Limited Randomness in Games, and Computational Perspectives in Revealed Preference

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To ma, pa, Sriram for their irrational love and support, and to D. who was blind to our faults.

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Abstract

In this dissertation, we explore two particular themes in connection with the study of games and general economic interactions: bounded resources and rationality. The rapidly maturing field of algorithmic game theory concerns itself with looking at the computational limits and effects when agents in such an interaction make choices in their "self-interest." The solution concepts that have been studied in this regard, and which we shall focus on in this dissertation, assume that agents are capable of *randomizing* over their set of choices. We posit that agents are *randomness-limited* in addition to being computationally bounded, and determine how this affects their equilibrium strategies in different scenarios.

In particular, we study three interpretations of what it means for agents to be randomness-limited, and offer results on finding (approximately) optimal strategies that are randomness-efficient:

- One-shot games with access to the support of the optimal strategies: for this case, our results are obtained by sampling strategies from the optimal support by performing a random walk on an expander graph.

- Multiple-round games where agents have no *a priori* knowledge of their payoffs: we significantly improve the randomness-efficiency of known *online algorithms* for such games by utilizing distributions based on *almost pairwise* independent random variables.

- Low-rank games: for games in which agents' payoff matrices have low rank, we devise "fixed-parameter" algorithms that compute strategies yielding approximately optimal payoffs for agents, and are polynomial-time in the size of the

input and the rank of the payoff tensors.

In regard to rationality, we look at some computational questions in a related line of work known as revealed preference theory, with the purpose of understanding the computational limits of inferring agents' payoffs and motives when they reveal their preferences by way of how they act. We investigate two problem settings as applications of this theory and obtain results about their intractability:

- Rationalizability of matchings: we consider the problem of rationalizing a given collection of bipartite matchings and show that it is NP-hard to determine agent *preferences* for which matchings would be *stable*. Further, we show, assuming $P \neq NP$, that this problem does not admit polynomial-time approximation schemes under two suitably defined notions of optimization.

- Rationalizability of network formation games: in the case of network formation games, we take up a particular model of connections known as the Jackson-Wolinsky model in which nodes in a graph have *valuations* for each other and take their valuations into consideration when they choose to build edges. We show that under a notion of stability, known as *pairwise stability*, the problem of finding valuations that rationalize a collection of networks as pairwise stable is NP-hard. More significantly, we show that this problem is hard even to approximate to within a factor 1/2 and that this is tight.

Our results on hardness and inapproximability of these problems use well-known techniques from complexity theory, and particularly in the case of the inapproximability of rationalizing network formation games, PCPs for the problem of satisfying the optimal number of linear equations in $\mathbb{Z}_+$, building on recent results of Guruswami and Raghavendra [GR07].

Contents

Chapter 1

Introduction

We didn't lose the game; we just ran out of time.
– Vince Lombardi.

1.1 Rationality and games

In this chapter, we introduce several key concepts and outline a platform that the rest of this dissertation will be based on. We also briefly preview our results and establish how they are related with the themes of bounded resources and rationality in economics.

Any kind of economic interaction between self-interested agents involves them choosing from a (possibly infinite) set of actions, or strategies, and deriving a value from a play of every agent's strategies. This value, or utility, is in turn based on a set of preferences the agents may have among the set of strategies. The interactions, also referred to as games, are best understood in an axiomatic framework in which we are able to express a relationship between the choices agents make when playing their strategies and the consequent utility they perceive by doing so.

Traditionally in economics, there are two schools of thought [MCWG95] in connection with the appropriate axiomatic framework to opt for: a preference-based approach wherein agents are assumed to have a partial order of *preferences* over the set of strategies they can choose from, or a choice-based approach where the observable quantity of interest is the actual choice behavior of the agents, i.e., the strategies they play in the interaction. In the former case, we would need to make the assumption that the preference relations of the agents satisfy some set of rationality axioms, whereas in the latter case we assume that the choices satisfy some consistency axioms.

1.1.1 Rationality and self-interest

As such, the utility that an agent derives from choosing her action in response to the other agents' actions in the game is intrinsic to the agent and is by no means determined solely by how she is impacted by the interaction. This means that, for example, the agent's utility is a non-decreasing function of both her own well-being and that of other agents participating in the interaction, i.e., the agent is *altruistic*. Or perhaps, the agent is *malicious* and therefore her utility is a non-increasing function of other agents in the interaction.

We say that agents are *self-interested* if their utility is solely determined with respect to their own well-being and not dependent on the utilities of other agents. Rationality and self-interest in agents are closely inter-connected; we think of a *rational* agent as one whose objective is to choose actions that would maximize her individual utility, i.e., a rational agent is interested in optimizing her own welfare. In this dissertation, we make the important assumption that we are always working with self-interested agents.

1.1.2 Solution concepts to reason about playing games

Fundamentally, given a game with knowledge of the agents' preferences (and in some cases, consequently their utility functions), and their available strategy space, we are interested in what strategies are feasible for each agent that would ensure the "best outcome" possible, holding down the assumption that agents are rational and, furthermore that every agent has complete knowledge of the utility functions of all other agents in addition to being guaranteed of their rational behavior. It is important to note here that when we refer to an agent choosing a strategy from her set of strategies, we are not restricting her to choose deterministically; the agent can choose to *randomize* over her strategy-set and sample a strategy that is drawn from some probability distribution over the set of all available strategies. We call such a randomized strategy a *mixed strategy*.

In this respect, there have been various *solution concepts* that have been widely investigated in the economics and game theory literature [FT91, Osb03]. In this dissertation, we will principally focus on the solution concept that is most widely used and is known as Nash equilibrium after John F. Nash [Nas51] who proposed it. Put simply, the Nash equilibrium in a game refers to the strategy each agent would play that would be her *best response* to the strategies played by all the other agents participating in the game. In other words, any deviation from the agent's Nash equilibrium strategy will at best result in no increase in her utility. This notion of no incentive for unilateral deviation is extremely powerful and is one of the many arguments in favor of considering the Nash equilibrium because it embodies other features like "expected" rational behavior in light of complete information of the game[1] and convergence of response to established social norms. However, what is even more remarkable is the assertion by Nash that in every game where agents have a finite set of strategies (i.e., finite games), there is always such an equilibrium present. Nash's proof for this result is based on some beautiful mathematical arguments flowing from the existence of Brouwer fixed-point theorems in analysis [Nas51].

1.2 Computational issues in game theory

A very natural question then arises: do efficient algorithms exist for finding Nash equilibria? It is worth observing here that for any such algorithm, there are at least two aspects of efficiency that are of interest from a theoretical computer science perspective: the running time of the algorithm, and the amount of randomness that is used in order to sample the (mixed) strategies of agents. The existence of a tractable algorithm would be of enormous significance since it would imply the ability for computers to model rational agents and determine equilibrium behavior efficiently. At

[1] Bayesian game theory which concerns itself with the study of games with incomplete information is beyond the scope of this work.

the same time and since randomness can also be considered a bounded computational resource, it is desirable that such algorithms use very little or no randomness.

We stress here that there is a distinction between the two aspects of efficiency: while the former pertaining to the tractability of finding Nash equilibria speaks to the hardness inherent in the problem, the latter implicitly assumes that the equilibrium mixed strategy is already available, albeit over a large support and therefore requiring to be optimized in terms of randomness.

1.2.1 PPAD completeness of finding Nash equilibria

Given the importance of time-efficiency in algorithms, it would not be unreasonable to assert that the complexity of Nash equilibria has been one of the pivotal questions that has fueled the growth of algorithmic game theory as a separate field of study within theoretical computer science over the last decade. There is an important distinction however; the class of NP is not the appropriate class to use in looking at the complexity of finding Nash equilibria, exactly because a Nash equilibrium always exists in finite games. To this end, Papadimitriou [Pap94] introduced a class of problems called Polynomial Parity Argument (Directed), abbreviated to PPAD, which captures the specific properties of the solution space of Nash equilibrium problems. An instance of the canonical problem in the PPAD class consists of an input which is a directed graph on an exponentially large set of vertices V (succinctly encoded in n bits) such that each vertex has in-degree and out-degree at most 1, and there is at least one vertex with out-degree 1. We call such a vertex that has in-degree 0 a *source* and correspondingly a vertex that has out-degree 0 a *sink*. A solution to this instance is to find a vertex that has in-degree 1. Note that a simple graph-theoretic property – a directed graph with a source must also have a sink – implies that such a vertex is guaranteed to exist.

This is the correct class of problems to be considering Nash equilibria complexity questions for other reasons as well; Papadimitriou [Pap94] showed that the problem of finding a Brouwer fixed point of a function on a compact set is complete for the PPAD class. This was followed by a string of results [DP05a, GP06, DGP06] ultimately culminating with Chen and Deng [CD05] showing that even in the case of 2-player games, finding a Nash equilibrium is PPAD-complete. To be sure, conjecturing that PPAD$\neq$ P is a much weaker statement than P$\neq$ NP. Nonetheless, it does not diminish the significance of the results above which say that finding a Nash equilibrium, even in two-player games, is as hard as finding a Brouwer fixed-point (notably, the latter problem has evaded a positive resolution thus far).

1.2.2 Approximate Nash equilibria

This leads us to ask how hard it is to find *approximate* Nash equilibria. We define an ϵ-approximate Nash equilibrium strategy in the additive sense (Our discussion of approximate equilibria in this dissertation is confined to additive equilibria, and subsequently, for this definition to work we assume that payoff matrices of players have values in $[-1, 1]$) to be one that, when played by an agent in response to the ϵ-approximate Nash equilibrium strategies of all other agents, achieves a pay-off that is at most ϵ away from the Nash equilibrium pay-off.[2]

Chen et al. [CDT06] showed that finding ϵ-approximate Nash equilibria is also PPAD-complete for ϵ inverse-polynomial in n. Tsaknakis and Spirakis [TS07] showed an algorithm that produced a 0.34-approximate Nash equilibrium, but the question of whether there is some ϵ such that finding approximate Nash equilibria for any fixed $\epsilon' < \epsilon$ is PPAD-hard (or conversely, whether there is a PTAS for finding Nash equilibria) remains a fascinating open problem in need of resolution.

1.2.3 Randomness as a computational resource

Randomness has been widely studied within theoretical computer science, alongside time and space, as a computational resource that is limited in availability. This carries over to game theory as well; agents in implementing their mixed strategies must rely on a random sample from their pure strategy-space.

[2]This is a weaker notion of approximate Nash equilibrium.

One motivation for studying the time-efficiency of finding Nash equilibria was because we could envision computers simulating agents in these economic interactions, and therefore a handle on the tractability of algorithms that are used was necessary. For that same reason, and given the fact that pure randomness as a computational resource is limited, exploring issues concerning randomness-efficiency of these algorithms assumes commensurate significance.

The study of the power (or lack thereof) of randomness in algorithms is a richly mined field with many important results and we refer the interested reader to a survey of the same [Tre06]. Indeed, this additionally motivates the need to understand and determine if many of the powerful tools and techniques from derandomization can carry over *mutatis mutandis* into the domain of algorithmic game theory.

In this dissertation, we embark on a study of randomness as a computational resource to be optimized in playing games. We provide some interpretations of what it means to play games when agents are constrained in their use of randomness and subsequently give algorithms for finding (approximate) equilibrium strategies with these limits in place.

1.2.4 Computational issues in revealed preference theory

The discussion on limited randomness in playing games constitutes the first part of this dissertation, and is premised on assumptions about rationality and explicit knowledge of agents' preferences. In the second part of the dissertation, we look at the computational questions pertaining to when we no longer have access to these preferences and are only privy to their *implications* by way of the choices they make and strategies they play.

Revealed preference theory has been well-studied within economics [Sam48, Afr67, Die73, Var82, FST04]. Our contribution in this part is towards investigating revealed preference theory from a computational perspective and we refer to this as *computational revealed preference theory.* The broad theme that underpins this is the following: given a dataset of observations made of agents' choices (strategies played), how hard is it computationally to:

1. *infer* and explicitly construct the underlying preferences that guide them to make these choices?

2. invalidate agents' choices as inconsistent with any preference relation?

1.2.4.1 Two perspectives from computer science.

There are at least two perspectives from theoretical computer science that we can suggest as components of computational revealed preference theory. We can think of the problem described above as a decision problem: do there exist preferences (utility functions) for all agents consistent with the choices observed in the given input? We say that such a preference relation, if it exists, *rationalizes*[3] the given set of observations. Outside of a natural interest for computer scientists to study them, there is motivation within economics also to study the complexity-theoretic questions relating to revealed preference problems. Deb [Deb08], in looking at the efficiency of rationalizability problems in a collective-household model, argues that determining the computational complexity of problems in revealed preference theory is of great value to empirical economists who design the experiments yielding data on consumer behavior.

At the same time, this is also a learning problem: we are interested in learning the preferences of agents that would be implied from a dataset of choices if agents were assumed to make those choices rationally and with the objective of maximizing their utility. Beigman and Vohra [BV06] looked into the PAC-learnability of revealed preferences in general. While we focus only on the complexity-theoretic approach to understanding revealed preference in this dissertation, we believe that the machine-learning angle is equally relevant and important to provide a better and more complete picture of 'rational' behavior in an economic setting.

We note here that some classes of data cannot always be explained, or *rationalized* by simple (say, linear) utility functions, or even any other "reasonable" class

[3]Note that our use of the term rationalizability is different from the standard game-theoretic notion (cf. pp. 50, [FT91]).

of utility functions. Such settings are interesting to economists, because it becomes possible then, in principle, to "test" various assumptions (e.g., that the players are maximizing a linear utility function). Several (classical and recent) results [Afr67, Var82, FST04, Ech08] in the economic literature establish criteria for when data is *always* rationalizable, thus delineating the limits of the "testable implications" of such data.

There is an important role for theoretical computer science in these questions, as the feasibility of performing such tests depends on being able to answer the rationalizability question *efficiently*. In other words, given a type of economic data, and a target form for an "explanation" (preference profile, a class of utility functions, etc.), we wish to understand the *complexity* of deciding whether the data can be rationalized by an explanation of the prescribed form. To our knowledge these sort of problems have not been studied before. Polynomial-time algorithms to infer preferences for choices made at equilibrium would imply that the process of learning and predicting rational behavior as defined for such settings is itself an efficiently computable phenomenon. This could have consequences, for instance, in problems where we are given a dataset about observed choices and are required to use it to predict future choices that agents would make given a different environment.

We will briefly describe below a few problems in computational revealed preference theory that we make contributions to in this dissertation. Broadly, any such problem will have the following attributes: a dataset that is a record of a number of interactions where agents have expressed a choice (strategy) in each such interaction, a particular solution concept (notion of stability) that is assumed to be in play in each interaction, and an unobservable quantity (typically, the utility function of the agents). We are required to infer this unobservable quantity that rationalizes the dataset.

In certain classical settings [Afr67], rationalization amounts to solving a linear program, and so it is immediately seen to be easy. Other settings have a more combinatorial feel, such as rationalizing matchings or network formation games, both of which we study in this dissertation.

In the remaining sections in this chapter, we will describe briefly our results pertaining to the topics discussed above. In Section 1.3, we describe our results on algorithms for agents playing games with restricted access to randomness. In Section 1.4, we look at the problem of rationalizability of matchings and in Section 1.5, we continue our analysis of computational revealed preference theory into the problem of rationalizability of network formation games.

1.3 Results on games with randomness-limited agents

As mentioned earlier, the trend of results in algorithmic game theory has been on investigating the limitations of the time-efficiency of algorithms for finding Nash equilibria and, with the possible exception of work by Lipton et al. [LMM03], there has not been a concerted effort to understand their *randomness-efficiency*. We offer some results in this direction in three separate settings.

1.3.1 Games with sparse-support strategies

In the first case where agents are playing a mixed strategy by sampling strategies from a probability distribution over their pure strategy set, e.g., by tossing a fair coin, we seek to reduce the randomness required. We consider the model of repeated zero-sum two-player games which are games played by agents over multiple rounds. In these games, players are not aware of their optimal strategies (e.g., games with incomplete information where players have limited access to their payoff matrices) and are required to *learn* them as they play strategies iteratively. This is an extremely powerful framework and is used most frequently to discuss online algorithms.

For these games, the amount of randomness that is required to play strategies at each round is at a premium. Freund and Schapire [FS99] proposed a multiplicative-weight adaptive algorithm for the case of two-player zero-sum repeated games that achieves an expected pay-off at most $O(\sqrt{\frac{\log n}{T}})$ away from that of the optimal mixed strategy over T rounds and n strategies. From a randomness-efficiency perspective, this algorithm is sub-optimal because it involves sampling strategies from the entire set of n strategies for each round, leading to $O(T \log n)$ random bits used. Applying insights from pseudorandomness, we show that using almost-pairwise independent random variables in lieu of fully independent random variables suffices for a small tradeoff in the expected payoff. Indeed, we show that we can make do with $O(\log n + \log\log T + \log(1/\epsilon))$ purely random bits in order to obtain a strategy whose payoffs is at most $O(\epsilon)$ away from that of the Freund-Schapire strategy.

For a restricted class of two-player single-round games known as zero-sum games, where the pay-offs for any pair of agents' strategies sum to zero, Lipton and Young [LY94] showed that for an agent with n strategies a random sampling of $O(\log n/\epsilon^2)$ strategies from a given Nash equilibrium strategy sufficed to obtain an ϵ-Nash equilibrium. Lipton et al. [LMM03] subsequently relaxed the assumption on the pay-off structure for the agents. In this dissertation, we improve these results in two ways: firstly, we completely derandomize the algorithm by giving a deterministic sparsification procedure. We do so by using the well-known technique in pseudorandomness of performing a random walk on an expander graph. We are also able to extend our result easily to games with more than two agents and show a similar sparsification procedure that obtains an approximate Nash equilibrium strategy with support $O(\ell \log n/\epsilon^2)$ for a game with ℓ players.

The significance of our improvements to these two problems lies in the fact that they can be used *orthogonally* to each other and therefore allow us to optimize on the use of randomness further. Therefore, in a scenario where agents are looking to play sparse strategies in a repeated game and are given access to the Nash equilibrium support in each round, we can use our first result to optimize on randomness used *across* rounds and our second result to optimize on randomness used *within* a single round.

1.3.2 Unbalanced games

In the second setting, we consider what happens when agents have *asymmetrically populated* pure strategy sets. We call such games *unbalanced*. For example, imagine a game where one player has access to n strategies whereas the other player's strategy set is $k << n$. In a sense, we can think of the latter player as perhaps being prevented from choosing large-support strategies because of limited access to randomness. How does this affect the other agents? Lipton et al. [LMM03] showed that for two-player games in which one player has only $k << n$ strategies to choose from while the other has access to a full complement of n strategies, there exists an exact equilibrium strategy for the latter player in at most $(k+1)$ strategies. This means that while one player needs $\log k$ random bits to play her strategy, the other would need $\log(k+1)$ random bits to play his.

We improve on this slight asymmetrical bias and show using similar techniques (involving application of a constructive version of Carathéodory's theorem) that given an equilibrium strategy for the player, there exists a deterministic polynomial-time algorithm to compute another equilibrium mixed strategy with support size k. While this result is perceivably less significant than our other results, it does lend insight on the role that randomness plays in agents choosing their equilibrium strategies.

1.3.3 Games of small rank

In the third setting in which we look at limited randomness in games, we consider low-rank games which we define to be games where the payoff matrices (can be higher-dimensional tensors) have small rank, say $k << n$. This manifestation of limited randomness is perhaps less explicit than in the other two frameworks we investigated above; intuitively, games with low-rank payoff matrices are a generalization of the other games we have seen before where the strategy spaces were sparse, and subsequently the payoff matrices had small rank.

Kannan and Theobald [KT07] considered another variant of low-rank games, in which the sum of the payoff matrices of the two players has small rank. In this setting, they gave an algorithm for computing approximate equilibria that is polynomial in the number of strategies and payoff range as long as the rank is fixed. However, their approach does not scale easily when the rank is specified as an input parameter to the algorithm. Specifically, the running-time of their algorithm is polynomial in

$n^k, 1/\epsilon, B$ where B translates to the number of bits of precision required to capture all the payoff values in the input matrices.

Our algorithm requires some ground assumptions that are not hard to justify. In particular, we assume that the payoff matrices of rank k are given in their decomposed form, where each decomposed matrix has entries lying within some range parameters $[-B, B]$. We contend that this is a setting common enough to capture a large class of games that model real-life scenarios, and we demonstrate by way of an example that some congestion games can be represented in this manner.

In this framework, we realize a "fixed-parameter" algorithm to compute approximate equilibria, i.e., polynomial in the number of strategies and some function of the rank and B in the worst case. We extend this to the general ℓ-player game and derive approximate equilibria that are also "fixed-parameter" in the sense as before.[4] To the best of our knowledge, our result is the first-ever that achieves this notion of tractability in this particular model. We believe that our approach in obtaining these results has the potential to be replicated even in general games with the view to finding strategies for approximate equilibria.

1.4 Results on rationalizability of matchings

Among rationalization problems, one can identify at least two broad classes of problems. Some, such as inferring utility functions from consumption data, are rather easily solved efficiently using linear programming [Afr67, Var82]. Others are more combinatorial in nature, and their complexity is not at all obvious. One recent example is the problem of inferring costs from observations of spanning trees being formed to distribute some service, say power [Özs06].

Among the combinatorial-type rationalization problems, one of the most natural is the matchings problem that we study in this dissertation. Here we are given a set of bipartite matchings, and we wish to determine if there are preferences for the

[4] Kannan and Theobald's result, by nature of the algorithm they use, appears to be confined to work only for the two-player setting.

nodes under which all of the given matchings are stable. Matchings, or more precisely "two-sided matching markets," are a central abstraction in economics, investigated in relation to the similar "marriage models" in auction and labor markets [RS90, Fle03, EO04, EY07] and from the point of view of mechanism design [Sön96] and related strategic issues [STT01]. At the same time, they are also a fundamental combinatorial abstraction from the computational perspective.

The simplest setting is the one-to-one model in which we are given two sets of agents, denoted "men" and "women." A matching then is a pairing between an agent from the set of men and an agent from the set of women. Each agent has an ordering that dictates his/her preferred partners. Given these orderings for all agents, the stable marriages problem is to find a pairing of all agents such that for every pair of agents not matched to each other, at least one of the agents in the pair prefers their current partner over the other agent (we refer to such pairs as *non-blocking pairs*).

The revealed preference problem mentioned above was first investigated by Echenique [Ech08] who asked if it was possible to characterize what preference orderings can be implied by a given dataset of matchings between men and women. Echenique gave necessary and sufficient conditions that would need to be satisfied by preference orders that rationalize a set of matchings.

Our contribution in this dissertation is to show that the problem of inferring rationalizable preference profiles is NP-complete. We also take up the hardness of approximation question for this class of problems, when no set of preferences exists that rationalizes all the matchings. Here, two competing notions of optimality for rationalizing a collection of matchings present themselves. In the first instance, our objective is to maximize the number of matchings whereas in the second instance, we are interested in maximizing the number of non-blocking pairs. We show that in either case, the problem is hard to approximate to within a constant factor. By way of an upper bound, we give a trivial 3/4-factor algorithm for the case of maximizing the number of non-blocking pairs.

1.5 Results on rationalizing network formation games

We turn our attention next to the rationalizability of network formation games. Network formation games have been extensively studied [WF94, Jac08] and there have been many solution concepts proposed [Chw94, DM97, JW96, DJ00, HM02, Jac03, Dem04, CAI04, Fer07]. These games are meant to capture a number of real-world scenarios such as the World Wide Web, social networks (e.g., Facebook, MySpace, LinkedIn), the Internet topology and so on.

Jackson and Wolinsky [JW96] proposed a "symmetric connections" model of network formation in which agents wish to form "connections" with other agents and must pay a price for each direct edge they choose to build. The choice of edges an agent builds constitutes her strategy. Any path between two agents has an associated latency. The requirement is that the resulting graph be connected, so that there is a path between any two agents. In addition, each agent, say Alice, has an "intrinsic value" for every other agent, say Bob, in the game which maps how important a path to Bob is for Alice. The utility accruing to an agent from the network formed takes into consideration the intrinsic values of all other agents in the network weighted by some measure of the latency of the respective paths, as well as the prices the agent must pay for building edges in her strategy.

The allied notion of stability for this game is *pairwise stability* wherein an edge present in the network is stable if the marginal utilities of the vertices involved in the edge are both non-negative, and an edge absent in the network is stable if the marginal utility of at least one of the vertices involved is non-positive.

There is considerable motivation to be looking at this class of problems from a sociological and economic perspective. These networks are common in day-to-day life among groups of people who ascribe a certain value ('friendship') to one another but establish connections with only those that they perceive to be most intrinsically valuable to them. If, for instance, everybody in the group was in close physical proximity to one another (they all went to the same high school or college) then the cost of connecting to any one person is insignificant compared to the value derived in return, no matter how small that may be. This would result in a clique as a stable network. However, once this group becomes geographically spread out, the network formed in 'equilibrium' can become sparser, such as a star network, where all connections are made to a single person since the cost of building mutual connections outweighs the utility. This illustrates that, holding the intrinsic value people in such a group have for one another to be invariant, temporal and spatial dynamics significantly affect the manner of how social networks coalesce and stabilize.

In this setting the observer cannot be expected to have access to the actual intrinsic value of each of the individuals in this group which are privately held beliefs. We are then confined to being able to observe the choices they make in who they choose to form connections with. Suppose we are given a series of snapshots taken over time of a single social network of individuals (when they are in equilibrium), it is therefore natural to ask whether it is possible to infer the valuations of individuals on the basis of their choices.

1.5.1 Results on rationalizing Jackson-Wolinsky network formation games

The prologue above sets us up for considering two different revealed preference problems that surface in connection with the Jackson-Wolinsky games. In the first case, denoted STABLE-PRICES, we assume that we do not have access to the prices agents must pay for the edges they choose to build and must infer a set of prices for all edges such that each edge is *stable* for the given dataset. The motivating rationale for this scenario is the setting of truthful mechanisms where the agents reveal their valuations by bidding them but it is not known what prices they are paying for the edges they build at equilibrium. Incidentally, for network formation games that we consider, this turns out to be an easy quantity to infer (when there exists a consistent set of edge-prices); each edge-price is determined exclusively by the two agents that the edge connects and therefore can be ascertained independently of one another.

In the second case, denoted STABLE-VALUES, we have no access to the intrinsic values agents have for one another and are required to infer them based on the observed dataset of networks formed. We show that STABLE-VALUES is NP-complete. The reduction for this problem is from a variant of inequality-satisfiability problems originally formulated by Hochbaum and Moreno-Centeno [HMC08], that we call I-SAT* (we show by a reduction from 3-SAT that I-SAT* is NP-complete). An instance of the inequality-satisfiability problem comprises a conjunction of *inequality-clauses* where each inequality-clause is a disjunction of linear inequalities over unknowns $x_1, \ldots, x_n$ drawn from the reals. The I-SAT* problem instance satisfies two additional constraints: (1) all of the coefficients in the inequalities are non-negative (and we are seeking a solution only in the non-negative reals), and (2) there is a partition of the variables into two sets S, T such that every inequality-clause is either the disjunction of two $\leq$ inequalities, one supported in S and one supported in T, or a conjunction of two $\geq$ inequalities, one supported in S and one in T.

1.5.2 Hardness of approximation

Turning to the hardness of approximation for rationalizing network formation games, we first take up the STABLE-PRICES problem when no consistent set of edge-prices exists to rationalize all the matchings in the dataset. The objective would be to maximize the number of edges that are stable with respect to the given dataset. This optimization problem is also tractable and the algorithm follows along the lines of the one used to solve the STABLE-PRICES problem.

The optimization problem for STABLE-VALUES however is more intricate to reason about. The notion of optimality here is to maximize the number of stable edges/non-edges across all graphs given. We show that this problem is hard to approximate beyond $(1/2 + \epsilon)$ (and this is tight). Our proof for this result is based on showing a hardness of approximation result for the problem MAX-LIN$_{\mathbb{R}_+}$ (and subsequently I-SAT*). Given a collection of linear equations over a set of unknowns and with co-efficients drawn from $\mathbb{R}_+$, the MAX-LIN$_{\mathbb{R}_+}$ problem asks to maximize the number of equations that can be satisfied with the solutions over $\mathbb{R}_+$. For this, we must rely on sophisticated techniques inspired from similar inapproximability results for MAX-LIN$_{\mathbb{F}_p}$ (Håstad [Hås01]) and MAX-LIN$_{\mathbb{R}}$ (Guruswami and Raghavendra [GR07]).

Our hardness results for STABLE-VALUES offer an interesting and potentially powerful connection between revealed preference problems and inequality-satisfiability problems. We contend that inequality-satisfiability problems are the abstract computational class that captures rationalization problems more generally; the "stability conditions" arising in a rationalization problem can be expressed by a finite Boolean formula whose inputs are inequalities in the (real) quantities being inferred. This is true, for instance, also for the matchings problem (the quantities being inferred are the values each "man" has for each "woman," and the stability condition for a non-blocking pair can be expressed as the disjunction of two inequalities involving these quantities).

1.6 Outline

We give an outline of the rest of this dissertation. Chapter 2 provides the background and context for the results discussed in this dissertation. Briefly, it goes over previous work in algorithms for playing games with small strategies, and introduces both an economics and a computational perspective to topics and questions in revealed preference theory.

Chapter 3 covers our results on playing equilibrium strategies for randomness-limited agents. We look at three specific settings in which these constraints are imposed and provide algorithms in each setting for obtaining approximate equilibria. The contents of Chapter 3 are largely based on work done with Umans [KU07].

Our foray into computational revealed preference theory begins with Chapter 4 in which we look into the complexity of rationalizability of matchings. We discuss our results both in terms of hardness and inapproximability of the problems. We first start by reviewing rationalizability as a topic deserving of further investigation, and subsequently review Echenique's work [Ech08] on which we base our results. The contents of this chapter are based on work done with Umans [KU08].

In Chapter 5, we continue with looking at these issues in the context of network formation games. We review in detail the Jackson-Wolinsky model and introduce the notion of pairwise stability. We consider the two revealed preference problems mentioned above and provide positive and negative results for the two problems respectively. The contents of this chapter are based on a paper with Umans [**?**].

Chapter 2

Background

In this chapter, we will establish the background necessary for discussing our results in this dissertation.

2.1 Computational issues in game theory

2.1.1 Nash equilibrium and the PPAD class

Game theory, ever since its inception in the 50s by Morgenstern and von Neumann [MvN44], and the result on existence of equilibria in finite games by Nash [Nas51], has played a seminal role in the formalization of economics. The following definition of a game is well-known and is borrowed from Fudenberg and Tirole [FT91]:

Definition 2.1.1 Game: *A game $\mathcal{G}$ in strategic form has three elements: a set of players labeled $\mathcal{P} = (1, \ldots, \ell)$, a* pure strategy space *S_i for $i \in \mathcal{P}$, and payoff functions $u_i : \times_j S_j \to \mathbb{R}$ that give the utility $u_i(s)$ for player i when a strategy profile, i.e., an ℓ-tuple $s = (s_1, \ldots, s_\ell)$ is observed to have been played with s_i played by player i.*

For this dissertation, we will restrict our focus only to games in which S_i are discrete sets. Before we introduce Nash equilibria and Nash's result on their existence, we must define mixed strategies:

Definition 2.1.2 Mixed strategy: *Given a pure strategy space S_i for player i, a mixed strategy σ_i is a probability distribution over S_i.*

In other words, a mixed strategy is a randomization over pure strategies in S_i (and trivially, a pure strategy is a mixed strategy with support of size 1). For the rest of this section, wherever it is clear, we will assume that a strategy refers by default to a mixed strategy. We denote $\Delta(S_i)$ to be the set of all strategies σ_i with support in S_i. The payoff for a mixed strategy σ_i is then simply the expected payoff over the joint distribution of the σ_i for each S_i. Given a strategy profile $\sigma = (\sigma_1, \ldots, \sigma_\ell)$, we will use the shorthand $\sigma = (\sigma_i, \sigma_{-i})$ where σ_{-i} denotes the profile of strategies over players $j \neq i$. We define a Nash equilibrium as follows.

Definition 2.1.3 Nash equilibrium: *For a game $\mathcal{G}$, a strategy profile σ^* is a Nash equilibrium if, for all $i \in \mathcal{P}$:*

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i', \sigma_{-i}^*) \textit{ for all } \sigma_i' \in \Delta(S_i)$$

For a player i, given a strategy profile σ_{-i} we will call a strategy σ_i i's *best response* to σ_{-i} if:

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i}) \text{ for all } \sigma_i' \in \Delta(S_i)$$

.

The following characterization is useful for some results we obtain in Chapter 3.

