_i$, and $\omega_i = g_{ji}^{-1}\omega_j g_{ji}$. Let $\tilde{\omega}_i = d\tilde{\theta}_i + \tilde{\theta}_i \wedge \tilde{\theta}_i$, then $\mathrm{ad}\tilde{\omega}_i = \omega_i$, and $\tilde{\omega}_i = \tilde{g}_{ji}^{-1}\tilde{\omega}_j\tilde{g}_{ji} + d\alpha_{ij}$. Let $\tilde{\Omega}_i = \psi_i\tilde{\omega}_i\psi_i^{-1}$, then $\tilde{\Omega}_i = \tilde{\Omega}_j + d\alpha_{ij}$. Since $d\alpha_{ij} + d\alpha_{jk} + d\alpha_{ki} = 0$, $d\alpha$ forms a 2-form valued cocycle, and since the sheaf of 2-forms is fine (or because of the existence of partition of unity on M), there exist $\beta_i \in \Omega^2(U_i)$ so that $2\pi i(\beta_i - \beta_j) = d\alpha_{ij}$. We can define a generalized 2-form by $\tilde{\Omega}_i - 2\pi i\beta_i$ on U_i, and it is globally well-defined.

Theorem 2.1. 1). *If A is a finite rank Azumaya bundle with connection ∇ and curvature Ω, then there is a unique traceless $\sigma(\Omega) \in \Omega^2(M, A)$ such that $\mathrm{ad}\sigma(\Omega) = \Omega$.*

2). *If A is an infinite rank Azumaya bundle associated to a principal $\mathrm{PU}(H)$-bundle P, with connection ∇ and curvature Ω, then there is a $\Gamma(P \times_{\mathrm{PU}(H)} \mathrm{B}(H))$-valued 2-form $\sigma(\Omega)$ so that $\mathrm{ad}\sigma(\Omega) = \Omega$. Here $\mathrm{PU}(H)$ acts on $\mathrm{B}(H)$ the same way as on $\mathrm{K}(H)$.*

Proof. $\sigma(\Omega)$, up to a scalar valued 2-form, can be defined by $\tilde{\Omega}_i - 2\pi i\beta_i$ as above the theorem. □

Theorem 2.2. *If A is an Azumaya bundle over M with connection ∇ and curvature Ω, then $-\frac{\nabla(\sigma(\Omega))}{2\pi i}$ represents the image of Dixmier-Douady class $\delta(A)$ in $H^3_{\mathrm{dR}}(M)$.*

Proof. First recall that the Čech-de Rham isomorphism between the third de Rham cohomology $H^3_{\mathrm{dR}}(M)$ and Čech cohomology $\check{H}^3(M, \mathbb{C})$ with constant coefficient sheaf $\mathbb{C}$ can be constructed as follows. For any closed 3-form $c \in \Omega^3(M)$, one can find $\beta(c)_i \in \Omega^2(U_i)$ so that $d\beta(c)_i = c|_{U_i}$. Since $d\beta(c)_i - d\beta(c)_j = 0$ one can find $\alpha(c)_{ij} \in \Omega^1(U_{ij})$ so that $\beta(c)_i - \beta(c)_j = d\alpha(c)_{ij}$. Since $d\alpha(c)_{ij} + d\alpha(c)_{jk} + d\alpha(c)_{ki} = 0$ one can find $\nu(c)_{ijk} \in C^\infty(U_{ijk})$ so that $(\partial\alpha(c))_{ijk} = d\nu(c)_{ijk}$ on U_{ijk}. Here ∂ denotes the coboundary operator on Čech cocycles. Likewise, since $d(\partial\nu(c))_{ijkl} = (\partial d\nu(c))_{ijkl} = 0$, one can find $\delta(c)_{ijkl} \in \mathbb{C}$ so that $(\partial\nu(c))_{ijkl} = \delta(c)_{ijkl}$. The Čech-de Rham isomorphism $H^3_{\mathrm{dR}}(M) \to \check{H}^3(M, \mathbb{C})$ is provided by the correspondence $c \mapsto \delta(c)$.

Now let c be the 3-form $-\frac{\nabla(\sigma(\Omega))}{2\pi i}$, then by Bianchi identity $-\frac{\nabla(\sigma(\Omega))}{2\pi i} = -\frac{1}{2\pi i}\nabla(\tilde{\Omega}_i - 2\pi i\beta_i) = d\beta_i$, thus we can choose $\beta(c)_i = \beta_i$, $\alpha(c)_{ij} = \alpha_{ij}, \nu(c)_{ijk} = \nu_{ijk}$, and $\delta(c)_{ijkl} = \delta_{ijkl}$. Therefore it follows

that c represents the image of $\delta(A)$ in $H^3_{\mathrm{dR}}(M)$. $\square$

Recall that if $\mathcal{B}$ is a pre-C*-algebra densely embedded in a C*-algebra $\mathbf{B}$, $K_0(\mathcal{B}) = K_0^{\mathrm{alg}}(\mathcal{B})$ is naturally isomorphic to $K_0(\mathbf{B})$. If $\mathcal{B}$ is a unital Fréchet algebra, $K_1(\mathcal{B})$ is defined to be the abelian group of the equivalence classes of $\mathrm{GL}_\infty(\mathcal{B})$. We say that $u, v \in \mathrm{GL}_\infty(\mathcal{B})$ are equivalent if there is a piecewise C^1-path in $\mathrm{GL}_\infty(\mathcal{B})$ joining u and v. The definition of $K_1(\mathcal{B})$ can be extended to the case of non-unital algebras so that the six-term exact sequence property always holds. For Azumaya algebras, $K_*(\mathcal{A})$ is naturally isomorphic to $K_*(\mathbf{A}) = K^*(M, \delta(\mathbf{A}))$. We refer to [6, 7, 18] for the definitions of Hochschild, cyclic and periodic cyclic homologies and cohomologies.

Definition 2.3. Following Gorokhovsky [15], define two maps

$$\mathrm{Chkr} : \bigoplus_{k \text{ even}} \overline{C}_k^{\mathrm{red}}(\mathcal{A}) \to \Omega^{\mathrm{ev}}(M) \quad \text{and} \quad \mathrm{Chkr} : \bigoplus_{k \text{ odd}} \overline{C}_k^{\mathrm{red}}(\mathcal{A}) \to \Omega^{\mathrm{odd}}(M)$$

by the JLO-type ([16]) formula

$$\mathrm{Chkr}(a_0, a_1, ..., a_k) = \int_{\boldsymbol{s} \in \Delta^k} \mathrm{tr}(a_0 e^{-s_0 \sigma(\Omega)} (\nabla a_1) e^{-s_1 \sigma(\Omega)} \cdots (\nabla a_k) e^{-s_k \sigma(\Omega)}) d\boldsymbol{s}. \tag{2.1}$$

Here $\overline{C}_0^{\mathrm{red}}(\mathcal{A}) = \mathcal{A}$ and $\overline{C}_j^{\mathrm{red}}(\mathcal{A}) = \mathcal{A}^+ \hat{\otimes} \mathcal{A}^{\hat{\otimes} j}$, for all $j \neq 0$, with $\mathcal{A}^+$ being the unitalization of $\mathcal{A}$ and $\hat{\otimes}$ the projective tensor product of locally convex topological algebras.

With the assumptions and notations above, the generalized CHKR theorem of Mathai-Stevenson's states that

Proposition 2.4 (Mathai-Stevenson [20])**.** 1). *The map* Chkr *in* (2.1) *induces a quasi-isomorphism between the two complexes*

$$\xrightarrow{\flat} \overline{C}_{\mathrm{ev}}^{\mathrm{red}}(\mathcal{A}) \xrightarrow{\flat} \overline{C}_{\mathrm{odd}}^{\mathrm{red}}(\mathcal{A}) \xrightarrow{\flat},$$

$$\xrightarrow{0} \Omega^{\mathrm{ev}}(M) \xrightarrow{0} \Omega^{\mathrm{odd}}(M) \xrightarrow{0};$$

and hence isomorphisms $HH_{\mathrm{ev}}(\mathcal{A}) \cong \Omega^{\mathrm{ev}}(M)$, $HH_{\mathrm{odd}}(\mathcal{A}) \cong \Omega^{\mathrm{odd}}(M)$.

2). *The map* Chkr *induces a quasi-isomorphism between the complex*

$$\xrightarrow{\flat+\mathrm{B}} \overline{C}^{\mathrm{red}}_{\mathrm{ev}}(\mathcal{A}) \xrightarrow{\flat+\mathrm{B}} \overline{C}^{\mathrm{red}}_{\mathrm{odd}}(\mathcal{A}) \xrightarrow{\flat+\mathrm{B}}, \text{ or equivalently, } \xrightarrow{\mathrm{B}} HH_{\mathrm{ev}}(\mathcal{A}) \xrightarrow{\mathrm{B}} HH_{\mathrm{odd}}(\mathcal{A}) \xrightarrow{\mathrm{B}}$$

and the twisted de Rham complex

$$\xrightarrow{d_c} \Omega^{\mathrm{ev}}(M) \xrightarrow{d_c} \Omega^{\mathrm{odd}}(M) \xrightarrow{d_c};$$

and hence an isomorphism

$$\mathrm{Chkr} : HP_*(\mathcal{A}) \xrightarrow{\cong} H^*_{\mathrm{dR}}(M, c), \tag{2.2}$$

where $c = -\frac{\nabla(\sigma(\Omega))}{2\pi i}$ *is a representative of the image of* $\delta(A)$ *in* $H^3_{dR}(M)$.

3). *The Connes-Chern character* $\mathrm{ch} : K_*(\mathcal{A}) \otimes \mathbb{C} \to HP_*(\mathcal{A})$ *and the* **twisted Chern character**

$$\mathrm{ch}_{\delta(A)} = \mathrm{Chkr} \circ \mathrm{ch} : \ K^*(M, \delta(A)) \otimes \mathbb{C} \to H^*_{\mathrm{dR}}(M, c)$$

are isomorphisms.

If $\delta(\mathcal{A})$ is a torsion class, then on the cyclic cycles level, there is an alternative way of constructing the maps Chkr, which we will see, is closely related to the relative Chern character ([3]) of Clifford modules.

