ANALYSIS OF OPTIMALIZING CONTROL SYSTEMS WITH SPECIAL REFERENCE TO NOISE INTERFERENCE EFFECTS

Thesis by

Sedat Serdengecti

In Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1955

ACKNOWLEDGEMENTS

The author wishes to express his sincere appreciation for the invaluable suggestions and criticisms of Dr. Hsue-Shen Tsien made during the course of this work. He is indebted to Dr. Frank E. Marble for introducing him to the present field of study.

ABSTRACT

The optimalizing control system is designed to search out automatically the optimum state of operation of the controlled system and to confine the operation to a region near this optimum state. The performance of the system is affected by the dynamics of the controlled system and by the noise interference.

The dynamic effects of the controlled system on the performance of a peak-holding optimalizing control is analyzed under the assumption that the controlled system dynamics may be represented by a first order input linear group and a first order output linear group. Design charts are constructed for determining the required input drive speed and consequent hunting loss with specified time constants of the input and output linear groups, the hunting period, and the critical indicated difference for input drive reversal.

The noise interference effects on the control system performance led to a new type of optimalizing control system which is a modification of the peak-holding optimalizing controller. Performance of the modified optimalizing controller is analyzed and several possible procedures are discussed for detecting and eliminating the incorrect operation modes. A statistical analysis is made to demonstrate the efficiency of a typical detection procedure, namely, the method of filtering through cross-correlation.

The modified optimalizing controller can utilize any periodically varying input. An example of this, a sinusoidal input controller, is analyzed to show the dynamic effects of the controlled system and to demonstrate the effect of noise interference on the performance of the modified optimalizing controller.

TABLE OF CONTENTS

TABLE OF CONTENTS (Cont'd)

NOMENCLATURE

A	amplitude of sinusoidal signal used for cross-correlation
a	amplitude of saw-tooth input
a'	amplitude of sinusoidal input
b	amount of incorrect input
c	critical indicated difference
D	hunting loss
F_i	transfer function of input linear group
F_o	transfer function of output linear group
f_2	function containing controlled system dynamic characteristics
g_1, g_2	functions containing noise characteristics
k	characteristic constant of the controlled system
l	dimensionless parameter indicating amount of incorrect input
m	number of input cycles which cross-correlation is carried out
N	input drive speed
$n(t)$	noise function
$R(\tau)$	cross-correlation function
T	hunting period of output in correct operating mode
t	time
x_o	physical input corresponding to maximum output
x^*	correct "potential" input referred to optimum point
x	correct "actual" input referred to optimum point
y_o	maximum physical output
y^*	correct "potential" output referred to optimum point
y	correct "actual" output referred to optimum point

α	coefficient in cross-correlation function
β	coefficient in cross-correlation function
Δ	hunting zone
ε	probability of error
ζ	statistical variable
λ	parameter containing noise effects
ξ	statistical variable
σ	standard deviation
τ	time displacement
τ_i	characteristic time of input linear group
τ_o	characteristic time of output linear group
$\Phi_{nn}(\omega)$	noise power spectrum
$\chi(\tau)$	cross-correlation function
ω	circular frequency
ω_i	circular input hunting frequency
ω_o	circular output hunting frequency

Subscripts

s	operation with sinusoidal input - correct mode
u	operation with saw-tooth form input - incorrect mode
α	associated with parameter α
β	associated with parameter β
ζ	associated with statistical variable ζ
ξ	associated with statistical variable ξ
σ	operation with sinusoidal input - incorrect mode

I. INTRODUCTION

Optimalizing control was invented by C. S. Draper, Y. T. Li, and H. Laning, Jr.[1][2] Their basic idea can be summarized as follows: In nearly all engineering systems, within the restrictions of operation, there is a state of the system for which optimum performance is obtained. For an internal combustion engine, producing the required load torque at a specified speed, the minimum fuel consumption is obtained at optimum settings for the manifold pressure and ignition timing. Another example is encountered when it is desired to cruise an airplane at maximum economy. At the engine cruising r.p.m. and assigned altitude, there is an optimum trim setting to obtain maximum cruising speed.

More important than the existence of an optimum operating state is the fact that, because of the natural changes in the environment of the engineering system, this state cannot be predicted exactly in advance. For the internal combustion engine it is the effects of temperature and humidity of the air that are unpredictable; for the airplane it is variations in the aerodynamic properties of the airplane and the performance of the engine that change the optimum operating state. Therefore, when the purpose is to operate always near the optimum state in spite of the "drift" of the system, then the control device for the engineering system must be so designed as to search out automatically the optimum state of operation and to confine the operation near to this state. This is the basic idea of optimalizing control.

The application of Draper's optimalizing control to the general cruise control of airplanes was discussed by J. R. Shull[3].

Shull emphasized the possible elimination of extensive flight testing of a new airplane for performance determination, because the optimalizing control will automatically measure the performance whenever the airplane is flown. Moreover, in critical circumstances, such as flight through an icing atmosphere, the ability of the optimalizing control to extract the best performance of a system, radically changed through ice deposition on the airplane, could be of utmost importance.

There are two fundamental problems in the theory of optimalizing control: One is to determine dynamic performance of the system in combination with the controller, the other is to minimize the effects of noise interference. These two problems are somewhat interrelated, because if the large deviations from the optimum state cannot be tolerated, then the signal to the controller will be small and noise interference will be critical. The basic aim in the design of an optimalizing control is to have the smallest loss or to operate as close to the optimum state as possible without the danger of having the control misled by noise interference. Both of these problems were recognized by the original inventors of the optimalizing control.

The aim of the present work is twofold. First the problem of dynamic performance is solved completely under the assumption that the dynamic properties of the controlled system can be approximated by a first order linear system. For this purpose the analysis was performed on the optimalizing control of the peak-holding type--the type which is least affected by noise interference, [1] [2]. Secondly, the problem of elimination of noise interference is considered. For this purpose, an output sensing instrument consisting of a cross-correlator and a computer are substituted for the conventional peak-holding optimalizing controller. This method is based on the idea that the noise interference can be eliminated more successfully in this manner than with the use of conventional band-pass filters. The resulting control system utilizes any convenient periodic input. Analyses for optimalizing controllers using a saw-tooth input and a sinusoidal input are carried out in detail.

II. DYNAMIC EFFECTS IN OPTIMALIZING CONTROL - THE PEAK-HOLDING TYPE

2.1 Principle of Operation

The essential element of the optimalizing control system is the nonlinear component which characterizes the optimum operating condition of the controlled system. For simplicity of discussion, it is assumed that the basic component has a single input and a single output. For the time being, the dynamic effects will be neglected and the output is assumed to be determined by the instantaneous value of the input. Since there is an optimum point, output as a function of input has a maximum value y_0 of the output at the input x_0, as shown in Fig. 2.1. It is convenient to refer the output and the input to values at the optimum point and designate the physical input as $x + x_0$, and the physical output as $y^* + y_0$. The optimum point is thus $x = y^* = 0$. The purpose of an optimalizing control is to search out this optimum point and to keep the system operating in the immediate neighborhood of this point. In this neighborhood, the relation between x and y^* can be represented as

$$y^* = -k x^2 \tag{2.1}$$

where k is a characteristic constant of the controlled system.

The operation of a peak-holding optimalizing control, neglecting the dynamic effects, then would be as follows: Say the input x is below the optimum value, and is thus negative. The input drive is then set to increase the input at a constant rate. At the time instant 1 (Fig. 2.2) the input changes from negative to positive and passes through the optimum point. The output y^* is thus maximum at the instant 1, and is decreasing after the instant 1. An output sensing instrument is designed so as to follow the output exactly while the output is increasing, and to hold the maximum value after the maximum is passed and the output starts to decrease. Then there will be a difference between the reading of this output sensing instrument and the output itself after the time instant 1. This difference is shown in the lower graph of Fig. 2.2. When this difference is built up to a critical value c at the time instant 2, the input drive unit is reversed. After the instant 2, the input decreases and the output increases until a maximum in output is again reached at the time instant 3. At time instant 3, the input again passes from positive to negative, and the indicated difference between the output sensing instrument and the output itself again builds up. At the time instant 4, the difference reaches the critical value c again and the input drive direction is again reversed. At the time instant 5, the input x becomes zero again, and another maximum of the output is reached. The period of the input variation is thus the time interval from the instant 1 to the instant 5; and the input, when plotted as a function of time, consists of a series of straight-line segments forming a saw-tooth variation. The period of output variation is the time interval from the instant 1 to the instant 3; and the output, when plotted as a function of time, consists of a series of parabolic arcs. The periodic variations of input and output are called the hunting of the system, and the period of output variation is called the hunting period T. The period of input variation is thus $2T$.

The extreme variation of output Δ (Fig. 2.2) is called the hunting zone. If a is the amplitude of the saw-tooth variation of the input (Fig. 2.2), then due to Eq. (2.1)

$$\Delta = ka^2 \tag{2.2}$$

The difference between the maximum output and the average output of the hunting system is called the hunting loss D (Fig. 2.2). Because of the fact that the output is a series of parabolic arcs,

$$D = \frac{1}{3}\Delta = \frac{1}{3}ka^2 \tag{2.3}$$

For this idealized case, the critical indicated difference c between the output sensing instrument and the output itself is equal to Δ. It is then clear that in order to reduce the hunting loss for better efficiency of the system, the amplitude of input variation, and hence the width of the hunting zone must be reduced. Unfortunately the critical indicated difference is also reduced by such modification; and since this quantity is employed as the signal to reverse the input drive, a limit is placed upon its reduction, namely, that noise interference should not generate a false signal for drive reversal.

Although the dynamic effects have been neglected so far, this is not possible in any physical system due to inertial and damping forces. The output y^* given by Eq. (2.1) must be considered then as the fictitious "potential" output but not the actual output measured by the output indicating and sensing instrument. The output y is equal to the "potential" output y^* only when the period of hunting, T, becomes very long. The relation between y^* and y is determined by the combined dynamics of the system and its controller. For the conventional engineering systems, these dynamical effects are determined by a linear relation. For an internal combustion engine, for instance, the "potential" output is essentially the indicated mean effective pressure generated in the engine cylinders, while the actual output is the brake mean effective pressure of the engine. The dynamical effects here are due mainly to the inertia of the piston, the crankshaft, and other moving parts of the engine. For small changes in the operating conditions of engine, such dynamical effects can be represented as a linear differential equation with constant coefficients.

Since the reference levels of input and output are taken to be the optimum input x_0 and the optimum output y_0, the potential physical output is $y^* + y_0$, and the actual physical output is $y + y_0$. Thus the relation between the potential physical output and the actual physical output can be written as an operator equation

$$y + y_0 = F_0\left(\frac{d}{dt}\right)(y^* + y_0) \tag{2.4}$$

where F_0 is generally the quotient of two polynomials in the differential operator $\frac{d}{dt}$. In the language of Laplace transform, then, $F_0(s)$ is the transfer function. Let the linear system which transforms the potential output to the actual output, be called the output linear group. Then $F_0(s)$ is, specifically, the transfer function of the output linear group. By implication however, when the dynamical effects are negligible, or when $s = 0$, the potential output is equal to the actual output. Therefore,

$$F_0(0) = 1 \tag{2.5}$$

Since the drift of the controlled system changes the optimum output y_0 at a rate very low with respect to the hunting frequency, y_0

can be taken as constant. Then the condition given by Eq. (2.5) simplifies Eq. (2.4) to

$$y = F_o\left(\frac{d}{dt}\right) y^* \tag{2.6}$$

In a similar manner, let x^* be the "potential" input which is actually the forcing function generated by the optimalizing control system. It is x^* which has the saw-tooth form shown in Fig. 2.2. The relationship between x, the actual input, and x^*, the potential input, is determined by the inertial and dynamical effects of the input drive system. This input drive system can be called the "input linear group" of the optimalizing control. The operator equation between the potential input x^* and the actual input x is

$$x = F_i\left(\frac{d}{dt}\right) x^* \tag{2.7}$$

The function $F_i(s)$ is thus the transfer function of the input linear group. Similar to Eq. (2.5), the meaning of the potential and the actual inputs implies

$$F_i(0) = 1 \tag{2.8}$$

Thus, a simple representative block diagram of the complete optimalizing control system can be drawn as shown in Fig. 2.3. The nonlinear components of the system are thus the optimalizing input drive and the controlled system itself.

2.2 Formulation of Mathematical Problem

The general relation between the input x and the output y is determined by the system of Eqs. (2.1), (2.6), and (2.7), with the potential input x^* specified as a saw-tooth curve with period $2T$ and amplitude a. Let ω_o be the output hunting frequency defined by

$$\omega_0 = 2\pi/T \tag{2.9}$$

Then x^* can be expanded into a Fourier series,

$$\begin{aligned} x^* &= \frac{8a}{\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\sin\frac{(2n+1)}{2}\omega_0 t \\ &= \frac{8a}{\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\frac{1}{2i}\left[e^{\frac{2n+1}{2}i\omega_0 t} - e^{-\frac{2n+1}{2}i\omega_0 t}\right] \end{aligned} \tag{2.10}$$

Therefore, with Eq. (2.7), the actual input x is given by

$$x = \frac{1}{i}\frac{4a}{\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\left[F_i\left(\frac{2n+1}{2}i\omega_0\right)e^{\frac{2n+1}{2}i\omega_0 t} - F_i\left(-\frac{2n+1}{2}i\omega_0\right)e^{-\frac{2n+1}{2}i\omega_0 t}\right] \tag{2.11}$$

and the potential output y^* is obtained by substituting Eq. (2.11) into Eq. (2.1),

$$\begin{aligned} y^* = \frac{16a^2k}{\pi^4}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{n+m}}{(2n+1)^2(2m+1)^2}&\Big[F_i\left(\tfrac{2n+1}{2}i\omega_0\right)F_i\left(\tfrac{2m+1}{2}i\omega_0\right)e^{(n+m+1)i\omega_0 t} - F_i\left(\tfrac{2n+1}{2}i\omega_0\right)F_i\left(-\tfrac{2m+1}{2}i\omega_0\right)e^{(n-m)i\omega_0 t} \\ &- F_i\left(-\tfrac{2n+1}{2}i\omega_0\right)F_i\left(\tfrac{2m+1}{2}i\omega_0\right)e^{-(n-m)i\omega_0 t} + F_i\left(-\tfrac{2n+1}{2}i\omega_0\right)F_i\left(-\tfrac{2m+1}{2}i\omega_0\right)e^{-(n+m+1)i\omega_0 t}\Big] \end{aligned} \tag{2.12}$$

Finally, using Eq. (2.6), the actual output y is given by

$$\begin{aligned} y = \frac{16a^2k}{\pi^4}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{n+m}}{(2n+1)^2(2m+1)^2}&\Big[F_0\{(n+m+1)i\omega_0\}F_i\left(\tfrac{2n+1}{2}i\omega_0\right)F_i\left(\tfrac{2m+1}{2}i\omega_0\right)e^{(n+m+1)i\omega_0 t} \\ &- F_0\{(n-m)i\omega_0\}F_i\left(\tfrac{2n+1}{2}i\omega_0\right)F_i\left(-\tfrac{2m+1}{2}i\omega_0\right)e^{(n-m)i\omega_0 t} - F_0\{-(n-m)i\omega_0\}F_i\left(-\tfrac{2n+1}{2}i\omega_0\right)F_i\left(\tfrac{2m+1}{2}i\omega_0\right)e^{-(n-m)i\omega_0 t} \\ &+ F_0\{-(n+m+1)i\omega_0\}F_i\left(-\tfrac{2n+1}{2}i\omega_0\right)F_i\left(-\tfrac{2m+1}{2}i\omega_0\right)e^{-(n+m+1)i\omega_0 t}\Big] \end{aligned} \tag{2.13}$$

Comparison of Eqs. (2.11) and (2.13) shows that the input has half of the frequency of the output. This is to be expected from the basic parabolic relation between input and output, specified by Eq. (2.1).

The time average of the actual output y, referred here to the optimum output y_0, gives the hunting loss D directly. Eq.

(2.13) shows that this average value is the sum of the terms with $n = m$ from the second and the third terms of that equation. Therefore, by using Eq. (2.5),

$$D = \frac{32 a^2 k}{\pi^4} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} F_i\left(\frac{2n+1}{2} i\omega_0\right) F_i\left(-\frac{2n+1}{2} i\omega_0\right) \tag{2.14}$$

This equation can be easily checked by observing that when the dynamic effects are absent $F_i = 1$ the series can be summed to give $D = \frac{1}{3} a^2 k$, as required by Eq. (2.3). Eq. (2.14) shows also that the average output and hence the hunting loss are independent of the output linear group. This agrees with one's physical understanding; for a conservative system only the detailed time variation of the output is modified by the dynamics of the output linear group. In case of an internal combustion engine, the average output specifies the power of the engine. The dynamics of the output linear group is determined by the inertia of the moving parts. The power of the engine is certainly independent of the inertia of the moving parts.

Eqs. (2.11) to (2.14) fully determine the performance of the optimalizing control system once the values of a, k, ω_0 are specified, and the transfer functions $F_i(s)$ and $F_o(s)$ of the input linear group and the output linear group are given. The following sections give the detailed calculations and results for the case of first order input and output groups.

2.3 First Order Input and Output Groups

The output hunting frequency ω_0 of the optimalizing control is usually quite low, so that important dynamic effects come from the inertia terms in the input and the output linear groups. Then these linear groups can be closely approximated by first order systems. In other words, their transfer functions are

$$F_i(i\omega) = \frac{1}{1+i\omega\tau_i} \tag{2.15}$$

and

$$F_o(i\omega) = \frac{1}{1+i\omega\tau_o} \tag{2.16}$$

where τ_i and τ_i are the characteristic time constants of the input linear group and the output linear group, respectively. It is evident that these transfer functions satisfy the conditions of Eqs. (2.5) and (2.8).

By substituting Eq. (2.15) into Eq. (2.11), the actual output x is given by

$$x = \frac{8a}{\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{2i(2n+1)^2}\left[\frac{e^{\frac{2n+1}{2}i\omega_o t}}{1+(2n+1)i\frac{\omega_o\tau_i}{2}} - \frac{e^{-\frac{2n+1}{2}i\omega_o t}}{1-(2n+1)i\frac{\omega_o\tau_i}{2}}\right]$$

$$= \frac{8a}{\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n\left[\sin(2n+1)\frac{\omega_o t}{2} - (2n+1)\frac{\omega_o\tau_i}{2}\cos(2n+1)\frac{\omega_o t}{2}\right]}{(2n+1)^2\left[1+(2n+1)^2\left(\frac{\omega_o\tau_i}{2}\right)^2\right]}$$

Resolving this expression into following partial fraction form:

$$x = \frac{8a}{\pi^2}\left\{\sum_{n=0}^{\infty}\frac{(-1)^n\sin(2n+1)\frac{\omega_o t}{2}}{(2n+1)^2} - \left(\frac{\omega_o\tau_i}{2}\right)^2\sum_{n=0}^{\infty}\frac{(-1)^n\sin(2n+1)\frac{\omega_o t}{2}}{\left[1+(2n+1)^2\left(\frac{\omega_o\tau_i}{2}\right)^2\right]} - \left(\frac{\omega_o\tau_i}{2}\right)\sum_{n=0}^{\infty}\frac{(-1)^n\cos(2n+1)\frac{\omega_o t}{2}}{(2n+1)}\right.$$

$$\left. + \left(\frac{\omega_o\tau_i}{2}\right)^3\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)\cos(2n+1)\frac{\omega_o t}{2}}{\left[1+(2n+1)^2\left(\frac{\omega_o\tau_i}{2}\right)^2\right]}\right\} \tag{2.17}$$

When the summations are carried out (see Appendix A), Eq. (2.17) yields the following equation for the input x :

$$x = (-1)^j NT\left\{\left(\frac{t}{T} - j\right) - \frac{\tau_i}{T}\left[1 - \frac{e^{-\frac{(\frac{t}{T}-j)}{\tau_i/T}}}{\cosh\frac{1}{2}\frac{T}{\tau_i}}\right]\right\} \quad \text{when} \quad -\frac{1}{2} \leq \left(\frac{t}{T} - j\right) \leq \frac{1}{2}$$
$$\text{where} \quad j = 0, 1, 2, \cdots \tag{2.18}$$

where N is the constant input drive speed, i.e.,

$$N = 2a/T \tag{2.19}$$

By use of Eq. (2.18), the variation of the actual input x with respect to time can be calculated for any specified data. Examples of such calculations are shown in Figs. 2.4 and 2.5 for $\tau_i/T = 0.1$ and $\tau_i/T = 0.4$, respectively. Both figures show the expected effects of rounding off the sharp corners of the saw-tooth curve and of a time delay. It is of interest to note that while the delay is very nearly equal to τ_i itself for small τ_i/T, the delay is less than τ_i for larger τ_i/T.