Theorem 2.1.4 *For any player i, given a mixed strategy profile σ_{-i}, a mixed strategy σ_i is a best response to σ_{-i} if and only if all pure strategies in the support of σ_i are each a best response to σ_{-i}.*

Proof. Let σ_i be a best response to σ_{-i}. This implies that there is at least one pure strategy that is a best response to σ_{-i} since the payoff on playing σ_i is nothing but an expectation over the distribution of pure strategies in the support of σ_i (holding down σ_{-i}). Suppose now that there is some strategy s in the support of σ_i which is not a best response to σ_{-i}. Then, the strategy obtained by shifting the probability weight from s to a pure strategy that is a best response to σ_{-i} is strictly better than σ_i against σ_{-i} which contradicts the assumption that σ_i is a best response.

To show the converse, suppose every pure strategy in the support of σ_i is a best response to σ_{-i}. Let σ_i' be an arbitrary best response to σ_{-i}. Since each pure strategy s in the support of σ_i is a best response to σ_{-i},

$$u_i(s, \sigma_{-i}) = u_i(\sigma_i', \sigma_{-i})$$

for all s in the support of σ_i. Therefore, σ_i, which is a positive convex combination of terms in the left-hand side of the above equation is also a best response to σ_{-i}. $\square$

Theorem 2.1.5 (Nash [Nas51]) *Every finite strategic-form game has a Nash equilibrium.*

As mentioned earlier, Theorem 2.1.5 is a seminal result and laid the foundations of game theory and its proof uses arguments flowing from existence of Brouwer fixed-point theorems, which is beyond the scope of this dissertation.

With the existence established, we are ready to look at the computational issues connected with Nash equilibria. Megiddo and Papadimitriou [MP91] argued easily that NP is not the correct complexity class to look at in considering the problem of finding a Nash equilibrium, denoted NASH. In general, by virtue of guaranteed existence of an equilibrium in finite games, NASH falls under the realm of an "umbrella-class" called TFNP, or Total Function Nondeterministic Polynomial-time which we define as the class of function problems that, for an input x and a polynomial-time computable predicate $P(x, y)$, output a y satisfying $P(x, y)$. Such a y is guaranteed to exist for all inputs x.

Papadimitriou [Pap94] defined a subclass of problems in TFNP, Polynomial Parity Argument Directed (or simply PPAD), which is informally defined as the set of all total functions for whom the existence is guaranteed by the following graph property: in a directed graph in which all vertices have indegree and outdegree at most 1, if there is a source (i.e., vertex with indegree 0) then there must exist a sink (i.e., vertex with outdegree 0). Daskalakis [Das08] provides a more formal circuit-based description for

PPAD; it is the set of all function problems that are polynomial-time reducible to the following canonical problem:

END OF THE LINE: Given two circuits S and P each with n input bits and n output bits, such that $P(0^n) = 0^n; S(0^n) \neq 0^n$, find an input $x \in \{0,1\}^n$ such that $P(S(x)) \neq x$ or $S(P(x)) \neq x$ and $x \neq 0^n$.

Papadimitriou [Pap94] first showed that a number of function problems involving finding topological fixed points such as BROUWER, SPERNER, etc. are PPAD-complete and conjectured that NASH was also PPAD-complete (at least for greater than 2 players). A string of results due to Daskalakis, Goldberg and Papadimitriou [DP05b, DGP06] established that 3-NASH, i.e., finding a Nash equilibrium in 3-player games, was PPAD-complete. This hardness result was subsequently strengthened to 2-NASH, or simply NASH, being PPAD-complete by Chen and Deng [CD05].

2.1.2 Finding approximate Nash equilibria

Given the results above, and assuming PPAD $\neq$ P, the next best thing we can hope for is efficient algorithms for finding *approximate* Nash equilibria. The most standard notion is that of *additive* approximate Nash equilibria, and unless explicitly mentioned, all the approximate Nash equilibria that we describe in this dissertation are by default additive Nash equilibria. We will also assume that the payoff functions are scaled to lie in $[0,1]$:

Definition 2.1.6 ϵ-approximate Nash equilibrium: *For a game G among ℓ players all of whose payoff functions take values in $[0,1]$, a mixed-strategy profile $(\sigma_1^*, \ldots, \sigma_\ell^*)$ is an ϵ-approximate Nash equilibrium if for every i:*

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*) - \epsilon \text{ for all } \sigma_i \in \Delta(S_i).$$

We could hope for a polynomial-time approximation algorithm for finding Nash equilibria for arbitrary ϵ, but Chen et al. [CDT06] showed that finding ϵ-approximate

Nash equilibria remains PPAD-hard for ϵ inverse-polynomial in the number of strategies n. Whereas a number of positive and negative results are known for fixed ϵ, the question of whether there exists an ϵ^* such that finding an ϵ-approximate Nash equilibrium is hard for all $\epsilon \leq \epsilon^*$ remains open. That is, the existence of a PTAS for finding Nash equilibria remains an open problem.

2.2 A brief overview of revealed preference theory

2.2.1 Historical context

The origins of revealed preference theory in economics are in consumer behavior theory. Revealed preference theory was originally proposed as a mathematical model in economics by Samuelson. In his Economica paper [Sam48], Samuelson gives a pithy description of the theory:

> The central notion underlying the theory of revealed preference, and indeed the whole modern economic theory of index numbers, is very simple. Through any observed equilibrium point, A, draw the budget-equation straight line with arithmetical slope given by the observed price ratio. Then all combinations of goods on or within the budget line could have been bought in preference to what was actually bought. But they weren't. Hence, they are all "revealed" to be inferior to A. No other line of reasoning is needed.

In other words, in a two-goods economy where a particular equilibrium is observed characterized by a pair P of goods consumed in quantities x_1, x_2 and at prices p_1, p_2 respectively, it must be the case that such a pair must be 'preferred' to any other pair P' of goods at the same price that cost at most the same as $(p_1x_1 + p_2x_2)$. We denote this by $P \succeq P'$ where the binary relation $\succeq$ is called a *preference relation*. By dint of choosing goods at equilibrium point (x_1, x_2) at prices (p_1, p_2) respectively therefore, the consumer is *revealing* such a preference for P over P' even though acquiring P'

would also satisfy the budget. This is expressed mathematically as the Weak Axiom of Revealed Preference:

Definition 2.2.1 Weak Axiom of Revealed Preference: *Given a set of m goods and two consumption vectors $\mathbf{x} = (x_1, \ldots, x_m), \mathbf{y} = (y_1, \ldots, y_m)$ consumed at price vectors $\mathbf{p} = (p_1, \ldots, p_m)$ and $\mathbf{p}' = (p'_1, \ldots, p'_m)$ respectively, if $\mathbf{x} \succeq \mathbf{y}$ then*

$$\sum p_i y_i \leq \sum p_i x_i \Rightarrow \sum p'_i y_i < \sum p'_i x_i$$

Informally, the Weak Axiom of Revealed Preference postulates that if a bundle of goods $\mathbf{x}$ is preferred to $\mathbf{y}$, then if $\mathbf{y}$ was affordable at the price vector $\mathbf{p}$ it cannot be the case that $\mathbf{x}$ was affordable at the price vector $\mathbf{p}'$. If it was, then $\mathbf{x}$ would have been consumed at $\mathbf{p}'$ because it is preferred to $\mathbf{y}$.

This axiom however only relates to the case when it is known that $\mathbf{x} \succeq \mathbf{y}$, i.e., when the ordered pair $(\mathbf{x}, \mathbf{y})$ is in the relation $\succeq$. The Strong Axiom of Revealed Preference relates to the transitive closure of a preference relation $\succeq$, denoted $\succeq_S$. To wit, for two bundles of goods $\mathbf{x}, \mathbf{y}$, we say that $\mathbf{x} \succeq_S \mathbf{y}$ if there exist some intermediate bundles $\mathbf{t}_1, \ldots, \mathbf{t}_n$ such that $\mathbf{x} \succeq \mathbf{t}_1, \mathbf{t}_1 \succeq \mathbf{t}_2, \ldots, \mathbf{t}_n \succeq \mathbf{y}$.

Definition 2.2.2 Strong Axiom of Revealed Preference: *Given two vectors $\mathbf{x} = (x_1, \ldots, x_m), \mathbf{y} = (y_1, \ldots, y_m)$ each of m goods consumed at price vectors $\mathbf{p} = (p_1, \ldots, p_m)$ and $\mathbf{p}' = (p'_1, \ldots, p'_m)$, if $\mathbf{x} \succeq_S \mathbf{y}$ then*

$$\sum p_i y_i \leq \sum p_i x_i \Rightarrow \sum p'_i y_i < \sum p'_i x_i$$

Note that under both such axioms, the consumer is expressing a preference implicitly for one bundle of goods over another even though the latter may be within her budget constraints. Therefore, revealed preference gives a partial description of the choice function the consumer is using to maximize her (unobservable) private utility function.

Subsequent work in revealed preference theory followed on Samuelson's ideas to generalize them even further. In particular, Houthakker [Hou50], Richter [Ric66] and Uzawa [Uza60] established deep connections between revealed preference theory and general consumer demand theory, and Hurwicz and Richter [HR71] showed that preference relations and choice functions could interchangeably be constructed from one another as long as they satisfied a set of axioms.

Samuelson [Sam47] showed that given a choice function that satisfied the Weak Axiom of Revealed Preference in addition to some other continuity properties, it is possible to construct a continuous, quasiconcave utility function that the consumer is assumed to maximize. Houthakker [Hou50] extended this result to construct a utility function when the demand function satisfied the Strong Axiom of Revealed Preference.

2.2.2 Preference relations and utility maximization

Having introduced the axioms of revealed preference and preference relations in the previous section, and having briefly touched on rationality earlier in Chapter 1, we now seek to tie the two concepts together. The following definition of *rationality* of preference relations is standard and is quoted from Mas-Colell et al. [MCWG95].

Definition 2.2.3 Rationality: *A preference relation $\succeq$ on a set of choices S is rational if it is:*

- *complete: for all $s, t \in S$ $s \succeq t$ or $t \succeq s$ or both.*
- *transitive: for all $s, t, v \in S$ if $s \succeq t$ and $t \succeq v$, then $s \succeq v$.*

Note that by assuming both completeness and transitivity as properties that must be satisfied by a rational preference relation, we are tacitly working under the Strong Axiom of Revealed Preference as defined in Definition 2.2.2.

Knowledge of an agent's preference relation, while handy, is unwieldy in reasoning about her choices. However, if the preference relation is "well-defined" on the choice set, then it can be shown that the preference relation corresponds to a payoff function for the agent that is also "well-behaved". By way of example, the following theorem due to Rader [Rad63] states conditions for the existence of a utility function corresponding to a preference relation over a choice set X:

Theorem 2.2.4 *(Rader [Rad63]) Let X be a separable metric space, and $\succeq$ a complete preference relation on X that is upper semicontinuous.*[1] *Then there exists a function $u : X \to [0,1]$ such that for any $x, y \in X$, $x \succeq y \iff u(x) \geq u(y)$.*

Rader's theorem sets up the corresponding utility maximization problem given a set of observations of consumer choices. This is a cardinal precept for our discussion on computational revealed preference theory, since it allows us continue our discussion by focusing on the utility function when convenient and avoid having to argue about preference relations.

2.2.3 Studying revealed preferences in the absence of access to consumer demand function

These results however pertain to the case when the continuous demand function is explicitly given to the observer (in other words, we are given access to an oracle who we can query infinitely often to obtain information about the consumer's demand). Afriat [Afr67] showed that given a finite set of observations, each observation being a price vector over all goods and the corresponding quantities of goods purchased at that price, it was possible to construct a piecewise linear, concave utility function that *rationalized* the observations. This was more explicitly set down by Diewert [Die73], and Varian [Var06] who observed that Afriat's approach was tantamount to using linear programming techniques to solve for the utility function. Fostel et al. [FST04] further explicitly addressed the complexity of determining whether the given observations were consistent with a utility function that consumers are seeking to maximize. They showed that the LP in Varian's algorithm to construct the piecewise linear utility function had a worst-case runtime of $O(n^3)$ where n is the number of

[1] $\succeq$ is upper semicontinuous on X if the set $L_{\succ}(x) = \{y \in X | x \succeq y, y \not\succeq x\}$ is an open subset of X for each $x \in X$.

observations made, and simplified this further to give an algorithm with $O(n^2)$ worst-case runtime.

2.2.4 Testable implications and refutability.

An important distinction between the economics approach to revealed preference theory and the computational approach pertains to the notion of *testable implications* and *refutability.* Specifically, in economics, if for a given dataset, there exists no consistent set of preference relations that would rationalize the data then the underlying assumptions governing the choices can be *refuted.* This is done by testing the implications of these assumptions on the choices observed in the dataset.

On the other hand, if a dataset of observed choices always has consistent preferences then any inferences drawn about the rationality of consumer behavior are rendered meaningless since every possible set of choices would have a plausible explanation. This is especially true for studies in empirical economics. In the computational approach to revealed preference theory however, we are mainly interested in studying the existence question from a computational tractability perspective. This means that even in the instance when every dataset has a plausible explanation in the form of consistent preferences, the computational interest persists in constructing efficient algorithms to find these preferences.

2.3 Computational perspectives on revealed preference theory

Theoretical computer science offers at least two different perspectives to revealed preference theory.

2.3.1 Learning-theoretic approach.

It is easy to see that the problem of constructing a utility function that is consistent with a set of observations and satisfying a set of revealed preference axioms is essentially a learning problem. This was first explored by Kalai [Kal03] who looked into the PAC-learnability of classes of choice functions. Beigman and Vohra [BV06] looked into learning from revealed preference datasets specifically. They cast out a learning-theoretic interpretation of Afriat's theorem and constructed a learning algorithm with zero error on the sample set of price-demand observations and returned a piecewise utility function. In addition, they use this learning algorithm to construct a forecasting algorithm which predicts consumer demand when the prices are unobserved. Finally, they shed light on the PAC-learnability of the class of demand functions in a supervised learning framework by giving lower and upper bounds on the sample complexity of any learning algorithm for different classes of demand functions. They show that for the general class of demand functions, the sample complexity is infinity but when the class of demand functions is more well-behaved (i.e., income-Lipschitz), the sample complexity is polynomially bounded.

2.3.2 Complexity-theoretic approach to revealed preference.

There has been some previous work on the question of the computational hardness of finding utility functions/preference relations that are consistent with a given set of observations of consumer choice. Galambos [Gal05] asked if it was possible to find transitive preferences over a set of observed choices under assumptions on the representation of the dataset such that each of the observations were pure Nash equilibrium strategies played by the agents. The assumptions in this model are key; the observations are assumed to be subsets of choices each agent makes so that the dataset includes every tuple in which strategies for each agent are chosen from the corresponding observed set. In this restricted model, Galambos showed that the problem of constructing preference relations consistent with the dataset is NP-complete.

A clearer and more direct connection between theoretical computer science and revealed preference theory from within the economics literature was recently investigated by Deb [Deb08]. In it, he explored a model in which the aggregate consumption behavior of a k-person household is observed and individual consumption is not known, and seeks to find k utility functions that rationalize the observed aggregate household consumption dataset. Deb showed that this problem is NP-complete.

2.3.3 Proving hardness of approximation results

The techniques used to show hardness of approximation for optimization problems are part of a seminal body of work in complexity theory covering drawing on elegant results from a diverse array of fields such as probabilistically checkable proof systems, algorithmic coding theory, algebraic and Fourier analysis methods. Below, we attempt to provide the barest minimum, self-contained introduction to this immensely sophisticated theory in the hope that our own results on hardness of approximation for the rationalizability problems described above are better understood. We harbor no illusion that this treatment is exhaustive and refer the interested reader to online coursework [Sud99, Tre06] and survey material [AL96, Aro98] for a more full-fledged and elaborate discussion of these techniques. The definitions we provide below are based on those provided in [AL96].

For an instance I of any optimization problem, we denote *the optimal value* $OPT(I)$ to be the value of the optimal solution to an instance of the optimization problem.

Definition 2.3.1 Approximation ratio: *Let P be an optimization problem and I an instance of P with size n and optimal value $OPT(I)$. An algorithm for P achieves an approximation ratio $\alpha = \alpha(n)$ if it produces a solution at least $\alpha \cdot OPT(I)$ if P is a maximization problem, or at most $OPT(I)/\alpha$ if P is a minimization problem.*

We say that a problem P has *hardness of approximation factor* α, or is *inapproximable to within* α, if it is NP-hard to achieve an approximation ratio of α for P.

The core idea behind proving *inapproximability* results is analogous to how we go about proving hardness results: we start with an instance of a problem P that is hard to approximate and produce an instance of the target problem P' that we are attempting to show is inapproximable. The one important property we require of this reduction is that it should be *gap-preserving*. This means, that if, say, both P, P' are maximization problems, then the reduction should ensure that for the respective instances I, I':

$$OPT(I) \geq c \Rightarrow OPT(I') \geq c'$$

$$OPT(I) \leq \rho \cdot c \Rightarrow OPT(I') \leq \rho' \cdot c'$$

where ρ, c, and ρ', c' are respectively parameters of the instances I, I'. Naturally, this means that the better parameters we can obtain in a gap-preserving reduction, the better the hardness of approximation factor would be. However, this assumes that we already have a problem with a known inapproximability result to begin with. The rest of this subsection explores obtaining these prototypical inapproximability results.

As mentioned at the beginning of the section, there is a deep connection between inapproximability and *probabilistically checkable proof systems* which we illuminate below, after a few preliminary definitions. Let $r, q : \mathbb{N} \to \mathbb{N}$ be some functions over the set of natural numbers.

Definition 2.3.2 Verifier: *Given a language L, an (r, q) verifier V is a probabilistic polynomial-time Turing machine which takes as input a string x of length n. V further has access to a proof string that it queries in $q(n)$ locations which are computed from the input string x and a set of $r(n)$ random bits. Finally, V accepts or rejects x after evaluating in polynomial time some function that takes as input x, the $r(n)$ random bits and the $q(n)$ query results.*

Definition 2.3.3 Probabilistically checkable proof system: *Let L be a language. An (r, q) probabilistically checkable proof system for L with completeness c and soundness s comprises an (r, q) verifier V with the following properties:*

- *for every input x in L, there exists a proof string Π_x for which V with input x and access to Π_x accepts with probability at least c,*
- *for an input x not in L, and every proof string Π_x, V with input x and access to Π_x will accept with probability at most s.*

We say that the language L is in $PCP(r,q)$ if there is an (r,q) probabilistically checkable proof system for L with some completeness c and soundness s bounded away from 0 and 1 respectively, and with $c > s$. The crucial result that heralded a new and influential approach to inapproximability was that NP was in $PCP(\log n, 1)$, and was a culmination of results due to Arora and Safra [AS98], and Arora et al. [ALM$^+$98]:

Theorem 2.3.4 *(Arora and Safra [AS98], Arora et al. [ALM$^+$98])*

$$NP = PCP(\log n, O(1)).$$

As an immediate consequence to the theorem above, known as the PCP theorem, the problem of maximizing the number of satisfiable 3SAT clauses, MAX-3SAT, was shown in [ALM$^+$98] to be inapproximable to within $1-\epsilon$ for some fixed ϵ. Inapproximability results for other problems such as MAX-CUT, MAX-CLIQUE, LABEL-COVER can all be shown based on the PCP theorem and devising PCP systems with the appropriate parameters. We refer the interested reader to other sources [AL96] for a more detailed treatment.

Chapter 3

Algorithms for playing games with limited randomness

Dennis "Cutty" Wise:	`The game done changed.`
Slim Pierce:	`The game's the same. Just got more fierce.`
	– *The Wire, Season 3.*

3.1 Background

The concept of randomness plays a central role in game theory and economics. For a Nash equilibrium to exist in a finite game, we crucially must rely on the fact that the strategies in consideration also include mixed strategies, i.e., probability distributions over the set of pure strategies.

There are different interpretations of what constitutes a mixed strategy. At its simplest, we can think of a mixed strategy as exactly a probability distribution over the set of strategies played in a one-shot game, with the pay-off then tabulated in expectation. In another characterization due to Rosenthal [Ros79] and Rosenthal and Landau [RL79], each player is thought of as coming from a population and the probability weight attached to each pure strategy is the fraction of the population that will play that pure strategy. The interaction then is between players drawn uniformly at random from their respective populations, playing out their corresponding pure strategy. In the entirety of this chapter, we concern ourselves with only the first interpretation.

3.1.1 Finding sparse-support equilibrium strategies

As stated in the introductory chapter, our treatment of randomness in this dissertation is two-fold. In the first case, we are looking at reducing the randomness required for players to play their equilibrium strategies. For two-player zero-sum games, Lipton and Young [LY94] showed that it was sufficient for players to play a random sample of $O(\log n/\epsilon^2)$ pure strategies drawn from their respective equilibrium mixed strategies to obtain a pay-off that was ϵ away from the value of the game. This was subsequently generalized to non-zero sum two-player games by Lipton et al. [LMM03], who gave a randomized procedure that produces small-support strategies for an ϵ-approximate Nash equilibrium given a Nash equilibrium with larger support using a simple algorithm: sample uniformly from the given equilibrium strategy. Their analysis applies Chernoff bounds to show that the sampled strategies present the opposing players with approximately (within ϵ) the same payoffs, and hence constitute an ϵ-equilibrium.

In fact, their result should be seen in the broader context of the quest for efficient algorithms that find approximate Nash equilibria. By showing the existence of ϵ-approximate Nash equilibrium strategies with support $O(\log n/\epsilon^2)$, Lipton et al. effectively proved an upper-bound of $O(n^{\log n/\epsilon^2})$ on the time-complexity of finding approximate equilibria. There has been extensive work on finding efficient algorithms for approximate equilibria [Alt94, DMP06, BBM07, FNS07, DMP07, TS07, PNS08] and many of them center around finding strategies with small support. Daskalakis et al. [DMP06] showed the existence of a simple algorithm to find a 2-support strategy that was a 1/2-approximate Nash equilibrium. Feder et al. [FNS07] showed that this was asymptotically optimal when only considering $O(\log n)$-support strategies.

3.1.2 Low-rank games

In addition to addressing the question of finding efficient algorithms for approximate Nash equilibria, [LMM03] gave sufficient conditions for existence of small-support exact Nash equilibria in two-player games when the payoff matrices have low rank.

They use this observation to show that games with pay-off matrices that can be approximated by low-rank matrices have approximate equilibria with small support.

Kannan and Theobald [KT07] approached the problem of low-rank games slightly differently. In their work, a low-rank game is defined as a generalization of zero-sum two-player games. Specifically, they require that the *sum* of the payoff matrices of the two players has small rank. In this setting, with n strategies for both players and the matrix sum having rank k, they give an algorithm that computes an ϵ-approximate Nash equilibrium and runs in time polynomial in $n^k, 1/\epsilon, B$ where B is the bit-length parameter determining the number of bits of precision to which the matrices (over reals) are represented.

Stein et al. [SOP08] consider low-rank games in the domain of continuous games. These are games where each agent's strategies are drawn from compact sets as opposed to discrete sets and the payoffs are given by continuous functions defined over these spaces. They define a subclass of continuous games called separable games in which the payoffs can be expressed as a sum-of-products so that a term in each product is dependent only on the strategy space of the corresponding agent. In the special case when the strategy spaces are subsets of $\mathbb{R}$ and the terms are monomials, the game is called a polynomial game. For separable games over n agents in general, the notion of rank as defined in [SOP08] is analogous to that defined when the strategies are over discrete sets and involves concepts in measure theory that are beyond the scope of this dissertation. It suffices to mention their observation that separable games have bounded rank. Furthermore, for the special class of two-player polynomial games, they show that an ϵ-approximate Nash equilibrium can be found in time polynomial in $1/\epsilon$ for fixed rank.

3.1.3 Our results

In this chapter, we study algorithms for finding equilibria and playing games *randomness-efficiently*. By "playing a game" we mean the actions a player must take to actually sample and play a strategy from a mixed strategy profile, which necessarily entails randomness. There is an immense body of literature in theoretical computer science devoted to addressing computational issues when limited or no randomness is available, and for the same reasons it makes sense to study games whose players have limited access to randomness. Moreover, as we will see below, limited randomness in games motivates some special classes of games that have arisen in other contexts thereby providing a viewpoint that we believe is helpful.

3.1.3.1 Sparse strategies in single-round games

We first look at single-round games. For the case of zero-sum games where both players have n available strategies, Lipton and Young [LY94] showed that a random sample of $O(\frac{\log n}{\epsilon^2})$ strategies was sufficient to approximate the value of the game by ϵ. Lipton et al. [LMM03] extended this to ϵ-equilibria for two-player nonzero-sum games. Indeed, they gave a randomized procedure that produced small-support strategies for an ϵ-equilibrium when given a Nash equilibrium with possibly large support. In the following theorem, we derandomize this procedure by using random walks on expander graphs:

Theorem 3.1.1 *Let $G = (R, C, n)$ be a two-player game, and let (p^*, q^*) be a Nash equilibrium for G. For every $\epsilon > 0$, there is a* deterministic *procedure P running in time poly$(|G|)^{1/\epsilon^2}$ such that the pair $(P(G, p^*, 1), P(G, q^*, 2))$ is an $O(\frac{\log n}{\epsilon^2})$-sparse 4ϵ-equilibrium for G.*

This can be viewed as a *deterministic* "sparsification" procedure for ϵ-equilibria in general two player games. In zero-sum games one can find optimal strategies efficiently, and as a result, we obtain a deterministic polynomial time algorithm to find sparse ϵ-equilibria for zero-sum games:

Corollary 3.1.2 *Let $G = (R, C, n)$ be a two-player* zero-sum *game. For every $\epsilon > 0$, there is a* deterministic *procedure running in time poly$(|G|)^{1/\epsilon^2}$ that outputs a pair (p^*, q^*) that an $O(\frac{\log n}{\epsilon^2})$-sparse ϵ-equilibrium for G.*

We point out that Freund and Schapire [FS96] obtained a similar result using an adaptive multiplicative-weight algorithm (discussed below) with $poly(|G|, 1/\epsilon)$ running time. We obtain the corollary above using a very different proof technique that flows from well-known derandomization techniques.

3.1.3.2 Reusing randomness in multiple round games

In single-round games a randomness-limited player requires sparse strategies, but in multiple-round games, we would like to be able to "reuse" randomness across rounds. This is an orthogonal concern to that of reducing randomness within a single round.

Freund and Schapire [FS99] proposed an adaptive online algorithm for a T-round two-player zero-sum game with n strategies available to each. Executing the mixed strategies produced by their algorithm uses $\Omega(T \log n)$ bits of randomness over T rounds, in the worst case. By making use of almost-pairwise independence, we show how to reuse randomness across rounds: it is possible to make do with just $O(\log n + \log\log T + \log(1/\epsilon))$ bits and achieve close to the same quality of approximation as in [FS99].

Theorem 3.1.3 *Let M be the $n \times n$-payoff matrix for a two-player zero-sum T-round game with entries in $\{0, 1\}$. For any $\epsilon < 1/2$ and constant δ, there exists an online randomized algorithm R using $O(\log n + \log\log T + \log(1/\epsilon))$ random bits with the following property: for any arbitrary sequence $Q_1, \ldots, Q_T$ of mixed strategies played (adaptively) by the column player over T rounds, R produces a sequence of strategies $S_1, \ldots, S_T$ such that with probability at least $1 - \delta$:*

$$\frac{1}{T}\sum_{i=1}^{T} M(S_i, Q_i) \le \frac{1}{T}\min_P \sum_{i=1}^{T} M(P, Q_i) + O\left(\sqrt{\frac{\log n}{T}} + \epsilon\right)$$

3.1.3.3 Games with sparse equilibria

As we have discussed, players with limited access to randomness can only achieve equilibria that are sparse. We saw before that in the general setting, we are able to deterministically "sparsify" if we are willing to settle for ϵ-equilibria. The sparseness cannot in general be less than $\log n$, though, so we are motivated to consider broad classes of games in which even sparser equilibria are guaranteed to exist.

Perhaps the simplest example is a 2-player games in which one player has only k available strategies, while the other player has $n \gg k$ available strategies. The results in the work of Lipton et al. [LMM03] imply there is a Nash equilibrium in this game with support size $k+1$. This is somewhat unsatisfying — it means that in a two-player game one player may need to choose from less-sparse strategies than his opponent (i.e., requiring slightly *more* randomness) to achieve equilibrium. Theorem 3.1.4 rectifies this asymmetry by showing that k-sparse strategies suffice for the opposing player.

Theorem 3.1.4 *Let $G = (R, C, k, n)$ be a two-player game. Given p^* for which there exists a q^* such that (p^*, q^*) is a Nash equilibrium, we can compute in deterministic polynomial time q' for which (p^*, q') is a Nash equilibrium and $|supp(q')| \leq k$.*

We also give a deterministic polynomial time algorithm to compute such a limited-support strategy for one player, given the k-sparse strategy of the other. We extend this further to the multiplayer case and show that for an ℓ-player game where players 1 through $\ell - 1$ have $k_1, \ldots, k_{\ell-1}$ pure strategies, respectively, the ℓ-th player need only play a $(\prod k_i)$-sparse strategy to achieve equilibrium:

Corollary 3.1.5 *Let $G = (T_1, T_2, \ldots, T_\ell, k_1, k_2, \ldots, k_{\ell-1}, n)$ be an ℓ-player game where $\prod_{i=1}^{\ell-1} k_i < n$. Given $G, p_1^*, \ldots, p_{\ell-1}^*$ there exists a deterministic polynomial-time procedure to compute $\widehat{p_\ell}$ such that $(p_1^*, p_2^*, \ldots, p_{\ell-1}^*, \widehat{p_\ell})$ is a Nash equilibrium for G and $|supp(\widehat{p_\ell})| \leq k = \prod_{i=1}^{\ell-1} k_i$.*

These bounds are tight.

3.1.3.4 Games of small rank

Perhaps the most significant technical contribution in this chapter pertains to a generalization of the "unbalanced" games that we saw above, namely, games of small rank. This is a broad class of games (encompassing some natural examples – see Section 3.6.3) for which sparse equilibria are known to exist.

For 2-player games with rank-k payoff matrices, Lipton et al. describe an enumeration algorithm that finds $(k+1)$-sparse strategies in $O(n^{k+1})$ time. We improve dramatically on this bound but we must relax the problem in two ways: first, we compute an ϵ-equilibrium rather than an exact one; second we require that the payoff matrices be presented as a low-rank decomposition, with entries up to precision B (limited precision makes sense since we are only considering ϵ-equilibria).

Theorem 3.1.6 *Let $G = (R, C, n)$ be a two player game such that R and C have rank at most k. Furthermore, let $R = R_1R_2$, $C = C_1C_2$ be a decomposition of R, C with R_1, R_2, C_1, C_2 containing integer entries in $[-B, B]$. Then, for every $\epsilon > 0$, there is a deterministic procedure P running in time $(4B^2k/\epsilon)^{2k}$poly$(|G|)$ that returns a 4ϵ-Nash equilibrium (p, q) with $|supp(p)|, |supp(q)| \leq k + 1$.*

To the best of our knowledge, Theorem 3.1.6 provides the first efficient "fixed-parameter" algorithm to this problem in the sense that the running time is polynomial in the input size n and some function $f(k, 1/\epsilon, B)$. The closest parallel to our result is by Kannan and Theobald [KT07] who consider a somewhat different definition of "rank" for two-player games: in their definition, the *sum* of the payoff matrices is required to have small rank. In that case, they present an algorithm that finds an ϵ-equilibrium in a rank k 2-player game in $O(n^{2k+o(1)}B^2)$ time. Their algorithm relies on results of Vavasis for solving indefinite quadratic programs [Vav92] and does not seem to generalize to $\ell > 2$ players.

Our algorithm is (arguably) simpler, and moreover, it easily generalizes to $\ell > 2$ players, where small rank games still are guaranteed to have sparse equilibria. In the ℓ-player setting, we give an $O(((2B)^\ell k\ell/\epsilon)^{k\ell(\ell-1)})$poly$(n^\ell)$ time deterministic procedure that computes such a sparse ϵ-equilibrium, when the payoff tensors are presented as a rank-k decomposition with entries up to precision B.

All of the algorithms for low-rank games rely on enumerating potential equilibria distributions in a basis dictated by the small rank decomposition. This seems like a technique that may be useful for algorithmic questions regarding low-rank games beyond those we have considered in this chapter.

The rest of the chapter is structured as follows. Section 4.2 goes over some preliminary definitions and lemmas. Section 3.3 presents algorithms for finding small-support strategies for single-round games, obtaining approximate Nash equilibria. In Section 3.4, we look at the multiple-round case and present a randomness-efficient variant of the adaptive online algorithm of Freund and Schapire. Games in which some players have very few strategies are discussed in Section 3.5 while our new algorithms for ϵ-equilibria in games with small rank are described in Section 3.6.

3.2 Preliminaries

Definition 3.2.1 *For a finite strategy set S the* **support** *of a mixed strategy $p \in \Delta(S)$ is the set supp(p) given by supp$(p) = \{s \in S | p_s > 0\}$. A mixed strategy p is* **k-sparse** *if $|supp(p)| = k$.*

In this chapter, we will concern ourselves with games that can be specified by payoff matrices (or tensors) whose entries denote the payoff upon playing the corresponding strategy tuple. We will also assume, unless otherwise specified, that these entries are bounded and can be scaled to lie in $[-1, 1]$. With this in mind, we give below an equivalent and more useful definition for a game.