Define $\psi_k : \mathcal{A}^{\hat{\otimes} k} \to \mathcal{A}\hat{\otimes}_{C^\infty(M)}\Omega^k(M)$ by letting

$$\psi_{-1} = 0, \quad \psi_0 = 1, \quad \psi_1(a_1) = \nabla a_1, \quad \psi_2(a_1, a_2) = (\nabla a_1)(\nabla a_2) + a_1\sigma(\Omega)a_2,$$

$$\psi_k(a_1, ..., a_k) = (\nabla a_1)\psi_{k-1}(a_2, ..., a_k) + a_1\sigma(\Omega)a_2\psi_{k-2}(a_3, ..., a_k), \quad \forall k \geq 2. \tag{2.3}$$

In other words, $\psi_k(a_1, ..., a_k)$ is obtained as follows: Consider all partitions π of the ordered set $\{a_1, ..., a_k\}$ into blocks, where each block contains either one or two elements. Assign to each block $\{a_i\}$ of π a term of the form ∇a_i, and to each block $\{a_j, a_{j+1}\}$ of π a term of the form $a_j\sigma(\Omega)a_{j+1}$.

Then let $\psi_{k,\pi}$ be the product of these terms, and $\psi_k(a_1, ..., a_k)$ be the sum of $\psi_{k,\pi}$ over all such partitions. So in its expansion, $\psi_k(a_1, ..., a_k)$ consists of a Fibonacci number of summands. Then let $\rho : C_k(\mathcal{A}) \to \Omega^k(M)$ be given by

$$\rho_k(a_0, ..., a_k) = \mathrm{tr}(a_0\psi_k(a_1, ..., a_k)). \tag{2.4}$$

Theorem 2.5. *If $\delta(\mathcal{A})$ is a torsion class, then the map ρ_k in* (2.4) *induces a homomorphism*

$$\rho : C_*^\lambda(\mathcal{A}) \to \Omega^*(M)/d(\Omega^{*-1}(M)),$$

where $C_*^\lambda(\mathcal{A})$ is the Connes complex of $\mathcal{A}$ (cf. [6], [7])*, and an isomorphism*

$$\rho : HP_*(\mathcal{A}) \xrightarrow{\cong} H^*_{\mathrm{dR}}(M)$$

which coincides with Chkr *in* (2.2)*.*

Proof. To see that the induced homomorphism $C_*^\lambda(\mathcal{A}) \to \Omega^*(M)/d(\Omega^{*-1}(M))$ is well-defined, we show that for all $k \geq 0$,

$$(-1)^{k-1}\rho_k(a_0, \ldots, a_k) + \rho_k(a_k, a_0, \ldots, a_{k-1}) = d\,\mathrm{tr}\big(a_0\,\psi_{k-1}(a_1, ...a_{k-1})\,a_k\big),$$

for all $a_i \in \mathcal{A}$. Noticing that $d \circ \mathrm{tr} = \mathrm{tr} \circ \nabla$, it suffices to show

$$(-1)^{k-1}a_0\psi_k(a_1, .., a_k) + \psi_k(a_0, ..., a_{k-1})a_k = \nabla\big(a_0\psi_{k-1}(a_1, ..., a_{k-1})a_k\big), \tag{2.5}$$

for all $a_i \in \mathcal{A}^+$. In fact, it is easy to see (2.5) is true for $k = 0, 1, 2$. Suppose (2.5) holds for all $k \leq m$ for some m. Then using $\nabla^2 = \mathrm{ad}\sigma(\Omega)$ and the Bianchi identity $\nabla(\sigma(\Omega)) = 0$, we have

$$
\begin{aligned}
&\nabla(a_0\psi_m(a_1,...,a_m)a_{m+1})\\
&= \nabla a_0\psi_m(a_1,...,a_m)a_{m+1} + a_0\nabla\psi_m(a_1,...,a_m)a_{m+1}\\
&\quad +(-1)^m a_0\psi_m(a_1,...,a_m)\nabla a_{m+1}\\
&= \nabla a_0\psi_m(a_1,...,a_m)a_{m+1} + a_0\nabla\big(\nabla a_1\psi_{m-1}(a_2,...,a_m)\big)a_{m+1}\\
&\quad +a_0\nabla\big(a_1\sigma(\Omega)a_2\psi_{m-2}(a_3,...,a_m)\big)a_{m+1} + (-1)^m a_0\psi_m(a_1,...,a_m)\nabla a_{m+1}\\
&= \psi_{m+1}(a_0,...,a_m)a_{m+1} - a_0a_1\sigma(\Omega)\psi_{m-1}(a_2,...,a_m)a_{m+1}\\
&\quad -a_0\nabla a_1\nabla\psi_{m-1}(a_2,...,a_m)a_{m+1} + a_0\nabla a_1\sigma(\Omega)a_2\psi_{m-2}(a_3,...,a_m)a_{m+1}\\
&\quad +a_0a_1\sigma(\Omega)\nabla a_2\psi_{m-2}(a_3,...,a_m)a_{m+1} + a_0a_1\sigma(\Omega)a_2\nabla\psi_{m-2}(a_3,...,a_m)a_{m+1}\\
&\quad +(-1)^m a_0\psi_m(a_1,...,a_m)\nabla a_{m+1}\\
&= \psi_{m+1}(a_0,...,a_m)a_{m+1} - a_0a_1\sigma(\Omega)\psi_{m-1}(a_2,...,a_m)a_{m+1}\\
&\quad +(-1)^m a_0\nabla a_1\psi_m(a_2,...,a_m,1)a_{m+1} - a_0\nabla a_1\psi_m(1,a_2,...,a_m)a_{m+1}\\
&\quad +a_0\nabla a_1\sigma(\Omega)a_2\psi_{m-2}(a_3,...,a_m)a_{m+1} + a_0a_1\sigma(\Omega)\nabla a_2\psi_{m-2}(a_3,...,a_m)a_{m+1}\\
&\quad +(-1)^m a_0a_1\sigma(\Omega)a_2\psi_{m-1}(a_3,...,a_m,1)a_{m+1}\\
&\quad +a_0a_1\sigma(\Omega)a_2\psi_{m-1}(1,a_3,...,a_m)a_{m+1}\\
&\quad +(-1)^m a_0\psi_m(a_1,...,a_m)\nabla a_{m+1}\\
&= \psi_{m+1}(a_0,...,a_m)a_{m+1} - a_0a_1\sigma(\Omega)\psi_{m-1}(a_2,...,a_m)a_{m+1}\\
&\quad +(-1)^m a_0\nabla a_1\psi_{m-2}(a_2,...,a_{m-1})a_m\sigma(\Omega)a_{m+1}\\
&\quad +a_0a_1\sigma(\Omega)\nabla a_2\psi_{m-2}(a_3,...,a_m)a_{m+1}\\
&\quad +(-1)^m a_0a_1\sigma(\Omega)a_2\psi_{m-3}(a_3,...,a_{m-1})a_m\sigma(\Omega)a_{m+1}\\
&\quad +a_0a_1\sigma(\Omega)a_2\sigma(\Omega)a_3\psi_{m-3}(a_4,...,a_m)a_{m+1}\\
&\quad +(-1)^m a_0\psi_m(a_1,...,a_m)\nabla a_{m+1}\\
&= \psi_{m+1}(a_0,...,a_m)a_{m+1} + (-1)^m a_0\nabla a_1\psi_{m-2}(a_2,...,a_{m-1})a_m\sigma(\Omega)a_{m+1}\\
&\quad +(-1)^m a_0a_1\sigma(\Omega)a_2\psi_{m-3}(a_3,...,a_{m-1})a_m\sigma(\Omega)a_{m+1}\\
&\quad +(-1)^m a_0\psi_m(a_1,...,a_m)\nabla a_{m+1}\\
&= \psi_{m+1}(a_0,...,a_m)a_{m+1} + (-1)^m a_0\psi_{m+1}(a_1,...,a_{m+1}).
\end{aligned}
$$

Thus, by induction, identity (2.5) is proved.

To show that the induced map $\rho : HP_*(\mathcal{A}) \to H^*_{\mathrm{dR}}(M)$ is well-defined: First note that $\rho_k \circ \flat = 0$ on $C_{k+1}(\mathcal{A})$. This means that the map $C^\lambda_*(\mathcal{A})/\flat(C^\lambda_{*+1}(\mathcal{A})) \to \Omega^*(M)/d(\Omega^{*-1}(M))$ is well-defined. Then it suffices to show that the images of $HP_*(\mathcal{A})$ under the map ρ are represented by closed forms. We prove this only for the even case, and the odd case is similar. Since $\mathrm{ch} : K_0(\mathcal{A}) \to HP_0(\mathcal{A})$ is an isomorphism, elements of $HP_0(\mathcal{A})$ are generated by $\mathrm{ch}[p]$ for $[p] \in K_0(\mathcal{A})$. Observe that $p(\nabla p)^{2i+1}p = 0$ for all idempotent p and $i \geq 0$, then

$$\rho_{2k}(\mathrm{ch}^\lambda_{2k}(p)) = (-1)^k \frac{(2k)!}{k!} \mathrm{tr}\big(p\ \psi_{2k}(p, ..., p)\big) = (-1)^k \frac{(2k)!}{k!} \mathrm{tr}\big(p\psi_2(p,p)^k\big),$$

because any term in the expansion of $p\ \psi_{2k}(p, ..., p)p$ that has a factor $p(\nabla p)^{2i+1}p$ vanishes. Since $\nabla(p\psi_2(p,p)) = 0$, it follows that $\rho(\mathrm{ch}[p])$ is a closed form for all $[p] \in K_0(\mathcal{A})$.

Finally, we can prove that $\rho : HP_*(\mathcal{A}) \to H^*_{\mathrm{dR}}(M)$ is an isomorphism identified with Chkr by an argument on the Čech-de Rham bicomplex of M, just similar to the argument used in Mathai-Stevenson [20]. $\square$

Chapter 3

Spectral Analysis of Spectral Triples

In this section we review the definition and some analytical properties of spectral triples. Note that a slight modification to the standard definition of spectral triple (cf. [10]) is made so that it will be more convenient to develop the theory in this paper. In fact, in definition 3.6 we require that the second entry $\mathcal{H}$ of a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ to be an $\mathcal{A}$-module as well as the smooth Sobolev domain of D, instead of the Hilbert space $\bar{\mathcal{H}}$. So in application in differential geometry, spectral triples defined this way operate directly with smooth sections of vector bundles. For a spectral triple in the conventional sense, that would be a strong requirement, as strong as the smoothness condition in Appendix B in [11].

Suppose that D is a densely defined self-adjoint operator on a Hilbert space H, and that D has compact resolvent. Let $\mu_1 > \mu_2 > \cdots$ be the list of eigenvalues of $(D^2+1)^{-1}$ in decreasing order, and $V_i \subset H$ be the eigenspace corresponding to μ_i for each i. Then every vector $v \in H$ can be uniquely represented as a sequence $(v_1, v_2, \dots)$ with $v_i \in V_i$ and $\sum_i \|v_i\|^2 < \infty$, and vice versa.