With the first order input transfer function given by Eq. (2.15), the hunting loss given by Eq. (2.14) becomes

$$D = \frac{32a^2k}{\pi^4}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^4\left[1 + (2n+1)^2\left(\frac{\omega_0\tau_i}{2}\right)^2\right]}$$

$$= \frac{32a^2k}{\pi^4}\left\{\sum_{n=0}^{\infty}\frac{1}{(2n+1)^4} - \left(\frac{\omega_0\tau_i}{2}\right)^2\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} + \left(\frac{\omega_0\tau_i}{2}\right)^4\sum_{n=0}^{\infty}\frac{1}{1 + (2n+1)^2\left(\frac{\omega_0\tau_i}{2}\right)^2}\right\}$$

Carrying out the summations in the last expression, the hunting loss is obtained as

$$D = N^2T^2k\left[\frac{1}{12} - \left(\frac{\tau_i}{T}\right)^2 + 2\left(\frac{\tau_i}{T}\right)^3\tanh\frac{1}{2}\frac{T}{\tau_i}\right] \tag{2.20}$$

Fig. 2.6 shows a dimensionless plot of this equation.

To calculate the actual output y, both Eqs. (2.15) and (2.16) are substituted into Eq. (2.13), i.e.,

$$y = \frac{4T^2N^2k}{\pi^4}\sum_{n=0}^{\infty}{}' \sum_{m=0}^{\infty}{}' \frac{(-1)^{n+m}}{(2n+1)^2(2m+1)^2}\left[\frac{e^{i(n+m+1)\omega_0 t}}{\{1+(n+m+1)i\omega_0\tau_0\}\{1+(2n+1)i\frac{\omega_0\tau_i}{2}\}\{1+(2m+1)i\frac{\omega_0\tau_i}{2}\}}\right.$$

$$-\frac{e^{i(n-m)\omega_0 t}}{\{1+(n-m)i\omega_0\tau_0\}\{1+(2n+1)i\frac{\omega_0\tau_i}{2}\}\{1-(2m+1)i\frac{\omega_0\tau_i}{2}\}} - \frac{e^{-i(n-m)\omega_0 t}}{\{1-(n-m)i\omega_0\tau_0\}\{1-(2n+1)i\frac{\omega_0\tau_i}{2}\}\{1+(2m+1)i\frac{\omega_0\tau_i}{2}\}}$$

$$\left. + \frac{e^{-i(n+m+1)\omega_0 t}}{\{1-(n+m+1)i\omega_0\tau_0\}\{1-(2n+1)i\frac{\omega_0\tau_i}{2}\}\{1-(2m+1)i\frac{\omega_0\tau_i}{2}\}}\right] \qquad (2.21)$$

To simplify the summation operations involved in Eq. (2.21) a new summation index $s = n+m+1$ is introduced for the first and fourth terms of this equation. For the resulting summations s takes integer values from 1 to ∞ and n takes integer values from 0 to $s-1$. With this change the first and fourth terms of Eq. (2.21) are respectively

$$\frac{4T^2N^2k}{\pi^4}\sum_{s=1}^{\infty}{}' \frac{(-1)^{s-1}e^{is\omega_0 t}}{\{1+is\omega_0\tau_0\}}\sum_{n=0}^{s-1}{}' \frac{1}{(2n+1)^2\{(2n+1)-2s\}^2\{1+(2n+1)i\frac{\omega_0\tau_i}{2}\}\{1-[(2n+1)-2s]i\frac{\omega_0\tau_i}{2}\}} \qquad (2.22a)$$

and

$$\frac{4T^2N^2k}{\pi^4}\sum_{s=1}^{\infty}{}' \frac{(-1)^{s-1}e^{-is\omega_0 t}}{\{1-is\omega_0\tau_0\}}\sum_{n=0}^{\infty}{}' \frac{1}{(2n+1)^2\{(2n+1)-2s\}^2\{1-(2n+1)i\frac{\omega_0\tau_i}{2}\}\{1+[(2n+1)-2s]i\frac{\omega_0\tau_i}{2}\}} \qquad (2.22b)$$

Similarly for the second and third terms of Eq. (2.21) the summation index $s = n-m$ is introduced. In this case, if $s \geqslant 0$, s takes integer values from 0 to ∞, and n takes integer values from s to ∞; and if $s<0$, s takes integer values from $-\infty$ to -1, and n takes integer values from 0 to ∞. With this change the second and third terms of Eq. (2.21) are respectively,

$$\frac{4T^2N^2k}{\pi^4}\left[-\sum_{n=0}^{\infty}{}'\frac{1}{(2n+1)^4\{1+(2n+1)^2(\frac{\omega_0\tau_i}{2})^2\}}+\sum_{s=1}^{\infty}{}'\frac{(-1)^{s-1}e^{is\omega_0 t}}{\{1+is\omega_0\tau_0\}}\sum_{n=s}^{\infty}{}'\frac{1}{(2n+1)^2\{(2n+1)-2s\}^2\{1+(2n+1)i\frac{\omega_0\tau_i}{2}\}\{1-[(2n+1)-2s]i\frac{\omega_0\tau_i}{2}\}}\right.$$

$$\left.+\sum_{s=-\infty}^{-1}{}'\frac{(-1)^{s-1}e^{is\omega_0 t}}{\{1+is\omega_0\tau_0\}}\sum_{n=0}^{\infty}{}'\frac{1}{(2n+1)^2\{(2n+1)-2s\}^2\{1+(2n+1)i\frac{\omega_0\tau_i}{2}\}\{1-[(2n+1)-2s]i\frac{\omega_0\tau_i}{2}\}}\right] \qquad (2.22c)$$

and

$$\frac{4T^2N^2k}{\pi^4}\left[-\sum_{n=0}^{\infty}{}'\frac{1}{(2n+1)^4\{1+(2n+1)^2(\frac{\omega_0\tau_i}{2})^2\}}+\sum_{s=1}^{\infty}{}'\frac{(-1)^{s-1}e^{-is\omega_0 t}}{\{1-is\omega_0\tau_0\}}\sum_{n=s}^{\infty}{}'\frac{1}{(2n+1)^2\{(2n+1)-2s\}^2\{1-(2n+1)i\frac{\omega_0\tau_i}{2}\}\{1+[(2n+1)-2s]i\frac{\omega_0\tau_i}{2}\}}\right.$$

$$\left.+\sum_{s=-\infty}^{-1}{}'\frac{(-1)^{s-1}e^{-is\omega_0 t}}{\{1-is\omega_0\tau_0\}}\sum_{n=0}^{\infty}{}'\frac{1}{(2n+1)^2\{(2n+1)-2s\}^2\{1-(2n+1)i\frac{\omega_0\tau_i}{2}\}\{1+[(2n+1)-2s]i\frac{\omega_0\tau_i}{2}\}}\right] \qquad (2.22d)$$

Combining Eqs. (2.22a), (2.22b), (2.22c), and (2.22d), Eq. (2.21) can be written as

$$y=\frac{4T^2N^2k}{\pi^4}\left[\sum_{s=-\infty}^{\infty}{}'\frac{(-1)^{s-1}e^{is\omega_0 t}}{\{1+is\omega_0\tau_0\}}\sum_{n=0}^{\infty}{}'\frac{1}{(2n+1)^2\{(2n+1)-2s\}^2\{1+(2n+1)i\frac{\omega_0\tau_i}{2}\}\{1-[(2n+1)-2s]i\frac{\omega_0\tau_i}{2}\}}\right.$$

$$\left.+\sum_{s=-\infty}^{\infty}{}'\frac{(-1)^{s-1}e^{-is\omega_0 t}}{\{1-is\omega_0\tau_0\}}\sum_{n=0}^{\infty}{}'\frac{1}{(2n+1)^2\{(2n+1)-2s\}^2\{1-(2n+1)i\frac{\omega_0\tau_i}{2}\}\{1+[(2n+1)-2s]i\frac{\omega_0\tau_i}{2}\}}\right]$$

or

$$y=\frac{8T^2N^2k}{\pi^4}\left[-\sum_{n=0}^{\infty}{}'\frac{1}{(2n+1)^4\{1+(2n+1)^2(\frac{\omega_0\tau_i}{2})^2\}}+\sum_{s=1}^{\infty}{}'\frac{(-1)^{s-1}e^{is\omega_0 t}}{(1+is\omega_0\tau_0)}\sum_{n=0}^{\infty}{}'\frac{\{(2n+1)^2+4s^2\}\{(1+i\omega_0\tau_i s)+(\frac{\omega_0\tau_i}{2})^2(2n+1)^2\}+8(\frac{\omega_0\tau_i}{2})^2s^2(2n+1)^2}{(2n+1)^2\{(2n+1)^2-4s^2\}^2\{1+(\frac{\omega_0\tau_i}{2})^2(2n+1)^2\}\{(1+i\omega_0\tau_i s)^2+(\frac{\omega_0\tau_i}{2})^2(2n+1)^2\}}\right.$$

$$\left.+\sum_{s=1}^{\infty}{}'\frac{(-1)^{s-1}e^{-is\omega_0 t}}{\{1-is\omega_0\tau_0\}}\sum_{n=0}^{\infty}{}'\frac{\{(2n+1)^2+4s^2\}\{(1-i\omega_0\tau_i s)+(\frac{\omega_0\tau_i}{2})^2(2n+1)^2\}+8(\frac{\omega_0\tau_i}{2})^2s^2(2n+1)^2}{(2n+1)^2\{(2n+1)^2-4s^2\}^2\{1+(\frac{\omega_0\tau_i}{2})^2(2n+1)^2\}\{(1-is\omega_0\tau_i)^2+(\frac{\omega_0\tau_i}{2})^2(2n+1)^2\}}\right]$$

$$(2.22)$$

The last two summations in Eq. (2.22) are complex conjugates of each other; thus

$$y=\frac{8T^2N^2k}{\pi^4}\left[-\sum_{n=0}^{\infty}{}'\frac{1}{(2n+1)^4\{1+(2n+1)^2(\frac{\omega_0\tau_i}{2})^2\}}+2\,\mathrm{Re}\sum_{s=1}^{\infty}{}'\frac{(-1)^{s-1}e^{is\omega_0 t}}{(1+is\omega_0\tau_0)}\sum_{n=0}^{\infty}{}'\frac{\{(2n+1)^2+4s^2\}\{(1+i\omega_0\tau_i s)+(\frac{\omega_0\tau_i}{2})^2(2n+1)^2\}+8(\frac{\omega_0\tau_i}{2})^2s^2(2n+1)^2}{(2n+1)^2\{(2n+1)^2-4s^2\}^2\{1+(\frac{\omega_0\tau_i}{2})^2(2n+1)^2\}\{(1+is\omega_0\tau_i)^2+(\frac{\omega_0\tau_i}{2})^2(2n+1)^2\}}\right]$$

$$(2.23)$$

where $\mathcal{R}\ell$ denotes the real part of the expression following it. In order to carry out the summation with respect to the index n, Eq. (2.23) is resolved into the following partial fraction form:

$$
\begin{aligned}
y = \frac{8T^2N^2k}{\pi^4}\Bigg[& -\sum_{n=0}^{\infty}{}' \frac{1}{(2n+1)^4\left\{1+(2n+1)^2\left(\frac{\omega_0\tau_i}{2}\right)^2\right\}} + 2\,\mathcal{R}\ell \sum_{s=1}^{\infty}{}' \frac{(-1)^{s-1}e^{is\omega_0 t}}{(1+is\omega_0\tau_0)}\Bigg\{\frac{1}{4s^2(1+is\omega_0\tau_i)}\sum_{n=0}^{\infty}{}'\frac{1}{(2n+1)^2} \\
& + \frac{\left(\frac{\omega_0\tau_i}{2}\right)^4}{2(1+is\omega_0\tau_i)^2\left(1+is\frac{\omega_0\tau_i}{2}\right)}\sum_{n=0}^{\infty}{}'\frac{1}{\left\{1+\left(\frac{\omega_0\tau_i}{2}\right)^2(2n+1)^2\right\}} + \frac{\left(\frac{\omega_0\tau_i}{2}\right)^4}{2(1+is\omega_0\tau_i)\left(1+is\frac{\omega_0\tau_i}{2}\right)}\sum_{n=0}^{\infty}{}'\frac{1}{\left\{(1+i\omega_0\tau_i s)^2+\left(\frac{\omega_0\tau_i}{2}\right)^2(2n+1)^2\right\}} \\
& - \frac{1+is\omega_0\tau_i + 4\left(\frac{\omega_0\tau_i}{2}\right)^2 s^2}{4s^2(1+is\omega_0\tau_i)^2}\sum_{n=0}^{\infty}{}'\frac{1}{\left\{(2n+1)^2+(2is)^2\right\}} + \frac{2}{(1+is\omega_0\tau_i)}\sum_{n=0}^{\infty}\frac{1}{\left\{(2n+1)^2+(2is)^2\right\}^2}\Bigg\}\Bigg]
\end{aligned}
\tag{2.24}
$$

By use of the summation formulae given in Appendix A, the sums with respect to n can be evaluated, and the result is,

$$
\begin{aligned}
y = \frac{8T^2N^2k}{\pi^4}\Bigg[& -\left\{\frac{\pi^4}{96} - \frac{\pi^2}{8}\left(\frac{\omega_0\tau_i}{2}\right)^2 + \frac{\pi}{4}\left(\frac{\omega_0\tau_i}{2}\right)^3\tanh\frac{\pi}{\omega_0\tau_i}\right\} + 2\,\mathcal{R}\ell\sum_{s=1}^{\infty}{}'\frac{(-1)^{s-1}e^{is\omega_0 t}}{(1+i\omega_0\tau_0 s)}\Bigg\{\frac{1}{4s^2(1+i\omega_0\tau_i s)}\frac{\pi^2}{8} \\
& + \frac{\left(\frac{\omega_0\tau_i}{2}\right)^4}{2(1+i\omega_0\tau_i s)^2\left(1+is\frac{\omega_0\tau_i}{2}\right)}\frac{\pi}{4\left(\frac{\omega_0\tau_i}{2}\right)}\tanh\left(\frac{\pi}{\omega_0\tau_i}+i\pi s\right) - \frac{(1+i\omega_0\tau_i s)+4\left(\frac{\omega_0\tau_i}{2}\right)^2 s^2}{4s^2(1+i\omega_0\tau_i s)^2}\frac{\pi}{4(i2s)}\tanh(i\pi s) \\
& + \frac{2}{(1+i\omega_0\tau_i s)}\left[\frac{\pi}{8}\frac{1}{(i2s)^3}\tanh i\pi s + \frac{\pi^2}{16}\frac{1}{(-4s^2)}(\tanh i\pi s - 1)\right]\Bigg\}\Bigg]
\end{aligned}
$$

and noting that $\tan \pi s = 0$ for integer values of s,

$$
y = \frac{8T^2N^2k}{\pi^4}\left[-\left\{\frac{\pi^4}{96} - \frac{\pi^2}{8}\left(\frac{\omega_0\tau_i}{2}\right)^2 + \frac{\pi}{4}\left(\frac{\omega_0\tau_i}{2}\right)^3\tanh\frac{\pi}{\omega_0\tau_i}\right\} + 2\,\mathcal{R}\ell\sum_{s=1}^{\infty}{}'\frac{(-1)^{s-1}e^{is\omega_0 t}}{(1+is\omega_0\tau_0)}\left\{\frac{\pi^2}{4}\frac{1}{(4s^2)(1+is\omega_0\tau_i)} + \frac{\pi}{4}\frac{\left(\frac{\omega_0\tau_i}{2}\right)^3\tanh\frac{\pi}{\omega_0\tau_i}}{(1+is\omega_0\tau_i)^2\left(1+is\frac{\omega_0\tau_i}{2}\right)}\right\}\right]
\tag{2.25}
$$

Eq. (2.25) is again resolved into partial fractions in order to carry out the summations with respect to s, viz.,

$$
\begin{aligned}
y = \frac{8T^2N^2k}{\pi^4}\Bigg[& -\left\{\frac{\pi^4}{96} - \frac{\pi^2}{8}\left(\frac{\omega_0\tau_i}{2}\right)^2 + \frac{\pi}{4}\left(\frac{\omega_0\tau_i}{2}\right)^3 \tanh\frac{\pi}{\omega_0\tau_i}\right\} + \frac{\pi}{2}\left\{\frac{\left(\frac{\omega_0\tau_0}{2}\right)^3}{\left(\frac{\omega_0\tau_0}{2} - \frac{\omega_0\tau_i}{2}\right)}\left[\frac{2\left(\frac{\omega_0\tau_i}{2}\right)^3 \tanh\frac{\pi}{\omega_0\tau_i}}{\left(\frac{\omega_0\tau_0}{2} - \frac{\omega_0\tau_i}{2}\right)\left(\omega_0\tau_0 - \frac{\omega_0\tau_i}{2}\right)} - \pi\right] \mathrm{R\ell}\sum_{s=1}^{\infty} \frac{(-1)^{s-1} e^{is\omega_0 t}}{(1+is\omega_0\tau_0)} \right. \\
& + \frac{\left(\frac{\omega_0\tau_i}{2}\right)^3}{\left(\frac{\omega_0\tau_0}{2} - \frac{\omega_0\tau_i}{2}\right)}\left[\pi - \frac{2\left(\frac{\omega_0\tau_i}{2}\right)^2 \tanh\frac{\pi}{\omega_0\tau_i}}{\left(\frac{\omega_0\tau_0}{2} - \frac{\omega_0\tau_i}{2}\right)}\right] \mathrm{R\ell}\sum_{s=1}^{\infty} \frac{(-1)^{s-1} e^{is\omega_0 t}}{(1+is\omega_0\tau_i)} - \frac{\pi}{2}\left(\frac{\omega_0\tau_0}{2} + \frac{\omega_0\tau_i}{2}\right) \mathrm{R\ell}\sum_{s=1}^{\infty} \frac{(-1)^{s-1}\, i\, e^{is\omega_0 t}}{s} \\
& \left. + \frac{\pi}{4}\mathrm{R\ell}\sum_{s=1}^{\infty} \frac{(-1)^{s-1} e^{is\omega_0 t}}{s^2} - \frac{\left(\frac{\omega_0\tau_i}{2}\right)^4 \tanh\frac{\pi}{\omega_0\tau_i}}{\left(\omega_0\tau_0 - \frac{\omega_0\tau_i}{2}\right)} \mathrm{R\ell}\sum_{s=1}^{\infty} \frac{(-1)^{s-1} e^{is\omega_0 t}}{\left(1+is\frac{\omega_0\tau_i}{2}\right)} - \frac{2\left(\frac{\omega_0\tau_i}{2}\right)^4 \tanh\frac{\pi}{\omega_0\tau_i}}{\left(\frac{\omega_0\tau_0}{2} - \frac{\omega_0\tau_i}{2}\right)} \mathrm{R\ell}\sum_{s=1}^{\infty} \frac{(-1)^{s-1} e^{is\omega_0 t}}{(1+is\omega_0\tau_i)^2}\right\}\Bigg]
\end{aligned}
$$

(2.26)

The result of carrying out the summations in Eq. (2.26) and simplifying the expression is,

$$
\begin{aligned}
y = 2T^2N^2k\Bigg\{ & -\left[\frac{1}{2}\left(\frac{t}{T}-j\right)^2 - \left(\frac{\tau_0}{T}+\frac{\tau_i}{T}\right)\left(\frac{t}{T}-j\right) + \frac{1}{2}\left(\frac{\tau_i}{T}\right)^2 + \frac{\tau_i\tau_0}{T^2} + \left(\frac{\tau_0}{T}\right)^2\right] - \frac{1}{2}\frac{\left(\frac{\tau_0}{T}\right)^2}{\left(\frac{\tau_0}{T}-\frac{\tau_i}{T}\right)}\left[\frac{2\left(\frac{\tau_i}{T}\right)^3 \tanh\frac{1}{2}\frac{T}{\tau_i}}{\left(\frac{\tau_0}{T}-\frac{\tau_i}{T}\right)\left(2\frac{\tau_0}{T}-\frac{\tau_i}{T}\right)} - 1\right]\frac{e^{-\frac{\left(\frac{t}{T}-j\right)}{\tau_0/T}}}{\sinh\frac{1}{2}\frac{T}{\tau_0}} \\
& + \left[\left(\frac{t}{T}-j\right) + \frac{\left(\frac{\tau_i}{T}\right)^2}{\left(\frac{\tau_0}{T}-\frac{\tau_i}{T}\right)}\right]\left[\frac{\left(\frac{\tau_i}{T}\right)^2 e^{-\frac{\left(\frac{t}{T}-j\right)}{\tau_i/T}}}{\left(\frac{\tau_0}{T}-\frac{\tau_i}{T}\right)\cosh\frac{1}{2}\frac{T}{\tau_i}} + \frac{1}{2}\frac{\left(\frac{\tau_i}{T}\right)^3 e^{-\frac{2\left(\frac{t}{T}-j\right)}{\tau_i/T}}}{\left(2\frac{\tau_0}{T}-\frac{\tau_i}{T}\right)\cosh^2\frac{1}{2}\frac{T}{\tau_i}}\right]\Bigg\}
\end{aligned}
$$

when $-\frac{1}{2} \leq \left(\frac{t}{T}-j\right) \leq \frac{1}{2}$

where $j = 0, 1, 2, \ldots$

(2.27)