Definition 3.2.2 *An* **ℓ-player finite game** *G is a tuple $(T_1, \ldots, T_\ell, n_1, n_2, \ldots, n_\ell)$ where T_i is the $(n_1 \times \ldots \times n_\ell)$ ℓ-dimensional payoff tensor with $T_i(s_1, \ldots, s_\ell)$ denoting the payoff to player i when the pure strategy ℓ-tuple $(s_1, \ldots, s_\ell)$ is played in the game.*

For ease of presentation, in the rest of this chapter we will often restrict ourselves to ℓ-player games where $n_1 = n_2 = \ldots = n_\ell = n$, which we denote by $G = (T_1, \ldots, T_\ell, n)$. We often refer to players by their payoff tensors. For example, for the two-player game $G = (R, C, n)$ we will refer to the row player as R and the column player as C. All vectors are thought of as row vectors.

Definition 3.2.3 *In an ℓ-player game $G = (T_1, T_2, \ldots, T_\ell, n_1, n_2, \ldots, n_\ell)$, we denote by $T_i(p_1, \ldots, p_\ell)$ the payoff to the i-th player when the ℓ players play mixed strategies $p_1, \ldots, p_\ell$, i.e.,*

$$T_i(p_1, \ldots, p_\ell) = \sum_{i_1 \in [n_1], \ldots, i_\ell \in [n_\ell]} p_{i_1} p_{i_2} \ldots p_{i_\ell} T_i(i_1, i_2, \ldots, i_\ell).$$

If we substitute some $a \in [n_j]$ for p_j we understand that to denote the distribution that places weight 1 on a and 0 everywhere else.

For a mixed strategy p with rational weights, we can alternatively represent it as a weighted multi-set $S_p \subseteq S$ with the multiplicity of an element $s \in S_p$ being directly proportional to the weight of s in p so that playing a mixed strategy p is equivalent to playing the uniform distribution over the corresponding multi-set. In this chapter we restrict our attention to mixed strategies that can be expressed in this manner (as the uniform distribution over a multiset).

Let G be an ℓ-player game. It is well-known that given the *supports* of the ℓ different p_i^* in a Nash equilibrium, one can find the actual *distributions* by linear programming. We will use a similar fact repeatedly:

Lemma 3.2.4 *Let $G = (T_1, T_2, \ldots, T_\ell, n)$ be an ℓ-player game, and let $(p_1^*, p_2^*, \ldots, p_\ell^*)$ be a Nash equilibrium. Given G and $p_1^*, p_2^*, \ldots, p_{\ell-1}^*$ one can find a distribution q in deterministic polynomial time for which $(p_1^*, p_2^*, \ldots, p_{\ell-1}^*, q)$ is also a Nash equilibrium.*

Proof. Once we know the distributions $p_1^*, \ldots, p_{\ell-1}^*$ in order to find a Nash equilibrium strategy q for player T_ℓ, we first determine the Nash equilibrium support for T_ℓ by considering the set

$$T = \{r \mid \forall r', T_\ell(p_1^*, \ldots, p_{\ell-1}^*, r) \geq T_\ell(p_1^*, \ldots, p_{\ell-1}^*, r')\}$$

We now find q satisfying the following linear program:

$$\sum_{i=1}^{n} q_i = 1$$

$$q_i \geq 0; i = 1, \ldots, n$$

$$q_i = 0; i \notin T$$

$$T_\ell(p_1^*, \ldots, p_{j-1}^*, i, p_{j+1}^*, \ldots, p_{\ell-1}^*, q) \geq T_\ell(p_1^*, \ldots, p_{j-1}^*, i', p_{j+1}^*, \ldots, p_{\ell-1}^*, q);$$

$$i \mid p_{j_i}^* > 0;$$

$$j = 1, \ldots, \ell - 1;$$

$$i' = 1, \ldots, n$$

From Nash's result [Nas51], we know that a solution to this linear program exists and this is the desired q. $\square$

3.3 Sparsifying Nash equilibria deterministically

In this section we give deterministic algorithms for "sparsifying" Nash equilibria (in the process turning them into ϵ-equilibria). In this way, a player with limited access to randomness, but who has access to an equilibrium mixed strategy, is able to produce a small strategy that can then be played.[1]

3.3.1 Two-players

Lipton et al. proved:

Theorem 3.3.1 (Lipton et al. [LMM03]) *Let $G = (R, C, n)$ be a two-player game, and let (p^*, q^*) be a Nash equilibrium for G. There is a polynomial-time randomized procedure P such that with probability at least 1/2, the pair $(P(G, p^*), P(G, q^*))$ is an $O(\log n/\epsilon^2)$-sparse ϵ-equilibrium for G.*

[1] The question of *how* the player may obtain an equilibrium mixed strategy is a separate and well-studied topic, but not the focus of this work.

The algorithm P is very simple: it amounts to sampling uniformly from the given equilibrium strategy. The analysis applies Chernoff bounds to show that the sampled strategies present the opposing players with approximately (within ϵ) the same weighted row- and column- sums, and hence constitute an ϵ-equilibrium. In our setting, since the players have limited randomness they cannot afford the above sampling (it requires at least as much randomness as simply playing (p^*, q^*)), so we derandomize the algorithm using an expander walk.

Theorem 3.1.1 (restated). *Let $G = (R, C, n)$ be a two-player game, and let (p^*, q^*) be a Nash equilibrium for G. For every $\epsilon > 0$, there is a* deterministic *procedure P running in time poly$(|G|)^{1/\epsilon^2}$ such that the pair $(P(G, p^*, 1), P(G, q^*, 2))$ is an $O(\log n/\epsilon^2)$-sparse 4ϵ-equilibrium for G.*

Before proving the theorem we will give a convenient characterization of ϵ-equilibrium:

Lemma 3.3.2 *Let $G = (R, C, n)$ be a 2-player game. Define*

$$\begin{aligned} T_p &= \{i|(pC)_i \geq \max_r (pC)_r - \epsilon\} \\ S_q &= \{j|(Rq^T)_j \geq \max_t (Rq^T)_t - \epsilon\}. \end{aligned}$$

If supp$(p) \subseteq S_q$ and supp$(q) \subseteq T_p$, then (p, q) is an ϵ-approximate Nash equilibrium for G.

Proof. Consider an arbitrary $p' \in \Delta([n])$. Since $p'Rq^T$ is a convex combination of the $(Rq)_j$, it is at most $\max_j (Rq)_j$. And, since p is a convex combination of the $(Rq)_j$, with supp$(p) \subseteq S_q$, we have $pRq^T \geq \max_j (Rq)_j - \epsilon$. Thus $p'Rq^T \leq pRq^T + \epsilon$. Similarly, for an arbitrary $q' \in \Delta([n])$, we have $pCq'^T \leq \max_i (pC)_i$, and $pCq^T \geq \max_i (pC)_i - \epsilon$ since supp$(q) \subseteq T_p$. Thus $pCq'^T \leq pCq^T + \epsilon$. These two conditions guarantee that (p, q) is an ϵ-approximate Nash equilibrium. □

We will use the Chernoff bound for random walks on an expander:

Theorem 3.3.3 (Gillman [Gil93]) *Let H be an expander graph with second largest eigenvalue λ and vertex set V, and let $f : V \to [-1, 1]$ be arbitrary with $E[f] = \mu$.*

Let $X_1, X_2, \ldots, X_t$ be the random variables induced by first picking X_1 uniformly in V and $X_2, \ldots, X_t$ by taking a random walk in H from X_1. Then

$$\Pr\left[\left|\frac{1}{t}\sum_i f(X_i) - \mu\right| > \delta\right] < e^{-O((1-\lambda)\delta^2 t)}.$$

Proof. [Proof of Theorem 3.1.1] When we are given G and p^*, we perform the following steps.

First, construct a multiset S of $[n]$ for which uniformly sampling from S approximates p^* to within ϵ/n. This can be done with $|S| \leq O(n/\epsilon)$. Denote by $\tilde{p}$ the distribution induced by sampling uniformly from S. We identify S with the vertices of a constant-degree expander H, and we can sample $S' \subseteq S$ by taking a walk of length $t = O(\log n/\epsilon^2)$ steps in H. Note that this requires $O(\log|S| + O(t)) = O(\log n/\epsilon^2)$ random bits. Let p' be the probability distribution induced by sampling uniformly from S'. By Theorem 3.3.3 (and using the fact that C has entries in $[-1, 1]$), for each fixed i,

$$\Pr[|(p'C)_i - (\tilde{p}C)_i| \geq \epsilon] \leq e^{-O(\epsilon^2 t)} < 1/n. \tag{3.1}$$

By a union bound $|(p'C)_i - (\tilde{p}C)_i| \leq \epsilon$ for all i with non-zero probability. This condition can be checked given G, p^*, and so we can derandomize the procedure completely by trying all choices of the random bits used in the expander walk.

When we are given G and q^*, we perform essentially the same procedure (with respect to R and q^*), and in the end we output a pair p', q' for which:

$$\begin{array}{rcl} |(p'C)_i - (\tilde{p}C)_i| & \leq & \epsilon \quad \forall i \\ |(Rq'^T)_j - (R\tilde{q}^T)_j| & \leq & \epsilon \quad \forall j \end{array}$$

We claim that any such (p', q') is an 4ϵ-equilibrium, assuming (p^*, q^*) are an equilibrium. Using the fact that C, R have entries in $[-1, 1]$, and the fact that our multiset approximations to p^*, q^* have error at most ϵ/n in each coordinate, we obtain:

$$\begin{aligned} |(\tilde{p}C)_i - (p^*C)_i| &\leq \epsilon \quad \forall i \\ |(R\tilde{q}^T)_j - (Rq^{*T})_j| &\leq \epsilon \quad \forall j \end{aligned}$$

Define (as in Lemma 3.3.2):

$$\begin{aligned} T_{p'} &= \{i|(p'C)_i \geq \max_i (p'C)_i - 4\epsilon\} \\ S_{q'} &= \{j|(Rq'^T)_j \geq \max_j (Rq'^T)_j - 4\epsilon\}. \end{aligned}$$

Now, $w \in \text{supp}(p')$ implies $w \in \text{supp}(p^*)$ which implies $(Rq^{*T})_w = \max_j (Rq^{*T})_j$ (since (p^*, q^*) is a Nash equilibrium). From above we have that $\max_j (Rq'^T)_j \leq \max_j (Rq^{*T})_j + 2\epsilon$ and that $(Rq'^T)_w \geq (Rq^{*T})_w - 2\epsilon$. So $(Rq'^T)_w \geq \max_j (Rq'^T)_j - 4\epsilon$, and hence w is in $S_{q'}$. We conclude that $\text{supp}(p') \subseteq S_{q'}$. A symmetric argument shows that $\text{supp}(q') \subseteq T_{p'}$. Applying Lemma 3.3.2, we conclude that (p', q') is a 4ϵ-equilibrium as required. □

Since an equilibrium can be found efficiently by Linear Programming in the two player zero-sum case, we obtain as a corollary:

Corollary 3.1.2 (restated). *Let $G = (R, C, n)$ be a two-player* zero-sum *game. For every $\epsilon > 0$, there is a* deterministic *procedure running in time poly$(|G|)^{1/\epsilon^2}$ that outputs a pair (p^*, q^*) that an $O(\log n/\epsilon^2)$-sparse ϵ-equilibrium for G.*

3.3.2 Three or more players

We extend the algorithm above to make it work for games involving three or more players.

Theorem 3.3.4 *Let $G = (T_1, T_2, \ldots, T_\ell, n)$ be an ℓ-player game, and let $(p_1^*, p_2^*, \ldots, p_\ell^*)$ be a Nash equilibrium for G. For every $\epsilon > 0$, there is a* deterministic *procedure P running in time poly$(|G|)^{1/\epsilon^2}$, such that the tuple $(P(G, p_1^*, 1), P(G, p_2^*, 2), \ldots, P(G, p_\ell^*, \ell))$ is an $O((\ell \log n)/\epsilon^2)$-sparse 4ϵ-equilibrium for G.*

Proof. The proof is almost identical to that of Theorem 3.1.1. When given $(G, p_1^*, 1)$ P samples $t = O(((\ell - 1)\log n + \log \ell)/\epsilon^2)$ strategies from player 1's multiset of strategies after identifying it with a constant-degree expander H and doing a t-step random walk on it. Let $\widetilde{p_1}$ be the distribution obtained by sampling uniformly from the original multiset of strategies and $\widehat{p_1}$ the distribution induced by sampling from the t strategies picked from the random walk. For some fixing of $(i_2, \ldots, i_\ell)$ and j the Chernoff bound in (3.1) now becomes

$$\Pr[|T_j(\widehat{p_1}, i_2, \ldots, i_\ell) - T_j(\widetilde{p_1}, i_2, \ldots, i_\ell)| \geq \epsilon] \leq e^{-O(\epsilon^2 t)} < 1/(\ell n^{\ell-1})$$

By a union bound on all $\ell n^{\ell-1}$ possible fixings for $(i_2, \ldots, i_\ell)$ and all j,

$$|T_j(\widehat{p_1}, i_2, \ldots, i_\ell) - T_j(\widetilde{p_1}, i_2, \ldots, i_\ell)| < \epsilon$$

with positive probability. As before, we can derandomize the procedure completely by trying all choices of the random bits used in the expander walk.

Essentially the same procedure gives us $\widehat{p_2}, \widehat{p_3}, \ldots, \widehat{p_\ell}$. We will first need a generalization of Lemma 3.3.2.

Lemma 3.3.5 *Let $G = (T_1, \ldots, T_\ell, n)$ be an ℓ-player game, and $(p_1, \ldots, p_\ell)$ an arbitrary mixed strategy. Define, for each i,*

$$S_i = \{j | T_i(p_1, \ldots, p_{i-1}, j, p_{i+1}, \ldots, p_\ell) \geq \max_r T_i(p_1, \ldots, p_{i-1}, r, p_{i+1}, \ldots, p_\ell) - \epsilon\}$$

If $supp(p_i) \subseteq S_i$ for all $i = 1, \ldots, \ell$, then $(p_1, \ldots, p_\ell)$ is an ϵ-approximate Nash equilibrium for G. In particular, $(p_1, \ldots, p_\ell)$ is a Nash equilibrium if it is a 0-approximate Nash equilibrium for G.

Proof. As before, for some player T_i we consider an arbitrary $p' \in \Delta([n])$. Note that $T_i(p_1, \ldots, p_{i-1}, p', p_{i+1}, \ldots, p_\ell)$ is a convex combination of the n terms

$$T_i(p_1, \ldots, p_{i-1}, j, p_{i+1}, \ldots, p_\ell)$$

where $j = 1, \ldots, n$. Therefore,

$$T_i(p_1, \ldots, p_{i-1}, p', p_{i+1}, \ldots, p_\ell) \leq \max_r T_i(p_1, \ldots, p_{i-1}, r, p_{i+1}, \ldots, p_\ell) \tag{3.2}$$

p_i is also a convex combination of the n terms above, with $supp(p_i) \subseteq S_i$, and so:

$$T_i(p_1, \ldots, p_{i-1}, p_i, p_{i+1}, \ldots, p_\ell) \geq \max_r T_i(p_1, \ldots, p_{i-1}, r, p_{i+1}, \ldots, p_\ell) - \epsilon \tag{3.3}$$

Combining (3.2) and (3.3), we have:

$$T_i(p_1, \ldots, p_{i-1}, p', p_{i+1}, \ldots, p_\ell) \leq T_i(p_1, \ldots, p_\ell) + \epsilon.$$

Applying an identical argument for all ℓ players, we have that $(p_1, \ldots, p_\ell)$ is an ϵ-approximate Nash equilibrium. □

To show that $(\widehat{p_1}, \ldots, \widehat{p_\ell})$ constitute a 4ϵ-equilibrium, consider the set $\mathcal{S}_1 = \{i \mid T_1(i, \widehat{p_2}, \ldots, \widehat{p_\ell}) \geq \max_i T_1(i, \widehat{p_2}, \ldots, \widehat{p_\ell}) - 4\epsilon\}$. Define $\mathcal{S}_j$ analogously with respect to T_j. Given Lemma 3.3.5, it suffices to show that $\mathrm{supp}(\widehat{p_j}) \subseteq \mathcal{S}_j$ for all j. We sketch the argument for $\widehat{p_1}$; symmetric arguments hold with respect to $\widehat{p_j}$ for all j.

By the same thread of reasoning as in the two-player case, for any $w \in \mathrm{supp}(\widehat{p_1})$, $T_1(w, \widehat{p_2}, \ldots, \widehat{p_\ell}) \geq T_1(w, p_2^*, \ldots, p_\ell^*) - 2\epsilon$ and since $\mathrm{supp}(\widehat{p_1}) \subseteq \mathrm{supp}(p_1^*)$,

$$T_1(w, p_2^*, \ldots, p_\ell^*) = \max_i T_1(i, p_2^*, \ldots, p_l^*) \geq \max_i T_1(i, \widehat{p_2}, \ldots, \widehat{p_\ell}) - 2\epsilon.$$

Combining the two inequalities, we get $T_1(w, \widehat{p_2}, \ldots, \widehat{p_\ell}) \geq \max_i T_1(i, \widehat{p_2}, \ldots, \widehat{p_\ell}) - 4\epsilon$. □

3.4 Limited randomness in repeated games

So far we have looked at optimizing the amount of randomness needed in single-round games where players execute their strategies only once. In this section, we investigate multiple-round games and in particular, the adaptive multiplicative weight algorithm of Freund and Schapire [FS99] for which we describe randomness-efficient modifications. In particular, we show that by using almost-pairwise independent random variables it is possible to achieve close to the same quality of approximation as in [FS99].

Note that we make crucial use of the full power of [FS99] – i.e., their guarantee on the performance of the row player's strategy (captured ahead in Lemma) still holds if the column player changes his play in response to the particular randomness-efficient sampling being employed by the row player:

Theorem 3.1.3 (restated). *Let M be the $n \times n$-payoff matrix for a two-player zero-sum T-round game with entries in $\{0,1\}$. For any $\epsilon < 1/2$ and constant δ, there exists an online randomized algorithm R using $O(\log n + \log\log T + \log(1/\epsilon))$ random bits with the following property: for any arbitrary sequence $Q_1, \ldots, Q_T$ of mixed strategies played (adaptively) by the column player over T rounds, R produces a sequence of strategies $S_1, \ldots, S_T$ such that with probability at least $1-\delta$:*

$$\frac{1}{T}\sum_{i=1}^{T} M(S_i, Q_i) \le \frac{1}{T}\min_P \sum_{i=1}^{T} M(P, Q_i) + O\left(\sqrt{\frac{\log n}{T}} + \epsilon\right)$$

Proof. Our randomized online algorithm R is a modification of Freund and Schapire's multiplicative-weight adaptive algorithm [FS99]. For a game with payoff matrix M where both players have n strategies belonging to a strategy-set S, and for a sequence of mixed strategies $(P_1, P_2, \ldots, P_T)$ over T rounds for the first player described by

$$P_{i+1}(s) = \left(\frac{\beta^{M(s,Q_t)}}{\sum_s p_i(s)\beta^{M(s,Q_t)}}\right) p_i(s) \tag{3.4}$$

where $\beta = 1/(1+\sqrt{2\log n/T})$, the Freund-Schapire algorithm offers the following guarantee on the expected payoff over T rounds:

Lemma 3.4.1 (Freund & Schapire [FS99])

$$\frac{1}{T}\sum_{t=1}^{T} M(P_t, Q_t) \;\leq\; \min_P \frac{1}{T}\sum_{t=1}^{T} M(P, Q_t) + O\left(\sqrt{\frac{\log n}{T}}\right) \tag{3.5}$$

Running the Freund-Schapire algorithm requires $\Omega(T \log n)$ random bits in order to select a strategy from each distribution but we can optimize on this by using almost pairwise independent random variables.

As we did in the proof of Theorem 3.1.1 in Section 3.3.1, we can approximate any distribution P_t by a uniform distribution S_t drawn from a multiset of size $O(n/\epsilon)$ that approximates P_t to within ϵ/n and suffer at most $O(\epsilon)$ error. Therefore, under the uniform distribution over vertices $s \in S_i$ for all $i = 1, \ldots, T$:

$$M(P_i, Q_i) - O(\epsilon) \leq E[M(S, Q_i)] \leq M(P_i, Q_i) + O(\epsilon)$$

Definition 3.4.2 *(Alon et al. [AGHP92]) Let $Z_n \subseteq \{0,1\}^n$ be a sample space and $X = x_1 \ldots x_n$ be chosen uniformly from Z_n. Z_n is (ρ, k)-**independent** if for any positions $i_1 < i_2 < \ldots < i_k$ and any k-bit string $t_1 \ldots t_k$, we have*

$$\left|\Pr_X[x_{i_1} x_{i_2} \ldots x_{i_k} = t_1 \ldots t_k] - 2^{-k}\right| \leq \rho$$

Alon et al. [AGHP92] give efficient constructions of (ρ, k)-independent random variables over $\{0,1\}^n$ that we can suitably adapt to obtain T $(\rho, 2)$-independent random variables $S_1, \ldots, S_T$ over a larger alphabet of size $O(n/\epsilon)$ using $O(\log n + \log(1/\rho) + \log(1/\epsilon) + \log\log T)$ random bits.

The following lemma is key to the remainder of the proof and a proof of the lemma follows immediately afterwards.

Lemma 3.4.3 *Let $S_1, \ldots, S_T$ be $(\rho, 2)$-independent random variables. Then, for any δ:*

$$\Pr_{S_1,\ldots,S_T}\left[\frac{1}{T}\sum_{i=1}^{T} M(S_i, Q_i) \geq \frac{1}{T} E\left[\sum_{i=1}^{T} M(S_i, Q_i)\right] + \sqrt{\frac{1}{\delta}\left(\frac{1}{T} + \frac{2\rho n^2}{\epsilon^2}\right)}\right] \leq \delta$$

Setting $\rho = O(\delta\epsilon^6/n^2)$ in Lemma 3.4.3 gives us:

$$\Pr_{S_1,\ldots,S_T}\left[\frac{1}{T}\sum_{i=1}^{T} M(S_i,Q_i) \geq \frac{1}{T}E\left[\sum_{i=1}^{T} M(S_i,Q_i)\right] + \epsilon\right] \leq \delta \tag{3.6}$$

Therefore with probability at least $1-\delta$ over the choice of randomness of $S_1,\ldots,S_T$

$$\frac{1}{T}\sum_{i=1}^{T} M(S_i,Q_i) \leq \frac{1}{T}E\left[\sum_{i=1}^{T} M(S_i,Q_i)\right] + \epsilon \leq \frac{1}{T}\sum_{i=1}^{T} M(P_i,Q_i) + O(\epsilon)$$

Finally by application of Lemma 3.4.1 we have with probability at least $1-\delta$

$$\frac{1}{T}\sum_{i=1}^{T} M(S_i,Q_i) \quad \leq \quad \frac{1}{T}\min_P \sum_{i=1}^{T} M(P,Q_i) + O\left(\sqrt{\frac{\log n}{T}}\right) + O(\epsilon) \tag{3.7}$$

Note that by our choice of ρ, we require $O(\log n + \log\log T + \log(1/\epsilon))$ random bits. This completes the proof of the theorem. $\square$

Proof. (Of Lemma 3.4.3) The proof is essentially a variation of the Chebyshev tail inequality for $(\delta, 2)$ independent random variables. Let $Z = \sum_i M(S_i, Q_i)$. Then,

$$\begin{aligned}
\Pr\left[\frac{1}{T}Z \geq \frac{1}{T}EZ + \lambda\right] &\leq \Pr\left[\left|\frac{1}{T}Z - \frac{1}{T}EZ\right| \geq \lambda\right] \\
&= \Pr\left[\frac{1}{T^2}\left(Z - E[Z]\right)^2 \geq \lambda^2\right] \\
&\leq \frac{1}{T^2}\frac{\sigma^2(Z)}{\lambda^2} \qquad (3.8)
\end{aligned}$$

We bound $\sigma^2(Z)$ as follows:

$$\begin{aligned}
\sigma^2 &= E\left[\left(\sum_T M(S_i,Q_i) - E[M(S_i,Q_i)]\right)^2\right] \\
&= \sum_T \sigma^2\left(M(S_i,Q_i)\right) + 2\sum_{i,j\in[T]} \left(E\left[M(S_i,Q_i)M(S_j,Q_j)\right]\right. \\
&\quad \left. -E\left[M(S_i,Q_i)\right]\cdot E\left[M(S_j,Q_j)\right]\right) \\
&= \sum_T \sigma^2\left(M(S_i,Q_i)\right) + \\
&\quad 2\sum_{i,j\in[T]}\sum_{s_i\in S_i, s_j\in S_j} M(s_i,Q_i)M(s_j,Q_j)\Pr[S_i=s_i,S_j=s_j] \\
&\quad -E\left[M(s_i,Q_i)\right]\cdot E\left[M(s_j,Q_j)\right] \\
&\leq \sum_T \sigma^2\left(M(S_i,Q_i)\right) + \\
&\quad 2\sum_{i,j\in[T]}\sum_{s_i\in S_i, s_j\in S_j} M(s_i,Q_i)M(s_j,Q_j)(\Pr[S_i=s_i]\Pr[S_j=s_j]+\delta) \\
&\quad -E\left[M(S_i,Q_i)\right]\cdot E\left[M(S_j,Q_j)\right] \\
&\quad \text{(By virtue of } (\delta,2)\text{-independence of } S_i,S_j) \\
&\leq \sum_T \sigma^2\left(M(S_i,Q_i)\right) + 2\sum_{i,j\in[T]} \delta n^2/\epsilon^2 + E\left[M(S_i,Q_i)\right]\cdot E\left[M(S_j,Q_j)\right] \\
&\quad -E\left[M(S_i,Q_i)\right]E\left[M(S_j,Q_j)\right] \\
&\quad \text{(There are at most } (n/\epsilon) \text{ possible strategies in the multisets } S_i,S_j) \\
&\leq T\max_j \sigma_j^2 + 2\delta n^2T^2/\epsilon^2
\end{aligned}$$

Let $\sigma_0 = \max_j \sigma_j$. Then,

$$\sigma^2(Z) \leq T\sigma_0^2 + 2\delta T^2n^2/\epsilon^2 \leq T + 2\delta T^2n^2/\epsilon^2 \tag{3.9}$$

Substituting in (3.8) and setting $\lambda = \sqrt{\frac{1}{\alpha}\left(\frac{1}{T}+\frac{2\delta n^2}{\epsilon^2}\right)}$ we get the desired inequality. □

3.5 Unbalanced games

In this section we will look at what happens when one of the players (perhaps as a consequence of having limited randomness) is known to have very few – k – available strategies, while the other player has $n \gg k$ available strategies. In such a game does there exist a k-sparse strategy for the second player? We prove that this is indeed the case. The main technical tool we will need is Carathéodory's Theorem. Since many of the results we obtain depend on the constructive feature of the theorem we record below such a proof.

Theorem 3.5.1 (Carathéodory's Theorem, constructive version) *Let $v_1, \ldots, v_n$ be vectors in a k-dimensional subspace of $\mathbb{R}^m$ where $n \geq k+1$, and suppose*

$$v = \sum_{i=1}^{n} \alpha_i v_i \quad \textit{with} \ \sum_i \alpha_i = 1 \ \textit{and} \ \alpha_i \geq 0 \ \textit{for all} \ i \tag{3.10}$$

Then there exist $\alpha'_1, \ldots, \alpha'_n$ for which $v = \sum_{i=1}^{n} \alpha'_i v_i$ with $\sum_i \alpha'_i = 1$ and $\alpha'_i \geq 0$ for all i, and $|\{i : \alpha'_i > 0\}| \leq k+1$. Moreover the α'_i can be found in polynomial time, given the α_i and the v_i.

Proof. We write a linear combination involving the $n-1 \geq k$ vectors $(v_1 - v_n), \ldots, (v_{n-1} - v_n)$ as follows

$$\sum_{j=1}^{n-1} t\beta_j (v_j - v_n) \;=\; 0 \tag{3.11}$$

for some arbitrary t that we shall define later. Let $\beta_l = -\sum_{j=1}^{n-1} \beta_j$. Adding (3.10), (3.11) we get

$$\sum_{j=1}^{n} (\alpha_j + t\beta_j) v_j \;=\; v \tag{3.12}$$

Now, we set $t = \min_j -\alpha_j/\beta_j$, say without loss of generality $-\alpha_1/\beta_1$. Rewriting (3.12),

$$\sum_{j=2}^{n}(\alpha_j - \alpha_1\beta_j/\beta_1)v_j \quad = \quad v \tag{3.13}$$

Furthermore, $(\alpha_j - \alpha_1\beta_j/\beta_1) > 0$ for all $j = 2, \ldots, l$ and $\sum_{j=2}^{n}(\alpha_j - \alpha_1\beta_j/\beta_1) = 1$. Hence, we have expressed v as a convex combination in $(n-1)$ vectors from $\{v_1, \ldots, v_n\}$. We proceed in this manner, until we are no longer able to write a linear combination in k vectors as was done in (3.11). This procedure takes poly(n) time. □

The main theorem in this section is below. We state it for two players for simplicity, and extend it to ℓ players later.

Theorem 3.1.4 (restated). *Let $G = (R, C, k, n)$ be a two-player game. Given p^* for which there exists a q^* such that (p^*, q^*) is a Nash equilibrium, we can compute in deterministic polynomial time q' for which (p^*, q') is a Nash equilibrium and $|supp(q')| \leq k$.*

Proof. We would like to apply Carathéodory's Theorem as was done to obtain a similar result in [LMM03] but we will need to modify our application since Carathéodory's Theorem applies to linear subspaces whereas the Nash equilibrium strategies define an affine subspace. By Lemma 3.2.4 given G, p^* we can construct a q such that (p^*, q) is also a Nash equilibrium. Consider the column vector u given by

$$u = Rq^T = \sum_{i=1}^{n} q_i m_i$$

where m_i is the i-th column vector in R. Since R is a $k \times n$ matrix, u is in the span of column vectors $m_1, \ldots, m_n$ that lie in a k-dimensional subspace of $\mathbb{R}^k$. Since (p^*, q) is a Nash-equilibrium, for all $i \in \text{supp}(q)$:

$$p^* m_i = \max_j p^* m_j = w^*$$

This is an additional linear constraint on the m_is and hence $m_1, \ldots, m_n$ lie in a $(k-1)$-dimensional affine subspace $\mathcal{A}$. Since u is a convex combination of the m_i's,

$$p^* u = p^* \left(\sum_{i=1}^{n} q_i m_i \right) = \sum_{i=1}^{n} q_i (p^* m_i) = w^* \sum_{q_i > 0} q_i = w^*$$

and hence u also lies in $\mathcal{A}$. Define $m_i' = m_i - w^* 1_k$, where 1_k is the all-ones column vector. Then for all $i \in \mathrm{supp}(q)$, $p^* m_i' = 0$. Therefore

$$u' = u - w^* 1_k = \sum_{q_i > 0} q_i m_i'$$

lies in a $(k-1)$-dimensional subspace of $\mathbb{R}^k$. Applying Carathéodory's theorem 3.5.1, u' can thus be rewritten as a convex combination:

$$u' = \sum_i q_i' m_i'$$

where $|\mathrm{supp}(q')| \leq k$. It follows that

$$u = u' + w^* 1_k = \sum_i q_i' m_i' + w^* 1_k = \sum_i q_i' (m_i' + w^* 1_k) = \sum_i q_i' m_i$$

We claim that (p^*, q') is the desired Nash equilibrium. This is true because $\mathrm{supp}(q') \subseteq \mathrm{supp}(q^*)$, and for each $j \in \mathrm{supp}(p^*)$, $R(q')^T = Rq^T$ and hence $\mathrm{supp}(p^*) \subseteq \{j : (R(q')^T)_j = \max_s (R(q')^T)_s\}$. $\square$

The following theorem extends the result above to the general ℓ-player game where $\ell - 1$ players play sparse strategies.