For every $s \geq 0$, consider the following subspaces of H,

$$W^s(D) = \{(v_1, v_2, \dots) \in H \mid \sum_i \mu_i^{-s} \|v_i\|^2 < \infty\},$$

with the norm $\|(v_1, v_2, \dots)\|_s = \sqrt{\sum_i \mu_i^{-s} \|v_i\|^2}$;

$$W^{s,p}(D) = \{(v_1, v_2, \dots) \in H \mid \sum_i \mu_i^{-sp/2} \|v_i\|^p < \infty\}, \quad \forall p > 0,$$

with the norm $\|(v_1, v_2, \dots)\|_{s,p} = \left(\sum_i \mu_i^{-sp/2} \|v_i\|^p\right)^{1/p}$; and

$$W^{s,\infty}(D) = \{(v_1, v_2, \dots) \in H \mid \sup_i \mu_i^{-s/2} \|v_i\| < \infty\},$$

with the norm $\|(v_1, v_2, \dots)\|_{s,\infty} = \sup_i \mu_i^{-s/2} \|v_i\|$. $W^s = W^{s,2}$ has a natural Hilbert space structure and

Proposition 3.1 (Rellich)**.** *For each $\epsilon > 0$, the inclusion $W^{s+\epsilon} \to W^s$ is compact.*

Proposition 3.2. *$W^1 \subset H$ is the domain of the self-adjoint operator D, and $D : W^1 \to H$ is a Fredholm operator.*

Let $W^\infty = \bigcap_{s>0} W^s$, then W^∞ is a Fréchet space with a family of norms $\|\cdot\|_s$. It is easy to see that restricted to W^∞, the mapping $D : W^\infty \to W^\infty$ is continuous with respect to the Fréchet space topology.

We say the operator D is *finitely summable* or has *spectral dimension* less than $2d$ (for some real number $d > 0$), if $(D^2 + 1)^{-d}$ is a trace class operator.

Theorem 3.3. *Suppose D has finite spectral dimension. If $T \in B(H)$ is a bounded operator that maps W^∞ into W^∞, then the restricted mapping $T : W^\infty \to W^\infty$ is also continuous.*

The theorem can be proved by the following lemmas.

Lemma 3.4 (Sobolev embeddings)**.** *If D has spectral dimension less than $2d$, then we have the following obvious estimate:*

$$\|v\|_{s,\infty} \le \|v\|_{s,p} \le \left(\sum_j \mu_j^d\right)^{1/p} \|v\|_{s+\frac{2d}{p},\infty}, \quad \forall v \in H, \forall s \ge 0, \forall p > 0,$$

i.e., there are bounded embeddings $W^{s+\frac{2d}{p},\infty} \subset W^{s,p} \subset W^{s,\infty}$.

Lemma 3.5. *Suppose D has finite spectral dimension.*

i) Let $T : W^\infty \to W^\infty$ be a continuous operator. Suppose for each j,

$$T(0, \dots, 0, v_j, 0, \dots) = (t_{1j}v_j, t_{2j}v_j, \dots), \quad \forall v_j \in V_j,$$

where (t_{ij}) is an infinite matrix with entries $t_{ij} \in \mathrm{Hom}(V_j, V_i)$. Then (t_{ij}) satisfies the property: for any $s > 0$ there exist C and $r > 0$ such that

$$\| \sum_i \mu_i^{-s} t_{ij} \| < C + \mu_j^{-r}, \quad \forall j.$$

ii) Conversely any matrix (t_{ij}) with entries $t_{ij} \in \mathrm{Hom}(V_j, V_i)$ satisfying the above property represents a continuous operator $T : W^\infty \to W^\infty$.

Proof. i) For any $v \in W^\infty$, we see that as $n \to \infty$, $\sum_{j=0}^n v_j \to v$ in W^s, therefore $\sum_{j=0}^n v_j \to v$ in W^∞. Because T is continuous, it follows that $T(\sum_{j=0}^n v_j) = (\sum_{j=0}^n t_{1j}v_j, \sum_{j=0}^n t_{2j}v_j, \dots) \to Tv$, hence $(Tv)_i = \sum_j t_{ij}v_j$. Suppose the claim is not true, then there must exist $s > 0$, such that for any C and n, there is $j(n, C)$ satisfying

$$\| \sum_i \mu_i^{-s} t_{ij(n,C)} \| > C + \mu_{j(n,C)}^{-n}.$$

Thus one may find an increasing sequence $\{j(n)\}$, such that

$$\| \sum_i \mu_i^{-s} t_{ij(n)} \| > \mu_{j(n)}^{-n}.$$

For each j, pick $u_j \in V_j$ so that $\|u_{j(n)}\| = \| \sum_i \mu_i^{-s} t_{ij(n)} \|^{-1} < \mu_{j(n)}^n$ and $\| \sum_i \mu_i^{-s} t_{ij(n)} u_{j(n)} \| = 1$, while $u_j = 0$ if $j \neq j(n), \forall n$. Then, because of the finite spectral dimension, $u = \sum_j u_j \in W^\infty$, but $T(\sum_{j=0}^n u_j)$ does not converge in W^{2s} as $n \to \infty$, which yields a contradiction.

ii) Suppose the matrix (t_{ij}) has that property. Define $T : W^\infty \to W^\infty$ by

$$(Tv)_i = \sum_j t_{ij} v_j, \quad \forall v \in W^\infty.$$

For any sequence $u(n) \in W^\infty$, we now prove that if $u(n) \to 0$ as $n \to \infty$ then $Tu(n) \to 0$. For any $s > 0$,

$$\|Tu(n)\|_{2s} = \|\sum_i \mu_i^{-s} \sum_j t_{ij} u_j(n)\| \leq \sum_j \|\sum_i \mu_i^{-s} t_{ij} u_j(n)\|$$

$$\leq \sum_j (C + \mu_j^{-r}) \|u_j(n)\| = C\|u(n)\|_{0,1} + \|u(n)\|_{r,1}.$$

Since $u(n) \to 0$ in W^∞ implies $\|u(n)\|_{s'} \to 0$ for all s', using Lemma 3.4, it follows that $\|Tu(n)\|_{2s} \to 0$ for all s. So this implies $Tu(n) \to 0$, and therefore $T : W^\infty \to W^\infty$ is a continuous operator.

□

For any pre-Hilbert space $\mathcal{H}$, we define the $*$-algebra $B(\mathcal{H}) = \{T \in B(\bar{\mathcal{H}}) \mid T(\mathcal{H}) \subset \mathcal{H}, T^*(\mathcal{H}) \subset \mathcal{H}\}$.

Definition 3.6. A triple $(\mathcal{A}, \mathcal{H}, D)$ is said to be a unital *spectral triple* if it is given by a unital pre-C^*-algebra $\mathcal{A}$, a pre-Hilbert space $\mathcal{H}$ with a norm-continuous unital $*$-representation $\mathcal{A} \to B(\mathcal{H})$, and a self-adjoint operator D on $\bar{\mathcal{H}}$ called *Dirac operator*, with the following properties:

i) D has compact resolvent,

ii) $W^\infty(D) = \mathcal{H}$,

iii) under the representation of $\mathcal{A}$, the commutator $[D, a] : \mathcal{H} \to \mathcal{H}$ is norm-bounded for each $a \in \mathcal{A}$.

Besides, we always assume that $\mathcal{A}$ is a locally convex topological $*$-algebra with a topology finer than the norm topology of $\mathcal{A}$, and the representation $\mathcal{A} \times \mathcal{H} \to \mathcal{H}$ is jointly continuous with respect to the locally convex topology of $\mathcal{A}$ and the Fréchet topology of $\mathcal{H}$.

Note that if $(\mathcal{A}, \mathcal{H}, D)$ is a unital spectral triple, then $W^1(D)$, the domain of D, also forms a left $\mathcal{A}$-module because of the last condition in the definition.

A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is said to be *even* if there is a $\mathbb{Z}_2$-grading on $\mathcal{H}$:

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-,$$

so that the grading operator

$$\gamma = \begin{bmatrix} \mathrm{id}_{\mathcal{H}_+} & 0 \\ 0 & -\mathrm{id}_{\mathcal{H}_-} \end{bmatrix} : \mathcal{H} \to \mathcal{H}$$

commutes with all $a \in \mathcal{A}$ and anti-commutes with D. Spectral triples equipped with no such gradings are said to be *odd*.

Two spectral triples $(\mathcal{A}_1, \mathcal{H}_1, D_1)$ and $(\mathcal{A}_2, \mathcal{H}_2, D_2)$ with the isomorphic algebras are said to be *unitarily equivalent* if there is a unitary operator $U : \bar{\mathcal{H}}_1 \to \bar{\mathcal{H}}_2$ intertwining the two representations of $\mathcal{A}_i$ and the two Dirac operators D_i in an obvious way. For the even case the unitary operator U also needs to be grade preserving.

Chapter 4

Morita Equivalence of Spectral Triples

In this section we introduce the notion of Morita equivalence of spectral triples. Let $\mathcal{A}$ be a pre-C^*-algebra. Recall that a right $\mathcal{A}$-module $\mathcal{S}$ is called a *pre-Hilbert* (or *Hermitian*) *right $\mathcal{A}$-module* if there is an $\mathcal{A}$-valued inner product $(\cdot,\cdot) : \mathcal{S} \times \mathcal{S} \to \mathcal{A}$, such that for all $x, y \in \mathcal{S}$, $a \in \mathcal{A}$,

i) $(x, y) = (y, x)^*$,

ii) $(x, ya) = (x, y)a$,

iii) $(x, x) \geq 0$, and $(x, x) = 0$ only if $x = 0$.

The norm on $\mathcal{S}$ is given by $\|x\| = \|(x, x)\|^{\frac{1}{2}}$, and its norm completion $\bar{\mathcal{S}}$ (which is automatically a pre-Hilbert right $\bar{\mathcal{A}}$-module) is called a *Hilbert right $\bar{\mathcal{A}}$-module*. Hilbert left modules can be defined in the same manner. In particular, every Hilbert space is a Hilbert $\mathbb{C}$-module.