Special care must be taken in evaluating Eq. (2.27) at the points $\frac{\tau_0}{T} = \frac{\tau_i}{T}$ and $2\frac{\tau_0}{T} = \frac{\tau_i}{T}$. By a simple limiting procedure it can be shown, for example, that when $\frac{\tau_0}{T} = \frac{\tau_i}{T}$,

$$y = 2T^2N^2k\left\{-\frac{1}{2}\left(\frac{t}{T}-j\right)^2+2\left(\frac{\tau_i}{T}\right)\left(\frac{t}{T}-j\right)-\frac{5}{2}\left(\frac{\tau_i}{T}\right)^2+\left[-\left(\frac{\tau_i}{T}\right)^2-\frac{1}{2}\left(\frac{t}{T}-j\right)^2+\left(\frac{\tau_i}{T}\right)\left(\frac{t}{T}-j\right)+\frac{1}{8}\right]\frac{e^{-\frac{\left(\frac{t}{T}-j\right)}{\tau_i/T}}}{\cosh\frac{1}{2}\frac{T}{\tau_i}}+\frac{3}{2}\frac{\tau_i}{T}\frac{e^{-\frac{\left(\frac{t}{T}-j\right)}{\tau_i/T}}}{\sinh\frac{1}{2}\frac{T}{\tau_i}}\right.$$

$$\left.+\frac{1}{2}\left(\frac{\tau_i}{T}\right)^2\frac{e^{-\frac{2\left(\frac{t}{T}-j\right)}{\tau_i/T}}}{\cosh^2\frac{1}{2}\frac{T}{\tau_i}}\right\} \qquad \text{when} \quad -\frac{1}{2}\leq\left(\frac{t}{T}-j\right)\leq\frac{1}{2}$$

$$\text{where} \quad j=0,1,2,\cdots \qquad (2.28)$$

and for $2\frac{\tau_0}{T}=\frac{\tau_i}{T}$

$$y = 2T^2N^2k\left\{-\frac{1}{2}\left(\frac{t}{T}-j\right)^2+\frac{3}{2}\frac{\tau_i}{T}\left(\frac{t}{T}-j\right)-\frac{5}{4}\left(\frac{\tau_i}{T}\right)^2-\left[\frac{\tau_i}{T}\left(\frac{t}{T}-j\right)+2\left(\frac{\tau_i}{T}\right)^2+\frac{1}{2}\frac{\tau_i}{T}\,\mathrm{ctnh}\,\frac{T}{\tau_i}+\frac{1}{8}\frac{\tau_i}{T}\,\mathrm{ctnh}\,\frac{1}{2}\frac{T}{\tau_i}\right]\frac{e^{-\frac{2\left(\frac{t}{T}-j\right)}{\tau_i/T}}}{\cosh^2\frac{1}{2}\frac{T}{\tau_i}}\right.$$

$$\left.-2\frac{\tau_i}{T}\left[\left(\frac{t}{T}-j\right)-2\frac{\tau_i}{T}\right]\frac{e^{-\frac{\left(\frac{t}{T}-j\right)}{\tau_i/T}}}{\cosh\frac{1}{2}\frac{T}{\tau_i}}\right\} \qquad \text{when} \quad -\frac{1}{2}\leq\left(\frac{t}{T}-j\right)\leq\frac{1}{2}$$

$$\text{where} \quad j=0,1,2,\cdots$$

$$(2.29)$$

These results can also be obtained by direct evaluation of Eq. (2.25).

The output $y(t)$ given by Eq. (2.27) is continuous across the boundaries of regions given by various values of j. By direct calculation it is easily shown that the values of the output y and its derivative with respect to t are the same at $\frac{t}{T}=\ell+\frac{1}{2}$, whether they are computed from Eq. (2.27) when $j=\ell$ or when $j=\ell+1$.

Now the "potential" output y^* can be obtained by letting $\frac{\tau_0}{T}=0$ in Eq. (2.27). Thus,

$$y^* = 2N^2T^2k\left\{-\frac{1}{2}\left(\frac{t}{T}-j\right)^2+\frac{\tau_i}{T}\left(\frac{t}{T}-j\right)-\frac{1}{2}\left(\frac{\tau_i}{T}\right)^2-\left[\left(\frac{t}{T}-j\right)-\frac{\tau_i}{T}\right]\frac{\tau_i}{T}\frac{e^{-\frac{\left(\frac{t}{T}-j\right)}{\tau_i/T}}}{\cosh\frac{1}{2}\frac{T}{\tau_i}}-\frac{1}{2}\left(\frac{\tau_i}{T}\right)^2\frac{e^{-\frac{2\left(\frac{t}{T}-j\right)}{\tau_i/T}}}{\cosh^2\frac{1}{2}\frac{T}{\tau_i}}\right\}$$

$$\text{when} \quad -\frac{1}{2}\leq\left(\frac{t}{T}-j\right)\leq\frac{1}{2}$$

$$\text{where} \quad j=0,1,2,\cdots \qquad (2.30)$$

This expression checks with the result of direct calculation of y^* from Eq. (2.12).

Figs. 2.7 and 2.8 show the dimensionless plots of actual output y and potential output y^* for the particular values of τ_o/T and τ_i/T. In these figures it is clearly seen that the dynamic effects not only decrease the output of the system but also introduce a time lag and lower the maximum output of the system. Fig. 2.8 with $\tau_i/T = 0.4$ and $\tau_o/T = 0.6$, has the maximum value of y almost at the very instants of input drive reversal points, $\frac{t}{T} = \ell + \frac{1}{2}$. This is indeed an extreme case.

2.4 Design Charts

From the principle of operation of the peak-holding optimalizing control, it is seen that the most important quantity to be specified for its design is the critical indicated difference c between the reading of the special output sensing instrument and the output itself. By definition, c is the difference of the maximum of the actual output y and the value of y at the tripping instant of the input drive. The instant of reversing the input drive is typified by $\frac{t}{T} = \frac{1}{2}$ when $\ell = 0$. If the corresponding instant of maximum y is t^*, then the critical indicated difference c is given as

$$c = y\left(\frac{t^*}{T}\right) - y\left(\frac{1}{2}\right) \qquad \text{when} \quad \ell = 0 \qquad (2.31)$$

by use of either Eqs. (2.27), (2.28) or (2.29). Since the instant of input drive reversal must come after the instant of maximum output, $\frac{t^*}{T} < \frac{1}{2}$.

To determine t^*, one may use the condition of zero slope, i.e., $dy/dt = 0$. Then Eq. (2.27) when $j = 0$ gives

$$-\left[\frac{t^*}{T} - \left(\frac{\tau_o}{T} + \frac{\tau_i}{T}\right)\right] + \frac{\left(\frac{\tau_o}{T}\right)}{2\left(\frac{\tau_o}{T} - \frac{\tau_i}{T}\right)}\left[\frac{2\left(\frac{\tau_i}{T}\right)^3 \tanh\frac{T}{2\tau_i}}{\left(\frac{\tau_o}{T} - \frac{\tau_i}{T}\right)\left(2\frac{\tau_o}{T} - \frac{\tau_i}{T}\right)} - 1\right]\frac{e^{-\frac{t^*/T}{\tau_o/T}}}{\sinh\frac{T}{2\tau_o}} + \left[1 - \frac{t^*}{T}\frac{T}{\tau_i} - \frac{\tau_i/T}{\frac{\tau_o}{T} - \frac{\tau_i}{T}}\right]\frac{\left(\frac{\tau_i}{T}\right)^2 e^{-\frac{t^*/T}{\tau_i/T}}}{\left(\frac{\tau_o}{T} - \frac{\tau_i}{T}\right)\cosh\frac{T}{2\tau_i}} - \frac{\left(\frac{\tau_i}{T}\right)^2 e^{-\frac{2t^*/T}{\tau_i/T}}}{\left(2\frac{\tau_o}{T} - \frac{\tau_i}{T}\right)\cosh^2\frac{T}{2\tau_i}} = 0 \qquad (2.32)$$

This transcendental equation for t^*/T may be solved by iteration. For instance, for small τ_o/T and τ_i/T, only terms within the first square brackets are of importance; then $t^*/T \simeq (\tau_o + \tau_i)/T$. This is already recognized by Draper and co-workers[1][2]. The complete results of the calculation are shown in Fig. 2.9, which shows that t^*/T is almost a function of $(\tau_o + \tau_i)/T$ alone with minor modifications from the parameter τ_o/τ_i, the ratio of the characteristic times of the output linear group and the input linear group. Values of t^*/T beyond $\frac{1}{2}$ are not shown because then the maxima of the output will occur after the corresponding input drive reversal points and proper operation of the control will be difficult if not impossible.

With t^*/T determined, Eqs. (2.31) gives c by evaluating Eq. (2.27) when $j = 0$. However, the specified quantities of an optimalizing control are k, the characteristics of the controlled system, and τ_o and τ_i, the characteristics of the linear groups. From considerations on the noise interference, the designer can make an appropriate choice of the period T and the critical indicated difference c for input drive reversal. Therefore the quantities which the designer wishes to know after he has the values of k, τ_i, τ_o, T and c, are N, the input drive speed, and D, the hunting loss. Thus the result of calculation with Eq. (2.31) should be written as follows:

$$\frac{TN}{\sqrt{c/k}} = \Bigg[\Big\{\frac{1}{4} \cdot \Big(\frac{t^*}{T}\Big)^2\Big\} + 2\frac{\tau_i}{T}\Big(\frac{\tau_o}{\tau_i}+1\Big)\Big(\frac{t^*}{T}-\frac{1}{2}\Big) - \frac{\left(\frac{\tau_o}{\tau_i}\right)^2\frac{\tau_i}{T}}{\left(\frac{\tau_o}{\tau_i}-1\right)\sinh\frac{1}{2}\frac{\tau_i}{\tau_o}\frac{T}{\tau_i}}\left\{\frac{2\frac{\tau_i}{T}\tanh\frac{T}{2\tau_i}}{\left(\frac{\tau_o}{\tau_i}-1\right)\left(2\frac{\tau_o}{\tau_i}-1\right)} - 1\right\}\left\{e^{-\frac{t^*/T}{(\tau_o/\tau_i)(\tau_i/T)}} - e^{-\frac{1/2}{(\tau_o/\tau_i)(\tau_i/T)}}\right\}$$

$$+\frac{2\frac{\tau_i}{T}}{\left(\frac{\tau_o}{\tau_i}-1\right)\cosh\frac{T}{2\tau_i}}\left\{\left(\frac{t^*}{T}+\frac{\tau_i/T}{\frac{\tau_o}{\tau_i}-1}\right)e^{-\frac{t^*/T}{\tau_i/T}} - \left(\frac{1}{2}+\frac{(\tau_i/T)}{\frac{\tau_o}{\tau_i}-1}\right)e^{-\frac{1/2}{\tau_i/T}}\right\} + \frac{\left(\frac{\tau_i}{T}\right)^2\left\{e^{-\frac{2t^*/T}{\tau_i/T}} - e^{-\frac{1}{\tau_i/T}}\right\}}{\left(2\frac{\tau_o}{\tau_i}-1\right)\cosh^2\frac{1}{2\tau_i}}\Bigg]^{-\frac{1}{2}} \tag{2.33}$$

When N is determined, Eq. (2.20) then gives the hunting loss D.

Figs. 2.10 and 2.11 are the design charts for peak-holding optimalizing control computed from the equations of preceding analysis. Fig. 2.10 gives $TN/\sqrt{c/k}$ as a function of τ_o/τ_i with $(\tau_o+\tau_i)/T$ as parameter. Fig. 2.11 gives the relative hunting loss D/c again as a function of τ_o/τ_i with $(\tau_o+\tau_i)/T$ as parameter. The peaks of curves near $\tau_o/\tau_i = 1$ indicate a sort of resonant effect between the input linear group and output linear group. The hunting loss for fixed $(\tau_o+\tau_i)/T$ and c is smaller for τ_o/τ_i away from unity. For fixed τ_o, τ_i, and c, clearly the way to reduce the hunting loss is to increase the period T.

2.5 Remarks on the Improvement of Control System

The preceding analysis gives the necessary input drive speed N and the hunting loss D for any specified hunting period T, time constants τ_i and τ_o for the input linear group and the output linear group, and the chosen critical indicated difference c. T and c are fixed by considerations on the noise interference. The analysis shows that whenever the hunting period is relatively short with respect to the time constant τ_i and τ_o, or whenever $(\tau_i+\tau_o)/T$ is relatively large, the hunting loss will be large, especially when

τ_i and τ_o are nearly equal. To avoid such an unfavorable condition, the designer should improve his input drive system so as to reduce the constant τ_i. The time constant τ_o is, however, a characteristic of the controlled system and is thus not at the disposal of the designer. However suppose that there is a compensating circuit between the output y and optimalizing input drive unit (Fig. 2.3), such that the effects of the output linear group are completely compensated. Then the effective signal for input drive reversal is not the actual output y, but the potential output y^*. In other words, the value of τ_o is effectively made to be zero. Even if complete compensation is not achieved, the effective value of τ_o can still be greatly reduced. For difficult cases, then, such a compensating unit should certainly be added to reduce the hunting loss. This will be just a minor complication when compared with the additional equipment required for satisfactory noise filtering.

III. OPTIMALIZING CONTROL WITH NOISE INTERFERENCE

In the Introduction, it was indicated that there are two fundamental problems in the theory of optimalizing control systems. One of the problems is to evaluate the effect of dynamics of the controlled system on the performance of the control system; the other is to evaluate the effect of noise interference. Under the assumption that the dynamic properties of the controlled system can be approximated by a first order linear system, the first problem of dynamic effects is solved completely in Part II. Now an attempt will be made to take account of noise interference on the control system performance.

It will be recalled that the input drive unit reverses the direction of input whenever a definite critical difference exists between the reading of the output sensing instrument and the output itself after the time instant when the optimum value of output occurs. The readings of the output sensing instrument are affected by the noise interference, especially during that part of the hunting cycle when the optimum value of the output must be located. Because of superimposed noise on the "actual" output, as illustrated in Fig. 3.1, the output sensing instrument will make serious errors in its decision for the optimum state. Moreover this error will become greater for the cases where the input-output characteristics of the controlled system is very flat around the optimum point. Thus the output sensing instrument will hold a value of output at random which is not necessarily the correct optimum value.

The noise interference will also randomize the time instants when input reversals must occur. The reversal instants will occur earlier or later depending on whether the output with superimposed noise is more or less negative than the "actual" output. For the cases when output with noise is more negative than the "actual" output in that part of the hunting cycle where input reversals theoretically expected to occur, there is the danger of a false reversal. To illustrate this effect more clearly, consider the time interval when input is still increasing just after the optimum value has been reached. Due to superimposed noise the input reversal command will be given earlier and the input will start to decrease. Thus the true optimum point of the controlled system is not reached, and the system fails in accomplishing the original purpose of the optimalizing control: to constrain the system near the optimum point.

3.1 Modified Peak-holding Optimalizing Control System

The proposed modification of peak-holding optimalizing control system can be summarized in the following manner: the output sensing instrument is abandoned in the control circuit and the input drive unit is permitted to function of its own accord with a preset hunting period of $2T$ and a drive speed of N. With this modification the random hunting of the control system due to noise interference is eliminated.

The correct operation mode of the control system is the one which will give a hunting loss predicted through the design charts, Figs. 2.10 and 2.11, as soon as a decision for hunting period T, input drive speed N, and critical difference c is made. It is unreasonable to expect the control system to function correctly if its starting time instant is random. Thus, first of all, the control system must be shifted to the correct operation mode; secondly it should be kept to operate there with minimum prescribed error due to incorrect mode of operation.

In order to point out significant differences between correct and various incorrect operation modes, a schematic illustration is presented in Figs. 3.2a through 3.2g. In this illustration, for the sake of simplicity, a controlled system without dynamic effects is chosen, since dynamic effects will not introduce any essential change. One of the most important features of this illustration is the fact that whether the incorrect input is greater or less than the correct input, the period of the "potential" output y_u^* is the same as the period of input x_u. Only in the correct operation mode does the output y_u^* have its period half as great as the period of input x_u. Therefore the criterion for shifting the control system toward the correct mode or for detection of the incorrect operation mode will be based on this simple fact. However, one must be aware of the fact that this criterion is based on the parabolic relation, given by Eq. (2.1), between the input and output in the neighborhood of the optimum state. In the present investigation this certainly is the case, since the optimalizing control system confines the operation to the optimum state as closely as possible. When deviation from the optimum state is quite large, then Eq. (2.1) should be modified as

$$y = -kx^2 + \epsilon x^3 \qquad (|\epsilon| \ll 1)$$

to include the possible asymmetric characteristics of the input-output relation. This modification will no doubt call for a new detection criterion.

Secondly, the incorrect operation mode will necessarily increase the hunting loss. It can be seen clearly from Fig. (3.2) that the magnitude of the time average of the output is always greater in the incorrect operation mode.

3.2 Detection Procedures

The difference between correct and incorrect modes of operation, as indicated in the previous section, is the occurrence of first harmonic component in the former and occurrence of both fundamental and first harmonic components in the output signal in the latter case. To illustrate detection procedures, then, one could assume the noise-free output signal to be

$$y = A + B \sin \omega_i t + C \cos \omega_i t + D \sin 2\omega_i t + E \cos 2\omega_i t$$

In Part IV of this work, it will be seen that the output y given by this equation is exactly the form of output from a sinusoidal input optimalizing controller in the incorrect operating mode. The coefficients D and E of the first harmonic component are only functions of the dynamic characteristics of the controlled system. The coefficients B and C of the fundamental component are proportional to the amount of incorrect input and are also functions of the dynamic characteristics of the controlled system. The d.c. component A is a function of both incorrect input and dynamic characteristics of the controlled system.

The criterion is first to devise a scheme for detecting the presence of the fundamental component in the output signal under noise interference; secondly, to drive the control system to the correct operation mode by eliminating, or at least by minimizing, the coefficients B and C . Since the detection process must be accomplished under conditions of noise interference, the detection scheme must reduce the noise interference effects as much as possible. Such detection procedures fall in one or the other of two distinct types.

(A) Direct Sampling and Averaging Technique: From observations on Fig. 3.2, for an ideal optimalizing control with input parameters a and b , one can summarize the following facts:

1. When $b<a$, and when $b=0$ (correct operation), two minimum points of the "potential" output occur during each input hunting cycle, and are located at the input reversal instants. In these cases, the maximum points of the "potential" output are located symmetrically with respect to the minimum points.
2. When $b \geq a$, there always occurs successively one minimum and one maximum point per input cycle, and these are located at the input reversal instants.

When the characteristic time constants τ_o and τ_i of the controlled system are small compared to the output hunting period, period T , the facts mentioned above remain unchanged, with the exception of the locations of the maximum and minimum points. However, one must give special care to the first case. For instance, when $b<a$, there occurs only one minimum and one maximum point per input cycle when the characteristic time constants τ_o and τ_i are of the same order of magnitude as T , or larger. Thus for small values of τ_o and τ_i , one

can conclude that the locations of minimum points of the output are independent of the amount of incorrect input b, and depend only on the dynamic characteristics of the controlled system. On the other hand, the locations of the maximum points are dependent on the amount of incorrect input b when $b < a$, and are independent of the amount of incorrect input b when $b \geq a$. Therefore, the method in this case would be to locate a minimum point of the output signal and to investigate whether or not a maximum point of the output occurs after $T/2$ seconds. When the maximum point of the output occurs $T/2$ seconds after the occurrence of the minimum point, the control system is in the correct operation mode. Otherwise the input level must be changed to produce the desired situation.