Corollary 3.1.5 (restated). *Let $G = (T_1, T_2, \ldots, T_\ell, k_1, k_2, \ldots, k_{\ell-1}, n)$ be an ℓ-player game where $\prod_{i=1}^{\ell-1} k_i < n$. Given $G, p_1^*, \ldots, p_{\ell-1}^*$ there exists a deterministic polynomial-time procedure to compute $\widehat{p_\ell}$ such that $(p_1^*, p_2^*, \ldots, p_{\ell-1}^*, \widehat{p_\ell})$ is a Nash equilibrium for G and $|supp(\widehat{p_\ell})| \leq k = \prod_{i=1}^{\ell-1} k_i$.*

Proof. The proof is similar to that of Theorem 3.1.4. Applying Lemma 3.2.4, we obtain p_ℓ such that $(p_1^*, \ldots, p_{\ell-1}^*, p_\ell)$ is a Nash equilibrium for G. Consider the $k \times n$ matrix $T_{-\ell}$ formed by choosing each of $T_1, \ldots, T_{\ell-1}$'s strategies. $T_{-\ell}$ is of rank at most $k < n$. As in proving Theorem 3.1.4, we observe that the column vector $u = T_{-\ell} p'_\ell$ is in the span of column vectors $m_1^{-\ell}, \ldots, m_n^{-\ell}$ that lie in a $(k-1)$-dimensional affine subspace following the same line of argument with respect to $(p_1^*, \ldots, p_{\ell-1}^*)$ imposing a constraint on the m_i's. Translating back and forth between the linear and affine subspace and applying Carathéodory's Theorem we get $p'_\ell, \mathrm{supp}(p'_\ell) \leq k$ the Nash equilibrium strategy for T_ℓ. The algorithm runs in time polynomial in n^ℓ. □

In Theorem 3.5.2, we show that these bounds are tight (for small values of k) in the case when T_ℓ's strategy is constrained to be a uniform probability distribution over the support. That is, we show examples of games for which there exist Nash equilibria where the ℓ-th player requires support $k = \prod_{i=1}^{\ell-1} k_i$ if her strategy is constrained to be uniformly distributed.

Theorem 3.5.2 *For every n, $\ell \geq 2$, and $k_1, \ldots k_{\ell-1}$ such that $k = \prod_{i=1}^{\ell-1} k_i < n$, one can construct an ℓ-player game $G = (T_1, T_2, \ldots, T_\ell, k_1, k_2, \ldots, k_{\ell-1}, n)$ for which there exists, for each $i, 1 \leq i \leq \ell - 1$, a strategy p_i^* with $|supp(p_i^*)| = k_i$ such that:*

1. *$(p_1^*, p_2^*, \ldots, p_\ell^*)$ is a Nash equilibrium where p_ℓ^* is a strategy with support $k < n$.*

2. *There is no strategy p_ℓ with uniform distribution over a support of size $|supp(p_\ell)| < \prod_{i=1}^{\ell-1} k_i = k$ such that $(p_1^*, p_2^*, \ldots, p_{\ell-1}^*, p_\ell)$ is a Nash equilibrium.*

Proof. For each T_i; $i = 1, \ldots, \ell$ we set values as follows. For some arbitrary ordering of $J = \{(v_1, \ldots, v_{\ell-1}) \mid 1 \leq v_r \leq k_r;\ \ 1 \leq r \leq \ell - 1\}$ given by $\alpha_1, \alpha_2, \ldots, \alpha_k$ we set for all $1 \leq i \leq \ell$:

$$T_i(v_1, \ldots, v_{\ell-1}, \alpha_j) = \begin{cases} 1 & \alpha_j = (v_1, \ldots, v_{\ell-1}); 1 \leq j \leq k, \\ 0 & \text{otherwise.} \end{cases} \tag{3.14}$$

Let $(p_1^*, p_2^*, \ldots, p_\ell^*)$ be the mixed strategy ℓ-tuple where for each $i = 1, \ldots, \ell - 1$ p_i^* is the uniform distribution on the set of strategies $\{v_i \mid 1 \leq v_i \leq k_i\}$ and p_ℓ^* be the uniform distribution on the set of strategies $\{\alpha_j \mid 1 \leq j \leq k\}$. Then, $(p_1^*, \ldots, p_\ell^*)$ specifies a Nash equilibrium for G. This is since the payoff to T_i upon playing p_i^* in response to $(p_1^*, \ldots, p_{i-1}^*, p_{i+1}^*, \ldots, p_\ell^*)$ is $1/k$ for $i = 1, \ldots, \ell$ and 0 for any other strategy. Hence, T_i has no incentive to deviate unilaterally to any other strategy. Since this holds true for all $i = 1, \ldots, \ell$ $(p_1^*, \ldots, p_\ell^*)$ is a Nash equilibrium.

For any of the $(\ell - 1)$ players $T_1, \ldots, T_{\ell-1}$, say T_i who plays pure strategy s, the payoff for the strategy tuple given by $(p_1^*, \ldots, p_{i-1}^*, s, p_{i+1}^*, \ldots, p_{\ell-1}^*, p_\ell)$ is

$$\begin{aligned} T_i(p_1^*, \ldots, p_{i-1}^*, s, p_{i+1}^*, \ldots, p_{\ell-1}^*, p_\ell) &= \sum_{a_1, \ldots, a_\ell} \Pr[a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_\ell] \times \\ &\quad T_i(a_1, \ldots, a_{i-1}, s, a_{i+1}, \ldots, a_{\ell-1}, a_\ell) \end{aligned}$$

where $\Pr[a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_\ell]$ is the probability that T_j plays a_j for all $j \neq i$. Note that given the choice of T_i, the above sum is only over tuples $(a_1, \ldots, a_{i-1}, s, a_{i+1}, \ldots, a_\ell)$ where $a_\ell = (a_1, \ldots, a_{i-1}, s, a_{i+1}, \ldots, a_{\ell-1})$.

Suppose that T_ℓ decides to play a strategy p_ℓ with uniform distribution over support S_ℓ of size, say $k' < k$. Without loss of generality, we may assume that $k' = k - 1$. This implies that there is some strategy, $\alpha = (s_1, \ldots, s_{\ell-1})$ say, which is not in S_ℓ. For $s \neq s_i$, we compute the payoff to player T_i using the expression above and the fact that all the mixed strategies are uniform distributions over their respective supports, and so we have:

$$\begin{aligned} T_i(p_1^*, \ldots, p_{i-1}^*, s, p_{i+1}^*, \ldots, p_{\ell-1}^*, p_\ell) &= \frac{1}{k_1 k_2 \ldots k_{i-1} k_{i+1} \ldots k_{\ell-1}} \times \frac{1}{k-1} \times \\ &\quad (k_1 k_2 \ldots k_{i-1} k_{i+1} \ldots k_{\ell-1}) \\ &= 1/(k-1) \end{aligned}$$

However, for $s = s_i$, since T_ℓ never plays the strategy $(s_1, \ldots, s_{\ell-1})$, the payoff to player T_i is:

$$\begin{aligned} T_i(p_1^*, \ldots, p_{i-1}^*, s, p_{i+1}^*, \ldots, p_{\ell-1}^*, p_\ell) &= \frac{1}{k_1 k_2 \ldots k_{i-1} k_{i+1} \ldots k_{\ell-1}} \times \frac{1}{k-1} \times \\ &\quad (k_1 k_2 \ldots k_{i-1} k_{i+1} \ldots k_{\ell-1} - 1) \\ &= 1/(k-1) - k_i/k(k-1) \end{aligned}$$

By Lemma 3.3.5 therefore, s_i cannot be in T_i's equilibrium strategy support. Hence, $(p_1^*, p_2^*, \ldots, p_{\ell-1}^*, p_\ell)$ is not a Nash equilibrium. □

3.6 Finding sparse ϵ-equilibria in low-rank games

We now consider games of rank k, which is a significant generalization of the "unbalanced" games in the previous section. Indeed, rank k games are perhaps the most general class of games for which sparse equilibria are guaranteed to exist. In this section we give algorithms to compute sparse ϵ-equilibria in this setting.

3.6.1 Two player case

Since we are computing ϵ-equilibria, we only expect the game specification to be given up to some fixed precision. We will be working with rank k matrices M expressed as $M_1 M_2$ (where M_1 is a $n \times k$ matrix and M_2 is a $k \times n$ matrix). Such a decomposition can be found efficiently via basic linear algebra. In the following theorem we take M_1 and M_2, specified up to fixed precision, as our starting point.[2] As the example in §3.6.3 illustrates, such a decomposition is already available for many natural games. Our convention for expressing fixed precision entries will be to require them to be integers in the range $[-B, B]$ for a bound B.

[2]We note that computing M_1, M_2 of fixed precision such that $M_1 M_2$ approximates M is not necessarily always possible or straightforward. We state our theorem in this way to avoid these complications, a detailed discussion of which would be beyond the scope of this dissertation.

Theorem 3.1.6 (restated). *Let $G = (R, C, n)$ be a two player game such that R and C have rank at most k. Furthermore, let $R = R_1R_2$, $C = C_1C_2$ be a decomposition of R, C with R_1, R_2, C_1, C_2 containing integer entries in $[-B, B]$. Then, for every $\epsilon > 0$, there is a deterministic procedure P running in time $(4B^2k/\epsilon)^{2k}poly(|G|)$ that returns a 4ϵ-Nash equilibrium (p, q) with $|supp(p)|, |supp(q)| \leq k + 1$.*

Lipton et al. showed that there *exist* $(k+1)$-sparse Nash equilibria in this setting and this implies an enumeration algorithm to find an equilibrium in time approximately $n^{k+1}\text{poly}(|G|)$. Our algorithm shows that the problem is "fixed parameter tractable" [Ces05, DF99, DFS97] where ϵ, the rank k and precision B are the parameters.

Proof. Note that the payoff to the row-player when (p, q) is the strategy tuple for the game which is given by pRq^T can now be written as $pR_1R_2q^T$ and likewise for the column player. The first step in our algorithm is to "guess" a collection of vectors to within $\delta = \epsilon/(2Bk)$ precision. We describe the "correct" guess relative to an (arbitrary) Nash equilibrium (p^*, q^*) for G. Let $p^{*'} = p^*C_1, q^{*'} = R_2q^{*T}$. Note that from our choice of C_1, R_2 it holds that $p^{*'}, q^{*'}$ satisfy $-B \leq p_i^{*'}, q_i^{*'} \leq B; i = 1, \ldots, k$. The algorithm is as follows:

1. Guess a $\tilde{p}'$ such that for all $i = 1, \ldots, k$ $|p_i^{*'} - \tilde{p}_i'| \leq \delta$. Similarly, guess $\tilde{q}'$ such that for all $i = 1, \ldots, k$ $|q_i^{*'} - \tilde{q}_i'| \leq \delta$.

2. Let $\alpha_s = (\tilde{p}'C_2)_s$ and $\beta_t = (R_1\tilde{q}'^T)_t$. Set $S = \{s | \max_r \alpha_r - 2\epsilon \leq \alpha_s \leq \max_r \alpha_r\}$ and $T = \{t | \max_r \beta_r - 2\epsilon \leq \beta_t \leq \max_r \beta_r\}$.

3. Find a feasible solution $\bar{p}$ to the following linear program

$$|(\bar{p}C_1)_j - \tilde{p}_j'| \leq \delta; \qquad j = 1, \ldots, k \tag{3.15}$$

$$\bar{p}_i \geq 0; \qquad i = 1, \ldots, n \tag{3.16}$$

$$\bar{p}_i = 0; \qquad i \notin T \tag{3.17}$$

$$\sum_{i=1}^{n} \bar{p}_i = 1 \tag{3.18}$$

and a feasible solution $\bar{q}$ to the analogous linear program in which the first set of constraints is

$$|(R_2\bar{q}^T)_j - \tilde{q}'_j| \leq \delta; j = 1, \ldots, k.$$

4. $v = \bar{p}C_1$ is a convex combination of the rows of C_1, all of which lie in a k-dimensional subspace. From Carathéodory's Theorem (Theorem 3.5.1), we can find $\hat{p}$ with $\mathrm{supp}(\hat{p}) \subseteq \mathrm{supp}(p)$ for which $|\mathrm{supp}(\hat{p})| \leq k+1$ and $v = \hat{p}C_1$.

5. Similarly $u = R_2\bar{q}^T$ is a convex combination of the columns of R_2, all of which lie in a k-dimensional subspace. Applying Carathéodory's Theorem again, we find $\hat{q}$ with $\mathrm{supp}(\hat{q}) \subseteq \mathrm{supp}(q)$ for which $|\mathrm{supp}(\hat{q})| \leq k+1$ and $u = R_2\hat{q}^T$.

6. Return $\hat{p}, \hat{q}$.

Correctness follows from the next two claims:

Claim 3.6.1 *A feasible solution to the linear programs formulated in step 3 of the algorithm exists.*

Proof. (Of Claim 3.6.1) We claim that if (p^*, q^*) is a Nash equilibrium for G then $\bar{p} = p^*$ is a feasible solution to the first LP and $\bar{q} = q^*$ is a feasible solution to the second LP.

Assume $\tilde{p}'$ and $\tilde{q}'$ are the correct guesses. We need to show that $\mathrm{supp}(p^*) \subseteq T$ and $\mathrm{supp}(q^*) \subseteq S$. Since $\tilde{p}'$ is a correct guess, $p^{*'} = p^*C_1$ satisfies $|p^{*'}_j - \tilde{p}'_j| \leq \delta$. Therefore, p^* satisfies (3.15). Similarly for $\tilde{q}'$ we have $|q^{*'}_j - \tilde{q}'_j| \leq \delta$ and so q^* satisfies (3.15) in the LP for q.

Consider $r \in \mathrm{supp}(p^*)$ and $(R_1\tilde{q}^{'T})_r$. Since $|\tilde{q}'_j - q^{*'}_j| \leq \epsilon/(2Bk)$ for all $j = 1, \ldots, k$, we get

$$(R_1 q^{*'T})_r - \epsilon \leq \quad (R_1\tilde{q}^{'T})_r \quad \leq (R_1 q^{*'T})_r + \epsilon \tag{3.19}$$

Moreover p^* is a Nash equilibrium strategy and $r \in \mathrm{supp}(p^*)$. Therefore,

$$\max_t (R_1\tilde{q}^{'T})_t - \epsilon \leq \quad (R_1 q^{*'T})_r = \max\nolimits_t (R_1 q^{*'T})_t \quad \leq \max_t (R_1\tilde{q}^{'T})_t + \epsilon \tag{3.20}$$

Combining (3.19) and (3.20),

$$\max_t (R_1\tilde{q}'^T)_t - 2\epsilon \leq \quad (R_1\tilde{q}'^T)_r \quad \leq \max_t (R_1\tilde{q}'^T)_t + 2\epsilon$$

and the right hand side can be simplified to yield:

$$\max_t (R_1\tilde{q}'^T)_t - 2\epsilon \leq \quad (R_1\tilde{q}'^T)_r \quad \leq \max_t (R_1\tilde{q}'^T)_t.$$

Hence $r \in T$ and $\text{supp}(p^*) \subseteq T$. Similarly, we can show that $\text{supp}(q^*) \subseteq S$. □

Claim 3.6.2 *$(\hat{p}, \hat{q})$ as returned by the algorithm is a 4ϵ-equilibrium.*

Proof. (Of Claim 3.6.2) We need to show that for any p', q', $(\hat{p}, \hat{q})$ satisfy

$$\begin{aligned} \hat{p}R\hat{q}^T &\geq p'R\hat{q}^T - 4\epsilon, \\ \hat{p}C\hat{q}^T &\geq \hat{p}Cq'^T - 4\epsilon \end{aligned}$$

Now,

$$\hat{p}C\hat{q}^T \quad = \quad ((\hat{p}C_1)C_2)\,\hat{q}^T$$

By our choice of $\hat{p}$ in step 4. of the algorithm, $\hat{p}C_1 = \bar{p}C_1$. So,

$$\begin{aligned} \hat{p}C\hat{q}^T &= (\bar{p}C_1)\,C_2\hat{q}^T \\ &\geq (\tilde{p}' - \delta 1_k)\,C_2\hat{q}^T \quad \text{from (3.15)} \\ &= \tilde{p}'C_2\hat{q}^T - \delta 1_k C_2\hat{q}^T \\ &\geq \tilde{p}'C_2\hat{q}^T - \epsilon \end{aligned} \tag{3.21}$$

Since $\text{supp}(\hat{q}) \subseteq \text{supp}(\bar{q})$ and $\text{supp}(\bar{q})$ contains only s for which $(\tilde{p}'C_2)_s \geq \max_r(\tilde{p}'C_2)_r -$

2ϵ we obtain

$$\begin{aligned}
\tilde{p}' C_2 \hat{q}^T &\geq \max_r (\tilde{p}' C_2)_r - 2\epsilon \\
&\geq \max_r \left((\bar{p} C_1 - \delta 1_k) C_2\right)_r - 2\epsilon \quad \text{from (3.15)} \\
&\geq \max_r \left(\hat{p} C_1 C_2\right)_r - 3\epsilon \\
&\geq \hat{p} C q'^T - 3\epsilon \quad \text{for any } q' \qquad (3.22)
\end{aligned}$$

Combining (3.21) and (3.22),

$$\hat{p} C \hat{q}^T \geq \hat{p} C q'^T - 4\epsilon$$

Similarly, $\hat{p} R \hat{q}^T \geq p' R \hat{q}^T - 4\epsilon$ for any p'. □

We analyze the run-time for the algorithm above in terms of the support parameter k and number of strategies n, and B. The first step of the algorithm where we "guess" $\tilde{p}'$ requires exhaustively going through all possible choices for each component of $\tilde{p}'$ in the interval $[-B, B]$ in steps of δ, and similarly for guessing $\tilde{q}'$. This takes time $(2B/\delta)^{2k} = (4B^2 k/\epsilon)^{2k}$. For each choice of $(\tilde{p}', \tilde{q}')$, we will need to solve the linear program above which takes poly$(|G|)$ time. The applications of Carathéodory's Theorem also take poly$(|G|)$ and so the running time is as claimed and this completes the proof of the theorem. □

3.6.2 Three or more players

In this section, we will look at obtaining approximate Nash equilibria for low-rank games with three or more players. This direction does not seem to have been studied before, and the previously known algorithms for low-rank games [KT07] don't seem to extend to more than 2 players.

We begin by stating some definitions related to tensor rank.

Definition 3.6.3 *An arity-l tensor T with dimension n has a p-***decomposition** *if* *it can be expressed as:* $T = \sum_{i=1}^{p} t_1^{(i)} \otimes t_2^{(i)} \otimes \ldots \otimes t_l^{(i)}$ *where each* $t_j^{(i)}$ *is an* $n \times 1$ *vector (i.e. it is the sum of* p *pure tensors).*

It will be convenient to aggregate the information in a p-decomposition of such a tensor T into a ℓ-tuple $(C_{i1}, C_{i2}, \ldots, C_{i\ell})$ where each C_{ij} is an $n \times p$ matrix whose columns are given by $\left(t_{ij}^{(1)}, \ldots, t_{ij}^{(p)}\right)$. We will refer to the r-th row of a matrix C_{ij} as $C_{ij}(r)$. We will also define $\odot$ to be the component-wise product of vectors:

Definition 3.6.4 *For two vectors* $v, w \in \mathbb{R}^n$*, define* $(v \odot w)$ *to be the vector* $(v_1 w_1, \ldots, v_n w_n) \in \mathbb{R}^n$*.*

The following inequality will be useful in obtaining a result for the extension to three or more players.

Lemma 3.6.5 *Let* $x_1, x_2, \ldots, x_\ell, \widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_\ell \in \mathbb{R}^k$ *be vectors satisfying* $|x_i - \widetilde{x}_i| \leq \delta$ *and furthermore, let* $|\widetilde{x}_i| \leq B$ *for all* $i = 1, \ldots, \ell$*. Then,*

$$\langle x_1, (\widetilde{x}_2 \odot \ldots \odot \widetilde{x}_\ell)\rangle - \delta\ell k(B+\delta)^{\ell-1} \leq \langle x_1, (x_2 \odot x_3 \odot \ldots \odot x_\ell)\rangle \leq \langle x_1, (\widetilde{x}_2 \odot \ldots \odot \widetilde{x}_\ell)\rangle + \delta\ell k(B+\delta)^{\ell-1}$$

Proof.

$$\begin{aligned}
\langle x_1, (x_2 \odot x_3 \odot \ldots \odot x_\ell)\rangle &\geq \langle x_1, (\widetilde{x}_2 - \delta 1_k) \odot x_3 \odot \ldots \odot x_\ell\rangle \\
&\geq \langle x_1, \widetilde{x}_2 \odot x_3 \odot \ldots \odot x_\ell\rangle - \delta(B+\delta)^{\ell-2}\langle x_1, 1_k\rangle \\
&\geq \langle x_1, \widetilde{x}_2 \odot x_3 \odot \ldots \odot x_\ell\rangle - \delta k(B+\delta)^{\ell-1} \\
&\vdots \\
&\geq \langle x_1, \widetilde{x}_2 \odot \ldots \odot \widetilde{x}_\ell\rangle - (\ell-1)\delta k(B+\delta)^{\ell-1} \\
&\geq \langle x_1, \widetilde{x}_2 \odot \ldots \odot \widetilde{x}_\ell\rangle - \delta\ell k(B+\delta)^{\ell-1}
\end{aligned}$$

The other side of the inequality can be similarly shown. $\square$

We begin by generalizing Lipton et al. [LMM03] to the multiplayer case. The following theorem shows that low rank games in the multiplayer case have sparse Nash equilibria; in the next section we give nontrivial algorithms to find approximate equilibria with this sparsity.

Corollary 3.6.6 *Let $G = (T_1, \ldots, T_\ell, n)$ be an ℓ-player game, and suppose T_i has rank k_i. Then there exists a Nash equilibrium $(p_1^*, \ldots, p_\ell^*)$ with $|supp(p_i^*)| \leq 1 + \sum_{j=1}^{\ell} k_j$ for all i.*

Proof. Let $(q_1, \ldots, q_\ell)$ be a Nash equilibrium for G and let $(C_{i,1}, \ldots, C_{i,\ell})$ be the tensor decomposition for T_i. Given $q_1, \ldots, q_\ell$, we can define vectors in $\mathbb{R}^n$:

$$
\begin{array}{lclcccccccccccl}
w_1 & = & (C_{1,1}(& & & q_2C_{1,2} & \odot & q_3C_{1,3} & \odot & \cdots & \odot & q_\ell C_{1,\ell} &)^T) & & \\
w_2 & = & (C_{2,2}(q_1C_{2,1} & & & & \odot & q_3C_{2,3} & \odot & \cdots & \odot & q_\ell C_{2,\ell} &)^T) & & \\
w_3 & = & (C_{3,3}(q_1C_{3,1} & \odot & & q_2C_{3,2} & & & \odot & \cdots & \odot & q_\ell C_{3,\ell} &)^T) & & (3.23) \\
 & \vdots & & & & & & & & & & & & & \\
w_\ell & = & (C_{\ell,\ell}(q_1C_{\ell,1} & \odot & & q_2C_{\ell,2} & \odot & \cdots & \odot & q_{\ell-1}C_{\ell,\ell-1} & & &)^T) & &
\end{array}
$$

By the definition of a Nash equilibrium, the following conditions are satisfied:

$$
\begin{array}{rcl}
\mathrm{supp}(q_1) & \subseteq & \{v : (w_1)_v = \max_u (w_1)_u\} \\
\mathrm{supp}(q_2) & \subseteq & \{v : (w_2)_v = \max_u (w_2)_u\} \\
 & \vdots & \\
\mathrm{supp}(q_\ell) & \subseteq & \{v : (w_\ell)_v = \max_u (w_\ell)_u\}.
\end{array}
$$

And indeed any $(\widehat{q_1}, \ldots, \widehat{q_\ell})$ satisfying these conditions (when the w_i are defined relative the the $\widehat{q_i}$) is a Nash equilibrium.

Set $s = \sum_{j=1}^{\ell} k_j$. Now, by Carathéodory's Theorem there exists $\widehat{q_1} \in \mathbb{R}^n$ with $\mathrm{supp}(\widehat{q_1}) \subseteq \mathrm{supp}(q_1)$ and $|\mathrm{supp}(\widehat{q_1})| \leq s + 1$ for which:

$$\widehat{q_1}\left(C_{2,1}|C_{3,1}|C_{4,1}|\cdots|C_{\ell,1}\right) = q_1\left(C_{2,1}|C_{3,1}|\cdots|C_{\ell,1}\right),$$

since the right-hand-side is a convex combination of vectors in $\mathbb{R}^{s'}$ for $s' \leq s$. Similarly, there exists $\widehat{q_2} \in \mathbb{R}^n$ with $\text{supp}(\widehat{q_2}) \subseteq \text{supp}(q_2)$ and $|\text{supp}(\widehat{q_2})| \leq s+1$ for which:

$$\widehat{q_2}\left(C_{1,2}|C_{3,2}|C_{4,2}|\cdots|C_{\ell,2}\right) = q_2\left(C_{1,2}|C_{3,2}|C_{4,2}|\cdots|C_{\ell,2}\right),$$

since the right-hand-side is a convex combination of vectors in $\mathbb{R}^{s'}$ for $s' \leq s$.

A symmetric argument gives us a sparse $\widehat{q_i}$ from each q_i. Moreover, these $\widehat{q_i}$ produce precisely the same vectors $w_1, w_2, \ldots, w_\ell$ via (3.23). And, since $\text{supp}(\widehat{q_i}) \subseteq \text{supp}(q_i)$ for all i, the strategies $\widehat{q_i}$ satisfy the above conditions for being a Nash equilibrium, assuming the original q_i did. $\square$

We now turn to algorithms for finding sparse approximate equilibria, with three or more players. We first consider the case when the tensor decomposition is known. This is not an unnatural assumption: in §3.6.3, we describe a class of games for which the tensor decomposition is naturally given by the description of the payoff functions.

Theorem 3.6.7 *Let $G = (T_1, \ldots, T_\ell, n)$ be an ℓ-player game, and suppose we are given a k-decomposition of $T_i = (C_{i1}, \ldots, C_{i\ell})$ where each of the C_{ij} is an $n \times k$ matrix with integer values in $[-B, B]$ for $i, j = 1, \ldots, \ell$. Then for every $\epsilon > 0$, there is a deterministic procedure P running in time*

$$\left((2B)^\ell k\ell/\epsilon\right)^{k(\ell-1)\ell} \textit{poly}(|G|)$$

that returns a 4ϵ-Nash equilibrium $(p_1, p_2, \ldots, p_\ell)$ with $|supp(p_i)| \leq 1 + \ell k$ for all i.

Proof. As in the two player case, our first step is to "guess" a collection of vectors to within $\delta = \frac{\epsilon}{k\ell(2B)^{\ell-1}}$ precision. We describe the "correct" guess relative to an (arbitrary) Nash equilibrium $(p_1, \ldots, p_\ell)$ for G.

1. Let $(p'_{11}, \ldots, p'_{1\ell}) = (p_1 C_{11}, p_2 C_{12}, \ldots, p_\ell C_{1\ell})$ be an ℓ-tuple of $1 \times k$ vectors. Note that since all entries in $(C_{11}, C_{12}, \ldots, C_{1\ell})$ lie in $[-B, B]$, entries in $(p'_{11}, \ldots, p'_{1\ell})$ also lie in $[-B, B]$.

2. Guess an $(\ell-1)$-tuple $(\widetilde{p'_{12}},\ldots,\widetilde{p'_{1\ell}})$ such that

$$|(p'_{1i})_j - \widetilde{(p'_{1i})_j}| \leq \delta \qquad i = 2,\ldots,\ell; j = 1\ldots,k$$

3. For $C_{11}(r)$, the r-th row in C_{11} let

$$\begin{aligned}\alpha_r &= \left(C_{11}\left(\widetilde{p'_{12}} \odot \widetilde{p'_{13}} \odot \ldots \odot \widetilde{p'_{1\ell}}\right)^T\right)_r ; r = 1,\ldots,n \\ \mathcal{S}_1 &= \left\{r | \max_t \alpha_t - 2\epsilon \leq \alpha_r \leq \max_t \alpha_t\right\}\end{aligned}$$

4. By repeating steps 1 through 3 on $T_2,\ldots,T_\ell$ we can similarly obtain $\mathcal{S}_2,\ldots,\mathcal{S}_\ell$.

5. Find $(\bar{p}_1,\ldots,\bar{p}_\ell)$ satisfying the following linear program

$$|(\bar{p}_j C_{mj})_r - (\widetilde{p_{mj}})_r| \leq \delta \qquad r = 1,\ldots,k; j = 1\ldots,\ell;$$

$$m = 1,\ldots,j-1,j+1,\ldots,\ell \tag{3.24}$$

$$(\bar{p}_j)i \geq 0; \qquad i = 1,\ldots,n; j = 1,\ldots,\ell \tag{3.25}$$

$$\sum_{i=1}^{n} (\bar{p}_j)_i = 1; \qquad j = 1,\ldots,\ell \tag{3.26}$$

$$(\bar{p}_j)_i = 0; \qquad i \notin \mathcal{S}_j; j = 1,\ldots,\ell \tag{3.27}$$

6. Let $D_i = (C_{1,i} \mid C_{2,i} \mid \ldots \mid C_{i-1,i} \mid C_{i+1,i} \mid \ldots \mid C_{\ell,i})$ for $i = 1,\ldots,\ell$. $v = \bar{p}_i D_i$ is a convex combination of column vectors in D_i that are in an ℓk-dimensional subspace. Apply Carathéodory's Theorem (Theorem 3.5.1) to obtain $\widehat{p_i}$ such that $v = \widehat{p_i} D_i$ for $i = 1,\ldots,\ell$ and $\mathrm{supp}(\widehat{p_i}) \leq \ell k + 1$.