If $\mathcal{S}$ is a pre-Hilbert right $\mathcal{A}$-module, then its *conjugate space*

$$\mathcal{S}^* = \{f_x := (x, \cdot) \mid x \in \mathcal{S}\}$$

is a pre-Hilbert left $\mathcal{A}$-module, and for all $x, y \in \mathcal{S}$, $a \in \mathcal{A}$,

$$af_x := a(x, \cdot) = f_{x(a^*)}, \quad (f_x, f_y) := (y, x), \quad (f_x, af_y) = a(f_x, f_y).$$

If $\mathcal{S}$ is a pre-Hilbert right $\mathcal{A}$-module, $B_{\mathcal{A}}(\bar{\mathcal{S}})$ denotes the $*$-algebra of all module homomorphisms $T : \bar{\mathcal{S}} \to \bar{\mathcal{S}}$ for which there is an adjoint module homomorphism $T^* : \bar{\mathcal{S}} \to \bar{\mathcal{S}}$ with $(Tx, y) = (x, T^*y)$ for all $x, y \in \bar{\mathcal{S}}$. Define

$$B_{\mathcal{A}}(\mathcal{S}) = \{T \in B_{\mathcal{A}}(\bar{\mathcal{S}}) \mid T(\mathcal{S}) \subset \mathcal{S}, T^*(\mathcal{S}) \subset \mathcal{S}\}.$$

In particular, if $\mathcal{A} = \mathbb{C}$, then $B_{\mathbb{C}}(\mathcal{S})$=$B(\mathcal{S})$. If $\mathcal{A}$ is unital and $\mathcal{S}$ is unital and finitely generated, then $B_{\mathcal{A}}(\mathcal{S})$ in fact consists of all $\mathcal{A}$-endomorphisms of S, i.e., $B_{\mathcal{A}}(\mathcal{S}) = \mathrm{End}_{\mathcal{A}}(\mathcal{S})$.

Suppose $\mathcal{A}$ and $\mathcal{B}$ are unital pre-C^*-algebras. Let $\mathcal{E}$ be a unital finitely generated projective right $\mathcal{A}$-module, then as a summand of a free module, one can endow $\mathcal{E}$ with a pre-Hilbert module structure (which is unique up to unitary $\mathcal{A}$-isomorphism). Suppose $\mathcal{B}$ acts on $\mathcal{E}$ on the left, and the representation $\mathcal{B} \to B_{\mathcal{A}}(\mathcal{E})$ is unital, $*$-preserving and norm-continuous. We also assume that $\mathcal{E}$ is also endowed with a topology induced from the locally convex topology of $\mathcal{A}$, that $\mathcal{B}$ is locally convex topological $*$-algebra with a topology finer than the norm topology, and that the representation $\mathcal{B} \times \mathcal{E} \to \mathcal{E}$ is jointly continuous with respect to the locally convex topologies on $\mathcal{B}$ and $\mathcal{E}$. We call such a $\mathcal{B}$-$\mathcal{A}$-bimodule with the above structure a *finite Kasparov $\mathcal{B}$-$\mathcal{A}$-module*, then we introduce the following

Definition 4.1. Suppose $\sigma = (\mathcal{A}, \mathcal{H}, D)$ is a unital spectral triple. A *σ-connection* on a finite Kasparov $\mathcal{B}$-$\mathcal{A}$-module $\mathcal{E}$ is a linear mapping

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} B(\mathcal{H})$$

with the following properties:

i) $\nabla(\xi a) = (\nabla\xi)a + \xi \otimes_{\mathcal{A}} [D, a]$,

ii) $(\xi, \nabla\varepsilon) - (\nabla\xi, \varepsilon) = [D, (\xi, \varepsilon)]$,

for all $a \in \mathcal{A}$, and $\xi, \varepsilon \in \mathcal{E}$.

Note that by the notation $(\nabla\xi, \varepsilon)$, which is a bit ambiguous, we really mean $(\varepsilon, \nabla\xi)^* \in B(\mathcal{H})$.

Since $\mathcal{E}$ is finitely generated, it follows that for each $b \in \mathcal{B}$, the commutator $[\nabla, b]$ corresponds to an element in $B(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{H})$. It also follows that any two σ-connections on $\mathcal{E}$ differ by a Hermitian element in $B(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{H})$.

From the above data $(\mathcal{A}, \mathcal{H}, D, \mathcal{B}, \mathcal{E}, \nabla)$, one can construct a new spectral triple $\sigma^{\mathcal{E}} = (\mathcal{B}, \mathcal{H}^{\mathcal{E}}, D^{\mathcal{E}})$ for $\mathcal{B}$ (cf. Connes [7, §VI.3]). The pre-Hilbert space $\mathcal{H}^{\mathcal{E}}$ is $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}$ with inner product given by

$$< \xi_1 \otimes_{\mathcal{A}} h_1, \xi_2 \otimes_{\mathcal{A}} h_2 >=< h_1, (\xi_1, \xi_2) h_2 >, \quad \forall \xi_i \in \mathcal{E}, h_i \in \mathcal{H}.$$

The Dirac operator $D^{\mathcal{E}}$ on $\overline{\mathcal{H}^{\mathcal{E}}}$ is given by

$$D^{\mathcal{E}}(\xi \otimes_{\mathcal{A}} h) = \xi \otimes_{\mathcal{A}} Dh + (\nabla \xi) h, \quad \forall \xi \in \mathcal{E}, h \in W^1(D).$$

It is easy to see that the commutator $[D^{\mathcal{E}}, b] = [\nabla, b] \in B(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{H})$ is bounded. To verify that $D^{\mathcal{E}}$ is self-adjoint with domain $\mathcal{E} \otimes_{\mathcal{A}} W^1(D)$, that $W^{\infty}(D^{\mathcal{E}}) = \mathcal{H}^{\mathcal{E}}$, and that $D^{\mathcal{E}}$ has compact resolvent, it is adequate to check choosing one particular σ-connection on $\mathcal{E}$, because bounded perturbations do not affect the conclusions.

Recall that a *universal connection* on a pre-Hilbert right $\mathcal{A}$-module $\mathcal{S}$ is a linear mapping $\nabla : \mathcal{S} \to \mathcal{S} \otimes_{\mathcal{A}} \Omega_u^1(\mathcal{A})$ that satisfies the Leibniz rule

$$\nabla(sa) = (\nabla s)a + \delta_u a,$$

and ∇ is said to be Hermitian if

$$(s, \nabla \varepsilon) - (\nabla s, \varepsilon) = \delta_u(s, \varepsilon).$$

Here $\Omega_u^1(\mathcal{A})$ is the space of universal 1-forms of $\mathcal{A}$, and involutions $(\delta_u a)^*$ are set to be $-\delta_u(a^*)$. Cuntz and Quillen [12] showed that only projective modules admit universal connections. Given a universal Hermitian connection on a finite Kasparov module $\mathcal{E}$, by sending 1-forms $\delta_u a$ to $[D, a]$, one

can associate with the universal connection a σ-connection on $\mathcal{E}$ for any spectral triple $\sigma = (\mathcal{A}, \mathcal{H}, D)$.

Let $p \in M_n(\mathcal{A})$ be a projection (i.e., a self-adjoint idempotent $n \times n$ matrix). Now we consider the right $\mathcal{A}$-module $p\mathcal{A}^n$. There is a canonical universal connection on $p\mathcal{A}^n$ which is given by the matrix $p\mathrm{diag}\{\delta_u, \ldots, \delta_u\}$ or simply written as $\nabla(p\boldsymbol{a}) = p\delta_u(p\boldsymbol{a}), \forall \boldsymbol{a} \in \mathcal{A}^n$, and this connection is Hermitian.

Definition 4.2. A universal connection ∇ on a pre-Hilbert right $\mathcal{A}$-module $\mathcal{S}$ is said to be *projectional* if there is a unitary $\mathcal{A}$-isomorphism $\phi : \mathcal{S} \to p\mathcal{A}^n$ for some n and some projection p, such that $\nabla = \phi^{-1} \circ p\delta_u \circ \phi$.

A projectional universal connection on each finite projective pre-Hilbert $\mathcal{A}$-module is unique up to unitary $\mathcal{A}$-isomorphism. If $\mathcal{E}$ is a finite Kasparov $\mathcal{B}$-$\mathcal{A}$-module admitting a universal connection $\nabla^{\mathcal{A}}$ and $\mathcal{F}$ is a finite Kasparov $\mathcal{C}$-$\mathcal{B}$-module admitting a universal connection $\nabla^{\mathcal{B}}$, then there is a twisted universal connection $\nabla^{\mathcal{B}} \circ \nabla^{\mathcal{A}}$ defined in an obvious way on the finite Kasparov $\mathcal{C}$-$\mathcal{A}$-module $\mathcal{F} \otimes_{\mathcal{B}} \mathcal{E}$. Furthermore if $\nabla^{\mathcal{A}}$ and $\nabla^{\mathcal{B}}$ are projectional, then so is $\nabla^{\mathcal{B}} \circ \nabla^{\mathcal{A}}$. This leads to a *category of noncommutative differential geometry* **NDG**, consisting of formal objects $X_{\mathcal{A}}$ for all unital pre-C^* algebras $\mathcal{A}$. The morphisms $X_{\mathcal{A}} \to X_{\mathcal{B}}$ in **NDG** are (isomorphism classes of) finite Kasparov $\mathcal{B}$-$\mathcal{A}$-modules with projectional universal connections. It is not difficult to extend morphisms of **NDG** to graded modules with super-connections in the sense of Quillen [22]; however in this paper, we only focus on finite Kasparov modules with trivial gradings.

For an even spectral triple $\sigma = (\mathcal{A}, \mathcal{H}, D, \gamma)$ with grading γ, σ-connections ∇ are required to be odd operators, then for each $(\mathcal{E}, \nabla)$, $\sigma^{\mathcal{E}}$ is also an even spectral triple with an obvious grading $\gamma^{\mathcal{E}}$.

Denote by $\mathrm{Sptr}(\mathcal{A}) = \mathrm{Sptr}^0(\mathcal{A}) \amalg \mathrm{Sptr}^1(\mathcal{A})$ the set of even spectral triples and odd spectral triples for $\mathcal{A}$ up to unitary equivalence, then Sptr yields a functor

$$\mathbf{NDG} \to \mathbf{Set} \quad \text{given by} \quad \mathbf{X}_{\mathcal{A}} \mapsto \mathrm{Sptr}(\mathcal{A}).$$

Remark 4.3. Baaj-Julg [2] established the theory of unbounded Kasparov modules and showed that every element in the bivariant KK-theory can be represented by an unbounded Kasparov module.