To locate either a minimum or a maximum of the output signal under noise interference, consider two sets of N samples. One set is taken at the time instants

$$t = 0 , 2T , 4T , \ldots , 2(N-1)T$$

and the other set is taken at the instants

$$t = \tau , \tau + 2T , \tau + 4T , \ldots , \tau + 2(N-1)T$$

where $\tau \leq T/2$. One can choose N such that when average values of these samples are considered, the noise interference effects are as small as desired. Denoting average of the first set of samples by $\bar{y}_{d1}$ and the second by $\bar{y}_{d2}$, one can distinguish the following three cases:

(i) When $\bar{y}_{d1} < \bar{y}_{d2}$, the maximum point is behind and the minimum point is ahead of the sampling instants nT.

(ii) When $\bar{y}_{d_1} > \bar{y}_{d_2}$, the maximum point is ahead and the minimum point is behind the sampling instants nT .

(iii) When $\bar{y}_{d_1} = \bar{y}_{d_2}$, there occurs either a maximum or a minimum point of the output which is located approximately at half way between two sampling instants.

By displacing the sampling instants, one can always obtain case (iii). The choice of τ must be such that two sets of samples are distinguishable when samples are taken as in the cases (i) and (ii), and that when case (iii) occurs the maximum or the minimum is very closely located half way between two sampling instants.

(B) Explicit Detection of the Fundamental Component: The following three methods may be considered as representative examples of this class:

1) Narrow Band-pass Filters.

Any convenient filter in this category can be utilized. The detection criterion in this case is to tune the filter circuit on the fundamental component frequency and to minimize the measured amplitude of the fundamental component by varying the input level.

2) Direct Integration Technique.

Starting at any time origin $t = 0$, the output is integrated with respect to time for a complete cycle of the first harmonic component. The next integration is performed ΔT seconds after the previous integration is terminated, and so on. Thus the integrations are performed starting at time instants

$$t = 0,\ T+\Delta T,\ 2(T+\Delta T),\ \ldots,\ (N-1)(T+\Delta T)$$

and the complete data are obtained when $N\Delta T = 2T$.

When the starting instants of the integration procedure are displaced, the resulting integrals have approximately the following form:

$$I_N = C \sin[\omega_i N(\Delta T)] \qquad \left(N = 0, 1, \ldots, \frac{2T}{\Delta T} - 1\right)$$

where C is a constant. With the integration process the noise interference effects are somewhat suppressed, and there are no contributions to the integrals from the first harmonic component. When N equivalent integrators are used for this process, a complete set of data are available in $2T$ seconds, since each integration process can start ΔT seconds after the preceding one.

The criterion in this case is to vary the input level such that the amplitude C of I_N is a minimum. The system in this condition is as close to the correct operation mode as possible.

3) Filtering by Correlation Method

This method uses the principle of correlating the output signal by a sinusoidal signal with the fundamental frequency ω_i . The details of this procedure will be given in the following sections.

The preceding discussions only indicated a few procedures out of perhaps a large number of possible ways of detecting and of eliminating the incorrect operation modes of the modified optimalizing control system. The primary objective in the following sections is to demonstrate a statistical analysis applied to assess the merit of the proposed filtering by the correlation method. The ultimate choice of the design has to be based upon such an analysis for each of the various possible methods and upon other engineering factors.

3.3 Mathematical Analysis of the Incorrect Operation Mode

To determine the dynamic effects of the controlled system on the performance of the modified optimalizing control system, neglecting noise interference effects, a modified optimalizing controller with an incorrect input will be analyzed.

In a similar manner as was done in Part II of this work, the potential input x_u^* specified as a saw-tooth curve with period $2T$, amplitude a, and an amount of incorrect operation b, (Fig. 3.2), can be expanded into a Fourier series as

$$x_u^* = b + \frac{8a}{\pi^2} \sum_{n=0}^{\infty}{}' \frac{(-1)^n}{(2n+1)^2} \frac{1}{2i} \left[e^{\frac{2n+1}{2} i\omega_0 t} - e^{-\frac{2n+1}{2} i\omega_0 t} \right] \tag{3.1}$$

where the hunting frequency ω_0 is defined by Eq. (2.9). The actual input x_u is determined by using the system of Eqs. (2.7), (2.8) and (3.1), thus

$$x_u = b + \frac{8a}{\pi^2} \sum_{n=0}^{\infty}{}' \frac{(-1)^n}{(2n+1)^2} \frac{1}{2i} \left[F_i\left(\frac{2n+1}{2} i\omega_0\right) e^{\frac{2n+1}{2} i\omega_0 t} - F_i\left(-\frac{2n+1}{2} i\omega_0\right) e^{-\frac{2n+1}{2} i\omega_0 t} \right] \tag{3.2}$$

The actual output y_u is obtained by using the system of Eqs. (3.2), (2.1), (2.5), and (2.6)

$$y_u = -kb^2 + \frac{8abk}{\pi^2}\, i \sum_{n=0}^{\infty}{}' \frac{(-1)^n}{(2n+1)^2}\Big[F_o\big(\tfrac{2n+1}{2} i\omega_0\big) F_i\big(\tfrac{2n+1}{2} i\omega_0\big) e^{\frac{2n+1}{2} i\omega_0 t} - F_o\big(-\tfrac{2n+1}{2} i\omega_0\big) F_i\big(-\tfrac{2n+1}{2} i\omega_0\big) e^{-\frac{2n+1}{2} i\omega_0 t}\Big]$$

$$+ \frac{16}{\pi^4}\, a^2 k \sum_{n=0}^{\infty}{}' \sum_{m=0}^{\infty} \frac{(-1)^{n+m}}{(2n+1)^2 (2m+1)^2}\Big[F_o\{(n+m+1) i\omega_0\} F_i\big(\tfrac{2n+1}{2} i\omega_0\big) F_i\big(\tfrac{2m+1}{2} i\omega_0\big) e^{(n+m+1) i\omega_0 t}$$

$$- F_o\{(n-m) i\omega_0\} F_i\big(\tfrac{2n+1}{2} i\omega_0\big) F_i\big(-\tfrac{2m+1}{2} i\omega_0\big) e^{(n-m) i\omega_0 t} - F_o\{-(n-m) i\omega_0\} F_i\big(-\tfrac{2n+1}{2} i\omega_0\big) F_i\big(\tfrac{2m+1}{2} i\omega_0\big) e^{-(n-m) i\omega_0 t}$$

$$+ F_o\{-(n+m+1) i\omega_0\} F_i\big(-\tfrac{2n+1}{2} i\omega_0\big) F_i\big(-\tfrac{2m+1}{2} i\omega_0\big) e^{-(n+m+1) i\omega_0 t}\Big] \tag{3.3}$$

Comparison of Eqs. (3.2) and (3.3) indicates that the output y_u has the same hunting period as of the input x_u. Further examination of Eq. (3.3) shows that the first summation is the term with the same period as the input and that the last double summation is the same as given by Eq. (2.13).

The magnitude of the time average of the actual output y_u is the hunting loss D_u of this incorrectly operating control system. The magnitude of the time average of Eq. (3.3) is

$$D_u = kb^2 + \frac{32 a^2 k}{\pi^4} \sum_{n=0}^{\infty}{}' \frac{1}{(2n+1)^4}\, F_i\big(\tfrac{2n+1}{2} i\omega_0\big) F_i\big(-\tfrac{2n+1}{2} i\omega_0\big) \tag{3.4}$$

The only difference between Eqs. (2.14) and (3.4) is the additional term kb^2 in the hunting loss of the incorrectly operating control system. Since the parameter b occurs as a square in this expression, the sign of b is immaterial, and hunting loss D_u is an increasing function of b . The minimum value of D_u occurs when $b=0$, that is, when the control system has the correct operation mode.

With the introduction of the transfer functions of the input and output linear groups given by Eqs. (2.15) and (2.16) and with a dimensionless parameter ℓ for incorrect operation defined as

$$\ell = \frac{b}{a} \tag{3.5}$$

the hunting loss D_u given by Eq. (3.4) now becomes

$$D_u = N^2T^2k\left[\frac{1}{4}\left(\ell^2+\frac{1}{3}\right)-\left(\frac{\tau_i}{T}\right)^2+2\left(\frac{\tau_i}{T}\right)^3 \tanh\frac{1}{2}\frac{T}{\tau_i}\right] \tag{3.6}$$

where N is again the input drive speed defined by Eq. (2.19).

The procedure for determination of the actual output y_u is the same as that used in Part II. Substituting Eqs. (2.15) and (2.16) into Eq. (3.3),

$$y_u = -\frac{N^2T^2k}{4}\ell^2 + \frac{2N^2T^2k\ell}{\pi^2}\, i\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\left[\frac{e^{\frac{2n+1}{2}i\omega_0 t}}{\left\{1+\frac{2n+1}{2}i\omega_0\tau_0\right\}\left\{1+\frac{2n+1}{2}i\omega_0\tau_i\right\}} - \frac{e^{-\frac{2n+1}{2}i\omega_0 t}}{\left\{1-\frac{2n+1}{2}i\omega_0\tau_0\right\}\left\{1-\frac{2n+1}{2}i\omega_0\tau_i\right\}}\right]$$

$$+ \text{Eq. } (2.21)$$

or

$$y_u = -\frac{N^2T^2k}{4}\ell^2 + \frac{4N^2T^2k\ell}{\pi^2}\left\{\sum_{n=0}^{\infty}\frac{(-1)^n\frac{(\tau_i+\tau_0)}{2}\omega_0\cos\frac{2n+1}{2}\omega_0 t}{(2n+1)\left[1+(2n+1)^2\left(\frac{\omega_0\tau_0}{2}\right)^2\right]\left[1+(2n+1)^2\left(\frac{\omega_0\tau_i}{2}\right)^2\right]}\right.$$

$$\left.-\sum_{n=0}^{\infty}\frac{(-1)^n\left[1-(2n+1)^2\left(\frac{\omega_0}{2}\right)^2\tau_i\tau_0\right]\sin(2n+1)\frac{\omega_0 t}{2}}{(2n+1)^2\left[1+(2n+1)^2\left(\frac{\omega_0\tau_0}{2}\right)^2\right]\left[1+(2n+1)^2\left(\frac{\omega_0\tau_i}{2}\right)^2\right]}\right\} + \text{Eq. } (2.21)$$

(3.7)

After resolving terms involving summation into partial fractions, Eq. (3.7) becomes

$$
\begin{aligned}
\frac{y}{y_u} = & -\frac{N^2T^2k}{4}\ell^2 + \frac{4N^2T^2k\ell}{\pi^2}\left\{\frac{\omega_0}{2}(\tau_0+\tau_i)\sideset{}{'}\sum_{n=0}^{\infty}\frac{(-1)^n\cos(2n+1)\frac{\omega_0 t}{2}}{(2n+1)} - \frac{\left(\frac{\omega_0\tau_0}{2}\right)^4}{\frac{\omega_0}{2}(\tau_0-\tau_i)}\sideset{}{'}\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)\cos(2n+1)\frac{\omega_0 t}{2}}{\left[1+(2n+1)^2\left(\frac{\omega_0\tau_0}{2}\right)^2\right]}\right. \\
& + \frac{\left(\frac{\omega_0\tau_i}{2}\right)^4}{\frac{\omega_0}{2}(\tau_0-\tau_i)}\sideset{}{'}\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)\cos(2n+1)\frac{\omega_0 t}{2}}{\left[1+(2n+1)^2\left(\frac{\omega_0\tau_i}{2}\right)^2\right]} - \sideset{}{'}\sum_{n=0}^{\infty}\frac{(-1)^n\sin(2n+1)\frac{\omega_0 t}{2}}{(2n+1)^2} + \frac{\left(\frac{\omega_0\tau_0}{2}\right)^3}{\frac{\omega_0}{2}(\tau_0-\tau_i)}\sideset{}{'}\sum_{n=0}^{\infty}\frac{(-1)^n\sin(2n+1)\frac{\omega_0 t}{2}}{\left[1+(2n+1)^2\left(\frac{\omega_0\tau_0}{2}\right)^2\right]} \\
& \left. - \frac{\left(\frac{\omega_0\tau_i}{2}\right)^3}{\frac{\omega_0}{2}(\tau_0-\tau_i)}\sideset{}{'}\sum_{n=0}^{\infty}\frac{(-1)^n\sin(2n+1)\frac{\omega_0 t}{2}}{\left[1+(2n+1)^2\left(\frac{\omega_0\tau_i}{2}\right)^2\right]}\right\} + \text{Eq. (2.21)}
\end{aligned}
\tag{3.8}
$$

The result of carrying out the summations in Eq. (3.8) and simplifying the expression is,

$$
\begin{aligned}
\frac{y}{y_u} = & N^2T^2k\left\{-\frac{\ell^2}{4} - \ell(-1)^j\left[\left(\frac{t}{T}-j\right)-\left(\frac{\tau_0}{T}+\frac{\tau_i}{T}\right) + \frac{\left(\frac{\tau_0}{T}\right)^2 e^{-\frac{\left(\frac{t}{T}-j\right)}{\tau_0/T}}}{\left(\frac{\tau_0}{T}-\frac{\tau_i}{T}\right)\cosh\frac{1}{2}\frac{T}{\tau_0}} - \frac{\left(\frac{\tau_i}{T}\right)^2 e^{-\frac{\left(\frac{t}{T}-j\right)}{(\tau_i/T)}}}{\left(\frac{\tau_0}{T}-\frac{\tau_i}{T}\right)\cosh\frac{1}{2}\frac{T}{\tau_i}}\right.\right. \\
& + 2T^2N^2k\left\{-\left[\frac{1}{2}\left(\frac{t}{T}-j\right)^2-\left(\frac{\tau_0}{T}+\frac{\tau_i}{T}\right)\left(\frac{t}{T}-j\right)+\frac{1}{2}\left(\frac{\tau_i}{T}\right)^2+\frac{\tau_0\tau_i}{T^2}+\left(\frac{\tau_0}{T}\right)^2\right] - \frac{1}{2}\frac{\left(\frac{\tau_0}{T}\right)^2}{\left(\frac{\tau_0}{T}-\frac{\tau_i}{T}\right)}\left[\frac{2\left(\frac{\tau_i}{T}\right)^3\tanh\frac{1}{2}\frac{T}{\tau_i}}{\left(\frac{\tau_0}{T}-\frac{\tau_i}{T}\right)\left(2\frac{\tau_0}{T}-\frac{\tau_i}{T}\right)}-1\right]\frac{e^{-\frac{\left(\frac{t}{T}-j\right)}{\tau_0/T}}}{\sinh\frac{1}{2}\frac{T}{\tau_0}}\right. \\
& \left. + \left[\left(\frac{t}{T}-j\right) + \frac{\left(\frac{\tau_i}{T}\right)^2}{\left(\frac{\tau_0}{T}-\frac{\tau_i}{T}\right)}\right]\frac{\left(\frac{\tau_i}{T}\right)^2 e^{-\frac{\left(\frac{t}{T}-j\right)}{\tau_i/T}}}{\left(\frac{\tau_0}{T}-\frac{\tau_i}{T}\right)\cosh\frac{1}{2}\frac{T}{\tau_i}} + \frac{1}{2}\frac{\left(\frac{\tau_i}{T}\right)^3}{\left(2\frac{\tau_0}{T}-\frac{\tau_i}{T}\right)}\frac{e^{-\frac{2\left(\frac{t}{T}-j\right)}{\tau_i/T}}}{\cosh^2\frac{1}{2}\frac{T}{\tau_i}}\right\}
\end{aligned}
$$

when $\quad -\frac{1}{2} \leq \left(\frac{t}{T}-j\right) \leq \frac{1}{2}$

where $\quad j = 0, 1, 2, \ldots$

(3.9)

Figs. (3.3) and (3.4) show the dimensionless plots of the actual output $\frac{y}{y_u}$ for several particular values of ℓ for the cases when $\frac{\tau_0}{T} = 0.15$ and $\frac{\tau_i}{T} = 0.10$. The plots in these figures have essentially the same shape as the corresponding cases illustrated in Fig. 3.2. The difference is again the rounding off the sharp corners and a shift in phase due to dynamic effects.

3.4 Detection and Elimination of Incorrect Operation Mode without Noise Interference.

One of the fundamental problems which is introduced by the modified peak-holding optimalizing control system is the detection of an incorrect mode of operation. The parameter ℓ , defined by Eq. (3.5), is introduced in such a manner that it is not only possible to detect the incorrect mode of operation, but it is also possible to determine magnitude and sign of the incorrect input $b = \frac{NT}{2}\ell$. Therefore the problem is to devise a scheme to determine the parameter ℓ. As explained in Section 3.2 one of the possible schemes is as follows: The actual output y_a will be cross-correlated with a sinusoid which has an amplitude A and a time displacement τ with respect to the actual output signal y_a .

In general cross-correlation of a function $f(t)$ with $g(t)$ is defined as

$$\chi(\tau) = \lim_{\theta \to \infty} \frac{1}{2\theta} \int_{-\theta}^{\theta} f(t+\tau)\,\overline{g(t)}\,dt \tag{3.10}$$

For periodic real-functions $f(t)$ and $g(t)$ the cross-correlation given by Eq. (3.10) reduces to

$$\chi(\tau) = \frac{1}{2\theta} \int_{-\theta}^{\theta} f(t+\tau)\,g(t)\,dt \tag{3.11}$$

where 2θ is the smallest common period of these two functions $f(t)$ and $g(t)$.

Choose,

$$f(t+\tau) = A \sin \omega_i (t+\tau) \tag{3.12}$$

and

$$g(t) = y_u(t) \tag{3.13}$$

where the circular input hunting frequency ω_i is defined as

$$\omega_i = \pi/T \tag{3.14}$$

By substitution of Eqs. (3.12) and (3.13) into Eq. (3.11), the cross-correlation coefficient $\chi(\tau)$ of the actual output $y_u(t)$ with a sinusoid is

$$\chi(\tau) = \frac{A}{2T}\int_0^{2T} y_u(t) \sin \omega_i (t+\tau)\, dt$$

or

$$\chi(\tau) = \frac{A}{2} \cos \omega_i \tau \int_0^2 y_u(t') \sin \pi t'\, dt' + \frac{A}{2} \sin \omega_i \tau \int_0^2 y_u(t') \cos \pi t'\, dt' \tag{3.15}$$

where the integration variable is changed to $t' = t/T$.

It is seen from Eq. (3.12) that the sinusoid chosen for cross-correlation has a period of $2T$. Therefore, when the parameter $\ell = 0$, the actual output $y_u(t)$ given by Eq. (3.9) has a period of T , and thus $\chi(\tau) = 0$. When $\ell \neq 0$, the actual output $y_u(t)$ has a period of $2T$, the cross-correlation coefficient $\chi(\tau)$ is not zero with the exception of a definite set of values of τ which will be indicated in the following analysis. In this scheme, then, the value of the cross-correlation is the basis of the detection criterion.