7. Return $(\widehat{p_1},\ldots,\widehat{p_\ell})$.

As before, in order to prove the correctness of the above algorithm we postulate the following claims:

Claim 3.6.8 *A feasible solution to the linear program formulated in step 5 of the algorithm exists.*

Proof. (Of Claim 3.6.8) We will show that ($\bar{p}_1 = p_1^*, \bar{p}_2, \ldots, \bar{p}_\ell = p_\ell^*$), the Nash equilibrium strategies for $T_1, \ldots, T_\ell$ satisfy the linear program. Since $p_i^{*'} = p_i^* C_{11}$ satisfies $|p_i^{*'}[j] - \widetilde{p'_{mi}}[j]| \leq \delta$ for $j = 1, \ldots, k$ and for $m = [\ell] - \{i\}$, p_i^* satisfies (3.24). (3.25) and (3.26) follow from $\bar{p}_i$ being a probability distribution. So, it suffices to show that $p_1^*, \ldots, p_\ell^*$ satisfy (3.27), i.e., for any $r \in \text{supp}(p_i^*)$, $r \in \text{supp}(\mathcal{S}_i)$. We will show this to be true for $i = 1$; it symmetrically follows for $i = 2, \ldots, \ell$. Note that $|\widetilde{p'_{1i}}| \leq B$ and $|p_i^* C_{1i} - \widetilde{p'_{1i}} C_{1i}| \leq \delta$. Then, applying Lemma 3.6.5 gives us

$$\begin{aligned}\left(C_{11}(p_2^* C_{12} \odot \ldots \odot p_\ell^* C_{1\ell})^T\right)_r - \epsilon \qquad &\leq \left(C_{11}(\widetilde{p'_{12}} \odot \ldots \odot \widetilde{p'_{1\ell}})^T\right)_r \\ &\leq \left(C_{11}(p_2^* C_{12} \odot \ldots \odot p_\ell^* C_{1\ell})^T\right)_r + \epsilon\end{aligned}$$

and likewise,

$$\begin{aligned}\max_t \left(C_{11}(\widetilde{p'_{12}} \odot \ldots \odot \widetilde{p'_{1\ell}})^T\right)_t - \epsilon \qquad &\leq \left(C_{11}(p_2^* C_{12} \odot \ldots \odot p_\ell^* C_{1\ell})^T\right)_r \\ &\leq \max\nolimits_t \left(C_{11}(\widetilde{p'_{12}} \odot \ldots \odot \widetilde{p'_{1\ell}})^T\right)_t + \epsilon\end{aligned}$$

Combining these two equations,

$$\begin{aligned}\max_t \left(C_{11}(\widetilde{p'_{12}} \odot \ldots \odot \widetilde{p'_{1\ell}})^T\right)_t - 2\epsilon \;\; &\leq \;\; \left(C_{11}(\widetilde{p'_{12}} \odot \ldots \odot \widetilde{p'_{1\ell}})^T\right)_r \\ &\leq \;\; \max_t \left(C_{11}(\widetilde{p'_{12}} \odot \ldots \odot \widetilde{p'_{1\ell}})^T\right)_t\end{aligned}$$

Hence $r \in \mathcal{S}_1$. $\square$

Claim 3.6.9 $(\widehat{p_1}, \ldots, \widehat{p_\ell})$ *as returned by the algorithm is a* 4ϵ*-equilibrium.*

Proof. (Of Claim 3.6.9) For the strategy tuple $(p_1, \ldots, p_\ell)$ and given a decomposition of $T_i = (C_{i1}, \ldots, C_{i\ell})$, the payoff to player i given by $T_i(p_1, \ldots, p_\ell)$ may be expressed as

$$T_i(p_1, \ldots, p_\ell) = p_i C_{ii} \left(p_1 C_{i1} \odot p_2 C_{i2} \odot \ldots \odot p_{i-1} C_{i(i-1)} \odot p_{i+1} C_{i(i+1)} \odot \ldots p_\ell C_{i\ell}\right)^T$$

Put this way, we need to show that for any $(p'_1, \ldots, p'_\ell)$, $\widehat{p}_1, \ldots, \widehat{p}_l$ returned by the algorithm satisfy

$$\widehat{p}_i C_{ii} \left(\widehat{p}_1 C_{i1} \odot \ldots \widehat{p_{i-1}} C_{i,i-1} \odot \widehat{p_{i+1}} C_{i,i+1} \ldots \odot \widehat{p}_\ell C_{i\ell}\right)^T \geq p'_i C_{ii} \left(\widehat{p}_1 C_{i1} \odot \ldots \odot \widehat{p}_\ell C_{i\ell}\right)^T - \epsilon$$

for $i = 1, \ldots, \ell$. We will prove this for $i = 1$, the proof follows symmetrically for $i = 2, \ldots, \ell$. First, we note that from the application of Carathéodory's theorem in step 6,

$$\widehat{p}_1 C_{11} \left(\widehat{p}_2 C_{12} \odot \ldots \odot \widehat{p}_\ell C_{1\ell}\right)^T = \widehat{p}_1 C_{11} \left(\bar{p}_2 C_{12} \odot \ldots \odot \bar{p}_\ell C_{1\ell}\right)^T \tag{3.28}$$

Then, applying Lemma 3.6.5

$$\widehat{p}_1 C_{11} \left(\bar{p}_2 C_{12} \odot \ldots \odot \bar{p}_\ell C_{1\ell}\right)^T \geq \widehat{p}_1 C_{11} \left(\widetilde{p'_{12}} \odot \ldots \odot \widetilde{p'_{1\ell}}\right)^T - \epsilon \tag{3.29}$$

By the same argument as before, since we picked only those r in $\text{supp}(\widehat{p}_1) \subseteq \mathcal{S}_1$ such that

$$\left(C_{11} \left(\widetilde{p'_{12}} \odot \ldots \odot \widetilde{p'_{1\ell}}\right)^T\right)_r \geq \max_r \left(C_{11} \left(\widetilde{p'_{12}} \odot \ldots \odot \widetilde{p'_{1\ell}}\right)^T\right)_r - 2\epsilon$$

we get

$$\widehat{p}_1 C_{11} \left(\widetilde{p'_{12}} \odot \ldots \odot \widetilde{p'_{1\ell}}\right)^T \geq \max_r C_{11}(r) \left(\widetilde{p'_{12}} \odot \ldots \odot \widetilde{p'_{1\ell}}\right)^T - 2\epsilon \tag{3.30}$$

Applying Lemma 3.6.5 and from Carathéodory's Theorem,

$$\begin{aligned}
\max_r \left(C_{11} \left(\widetilde{p'_{12}} \odot \ldots \odot \widetilde{p'_{1\ell}}\right)^T\right)_r &\geq \max_r \left(C_{11} \left(\bar{p}_2 C_{12} \odot \ldots \odot \bar{p}_\ell C_{1\ell}\right)^T\right)_r - \epsilon \\
&= \max_r \left(C_{11} \left(\widehat{p}_2 C_{12} \odot \ldots \odot \widehat{p}_\ell C_{1\ell}\right)^T\right)_r - \epsilon \\
&\geq p'_1 C_{11} \left(\widehat{p}_2 C_{12} \odot \ldots \odot \widehat{p}_\ell C_{1\ell}\right)^T - \epsilon \qquad (3.31)
\end{aligned}$$

for any p'_1

Combining (3.28)–(3.31),

$$\widehat{p_1}C_{11}\left(\widehat{p_2}C_{12}\odot\ldots\odot\widehat{p_\ell}C_{1\ell}\right)^T \geq p_1'C_{11}\left(\widehat{p_2}C_{12}\odot\ldots\odot\widehat{p_\ell}C_{1\ell}\right)^T - 4\epsilon \quad \text{for any } p_1'$$

□

The total number of guesses made in step 2. of the procedure is $k(\ell-1)\ell$ (k guesses for each component of the $(\ell-1)$ different vectors in step 2, repeated ℓ times in step 4.). Each component is within $[-B,B]$ and is guessed to δ accuracy, so the guessing takes time

$$(2B/\delta)^{k(\ell-1)\ell} = (2B(k\ell(2B)^{\ell-1})/\epsilon)^{k(\ell-1)\ell} = \left((2B)^\ell k\ell/\epsilon\right)^{k(\ell-1)\ell}$$

For each guess, we solve a linear program, taking time poly($|G|$), for a total running time as claimed. This completes the proof of the theorem. □

3.6.3 An example of games with known low-rank tensor decomposition

Many natural games are specified implicitly (rather than by explicitly giving the tensors) by describing the payoff function, which itself is often quite simple. In such cases, the tensor ranks may be significantly smaller than n, and moreover, a low-rank decomposition into components with bounded entries can often be derived from the payoff functions.

One prominent example is simple ℓ-player congestion games as described in [FPT04, Pap05]. Such a game is based on a graph $\mathcal{G}(V,E)$ with n vertices and m edges. Each player's strategy set corresponds to a subset $S_p \subseteq 2^E$, the set of all subsets of edges. We define the payoff accruing to some strategy l-tuple $(s_1,\ldots,s_\ell)$ as $U(s_1,\ldots,s_\ell) = -\sum_e c_e(s_1,\ldots,s_\ell)$ where $c_e(s_1,\ldots,s_\ell) = |\{i|e \in s_i, 1 \leq i \leq \ell\}|$ is thought of as the *congestion* on paths $s_1,\ldots,s_\ell$. Let $G = (T_1,\ldots,T_\ell, N = 2^m)$ be the game corresponding to the situation described above where for $i = 1,\ldots,\ell$ and strategy tuple $(s_1, \ldots, s_\ell)$, $T_i(s_1, \ldots, s_\ell) = -\sum_e c_e(s_1, \ldots, s_\ell)$.

Theorem 3.6.10 *For $i = 1, \ldots, \ell$ T_i as defined above is of rank at most ℓm. Furthermore, an explicit ℓm-decomposition $(C_{i1}, C_{i2}, \ldots, C_{i\ell})$ for T_i exists where C_{ij} are $n \times k$ matrices with entries in $\{-1, 0, 1\}$.*

Proof. In order to give an ℓm-decomposition of T_1 say, we need to construct $(C_1, \ldots, C_\ell)$ where $C_i, i = 1, \ldots, \ell$ are $n \times (\ell m)$ matrices. Consider the tensors $\{T_{i,j}\}_{i=1,\ldots,\ell,j=1,\ldots,m}$ that are described as follows: fix some s_i to be the strategy for player i. For all $e_j \in E(\mathcal{G}), j = 1, \ldots, m$ such that $e_j \in s_i$ let

$$T_{i,j}(s'_1, \ldots, s'_{i-1}, s_i, s'_{i+1}, \ldots, s'_\ell) = -1$$

where $s'_1, \ldots, s'_{i-1}, s'_{i+1}, \ldots, s'_\ell \in 2^E$. Then, $T_1 = \sum_{i=1}^{\ell} \sum_{j=1}^{m} T_{i,j}$. This follows easily since each tuple $(s_1, \ldots, s_\ell)$ in some $T_{i,j}$ contributes -1 to $T_1(s_1, \ldots, s_\ell)$ iff $e_j \in s_i$. Summing over all $T_{i,j}$ for $i = 1, \ldots, \ell$, e_j contributes exactly $-c_{e_j}(s_1, \ldots, s_\ell)$ to $T_1(s_1, \ldots, s_\ell)$ from the definition of $c_{e_j}(s_1, \ldots, s_\ell)$ above. Summing over all $j = 1, \ldots, m$ we obtain the total contribution from all the edges. Next, we claim that each $T_{i,j}$ is a rank-1 tensor. Indeed, $T_{i,j} = v_{1,j} \otimes \ldots \otimes v_{\ell,j}$ where $v_{k,j} = 1_N$ the all-ones $N \times 1$ column vector for $k = 1, \ldots, i-1, i+1, \ldots, \ell$ and $v_{i,j}$ is the $N \times 1$ column vector given by:

$$v_{i,j}[k] = \begin{cases} -1 & e_j \in s_k; \\ 0 & \text{otherwise.} \end{cases}$$

for $k = 1, \ldots, N$. $\square$

3.7 Conclusions

There are many other interesting questions that are raised by viewing game theory through the lens of requiring players to be randomness-efficient. In this chapter, we have framed some of the initial questions that arise and have provided answers to several of them. In particular, we have exploited the extensive body of work in derandomization to construct deterministic algorithms for finding sparse ϵ-equilibria (which can be played with limited randomness), and for playing repeated games while reusing randomness across rounds. The efficient fixed-parameter algorithms we describe for finding ϵ-equilibria in games of small rank significantly improve over the standard enumeration algorithm, and to the best of our knowledge, they are the first such results for games of small rank.

The notion of *resource-limited* players has been an extremely useful one in game theory, and we think that it is an interesting and natural question in this context to consider the case in which the limited computational resource is *randomness*. These considerations expose a rich and largely untapped area straddling complexity theory and game theory.

Chapter 4

Rationalizability of matchings

To economize is to choose.

– Marcel K. Richter.

4.1 Background

In this chapter, we focus on one of the specific problems in rationalizability mentioned earlier. Matchings in economics have been studied from the viewpoint of understanding labor markets, best exemplified by the canonical hospital-intern assignment problem. The problem was first[1] studied by Gale and Shapley [GS62] who looked at the one-one matchings between two disjoint sets, say men and women. A matching is said to be Gale-Shapley stable if there was no man-woman (m, f) such that m, f are not married to each other and neither would prefer the other to their current partners. For this model, Gale and Shapley proposed a simple algorithm and Fleiner [Fle03]. that runs in $O(n^2)$ time and constructs a matching that is Gale-Shapley stable. Fleiner [Fle03] pointed out interesting connections between stable matchings and lattice fixed-point theorems, and used them to develop further matroid-based generalizations of the stable matchings model which are perhaps even more of interest to theoretical computer scientists. Further extensions to matchings include looking at many-to-one matchings in connection with the college admissions problem [Rot85], finding core matchings by fixed-point methods [EO04] and finding

[1] The exact historical beginnings of investigations into matchings are somewhat open to interpretation. See Roth [Rot82].

core matchings when preferences are expressed [EY07]. Roth and Sotomayor [RS90] give an exhaustive survey of two-sided matchings.

4.1.1 Revealed preference theory and matchings

Echenique [Ech08] looked into the problem of finding preferences that rationalize a collection of one-to-one matchings. Echenique specifically asked about the testable implications of matchings, and showed that the theory is falsifiable (in other words, there exist collections of matchings that are not rationalizable by any preference profile). More significantly, he showed necessary and sufficient conditions that must be satisfied for a collection of matchings to be rationalized by a set of preference profiles. Broadly, these conditions involve the so-called coincidence/opposition property that need to be satisfied amongst any pair of agents belonging to one set and matched to one common agent in the other set in at least one matching. This property stipulates that the preferences of such a pair will be in lockstep in the set of matchings in which they were matched to one agent in common. Correspondingly, for the agents belonging to the other set, the preferences are in *opposition* to those of the former pair.

This particular setting does not extend to situations where there are monetary transfers involved, such as housing markets. In some recent work, Chambers and Echenique [CE08] look at the rationalizability of assignment games which are two-sided markets with monetary transfers and obtain similar characterizations for the testable implications of the observed data.

4.1.2 Stable matchings and preference profiles

Given two sets of nodes, M ("men") and W ("women"), together with preferences for each node, the famous algorithm of Gale and Shapley [GS62] obtains a *stable matching*. We will be interested in the "reverse" question: given a set of matchings, are there preferences under which they are simultaneously stable? One may wonder why we should be given a collection of matchings instead of a single instance of a matching between the set of men and women. Indeed, we think of the men (and women) as representing instances of different *types* or populations that are matched differently in each matching and we are interested in determining the preference profiles that define these types based on the observed set of matchings. Before stating our results, we formalize the problem and introduce some terminology.

Definition 4.1.1 *Let M, W be disjoint sets of equal cardinality. A* **one-one matching** μ *is a bijection $\mu : M \cup W \to M \cup W$, such that for all $m \in M$, $\mu(m) \in W$, for all $w \in W$, $\mu(w) \in M$, and for all $m \in M, w \in W$, $\mu(m) = w \Leftrightarrow \mu(w) = m$.*

In the problems we consider, we will be seeking preferences for the elements of M and W, which are expressed as follows:

Definition 4.1.2 *A* **preference order** *for $m \in M$ (resp. $w \in W$) is a linear ordering of W (resp. M). We write $m : w > w'$ to mean that w occurs before w' in the preference order for m. A* **preference profile** *is a collection of preference orders for each $m \in M$ and $w \in W$.*

The "stability" of a matching with respect to a preference profile depends on the crucial notion of *blocking pair*:

Definition 4.1.3 *A* **blocking pair** *with respect to a matching μ and a preference profile $\mathcal{P}$ is a pair $(m, w) : m \in M, w \in W$ such that $\mu(m) \neq w$ and*

$$m : w > \mu(m) \text{ and } w : m > \mu(w).$$

Matching μ is **stable** *with respect to $\mathcal{P}$ if there is no blocking pair with respect to μ and $\mathcal{P}$.*

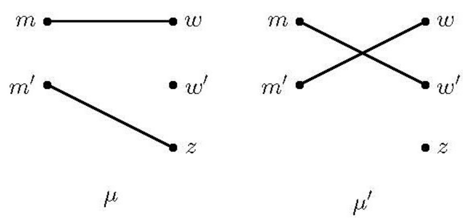
In other words, in a blocking pair (m, w) with respect to μ and $\mathcal{P}$, both people are "unhappy" with their current partner in μ and would instead prefer to be matched to each other.

4.1.3 Our results

Our first result is that rationalizing matchings is hard.

Theorem 4.1.4 *Given a collection of one-one matchings $\mathcal{H}$ on the sets M and W, it is NP-complete to determine if there exists a preference profile $\mathcal{P}$ such that every $\mu \in \mathcal{H}$ is stable with respect to $\mathcal{P}$.*

We call such a preference profile a *rationalization* of the matchings $\mathcal{H}$. The main gadget we use in the reduction is distilled from some fairly involved necessary and sufficient conditions for a preference profile to be a rationalization, discovered by Echenique [Ech08]. We describe the full conditions in Section 4.2. Our gadget is a configuration across two matchings, that looks like this:



A preference profile $\mathcal{P}$ rationalizes the matchings containing this configuration only if either $m : w > w'$ and $m' : z > w$, or $m : w' > w$ and $m' : w > z$. Conversely, if these conditions hold (together with additional conditions concerning the remainder of the matchings) then $\mathcal{P}$ rationalizes the set of matchings. We use this gadget fundamentally as a Boolean choice gadget (either m prefers w over w' or w' over w), and as part of a scheme to ensure consistency (since the choice of m is tied to the choice of m').

Having ascertained that rationalizing a collection of matchings is NP-complete, we would next want to know how hard it is to solve the problem approximately. In this context, we first need to decide what exactly we mean by 'approximate' rationalization. Two notions are of particular interest: on the one hand, we can think of identifying a preference profile that rationalizes the maximum number of matchings.

Problem 1 (MAX-STABLE-MATCHINGS) *Given a collection of matchings $\mathcal{H}$ on sets M, W, find a preference profile P that maximizes the number of matchings in $\mathcal{H}$ that are simultaneously rationalized by P.*

This problem is hard to approximate to within some constant factor:

Theorem 4.1.5 *There is a constant $\epsilon > 0$ for which it is NP-hard to approximate* MAX-STABLE-MATCHINGS *to within a factor of $(1 - \epsilon)$.*

A second natural notion of approximation attempts to maximize "stability" among the given set of matchings at a more fine-grained level, by maximizing the number of non-blocking pairs across all matchings.

Some effort is required to make this notion of approximation meaningful. In a typical instance there will be many pairs (m, w) for which m is not matched to w in *any* of the given matchings. We say such a pair is *non-active* and pairs that are matched in some matching are *active*. It is easy to ensure that all non-active pairs are non-blocking pairs with respect to any matching, by requiring the preference profile to be *valid*:

Definition 4.1.6 *A preference profile $\mathcal{P}$ is* valid *with respect to a collection of matchings $\mathcal{H}$ if for every $m \in M$, $m : w > w'$ if (m, w) is active and (m, w') is not active, and for every $w \in W$, $w : m > m'$ if (m, w) is active and (m', w) is not active.*

In other words, each man m prefers women that he is matched to in some matching over women that he is never matched to, and similarly for each women w. We argue that to have a meaningful notion of maximizing non-blocking pairs, one should consider only valid preference profiles, and therefore attempt to maximize the number of non-blocking pairs *among the active pairs* (since a valid preference profile automatically takes care of all of the non-active pairs). We are led to define the following optimization problem:

Problem 2 (MAX-STABILITY) *Given a collection of matchings $\mathcal{H}$ on sets M, W, find a valid preference profile P for M, W that maximizes:*

$$|\{(m, w, \mu) : (m, w) \text{ is active}$$

$$\text{and is not a blocking pair with respect to } \mu, P\}|.$$

This problem is also hard to approximate to within some constant factor:

Theorem 4.1.7 *There is a constant $\epsilon > 0$ for which it is NP-hard to approximate* MAX-STABILITY *to within a factor of $(1 - \epsilon)$.*

Our proof uses the overall structure of the reduction used to prove Theorem 4.1.4 together with an explicit constant-degree expander to make aspects of the reduction robust enough to be gap-preserving.

An approximation of $3/4$ is achievable (in expectation) for this problem by a simple randomized assignment of preferences. Derandomizing via the method of conditional expectations yields:

Theorem 4.1.8 *There is a deterministic, polynomial-time approximation algorithm for* MAX-STABILITY *that achieves an approximation factor of 3/4.*

Finally, we turn to a generalization of the one-one matchings we have been considering:

Definition 4.1.9 *Let F, W be disjoint sets. A* **one-many matching** *is a pair of functions (μ, τ) with $\mu : F \to 2^W$, and $\tau : W \to F$ for which*

$$\forall w \in \mu(f), \tau(w) = f \text{ and } \forall w \in W, w \in \mu(\tau(w)).$$

Typically in economics literature, one-to-many matchings are spoken of in reference to firms and workers (or, similarly, hospitals and interns) and hence the notation of F, W is more prevalent. However, since this problem is so closely tied in with our discussion of one-to-one matchings we will continue to use the notation of "men" M and "women" W when we mention one-to-many matchings in the rest of the paper.

One-many matching models have been widely studied [Rot82, Rot85]. In a one-many matching, preference order and preference profile are defined in the same way as for one-one matchings, except that each m has a linear ordering of 2^W instead of just W. Also analogous to the blocking pair for one-to-one matchings, we can define a *blocking set* and a notion of stability [EO04] for one-to-many matchings:

Definition 4.1.10 *A* **blocking set** *with respect to a one-many matching* (μ, τ) *and a preference profile* $\mathcal{P}$ *is a pair* $(m, B) : m \in M, B \subseteq W$ *such that* $\mu(m) \cap B = \emptyset$ *and*

$$\exists A \subseteq \mu(m) \text{ such that}$$

$$m : A \cup B > \mu(m) \text{ and } \forall w \in B \;\; w : m > \tau(w).$$

Matching (μ, τ) *is* **stable*** *with respect to* $\mathcal{P}$ *if there is no blocking set with respect to* (μ, τ) *and* $\mathcal{P}$.

The rationalization problem for one-many matchings is not likely to even be *in* NP, because a witness (preference profile) entails listing preference over 2^W, which is exponentially large. We are then led to consider a restricted version of the problem in which we only allow $m \in M$ to be matched to a set of cardinality at most some constant parameter ℓ. We call such matchings **one-ℓ** matchings.

The resulting rationalization problem is in NP and, we show, NP-complete:

Theorem 4.1.11 *For every fixed* ℓ*, given a collection of one-*ℓ *matchings* $\mathcal{H}$ *on the sets* M *and* W*, it is NP-complete to determine if there exists a preference profile* $\mathcal{P}$ *such that every* $\mu \in \mathcal{H}$ *is stable** *with respect to* $\mathcal{P}$.

We can define the notion of an active pair (m, B) for one-ℓ matchings in analogy with active pairs, and also valid preference profiles as in Definition 4.1.6.

The two approximation problems arising with respect to one-ℓ matchings are hard to approximate to within some constant factor, just as in the one-one case:

Theorem 4.1.12 *There is a constant $\epsilon > 0$ for which it is NP-hard to approximate* MAX-STABLE-ONE-ℓ-MATCHINGS *to within a factor of* $(1 - \epsilon)$.

Theorem 4.1.13 *There is a constant $\epsilon > 0$ for which it is NP-hard to approximate* MAX-ONE-ℓ-STABILITY *to within a factor of* $(1 - \epsilon)$.

4.2 Preliminaries

In this section, we encapsulate the working of the result for one-one matchings due to Echenique [Ech08] and provide the necessary and sufficient conditions for the existence of a preference profile that rationalizes a given collection of matchings. We start with some definitions and notations.

Definition 4.2.1 *For any two matchings $\mu, \mu' \in \mathcal{H}$, a (μ, μ')-***pivot** *is a $w \in W$ such that there exist some $m_k, m_\ell \in M$ such that $\mu(m_k) = \mu'(m_\ell) = w$.*

The key to proving Theorem 4.1.4 is a result due to Echenique [Ech08] which we encapsulate in Lemma 4.2.3 which sets down necessary and sufficient conditions for the existence of a preference profile that rationalizes a given collection of matchings. We first introduce some notation that will be necessary to describe Lemma 4.2.3. Consider the directed graph G_{ij} with M as vertex set and E_{ij} as edge-set where $(m, m') \in E_{ij}$ if $\mu_i(m) = \mu_j(m')$. Let $\mathbf{C}(\mu_i, \mu_j)$ denote the set of all connected components of G_{ij}. We will denote the analogous graph obtained by considering as vertex set W as H_{ij}. The following proposition now follows from our notation and establishes a correspondence between G_{ij} and H_{ij}.

Proposition 4.2.2 *(Echenique [Ech08]) C is a connected component of G_{ij} iff $\mu_i(C)$ is a connected component of H_{ij}. Furthermore, $\mu_i(C) = \mu_j(C)$.*

Echenique [Ech08] showed the following lemma to be true.

Lemma 4.2.3 *(Echenique [Ech08]) Let $\mathcal{H} = \{\mu_1, \ldots, \mu_\ell\}$ be rationalized by preference profile $\mathcal{P}$. Consider, for all $\mu_i, \mu_j \in \mathcal{H}$ the graph G_{ij} and all $C \in \mathbf{C}_{ij}$. Then,* *exactly one of (4.1) or (4.2) must be true:*

$$\begin{aligned} &m : \mu_i(m) > \mu_j(m) \textit{ for all } m \in C \textit{ and} \\ &\quad w : \mu_j(w) > \mu_i(w) \textit{ for all } w \in \mu_i(C) \qquad (4.1) \\ &m : \mu_i(m) < \mu_j(m) \textit{ for all } m \in C \textit{ and} \\ &\quad w : \mu_j(w) < \mu_i(w) \textit{ for all } w \in \mu_i(C) \qquad (4.2) \end{aligned}$$

Conversely, if $\mathcal{P}$ *is a preference profile such that for all* $\mu_i, \mu_j \in \mathcal{H}$ *and* $C \in \mathbf{C}(\mu_i, \mu_j)$*, exactly one of (4.1) or (4.2) holds, then* $\mathcal{P}$ *rationalizes* $\mathcal{H}$*.*

4.3 Hardness of rationalizability of matchings

We are given two sets M, W with $|M| = |W| = N$ and a set $\mathcal{H}$ of s matchings $\mu_1, \ldots, \mu_s : M \to W$. We show that the problem of determining whether there exists a preference profile that rationalizes $\mathcal{H}$ is NP-complete by reducing from NAE-3SAT.

4.3.1 Proof outline

We give below a broad overview of the reduction used to prove Lemma 4.3.2. Our objective is to start with a set of clauses and construct matchings corresponding to them in such a way that the all-equal assignment to variables in a clause would lead to a conflicting preference relation for some element in the set of matchings. With this in mind, we build 'matching gadgets' corresponding to a given Boolean formula.

By way of example, consider a single clause $C_1 = (x_1 \vee \bar{x}_2 \vee \bar{x}_3)$. We associate with each variable x_i, the elements $m_{1i} \in M_1, w_{1i}, w'_{1i} \in W_1$. We will subsequently pad M_1 with dummy elements to ensure that $|M_1| = |W_1|$. For such a clause, we look up Table 4.4 to construct 10 partial matchings $\mu_1, \ldots, \mu_{10}$ involving $M_1 = \{m_{1i} | i = 1, 2, 3\} \cup \{u_1\}$ and $W_1 = \{w_{1i}, w'_{1i} | i = 1, 2, 3\} \cup \{y_1, z_1\}$. Our encoding of the truth assignment to a variable x_i in clause C_1 will then correspond to m_{1i} preferring w'_{1i} over w_{1i}, i.e. $m_{1i} : w'_{1i} > w_{1i}$ iff $x_i = 1$. The claim below gives a flavor of how the entire reduction works.

Claim 4.3.1 *There exists a rationalizable preference profile for M_1, W_1 for the matchings described in Table 4.4 iff there exists a not-all-equal satisfying assignment for C_1.*

Proof. (Sketch) Suppose there exists a not-all-equal satisfiable assignment to C_1. Then, in order to show that the corresponding preference profile obtained is rationalizable, we will show that it satisfies the conditions in Lemma 4.2.3. We fix the preference for each m_{1i} between w_{1i} and w'_{1i} based on the assignment to x_i for $i = 1, 2, 3$. We set $m_{1i} : w'_{1i} > w_{1i}$ if $x_i = 1$ and $m_{1i} : w_{1i} > w'_{1i}$ otherwise. Note that since an assignment $(0, 1, 1)$ or $(1, 0, 0)$ to (x_1, x_2, x_3) is ruled out, the matchings in Table 4.4 ensure that there will be no "cycles" in the preference orders of m_{11}, m_{12}, m_{13}. Furthermore, an assignment to x_1, x_2, x_3 only fixes a preference order for all $m \in M_1$ and so we can fix a preference order for $w \in W_1$ so that there is no conflict in the preference orders for all m, w and that the conditions in Lemma 4.2.3 are satisfied.

The converse is immediate because for a rationalizable preference profile for $m \in M_1, w \in W_1$, Lemma 4.2.3 holds and hence an all-equal assignment to C_1 is not allowed. For instance, suppose (x_1, x_2, x_3) were assigned $(0, 1, 1)$ then using Lemma 4.2.3 to draw up all the preference relations we would obtain a conflict, i.e. $m_{11} : w_{12} > w'_{11}$ (applying Lemma 4.2.3 to $\mu_{11}, \ldots, \mu_{18}$) and $m_{11} : w_{12} < w'_{11}$ (applying Lemma 4.2.3 to μ_{19}, μ_{110}). Therefore, setting each of the x_i to the values obtained depending on the preference relation for m_{1i} between w_{1i} and w'_{1i} as delineated above is a not-all-equal satisfying assignment. An identical argument goes through when (x_1, x_2, x_3) is assigned each of the other 7 Boolean assignments. □

In a Boolean formula with m clauses, we repeat the exercise above but use disjoint sets M_ℓ, W_ℓ for each clause C_ℓ to avoid conflicting preference orders *across* clauses. This makes it necessary for us to enforce consistency between the preference relations for $m_{\ell i}$ and $w_{\ell i}, w'_{\ell i}$ for all $\ell = 1, \ldots, m$ and the assignment to x_i. To this end, we use additional matching gadgets from Table 4.5 and an auxiliary element v_i. Again applying Lemma 4.2.3, we see that for x_1 occurring in clauses C_1, C_2 say, we must have that $m_{11} : w'_{11} > w_{11} \iff m_{21} : w'_{21} > w_{21}$.

Note that in the manner our construction of matching gadgets is set up, it is necessary for our purposes to reduce from NAE-3SAT as opposed to 3SAT because, if an all-false assignment to a clause were to lead to a conflict in preference relation for some m, w, w', then by symmetry an all-true assignment would also lead to a contradictory preference relation.

4.3.2 Proof of Theorem 4.1.4.

The proof for Theorem 4.1.4 automatically follows from Lemma 4.3.2 which we formally state and prove below.

Lemma 4.3.2 *Let $\mathcal{Z}$ be an instance of* NAE-3SAT *over n variables $x_1, \ldots, x_n$ and m clauses $C_0, \ldots, C_{m-1}$. Then, there exists an instance $\mathcal{Z}'$ of $O(m)$ matchings between sets M and W, $|M| = |W| = O(m+n)$ such that there exists a rationalizable preference profile for all $m \in M, w \in W$ iff there exists a not-all-equal satisfiable assignment to $x_1, \ldots, x_n$. Furthermore, these matchings can be constructed in polynomial time.*

Proof. Consider a clause C_ℓ involving x_i, x_j, x_k. For C_ℓ, we consider the following sets of men and women: $\mathcal{M}_\ell = M_\ell \cup M'_\ell \cup B_\ell \cup U_\ell \cup V_\ell \cup T_\ell, \mathcal{W}_\ell = W_\ell \cup W'_\ell \cup G_\ell \cup Y_\ell \cup V'_\ell \cup Z_\ell$. Each of M_ℓ, W_ℓ comprises 3 men and women $\{m_{\ell i}, m_{\ell j}, m_{\ell k}\}$ and $\{w_{\ell i}, w_{\ell j}, w_{\ell k}\}$ respectively. The remaining sets are similarly constructed with each containing 3 elements. We then look up the corresponding table from Tables 4.1 through 4.4 and construct 10 partial matchings. In addition, we consider the singleton element v_ℓ which is used in matchings in Table 4.5. Note that each $m \in M_\ell$ corresponds to a variable occurring in C_ℓ. We will use $v_{\ell i}$ to match, say, $m_{\ell i} \in M_\ell$ for consistency in the assignment made to the variable x_i occurring in the first clause $C_r, r > \ell$. This gives rise to 4 matchings for each clause. Let $M = \cup_{\ell=1}^{m} \mathcal{M}_\ell, W = \cup_{\ell=1}^{m} \mathcal{W}_\ell$. Furthermore, we will denote $\mathcal{R}(C_\ell)$ to be the set of all matchings μ associated with clause C_ℓ as described above.

We now describe in detail the complete set of matchings between $\mathcal{M}_\ell$ and $\mathcal{W}_\ell$. The idea is to make sure that every element $m \in \mathcal{M}_\ell$ not already matched according to the tables is matched to some $w \in \mathcal{W}_\ell$. We use the following rules:

Table 4.1: For $C_\ell = (x_i + x_j + x_k), (\bar{x}_i + \bar{x}_j + \bar{x}_k)$:

$\mu_{\ell 1}$:	(m_i, w_i') (m_j, w_i)
$\mu_{\ell 2}$:	(m_i, w_i) (m_j, y_ℓ)
$\mu_{\ell 3}$:	(m_j, w_j') (m_k, w_j)
$\mu_{\ell 4}$:	(m_j, w_j) (m_k, z_ℓ)
$\mu_{\ell 5}$:	(m_k, w_k') (u_ℓ, w_k)
$\mu_{\ell 6}$:	(m_k, w_k) (u_ℓ, w_j)
$\mu_{\ell 7}$:	(u_ℓ, w_k) (m_i, w_j)
$\mu_{\ell 8}$:	(u_ℓ, w_j) (m_i, w_i')
$\mu_{\ell 9}$:	(m_k, z_ℓ) (m_i, w_j)
$\mu_{\ell 10}$:	(m_k, w_j) (m_i, w_i)

Table 4.2: $C_\ell = (x_i + x_j + \bar{x}_k), (\bar{x}_i + \bar{x}_j + x_k)$

$\mu_{\ell 1}$:	(m_i, w_i') (m_j, w_i)
$\mu_{\ell 2}$:	(m_i, w_i) (m_j, y_ℓ)
$\mu_{\ell 3}$:	(m_j, w_j') (m_k, w_j)
$\mu_{\ell 4}$:	(m_j, w_j) (m_k, z_ℓ)
$\mu_{\ell 5}$:	(m_k, w_k') (u_ℓ, w_k)
$\mu_{\ell 6}$:	(m_k, w_k) (u_ℓ, w_i')
$\mu_{\ell 7}$:	(u_ℓ, w_k) (m_i, w_i')
$\mu_{\ell 8}$:	(u_ℓ, w_i') (m_i, w_k)
$\mu_{\ell 9}$:	(m_k, z_ℓ) (m_i, w_j)
$\mu_{\ell 10}$:	(m_k, w_j) (m_i, w_i)

1. For $m_{\ell i}, \mu(m_{\ell i}) = \phi$, we match $m_{\ell i}$ to $g_{\ell i} \in G_\ell$ and $w_{\ell i}$ to $b_{\ell i} \in B_\ell$.