It would be nice if morphisms of **NDG** could be enlarged to unbounded (even) Kasparov modules $(E, \mathscr{D}, \Gamma)$ with a grading Γ and an appropriate universal connection ∇, and thereby $D^{E,\mathscr{D},\Gamma}(\xi \otimes_{\mathcal{A}} h) = \mathscr{D}\xi \otimes_{\mathcal{A}} h + \Gamma\xi \otimes_{\mathcal{A}} Dh + \Gamma(\nabla\xi)h$. The notion of connection for bounded Kasparov modules introduced by Connes-Skandalis was well-known, whereas theory of connection for unbounded Kasparov modules has been developed in a recent work by Mesland [21].

Definition 4.4. Two unital spectral triples $\sigma_1 = (\mathcal{A}_1, \mathcal{H}_1, D_1)$ and $\sigma_2 = (\mathcal{A}_2, \mathcal{H}_2, D_2)$ are said to be *Morita equivalent* if $\mathcal{A}_1$ and $\mathcal{A}_2$ are Morita equivalent as algebras and there is a finite Kasparov $\mathcal{A}_2$-$\mathcal{A}_1$-module $\mathcal{E}$ with a σ_1-connection, such that $\mathcal{E}$ is an equivalence bimodule and σ_2 is unitarily equivalent to $\sigma_1^{\mathcal{E}}$.

Theorem 4.5. *The Morita equivalence between spectral triples is an equivalence relation.*

Proof. Reflexivity: $(\mathcal{A}, \mathcal{H}, D)$ is Morita equivalent to itself via the trivial connection $\nabla a = [D, a]$ on $\mathcal{A}$.

Transitivity: it is straight forward by definition.

Symmetry: suppose $(\mathcal{B}, \mathcal{H}^{\mathcal{E}}, D^{\mathcal{E}})$ is Morita equivalent to $\sigma = (\mathcal{A}, \mathcal{H}, D)$ via a σ-connection ∇ on $\mathcal{E}$. Let $\mathcal{F} = \mathcal{E}^*$. Define $\nabla^{\mathcal{F}} : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{B}} B(\mathcal{H}^{\mathcal{E}})$ by

$$(\nabla^{\mathcal{F}} f)(\xi \otimes_{\mathcal{A}} h) = -f(\nabla\xi) \otimes_{\mathcal{A}} h + [D, f\xi]h, \quad \forall f \in \mathcal{F}, \xi \in \mathcal{E}, h \in \mathcal{H}.$$

Here we use a map $\mathcal{H}^{\mathcal{E}} \to \mathcal{H}$ to represent an element in $\mathcal{F} \otimes_{\mathcal{B}} B(\mathcal{H}^{\mathcal{E}})$. One can verify $\nabla^{\mathcal{F}}$ is a $(\mathcal{B}, \mathcal{H}^{\mathcal{E}}, D^{\mathcal{E}})$-connection. Now we check $(D^{\mathcal{E}})^{\mathcal{F}} = D$ as follows,

$$\begin{aligned}
(D^{\mathcal{E}})^{\mathcal{F}}(f \otimes_{\mathcal{B}} \xi \otimes_{\mathcal{A}} h) &= f \otimes_{\mathcal{B}} D^{\mathcal{E}}(\xi \otimes_{\mathcal{A}} h) + (\nabla^{\mathcal{F}} f)(\xi \otimes_{\mathcal{A}} h) \\
&= f \otimes_{\mathcal{B}} \xi Dh + f(\nabla\xi)h + (\nabla^{\mathcal{F}} f)(\xi \otimes_{\mathcal{A}} h) \\
&= f\xi Dh + f(\nabla\xi)h - f(\nabla\xi)h + [D, f\xi]h \\
&= D(f\xi h).
\end{aligned}$$

(An alternative way to prove the symmetry property is using bounded perturbations of the σ-

connection and $\sigma^{\mathcal{E}}$-connection that are associated to projectional universal connections.)

In conclusion, Morita equivalence of spectral triples is an equivalence relation. $\square$

Remark 4.6. As a special case, $\mathcal{A}$ is Morita equivalent to itself via $\mathcal{E} = \mathcal{A}$. Using universal connections on $\mathcal{A}$, one can construct new spectral triples which are called inner fluctuations of spectral triples [9]. Morita equivalence of spectral triples $(\mathcal{A}, \mathcal{H}, D_1)$ and $(\mathcal{A}, \mathcal{H}, D_2)$ with the same $\mathcal{A}$ and $\mathcal{H}$ is different from inner fluctuation of spectral triples, as the latter is generally not an equivalence relation when $\mathcal{A}$ is noncommutative. For instance, any spectral triple $(\mathcal{A}, \mathcal{H}, D)$ with a finite dimensional matrix algebra $\mathcal{A}$ and a finite dimensional Hilbert space $\mathcal{H}$, has an inner fluctuation $(\mathcal{A}, \mathcal{H}, 0)$, whereas $(\mathcal{A}, \mathcal{H}, D)$ is not an inner fluctuation of $(\mathcal{A}, \mathcal{H}, 0)$ as long as $D \neq 0$. In general, if we confine ourselves to equivalence bimodules with universal connections, the symmetry property in the above proof does not hold. However if we restrict the universal connections on equivalence bimodules to the projectional ones, the symmetry property holds again.

Similar to $\mathrm{Sptr}(\mathcal{A})$, let $\mathrm{Sptr}_{\mathrm{M}}(\mathcal{A})$ denote the set of spectral triples over $\mathcal{A}$ up to Morita equivalence. In other words, $\mathrm{Sptr}_{\mathrm{M}}(\mathcal{A})$ is the set of spectral triples over $\mathcal{A}$ up to unitary equivalence and bounded perturbations of Dirac operators. $\mathrm{Sptr}_{\mathrm{M}}$ yields a functor

$$\mathbf{NDG} \to \mathbf{Set}, \quad \mathbf{X}_{\mathcal{A}} \mapsto \mathrm{Sptr}_{\mathrm{M}}(\mathcal{A}),$$

and the induced morphisms $\mathrm{Sptr}_{\mathrm{M}}(\mathcal{E}, \nabla)$ do not depend on the choice of spectral-triple-connections ∇ on $\mathcal{E}$.

Suppose $(\mathcal{E}, \nabla)$ is a morphism in $\mathrm{Hom}_{\mathbf{NCG}}(X_{\mathcal{A}}, X_{\mathcal{B}})$, then there is a free right $\mathcal{A}$-module $\mathcal{A}^m$ and a projection p in $\mathrm{M}_m(\mathcal{A})$ such that $\mathcal{E} \cong p\mathcal{A}^m$. Let $\{e_1, ..., e_m\}$ be the standard generators of $\mathcal{A}^m$. For each b in $\mathcal{B}$, one can find a matrix $\alpha(b)$ in $p\mathrm{M}_m(\mathcal{A})p$, such that $bpe_i = \sum_j e_j \alpha_{ji}(b)$.

The K-theory, Hochschild (co)homology and (periodic) cyclic (co)homology are all functors on the category **NDG**. For instance, suppose $(b_0, ..., b_n)$ is a Hochschild n-cycle representing an element

$\Phi \in HH_*(\mathcal{B})$, then $\mathcal{E}(\Phi) \in HH_*(\mathcal{A})$ is the Hochschild n-cycle given by the Dennis trace map

$$\mathrm{tr}(\alpha(b_0) \otimes \cdots \otimes \alpha(b_n)) = \sum_{i_0,\ldots,i_n} (\alpha_{i_0 i_1}(b_0), \alpha_{i_1 i_2}(b_1), \ldots, \alpha_{i_n i_0}(b_n)).$$

Note that α depends on the choice of the isomorphism $\mathcal{E} \cong p\mathcal{A}^m$; however, it does induce the well-defined morphisms $\mathcal{E} : HH_*(B) \to HH_*(\mathcal{A})$.

Furthermore, the Connes-Chern characters [6] are natural transformations from the K-theory functor $X_{\mathcal{A}} \mapsto K_0(\mathcal{A})$ to periodic cyclic homology functor $X_{\mathcal{A}} \mapsto HP_0(P)$, and from functors $X_{\mathcal{A}} \mapsto \mathrm{Sptr}^0(\mathcal{A})$, $\mathrm{Sptr}^0_{\mathrm{M}}(\mathcal{A})$ and $K^0(\bar{\mathcal{A}})$ to the periodic cyclic cohomology functor $X_{\mathcal{A}} \mapsto HP^0(\mathcal{A})$. The naturalness of Connes-Chern characters is illustrated in the following commutative diagrams

$$\begin{array}{ccc} K_0(\mathcal{B}) & \xrightarrow{\mathcal{E}} & K_0(\mathcal{A}) \\ {\scriptstyle \mathrm{ch}}\downarrow & & {\scriptstyle \mathrm{ch}}\downarrow \\ HP_0(\mathcal{B}) & \xrightarrow{\mathcal{E}} & HP_0(\mathcal{A}), \end{array} \qquad \begin{array}{ccc} \mathrm{Sptr}^0(\mathcal{A}) & \xrightarrow{(\mathcal{E},\nabla)} & \mathrm{Sptr}^0(\mathcal{B}) \\ {\scriptstyle \mathrm{ch}}\downarrow & & {\scriptstyle \mathrm{ch}}\downarrow \\ HP^0(\mathcal{A}) & \xrightarrow{\mathcal{E}} & HP_0(\mathcal{B}). \end{array}$$

Proposition 4.7. *The following diagrams*

$$\begin{array}{ccccc} K_0(\mathcal{A}) & \times & \mathrm{Sptr}^0(\mathcal{A}) & \xrightarrow{\mathrm{Ind}} & \mathbb{Z} \\ \uparrow{\scriptstyle \mathcal{E}} & & \downarrow{\scriptstyle (\mathcal{E},\nabla)} & & \| \\ K_0(\mathcal{B}) & \times & \mathrm{Sptr}^0(\mathcal{B}) & \xrightarrow{\mathrm{Ind}} & \mathbb{Z}, \end{array} \qquad \begin{array}{ccccc} HP_0(\mathcal{A}) & \times & HP^0(\mathcal{A}) & \longrightarrow & \mathbb{C} \\ \uparrow{\scriptstyle \mathcal{E}} & & \downarrow{\scriptstyle \mathcal{E}} & & \| \\ HP_0(\mathcal{B}) & \times & HP^0(\mathcal{B}) & \longrightarrow & \mathbb{C}, \end{array}$$

and

$$\begin{array}{ccccc} K_0(\mathcal{A}) & \times & \mathrm{Sptr}^0(\mathcal{A}) & \xrightarrow{\mathrm{Ind}} & \mathbb{Z} \\ \downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} & & \downarrow \\ HP_0(\mathcal{A}) & \times & HP^0(\mathcal{A}) & \longrightarrow & \mathbb{C}, \end{array}$$

commute.