In order to compute the cross-correlation coefficient $\chi(\tau)$, defined by Eq. (3.15), for an incorrectly operating control system, Eq. (3.9) can be written in the following manner:

$$y_u(t') = A_1 - A_2 t' - A_3 e^{-b_1 t'} + A_4 e^{-a_1 t'} + Eq.(2.27) \qquad 0 \leq t' \leq \tfrac{1}{2} \tag{3.16}$$

$$y_u(t') = A_1' + A_2(t'-1) + A_3 e^{-b_1(t'-1)} - A_4 e^{-a_1(t'-1)} + Eq.(2.27) \qquad \tfrac{1}{2} \leq t' \leq \tfrac{3}{2} \tag{3.17}$$

$$y_u(t') = A_1 - A_2(t'-2) - A_3 e^{-b_1(t'-2)} + A_4 e^{-a_1(t'-2)} + Eq.(2.27) \qquad \tfrac{3}{2} \leq t' \leq 2 \tag{3.18}$$

where A_1, A_1', A_2, A_3, A_4, a_1 and b_1 are shorthand notations used for the time invariant coefficients in Eq. (3.9) and are defined as,

$$\left.\begin{aligned}
A_1 &= N^2 T^2 k\left[-\frac{\ell^2}{4} + \ell\left(\frac{\tau_0}{T} + \frac{\tau_1}{T}\right)\right] \\
A_1' &= N^2 T^2 k\left[-\frac{\ell^2}{4} - \ell\left(\frac{\tau_0}{T} + \frac{\tau_1}{T}\right)\right] \\
A_2 &= N^2 T^2 k \ell \\
A_3 &= N^2 T^2 k \ell \frac{\left(\frac{\tau_0}{T}\right)^2}{\left(\frac{\tau_0}{T} - \frac{\tau_1}{T}\right)\cosh \frac{1}{2}\frac{T}{\tau_0}} \\
A_4 &= N^2 T^2 k \ell \frac{\left(\frac{\tau_1}{T}\right)^2}{\left(\frac{\tau_0}{T} - \frac{\tau_1}{T}\right)\cosh \frac{1}{2}\frac{T}{\tau_1}} \\
a_1 &= T/\tau_1 \\
b_1 &= T/\tau_0
\end{aligned}\right\} \tag{3.19}$$

By substitution of Eqs. (3.16), (3.17), and (3.18) into Eq. (3.15), the cross-correlation $\chi(\tau)$ can now be written,

$$\chi(\tau) = \frac{A}{2}\cos\omega_c\tau\Big\{\int_0^{1/2}[A_1 - A_2 t' - A_3 e^{-b_1 t'} + A_4 e^{-a_1 t'}]\sin\pi t'\,dt' + \int_{\frac{1}{2}}^{3/2}[A_1' + A_2(t'-1) + A_3 e^{-b_1(t'-1)} - A_4 e^{-a_1(t'-1)}]\sin\pi t'\,dt'$$

$$+\int_{\frac{3}{2}}^{2}[A_1 - A_2(t'-2) - A_3 e^{-b_1(t'-2)} + A_4 e^{-a_1(t'-2)}]\sin\pi t'\,dt'\Big\} + \frac{A}{2}\sin\omega_c\tau\Big\{\int_0^{1/2}[A_1 - A_2 t' - A_3 e^{-b_1 t'} + A_4 e^{-a_1 t'}]\cos\pi t'\,dt'$$

$$+\int_{\frac{1}{2}}^{3/2}[A_1' + A_2(t'-1) + A_3 e^{-b_1(t'-1)} - A_4 e^{-a_1(t'-1)}]\cos\pi t'\,dt' + \int_{3/2}^{2}[A_1 - A_2(t'-2) - A_3 e^{-b_1(t'-2)} + A_4 e^{-a_1(t'-2)}]\cos\pi t'\,dt'\Big\}$$

or by changing the limits of integration, and simplifying

$$\chi(\tau) = \frac{A}{2}\cos\omega_c\tau\Big[0 - 2A_2\int_{-\frac{1}{2}}^{\frac{1}{2}} t'\sin\pi t'\,dt' - 2A_3\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-b_1 t'}\sin\pi t'\,dt' + 2A_4\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-a_1 t'}\sin\pi t'\,dt'\Big]$$

$$+\frac{A}{2}\sin\omega_c\tau\Big[\frac{2}{\pi}(A_1 - A_1') - 2A_2\int_{-\frac{1}{2}}^{\frac{1}{2}} t'\cos\pi t'\,dt' - 2A_3\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-b_1 t'}\cos\pi t'\,dt' + 2A_4\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-a_1 t'}\cos\pi t'\,dt'\Big]$$

Evaluation of the definite integrals yields,

$$\chi(\tau) = 2A\Big[-\frac{A_2}{\pi^2} + \frac{b_1 A_3\cosh\frac{b_1}{2}}{b_1^2 + \pi^2} - \frac{a_1 A_4\cosh\frac{a_1}{2}}{a_1^2 + \pi^2}\Big]\cos\omega_c\tau + 2A\Big[-\frac{1}{2A}(A_1 - A_1') - \frac{\pi A_3\cosh\frac{b_1}{2}}{b_1^2 + \pi^2} + \frac{\pi A_4\cosh\frac{a_1}{2}}{a_1^2 + \pi^2}\Big]\sin\omega_c\tau$$

After substitution of the expressions defined by Eq. (3.19) back into this equation and after simplification, the cross-correlation function $\chi(\tau)$ is obtained as

$$\chi(\tau) = \frac{2AN^2 k \ell}{\omega_i^2 [1+(\omega_i\tau_o)^2][1+(\omega_i\tau_i)^2]} \left\{ (\omega_i^2 \tau_o \tau_i - 1) \cos \omega_i \tau + \omega_i (\tau_o + \tau_i) \sin \omega_i \tau \right\} \tag{3.20}$$

With a specified set of dynamic characteristics of the controlled system, operating characteristics of the controller, amplitude A of the sinusoid, and value of cross-correlation function $\chi(\tau)$ for a given value of τ, the parameter ℓ can be computed by using Eq. (3.20). Then the amount of incorrect input which must be eliminated is just $b = \frac{NT}{2} \ell$.

There is one unfortunate feature of Eq. (3.20), namely, there are certain values of time displacement τ which make the cross-correlation functions $\chi(\tau)$'s vanish regardless of the values of the parameter ℓ. Since these values of τ will not give any information about the incorrect operation mode, they cannot be utilized. The forbidden values of τ are found by setting Eq. (3.20) equal to zero, thus

$$(\omega_i^2 \tau_o \tau_i - 1) \cos \omega_i \tau + \omega_i (\tau_o + \tau_i) \sin \omega_i \tau = 0$$

or

$$(\omega_i \tau \pm n\pi) = \arctan \frac{(1 - \omega_i^2 \tau_o \tau_i)}{\omega_i (\tau_o + \tau_i)} \qquad (n = 0, 1, 2, \cdots) \tag{3.21}$$

The significant values of $\chi(\tau)$ for accurate computation of the parameter ℓ will be obtained when τ is chosen such that the absolute value of the cross-correlation is the largest. By differentiating $\chi(\tau)$ with respect to τ and setting it equal to zero

$$\frac{d\chi(\tau)}{d\tau} = 0 = (\text{Const.}) \left\{ -[\omega_i^2 \tau_o \tau_i - 1] \sin \omega_i \tau + \omega_i (\tau_o + \tau_i) \cos \omega_i \tau \right\}$$

or

$$(\omega_i \tau \pm n\pi) = \arctan \frac{\omega_i(\tau_o + \tau_i)}{[\omega_i^2 \tau_o \tau_i - 1]} \qquad (n = 0, 1, 2, \cdots) \tag{3.22}$$

which are the values of τ that give $\chi(\tau)$'s of large magnitude for accurate determination of the parameter ℓ in Eq. (3.20).

Fig. 3.5 shows dimensionless plots of cross-correlation functions for $\ell = +1$ and for various values of $\frac{\tau_o}{T}$ and $\frac{\tau_i}{T}$.

When the only information available is the input hunting frequency ω_i, Eq. (3.20) can be written as

$$\chi(\tau) = \alpha \cos \omega_i \tau + \beta \sin \omega_i \tau \tag{3.23}$$

where α and β are defined as

$$\alpha = \frac{2AN^2 k\ell}{\omega_i^2} \frac{(\omega_i^2 \tau_o \tau_i - 1)}{[1 + (\omega_i \tau_o)^2][1 + (\omega_i \tau_i)^2]} \tag{3.24}$$

$$\beta = \frac{2AN^2 k\ell}{\omega_i^2} \frac{\omega_i(\tau_o + \tau_i)}{[1 + (\omega_i \tau_o)^2][1 + (\omega_i \tau_i)^2]} \tag{3.25}$$

From Eqs. (3.24) and (3.25) it is clear that the coefficient β will always have the same sign as the parameter ℓ, and the coefficient α will have the same sign as ℓ when $\omega_i^2 \tau_o \tau_i > 1$, but the opposite sign of ℓ when $\omega_i^2 \tau_o \tau_i < 1$. When only two distinct values of cross-correlation function $\chi(\tau)$ are known for two given values of time displacements, except when $\tau_1 - \tau_2 = \frac{n\pi}{\omega_i}$, it is possible to determine the coefficients α and β from Eq. (3.23). Although there is not enough information to determine the value of the parameter ℓ with either Eq. (3.24) or Eq. (3.25), one knows that the sign of ℓ is the same as the coefficient β . Since the characteristic time constants τ_o and τ_i of the controlled system are not available, the information obtained through the coefficient α is unreliable. Thus knowing only ω_i , the input hunting frequency, and evaluating two distinct values of cross-correlation function for two values of time displacement, it is still possible to detect the incorrect operation mode and also the direction in which it occurs. To eliminate the incorrect operation mode, the input level can be varied by increments in accordance with the sign of the coefficient β . After each stepwise variation of input, two new cross-correlation processes must be carried. This elimination of the incorrect operation mode through an iteration technique is continued until both cross-correlations obtained vanish, that is, $\ell = 0$ or $\alpha = \beta = 0$. This operation mode is then the correct operation mode.

3.5 Detection and Elimination of An Incorrect Operation Mode with Noise Interference

In Section 3.4, the detection and elimination criterion of the incorrect operation mode was formulated for a modified optimalizing control system free from noise interference effects. Needless to say, noise from both external and internal sources is present and affects the performance of the control system. However, when the noise interference effects are considered, the fundamental concept of the detection and elimination criterion introduced in Section 3.4 is still valid. The method is again to determine a parameter ℓ' which is an indication of the amount of incorrect operation mode.

For simplicity of analysis, the random noise function $n(t)$ is assumed to be superimposed on the actual output $y_a(t)$; thus the signal $F(t)$ indicated by an output measuring instrument is of the form

$$F(t) = y_a(t) + n(t) \tag{3.26}$$

Then the cross-correlation function $R(\tau)$ of $F(t)$ with a sinusoid of the form $A \sin \omega_i (t+\tau)$ is

$$R(\tau) = \lim_{\theta \to \infty} \frac{A}{2\theta} \int_{-\theta}^{\theta} F(t) \sin \omega_i (t+\tau)\, dt$$

With the results of Section 3.4 and Eq. (3.26), the cross-correlation function $R(\tau)$ can approximately be written as:

$$R(\tau) = \chi(\tau) + \frac{A}{2mT} \int_0^{2mT} n(t) \sin \omega_i (t+\tau)\, dt \tag{3.27}$$

where m takes positive integer values and $\chi(\tau)$ is given by Eq. (3.20).

When a value of the cross-correlation function $R(\tau)$, the dynamic characteristics of the controlled system, the operating characteristics of the controller, and the amplitude A of the sinusoid are available, a parameter ℓ' , an indication of the incorrect operation mode, can be computed approximately by replacing $\chi(\tau)$ with $R(\tau)$ in Eq. (3.20). Thus

$$\ell' = \frac{\omega_i^2 R(\tau)[1+\omega_i^2\tau_o^2][1+\omega_i^2\tau_i^2]}{2AN^2 k\{(\omega_i^2\tau_o\tau_i - 1)\cos\omega_i\tau + \omega_i(\tau_o+\tau_i)\sin\omega_i\tau\}} \tag{3.28}$$

The amount of incorrect input which must be eliminated is then $b' = \frac{NT}{2} \ell'$. Since the cross-correlation process is carried out for a finite value of m, the noise interference effects are not completely eliminated, and b' is not the actual amount of incorrect input. For example, $\ell' = 0$ may not mean that the system is operating in the correct mode. Thus, there is a certain probability of error associated with the value of the parameter ℓ'.

When the cross-correlation process is carried out for two distinct values of time displacements and only the input hunting frequency ω_i is known, it is still possible to detect and to eliminate the incorrect operation mode with a certain probability of error due to noise interference effects. In this case the cross-correlation function $R(\tau)$ can be written as

$$R(\tau) = \alpha' \cos \omega_i \tau + \beta' \sin \omega_i \tau \tag{3.29}$$

With two distinct values of $R(\tau)$, one can determine the coefficients α' and β' from Eq. (3.29). Again, with a certain probability of error, one can say that the sign of the parameter ℓ is the same as that of the coefficient β', and one can vary the input level in increments in order to eliminate the incorrect operation mode. In this case the information obtained by using the coefficient α' is again unreliable.

To refine the process of elimination of the incorrect operation mode, it is necessary to determine the parameter ℓ' as accurately as possible. This in turn requires that the cross-correlation function $R(\tau)$, given by Eq. (3.27), be as free as possible from noise interference effects. A possible way to reduce the noise interference effects is to choose m to be large; that is, to carry out the cross-correlation process for the value of time displacement τ for a large number of cycles, and thus reduce the noise interference effects to a minimum by an averaging process. However, the larger the value of m is chosen, the longer time it will take to obtain the desired information.

Another possible way to reduce the noise interference effects is to choose a reasonable value of m, however small, and to carry out the cross-correlation process for several values of time displacements. With these several values of cross-correlation functions, it is possible to produce a cross-correlation function of the form

$$R^*(\tau) = \alpha^* \cos \omega_i \tau + \beta^* \sin \omega_i \tau \tag{3.30}$$

such that it is the best estimation of Eq. (3.23). When only one cross-correlator is used, this process is again time consuming, but when it is feasible to use a number of equivalent cross-correlators this process takes a rather short time.

One of perhaps several ways of evaluating Eq. (3.30), through a cross-correlation process for several values of time displacement, can be outlined as follows:

Consider a set of values of cross-correlation functions

$$R_0\ ,\ R_1\ ,\ R_2\ ,\ \ldots\ ,\ R_j\ ,\ R_{j+1}\ ,\ \ldots\ ,\ R_k$$

which correspond to time displacements

$$0 = \tau_0\ ,\ \tau_1\ ,\ \tau_2\ ,\ \ldots\ ,\ \tau_j\ ,\ \tau_{j+1}\ ,\ \ldots\ ,\ \tau_k = \frac{2\pi}{\omega_i}$$

where each of these time displacements differs by an equal increment

$\Delta\tau$ from the preceding one. For the value of time displacement τ_j, the corresponding error of estimation in cross-correlation function is given by

$$R_j - R^*(\tau_j)$$

Since the problem is to fit a simple curve to given data, the method of least squares is considered to be satisfactory. The method of least squares requires that the sum of squares of the errors be a minimum. The sum of the squares of the errors G is given by

$$G = \sum_{j=0}^{k}{}' [R_j - R^*(\tau_j)]^2$$

or

$$G = \sum_{j=1}^{k}{}' [R_j - (\alpha^* \cos\omega_i\tau_j + \beta^* \sin\omega_i\tau_j)]^2 \tag{3.31}$$

Inspection of Eq. (3.31) shows that the sum of the squares of the errors G is a function of α^* and β^*; thus the coefficients α^* and β^* must be determined such that G is a minimum. The requirement for the minimum will be satisfied when the partial derivatives of G with respect to both α^* and β^* vanish; thus

$$\frac{\partial G}{\partial \alpha^*} = 0 = -2\sum_{j=1}^{k}{}' R_j \cos\omega_i\tau_j + 2\alpha^* \sum_{j=1}^{k}{}' \cos^2\omega_i\tau_j + 2\beta^* \sum_{j=1}^{k}{}' \cos\omega_i\tau_j \sin\omega_i\tau_j$$

$$\frac{\partial G}{\partial \beta^*} = 0 = -2\sum_{j=1}^{k}{}' R_j \sin\omega_i\tau_j + 2\alpha^* \sum_{j=1}^{k}{}' \cos\omega_i\tau_j \sin\omega_i\tau_j + 2\beta^* \sum_{j=1}^{k}{}' \sin^2\omega_i\tau_j$$

or

$$\alpha^* \sum_{j=1}^{k}{}' \cos^2\omega_i\tau_j + \beta^* \sum_{j=1}^{k}{}' \cos\omega_i\tau_j \sin\omega_i\tau_j = \sum_{j=1}^{k}{}' R_j \cos\omega_i\tau_j$$

$$\alpha^* \sum_{j=1}^{k}{}' \cos\omega_i\tau_j \sin\omega_i\tau_j + \beta^* \sum_{j=1}^{k}{}' \sin^2\omega_i\tau_j = \sum_{j=1}^{k}{}' R_j \sin\omega_i\tau_j$$

With the list of summations given in Appendix A, the coefficients α^* and β^* are obtained as

$$\left.\begin{aligned} \alpha^* &= \frac{2}{k}\sum_{j=1}^{k} R_j \cos\frac{2\pi}{k} j \\ \beta^* &= \frac{2}{k}\sum_{j=1}^{k} R_j \sin\frac{2\pi}{k} j \end{aligned}\right\} \tag{3.32}$$

As was done for the case shown by Eq. (3.28), a parameter ℓ^* may be computed to be

$$\ell^* = \frac{\omega_i^2 R^*(\tau)\left[1+\omega_i^2\tau_0^2\right]\left[1+\omega_i^2\tau_1^2\right]}{2AN^2k\left\{(\omega_i^2\tau_0\tau_1 - 1)\cos\omega_i\tau + \omega_i(\tau_0+\tau_1)\sin\omega_i\tau\right\}} \tag{3.33}$$

which is a better indication of the incorrect operation mode than the parameter ℓ'.

The criterion for the detection and elimination of the incorrect operation mode under noise interference effects can now be summarized as follows:

Case 1. <u>Controlled and control system characteristics are completely known</u>

(a) Choose a satisfactory value of m and carry out the cross-correlation process for one particular value of time displacement τ, preferably for a value of given by Eq. (3.22). Then with a certain probability of error due to noise interference effects, a parameter ℓ', which is proportional to incorrect input, can be computed by Eq. (3.28).

(b) Choose a value of m and carry out the cross-correlation process for several values of time displacements in order to obtain Eq. (3.30). Then with a certain probability of error due to noise interference effects, a parameter ℓ^*, which is proportional to incorrect input, can be computed by Eq. (3.28), preferably by using a value of τ given by Eq. (3.22).

Case 2. Only the input hunting frequency ω is known

(a) Choose a value of m and carry out the cross-correlation process for two particular values of time displacement, preferably for values of τ which are given by Eq. (3.22). By using Eq. (3.29), determine values of the coefficients α' and β'. With a certain probability of error due to noise interference effects, one can say that the sign of the parameter ℓ is the same as that of coefficient β' (see Eq. (3.25)). The information obtained from coefficient α' is unreliable. Thus assuming the direction of occurrence of incorrect operation mode, as indicated by the coefficient β', the input level is corrected in steps. After each stepwise correction of the input level, the process must be repeated until the condition $\beta' \cong 0$ is reached.

(b) Choose a value of m and carry out the cross-correlation process for several values of time displacements in order to obtain Eq. (3.32). Then, with β^* known, the elimination process is the same as the procedure given for β'.

Fig. (3.6) shows a complete block diagram of a modified optimalizing control system using a cross-correlator, a variable time displacement sine wave generator, and a computer to filter out the noise interference effects. This control system can be operated to produce either the parameter ℓ' by using only one value of time displacement or the parameter ℓ^* by using several values of the time displacement. Thus the output R_j of the cross-correlator corresponding to time displacement τ_j can be used either to compute the parameter ℓ' directly or to store it in a memory unit of the computer for the computation of the parameter ℓ^*. When several values of R_j's corresponding to several values of time displacements, as required by Eq. (3.32), are available, the parameter ℓ^* is computed according to Eq. (3.33). In order to reduce the time required to compute the parameter ℓ^*, several cross-correlators and sine-wave generators should be used. With a proper modification of the computer, the same control system can be utilized to perform stepwise elimination of the incorrect mode.

3.6 Probable Error in Detection and Elimination of Incorrect Operation Mode under Noise Interference Effects

In Section 3.5 it is shown that with noise interference effects, the detection and elimination criterion of incorrect operation mode can be based either on the parameter ℓ' (or ℓ^*) or on the coefficient ρ' (or ρ^*)* depending on how much information is available about the controlled and control system. Since the parameter ℓ' or the coefficient ρ' is computed in a finite length of time,

----

*In this section the analysis will be carried out for primed quantities. However, the results can be used equivalently for starred quantities.

the noise interference effects are not completely eliminated, and it is conceivable that the parameter ℓ' or the coefficient β' will contain the following two types of error:

Type I error: Error in the magnitude of the parameter ℓ' or β'. For example, ℓ' and ℓ have the same algebraic signs but their magnitudes differ.

Type II error: Error in the algebraic sign of the parameter ℓ' or β'. For example, to obtain ℓ' as a negative quantity while ℓ really is a positive quantity, or vice versa.

Type I error is not a serious type of error. After all, the input level is corrected in the right direction. When ℓ' is large in magnitude, Type I error is expected to occur more frequently than Type II error. When the magnitude of ℓ' is reduced in order to reach the correct operation mode, both Type I and Type II errors are expected to occur frequently. Type II error is more serious than Type I error, since the occurrence of Type II error is a false alarm, and the input level is changed in the wrong direction. Therefore confining the system to the correct operation mode as closely as possible results in the frequent occurrence of Type II error. Thus the control system designer must decide upon a certain probability of occurrence of Type II error as being satisfactory, and compute a limiting value of the parameter ℓ' or β' as the best operation mode. For example, denoting this limiting value of ℓ' by ℓ'^*, then the computer should be equipped with a device such that whenever $|\ell'| \leq |\ell'^*|$, the input level is left unchanged; otherwise alteration of input level is carried on.

In the following analysis, the principal objective is to determine the probability of occurrence of Type II error in the detection and elimination criterion introduced in Section 3.5. Then a theoretical illustration will be presented so as to demonstrate how the limiting parameters ℓ'^{*} or β'^{*} can be computed.