2. For $m_{\ell' i}, \ell' \neq \ell$ match $m_{\ell' i}$ to $g_{\ell' i} \in G_{\ell'}$ and $w_{\ell' i}$ to $b_{\ell' i} \in B_{\ell'}$. Match $m'_{\ell' i} \in \mathcal{M}'_{\ell'}$ to $w'_{\ell' i} \in \mathcal{W}'_{\ell'}$. Match $u_{\ell' i}$ to $y_{\ell' i}$, $v_{\ell' i}$ to $v'_{\ell' i}$ and $t_{\ell' i}$ to $z_{\ell' i}$.

3. Let $B'_\ell = \{b_{\ell k} | \mu(b_{\ell k}) = \phi\}, G'_\ell = \{g_{\ell r} | \mu(g_{\ell r}) = \phi\}$. Note that by the structure of our matching rules in Tables 4.1 through 4.4, $1 \leq |B'_\ell| \leq |G'_\ell| \leq 2$. For each $b_{\ell k} \in B_\ell$ we match to $g_{\ell r} \in G_\ell$ in ascending order of k, r.

4. If after (3), there is some $g_{\ell r} \in G_\ell, \mu(g_{\ell r}) = \phi$ match the first $m'_{\ell k} \in M'_\ell, \mu(m'_{\ell k}) = \phi$ to $g_{\ell r}$.

5. For all $m'_{\ell i} \in M'_\ell, \mu(m'_{\ell i}) = \phi$, match $m'_{\ell i}$ to the first $w'_{\ell j}, \mu(w'_{\ell j}) = \phi$. Similarly with $u_{\ell i}, t_{\ell i}$ and $z_{\ell i}, y_{\ell i}$.

Table 4.3: For $C_\ell = (x_i + \bar{x}_j + x_k), (\bar{x}_i + x_j + \bar{x}_k)$

$\mu_{\ell 1}$:	(m_i, w_i') (m_j, w_i)
$\mu_{\ell 2}$:	(m_i, w_i) (m_j, y_ℓ)
$\mu_{\ell 3}$:	(m_j, w_j') (m_k, w_j)
$\mu_{\ell 4}$:	(m_j, w_j) (m_k, z_ℓ)
$\mu_{\ell 5}$:	(m_k, w_k') (u_ℓ, w_k)
$\mu_{\ell 6}$:	(m_k, w_k) (u_ℓ, w_i)
$\mu_{\ell 7}$:	(u_ℓ, w_i) (m_i, w_k)
$\mu_{\ell 8}$:	(u_ℓ, w_k) (m_i, w_i)
$\mu_{\ell 9}$:	(m_k, z_ℓ) (m_i, w_j)
$\mu_{\ell 10}$:	(m_k, w_j) (m_i, w_i')

Table 4.4: For $C_\ell = (x_i + \bar{x}_j + \bar{x}_k), (\bar{x}_i + x_j + x_k)$:

$\mu_{\ell 1}$:	(m_i, w_i') (m_j, w_i)
$\mu_{\ell 2}$:	(m_i, w_i) (m_j, y_ℓ)
$\mu_{\ell 3}$:	(m_j, w_j') (m_k, w_j)
$\mu_{\ell 4}$:	(m_j, w_j) (m_k, z_ℓ)
$\mu_{\ell 5}$:	(m_k, w_k') (u_ℓ, w_k)
$\mu_{\ell 6}$:	(m_k, w_k) (u_ℓ, w_j)
$\mu_{\ell 7}$:	(u_ℓ, w_k) (m_i, w_j)
$\mu_{\ell 8}$:	(u_ℓ, w_j) (m_i, w_i)
$\mu_{\ell 9}$:	(m_k, z_ℓ) (m_i, w_j)
$\mu_{\ell 10}$:	(m_k, w_j) (m_i, w_i')

Table 4.5: Consistency matching for x_p occurring in clauses C_i, C_j:

μ_{p1}':	(m_{ip}, w_{ip}') (v_{ip}, w_{ip})
μ_{p2}':	(m_{ip}, w_{ip}) (v_{ip}, w_{jp}')
μ_{p3}':	(v_{ip}, w_{ip}) (m_{jp}, w_{jp}')
μ_{p4}':	(v_{ip}, w_{jp}') (m_{jp}, w_{jp})

6. Finally, for all $v_{\ell i}, \mu(v_{\ell i}) = \phi$ match $v_{\ell i}$ to $v'_{\ell i}$.

This specifies a complete matching $\mu : M \rightarrow W$. We have 10 such matchings for each clause, and at most 4 matchings for each variable in a clause to ensure consistency of assignment. Therefore, the total number of matchings is at most $22m$. The claims below demonstrate how our reduction works.

Claim 4.3.3 *Suppose there exists a not-all-equal satisfying assignment to an instance in m clauses $C_1, \ldots, C_m$ and n variables $x_1, \ldots, x_n$. Then, there exists a rationalizing preference profile $\mathcal{H}$ for the corresponding instance of matchings between M and W.*

Proof.

We construct a valid preference profile and hence will only consider active pairs. Note that by the structure of our reduction setting up the matchings, each $m \in \mathcal{M}_\ell, w \in \mathcal{W}_\ell$ has at most five elements that it is matched to. In order to satisfy conditions in Lemma 4.2.3 we will construct these preference orders so that for every active pair, one of (4.1) or (4.2) holds.

Note that the only connected components possible in any graph $G_{\mu_1\mu_2}$ constructed from matchings μ_1, μ_2 are either a cycle or a self-loop (when an element m is matched to the same w in both μ_1 and μ_2).

Consider the variable x_j and the set of matchings μ, μ' where $m_{\ell j}$ is matched to $w_{\ell j}$ and $w'_{\ell j}$ respectively. Note that by consequence of our construction of the matchings, for any element $m \in \mathcal{M}_\ell$ (resp. $w \in \mathcal{W}_\ell$) m (resp. w) occurs in a cycle in only those graphs involving at least one of μ, μ'. For all other such pairs of matchings, m occurs in a self-loop because m is connected to the same element in both such matchings. We look at the graph $G_{\mu\mu'}$.

For a cycle C in $G_{\mu\mu'}$ involving $m_{\ell j}$, the preference order is dictated by x_j's assignment: $x_j = 1 \Leftrightarrow m_{\ell j} : w'_{\ell j} > w_{\ell j}$. To satisfy Lemma 4.2.3, we will ensure in the preference order for all elements m occurring in C that $m : \mu'(m) > \mu(m)$ and similarly, for all elements w occurring in $\mu(C)$ in the graph $H_{\mu\mu'}$ that $w : \mu(w) > \mu'(w)$.

A preference order constructed as above will lead to a conflict in two possible ways. Firstly, there may exist a blocking pair (m, w) for some μ. Since our preference profile is a valid preference profile, there must exist some μ' such that $\mu'(m) = w$. Then, w is a (μ, μ')-pivot for m and $\mu(w) = m'$ say. But we ensured that for such a pair of matchings (μ, μ') either $m : w > \mu(m)$ and $w : \mu'(w) > m$ or $m : \mu(m) > w$ and $w : m > \mu'(w)$ and hence (m, w) cannot be a blocking pair.

Secondly, there may exist some $m \in \mathcal{M}_\ell$ (resp. $w \in \mathcal{W}_\ell$) for which some preference is contradictory, i.e. for instance when $m : w > w'$ and $m : w' > w$. For a not-all-equal satisfiable assignment to any clause C_ℓ containing x_j, it is easy to check given Tables 4.1 through 4.5 exhaustively amongst all w that m can be matched to that this is not the case. Furthermore, since each clause C_ℓ has a different set of $\mathcal{M}_\ell, \mathcal{W}_\ell$ from which the matchings are constructed, no contradictory preference order exists *across* any two clauses.

Finally, we remark since we wish to construct a valid preference profile, for all elements w for which (m, w) is not active, our preference order for m will have $m : w' > w$ for all w' such that (m, w') is active. This completes the proof of the claim.

□

Claim 4.3.4 *Let $\mathcal{H}$ be a rationalizing preference profile for the above instance of matchings. Then, the assignment*

$$x_i = \begin{cases} 1 & \forall \ell, m_{\ell i} : w'_{\ell i} > w_{\ell i} \\ 0 & \textit{otherwise.} \end{cases}$$

for all i is a not-all-equal satisfying assignment.

Proof. We first point out that the consistency matchings involving v_ℓ and $m_{\ell i}, i = 1, \ldots, n$ ensure that any rationalizing preference profile $\mathcal{H}$ must satisfy either $(m_{\ell i} : w'_{\ell i} > w_{\ell i})$ or $(m_{\ell i} : w_{\ell i} > w'_{\ell i})$ *for all* $\ell = 1, \ldots, m$. This means that a truth assignment to $x_1, \ldots, x_n$ will be consistent in all clauses $C_1, \ldots, C_m$.

Consider an arbitrary clause C_ℓ. We show that if $\mathcal{H}$ is a rationalizing preference profile, then it is not possible to have an all-equal assignment made to variables in some C_ℓ. Suppose, by way of contradiction that there existed such an assignment. Depending on the order and number of variables that appear negated in C_ℓ, we look up one of Tables 4.1 through 4.4. Then, as illustrated in Claim 4.3.1, we would obtain a conflict in preference orders for some m thereby giving a contradiction. $\square$

This completes the proof of the Lemma. $\square$

4.4 Hardness of approximate rationalizability of matchings

Our next step in exploring the computational aspects of rationalizability of matchings will be to look at the complexity of 'approximate' rationalizability.

4.4.1 Maximizing the number of rationalizable matchings

In the first setting, we wish to maximize the number of matchings that can be completely rationalized as stable by a preference profile. Theorem 4.1.5 states that this is hard to approximate within a constant factor.

Theorem 4.1.5 (restated). *There is a constant $\epsilon > 0$ for which it is NP-hard to approximate* MAX-STABLE-MATCHINGS *to within a factor of* $(1 - \epsilon)$.

To prove the theorem we show that it is NP-hard to rationalize any fixed set of matchings as captured in the lemma below.

Lemma 4.4.1 *Given a collection of matchings $\mathcal{H} = \{\mu_1, \ldots, \mu_k\}$ between M and W where k is some fixed constant, it is NP-hard to determine if there exists preferences for $m \in M, w \in W$ for which each of $\mu \in \mathcal{H}$ is a stable matching.*

In order to prove Lemma 4.4.1 we proceed as before by reducing from NAE-3SAT but we will use a special variant of the NAE-3SAT problem: NAE-3SAT(k') which has the property that every variable in the Boolean formula occurs in exactly k' clauses where $k' \geq 29$ is a constant. Lemma 4.4.2 below captures this reduction.

Lemma 4.4.2 *Let $\mathcal{Z}$ be an instance of* NAE-3SAT(k') *over n variables $x_1, \ldots, x_n$ and m clauses $C_0, \ldots, C_{m-1}$ where k' is some fixed constant. Then, there exists an instance $\mathcal{Z}'$ of $(10 + k')$ matchings between sets M and W, $|M| = |W| = O(m + n)$ such that there exists a rationalizable preference profile for all $m \in M, w \in W$ iff there exists a not-all-equal satisfiable assignment to $x_1, \ldots, x_n$. Furthermore, these matchings can be constructed in polynomial time.*

The following claim is key to proving Lemma 4.4.2.

Claim 4.4.3 *Let $M_1, \ldots, M_k; W_1, \ldots, W_k$ be respectively k disjoint sets of men and women and $\mu_1, \ldots, \mu_k$ a collection of matchings with $\mu_i : M_i \cup W_i \to M_i \cup W_i$. There exists a set of preference orders $\mathcal{P}$ for M_i, W_i for $i = 1, \ldots, k$ that rationalizes $\mu_1, \ldots, \mu_k$ iff there exists a set of preference orders $\mathcal{P}'$ that rationalizes μ, where $\mu : M_1 \cup \ldots M_k \cup W_1 \cup \ldots W_k \to M_1 \cup \ldots M_k \cup W_1 \cup \ldots W_k$ is the matching obtained by setting $\mu(m) = \mu_i(m)$ for all $m \in M_i$; $i = 1, \ldots, k$.*

Proof. Suppose there exists a preference profile $\mathcal{P}$ for $\mu_1, \ldots, \mu_k$. Then, for μ we construct $\mathcal{P}'$ by assigning for $m \in M_i$ as sub-ordering over W_i, the corresponding preference order for m in $\mathcal{P}$. We complete the preference order for m by ranking all other $w \in W_1 \cup \ldots W_{i-1} \cup W_{i+1} \ldots W_k$ below the sub-ordering for $w \in W_i$. Conversely, for every preference order corresponding to some $m \in M_i$ in a rationalizing preference profile $\mathcal{P}'$ for μ, we obtain a preference order in $\mathcal{P}$ for μ_i by restricting the order over only W_i. $\square$

Proof. (Of Lemma 4.4.2) The proof essentially follows the same technique as that of Lemma 4.3.2 except that we need to be careful in our reduction to maintain the number of matchings at a constant. To this end, we revisit the matchings shown in Tables 4.1-4.4. Note that in our previous reduction, we required disjoint sets of M_ℓ, W_ℓ to correspond with each clause C_ℓ. Furthermore, each clause corresponds to 10 matchings (not counting those required to ensure consistency). The following proposition allows us to maintain the overall number of matchings at a constant by merging each of the 10 matchings *across* all the clauses $C_0, \ldots, C_{m-1}$.

Claim 4.4.3 tells us that we can merge all $\mu_{11}, \mu_{21}, \ldots, \mu_{m1}$ into one single matching μ'_1. We repeat this for all matchings $\mu_{\ell i}, \ell = 1, \ldots, m; i = 2, \ldots, 10$ to obtain 10 new matchings $\mu'_1, \ldots, \mu'_{10}$.

We now focus on the consistency matchings. We will exploit the fact that each variable x_i in Z occurs in at most k' clauses. Therefore, each x_i will correspond to at most $2k'$ matchings. Appealing once again to Claim 4.4.3, we can merge each of these matchings into a collection of $2k'$ matchings because each x_i is associated to a disjoint set of 'linking' elements $v_{i1}, \ldots, v_{ik'}$. Claims 4.3.3 and 4.3.4 go through with their proofs unchanged. This completes the proof for Lemma 4.4.2, and consequently Lemma 4.4.1. □

From Lemma 4.4.1 it follows that it is NP-hard to approximate MAX-STABLE-MATCHINGS for $\mathcal{H}$ to within a factor of $(1 - \epsilon)$ where $\epsilon = 1/(k+1)$.

Note that given a collection $\mathcal{H}$ of any two matchings, it is trivial to construct a (valid) preference profile that rationalizes $\mathcal{H}$ by arbitrarily assigning a preference for each element in M matched to W in one matching over the other and correspondingly assigning the reverse preference for elements in W.

4.4.2 Maximizing the number of non-blocking pairs

We look at the MAX-STABILITY problem. The motivation in considering this problem as a notion of approximate rationalizability is that we are now striving to ensure that given a collection of matchings between two sets M and W, there are optimally many different pairs (m, w) for which at least one of them is happy with their current partner and has no incentive to be matched to the other.

As a preliminary exercise, we ask how well would a simple randomized assignment of preferences to $m \in M, w \in W$ perform. It turns out that this would achieve a 3/4-approximate solution. This is the content of Theorem 4.1.8.

Theorem 4.1.8 (restated). *There is a deterministic, polynomial-time approximation algorithm for* MAX-STABILITY *that achieves an approximation factor of 3/4.*

Proof. Note that since we are only interested in finding valid preference profiles, we will automatically accord the least preference for all $w \in W$ that $m \in M$ is not matched with in any of the matchings. Subsequently, each such (m, w) is by default a stable pair and is excluded from our estimation. Let P denote the total number of all remaining pairs for which no preference has been allocated as yet.

We start with an equivalent formulation of the problem. We are given sets M, W such that $|M| = |W| = n$, and a collection $\mathcal{H}$ of ℓ matchings $\mu_1, \ldots, \mu_\ell$. For some m (similarly, w), we associate a 'rank' function $r_m : W \to [n]$ (similarly, $r_w : M \to [n]$) which would completely describe m's (similarly w's) preference order with $r_m(w) < r_m(w')$ implying that $m : w > w'$. A pair (m, w) then is stable for some μ if either $r_m(w) > r_m(\mu(m))$ or $r_w(m) > r_w(\mu(w))$ is true. Let $S = \{(m, w, \mu) | (m, w) \text{ is a stable pair for } \mu\}$. Our objective then is to maximize $|S|$.

Consider the following scheme: for each $m \in M, w \in W$ we construct the rank function by assigning ranks uniformly at random to all $w' \in W$ and $m' \in W$ respectively. The probability that a pair (m, w) is stable for μ is $3/4$ and hence, the expected number of stable pairs denoted by $E[|S|]$ is $3P/4$. Furthermore, we can easily derandomize this scheme by the method of conditional expectations. At every step when a value is to be assigned to $r_m(w)$, we can efficiently calculate the conditional expectation of the number of non-blocking pairs given the previous values assigned to all $w \in W$ and all other rank functions $r_{m'}$ (there are only a polynomial number of $w \in W$ and rank functions $r_{m'}$ to consider) and fix $r_m(w)$ to be the value that maximizes the conditional expectation. □

It suffices to mention here that a simple randomized preference order for all $m \in M, w \in W$ achieves the 3/4-approximation factor in expectation and can subsequently be derandomized. How much better can we do than just a random assignment of preferences? Theorem 4.1.7 tells us that a constant-factor approximation is all we can hope for.

Theorem 4.1.7 (restated). *There is a constant $\epsilon > 0$ for which it is NP-hard to approximate* MAX-STABILITY *to within a factor of* $(1 - \epsilon)$.

To prove the theorem, we once again construct matchings corresponding to each clause in MAX-NAE-3SAT instance Z. Recall that in proving Lemma 4.3.2 we needed to construct auxiliary matchings to ensure consistency of assignment to the variables in accordance with the preferences of the corresponding elements in the matchings. To prove hardness of approximation, we will need to establish a gap-preserving reduction by boosting the *robustness* of these consistency gadgets. We do so by augmenting the number of matchings corresponding to the consistency and argue subsequently that if there exists a preference profile that achieves at least a $(1 - \epsilon')$ fraction of stable pairs, then there exists an assignment that would satisfy at least a $(1 - \epsilon)$ fraction of the clauses. Theorem 4.1.7 then follows from the following Lemma:

Lemma 4.4.4 *Let Z be an instance of* MAX-NAE-3SAT *over n variables $x_1, \ldots, x_n$ and m clauses $C_0, \ldots, C_{m-1}$ where k' is some fixed constant. Then, there exists a $\epsilon' < 1$ and a polynomial time reduction to an instance Z' of* MAX-STABILITY *of matchings between sets M and W, $|M| = |W| = O(m)$ such that the following is true:*

$$opt(Z) = 1 \implies opt(Z') = 1 \tag{4.3}$$

$$opt(Z) < 1 - \epsilon \implies opt(Z') < 1 - \epsilon' \tag{4.4}$$

Proof. The reduction is similar to what we used to prove Lemma 4.3.2. We set up matchings corresponding to the clauses $C_0, \ldots, C_{m-1}$ as before, but now we need to work harder to boost the robustness of the consistency gadgets. Previously, we used Table 4.5 to construct additional matchings using auxiliary elements to 'link' different copies of $m_{ji}; j = 1, \ldots, m$ corresponding to a single variable x_i. It will help to conceptualize this as a graph.

For a variable x_i which occurs in some t clauses $C_{j_1}, \ldots, C_{j_t}$, we associate elements from $M, m_{j_1 i}, \ldots, m_{j_t i}$ and define the consistency graph for x_i, G_i to comprise vertex set $V_i = \{m_{j_1 i}, \ldots, m_{j_t i}\}$. An edge exists between any two vertices $(m_{j_p i}, m_{j_q i})$ if they are 'linked' together by an auxiliary element.

Then, the consistency matchings described above in Lemma 4.3.2 correspond to a path in G_i. In order to boost the robustness, we will now replace the path in G_i by a constant-degree expander graph on t vertices. We make use of the edge expansion notion to define an expander graph: an (n, d, λ) expander graph is a d-regular graph on n vertices with the property that $|\partial(Y)|/|Y| \geq d(1-\lambda)/2$ where $Y \subseteq V_i, |Y| \leq |V_i|/2$, $\partial(Y)$ is the set of all edges with exactly one end-point in Y and λ is the spectral expansion parameter of the graph. In particular, the following lemma will be useful (the proof can be found in [DH05]):

Lemma 4.4.5 *For a (t, d, λ) expander graph G and all $\delta \leq (1-\lambda)/12$, upon removing $2\delta t$ vertices from G, there exists a connected component of size at least*

$$\left(1 - \frac{4\delta}{1-\lambda}\right) t$$

Note that the total number of occurrences of variables in all the clauses is at most $3m$, and further, that in each clause a variable corresponds to an element m matched to at most an $O(1)$ elements in W. Therefore, the total number of pairs for which a matching exists is at most $O(m)$. Since we only consider valid preference profiles, this means that the number of active pairs under consideration is also $O(m)$ say. Additionally, the total number of auxiliary elements required to construct the expander graphs in the consistency gadgets is also at most $O(m)$ and hence $|M| = O(m)$.

Claim 4.3.3 from earlier goes through unchanged since our reduction is unchanged in how a satisfying assignment will correspond to a rationalizing preference profile (and hence, all stable pairs). It remains to show that (4.4) holds.

We shall show that if there is a valid preference profile for Z' such that there are at most an ϵ' fraction of blocking pairs, then there exists an assignment that fails to satisfy at most ϵ fraction of clauses in Z.

Suppose that there is a valid preference profile that allows at most $\epsilon' m$ blocking pairs. Note that if a pair (m, w) is a blocking pair for some matching μ, then Lemma 4.2.3 breaks down for μ. Since each matching in Z' can be identified with a clause, a blocking pair could result in the clause being unsatisfied.

For a blocking pair (m, w) for some matching μ in our reduction, we evaluate how many clauses are affected. Suppose μ corresponds to one of the matchings for clause C_ℓ. If $m \in M_\ell$ then m must be associated with some variable x_i occurring in C_ℓ, and we will label C_ℓ unsatisfiable. Otherwise, (m, w) has no effect on the satisfiability of C_ℓ.

Suppose μ corresponds to a matching constructed to ensure consistency. If $m \in M_\ell$ for some clause C_ℓ and x_i, then we delete the node $m_{\ell i}$ in G_i and as before label C_ℓ as unsatisfiable. However, now we also need to argue that (m, w) does not cause too many other clauses to be labeled unsatisfiable.

From Lemma 4.4.5 we know that deleting at most a constant fraction of vertices from G_i will result in a connected component of size at least $(1 - \frac{4\delta}{(1-\lambda)})t$. Taking the aggregate for every variable x_i and after deleting at most $\epsilon' m$ vertices from all the consistency graphs G_i together, the total sum of the largest connected components amongst all G_i will be some $(1 - \epsilon)m$ where ϵ is determined by ϵ', λ and the total number of occurrences of all variables in all the clauses. Therefore, at most ϵm of these occurrences in clauses will be discarded and the corresponding ϵm clauses labeled as unsatisfiable.

□

MAX-NAE-3SAT is known to be APX-complete [PY91] and not approximable to within 0.917 [Zwi98].

4.5 Rationalizing one-many matchings

For the generalized instance of rationalizing one-many matchings, the problem seems considerably harder. To begin with, since the preference order for any $m \in M$ is over

2^W, given sets of length n, expressing the preference order alone takes exponential time.

However, for a specific restriction of the problem where we allow $m \in M$ to be matched with at most ℓ elements $w \in W$ the problem is in NP and, in fact, NP-complete.

Theorem 4.1.11 (restated). *For every fixed ℓ, given a collection of one-ℓ matchings $\mathcal{H}$ on the sets M and W, it is NP-complete to determine if there exists a preference profile $\mathcal{P}$ such that every $\mu \in \mathcal{H}$ is stable* with respect to $\mathcal{P}$.*

Proof. Let Z be an instance of a collection $\mathcal{H}_Z = \{\mu_1, \ldots, \mu_r\}$ of one-to-one matchings between M_Z and W_Z. We need to construct an instance Z' of many-to-one matchings such that a stable preference profile for Z exists iff a stable* preference profile exists for Z'. Indeed, we show that $Z' = Z$ is itself such an instance. In other words, $M_{Z'} = M_Z; W_{Z'} = W_Z; \mathcal{H}_{Z'} = \mathcal{H}_Z$.

Claim 4.5.1 *Suppose there exists a stable preference profile for Z, then there exists a stable* preference profile for Z'.*

Proof. A stable preference profile for Z gives preference orders for all $m \in M_Z$ ($w \in W_Z$) over $w \in W_Z$ ($m \in M_Z$). Consider the following preference profile for Z': for each $m \in M_{Z'}$, we construct a preference order over all $B \subseteq W_{Z'}$ where $|B| \leq \ell$ as follows: we look at all singleton sets $B \subseteq W_{Z'}$ and affix preferences identical to the preference order for $m \in M_Z$ over $w \in W_Z$. Therefore, for $m \in Z'$, $m : w_1 > w_2 \Leftrightarrow$ for $m \in Z$, $m : \{w_1\} > \{w_2\}$. We fix preference for all other subsets $B \subseteq W_{Z'}$ below the singleton sets and in some consistent order (say lexicographic). It is not hard to see that by virtue of our construction, the preference profile outlined above for $m \in M_{Z'}$ is stable* if the corresponding preference profile for $m \in M_Z$ is stable. □

Claim 4.5.2 *If there exists a stable* preference profile for Z', then there exists a stable preference profile for Z.*

Proof. We construct the preference order for $m \in M_Z$ as follows: we look at the preference order of the corresponding $m \in M_{Z'}$ and extract the partial order comprising m's preference for all $\{w\} \subseteq W_{Z'}$. Suppose that there is a blocking pair (m', w') in Z. Then, this would imply that $(m', \{w'\})$ is a blocking set in Z' which is a contradiction. □

Claims 4.5.1 and 4.5.2 give us Theorem 4.1.11. □

Given how the two problems of rationalizability are so naturally related, it is not surprising then to observe that the one-ℓ matchings problem would have a similar hardness of approximation performance with respect to both analogs of the optimization problem in the case of the one-one matchings.

Theorem 4.1.12 (restated). *There is a constant $\epsilon > 0$ for which it is NP-hard to approximate* MAX-STABLE-ONE-ℓ-MATCHINGS *to within a factor of* $(1 - \epsilon)$.

The proof follows immediately by combining Lemma 4.4.1 and Theorem 4.1.11.

Theorem 4.1.13 (restated). *There is a constant $\epsilon > 0$ for which it is NP-hard to approximate* MAX-ONE-ℓ-STABILITY *to within a factor of* $(1 - \epsilon)$.

The theorem follows from the lemma below.

Lemma 4.5.3 *Let Z be an instance of the* MAX-STABILITY *problem for a collection of matchings. Then, there exists an $\epsilon < 1$ and a polynomial-time reduction to an instance Z' of* MAX-ONE-ℓ-STABILITY *of one-ℓ matchings such that the following is true:*

$$\begin{aligned} opt(Z) = 1 &\implies opt(Z') = 1 \\ opt(Z) < 1 - \epsilon &\implies opt(Z') < 1 - \epsilon \end{aligned}$$

Proof. As in proving Theorem 4.1.11, we will use exactly Z as our instance for the MAX-ONE-ℓ-STABILITY problem. This means automatically that

$$opt(Z) = 1 \implies opt(Z') = 1$$

Note that we are looking at valid preference profiles. Since Z' matches all $m \in M$ exclusively to singleton elements in 2^W, these singleton elements are assigned preference over subsets $B \subseteq W, |B| \geq 2$. Hence, our estimate of the optimal number of stable sets will only include the pairs $(m, \{w\})$ which is the same as the optimal number of stable pairs in Z.

Suppose there exists a valid preference profile for Z' for which there are at most ϵ fraction of blocking sets. Then, each of these blocking sets also corresponds exactly to a blocking pair in Z and there cannot be any blocking pair in Z that does not have an equivalent blocking set in Z' for the same reasons as mentioned above in proving Theorem 4.1.11. Therefore, there are at most ϵ fraction of blocking pairs in Z hence giving us a contradiction and completing the proof to the lemma and the theorem.

□

4.6 Conclusions and Future work

There are many interesting opportunities for extensions to our work on the rationalization problem for matchings. It would be interesting to tighten the constant factor in Lemma 4.4.1: is it hard even to rationalize three matchings? It would also be satisfying to tighten the hardness of approximation result in Theorem 4.1.7. We can additionally look at other (restricted) variants of the matchings problem such as many-many matchings and pose the related complexity questions.

Chapter 5

Rationalizability of network formation games

5.1 Introduction

The rationale for understanding how social networks form is motivated by the prevalence of such networks in society. These are groups that can form and disband on an ad-hoc basis, or over a sustained time period, with the broad purpose of information exchange. Such an exchange of information can be unidirectional via a process of diffusion and dissemination, where information is propagated from a source (e.g., adoption of a new product), or bidirectional with everybody communicating with each other (e.g., online communities such as Facebook and MySpace). One way to study the structure of these networks is to make assumptions on the growth processes underlying how the networks are formed. Not surprisingly, this relies crucially on literature covering random graphs and spectral graph theory. The interested reader is referred to Jackson [Jac08], and Wasserman and Faust [WF94] for an extensive overview and coverage of social and economic networks, and methods used to analyze them.

5.1.1 Algorithmic game theoretic perspective

Network formation games have also been extensively studied from within algorithmic game theory in connection with the broader goal for understanding the efficiency of equilibria. There are two contrasting frameworks that have been investigated. In the first setup, called the global connection game and proposed by Anshelevich et al. [ADK+08], there are k agents each of whom has a corresponding source-sink pair that they wish to connect. To this end, they can build edges anywhere in the network. Each edge however has a cost associated in order for it to get built. With a cost-sharing mechanism in place, this price is split up among the agents whose strategy sets include the edge. For example, in the case of the Shapley or fair cost-sharing mechanism, the price is evenly divvied up among the associated agents.

In the other variant proposed by Fabrikant et al. [FLM+03], known as the local connection game, the nature of network formation is more organic. Agents wish to form connections with each other and in order to do so, they choose as their strategies neighbors that they unilaterally build edges to. The ambiguity that can arise here is when both agents choose to build edges unilaterally to one another. In such a case, ties are assumed to be broken arbitrarily.

5.1.2 The network formation model and two rationalization problems

In this chapter, we consider the rationalization problem for network formation games. We study two variants of a seminal network formation model due to Jackson and Wolinsky [JW96] that describes how selfish parties (the vertices) produce a graph by making individual decisions to form or not form incident edges. The model is equipped with a notion of stability (or equilibrium), and we observe a set of "snapshots" of graphs that are assumed to be stable. From this we would like to infer some unobserved data about the system: in one variant we are interested in edge prices; in the other, we are interested in how much each vertex values short paths to each other vertex. Both variants resemble the settings in which the rationalization problem can be solved using linear programming (in the sense that the equilibrium conditions can be expressed as linear inequalities), and yet they have a combinatorial component because the participants' total utility depends on the length of various shortest paths in the network.

We show an interesting contrast: inferring "per-edge" quantities (i.e., prices) is easy, while inferring "end-to-end" quantities (i.e., the value each vertex u assigns to having a short path to each other vertex v) is hard. In the latter case we show a tight $(1/2 + \delta)$ inapproximability result (and this is our most technically significant contribution). The 1/2 ratio implies that the trivial approximation algorithm that sets everyone's valuations to infinity (which rationalizes all the edges present in the input graphs) or to zero (which rationalizes all the non-edges present in the input graphs) is the best possible assuming P $\neq$ NP.

In the Jackson-Wolinsky model, there are n vertices, and each pair (u, v) ("potential edge") has an associated price and a distance. The strategy of each player (vertex) is a subset of incident edges which should be thought of as the edges it wants to form. The graph that arises given the vertex strategies has an edge (u, v) iff at least it appears in u's or v's subset. The utility that accrues to each vertex v depends on two additional features of the model: (1) a non-increasing function f from distances to the non-negative reals (think of $f(d)$ as representing the value of having a connection of length d), and (2) "intrinsic values" of vertex u by v for each $u \neq v$. The utility realized by vertex v is then the *aggregate distance* minus the price of the edges in v's subset, where the *aggregate distance* is the sum over vertices u of v's intrinsic value of u times f applied to the shortest path length in G to each u.

We remark that the equilibrium concept here is *not* a Nash equilibrium[1], but rather a simpler notion of pairwise stability; the vertex strategies are *stable* if (1) for each edge (u, v) in G, both v's and u's marginal utility of forming edge (u, v) is non-negative, and (2) for each non-edge (u, v) in G either u's or v's marginal utility of forming edge (u, v) is non-positive.