Chapter 5

Gluing Local Spin Structures on Riemannian Manifolds via Morita Equivalence

We can apply the above theory to spectral triples on Riemannian manifolds. By gluing local pieces of spectral triples via Morita equivalence, we construct a so called projective spectral triple, the Dirac operator of which was defined in a formal sense by Mathai-Melrose-Singer [19].

Let X be a closed oriented Riemannian manifold of dimension n. Suppose X is spin or spin^c. Let $\mathbb{C}l_n$ denote the complex Clifford algebra of $\mathbb{R}^n$. In this paper we use the following convention for the definition of Clifford algebras

$$\mathbb{C}l_n :=< u \in \mathbb{R}^n | uv + vu = -2(u,v), \forall u, v \in \mathbb{R}^n > .$$

Decomposed by the parity of degree, $\mathbb{C}l_n = \mathbb{C}l_n^0 \oplus \mathbb{C}l_n^1$. Write

$$B_n = \begin{cases} \mathbb{C}l_n \cong M_{2^m}(\mathbb{C}) \\ \mathbb{C}l_n^0 \cong M_{2^m}(\mathbb{C}) \end{cases} \quad \text{and} \quad B_x = \begin{cases} \mathbb{C}l(T_x^*X) & \text{when } n = 2m \text{ is even} \\ \mathbb{C}l^0(T_x^*X) & \text{when } n = 2m+1 \text{ is odd.} \end{cases} \tag{5.1}$$

Denote by $\mathbb{C}l(X)$ and $B(X)$ the vector bundles over X whose fibers at a point $x \in X$ are $\mathbb{C}l(T_x^*X)$ and B_x. Let $S_n = \mathbb{C}^{2^m}$ be the standard spinor vector space, and we fix a specific isomorphism

$$c : B_n \to \mathrm{End}_{\mathbb{C}}(S_n).$$

Let $\omega_n = i^{[\frac{n+1}{2}]} e_1 \cdots e_n \in \mathbb{C}l_n$, then when n is odd, $B_n = \omega_n \mathbb{C}l_n^1$. This indicates a homomorphism $c : \mathbb{C}l_n \to \mathrm{End}_{\mathbb{C}}(S_n)$. When n is even, $S_n = S_{n+} \oplus S_{n-}$ is graded by the eigenspaces of ω_n, which are invariant under the action of $\mathrm{Spin}(n)$.

Let $P_{\mathrm{Fr}}(X)$ and $P_{\mathrm{Spin}(^c)}(X)$ denote the orthonormal oriented frame bundle and the principal $\mathrm{Spin}(n)$ or $\mathrm{Spin}^c(n)$-bundle over X respectively. Then

$$\mathbb{C}l(X) = P_{\mathrm{Fr}}(X) \times_{\mathrm{SO}(n)} \mathbb{C}l_n = P_{\mathrm{Spin}(^c)}(X) \times_{\mathrm{Ad}} \mathbb{C}l_n.$$

The spinor bundle over X is the associated $\mathrm{Spin}(^c)(n)$-bundle

$$S_X = P_{\mathrm{Spin}(^c)}(X) \times_c S_n.$$

The Clifford algebra bundle acts naturally on the spinor bundle, $c : \mathbb{C}l(X) \times_X S_X \to S_X$, which is given by

$$(p, \xi) \times (p, s) \mapsto (p, c(\xi)s), \quad \forall p \in P_{\mathrm{Spin}(^c)}(X), \xi \in \mathbb{C}l_n, s \in S_n.$$

When n is even, ω_n induces a grading operator ω on $S_X = S_{X+} \oplus S_{X-}$.

Denote by $\not{D}$ the Dirac operator on S_X. Let E be a Hermitian vector bundle over M with a Hermitian connection ∇^E. Let $\mathcal{A} = C^\infty(X)$, $\mathcal{B} = \Gamma(B(X))$, and $\mathcal{E} = \Gamma(E)$. Then the well-known spin spectral triple

$$(\mathcal{A}, \mathcal{H}, D) = (C^\infty(X), \Gamma(S_X), \not{D}) \tag{5.2}$$

is Morita equivalent to the following spectral triple with a noncommutative algebra

$$(\mathcal{B}, \mathcal{H}^{\mathcal{E}}, D^{\mathcal{E}}) = (\Gamma(\mathrm{End}(E)), \Gamma(E \otimes S_X), \not{D}^E), \tag{5.3}$$

via the finite Kasparov module $\mathcal{E}$ with $(\mathcal{A}, \mathcal{H}, D)$-connection associated to ∇^E. Here $\not{D}^E$ denotes the twisted Dirac operator on the vector bundle $E \otimes S_X$.

Now interesting things happen when a manifold has no spin^c structure: The spinor bundle does

not exist, and neither does the spin spectral triple. However, as constructed later in this section, for any closed oriented Riemannian manifold, not necessarily spin^c, there is a canonical noncommutative spectral triple:

Definition 5.1. The *projective spectral triple* of a closed oriented Riemannian manifold M is defined to be

$$(\mathcal{A}_{W_3}, \mathcal{H}_{W_3}, D_{W_3}) := (\Gamma(B(M)), (1+*)\Omega(M), (d-d^*)(-1)^{\deg}), \tag{5.4}$$

if M is odd dimensional, and

$$(\mathcal{A}_{W_3}, \mathcal{H}_{W_3}, D_{W_3}, \gamma_{W_3}) := (\Gamma(B(M)), \Omega(M), (d-d^*)(-1)^{\deg}, *(-1)^{\frac{\deg(\deg+1)}{2}-\frac{n}{4}}), \tag{5.5}$$

if M is even dimensional.

We can consider that projective spectral triples are obtained by gluing local spin spectral triples in the following way:

Let M be a closed oriented Riemannian manifold of dimension n, not necessarily spin^c. Let $\{U_i\}$ be a good covering of M. Then on each local piece we have the principal Spin(n)-bundle $P_i = P_{\mathrm{Spin}}(U_i)$, the associated spinor bundle $S_i = S_{U_i}$, the spin connection ∇_i, and the Dirac operator $\not{D}_i$. Over each intersection $U_{ij} = U_i \cap U_j \neq \emptyset$, there is up to $\pm 1 \in \mathrm{Spin}(n)$ a unique homeomorphism of principal bundles

$$\alpha_{ij} : P_i|_{U_{ij}} \to P_j|_{U_{ij}}, \text{ such that } \alpha_{ij}(p_i g) = \alpha_{ij}(p_i)g, \ \forall p_i \in P_i|_{U_{ij}}, \ g \in \mathrm{Spin}(n).$$

This homeomorphism induces up to ± 1 a Clifford module homomorphism $\alpha_{ij} : S_i|_{U_{ij}} \to S_j|_{U_{ij}}$,

$$\alpha_{ij}(p,s) = (\alpha_{ij}(p), s), \quad \forall p \in P_i|_{U_{ij}}, s \in S_n.$$

For each triple overlap $U_{ijk} = U_i \cap U_j \cap U_k \neq \emptyset$, write

$$\sigma_{ijk} = \alpha_{ki} \circ \alpha_{jk} \circ \alpha_{ij}|_{U_{ijk}},$$

then $\{\sigma_{ijk}\}$ represents a Čech cocycle in $\check{H}^2(M, \mathbb{Z}_2)$, and the corresponding singular class $w_2(M) \in H^2(M, \mathbb{Z}_2)$ is the second Stiefel-Whitney class. Let $L_{ij} = \mathrm{Hom}_{\mathbb{C}l(U_{ij})}(S_i|_{U_{ij}}, S_j|_{U_{ij}})$ denote the line bundle of the Clifford module homomorphisms of S_i and S_j over U_{ij}, then α_{ij} is the canonical section of L_{ij}. $\{L_{ij}\}$ forms a gerbe of line bundles over M characterized by the Dixmier-Douady class $\delta(B(M)) = W_3(M)$ (the third integral Stiefel-Whitney class). We refer [4] for (twisted) K-theory of bundle gerbes.

If M is spin, then for each U_{ij} there is $\beta_{ij} \in \mathbb{Z}_2$, such that $\{\beta_{ij}\alpha_{ij}\}$ satisfies the cocycle condition. That means the local spinor bundles S_{ij} can be glued together as a global spinor bundle via the Clifford module isomorphisms $\beta_{ij}\alpha_{ij}$. The difference between two different sets of such $\{\beta_{ij}\}$ is a cocycle in $\check{H}^1(M, \mathbb{Z}_2)$ which parameterizes distinct spin structures on M.

If M is spin^c, then for each U_{ij} there is $\beta_{ij} \in C^\infty(U_{ij}, U(1))$, such that $\{\beta_{ij}\alpha_{ij}\}$ satisfies the cocycle condition. That means the local spinor bundles S_{ij} can be glued together as a global spinor bundle via the Clifford module isomorphisms $\beta_{ij}\alpha_{ij}$. β_{ij}^2 also satisfies the cocycle condition, and the Čech cocycle $\{\beta_{ij}^2\} \in \check{H}^1(M, U(1))$ corresponds to the canonical line bundle $\mathscr{L}$ of a spin^c structure. The difference between two different sets of such $\{\beta_{ij}\}$ is a cocycle in $\check{H}^1(M, U(1))$ which parameterizes distinct spinor bundles on M.

Denote by $C^\infty(\bar{U}_i)$ the space of smooth functions on U_i that can be extended to a smooth function on a small open neighborhood V_i of $\bar{U}_i$, and likewise denote by $\Gamma(\bar{U}_i, \cdot)$ for extendable smooth sections. Take $\mathcal{E}_i = \Gamma(\bar{U}_i, S_i)$, then the "local spin spectral triple"

$$(\mathcal{A}_i, \mathcal{H}_i, D_i) = (C^\infty(\bar{U}_i), \Gamma(\bar{U}_i, S_i), \not{D}_i) \tag{5.6}$$

is Morita equivalent to the local spectral triple

$$(\mathcal{B}_i, \mathcal{H}_i^{\mathcal{E}_i}, D_i^{\mathcal{E}_i}) = (\Gamma(\bar{U}_i, B(U_i)), \Gamma(\bar{U}_i, S_i \otimes S_i), \not{D}_i^{S_i}). \tag{5.7}$$

Remark 5.2. We use here the triple (5.6) to formulate the spin structure of the open subspace U_i, but it is by definition not a spectral triple, for the compact resolvent condition fails. However, it can naturally act on the relative K-theory for the pair of spaces (V_i, U_i) or $(Y, \iota(U_i))$ to get an index, where Y is any compact Riemannian spin manifold that admits an isometric embedding $\iota : V_i \to Y$. By excision property, the choice of V_i is irrelevant. In this sense the triple (5.6) represents a relative K-cycle. One may also think of the standard treatment using the nonunital spectral triple $(C_c^\infty(U_i), L^2(U_i, S_i), \not{D}_i)$ with the algebra of smooth functions with compact support. See Gayral-Gracia-Bondía-Iochum-Schücker-Várilly [14] for a set of axioms for nonunital spectral triples which is proposed to set up the notion of noncompact noncommutative spin manifolds. This, however, will cause some subtleties when considering Morita equivalence and smoothness condition.