The probability of occurrence of Type II error is equivalent to the probability of having $\ell'\ell < 0$ or $\beta'\beta < 0$. From Eqs. (3.20) and (3.28) it is clear that the probability of having $\ell'\ell < 0$ is equivalent to the probability of having $\chi(\tau)\mathcal{R}(\tau) < 0$. To compute the probability that $\beta'\beta < 0$ or $\chi(\tau)\mathcal{R}(\tau) < 0$, consider the following relations

$$\zeta = \beta'\beta = \beta'^{2} - (\beta'-\beta)\beta' \tag{3.34}$$

$$\xi = \chi(\tau)\mathcal{R}(\tau) = [\mathcal{R}(\tau)]^{2} - [\mathcal{R}(\tau) - \chi(\tau)]\mathcal{R}(\tau) \tag{3.35}$$

From Eqs. (3.23), (3.27), and (3.29) one can show that

$$\left.\begin{aligned} \beta' &= \frac{1}{T}\int_0^{2T} \mathcal{R}(\tau)\sin\omega_i\tau\, d\tau \\ (\beta'-\beta) &= \frac{A}{2Tm}\int_0^{2Tm} n(t)\cos\omega_i t\, dt \end{aligned}\right\} \tag{3.36}$$

By substitution of the expressions given by Eqs. (3.27) and (3.36) into Eqs. (3.34) and (3.35), the relations for the variable ζ and ξ are found to be

$$\zeta = \left[\frac{1}{T}\int_0^{2T} \mathcal{R}(\tau)\sin\omega_i\tau\, d\tau\right]^{2} - \left[\frac{1}{T}\int_0^{2T} \mathcal{R}(\tau)\sin\omega_i\tau\, d\tau\right]\left[\frac{A}{2Tm}\int_0^{2Tm} n(t)\cos\omega_i t\, dt\right] \tag{3.37}$$

$$\xi = [R(\tau)]^2 - [R(\tau)]\left[\frac{A}{2Tm}\int_0^{2Tm} n(t)\sin\omega_i(t+\tau)\,dt\right] \tag{3.38}$$

With the assumption that the noise considered in this analysis is a stationary random function, then mean values and mean deviations σ of the statistical variables ζ and ξ are, respectively

$$\bar{\zeta} = \left[\frac{1}{T}\int_0^{2T} R(\tau)\sin\omega_i\tau\,d\tau\right]^2 - \left[\frac{1}{T}\int_0^{2T} R(\tau)\sin\omega_i\tau\,d\tau\right]\left[\frac{A}{2mT}\int_0^{2Tm} \overline{n(t)}\cos\omega_i t\,dt\right] \tag{3.40}$$

$$\bar{\xi} = [R(\tau)]^2 - [R(\tau)]\left[\frac{A}{2Tm}\int_0^{2Tm} \overline{n(t)}\sin\omega_i(t+\tau)\,dt\right] \tag{3.41}$$

and

$$\sigma_\zeta = \left[\frac{1}{T}\int_0^{2T} R(\tau)\sin\omega_i\tau\,d\tau\right]\left\{\frac{A^2}{(2Tm)^2}\int_0^{2Tm} dt''\int_0^{2Tm} dt'\,\overline{n(t')n(t'')}\cos\omega_i t'\cos\omega_i t'' - \left[\frac{A}{2Tm}\int_0^{2T} \overline{n(t)}\cos\omega_i t\,dt\right]^2\right\}^{1/2} \tag{3.42}$$

$$\sigma_\xi = R(\tau)\left\{\frac{A^2}{(2Tm)^2}\int_0^{2Tm} dt''\int_0^{2Tm} dt'\,\overline{n(t')n(t'')}\sin\omega_i(t'+\tau)\sin\omega_i(t''+\tau) - \left[\frac{A}{2Tm}\int_0^{2Tm} \overline{n(t)}\sin\omega_i(t+\tau)\,dt\right]^2\right\}^{1/2} \tag{3.43}$$

It is expected that the time average of the noise function $n(t)$ will vanish. Thus letting $\overline{n(t)} = 0$ in Eqs. (3.40), (3.41), (3.42) and (3.43), one obtains

$$\bar{\zeta} = \beta'^2 = \left[\frac{1}{T}\int_0^{2T} R(\tau)\sin\omega_i\tau\right]^2 \tag{3.44}$$

$$\bar{\xi} = [R(\tau)]^2 \tag{3.45}$$

$$\sigma_\zeta = \left[\frac{1}{T}\int_0^{2T} R(\tau)\sin\omega_i\tau\right]\left\{\frac{A^2}{(2Tm)^2}\int_0^{2Tm} dt''\int_0^{2Tm} dt'\,\overline{n(t')n(t'')}\cos\omega_i t'\cos\omega_i t''\right\}^{1/2} \tag{3.46}$$

$$\sigma_{\bar{\xi}} = R(\tau)\left\{\frac{A^2}{(2mT)^2}\int_0^{2Tm} dt'' \int_0^{2Tm} dt' \; \overline{n(t')n(t'')} \; \sin\omega_i(t'+\tau)\sin\omega_i(t''+\tau)\right\}^{1/2} \tag{3.47}$$

For a control engineer, the power spectrum of the noise, rather than its cross-correlation function, is more readily available. The cross-correlation function of noise, $\overline{n(t')n(t'')}$, is related to the power-spectrum of the noise, $\Phi_{nn}(\omega)$, by

$$\overline{n(t')n(t'')} = \int_0^{\infty} \Phi_{nn}(\omega)\cos\omega(t'-t'')\,d\omega \tag{3.48}$$

Substituting Eq. (3.48) into Eqs. (3.46) and (3.47), and carrying out the integrations with respect to t' and t'' (for details see Appendix B) the mean deviations σ_{ζ} and $\sigma_{\bar{\xi}}$ now become

$$\sigma_{\zeta} = \left[\frac{1}{T}\int_0^{2T} R(\tau)\sin\omega_i\tau\,d\tau\right]\left\{\frac{A^2}{\pi^2 m^2}\int_0^{\infty}\Phi_{nn}(\omega)\,\frac{\left(\frac{\omega}{\omega_i}\right)^2\sin^2\pi m\frac{\omega}{\omega_i}}{\left[\left(\frac{\omega}{\omega_i}\right)^2-1\right]^2}\,d\omega\right\}^{1/2} \tag{3.49}$$

$$\sigma_{\bar{\xi}} = R(\tau)\left\{\frac{A^2}{\pi^2 m^2}\int_0^{\infty}\Phi_{nn}(\omega)\,\frac{\sin^2\pi m\frac{\omega}{\omega_i}}{\left[\left(\frac{\omega}{\omega_i}\right)^2-1\right]^2}\,d\omega + \frac{A^2}{\pi^2 m^2}(\sin\omega_i\tau)^2\int_0^{\infty}\Phi_{nn}(\omega)\,\frac{\sin^2\pi m\frac{\omega}{\omega_i}}{\left[\left(\frac{\omega}{\omega_i}\right)^2-1\right]}\,d\omega\right\}^{1/2} \tag{3.50}$$

When the first probability distributions of the statistical variables ζ and $\bar{\xi}$ are not known, it is possible to make a broad approximation of the errors ε_{ζ} and $\varepsilon_{\bar{\xi}}$, the probability of finding $\beta'\beta < 0$, and the probability of finding $\ell'\ell < 0$. First, for this purpose the probability of deviation from the mean will be estimated.

Assume, for example, that the ζ's distribute themselves around the mean $\bar{\zeta}$ symmetrically and that $\bar{\zeta}$ is positive and sufficiently large. Then the probability of the occurrence of large deviations from the mean is given by Bienaymé-Chebyshev's inequality[4]

$$P[|\zeta-\bar{\zeta}| \geq K\sigma_{\zeta}] \leq \frac{1}{K^2} \tag{3.51}$$

The error ε_{ζ} is one half of the probability given by Eq. (3.51) when $K\sigma_{\zeta}$ is replaced by $\bar{\zeta}$. Thus

$$\varepsilon_{\zeta} = \frac{1}{2} P[|\zeta-\bar{\zeta}| \geq \bar{\zeta}] \leq \frac{1}{2} \frac{\sigma_{\zeta}^2}{\bar{\zeta}^2} \tag{3.52}$$

Similarly, the error ε_{ξ}, the probability of finding $\ell\ell' < 0$, can be approximated as

$$\varepsilon_{\xi} \leq \frac{1}{2} \frac{\sigma_{\xi}^2}{\bar{\xi}^2} \tag{3.53}$$

A sharper approximation of the errors ε_{ζ} and ε_{ξ} will be obtained when the probability distribution is unimodal. Then the Gauss inequality[4] can be used in place of the Bienaymé-Chebyshev's inequality. For example, the Gauss inequality for unimodal distribution is given as

$$P[|\zeta-\bar{\zeta}| \geq K\sigma_{\zeta}] \leq \frac{4}{9} \frac{1}{K^2} \tag{3.54}$$

and the errors ε_{ζ} and ε_{ξ} in this case are

$$\varepsilon_{\zeta} \leq \frac{2}{9} \frac{\sigma_{\zeta}^2}{\bar{\zeta}^2} \tag{3.55}$$

and

$$\varepsilon_{\xi} \leq \frac{2}{9} \frac{\sigma_{\xi}^2}{\bar{\xi}^2} \tag{3.56}$$

When first probability distributions of the statistical variables ζ and ξ are known, it is possible to compute the errors ε_ζ and ε_ξ exactly. For example, since ζ is a stationary random variable, its first probability distribution W_1 is a function of ζ only. Then the probability that $\zeta < 0$ is simply the error ε_ζ; thus

$$\varepsilon_\zeta = \int_{-\infty}^{0} W_1(\zeta)\, d\zeta \tag{3.57}$$

Assuming that the first probability distribution of ζ is Gaussian, that is

$$W_1(\zeta) = \frac{1}{\sigma_\zeta \sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{\zeta - \bar{\zeta}}{\sigma_\zeta}\right)^2} \tag{3.58}$$

there the error ε_ζ is computed by substituting Eq. (3.58) into Eq. (3.57). Thus

$$\varepsilon_\zeta = \frac{1}{\sigma_\zeta \sqrt{2\pi}} \int_{-\infty}^{0} e^{-\frac{1}{2}\left(\frac{\zeta - \bar{\zeta}}{\sigma_\zeta}\right)^2} d\zeta$$

or

$$\varepsilon_\zeta = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\frac{\bar{\zeta}}{\sigma_\zeta}} e^{-\frac{1}{2}\eta^2}\, d\eta$$

The evaluation of this integral yields the familiar error-function. Thus

$$\varepsilon_\zeta = \frac{1}{2}\left[1 - \operatorname{erf} \frac{1}{\sqrt{2}} \frac{\bar{\zeta}}{\sigma_\zeta}\right] \tag{3.59}$$

Similarly, the error ε_ξ is obtained as

$$\varepsilon_{\xi} = \frac{1}{2}\left[1 - \operatorname{erf}\frac{1}{\sqrt{2}}\frac{\bar{\xi}}{\sigma_{\xi}}\right] \tag{3.60}$$

It is to be noted that whether the probability of the occurrence of Type II error is approximated, for example, by one of Eqs. (3.53) or (3.56), or is computed by assuming a first probability distribution, Eq. (3.60), ε_{ξ} depends only on the quotient $\bar{\xi}/\sigma_{\xi}$. Fig. 3.7 shows the probability of the occurrence of Type II error as a function of the quotient $\bar{\xi}/\sigma_{\xi}$ or $\bar{\zeta}/\sigma_{\zeta}$.

In order to demonstrate a possible way of choosing the limiting parameters ℓ'^{*} or ρ'^{*} when the probability of Type II error is assigned and the statistical properties of the noise interference are given, one can form the following quotients:

$$\rho' \frac{\sigma_{\zeta}}{\bar{\zeta}} = \sqrt{\overline{n^2(t)}}\, A \left\{ \frac{1}{\pi^2 m^2} \frac{1}{\overline{n^2(t)}} \int_0^{\infty} \bar{\Phi}_{nn}(\omega) \frac{\left(\frac{\omega}{\omega_i}\right)^2 \sin^2 \pi m \frac{\omega}{\omega_i}}{\left[\left(\frac{\omega}{\omega_i}\right)^2 - 1\right]^2} d\omega \right\}^{1/2} \tag{3.61}$$

$$\ell' \frac{\sigma_{\xi}}{\bar{\xi}} = \frac{\sqrt{\overline{n^2(t)}}}{D} \frac{\pi^2}{2} \frac{\left[\frac{1}{12} - \left(\frac{\tau_i}{T}\right)^2 + 2\left(\frac{\tau_i}{T}\right)^3 \tanh \frac{1}{2}\frac{T}{\tau_i}\right]\left[1+\omega_i^2\tau_0^2\right]\left[1+\omega_i^2\tau_i^2\right]}{\left\{(\omega_i^2\tau_0\tau_i - 1)\cos\omega_i\tau + \omega_i(\tau_0+\tau_i)\sin\omega_i\tau\right\}} \frac{1}{\sqrt{\overline{n^2(t)}}} \left\{ \frac{1}{\pi^2 m^2} \int_0^{\infty} \frac{\bar{\Phi}_{nn}(\omega)\sin^2 \pi m \frac{\omega}{\omega_i}}{\left[\left(\frac{\omega}{\omega_i}\right)^2 - 1\right]^2} d\omega + \frac{\sin^2\omega_i\tau}{\pi^2 m^2} \int_0^{\infty} \frac{\bar{\Phi}_{nn}(\omega)\sin^2 \pi m \frac{\omega}{\omega_i}}{(\omega/\omega_i)^2 - 1} d\omega \right\}^{1/2} \tag{3.62}$$

where Eq. (3.61) is obtained from Eqs. (3.44) and (3.49), and Eq. (3.62) is obtained from Eqs. (2.20), (3.28), (3.45), and (3.50).

The limiting value of ρ'^{*} for a given value of the probability of occurrence of Type II error, ε_{ζ}, can be computed from Eq. (3.61) when the power spectrum of the noise, $\bar{\Phi}_{nn}(\omega)$, and the amplitude A of the sinusoid used for correlation are known. For example, with an assigned value of ε_{ζ}, the quotient $\sigma_{\zeta}/\bar{\zeta}$ is obtained from Fig. 3.7. Thus by use of Eq. (3.61), β'^{*} is given as

$$\beta'^{*} = A\sqrt{\overline{n^2(t)}}\,\frac{\ddot{\xi}}{\sigma_{\xi}}\,g_1 \qquad (3.62)$$

where the characteristic noise function g_1 is defined as

$$g_1 = \left\{\frac{1}{\pi^2 m^2}\,\frac{1}{\overline{n^2(t)}}\int_0^{\infty}\Phi_{nn}(\omega)\,\frac{\left(\frac{\omega}{\omega_i}\right)^2\sin^2\pi m\frac{\omega}{\omega_i}}{\left[\left(\frac{\omega}{\omega_i}\right)^2-1\right]^2}\,d\omega\right\}^{1/2} \qquad (3.63)$$

The coefficient β', and hence its limiting value β'^{*}, are obtained without any knowledge of controlled and control system characteristics. For the purpose of illustration, Figs. 3.8 and 3.9 show g_1 as a function of the ratio ω_n/ω_i with m as parameter for typical noise power spectrums, $\Phi_{nn}(\omega) = \nu^2/(\omega^2+\omega_n^2)$ and $\Phi_{nn}(\omega) = \nu^4/(\omega^4+\omega_n^4)$, respectively. Details are presented in Appendix C.

When controlled and control system characteristics are available and when the power spectrum of the noise can be estimated, then ℓ'^{*} is obtained from Eq. (3.62) by using the value of the quotient $\sigma_{\xi}/\ddot{\xi}$ obtained from Fig. 3.7 using the assigned value of ϵ_{ξ}. To illustrate a typical procedure, choose a time displacement $\tau = 0$; then Eq. (3.62) becomes

$$\ell'^{*} = \frac{\sqrt{\overline{n^2(t)}}}{D}\cdot\frac{\ddot{\xi}}{\sigma_{\xi}}\cdot g_2\cdot f_2 \qquad (3.64)$$

where

$$g_2 = \left\{\frac{1}{\pi^2 m^2}\cdot\frac{1}{\overline{n^2(t)}}\int_0^{\infty}\Phi_{nn}(\omega)\,\frac{\sin\pi m\frac{\omega}{\omega_i}}{\left[\left(\frac{\omega}{\omega_i}\right)^2-1\right]^2}\,d\omega\right\}^{1/2} \qquad (3.65)$$

and

$$f_2 = \frac{\pi^2}{2} \frac{\left[\frac{1}{12} - \left(\frac{\tau_i}{T}\right)^2 + 2\left(\frac{\tau_i}{T}\right)^3 \tanh \frac{1}{2}\frac{T}{\tau_i}\right]\left[1 + \omega_i^2 \tau_o^2\right]\left[1 + \omega_i^2 \tau_i^2\right]}{\left(\omega_i^2 \tau_o \tau_i - 1\right)} \tag{3.66}$$

Figs. 3.10 and 3.11 show the characteristic noise function g_2 as a function of the ratio ω_n/ω_i with m as parameter for typical noise power spectrums, $\Phi_{nn}(\omega) = \nu^2/(\omega^2 + \omega_n^2)$ and $\Phi_{nn}(\omega) = \nu^4/(\omega^4 + \omega_n^4)$, respectively. Details are presented in Appendix C. From Figs. (3.8), (3.9), (3.10), and (3.11) it is observed that in the neighborhood of unity, for constant values of the ratio ω_n/ω_i, the functions g_1 and g_2 decrease in magnitude very rapidly for the first few values of the parameter m. For the values of the quotient ω_n/ω_i away from unity, the effect of the parameter m on the functions g_1 and g_2 is unimportant. Fig. 3.12 shows the controlled system dynamic characteristics function f_2 as a function of τ_o/τ_i with $(\tau_o + \tau_i)/T$ as parameter. Figs. 3.13 through 3.15 show the limiting value ℓ'^* as a function of the dimensionless noise level $\sqrt{\overline{n^2(t)}}/D$ with m, the number of input cycles for which the correlation process is carried out, as a parameter. On each figure two values of ℓ'^* are presented which correspond to five and ten per cent probability of the occurrence of Type II error due to noise interference effects, which are characterized by a power spectrum of the form $\Phi_{nn}(\omega) = \nu^2/(\omega^2 + \omega_n^2)$ and characteristic frequency $\omega_n = \omega_i$. Thus Figs. 3.7 through 3.15 are typical illustrations of the analysis which must be carried out to determine the parameter ℓ'^*.

IV. SINUSOIDAL INPUT OPTIMALIZING CONTROLLER

In Part II of this work, it was indicated that the saw-tooth form of input is the simplest type that is suitable for a peak-holding optimalizing control. However, in Part III of this work, the optimalizing control scheme is changed in such a manner that any periodic signal can be used as the input. With the principle introduced there, the resulting control system will again hunt around the optimum state. In this part, because of the ease with which it may be generated, a simple sinusoidal form of "potential" input will be considered as an illustration.

4.1 Dynamic Effects

The dynamic effects in this particular case can be taken into account as was done in Part II. In order to avoid repetition, most of the derivations will be omitted and the final results will be given directly.

The sinusoidal "potential" input x_s^* with amplitude a' and period $2T$ is given as

$$x_s^* = a' \sin \omega_i t \tag{4.1}$$

where $\omega_i = \pi/T$. In the operator form, the actual input x_s and the actual output y_s are given by

$$x_s = -\frac{a'}{2} i \left[F_i(i\omega_i) e^{i\omega_i t} - F_i(-i\omega_i) e^{-i\omega_i t} \right] \tag{4.2}$$

and

$$y_s = \frac{a'^2 k}{4} \left\{ F_o(i2\omega_i) [F_i(i\omega_i)]^2 e^{i2\omega_i t} - 2 F_i(i\omega_i) F_i(-i\omega_i) + F_o(-i2\omega_i) [F_i(-i\omega_i)]^2 e^{-i2\omega_i t} \right\} \tag{4.3}$$

The hunting loss D_s is simply given by

$$D_s = \frac{a'^2 k}{2} |F_i(i\omega_i)|^2 \tag{4.4}$$

If it is assumed that the input and output linear groups can be closely approximated by first order systems, then the actual input x_s, the actual output y_{ds}, and the hunting loss D_s become

$$x_s = \frac{a'}{1+\omega_i^2\tau_i^2}\left\{\sin\omega_i t - \omega_i\tau_i\cos\omega_i t\right\} \tag{4.5}$$

$$y_{ds} = \frac{a'^2 k}{2}\left\{\frac{[1-(\omega_i\tau_i)^2 - 4\omega_i^2\tau_o\tau_i]\cos 2\omega_i t + 2[\omega_i\tau_o(1-\omega_i^2\tau_i^2)+\omega_i\tau_i]\sin 2\omega_i t}{[1+(2\omega_i\tau_o)^2][1+(\omega_i\tau_i)^2]^2} - \frac{1}{1+\omega_i^2\tau_i^2}\right\} \tag{4.6}$$

and

$$D_s = \frac{a'^2 k}{2}\,\frac{1}{1+\omega_i^2\tau_i^2} \tag{4.7}$$

4.2 Comparison of Sinusoidal Input Optimalizing Controller Performance with Peak-holding Optimalizing Controller

The basis for comparison of the performance of the sinusoidal input optimalizing controller and the saw-tooth input optimalizing controller is chosen to be the equality of the root mean square, or effective value, of the "potential" input for both types of controllers. This is to say that the average power-producing capacity of the "potential" input per cycle is the same for both controllers. It is also understood that the hunting periods of these controllers are the same.