We consider two rationalization problems arising in this network formation model. In the first, which we call STABLE-PRICES, we are trying to infer edge prices, and we assume the other data (distances, the function f, and the pairwise "intrinsic values")

[1] This is because we only consider unilateral deviations of a player to an *adjacent* strategy – one in which a single edge has been added or removed – instead of to *any* alternative strategy.

are fixed or given. Specifically, we are given a collection of distance-weighted graphs $G_1, G_2, \ldots, G_m$ on the same underlying vertex set, that arise from equilibrium play. In addition, we are given the function f (as a circuit computing it), and the pairwise intrinsic values (which are the same across the different graphs). We do not observe the (potential) edge prices (which are the same across the different graphs). We are interested in determining edge prices that *rationalize* $G_1, G_2, \ldots, G_m$; i.e., for which there exist vertex strategies for each i that give rise to the graph G_i, and that constitute an equilibrium in the above sense.

In the second rationalization problem under consideration, which we call STABLE-VALUES, we are trying to infer the pairwise "intrinsic values," and we assume the other data (latencies, the function f, and the edge prices) are given. Specifically, as above, we are given a collection of distance-weighted graphs $G_1, G_2, \ldots, G_m$ on the same underlying vertex set, that arise from equilibrium play. In addition, we are given the function f (as a circuit[2]), and the edge prices (which are the same across the different graphs); we do not observe the pairwise "intrinsic values" (which are the same across the different graphs). We are interested in determining pairwise intrinsic values that *rationalize* $G_1, G_2, \ldots, G_m$; i.e., for which there exist vertex strategies for each i that give rise to the graph G_i, and that constitute an equilibrium.

For concreteness, we briefly describe an example scenario in which this rationalization problem naturally arises. Social networks are formed among groups of people who ascribe a certain value ("friendship") to one another but establish connections with only those that they perceive to be most intrinsically valuable to them. If, for instance, everybody in the group was in close physical proximity to one another (they all went to the same high school or college) then the price of connecting to any one person is insignificant compared to the value derived in return, no matter how small that may be. This would result in a clique as a stable network. However, once this group becomes geographically spread out, the network formed in equilibrium can become sparser, such as a star network, where all connections are made to a single

[2]This permits un-natural functions f, but note that all of our reductions produce instances with very simple piecewise linear and non-increasing f that one can easily envision occurring in the real world.

person since the cost of building mutual connections outweighs the utility gained. This illustrates that (when holding the intrinsic value people in such a group have for one another to be invariant) temporal and spatial dynamics affect the manner of how social networks coalesce and stabilize. While prices and distances might be readily observable, the intrinsic value each individual has for each other individual is generally private. The problem STABLE-VALUES in this chapter asks to infer these values given (say) a series of snapshots taken over time of a single social network of individuals.

5.1.3 Hardness of approximation

We also consider an optimization version of STABLE-VALUES. In it, we are seeking pairwise "intrinsic values" that maximize the number of stable edges/non-edges across all m input graphs *among active pairs*. We deem a pair (u, v) *active* unless (1) it is an edge in all of the input graphs, with price zero (which means effectively that edge (u, v) is present and fixed no matter how the other quantities are varied) or (2) it is a non-edge in all of the input graphs, with price infinity (which means effectively that edge (u, v) is permanently absent regardless of the other relevant quantities). Non-active pairs are "part of the landscape" and intuitively do not contribute to the stability of the system. After this consideration, our optimization problem is to infer intrinsic values with the maximum explanatory power (and note that edges/non-edges are counted separately for each graph in which they appear).

5.1.4 Rationalization problems and I-SAT

As mentioned above, we show that STABLE-PRICES is easy, while STABLE-VALUES is hard. Our hardness result is based on a reduction from a variant of the Inequality Satisfiability problem (abbreviated as I-SAT) introduced recently by Hochbaum and Moreno-Centeno [HMC08]. An instance of I-SAT is a conjunction of *inequality-clauses*, where each inequality-clause is a disjunction of linear inequalities over n real variables $x_1, x_2, \ldots, x_n$. The instance is a "yes" instance iff there exists an assignment of real values to the variables simultaneously satisfying all of the inequality-clauses. Hochbaum and Moreno-Centeno showed by a simple reduction from 3-SAT that this class of problems is NP-complete even in the case when each inequality-clause is a disjunction of only two inequalities.

The variant of I-SAT that we need for our reduction satisfies two additional constraints: (1) all of the coefficients are non-negative (and we a seeking a solution only in the non-negative reals), and (2) there is a partition of the variables into two sets S, T such that every inequality-clause is either the disjunction of two $\leq$ inequalities, one supported in S and one supported in T, or a conjunction of two $\geq$ inequalities, one supported in S and one in T. We call this variant I-SAT*.

To achieve our main hardness results, we show that I-SAT* is NP-complete, and that the optimization version (maximize the number of inequality-clauses simultaneously satisfied) is NP-hard to approximate to within $(1/2 + \delta)$. Note that, just as it is trivial to achieve approximation ratio 1/2 in the rationalization problem to which we reduce, it is trivial to achieve approximation ratio 1/2 here by either setting all variables to zero (satisfying all the inequality-clauses of the first type) or setting all variables sufficiently large (satisfying all the inequality-clauses of the second type).

The ease of translating between these problems brings us to an important observation. Not only is I-SAT useful as a starting point for reductions involving the hard rationalization problem in this chapter, but we contend it is the abstract computational problem that captures rationalization problems more generally. It is common for the "stability conditions" arising in a rationalization problem to be expressible by a finite Boolean formula whose inputs are inequalities in the (real) quantities being inferred. This is true, e.g., for the bipartite matchings problem studied in [KU08] (the quantities being inferred are the values each left node has for each right node, and the familiar stability condition for stable matchings is expressible as the disjunction of two inequalities involving these quantities), and for the rationalization problems studied here, and those mentioned in the introduction. Even the positivity constraint we add arises naturally in many such settings, as utilities, prices, etc. are often assumed to be non-negative.

Thus we expect that a more complete understanding of the approximability of I-SAT (which to our knowledge has not been studied prior to this chapter) can serve as a useful starting point for understanding the approximability of rationalization problems more generally, and we view this as an important contribution of this chapter.

5.1.5 Hardness of approximating I-SAT* via Max-Lin$_{\mathbb{R}_+}$

An instance of the Max-Lin$_{\mathbb{R}}$ problem is a collection of linear equations over the reals with the solution being an assignment that maximizes the number of equations satisfied. For the general I-SAT problem, there is an easy reduction from Max-Lin$_{\mathbb{R}}$. Namely, for each equation $\sum_i a_i x_i = b$, we produce the pair of I-SAT clauses $\sum_i a_i x_i \leq b$ and $\sum_i a_i x_i \geq b$.

Max-Lin$_{\mathbb{R}}$ was (only recently) shown to have a PCP system with $(1-\epsilon)$ completeness and γ soundness [GR07] (with ϵ, γ close to 0), which gives rise to $(1/2+\delta)$-inapproximability for the general I-SAT problem via this reduction (although, the non-perfect completeness means this gap is between unsatisfiable instances, which is a minor drawback).

We need a similar hardness result for our variant, I-SAT*, which crucially entails a positivity constraint. In the [GR07] inapproximability result (and similar inapproximability results using the basic framework of Hastad [Hås01]), the equations all have the form $x_i + x_j - x_k = 0$ since they arise from linearity tests performed by the verifier in the PCP system. Thus, they are not suitable for proving inapproximability for I-SAT*. Simple transformations like translating the origin do not work, and the natural idea of introducing new variables x_i' and the constraints $x_i + x_i' = 0$ (and using x_i' in place of $-x_i$ to remove the negative coefficients) does not preserve the inapproximability.

It is also important to note that while Hastad's inapproximability results for Max-Lin$_{\mathbb{F}_p}$ can be easily transformed into similar inapproximability results for Max-Lin$_{\mathbb{Z}}$, this transformation introduces large coefficients (of magnitude p), which prohibit the clever trick in [GR07] that is used to argue that the inapproximability carries over to the reals.

So our hands are somewhat tied: to obtain the $(1/2 + \delta)$ inapproximability for I-SAT*, we really need an exact analog of [GR07], but one that produces equations with positive coefficients. In Section 5.5, we give such a result for $\textsc{Max-Lin}_{\mathbb{R}_+}$, showing that it is NP-hard to distinguish between an instance with a $(1 - \epsilon)$ fraction satisfiable assignment and one with at most δ fraction satisfiable, and in turn a $(1/2+\delta)$ inapproximability result for I-SAT*. Doing so requires more than a superficial modification of the proof in [GR07]. In stating our results, we abstract properties of the distribution used for the verifier's queries that are sufficient for the general proof strategy of [GR07] to work, and then utilize a different distribution (and some minor changes in the Fourier analysis) to eventually produce equations with all coefficients $+1$. This result is our most significant technical contribution.

5.1.6 Related work

Rationalizability has been well-studied under the domain of revealed preference theory and social choice theory by economists [Sam48, Var82, Spr00, FST04, BV06, Var06, Ech08]. Traditionally, the questions have been connected with characterizing the implications of various solution concepts to games and market settings, and whether these implications can be tested based on data obtained from consumer choices.

In connection with studying network formation games, while the question of understanding the properties and limitations of equilibria is not new [JW96, DM97, JW01, DJ03, FLM+03], to the best of our knowledge there is no previous work done with respect to either the rationalizability question for these games in general, or the Jackson-Wolinsky model of network formation in particular.

5.1.7 Outline

In Section 5.3 we formally define STABLE-PRICES, and observe that it is easy (and even the optimization variant is easy to solve exactly). In Section 5.4 we define

STABLE-VALUES and give a reduction from I-SAT* to it. We then show that I-SAT* is NP-complete (this is not subsumed by our eventual inapproximability result, since this reduction has perfect completeness). In Section 5.5 we state an approximation preserving reduction from MAX-LIN$_{\mathbb{R}_+}$ to I-SAT*, and we then describe the PCP system (based on a non-trivial modification of [GR07]) that implies ϵ inapproximability for it. This yields the tight $(1/2 + \delta)$ inapproximability for STABLE-VALUES.

5.2 Jackson-Wolinsky model for network formation games

We describe formally the seminal model for network formation games as formulated by Jackson and Wolinsky [JW96]. The model comprises:

- n agents V
- pairwise distance function $d : V \times V \to \mathbb{R}_+$
- pairwise intrinsic value function $w : V \times V \to \mathbb{R}_+$
- a function $f : \mathbb{R}_+ \to \mathbb{R}$ that defines the contribution of a path of length d
- price profile $c : V \times V \to \mathbb{R}_+$

The strategy for agent $i \in V$ is a set $\Gamma(i) \subseteq V$ of agents that she wishes to build edges to. Given a strategy for each agent i, the graph formed is $G = \cup_{i,j\in\Gamma(i)}(i, j)$. For a graph G and any two $i, j \in V$, we define $d_G(i, j)$ to be the distance of the shortest path $P(i, j)$ from i to j given by $\sum_{(u,v)\in P(i,j)} d(u, v)$. The utility agent i derives from the graph formed, $u_i(G)$ is given by:

$$u_i(G) = \sum_{j\in V} f(d_G(i, j))w(i, j) - \sum_{k\in\Gamma(i)} c(i, k)$$

Definition 5.2.1 *For a collection of n agents V, a set of strategies* $(\Gamma(1), \ldots, \Gamma(n))$, *is said to be pairwise stable with respect to d, w, f, c if in the ensuing graph G that is formed:*

1. for all $i, j \in V$ *such that* $(i, j) \in G$,

$$u_i(G) \geq u_i(G - (i,j)) \text{ and } u_j(G) \geq u_j(G - (i,j))$$

2. for all $i, j \in V$ *such that* $(i, j) \notin G$,

$$u_i(G) \geq u_i(G + (i,j)) \text{ or } u_j(G) \geq u_j(G + (i,j))$$

For the sake of convenience, we have an equivalent definition for pairwise stability with respect to the graph G formed, that we will use for the rest of the chapter.

Definition 5.2.2 *A graph G formed from a set of strategies* $(\Gamma(1), \dots, \Gamma(n))$ *is said to be pairwise stable with respect to* d, w, f, c *if* $(\Gamma(1), \dots, \Gamma(n))$ *is pairwise stable with respect to* d, w, f, c.

Note that we can use this abuse of notation and claim that a graph is pairwise stable because given a set of strategies $(\Gamma(1), \dots, \Gamma(n))$ that is pairwise stable, we can construct the corresponding graph where we assign edges as being built by exactly one of the end-points (breaking ties arbitrarily when an edge belongs to the strategies of both).

5.3 Finding stable prices when intrinsic values are known

In the first rationalization problem that we will call STABLE-PRICES, we consider a scenario where the intrinsic values are known but the edge-prices are not. We are given a collection of undirected graphs $G_1, \dots, G_m$ all of which are formed over a common set of vertices V. In addition, we are given the pairwise distance functions for each G_i, $d_i : V \times V \to \mathbb{R}_+$. The rationalizability question entails inferring the prices that players in each of the network graphs would have to pay given that the graphs are in pairwise equilibrium.

Problem 1. STABLE-PRICES

Given: Collection of graphs $G_1, \ldots, G_m$ over common set of vertices V
Pairwise distance functions $d_i : V \times V \to \mathbb{R}_+$ for $i = 1, \ldots, m$
Intrinsic value function $w : V \times V \to \mathbb{R}_+$
Path distance contribution function $f : \mathbb{R}_+ \to \mathbb{R}$.

Find: Price profile c that rationalizes $G_1, \ldots, G_m$, i.e. for which G_i is stable w.r.t. d_i, w, f, c for all i, if it exists.

Here, we assume that f is given to us in the form of a circuit that takes as input numbers represented with some number of bits of precision that is polynomial in the size of the rest of the input. This precision is sufficient to exactly express the shortest path distance to which f is applied.

Theorem 5.3.1 *There is a polynomial-time algorithm for* STABLE-PRICES.

Proof. The algorithm is based on the simple intuition that for any edge to exist between two vertices v, w it must be the case that the *marginal utility* that v derives by building that edge must at least be equal to the price v pays to build the edge.

Claim 5.3.2 *For any $e = (v, w)$, the cost for building the edge $c(v, w)$ must satisfy:*

$$\sum_{x \in V} \left(f(d_{G_i}(v,x)) - \sum_{x \in V} f(d_{G_i - e}(v,x)) \right) \cdot w(v,x) \geq c(e) \text{ and}$$
$$\sum_{x \in V} f(d_{G_i}(w,x)) - \sum_{x \in V} (f(d_{G_i - e}(w,x))) \cdot w(v,x) \geq c(e) \tag{5.1}$$

for $i : e \in E_i$ where $G_i - e = (V, E_i - e)$, and

$$\sum_{x \in V} \left(f(d_{G_j + e}(v,x)) - \sum_{x \in V} f(d_{G_j}(v,x)) \right) \cdot w(v,x) \leq c(e) \text{ or}$$
$$\sum_{x \in V} \left(f(d_{G_j + e}(w,x)) - \sum_{x \in V} f(d_{G_j}(w,x)) \right) \cdot w(v,x) \leq c(e) \tag{5.2}$$

for $j : e \notin E_j$ and $G_j + e = (V, E_j + e)$.

Proof. Let $w \in \Gamma_{G_i}(v)$ for some G_i in M. For any $e \in G$, it must satisfy:

$$u_v(G_i) \geq u_v(G_i - e) \text{ and } u_w(G_i) \geq u_w(G_i - e)$$

Expanding terms for the case of v alone,

$$\sum_{x\in V} f(d_{G_i}(v,x))\cdot w(v,x) - \sum_{y\in\Gamma_v(G_i)} c(v,y) \geq \sum_{x\in V} f(d_{G_i-e}(v,x))\cdot w(v,x) - \sum_{y\in\Gamma_v(G_i-e)} c(v,y)$$

Therefore,

$$\sum_{x\in V}\left(f(d_{G_i}(v,x)) - \sum_{x\in V} f(d_{G_i-e}(v,x))\right)\cdot w(v,x) \geq c(e)$$

Similarly,

$$\sum_{x\in V}\left(f(d_{G_i}(w,x)) - \sum_{x\in V} f(d_{G_i-e}(w,x))\right)\cdot w(v,x) \geq c(e)$$

Applying the pairwise stability conditions for a graph G_j not containing e, we would get:

$$\left(\sum_{x\in V}\left(f(d_{G_j+e}(v,x)) - \sum_{x\in V} f(d_{G_j}(v,x))\right)\cdot w(v,x) \leq c(e)\right) \text{ or}$$
$$\left(\sum_{x\in V}\left(f(d_{G_j+e}(w,x)) - \sum_{x\in V} f(d_{G_j}(w,x))\right)\cdot w(v,x) \leq c(e)\right) \tag{5.3}$$

□

Clearly, as long as the greatest lower bound on $c(e)$ over all graphs as shown in (5.2) is less than the least upper bound as in (5.1), e can be priced in such a way as to rationalize all the graphs in the collection. Setting up these constraints and solving them for the prices for each pair of vertices gives us the entire price profile in polynomial-time.

□

We think of a price profile c as *rationalizing a pair* (u, v) in G_i if the conditions in Definition 5.2.1 hold for (u, v). An optimization version of the STABLE-PRICES problem is to find, given a collection of graphs $G_1, \ldots, G_m$, a price profile that rationalizes the maximum number of $((u, v), G_i)$. It is easy to see that the same algorithm described above will also work to find a price profile that would satisfy the optimization problem for STABLE-PRICES exactly.

Corollary 5.3.3 *Given an instance of* STABLE-PRICES *there exists a polynomial-time algorithm to construct a price profile c that rationalizes the maximum number of pairs $((u, v), G)$.*

5.4 Finding stable intrinsic values when prices are known

In the problem of STABLE-VALUES, we consider the scenario where the edge-prices are known but the intrinsic values function w is unknown. We define STABLE-VALUES below. We show that STABLE-VALUES is NP-complete via a reduction from a variant

Problem 2. STABLE-VALUES

Given: Collection of graphs $G_1, \ldots, G_m$ over common set of vertices V
Pairwise distance functions $d_i : V \times V \to \mathbb{R}_+$ for $i = 1, \ldots, m$
Path distance contribution function $f : \mathbb{R}_+ \to \mathbb{R}$ where f is non-increasing
Price profile for building edges $c : V \times V \to \mathbb{R}_+$

Find: Intrinsic values function $w : V \times V \to \mathbb{R}_+$ that rationalizes $G_1, \ldots, G_m$, i.e. for G_i is pairwise stable w.r.t. d_i, w, f, c for all i, if it exists.

of I-SAT that we call I-SAT* defined below.

Lemma 5.4.1 I-SAT* *is NP-complete.*

Proof. We give a reduction from 3-SAT. Consider an instance of 3-SAT given by n variables $x_1, \ldots, x_n$ and m clauses $C_1, \ldots, C_m$. For each variable x_i our instance of I-SAT* has two variables z_i^t, z_i^f and for each clause C_j, we have three auxiliary

Problem 3. I-SAT*

Given: n variables $x_1, \ldots, x_n$, and a partition $S \cup T$ of $[n]$.
m clauses that are either of type 1: $\left(\sum_{i\in S} a_i x_i \leq 1\right)$ OR $\left(\sum_{j\in T} c_j x_j \leq 1\right)$
or of type 2: $\left(\sum_{i\in S} a'_i x_i \geq 1\right)$ AND $\left(\sum_{j\in T} c'_j x_j \geq 1\right)$,
where all $a_i, c_j, a'_i, c'_j \geq 0$.

Find: An assignment for $x_1, \ldots, x_n \in \mathbb{R}_+$ that satisfies all m clauses

variables a_j, b_j, c_j. In total, we will have $2n + 3m$ unknowns in our I-SAT* instance and we partition this into $S = \{z_i^t, z_i^f | i = 1, \ldots, n\}$ and $T = \{a_j, b_j, c_j | j = 1, \ldots, 3m\}$.

Suppose $C_j = (x_p + \bar{x}_q + x_r)$ by way of example. To correspond with each variable x_p, x_q, x_r in the 3-SAT instance, we have a clause:

$$\left(\frac{z_i^t}{3} + \frac{z_i^f}{3} \geq 1\right) \text{ AND } \left(\frac{a_j}{4} + \frac{b_j}{4} + \frac{c_j}{4} \geq 1\right)$$

and to correspond to C_j, we have the following three type 1 clauses

$$\begin{array}{ccc} \left(z_p^t \leq 1\right) & \text{OR} & (a_j \leq 1) \\ \left(z_q^f \leq 1\right) & \text{OR} & (b_j \leq 1) \\ \left(z_r^t \leq 1\right) & \text{OR} & (c_j \leq 1) \end{array}$$

Suppose that there exists a satisfiable assignment for the 3-SAT instance. Then, for each x_i that is true in this assignment we let $z_i^t = 1, z_i^f = 2$ and conversely for x_i that is false, we let $z_i^t = 2, z_i^f = 1$. These assignments to z_i^t, z_i^f will satisfy their corresponding type 2 clause (5.4) in the I-SAT* instance. For a clause C_j as depicted above, since there is always at least one true literal say x_p, we will let $a_j = 2$. For false literals $\bar{x}_q, x_r$ we let $b_j = c_j = 1$. This ensures that all the corresponding clauses are satisfied.

Conversely, let there be an assignment to variables $z_i^t, z_i^f, a_j, b_j, c_j$ for $i = 1, \ldots, n; j = 1, \ldots, m$. Note that for any z_i^t, z_i^f, if the original boolean variable x_i appears in some clause, then it must be true that exactly one of them is at most 1 and the other is at least 2. We let x_i to be true if $z_i^t \leq 1$ and false if $z_i^t \geq 2$.

Since $a_j + b_j + c_j \geq 4$, a_j, b_j, c_j cannot all be simultaneously ≤ 1. Suppose $b_j > 1$ in the clause C_j above. But this means that $z_q^f \leq 1$ and $z_q^t \geq 2$. By our assignment to the boolean variables therefore, x_q is false and hence C_j is satisfied. □

Theorem 5.4.2 STABLE-VALUES *is NP-complete.*

Proof.

Our proof is by reduction from I-SAT*. Suppose we are given an instance of I-SAT* with m clauses $C_1, \ldots, C_m$ over n unknowns $x_1, \ldots, x_n$ where each clause C_i is of type 1:

$$\left(\sum_{i \in S} a_i x_i \leq 1\right) \text{ OR } \left(\sum_{j \in T} c_j x_j \leq 1\right)$$

or of type 2

$$\left(\sum_{i \in S} a'_i x_i \geq 1\right) \text{ AND } \left(\sum_{j \in T} c'_j x_j \geq 1\right)$$

and all $a_i, c_j, a'_i, c'_j \geq 0$ and $S \cup T$ is a partition of $[n]$.

We construct m edge-weighted graphs $G_1, \ldots, G_m$ on $(n+3)$ vertices labeled $v_1, \ldots, v_n, u, s, t$.Fix L to be some suitably large number such that $L > \max_{i,j}\{a_i, c_j, a'_i, c'_j\}$. We crucially use the special structure of the I-SAT* problem. Consider a type 1 clause, say C. For any $i \in S$, G contains (s, v_i) with weight $d(s, v_i) = L + a_i$ and (t, v_i) with weight $d(t, v_i) = L$. Similarly for any $j \in T$, G contains (s, v_j) with weight $d(s, v_j) = L$, and the edge (t, v_j) with weight $d(t, v_j) = L + c_j$. Finally, we build edges (s, u), (u, t) with weights $d(s, u) = d(t, u) = L/2$.We set the cost of an edge from s to t denoted by $c(s, t)$ to be $c(s, t) = 1$. For any pair of vertices (p, q) other than (s, t), we set $c(p, q) = \infty$ if $(p, q) \notin G$ and $c(p, q) = 0$ if $(p, q) \in G$.

Suppose C is of type 2. We repeat the construction as before, i.e. all the edges continue to exist but with different weights for each edge. This is captured in Table 5.3: (s, v_i) has weight $d(s, v_i) = L + a'_i$ for $i \in S$ and (s, v_j) has weight $d(s, v_j) = L$ for $j \in T$. (t, v_j) has weight $d(t, v_j) = L + c'_j$ for $j \in T$ and (t, v_i) has weight $d(t, v_i) = L$ for $i \in S$.

Crucially, the main distinction between the two constructions described above is that for a type 2 clause,the graph will have the edge (s,t) with weight 0.

Finally, we define the piecewise-linear function f as follows:

$$f(x) = \begin{cases} -L & 0 \leq x \leq L \\ -x & x > L \end{cases}$$

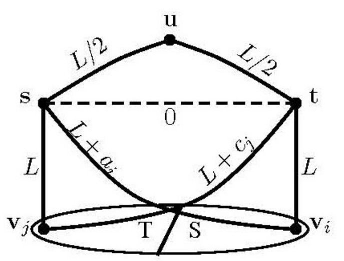


Figure 5.1: G from clause of type 1

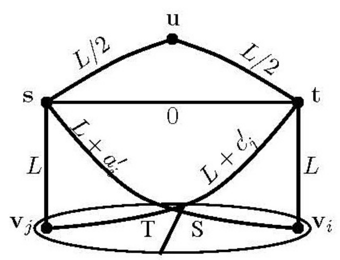


Figure 5.2: G from clause of type 2

Claim 5.4.3 *Solution to* I-SAT* $\Rightarrow$ *Solution to* STABLE-VALUES

Proof. Suppose there is a solution to the unknowns $x_1, \ldots, x_n$ in the I-SAT* instance. We set the intrinsic valuations as shown in Table 5.1.

Table 5.1: Intrinsic valuations

(p,q)	$w(p,q)$	$w(q,p)$
(s,i) $i \in S$	x_i	0
(t,j) $j \in T$	x_j	0
any other pair	0	0

Note that we have set up our instance in such a way as to render all stability conditions not involving (s,t) to be trivially satisfied. For any other pair (p,q) if the edge (p,q) is present in G_i then it is stable since the cost $c(p,q) = 0$ and therefore both p and q are indifferent to keeping the edge. Considering the case where $(p,q) \in G_i$, we look at the stability criteria for p, without loss of generality. Again, since we set the price $c(p,q)$ to be infinity, both p and q are indifferent to building the edge. Hence, such a pair (p,q) is also stable given the intrinsic valuations. Therefore, we need only focus on the stability conditions for (s,t).

We denote, for any graph G, edge e and pair of vertices u, v,

$$\Delta f_{+e}(u,v) = f(d_{G+e}(u,v)) - f(d_G(u,v))$$

to be the marginal utility contribution by adding e to the graph G for u and v. Similarly, we use

$$\Delta f_{-e}(u,v) = f(d_G(u,v)) - f(d_{G-e}(u,v))$$

for the marginal utility contribution by deleting e from the graph G for u and v. Let us suppose that the edge is not present in G_i. For C_i the pairwise stability conditions are:

$$u_s(G_i + (s,t)) \leq u_s(G_i) \text{ OR } u_t(G_i + (s,t)) \leq u_t(G_i).$$

Expanding and rearranging terms, this becomes:

$$\sum_v \Delta f_{+(s,t)}(s,v) \cdot w(s,v) \leq c(s,t) \text{ OR } \sum_v \Delta f_{+(s,t)}(t,v) \cdot w(t,v) \leq c(s,t)$$

From Table 5.1 above, we observe that for all $v = v_j, j \in T$ and $v = u$, $w(s,v) = 0$ and similarly for all $v = v_i, i \in S$ and $v = u$, $w(t,v) = 0$. This allows us to further simplify the expression we must satisfy:

$$\sum_{i \in S} \Delta f_{+(s,t)}(s,v_i) \cdot w(s,v_i) \leq 1 \text{ OR} \sum_{j \in T} \Delta f_{+(s,t)}(t,v_j) \cdot w(t,v_j) \leq 1$$

Substituting terms from Table 5.2 that correspond to the coefficients of $w(s,.), w(t,.)$, and setting $w(s,v_i) = x_i$ for $i \in S$ and $w(t,v_j) = x_j$ for $j \in T$, we have exactly the clause C which is satisfied by the solution to $x_1, \ldots, x_n$.

Suppose now that the edge is present and G corresponds to an inequality of type 2.

Table 5.2: Shortest path distance for G constructed from a type 1 clause

(p, q)	$d_G(p, q)$	$d_{G+(s,t)}(p, q)$	$\Delta f_{+(s,t)}(p, q)$
(s, v_i) $i \in S$	$L + a_i$	L	a_i
(t, v_i) $i \in S$	L	L	0
(s, v_j) $j \in T$	L	L	0
(t, v_j) $j \in T$	$L + c_j$	L	c_j
(s, t)	L	0	0
$(s, u), (t, u)$	$L/2$	$L/2$	0

Table 5.3: Shortest path distance for G constructed from a type 2 clause

(p, q)	$d_{G-(s,t)}(p, q)$	$d_G(p, q)$	$\Delta f_{-(s,t)}(p, q)$
(s, v_i) $i \in S$	$L + a'_i$	L	a'_i
(t, v_i) $i \in S$	L	L	0
(t, v_j) $j \in T$	$L + c'_j$	L	c'_j
(s, v_j) $j \in T$	L	L	0
(s, t)	L	0	0
$(s, u), (t, u)$	$L/2$	$L/2$	0

We would need to satisfy the stability conditions:

$$u_s(G - (s, t)) \geq u_s(G) \text{ AND } u_t(G - (s, t)) \geq u_t(G).$$

As before, this becomes:

$$\sum_v \Delta f_{-(s,t)}(s, v) \cdot w(s, v) \geq c(s, t) \text{ AND } \sum_v \Delta f_{-(s,t)}(t, v) \cdot w(t, v) \geq c(s, t)$$

and after eliminating terms that do not contribute to the total marginal utility in the case of $v = v_j, j \in T, t, u$ for s and $v = v_i, i \in S, s, u$ for t, we will have:

$$\sum_{i \in S} \Delta f_{-(s,t)}(s, v_i) \cdot w(s, v_i) \geq c(s, t) \text{ AND } \sum_{j \in T} \Delta f_{-(s,t)}(t, v_j) \cdot w(t, v_j) \geq c(s, t)$$

Again, looking at terms in Tables 5.1 and 5.3 we have:

$$\left(\sum_{i\in S} a_i' x_i \geq 1\right) \text{ AND } \left(\sum_{j\in T} c_j' x_j \geq 1\right)$$

which is exactly the original clause we started out assuming was true.

□

Claim 5.4.4 *Solution to* STABLE-VALUES $\Rightarrow$ *solution to* I-SAT*.

Proof. Suppose there is a solution to the STABLE-VALUES instance. If the intrinsic valuations for s, t are such that $w(s,p) = w(t,p) = 0$ for $p \notin [n]$, then the pairwise stability conditions would correspond to what we obtained in the completeness case and reciprocally setting $x_i = w(s,i)$ for $i \in S$ and $x_j = w(t,j)$ for $j \in T$ would be a feasible assignment satisfying all the clauses.

Suppose this were not the case. Note however from Tables 5.2 and 5.3, that the marginal utility contribution for (s,p) to $w(s,p)$ would be 0 for all $p \notin [n]$ and so the total contribution would still vanish leaving behind the same inequality as in the original clause C_i and returning us to the case above. □

This completes the proof of Theorem 5.4.2.

□

After this warmup to establish the basic problems and their complexity, we get to the meat of the chapter, where we give a tight inapproximability result for an optimization version of STABLE-VALUES.

5.5 A tight inapproximability result for STABLE-VALUES

Before defining the optimization version of STABLE-VALUES, we need to define the notion of *active pairs*:

Definition 5.5.1 *Given an instance of* STABLE-VALUES *containing a collection of graphs $G_1, \ldots, G_m$ all over a set of vertices V and a price profile $c : V \times V \to \mathbb{R}_+$, a pair $(u, v) \in V \times V$ is said to be an* active pair *if it is not the case that i) $(u, v) \in G_i$ for all i AND $c(u, v) = 0$, or ii) $(u, v) \notin G_i$ for any i AND $c(u, v) = \infty$.*

As explained in the introduction, a pair (u, v) that is not an active pair effectively "comes for free" since regardless of what the intrinsic values for u, v are, the pairwise stability conditions are trivially satisfied because the $c(u, v) = 0$ if $(u, v) \in G_i$ for all i and $c(u, v) = \infty$ if $(u, v) \notin G_i$ for any i. We observed in the introduction that there is a trivial 1/2-factor approximation algorithm. We prove in this section that this is tight assuming $P \neq NP$.

As before, we think of an intrinsic values function w as rationalizing an active pair (u, v) in G_i if the stability conditions in Definition 5.2.1 hold for (u, v). Then, the MAX-STABLE-VALUES problem is to seek intrinsic values that rationalize the *maximum* number of *active pairs* (u, v), counted separately for each G_i.

To our end of showing a hardness result for MAX-STABLE-VALUES we show an inapproximability result for MAX-LIN$_{\mathbb{Z}_+}$, which is the main technical contribution of this chapter. Although MAX-STABLE-VALUES is actually defined over the reals and would admittedly require us to show a hardness result for MAX-LIN$_{\mathbb{R}_+}$, we are able to employ a clever trick shown in [GR07] that makes it sufficient for us to work with MAX-LIN$_{\mathbb{Z}_+}$ and then carry the result over to the reals as long as we can ensure that the co-efficients in the MAX-LIN$_{\mathbb{Z}_+}$ instance we obtain are bounded and the equations have sparse support.