Because the collection of maps $\{\alpha_{ij} \otimes \alpha_{ij}\}$ satisfy the cocycle condition, the vector bundles $S_i \otimes S_i$ and Dirac operators $D_i^{\mathcal{E}_i}$ can be glued together to form a vector bundle N over M and a Dirac operator D_N on N, so that $(\mathcal{B}|_{U_i}, \Gamma(M, N)|_{U_i}, D_N|_{U_i})$ are unitarily equivalent to $(\mathcal{B}_i, \mathcal{H}_i^{\mathcal{E}_i}, D_i^{\mathcal{E}_i})$, where $\mathcal{B} = \Gamma(M, B(M))$. Thus we succeed to construct a globally well-defined spectral triple $(\mathcal{B}, \Gamma(N), D_N)$ on M.

Proposition 5.3. *The global vector bundle N is isomorphic to $B(M)$, and $\Gamma(N)$ is isomorphic to $\Gamma(B(M))$ as both $\Gamma(B(M))$-modules and pre-Hilbert spaces.*

Proof. Let S_n^* be the dual vector space of the standard spinor vector space S_n. We endow S_n^* with a left B_n-module structure by $\gamma^*(b) f_x := f_{\bar{b}x} = (x, \bar{b}^* \cdot)$, for all $x \in S_n, b \in B_n$, where $\bar{b}$ is the complex conjugation of b, and b^* is the adjoint of $b \in B_n \cong M_{2^m}(\mathbb{C})$. Since B_n is a simple algebra, up to a scalar there is a unique B_n-module isomorphic from S_n to S_n^*. We fix one specific unitary B_n-module isomorphism $T_n : S_n \to S_n^*$. Then T_n induces a Clifford module isomorphism of bundles $T : S_i \to S_i^*$, given by $(p, s) \mapsto (p, T_n s)$, for all $p \in P_i, s \in S_n$. Let $S_i^* = P_i \times_{\gamma^*} S_n^*$, and let $\alpha_{ij}^* : S_i^* \to S_j^*$ denote

the Clifford module isomorphism given by $\alpha_{ij}^*(p,f)) = (\alpha_{ij}p, f)$, for all $f \in S_i^*$. The mappings T on U_i and U_j are compatible, namely the diagram below commutes on U_{ij}.

$$\begin{tikzcd} S_i \arrow[r, "T"] \arrow[d, "\alpha_{ij}"'] & S_i^* \arrow[d, "\alpha_{ij}^*"] \\ S_j \arrow[r, "T"] & S_j^* \end{tikzcd}$$

Then one can glue the bundles $S_i^* \otimes S_i \cong \mathrm{End}_{\mathbb{C}}(S_i^*)$ together as a global bundle $B(M)$ via the maps $\alpha_{ij}^* \otimes \alpha_{ij}$, and T induces an isomorphism from N to $B(M)$. Then it is easy to verify the proposition. □

Corollary 5.4. *When M is odd dimensional, one can take the bundle $N = (1+*)\bigwedge^*(T_{\mathbb{C}}^*M)$, then the spectral triple $(\mathcal{B}, \Gamma(N), D_N)$ is the projective spectral triple*

$$(\mathcal{A}_{W_3}, \mathcal{H}_{W_3}, D_{W_3}) = (\mathcal{B}, (1+*)\Omega(M), (d-d^*)(-1)^{\deg}); \tag{5.8}$$

and when M is even dimensional, take $N = \bigwedge^*(T_{\mathbb{C}}^*M)$, then the spectral triple $(\mathcal{B}, \Gamma(N), D_N)$ with the grading on $\Gamma(N)$ obtained from the grading on S_i is the even projective spectral triple

$$(\mathcal{A}_{W_3}, \mathcal{H}_{W_3}, D_{W_3}, \gamma_{W_3}) = (\mathcal{B}, \Omega(M), (d-d^*)(-1)^{\deg}, *(-1)^{\deg(\deg+1)/2-n/4}), \tag{5.9}$$

and its center $(\mathcal{A}, \mathcal{H}_{W_3}, D_{W_3}, \gamma_{W_3})$ is unitarily equivalent to the spectral triple for Hirzebruch signature. For any $a \in \mathcal{A}$, the commutator $[D_{W_3}, a]$ is just the right Clifford action of da on $\mathcal{H}_{W_3}$.

Theorem 5.5. *If M is spin^c, the projective spectral triple on M is Morita equivalent to the spin spectral triple.*

Proof. Let S be the global spinor bundle over M. There exist local half line bundles $\mathscr{L}_i^{1/2}$, such that $\mathscr{L}_i^{1/2}$ are characterized by $\{\beta_{ij}\}$ as local transition functions, $\mathscr{L}_i^{1/2} \otimes \mathscr{L}_i^{1/2} = \mathscr{L}|_{U_i}$, and $S|_{U_i} = \mathscr{L}_i^{1/2} \otimes S_i$. There also exists a hermitian connection ∇_1 on $\mathscr{L}^{1/2}$ such that the Spin^c-connection $\nabla|_{U_i} = 1 \otimes \nabla_i + \nabla_1 \otimes 1$, where ∇_i is the Spin-connection on S_i. Then follows the Morita

equivalence of the spectral triples

$$(C^\infty(M), \Gamma(M,S), \not{D}) = (\mathcal{A}, \mathcal{H}, D) \sim (\mathcal{A}, \mathcal{H}^{\mathscr{L}^{-1}}, D^{\mathscr{L}^{-1}})$$

$$\sim (\mathcal{B}, (\mathcal{H}^{\mathscr{L}^{-1}})^S, (D^{\mathscr{L}^{-1}})^S) = (\mathcal{A}_{W_3}, \mathcal{H}_{W_3}, D_{W_3}).$$

□

We see that the projective spectral triple is defined for any closed oriented Riemannian manifold regardless of whether the manifold is spin^c or not. The projective spectral triple depends only on the metric and orientation of M and does not depend on the choice of the local spinor bundles S_i.

Chapter 6

Projective Spectral Triple as Fundamental Class in $K_0(M, W_3)$

In this section we see how projective spectral triples represent the fundamental classes in the twisted K-homology $K^0(\mathcal{A}_{W_3}) \cong K_0(M, W_3)$.

Denote by $\mathbf{C}_{\mathrm{gr}}$ the $\mathbb{Z}_2$-graded algebra of sections of Clifford bundle $\mathbb{C}l(T^*M)$ over M (even dimensional only), then every Clifford module E over M can be considered as a finitely generated projective right $\mathbf{C}_{\mathrm{gr}}^{\mathrm{op}}$-module, and a Clifford connection ∇^E on E gives rise to a Dirac operator D^E on E. Then $E \mapsto \operatorname{Ind} D^E$ defines a canonical homomorphism

$$K_0(\mathbf{C}_{\mathrm{gr}}^{\mathrm{op}}) \xrightarrow{\mathrm{Ind}} \mathbb{Z}.$$

By Morita equivalence, $K_0(\mathbf{C}_{\mathrm{gr}}^{\mathrm{op}})$ can be replaced by the K-theory of an ungraded algebra, $K_0(\mathcal{A}_{W_3})$, and the homomorphism Ind becomes the operation of pairing with the projective spectral triple.

Theorem 6.1 (Poincaré duality)**.** *For an even dimensional closed oriented manifold M, the projective spectral triple $\varsigma = (\mathcal{A}_{W_3}, \mathcal{H}_{W_3}, D_{W_3}, \gamma_{W_3})$ represents the twisted K-orientation as a cycle of the twisted K-homology $K^0(\overline{\mathcal{A}_{W_3}}) \cong K_0(M, W_3)$, and hence gives rise to the Poincaré duality*

$$K^0(M, W_3 - c) \xrightarrow[\cong]{\frown[\varsigma]} K_0(M, c), \quad \textit{or} \quad K^*(M, c) \times K^*(M, W_3 - c) \xrightarrow[\text{pairing}]{\text{nondegenerate}} \mathbb{Z},$$

for all $c \in H^3(M, \mathbb{Z})$. Here the cap product can be defined by $[\mathcal{E}] \frown [\varsigma] = [\varsigma^{\mathcal{E}}]$ for any finite Kasparov

$C^\infty(M)$-$\mathcal{A}_{W_3}$-module $\mathcal{E}$.

For odd dimensional M, the Poincaré duality reads

$$K^*(M,c) \times K^{*+1}(M, W_3 - c) \xrightarrow[\text{pairing}]{\text{nondegenerate}} \mathbb{Z}, \quad \forall c \in H^3(M,\mathbb{Z}).$$

See Kasparov [17], Carey-Wang [5], and Wang [24] for details. When c is 0, this is a special case of the second Poincaré duality theorem [17] in KK-theory.

Chapter 7

Local Index Formula for Projective Spectral Triples

In this section we present a local index formula associated to the projective spectral triple for every closed oriented Riemannian manifold M of dimension $2n$. Let $\mathcal{A} = C^\infty(M)$. Denote by

$$\varsigma = (\mathcal{B}, \mathcal{H}, D, \gamma) = (\mathcal{A}_{W_3}, \mathcal{H}_{W_3}, D_{W_3}, \gamma_{W_3})$$

the projective spectral triple of M defined in the preceding sections. Suppose a K-class $[p]$ or $[\mathcal{E}]$ in $K_0(\mathcal{B})$ is represented by a projection matrix $p = (p_{ij}) \in \mathrm{M}_m(\mathcal{B})$ or by a right $\mathcal{B}$-module $\mathcal{E} = p\mathcal{B}^m$. Let $D^{\mathcal{E}}$ denote the twisted Dirac operator on $\mathcal{H}^{\mathcal{E}} = \mathcal{E} \otimes_{\mathcal{B}} \mathcal{H} = p\mathcal{H}^m$ associated to the projective universal connection $\nabla^{\mathcal{E}} : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{B}} \Omega^1_u(\mathcal{B})$ on $\mathcal{E}$, namely $\nabla^{\mathcal{E}}(p\boldsymbol{b}) = p\delta_u(p\boldsymbol{b})$ and $D^{\mathcal{E}}(p\boldsymbol{h}) = pD(p\boldsymbol{h})$, $\forall \boldsymbol{b} \in \mathcal{B}^m$, $\forall \boldsymbol{h} \in \mathcal{H}^m$.