The root mean square value of "potential" input x^* for a saw-tooth type input is given by

$$a/\sqrt{3} \tag{4.8}$$

where a is the amplitude of saw-tooth referred to optimum level. The root mean square of the "potential" input x_s^* for sinusoidal input is given by

$$a'/\sqrt{2} \tag{4.9}$$

where a' is the amplitude of the sinusoid. Equating Eq. (4.8) to Eq. (4.9), the amplitude of the sinusoid is found to be

$$a' = \sqrt{\frac{2}{3}}\, a \tag{4.10}$$

Substituting the value of a' from Eq. (4.10) into Eqs. (4.1), (4.5), (4.6), and (4.7), it is now possible to compare the performances of these two types of control systems. The results of the comparison are presented with Figs. 4.2 through 4.5 for a controlled system with input and output linear group time constants of $\frac{\tau_i}{T}$ = 0.10 and $\frac{\tau_o}{T}$ = 0.15 respectively. The deviations of actual inputs and actual outputs for both systems, presented in Figs. 4.3 and 4.4 respectively, are rather small. The hunting losses of both systems are the same, as is shown in Fig. 4.5. This is due to the fact that for both systems the root mean square of "potential" inputs have been chosen to be equal. For engineering analysis it is far more preferable to work with the simple equations that are found for the case of sinusoidal input. However, it may be necessary for far more complicated equipment to generate a sinusoidal input.

4.3 Incorrect Operation Mode and Noise Interference

The analysis carried out in Part III for the detection of an incorrect operation mode and the elimination of noise interference holds for this case without any substantial change. Assuming again an amount of incorrect input b, the "potential" input x_{σ}^* in this case is given by

$$x_{\sigma}^* = b + a' \sin \omega_i t \tag{4.11}$$

The actual input x_σ , the actual output y_σ , and the hunting loss D_σ in the incorrect operation mode are found to be

$$x_\sigma = b + \frac{1}{1+\omega_i^2\tau_i^2}\left\{\sin\omega_i t - \omega_i\tau_i\cos\omega_i t\right\} \tag{4.12}$$

$$y_\sigma = -kb^2 - 2a'bk\,\frac{[1-\omega_i^2\tau_i\tau_0]\sin\omega_i t - \omega_i(\tau_0+\tau_i)\cos\omega_i t}{[1+\omega_i^2\tau_0^2][1+\omega_i^2\tau_i^2]}$$

$$+\frac{a'^2k}{2}\left\{\frac{[1-\omega_i^2\tau_i^2-4\omega_i^2\tau_i\tau_0]\cos 2\omega_i t + 2[\omega_i\tau_0(1-\omega_i^2\tau_i^2)+\omega_i\tau_i]\sin 2\omega_i t}{[1+(2\omega_i\tau_0)^2][1+(\omega_i\tau_i)^2]^2} - \frac{1}{1+(\omega_i\tau_i)^2}\right\} \tag{4.13}$$

and

$$D_\sigma = kb^2 + \frac{a'^2k}{2}\,\frac{1}{1+(\omega_i\tau_i)^2} \tag{4.14}$$

Similarly the noise interference-free cross-correlation function $\chi_\sigma(\tau)$ is obtained as

$$\chi_\sigma(\tau) = \frac{Aa'bk}{[1+\omega_i^2\tau_0^2][1+\omega_i^2\tau_i^2]}\left\{[(\omega_i^2\tau_i\tau_0-1)]\cos\omega_i\tau + \omega_i(\tau_0+\tau_i)\sin\omega_i\tau\right\} \tag{4.15}$$

It is observed that the form of Eq. (4.15) is exactly the same as that of Eq. (3.20). The only difference is a multiplicative constant. Therefore it is unnecessary to carry out the remainder of the analysis for this type of control system. The fundamental concepts introduced in Part III are perfectly valid in the present analysis.

REFERENCES

1. C. S. Draper and Y. T. Li, "Principles of Optimalizing Control Systems and an Application to the Internal Combustion Engine", American Society of Mechanical Engineers Publication, (September 1951).

2. Y. T. Li, "Optimalizing System for Process Control", Instruments, Vol. : 25, (January 1952), pp. 72-77; (February 1952), pp. 190-193, 228; (March 1952), pp. 324-327, 350-352.

3. R. Shull, Jr., "An Automatic Cruise Control Computer for Long Range Aircraft", Trans. I.R.E., Professional Group on Electronic Computers, (December 1952), pp. 47-51.

4. H. Cramér, "Mathematical Methods of Statistics", Princeton University Press, 1951, p. 183.

APPENDIX A

TYPICAL SUMMATION FORMULAE

The following table contains a complete list of summation formulae used throughout this text. Most of these formulae are well known and can be found in standard texts. For the sake of illustration of the summation technique used, in this appendix summations of two typical infinite series, namely S_{11}' and S_{12}', are also carried out in detail.

TABLE I

$$S_1' = \sum_{n=0}^{\infty}{}' \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$$

$$S_2' = \sum_{n=0}^{\infty}{}' \frac{1}{(2n+1)^4} = \frac{\pi^4}{96}$$

$$S_3' = \sum_{n=0}^{\infty}{}' \frac{1}{\{1+(2n+1)^2 a^2\}} = \frac{\pi}{4a} \tanh \frac{\pi}{2a}$$

$$S_4' = \sum_{n=0}^{\infty}{}' \frac{1}{\{1+(2n+1)^2 a^2\}^2} = \frac{\pi}{8a}\left\{\tanh \frac{\pi}{2a} + \frac{\pi}{2a}\left(\tanh^2 \frac{\pi}{2a} - 1\right)\right\}$$

$$S_5' = \sum_{n=1}^{\infty}{}' \frac{(-1)^{n-1} \sin 2n\theta}{n} = (\theta - j\pi)$$ when $-\frac{\pi}{2} < (\theta - j\pi) < \frac{\pi}{2}$ where $j = 0, 1, 2, \cdots$

$$S_6' = \sum_{n=1}^{\infty}{}' \frac{(-1)^{n-1} \cos 2n\theta}{n^2} = \frac{\pi^2}{12} - (\theta - j\pi)$$ when $-\frac{\pi}{2} < (\theta - j\pi) < \frac{\pi}{2}$ where $j = 0, 1, 2, \cdots$

$$S_7' = \sum_{n=0}^{\infty}{}' \frac{(-1)^{n} \cos (2n+1)\theta}{(2n+1)} = (-1)^{j}\frac{\pi}{4}$$ when $-\frac{\pi}{2} < (\theta - j\pi) < \frac{\pi}{2}$ where $j = 0, 1, 2, \cdots$

$$S_8' = \sum_{n=0}^{\infty}{}' \frac{(-1)^{n} \sin (2n+1)\theta}{(2n+1)^2} = (-1)^{j}\frac{\pi}{4}(\theta - j\pi)$$ when $-\frac{\pi}{2} \leq (\theta - j\pi) \leq \frac{\pi}{2}$ where $j = 0, 1, 2, \cdots$

$$S_9 = \sum_{n=0}^{\infty}{}' \frac{(-1)^n \sin(2n+1)\theta}{[1+(2n+1)^2 a^2]} = (-1)^j \frac{\pi}{4a} \frac{\sinh \frac{1}{a}(\theta - j\pi)}{\cosh \frac{\pi}{2a}}$$

when $-\frac{\pi}{2} < (\theta - j\pi) < \frac{\pi}{2}$

where $j = 0, 1, 2, \cdots$

$$S_{10} = \sum_{n=0}^{\infty}{}' \frac{(-1)^n (2n+1) \cos(2n+1)\theta}{[1+(2n+1)^2 a^2]} = (-1)^j \frac{\pi}{4a^2} \frac{\cosh \frac{1}{a}(\theta - j\pi)}{\cosh \frac{\pi}{2a}}$$

when $-\frac{\pi}{2} < (\theta - j\pi) < \frac{\pi}{2}$

where $j = 0, 1, 2, \cdots$

$$S_{11}^{(*)} = \mathcal{R}\ell \sum_{n=1}^{\infty}{}' \frac{(-1)^{n-1} e^{i2n\theta}}{(1+i2na)} = \frac{1}{2} - \frac{\pi}{4a} \frac{e^{-\frac{1}{a}(\theta - j\pi)}}{\sinh \frac{\pi}{2a}}$$

when $-\frac{\pi}{2} < (\theta - j\pi) < \frac{\pi}{2}$

where $j = 0, 1, 2, \cdots$

$$S_{12} = \mathcal{R}\ell \sum_{n=1}^{\infty}{}' \frac{(-1)^{n-1} e^{i2n\theta}}{(1+i2na)^2} = \frac{1}{2} - \frac{\pi}{4a^2} \frac{e^{-\frac{1}{a}(\theta - j\pi)}}{\sinh \frac{\pi}{2a}} \left[(\theta - j\pi) + \frac{\pi}{2} \coth \frac{\pi}{2a}\right]$$

when $-\frac{\pi}{2} < (\theta - j\pi) < \frac{\pi}{2}$

where $j = 0, 1, 2, \cdots$

$$S_{13} = \mathcal{R}\ell \sum_{n=1}^{\infty}{}' \frac{(-1)^{n-1} e^{i2n\theta}}{(1+i2na)^3} = \frac{1}{2} - \frac{\pi}{8a^3} \frac{e^{-\frac{1}{a}(\theta - j\pi)}}{\sinh \frac{\pi}{2a}} \left[(\theta - j\pi)^2 + \pi(\theta - j\pi)\coth \frac{\pi}{2a} + \frac{\pi^2}{4}\left(2\coth^2 \frac{\pi}{2a} - 1\right)\right]$$

when $-\frac{\pi}{2} < (\theta - j\pi) < \frac{\pi}{2}$

where $j = 0, 1, 2, \cdots$

$$S_{14} = \sum_{n=1}^{k}{}' \sin^2 \frac{2\pi}{k} n = \frac{k}{2}$$

$$S_{15} = \sum_{n=1}^{k}{}' \cos^2 \frac{2\pi}{k} n = \frac{k}{2}$$

$$S_{16} = \sum_{n=1}^{k}{}' \sin \frac{2\pi}{k} n \cos \frac{2\pi}{k} n = 0$$

<u>Evaluation of Infinite Series,</u> $S_{11} = \mathcal{R}\ell \sum_{n=1}^{\infty}{}' \frac{(-1)^{n-1} e^{i2n\theta}}{(1+i2na)}$

Consider the following infinite series

$$S = \sum_{n=1}^{\infty}{}' \frac{(-1)^{n-1} e^{i2n\theta}}{(1+i2na)} \quad \text{(A-1)}$$

where a is a positive real number. The series given by Eq. (A-1) can also be written as

$$S = \lim_{N \to \infty} \sum_{n=1}^{N}{}' (-1)^{n-1} \int_0^{\infty} e^{i2n\theta - (1+i2na)\xi} \, d\xi \quad \text{(A-2)}$$

---

* $\mathcal{R}\ell$ means the "real part of" of the expression following it.

Since the series given in Eq. (A-2) is finite, it is perfectly legitimate to perform the summation operation before integration operation, thus

$$S' = \lim_{N\to\infty}\int_0^\infty d\xi\, e^{-\xi}\sum_{n=1}^{N}(-1)^{n-1}\, e^{-i2n(a\xi-\theta)} \tag{A-3}$$

The series in the integrand of Eq. (A-3) is a typical finite geometric series and can be summed as,

$$\sum_{n=1}^{N}(-1)^{n-1}\, e^{-i2n(a\xi-\theta)} = \frac{e^{-i(a\xi-\theta)} - (-1)^N e^{-i(2N+1)(a\xi-\theta)}}{e^{-i(a\xi-\theta)} + e^{i(a\xi-\theta)}}$$

$$= \left\{\frac{1}{2} - \frac{(-1)^N \cos(2N+1)(a\xi-\theta)}{2\cos(a\xi-\theta)}\right\} + i\left\{\frac{(-1)^N \sin(2N+1)(a\xi-\theta) - \sin(a\xi-\theta)}{2\cos(a\xi-\theta)}\right\} \tag{A-4}$$

By substitution of Eq. (A-4) into Eq. (A-3), series S' can also be written as,

$$S' = \lim_{N\to\infty}\int_0^\infty d\xi\, e^{-\xi}\left\{\frac{1}{2} - \frac{(-1)^N \cos(2N+1)(a\xi-\theta)}{2\cos(a\xi-\theta)}\right\} + i\lim_{N\to\infty}\int_0^\infty d\xi\, e^{-\xi}\left\{\frac{(-1)^N \sin(2N+1)(a\xi-\theta) - \sin(a\xi-\theta)}{2\cos(a\xi-\theta)}\right\} \tag{A-5}$$

But the infinite series S'_{11} is just the real part of the expression for S' given by Eq. (A-5), thus

$$S'_{11} = Re\sum_{n=1}^{\infty}\frac{(-1)^{n-1} e^{i2n\theta}}{(1+i2na)} = \lim_{N\to\infty}\int_0^\infty d\xi\, e^{-\xi}\left\{\frac{1}{2} - \frac{(-1)^N \cos(2N+1)(a\xi-\theta)}{2\cos(a\xi-\theta)}\right\}$$

$$= \frac{1}{2} - \frac{1}{2}\lim_{N\to\infty}\int_0^\infty \frac{(-1)^N e^{-\xi}\cos(2N+1)(a\xi-\theta)}{\cos(a\xi-\theta)}\, d\xi \tag{A-6}$$

The factor $\frac{\cos(2N+1)(a\xi-\theta)}{\cos(a\xi-\theta)}$ in the integrand of Eq. (A-6) has a period of π, and the integration can be carried out piecewise with proper modification of factor $e^{-\xi}$, thus

$$S'_{II} = \frac{1}{2} - \frac{1}{2a} \lim_{N\to\infty} \int_0^{\pi} e^{-\frac{\eta}{a}} \left(\sum_{m=0}^{\infty}{}' e^{-m\frac{\pi}{a}} \right) \frac{(-1)^N \cos(2N+1)(\eta-\theta)}{\cos(\eta-\theta)} d\eta \qquad \text{(A-7)}$$

where η is substituted in place of $a\xi$. The infinite series $\sum_{m=0}^{\infty}{}' e^{-m\frac{\pi}{a}}$ in the integrand of Eq. (A-7) is an infinite geometric series and can be summed as,

$$\sum_{m=0}^{\infty}{}' e^{-m\frac{\pi}{a}} = \frac{e^{\frac{\pi}{2a}}}{2 \sinh \frac{\pi}{2a}} \qquad \text{(A-8)}$$

By substitution of Eq. (A-8) into Eq. (A-7), series S'_{II} becomes

$$S'_{II} = \frac{1}{2} - \frac{e^{\frac{\pi}{2a}}}{4a \sinh \frac{\pi}{2a}} \lim_{N\to\infty} \int_0^{\pi} e^{-\frac{\eta}{a}} \frac{(-1)^N \cos(2N+1)(\eta-\theta)}{\cos(\eta-\theta)} d\eta$$

or

$$S'_{II} = \frac{1}{2} - \frac{e^{\frac{\pi}{2a}-\frac{\theta}{a}}}{4a \sinh \frac{\pi}{2a}} \lim_{N\to\infty} \int_{-\theta}^{\pi-\theta} e^{-\frac{\varphi}{a}} \frac{(-1)^N \cos(2N+1)\varphi}{\cos\varphi} d\varphi \qquad \text{(A-9)}$$

where φ is substituted in place of $(\eta-\theta)$. The factor $\frac{\cos(2N+1)\varphi}{\cos\varphi}$ in the integrand of Eq. (A-9) acts as a delta-function with strength π located at the values of $\varphi = \frac{\pi}{2} \mp k$ (where k = 0, 1, 2, ...). Therefore in the interval $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, as $N\to\infty$ the only contribution from the integral given in Eq. (A-9) is $\pi e^{-\frac{\pi}{2a}}$. Thus Eq. (A-9) becomes

$$S'_{II} = \frac{1}{2} - \frac{\pi}{4a} \frac{e^{-\frac{\theta}{a}}}{\sinh \frac{\pi}{2a}} \qquad \text{when} \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2} \qquad \text{(A-10)}$$

or

$$S'_{11} = \frac{1}{2} - \frac{\pi}{4a} \frac{e^{-\frac{1}{a}(\theta - j\pi)}}{\sinh \frac{\pi}{2a}} \qquad \text{when } -\frac{\pi}{2} < (\theta - j\pi) < \frac{\pi}{2} \quad \text{where } j = 0, 1, 2, \cdots \tag{A-11}$$

Evaluation of Infinite Series, $S'_{12} = \mathcal{R}\ell \sum_{n=1}^{\infty}{}' \frac{(-1)^{n-1} e^{i2n\theta}}{(1 + i2na)^2}$

The infinite series S'_{12} can also be written as

$$S'_{12} = -b^2 \mathcal{R}\ell \sum_{n=1}^{\infty}{}' \frac{\partial}{\partial b} \frac{(-1)^{n-1} e^{i2n\theta}}{(b + i2n)} \tag{A-12}$$

where $b = 1/a$. It can be shown that the infinite series S'_{12} and $\sum_{n=1}^{\infty}{}' \frac{(-1)^{n-1} e^{i2n\theta}}{(b + i2n)}$ are both uniformly convergent; therefore it is perfectly legitimate to perform the summation operation in Eq. (A-12) first and then to differentiate with respect to b. Thus

$$S'_{12} = -b^2 \frac{\partial}{\partial b} \left\{ \mathcal{R}\ell \sum_{n=1}^{\infty}{}' \frac{(-1)^{n-1} e^{i2n\theta}}{(b + i2n)} \right\} \tag{A-13}$$

The infinite series in the brace is the typical summation S'_{11} given in Table I, thus

$$\mathcal{R}\ell \sum_{n=1}^{\infty}{}' \frac{(-1)^{n-1} e^{i2n\theta}}{(b + i2n)} = \frac{1}{2b} - \frac{\pi}{4} \frac{e^{-b(\theta - j\pi)}}{\sinh \frac{\pi}{2} b} \qquad \text{when } -\frac{\pi}{2} < (\theta - j\pi) < \frac{\pi}{2} \quad \text{where } j = 0, 1, 2, \cdots \tag{A-14}$$

By substitution of Eq. (A-14) into Eq. (A-13), the infinite series S'_{12} becomes

$$S'_{12} = \frac{1}{2} - \frac{\pi}{4} b^2 \frac{e^{-b(\theta - j\pi)}}{\sinh \frac{\pi b}{2}} \left[(\theta - j\pi) + \frac{\pi}{2} \coth \frac{\pi b}{2} \right]$$

or

$$S'_{12} = \frac{1}{2} - \frac{\pi}{4a^2} \frac{e^{-\frac{1}{a}(\theta - j\pi)}}{\sinh \frac{\pi}{2a}} \left[(\theta - j\pi) + \frac{\pi}{2} \coth \frac{\pi}{2a} \right] \qquad \text{when } -\frac{\pi}{2} < (\theta - j\pi) < \frac{\pi}{2} \quad \text{where } j = 0, 1, 2, \cdots \tag{A-15}$$

APPENDIX B

EVALUATION OF THE INTEGRALS GIVEN IN EQS. (3.46) AND (3.47)

The integrals to be evaluated are the following:

$$I_1 = \frac{A^2}{(2Tm)^2} \int_0^{2Tm} dt'' \int_0^{2Tm} dt' \, \overline{n(t')\, n(t'')} \, \cos\omega_i t' \cos\omega_i t'' \tag{B-1}$$

and

$$I_2 = \frac{A^2}{(2Tm)^2} \int_0^{2Tm} dt'' \int_0^{2Tm} dt' \, \overline{n(t')\, n(t'')} \, \sin\omega_i(t'+\tau) \sin\omega_i(t''+\tau) \tag{B-2}$$