Problem 4. MAX-LIN$_{\mathbb{Z}_+}$

Given: n variables $x_1, \ldots, x_n$
m equations, each of which is of the type
$\sum_i a_i x_i = b$
where $a_i, b \in \mathbb{Z}_+$ for all $i = 1, \ldots, n$

Find: An assignment for $x_1, \ldots, x_n \in \mathbb{Z}_+$ that satisfies the maximum number of equations.

Theorem 5.5.2 *Given an instance of* MAX-LIN$_{\mathbb{Z}_+}$*, for all positive constants* ϵ, δ *it is NP-hard to distinguish between the following two cases:*

- *There exists a solution satisfying at least a* $(1-\epsilon)$ *fraction of the linear equations in the instance.*
- *Every solution satisfies at most a* δ *fraction of the linear equations.*

The above promise problem will be referred to as MAX-LIN$_{\mathbb{Z}_+}(1-\epsilon, \delta)$. Our proof follows the outline of the proof for Theorem 3.4 in [GR07]. In the remainder of this section, we go through the proof of Theorem 5.5.2, pointing out the crucial points where our proof needs to differ from [GR07]. We define the LABEL-COVER problem below.

Definition 5.5.3 *An instance of the* LABEL-COVER(c, s) *problem comprises a bipartite graph* $\mathcal{H} = (\mathcal{A}, \mathcal{B}, \mathcal{E})$*, a set of labels* Σ *and a set of projection mappings* $\pi_e : \Sigma \to \Sigma$ *for each edge* $e \in \mathcal{E}$*. An assignment* $A : (\mathcal{A} \cup \mathcal{B}) \to \Sigma$ *is a mapping from the set of vertices onto the set of labels and is* legal *for an edge* $e = (u, v)$ *if* $\pi_e(A(u)) = A(v)$*. We wish to ascertain for this instance of the problem if*

- *there exists an assignment* A *that is legal for at least a* c *fraction of edges, or*
- *every assignment is legal for at most an* s *fraction of the edges.*

The following theorem due to Håstad [Hås01] gives a hardness of approximation result for LABEL-COVER.

Theorem 5.5.4 *([Hås01]) It is NP-hard to distinguish between an instance of* LABEL-COVER *that has an assignment legal for all edges and one for which every assignment is legal for at most* $1/|\Sigma|^\gamma$ *fraction of edges.*

As we noted in Section 1.3, there doesn't seem to be an easy reduction from MAX-LIN$_{\mathbb{Z}}$ (shown to be hard to approximate in [GR07]) or from MAX-LIN$_{\mathbb{F}_p}$ (shown to be hard to approximate in [Hås01]). Both those results are obtained by reductions from Label-Cover and involve constructing equations of the form $x + y - z = c$. Most of our effort in our proof is spent on giving an alternative reduction that gives rise to equations with coefficients in $\mathbb{Z}_+$. Specifically, our proof abstracts properties of the verifier query distribution that are sufficient for main steps of the [GR07] proof. We then specify a different distribution than the one in [GR07] that satisfies these properties, as well as an additional symmetry property that is key to our final PCP system for MAX-LIN$_{\mathbb{Z}_+}$.

5.5.1 Proof of Theorem 5.5.2

This subsection is devoted to the proof of Theorem 5.5.2. Consider an instance of the Label-Cover$(1, \delta)$ problem comprising the bipartite graph $G(U, V, E)$ over n vertices and m edges, a set of labels Σ and constraint relations $\pi_e : \Sigma \to \Sigma$. We assume that the finite set of labels $\Sigma = \{\ell_1, \ldots, \ell_h\}$ can also be interchangeably represented as a set of integers $\{1, \ldots, h\}$.

In our PCP system, the proof comprises the labels for all vertices encoded using the Long code we define below.

Definition 5.5.5 *[GR07] For a label $r \in [h]$, the codeword $\mathcal{C}(r)$ is an evaluation of the projection function $f_r : \mathbb{Z}_+^h \to \mathbb{Z}_+$ given by $f_r((z_1, \ldots, z_h)) = z_r$ over $\mathbb{Z}_+^h$. In other words, $\mathcal{C}(r)[x] = x_r$.*

In other words, the proof is given to the verifier as a sequence $(\mathcal{C}(A(v_1)), \ldots, \mathcal{C}(A(v_n)))$ where A is the purported legal assignment. The verifier makes queries to the proof at three locations. These locations are chosen based on probability distributions P_1, P_2, Q where P_1, P_2 will have the following two properties:

Definition 5.5.6 *P is said to be (M, δ)-heavy if*

$$\sum_{x \in [M]^h} P(x) \geq (1 - \delta)$$

Definition 5.5.7 *P is said to be (δ, L)-*decay-resilient *if for all $x \in [L]^h$ and any $y \in \mathbb{Z}_+^h$*

$$\frac{P(y+x)}{P(y)} \geq \delta.$$

We note that Property 5.5.7 is the crucial property which must be satisfied by a probability distribution P the verifier uses to query the proof, in order for the soundness analysis to go through. We are now ready to define P_1, P_2.

Definition 5.5.8 *Fix some prime p. For $j = 1, 2$ we define the probability distributions P_j over $\mathbb{Z}_+^h$ to be*

$$P_j((x_1, \ldots, x_h)) = \Gamma_j \prod_{i=1}^{h} e^{-c_j |x_i - p/2|}$$

where Γ_1, Γ_2 are normalization constants.

Our choice for P_1, P_2 is dictated by the following useful proposition that is easy to verify.

Proposition 5.5.9 *There exist constants $\Gamma_1, \Gamma_2, c_1, c_2$ and some large integer $M = M(h, \delta)$ for which P_1 and P_2 are (M, δ)-heavy.*

Let X_1, X_2 denote random variables drawn from $\mathbb{Z}^h$ using some probability distributions to be specified later. We will use some suitably large integer M as a parameter of our reduction that is obtained from Proposition 5.5.9. We fix a prime $p >> 3M$ and denote $\mathbf{p} = (p, \ldots, p)$. We use $x \circ \pi_e$ to denote a permutation of π_e applied to the co-ordinates of $x \in \mathbb{Z}^h$. In other words $(x \circ \pi_e)_i = x_{\pi_e(i)}$. We denote μ to be some random noise generated by picking each co-ordinate to be 0 with probability $(1 - \epsilon)$ and some integer k chosen randomly from $[t]$ with probability ϵ where $t > h^2/\delta$. We denote Q to be the induced probability distribution with which μ is chosen. Where necessary, we will use the shorthand X_3 to denote the random variable $\mathbf{p} - (X_1 \circ \pi_e + X_2 + \mu)$. Finally, for any two $x, y \in \mathbb{Z}^h$ we denote $\langle x, y \rangle$ to mean the inner product: $\langle x, y \rangle = \sum_{i=1}^{h} x_i y_i$. The verifier picks X_1, X_2 using probability distributions P_1, P_2 respectively. The test that the verifier checks is:

$$\mathcal{C}(A(u))[X_1] + \mathcal{C}(A(v))[X_2] + \mathcal{C}(A(v))[X_3] \quad = \quad p \tag{5.4}$$

Lemma 5.5.10 *The PCP system for* $\text{Max-Lin}_{\mathbb{Z}_+}$ *described above has* $(1-\epsilon)$ *completeness.*

Proof. Suppose A is indeed a legal assignment for all edges $e \in E(G)$. This means that for any edge $e = (u, v)$, $\pi_e(A(v)) = A(u)$. Therefore,

$$\begin{aligned}
\mathcal{C}(A(u))[X_1] + \mathcal{C}(A(v))[X_2] + \mathcal{C}(A(v))[X_3] \quad &= \quad (X_1)_{A(u)} + (X_2)_{A(v)} + \mathbf{p}_{A(v)} \\
&\quad\quad - (X_1 \circ \pi_e + X_2 + \mu)_{A(v)} \\
&= \quad (X_1)_{A(u)} + (X_2)_{A(v)} + p \\
&\quad\quad - (X_1)_{\pi_e(A(v))} - (X_2)_{A(v)} - \mu_{A(v)} \\
&= \quad p - \mu_{A(v)}
\end{aligned}$$

Recalling how we picked μ, we know that $\mu_{A(v)}$ is 0 with probability exactly $(1-\epsilon)$ and hence, (5.4) is satisfied with probability $(1-\epsilon)$. $\square$

Lemma 5.5.11 *The PCP system for* $\text{Max-Lin}_{\mathbb{Z}_+}$ *described above has* 19δ *soundness error.*

Proof. To argue for soundness, suppose A is an assignment that causes the verifier to accept with probability at least $\delta' = 19\delta$. This means that over all $e = (u, v)$ chosen uniformly at random from $E(G)$, and $\mathbf{x}_1, \mathbf{x}_2, \mu$ chosen according to their respective probability distributions P_1, P_2, Q from $\mathbb{Z}_+^h$:

$$\Pr_{e, X_1, X_2, \mu}\left[\mathcal{C}(A(u))[X_1] + \mathcal{C}(A(v))[X_2] + \mathcal{C}(A(v))[\mathbf{p} - (X_1 \circ \pi_e + X_2 + \mu)] = p\right] \quad \geq \quad 19\delta$$

The following fact is handy:

Fact 5.5.12 *Let P be a $(1/4, (M+t))$-decay-resilient probability distribution over $\mathbb{Z}_+^h$. Then, for any $y \in [M+t]^h$ and all $x \in \mathbb{Z}_+^h$:*

$$P(x) \leq 2\sqrt{P(x+y) \cdot P(x)}$$

Proof. Since P is $(1/4, (M+t))$-decay-resilient and symmetric around $(p/2, \ldots, p/2)$ it satisfies the following inequality:

$$\begin{aligned} P(x) &\leq 4P(x+y) \\ &\leq 2\sqrt{P(x+y) \cdot P(x)} \end{aligned}$$

□

The following lemma is based on the first step of the proof technique used in [GR07] applied to our setting:

Lemma 5.5.13 *Let P_1, P_2 be probability distributions over $\mathbb{Z}_+^h$ and Q be a probability distribution over $[t]^h$ such that P_1, P_2, Q satisfy the following properties:*

1. *P_1, P_2 are (M, δ)-heavy.*
2. *P_2 is $(1/4, M+t)$-decay-resilient.*
3. *P_1, P_2, Q are $(p/3, \delta)$-heavy.*
4. *P_2 is symmetric around $(p/2, \ldots, p/2)$, i.e. $P_2(x) = P_2(\mathbf{p} - x)$*

Suppose that with X_1, X_2, μ chosen respectively from distributions P_1, P_2, Q and $e = (u, v)$ chosen uniformly at random:

$$\Pr_{e, X_1, X_2, \mu}[\mathcal{C}(A(u))[X_1] + \mathcal{C}(A(v))[X_2] + \mathcal{C}(A(v))[\mathbf{p} - (X_1 \circ \pi_e + X_2 + \mu)] = p] \geq 19\delta$$

Let $\Upsilon_p^{(u,v)}(X_1, X_2, \mu)$ be the indicator variable for the event

$$\mathcal{C}(A(u))[X_1] + \mathcal{C}(A(v))[X_2] + \mathcal{C}(A(v))[\mathbf{p} - (X_1 \circ \pi_e + X_2 + \mu)] = 0 \mod p$$

Then,

$$E_{(u,v)}\left[\sum_{x_1,x_2,x_3\in[p]^h} P_1(x_1)\sqrt{P_2\left(x_2\right)\cdot P_2(x_3)}Q(\mu)\Upsilon_p^{(u,v)}(x_1,x_2,\mu)\right] \;\geq\; 8\delta$$

Proof. Since P_1 was chosen to be (M,δ)-heavy and μ is by default chosen from $[t]^h$, with probability at most δ our choice of X_1 will lie outside $[M]^h$ and so:

$$\Pr_{e,X_1,X_2,\mu}[\mathcal{C}(A(u))[X_1]+\mathcal{C}(A(v))[X_2]+\mathcal{C}(A(v))[\mathbf{p}-(X_1\circ\pi_e+X_2+\mu)] \text{ is equal to } p|X_1\in[M]^h] \;\geq\; 18\delta$$

Denoting $\Upsilon^e(x_1,x_2,\mu)$ to be the indicator variable for the event:

$$\mathcal{C}(A(u))[x_1]+\mathcal{C}(A(v))[x_2]+\mathcal{C}(A(v))[\mathbf{p}-(x_1\circ\pi_e+x_2+\mu)]=p$$

we can rewrite the left-hand side above in terms of an expectation over all edges $e(u,v)$:

$$E_e\left[\sum_{x_1\in[M]^h,x_2\in\mathbb{Z}_+^h,\mu\in[t]^h} P_1(x_1)P_2(x_2)Q(\mu)\Upsilon^{(u,v)}(x_1,x_2,\mu)\right] \;\geq\; 18\delta \qquad (5.5)$$

Combining (5.5) with Property 2 and Fact 5.5.12, we get:

$$E_e\left[\sum_{x_1\in[M]^h,x_2\in\mathbb{Z}_+^h,\mu\in[t]^h} P_1(x_1)\sqrt{P_2\left(x_2\right)\cdot P_2(x_2+x_1\circ\pi_e+\mu)}Q(\mu)\Upsilon^{(u,v)}(x_1,x_2,\mu)\right] \;\geq\; 9\delta$$

Since P_2 is symmetric around $(p/2,\ldots,p/2)$, $P_2(x_2+x_1\circ\pi+\mu)=P_2(\mathbf{p}-(x_2+x_1\circ\pi+\mu))$ and hence the above inequality becomes:

$$E_e[\sum_{x_1\in[M]^h,x_2\in\mathbb{Z}_+^h,\mu\in[t]^h} P_1(x_1)\sqrt{P_2\left(x_2\right)\cdot P_2(\mathbf{p}-(x_1\circ\pi_e+x_2+\mu))}Q(\mu)\Upsilon^{(u,v)}(x_1,x_2,\mu)] \;\geq\; 9\delta$$

Observing that $\Upsilon_p^{(u,v)}(x_1, x_2, \mu) \geq \Upsilon^{(u,v)}(x_1, x_2, \mu)$, we have:

$$E_e[\sum_{x_1 \in [M]^h, x_2 \in \mathbb{Z}_+^h, \mu \in [t]^h} P_1(x_1)\sqrt{P_2(x_2) \cdot P_2(\mathbf{p} - (x_1 \circ \pi_e + x_2 + \mu))} \cdot$$
$$Q(\mu)\Upsilon_p^{(u,v)}(x_1, x_2, \mu)] \geq 9\delta$$

Since P_1, P_2, Q are $(p/3, \delta)$-heavy:

$$E_{(u,v)}[\sum_{x_1, x_2, \mu \in [p/3]^h} P_1(x_1)\sqrt{P_2(x_2) \cdot P_2(\mathbf{p} - (x_1 \circ \pi_e + x_2 + \mu))} \cdot$$
$$Q(\mu)\Upsilon_p^{(u,v)}(x_1, x_2, \mu)] \geq 8\delta \quad (5.6)$$

□

For the rest of the proof, we will use the shorthand $x_3 = \mathbf{p} - (x_1 \circ \pi_e + x_2 + \mu)$ as shorthand for simplicity of representation. Note that the function $\Upsilon_p^{(u,v)} : [p]^{3h} \to \{0, 1\}$ is given by

$$\Upsilon_p^{(u,v)}(x_1, x_2, \mu) = \begin{cases} 1 & \mathcal{C}(A(u))[x_1] + \mathcal{C}(A(v))[x_2] + \mathcal{C}(A(v))[x_3] = 0 \mod p, \\ 0 & \text{otherwise.} \end{cases}$$

$\Upsilon_p^{(u,v)}$ can equivalently be written as below:

$$\Upsilon_p^{(u,v)}(x_1, x_2, \mu) = \frac{1}{p}\sum_{k=0}^{p-1} e^{\frac{2\pi i k}{p}(\mathcal{C}(A(u))[x_1] + \mathcal{C}(A(v))[x_2] + \mathcal{C}(A(v))[x_3])}$$

Substituting this in (5.6), and using $x_3 = \mathbf{p} - (x_1 \circ \pi_e + x_2 + \mu)$ as shorthand, the left-hand side becomes:

$$E_{(u,v)}\left[\frac{1}{p}\sum_{x_1, x_2, \mu \in [\frac{p}{3}]^h} P_1(x_1)\sqrt{P_2(x_2) \cdot P_2(x_3)}Q(\mu)\left(\sum_{k=0}^{p-1} e^{\frac{2\pi i k}{p}(\mathcal{C}(A(u))[x_1] + \mathcal{C}(A(v))[x_2] + \mathcal{C}(A(v))[x_3])}\right)\right]$$

We further simplify the term within the expectation:

$$\frac{1}{p}\sum_{x_1,x_2,\mu\in[\frac{p}{3}]^h} P_1(x_1)\sqrt{P_2(x_2)\cdot P_2(x_3)}Q(\mu)\left(\sum_{k=0}^{p-1} e^{\frac{2\pi ik}{p}(\mathcal{C}(A(u))[x_1]+\mathcal{C}(A(v))[x_2]+\mathcal{C}(A(v))[x_3])}\right)$$

$$= \frac{1}{p}\sum_{k=0}^{p-1}\sum_{x_1,x_2,\mu\in[\frac{p}{3}]^h} Q(\mu)\left(P_1(x_1)e^{\frac{2\pi ik}{p}\mathcal{C}(A(u))[x_1]}\right)\left(\sqrt{P_2(x_2)}e^{\frac{2\pi ik}{p}\mathcal{C}(A(v))[x_2]}\right)\cdot \left(\sqrt{P_2(x_3)}e^{\frac{2\pi ik}{p}\mathcal{C}(A(v))[x_3]}\right) \tag{5.7}$$

Setting $\mathcal{U}(x) = P_1(x)e^{\frac{2\pi ik}{p}\mathcal{C}(A(u))[x]}$ and $\mathcal{V}(x) = \sqrt{P_2(x)}e^{\frac{2\pi ik}{p}\mathcal{C}(A(v))[x]}$, (5.6) now simplifies to:

$$E_{(u,v)}\left[\frac{1}{p}\sum_{k=0}^{p-1}\sum_{x_1,x_2,\mu\in[\frac{p}{3}]^h} Q(\mu)\mathcal{U}(x_1)\mathcal{V}(x_2)\mathcal{V}(x_3)\right] \geq 8\delta \tag{5.8}$$

This is where our proof technique has a crucial point of departure from that used in [GR07]. Since our test has only positive co-efficients, we do not have the luxury to make the substitution

$$\sqrt{P_2(x')}e^{-\frac{2\pi ik}{p}\mathcal{C}(A(v))[x']} = \overline{\mathcal{V}(x')}$$

that is made in [GR07] which simplifies their analysis.

Consider the Fourier expansion for $\mathcal{U}$ described below:

$$\mathcal{U}(x) = \sum_{w\in[p]^h}\widehat{\mathcal{U}}(w)e^{\frac{2\pi i}{p}\langle w,x\rangle}$$

where

$$\widehat{\mathcal{U}}(w) = \frac{1}{p^h}\sum_{x\in[p]^h}\mathcal{U}(x)e^{-\frac{2\pi i}{p}\langle w,x\rangle}$$

We substitute this and a similar Fourier expansion for $\mathcal{V}$ back in (5.8):

$$E_{(u,v)}[\frac{1}{p}\sum_{k=0}^{p-1}\sum_{x_1,x_2,\mu} Q(\mu)\sum_{w_1}\widehat{\mathcal{U}}(w_1)e^{\frac{2\pi i}{p}\langle w_1,x_1\rangle}\sum_{w_2}\widehat{\mathcal{V}}(w_2)e^{\frac{2\pi i}{p}\langle w_2,x_2\rangle}\cdot \sum_{w_3}\widehat{\mathcal{V}}(w_3)e^{\frac{2\pi i}{p}\langle w_3,x_3\rangle}] \geq 8\delta \tag{5.9}$$

Substituting $x_3 = (\mathbf{p} - (x_1 \circ \pi_e + x_2 + \mu))$ the left-hand side becomes:

$$
\begin{aligned}
& E_{(u,v)}[\frac{1}{p}\sum_{k=0}^{p-1} \sum_{x_1,x_2,\mu} Q(\mu) \sum_{w_1} \widehat{\mathcal{U}}(w_1) e^{\frac{2\pi i}{p}\langle w_1, x_1\rangle} \sum_{w_2} \widehat{\mathcal{V}}(w_2) e^{\frac{2\pi i}{p}\langle w_2, x_2\rangle} \cdot \\
& \sum_{w_3} \widehat{\mathcal{V}}(w_3) e^{\frac{2\pi i}{p}\langle w_3, (\mathbf{p}-(x_1\circ\pi_e + x_2 + \mu))\rangle}] \\
= \; & E_{(u,v)}[\frac{1}{p}\sum_{k=0}^{p-1} \sum_{x_1,x_2,\mu} Q(\mu) \sum_{w_1} \widehat{\mathcal{U}}(w_1) e^{\frac{2\pi i}{p}\langle w_1, x_1\rangle} \sum_{w_2} \widehat{\mathcal{V}}(w_2) e^{\frac{2\pi i}{p}\langle w_2, x_2\rangle} \cdot \\
& \sum_{w_3} \widehat{\mathcal{V}}(w_3) e^{-\frac{2\pi i}{p}\langle w_3, (x_1\circ\pi_e + x_2 + \mu)\rangle}] \\
= \; & E_{(u,v)}[\frac{1}{p}\sum_{k=0}^{p-1} \sum_{w_1,w_2,w_3} \widehat{\mathcal{U}}(w_1)\widehat{\mathcal{V}}(w_2)\widehat{\mathcal{V}}(w_3) \sum_{x_1} e^{\frac{2\pi i}{p}\langle (w_1 - w_3\circ\pi_e^{-1}), x_1\rangle} \sum_{x_2} e^{\frac{2\pi i}{p}\langle (w_2 - w_3), x_2\rangle} \cdot \\
& \sum_{\mu} Q(\mu) e^{-\frac{2\pi i}{p}\langle w_3, \mu\rangle}]
\end{aligned}
$$

where $w_3 \circ \pi_e^{-1}$ denotes the vector obtained by setting $(w_3 \circ \pi_e^{-1})_i = \sum_{j\in\pi_e^{-1}(i)} w_{3j}$ for $i = 1, \ldots, h$. Note that for $w_1 \neq w_3 \circ \pi_e^{-1}$, $\sum_{x_1} e^{\frac{2\pi i}{p}\langle (w_1 - w_3\circ\pi_e^{-1}), x_1\rangle} = 0$ and similarly, for $w_2 \neq w_3$, $\sum_{x_2} e^{\frac{2\pi i}{p}\langle (w_2 - w_3), x_2\rangle} = 0$. Setting $w = w_3$, the overall inequality simplifies to:

$$
E_{(u,v)}\left[\frac{1}{p}\sum_{k=0}^{p-1}\sum_{w}\left(p^h\widehat{\mathcal{U}}(w\circ\pi_e^{-1})\right)\left(p^h\widehat{\mathcal{V}}(w)^2\right)\sum_{\mu} Q(\mu)e^{-\frac{2\pi i}{p}\langle w,\mu\rangle}\right] \geq 8\delta \quad (5.10)
$$

Also, note that $|\widehat{\mathcal{V}}(w)^2| = \sqrt{\left(\widehat{\mathcal{V}}(w)^2 \cdot \overline{\widehat{\mathcal{V}}(w)^2}\right)} = \sqrt{\left(\widehat{\mathcal{V}}(w) \cdot \overline{\widehat{\mathcal{V}}(w)}\right)^2} = |\widehat{\mathcal{V}}(w)|^2$ using the simple identity that $\mathbf{z}^2 \cdot \overline{\mathbf{z}^2} = (\mathbf{z} \cdot \overline{\mathbf{z}})^2$ for any complex number $\mathbf{z}$.

Substituting back in (5.10), we obtain:

$$
E_{(u,v)}\left[\frac{1}{p}\sum_{k=0}^{p-1}\sum_{\omega}\left(p^h\widehat{\mathcal{U}}(w\circ\pi_e^{-1})\right)\left(|p^h\widehat{\mathcal{V}}(w)|^2\right)\sum_{\mu} Q(\mu)e^{-\frac{2\pi i}{p}\langle w,\mu\rangle}\right] \geq 8\delta \quad (5.11)
$$

We are now ready to use the following lemma, again from [GR07] concerning probability distributions P_1, P_2, Q and some assignment A of labels to vertices in G satisfying (5.11).

Lemma 5.5.14 *([GR07]) Let $P_1, P_2, Q : \mathbb{Z}_+^h \to [0,1]$ be probability distributions and $A : V(G) \to [h]$ some assignment of labels to vertices in G satisfying (5.11). Then, there exists a constant C such that*

$$\Pr_{(u,v)} [A \textit{ is legal for } (u,v)] \geq \delta^4/96C^2$$

By choosing our original instance of LABEL-COVER$(1,\delta)$ to be such that h is large enough, we can ensure that $\delta^4/96C^2 \geq 1/h^\gamma$. This gives us a soundness of 19δ as required. □

5.5.2 Tying it all together

Theorem 5.5.15 *For all constants $\epsilon, \delta > 0$, the problem* MAX-LIN$_{\mathbb{R}_+}(1-\epsilon,\delta)$ *is NP-hard.*

Proof. We just need the following argument from [GR07]:

Theorem 5.5.16 *(Theorem 3.5, [GR07]) For all constants $\epsilon, \delta > 0$, the problem* MAX-LIN$_{\mathbb{R}}(1-\epsilon,\delta)$ *is NP-hard.*

We give a reduction from MAX-LIN$_{\mathbb{Z}_+}(1-\epsilon,\delta/8)$. Note that our reduction in proving that this problem was hard produced an instance of MAX-LIN$_{\mathbb{Z}_+}$ where each equation consists of three variables:

$$x_i + x_j + x_k = c$$

where $x_i, x_j, x_k, c \in \mathbb{Z}_+$. The MAX-LIN$_{\mathbb{R}_+}$ instance we construct will have exactly the same set of these equations:

$$x'_i + x'_j + x'_k = c$$

with x'_i, x'_j, x'_k chosen from $\mathbb{R}_+$. A solution in integers to the original MAX-LIN$_{\mathbb{Z}_+}$ instance is automatically a solution to the MAX-LIN$_{\mathbb{R}_+}$ instance.

Suppose that with probability at least δ over the choice of equations in the instance, a solution is feasible. Then, for any such equation:

$$x'_i + x'_j + x'_k = c$$

by choosing each variable x_s to be either $\lfloor x'_s \rfloor$ or $\lceil x'_s \rceil$ uniformly at random for $s = i, j, k$ we will have satisfied the equation in the corresponding MAX-LIN$_{\mathbb{Z}_+}$ instance with probability at least $\delta/8$ thereby contradicting the hardness assumption we made for the instance. □

Theorem 5.5.17 *For all $\epsilon, \delta > 0$, the problem* I-SAT$^*(1-\epsilon, 1/2 + 76\delta)$ *is NP-hard*

Proof. The proof is by reduction from MAX-LIN$_{\mathbb{R}_+}(1-\epsilon, 19\delta)$ over a set of unknowns $x_1, \ldots, x_n$. Our I-SAT* instance will have variables $y_1, \ldots, y_n; y'_1, \ldots, y'_n$. We define a partition $S \cup T$ where $S = \{y_i | i = 1, \ldots, n\}$ and $T = \{y'_j | j = 1, \ldots, n\}$. For each equation over the reals $\sum_i a_i x_i = b$ in an instance of MAX-LIN$_{\mathbb{R}_+}$, we construct the I-SAT* clauses:

$$\begin{pmatrix} \sum_{i\in S} a_i y_i \leq b \\ \text{OR} \\ \sum_{j \in T} a_j y'_j \leq b \end{pmatrix}, \begin{pmatrix} \sum_{i\in S} a_i y_i \geq b \\ \text{AND} \\ \sum_{j \in T} a_j y'_j \geq b \end{pmatrix} \tag{5.12}$$

Let $x_1, \ldots, x_n$ be a solution for MAX-LIN$_{\mathbb{Z}_+}$. We set $y_i = y'_i = x_i$ for all $i = 1, \ldots, n$. A solution satisfying at least a $(1-\epsilon)$ fraction of the equations in the MAX-LIN$_{\mathbb{R}_+}$ instance will satisfy at least a $(1-\epsilon)$ fraction of the inequality-clauses in the I-SAT* instance. Conversely, suppose there exists a solution satisfying at least a $(1/2 + 76\delta)$ fraction of the inequality-clauses. Then there is at least a 38δ fraction of inequality-clause pairs of type (5.12) which must be satisfied. For each such pair, the type 2 clause implies that both $(\sum_{i\in S} a_i y_i \geq b)$ and $(\sum_{j\in T} a_j y'_j \geq b)$ and the type 1 clause implies that one of $\sum_{i\in S} a_i y_i \leq b$ and $\sum_{j\in T} a_j y'_j \geq b$ must hold. Therefore, for each such pair one of $\sum_{i\in S} a_i y_i = b$ or $\sum_{i\in T} a_i y'_i = b$ must hold. Setting x_i to be y_i for all i if there are more pairs for which $\sum_{i\in S} a_i y_i \leq b$ and y'_i otherwise, guarantees that at least a 19δ fraction of equations $\sum_i a_i x_i = b$ must be satisfied in the original MAX-LIN$_{\mathbb{Z}_+}$ instance thereby giving us the necessary gap reduction. This completes the proof of the theorem.

□

Corollary 5.5.18 *For all $\epsilon, \delta > 0$ the problem* MAX-STABLE-VALUES$(1-\epsilon, 1/2+\delta)$ *is NP-hard.*

Proof. We argue that the reduction from I-SAT* shown in the proof of Theorem 5.4.2 is also a gap-preserving reduction and reduce from I-SAT$^*(1-\epsilon, 1/2+\delta)$. Arguing first for $(1-\epsilon)$ completeness, we note that if a clause in the I-SAT* instance is satisfied then the corresponding pairwise stability condition is also satisfied.

Suppose now, that $(1/2+\delta)$ fraction of the stability conditions for the active pairs are satisfied. But each such condition exactly corresponds to a clause being satisfied in the original I-SAT* instance thereby giving us the required $(1/2+\delta)$ soundness for STABLE-VALUES. □

Epilogue

In this dissertation we looked at questions concerning two aspects of any game: randomness and rationality. In the former case, we considered various scenarios in which agents are constrained in their access to randomness and must perforce play strategies that must make do with little or no randomness. We investigated three interesting instances of such a framework and offered insight into how techniques from pseudorandomness can be brought to bear upon problems in algorithmic game theory. Our results can also be viewed in the larger context of the ongoing attempts to find PTAS for Nash equilibria. We believe that our algorithm for efficiently finding approximate equilibria in low-rank games (even in the multi-player case) can serve as a reference point for future analyses of Nash equilibria in games where agents have sparse payoff matrices.

In the latter case, we argued for a computational approach to investigating problems in revealed preference theory. We believe that this would be of mutual interest to both theoretical computer scientists as well as economists. There are several problems in microeconomic theory that have the combinatorial structure that lends itself to investigation through a theoretical computer scientific lens and opens yet another front in the fast maturing field of algorithmic game theory. Results on intractability as well as finding feasible algorithms for constructing utility functions that are consistent with consumer choice would be of immense benefit to studies and field experiments in empirical economics.

It is hoped that the two problems in connection with rationalizability of matchings and network formation games that we looked into in this dissertation clarify the outlines of computational revealed preference theory. Needless to say, these are only two of a large cachet of possible problem settings in this area. Moreover, we believe that there is a deep and fundamental connection between a large body of problems in revealed preference theory and the class of inequality-satisfiability problems. We demonstrated this connection in both the settings that we considered. It would be heartening to see if this line of enquiry is pursued further and a stronger nexus can be established between two seemingly disparate classes of problems originating from economics and theoretical computer science respectively.

The complexity-theoretic perspective to revealed preference theory we offered in this dissertation is but one dimension. Yet another dimension concerns applying techniques from statistical learning theory to understand and model the learnability of consumer choice functions, and using them to set up forecasting models that predict consumer demand in the face of different choice-sets or budget constraints. We reviewed some results that are currently known in this regard, but we believe again that there is scope for more interesting research that can fully exploit what is known from machine-learning and deploy that machinery towards problems in revealed preference theory and econometrics.

In addition to computational revealed preference theory, we believe that another subject within economics that is ripe for theoretical computer science to contribute to is that of bounded rationality. Indeed, if anything, the notion of bounded rationality offers an even more direct connection since the assumptions underlying bounded rationality concern the limited computational power (both time and space) of agents when they play their optimal strategies. In one sense, it can be argued that revealed preference theory is intended to capture an ex-post facto analysis of bounded rationality since the theory argues and rationalizes consumer choices after the fact. We believe that there are significant and exciting questions to be looked into and answered in this regard.

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