The left $\mathcal{B}$-module $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ is $\mathbb{Z}_2$-graded and so is $\mathcal{H}^{\mathcal{E}} = \mathcal{H}^{\mathcal{E}}_+ \oplus \mathcal{H}^{\mathcal{E}}_-$. Denote by $D^{\mathcal{E}}_\pm$ the restrictions of $D^{\mathcal{E}}$ to $\mathcal{H}^{\mathcal{E}}_\pm \to \mathcal{H}^{\mathcal{E}}_\mp$. The index of $D^{\mathcal{E}}$ is

$$\mathrm{Ind}(D^{\mathcal{E}}) = \dim\ker D^{\mathcal{E}}_+ - \dim\ker D^{\mathcal{E}}_-.$$

Using the well-known local index formula (cf.[3]), we have

$$\mathrm{Ind}(D^{\mathcal{E}}) = \int_M \hat{A}(M)\mathrm{ch}(\mathcal{H}^{\mathcal{E}}/\mathcal{S}).$$

$\hat{A}(M)$ is the $\hat{A}$-genus of the manifold M,

$$\hat{A}(M) = \det{}^{1/2}\left(\frac{R/2}{\sinh(R/2)}\right) \in \Omega^{\mathrm{ev}}(M).$$

The relative Chern character $\mathrm{ch}(\mathcal{H}^{\mathcal{E}}/\mathcal{S})$ is explained as follows. We consider $\mathcal{H}$, and $\mathcal{H}^{\mathcal{E}}$ as well, as right Clifford modules with right Clifford actions c_R. The connection $\nabla : \mathcal{H} \to \mathcal{H} \otimes_{\mathcal{A}} \Omega^1(M)$ on $\mathcal{H}$ induced by the Levi-Civita connection on M is a right Clifford connection. We can define a right Clifford connection $\nabla^{\mathcal{H}^{\mathcal{E}}} : \mathcal{H}^{\mathcal{E}} \to \mathcal{H}^{\mathcal{E}} \otimes_{\mathcal{A}} \Omega^1(M)$ on $\mathcal{H}^{\mathcal{E}}$ by $\nabla^{\mathcal{H}^{\mathcal{E}}}(p\boldsymbol{h}) = p\nabla(p\boldsymbol{h})$. Denote by $R^{\mathcal{H}^{\mathcal{E}}} \in \mathrm{End}_{\mathcal{A}}(\mathcal{H}^{\mathcal{E}}) \otimes_{\mathcal{A}} \Omega^2(M)$ the curvature of the connection $\nabla^{\mathcal{H}^{\mathcal{E}}}$,

$$R^{\mathcal{H}^{\mathcal{E}}} = \nabla^{\mathcal{H}^{\mathcal{E}}}\nabla^{\mathcal{H}^{\mathcal{E}}} = p(\nabla p)(\nabla p) + p\nabla^2 \circ p,$$

and denote by T the twisting curvature, that is $T = R^{\mathcal{H}^{\mathcal{E}}} - R^{\mathcal{S}}$, where

$$R^{\mathcal{S}} = c_R(R) = \frac{1}{4}R_{ijkl}c_R(e_l)c_R(e_k)e_i \wedge e_j,$$

and R_{ijkl} are the components of the Riemannian curvature tensor on M under an orthonormal frame $\{e_i\}$. One can verify that $T = p(\nabla p)(\nabla p) - pc_L(R)p$. With the above notations, the relative Chern character is

$$\mathrm{ch}(\mathcal{H}^{\mathcal{E}}/\mathcal{S}) = 2^{-n}\mathrm{tr}\exp(-T).$$

So we have an explicit local index formula

$$\mathrm{Ind}(D^{\mathcal{E}}) = 2^{-n}\int_M \hat{A}(M)\mathrm{tr}\ \exp(-p(\nabla p)(\nabla p) + pc_L(R)p). \tag{7.1}$$

From the viewpoint of noncommutative geometry,

$$\mathrm{Ind}(D^{\mathcal{E}}) =< [p], [\varsigma] >=< \mathrm{ch}[p], \mathrm{ch}[\varsigma] >,$$

where $\mathrm{ch}[p] \in HP_0(\mathcal{B})$ and $\mathrm{ch}[\varsigma] \in HP^0(\mathcal{B})$ are the periodic Connes-Chern characters of $[p]$ and $[\varsigma]$ respectively. On the other hand, in terms of twisted Chern characters, as defined below,

$$\mathrm{ch}_{W_3}[p] := \mathrm{Chkr}(\mathrm{ch}[p]) \in H^{\mathrm{ev}}(M, \mathbb{C}), \quad \mathrm{ch}_{W_3}[\varsigma] := (\mathrm{Chkr}^*)^{-1}(\mathrm{ch}[\varsigma]) \in H_{\mathrm{ev}}(M, \mathbb{C}), \tag{7.2}$$

the index pairing can be written as

$$\mathrm{Ind}(D^{\mathcal{E}}) =< [p], [\varsigma] >=< \mathrm{ch}_{W_3}[p], \mathrm{ch}_{W_3}[\varsigma] > .$$

We now try to give local expressions of $\mathrm{ch}[p]$, $\mathrm{ch}[\varsigma]$, $\mathrm{ch}_{W_3}[p]$, and $\mathrm{ch}_{W_3}[\varsigma]$ as well as their relation (7.2) explicitly. The periodic Connes-Chern character $\mathrm{ch}[p]$ is represented by a sequence of cyclic cycles $\{\mathrm{ch}_0^\lambda(p), \mathrm{ch}_2^\lambda(p), ...\}$, where

$$\mathrm{ch}_{2m}^\lambda(p) = (-1)^m \frac{(2m)!}{m!} \mathrm{tr}(p^{\otimes 2m+1}) \in C_{2m}^\lambda(\mathcal{B}).$$

This sequence satisfies the periodicity condition $S(\mathrm{ch}_{2m+2}^\lambda(p)) = \mathrm{ch}_{2m}^\lambda(p)$. An alternate way to represent $\mathrm{ch}[p]$ is to use normalized $(\flat, \mathrm{B})$-cycles, that is

$$\mathrm{ch}_{2m}^{(\flat,\mathrm{B})}(p) = (-1)^m \frac{(2m)!}{m!} \mathrm{tr}((p - \frac{1}{2}) \otimes p^{\otimes 2m}).$$

As for the Connes-Chern character of ς, one can apply Connes-Moscovici [11] local index formula to get a normalized $(\flat, \mathrm{B})$-cocycle. However, when trying to derive from Connes-Moscovici's formula an expression in terms of integrals of differential forms on M, one will be confronted with a very much involved calculation of Wodzicki residues of various pseudo-differential operators. On the other hand, based on the appearance of formula (7.1), one can get a C_λ-cocycle $\mathrm{ch}_\lambda(\varsigma) = \sum_m \mathrm{ch}_\lambda^{2m}(\varsigma)$ as follows:

Let $T(b_1, b_2) = (\nabla b_1)(\nabla b_2) - b_1 c_L(R) b_2$, and define $\rho_{2m}^0 : \mathcal{B}^{\otimes 2m+1} \to \Omega^{2m}(M)$ by

$$\rho_{2m}^0(b_0, ..., b_{2m}) = \mathrm{tr}(b_0 T(b_1, b_2) \cdots T(b_{2m-1}, b_{2m})).$$

Then the relative Chern character $\mathrm{ch}(\mathcal{H}^{\mathcal{E}}/\mathcal{S}) = \frac{1}{2^n(2m)!}\rho^0_{2m}(\mathrm{ch}^\lambda_{2m}[p])$.

It is easily seen that $\frac{1}{2^n(2m)!}\int_M \hat{A}(M)\rho^0_{2m}(b_0, ..., b_{2m})$ is a Hochschild cocycle but not cyclic cocycle if $m \geq 2$. By applying Theorem 2.5, we know that

$$\rho_{2m}(b_0, ..., b_{2m}) = \mathrm{tr}(b_0\psi_{2m}(b_1, ..., b_{2m})) \tag{2.4}$$

is a cyclic cocycle, and that $\rho_{2m}(\mathrm{ch}^\lambda_{2m}(p)) = \rho^0_{2m}(\mathrm{ch}^\lambda_{2m}(p))$ for all p with $[p] \in K_0(\mathcal{B})$. Thus by Theorem 2.5 and the duality theorem (Thm. 6.1), we have the following conclusions:

Theorem 7.1. *The cyclic cocycle* $\mathrm{ch}_\lambda(\varsigma) = \sum_m \mathrm{ch}^{2m}_\lambda(\varsigma)$, *where*

$$\mathrm{ch}^{2m}_\lambda(\varsigma)(b_0, ..., b_{2m}) = \frac{1}{2^n(2m)!}\int_M \hat{A}(M)\rho_{2m}(b_0, ..., b_{2m}), \quad \forall b_i \in \mathcal{B},$$

represents the Connes-Chern character $\mathrm{ch}[\varsigma]$ *of the projective spectral triple* ς.

Theorem 7.2. *The Connes-Chern character and the twisted Chern character are related by*

$$\mathrm{ch}[\varsigma] = \mathrm{ch}_{W_3}[\varsigma] \circ \sum_m \rho_{2m}, \quad \textit{and} \quad \mathrm{ch}_{W_3}[p] = \sum_m \rho_{2m}(\mathrm{ch}^\lambda_{2m}[p])$$

as identical periodic cyclic cohomology classes and de Rham cohomology classes respectively.

Corollary 7.3. *The twisted Chern characters of* $[p]$ *and* $[\varsigma]$ *can be represented by*

$$\mathrm{ch}_{W_3}[p] = 2^n(\deg)!\,\mathrm{ch}(\mathcal{H}^{\mathcal{E}}/\mathcal{S}), \quad \mathrm{ch}_{W_3}[\varsigma] = \frac{1}{2^n(\deg)!}[\hat{A}(M)] \frown [M]$$

respectively.

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