By introduction of the noise-power spectrum $\Phi_{nn}(\omega)$ into these expressions for I_1 and I_2,

$$I_1 = \frac{A^2}{(2Tm)^2} \int_0^{2Tm} dt'' \int_0^{2Tm} dt' \int_0^{\infty} d\omega \, \Phi_{nn}(\omega) \cos\omega(t'-t'') \cos\omega_i t' \cos\omega_i t'' \tag{B-3}$$

and

$$I_2 = \frac{A^2}{(2Tm)^2} \int_0^{2Tm} dt'' \int_0^{2Tm} dt' \int_0^{\infty} d\omega \, \Phi_{nn}(\omega) \cos\omega(t'-t'') \sin\omega_i(t'+\tau) \sin\omega_i(t''+\tau) \tag{B-4}$$

Since the integrals I_1 and I_2 have essentially the same character, the evaluation of integral I_1 only will be carried out in detail. When the order of integration in Eq. (B-3) is changed, I_1 can also be written as

$$I_1 = \frac{A^2}{(2Tm)^2}\int_0^\infty \bar{\Phi}_{nn}(\omega)\,d\omega \int_0^{2Tm} dt'' \int_0^{2Tm} dt' \cos\omega(t'-t'')\cos\omega_i t'\cos\omega_i t''$$

$$= \frac{A^2}{(2Tm)^2}\int_0^\infty \bar{\Phi}_{nn}(\omega)\left\{\left[\int_0^{2Tm}\cos\omega t\cos\omega_i t\,dt\right]^2 + \left[\int_0^{2Tm}\sin\omega t\cos\omega_i t\right]^2\right\}d\omega \qquad \text{(B-5)}$$

The integrations with respect to t can be carried out easily, i.e.,

$$\int_0^{2Tm}\cos\omega t\cos\omega_i t\,dt = \left[\frac{\sin(\omega-\omega_i)t}{2(\omega-\omega_i)}+\frac{\sin(\omega+\omega_i)t}{2(\omega+\omega_i)}\right]_0^{2Tm} = \frac{\omega}{\omega^2-\omega_i^2}\sin 2\pi m\frac{\omega}{\omega_i} \qquad \text{(B-6)}$$

and

$$\int_0^{2Tm}\sin\omega t\cos\omega_i t\,dt = -\frac{1}{2}\left[\frac{\cos(\omega-\omega_i)t}{(\omega-\omega_i)}+\frac{\cos(\omega+\omega_i)t}{(\omega+\omega_i)}\right]_0^{2Tm} = -\frac{\omega}{\omega^2-\omega_i^2}\left[\cos 2\pi\frac{\omega}{\omega_i}-1\right] \qquad \text{(B-7)}$$

Substituting the results given in Eqs. (B-6) and (B-7) into Eq. (B-5) and simplifying I_1 is obtained as

$$I_1 = \frac{A^2}{\pi^2 m^2}\int_0^\infty \bar{\Phi}_{nn}(\omega)\,\frac{\left(\frac{\omega}{\omega_i}\right)^2\sin^2\pi m\frac{\omega}{\omega_i}}{\left[\left(\frac{\omega}{\omega_i}\right)^2-1\right]^2}\,d\omega \qquad \text{(B-8)}$$

Through a similar procedure I_2 is obtained as

$$I_2 = \frac{A^2}{\pi^2 m^2}\int_0^\infty \bar{\Phi}_{nn}(\omega)\,\frac{\sin^2\pi m\frac{\omega}{\omega_i}}{\left[\left(\frac{\omega}{\omega_i}\right)^2-1\right]^2}\,d\omega + \frac{A^2}{\pi^2 m^2}(\sin\omega\tau)^2\int_0^\infty \bar{\Phi}_{nn}(\omega)\,\frac{\sin^2\pi m\frac{\omega}{\omega_i}}{\left[\left(\frac{\omega}{\omega_i}\right)^2-1\right]}\,d\omega \qquad \text{(B-9)}$$

APPENDIX C

EVALUATION OF PARAMETERS λ_α AND λ_β FOR SOME REPRESENTATIVE VALUES OF NOISE POWER SPECTRUM

$$\lambda_\alpha = \left\{ \frac{A^2}{m^2\pi^2} \int_0^\infty \Phi_{nn}(\omega) \frac{\sin^2 \pi m \frac{\omega}{\omega_i}}{\left[\left(\frac{\omega}{\omega_i}\right)^2 - 1\right]^2} \, d\omega \right\}^{1/2}$$

$$\lambda_\beta = \left\{ \frac{A^2}{m^2\pi^2} \int_0^\infty \Phi_{nn}(\omega) \frac{\left(\frac{\omega}{\omega_i}\right)^2 \sin^2 \pi m \frac{\omega}{\omega_i}}{\left[\left(\frac{\omega}{\omega_i}\right)^2 - 1\right]^2} \, d\omega \right\}^{1/2}$$

The following is a list of values of the parameters λ_α and λ_β evaluated for some representative values of the noise power spectrum. For the sake of illustration, two typical evaluations are carried out in detail.

I. $\Phi_{nn}(\omega) = \gamma^2$ (White Noise)

$$\lambda_\alpha = \frac{A\gamma}{2}\sqrt{\frac{\omega_i}{m}} \qquad \text{and} \qquad \lambda_\beta = \frac{A\gamma}{2}\sqrt{\frac{\omega_i}{m}}$$

II. $\Phi_{nn}(\omega) = \gamma^2/(\omega^2 + \omega_n^2)$

$$\lambda_\alpha = \frac{A}{\left[\left(\frac{\omega_i}{\omega_n}\right)^2 + 1\right]} \sqrt{\frac{\overline{n^2(t)}}{2\pi m} \frac{\omega_i}{\omega_n}} \left\{ \left[\left(\frac{\omega_i}{\omega_n}\right)^2 + 1\right] + \frac{1}{\pi m}\left(\frac{\omega_i}{\omega_n}\right)^3 \left(1 - e^{-2\pi m \frac{\omega_n}{\omega_i}}\right) \right\}^{1/2}$$

$$\lambda_\beta = \frac{A}{\left[\left(\frac{\omega_i}{\omega_n}\right)^2 + 1\right]} \sqrt{\frac{\overline{n^2(t)}}{2\pi m} \frac{\omega_i}{\omega_n}} \left\{ \left[\left(\frac{\omega_i}{\omega_n}\right)^2 + 1\right] - \frac{1}{\pi m}\left(\frac{\omega_i}{\omega_n}\right) \left(1 - e^{-2\pi m \frac{\omega_n}{\omega_i}}\right) \right\}^{1/2}$$

III. $\Phi_{nn}(\omega) = \gamma^4/(\omega^4+\omega_n^4)$

$$\lambda_\alpha = \frac{A}{\left[\left(\frac{\omega_i}{\omega_n}\right)^4+1\right]}\sqrt{\frac{\overline{n^2(t)}}{2\pi m}\,\frac{\omega_i}{\omega_n}}\left\{\sqrt{2}\left[\left(\frac{\omega_i}{\omega_n}\right)^4+1\right]-\frac{1}{\pi m}\left(\frac{\omega_i}{\omega_n}\right)^3\left[\left(\frac{\omega_i}{\omega_n}\right)^4-2\left(\frac{\omega_i}{\omega_n}\right)^2-1\right]\left(e^{-\sqrt{2}\pi m\frac{\omega_n}{\omega_i}}\sin\sqrt{2}\pi m\frac{\omega_n}{\omega_i}\right)\right.$$
$$\left.+\frac{1}{\pi m}\left(\frac{\omega_i}{\omega_n}\right)^3\left[\left(\frac{\omega_i}{\omega_n}\right)^4+2\left(\frac{\omega_i}{\omega_n}\right)^2-1\right]\left(1-e^{-\sqrt{2}\pi m\frac{\omega_n}{\omega_i}}\cos\sqrt{2}\pi m\frac{\omega_n}{\omega_i}\right)\right\}^{1/2}$$

$$\lambda_\beta = \frac{A}{\left[\left(\frac{\omega_i}{\omega_n}\right)^4+1\right]}\sqrt{\frac{\overline{n^2(t)}}{2\pi m}\,\frac{\omega_i}{\omega_n}}\left\{\sqrt{2}\left[\left(\frac{\omega_i}{\omega_n}\right)^4+1\right]+\frac{1}{\pi m}\left(\frac{\omega_i}{\omega_n}\right)\left[\left(\frac{\omega_i}{\omega_n}\right)^4+2\left(\frac{\omega_i}{\omega_n}\right)^2-1\right]\left(e^{\sqrt{2}\pi m\frac{\omega_n}{\omega_i}}\sin\sqrt{2}\pi m\frac{\omega_n}{\omega_i}\right)\right.$$
$$\left.+\frac{1}{\pi m}\left(\frac{\omega_i}{\omega_n}\right)\left[\left(\frac{\omega_i}{\omega_n}\right)^4-2\left(\frac{\omega_i}{\omega_n}\right)^2-1\right]\left(1-e^{-\sqrt{2}\pi m\frac{\omega_n}{\omega_i}}\cos\sqrt{2}\pi m\frac{\omega_n}{\omega_i}\right)\right\}^{1/2}$$

In the above γ is a positive real constant, A is the amplitude of the sinusoid used for correlation, ω_i is the circular input hunting frequency, ω_n is a circular characteristic frequency associated with noise, and m is the number of input cycles which the correlation process is carried out.

<u>Evaluation of the parameter</u> $\lambda_\beta = \left\{\frac{A^2}{m^2\pi^2}\int_0^\infty \Phi_{nn}(\omega)\frac{\left(\frac{\omega}{\omega_i}\right)^2\sin^2\pi m\frac{\omega}{\omega_i}}{\left[\left(\frac{\omega}{\omega_i}\right)^2-1\right]^2}\,d\omega\right\}^{1/2}$ <u>for the value of</u> $\Phi_{nn} = \gamma^2/(\omega^2+\omega_n^2)$.

In order to have a measurable statistical property of noise as a parameter in the expression for λ_β, the mean square of the noise function $\overline{n^2(t)}$ is introduced in the following manner:

$$\overline{n^2(t)} = \int_0^\infty \Phi_{nn}(\omega)\,d\omega \qquad \text{(C-1)}$$

Substituting the expression for power spectrum of noise into Eq.

(C-1) and carrying out the integration, $\overline{n^2(t)}$ is found to be

$$\overline{n^2(t)} = \frac{\pi}{2}\frac{\gamma^2}{\omega_n} \tag{C-2}$$

By use of Eq. (C-2) the power spectrum of noise can be written as

$$\Phi_{nn}(\omega) = \frac{2\omega_n}{\pi}\frac{\overline{n^2(t)}}{\omega^2+\omega_n^2} \tag{C-3}$$

and the expression for the parameter λ_β is

$$\lambda_\beta = \left\{\frac{2A^2}{\pi^3 m^2}\, a\, \overline{n^2(t)}\int_0^\infty \frac{x^2 \sin^2\pi m x}{(a^2x^2+1)(x^2-1)^2}\,dx\right\}^{1/2} \tag{C-4}$$

where $a = \omega_i/\omega_n$.

The integral to be evaluated is in the following form:

$$I = \int_0^\infty \frac{x^2 \sin^2\pi m x}{(a^2x^2+1)(x^2-1)^2}\,dx$$

Since the integrand is an even function of x , I can also be written as

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2 \sin^2\pi m x}{(a^2x^2+1)(x^2-1)^2}\,dx = \frac{1}{4}\int_{-\infty}^{\infty} \frac{x^2(1-\cos 2\pi m x)}{(a^2x^2+1)(x^2-1)^2}\,dx \tag{C-5}$$

In the above integral $\cos 2\pi m x$ can be replaced by $e^{i2\pi m x}$, since the odd part $\sin 2\pi m x$ of the exponential will not contribute to this integral. Thus

$$I = \frac{1}{4}\int_{-\infty}^{\infty} \frac{x^2(1-e^{i2\pi m x})}{(a^2x^2+1)(x^2-1)^2}\,dx$$

The integrand of this integral contains four simple poles located at

$x = \mp \frac{i}{a}$ and $x = \mp 1$. It is clear that the poles at $x = \mp \frac{i}{a}$ are simple poles, but it is not obvious that the poles at $x = \mp 1$ are simple poles. It is observed that $x = \mp 1$ are also first order zeros of the numerator of the integrand. Thus the order of the poles at $x = \mp 1$ of the integrand must be reduced by one. The result is then the existence of simple poles at $x = \mp 1$. To show rigorously that the poles at $x = \mp 1$ are simple poles let $F(x)$ denote the integrand. Now if, for example, $x = +1$ is a simple pole, $\lim_{\epsilon \to 0} F(1+\epsilon)$ must approach ∞ as $O(\frac{1}{\epsilon})$.

$$F(1+\epsilon) = \frac{(1+\epsilon)^2 \left[1 - e^{i(2\pi m)(1+\epsilon)}\right]}{\left[a^2(1+\epsilon)^2 + 1\right]\left[(1+\epsilon)^2 - 1\right]^2}$$

$$= \frac{1}{\epsilon}\left[-\frac{i\pi m}{2(1+a^2)}\right]\left[1 + \left(i\pi m + \frac{1-a^2}{1+a^2}\right)\epsilon + O(\epsilon^2)\right] \tag{C-6}$$

It is clear from Eq. (C-6) that as $\epsilon \to 0$, $F(1+\epsilon) \to \infty$ as $O(\frac{1}{\epsilon})$.

The integral I can now be evaluated with the standard contour integration technique. The contour of integration is chosen as shown in Fig. (C-1). Consider the integration of the following inte-

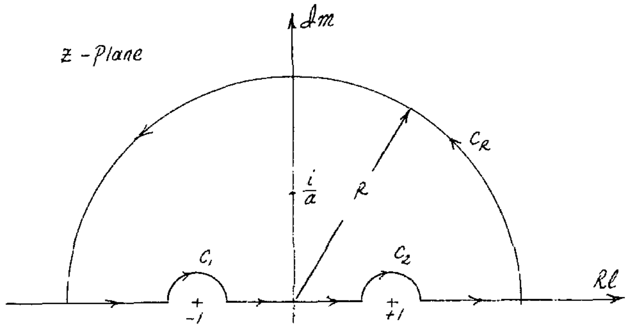

Fig. C-1

gral around that closed contour:

$$I' = \oint \frac{z^2(1-e^{i2\pi m z})}{(a^2z^2+1)(z^2-1)^2}\,dz$$

Because of the particular choice of this contour, it is clear that the contribution to I' from the semi-circle C_R is zero as $R \to \infty$. Since only the singularity at $z = \frac{i}{a}$ is included in this closed contour

$$4I + I_1 + I_2 = 2\pi i \text{ (Residue of } F(z) \text{ at } z = \tfrac{i}{a})$$

where I_1 and I_2 denote the values of the integral I' around the partial contours C_1 and C_2 respectively, and $F(z)$ denotes the integrand of I'. Since the poles at $z = \mp 1$ are simple poles, the integrals I_1 and I_2 are simply

$$-I_1 = \pi i \,[\text{ Residue of } F(z) \text{ at } z = -1]$$

$$-I_2 = \pi i \,[\text{ Residue of } F(z) \text{ at } z = +1]$$

Thus the integral I can now be expressed as

$$I = \frac{1}{4}\left\{2\pi i\,[\text{Res. of } F(z) \text{ at } z = \tfrac{i}{a}] + \pi i\,[\text{Res. of } F(z) \text{ at } z = -1] + \pi i\,[\text{Res. of } F(z) \text{ at } z = 1]\right\} \quad \text{(C-7)}$$

<u>Residue of $F(z)$ at $z = \frac{i}{a}$</u>

$$\left[(z-\tfrac{i}{a})F(z)\right]_{z=\frac{i}{a}} = \left.\frac{z^2(1-e^{i2\pi m z})}{a^2(z+\frac{i}{a})(z^2-1)^2}\right]_{z=\frac{i}{a}} = \frac{ia}{2}\,\frac{(1-e^{-\frac{2\pi m}{a}})}{(1+a^2)^2}$$

<u>Residue of $F(z)$ at $z = \mp 1$</u>

With the result given in Eq. (C-6), the residue at $z = \mp 1$ is simply

$$\lim_{\epsilon \to 0} \epsilon F(1+\epsilon) = -\frac{i\pi m}{2(1+a^2)}$$

Substituting the values of the residues into Eq. (C-7) and simplifying

$$I = \frac{\pi^2 m}{4(1+a^2)^2}\left\{(1+a^2) - \frac{a}{\pi m}\left(1 - e^{-\frac{2\pi m}{a}}\right)\right\} \tag{C-8}$$

With this value of I , the expression for λ_β is

$$\lambda_\beta = \frac{A}{[a^2+1]}\sqrt{\frac{\overline{n^2(t)}}{2\pi m}\,a}\left\{(a^2+1) - \frac{1}{\pi m}\,a\left(1 - e^{-\frac{2\pi m}{a}}\right)\right\}^{1/2}$$

or

$$\lambda_\beta = \frac{A}{[(\frac{\omega_i}{\omega_n})^2+1]}\sqrt{\frac{\overline{n^2(t)}}{2\pi m}\,\frac{\omega_i}{\omega_n}}\left\{[(\frac{\omega_i}{\omega_n})^2+1] - \frac{1}{\pi m}\left(\frac{\omega_i}{\omega_n}\right)\left(1 - e^{-2\pi m\frac{\omega_n}{\omega_i}}\right)\right\}^{1/2} \tag{C-9}$$

<u>Evaluation of the Parameter</u> $\lambda_\alpha = \left\{\frac{A^2}{m^2\pi^2}\int_0^\infty \Phi_{nn}(\omega)\,\frac{\sin^2 \pi m\frac{\omega}{\omega_i}}{[(\frac{\omega}{\omega_i})^2-1]^2}\,d\omega\right\}^{1/2}$ <u>for the value</u> of $\Phi_{nn}(\omega) = \nu^4/(\omega^4+\omega_n^4)$.

The evaluation of the parameter λ_α is carried out in exactly the same manner as the parameter λ_β . In this case the noise power spectrum is given as

$$\Phi_{nn}(\omega) = \frac{2\sqrt{2}}{\pi}\,\omega_n^3\,\frac{\overline{n^2(t)}}{\omega^4+\omega_n^4} \tag{C-10}$$

Substitution of $\Phi_{nn}(\omega)$ into the expression for λ_α , gives

$$\lambda_\alpha = \left\{\frac{2\sqrt{2}\,A^2}{m^2\pi^3}\,a\,\overline{n^2(t)}\int_0^\infty \frac{\sin^2 \pi m x}{(a^4x^4+1)(x^2-1)^2}\,dx\right\}^{1/2} \tag{C-11}$$

The integral that has to be evaluated is

$$I = \frac{1}{4}\int_{-\infty}^{\infty} \frac{(1-e^{i2\pi m x})}{(a^4x^4+1)(x^2-1)^2}\,dx \tag{C-12}$$

The integrand of I has six simple poles located at $x = \mp 1$, $x = \frac{e^{\mp i\pi/4}}{a}$, and $x = \frac{e^{\mp i\frac{3\pi}{4}}}{a}$. The contour of integration is chosen as shown in Fig. (C-2). Consider the following integral

$$I' = \oint \frac{(1 - e^{i2\pi m z})}{(a^4 z^4 + 1)(z^2 - 1)^2}\, dz$$

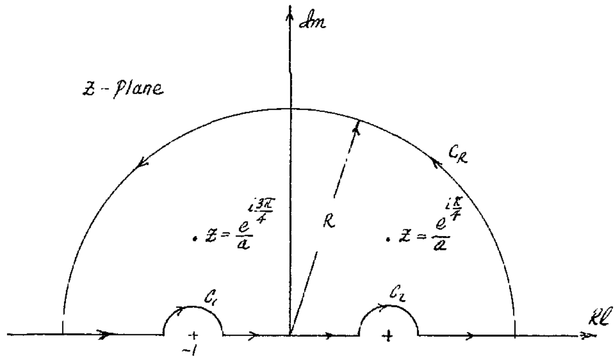

Fig. C-2

Thus

$$4I + I_1 + I_2 = 2\pi i\left[\text{Res. of } F(z) \text{ at } z = \frac{e^{i\frac{\pi}{4}}}{a}\right] + 2\pi i\left[\text{Res. of } F(z) \text{ at } z = \frac{e^{i\frac{3\pi}{4}}}{a}\right]$$

where I_1 and I_2 again denote the values of the integral I' around the partial contours C_1 and C_2 respectively, and $F(z)$ denotes the integrand of I'. I_1 and I_2 can also be written as

$$-I_1 = \pi i\left[\text{Res. of } F(z) \text{ at } z = -1\right]$$

$$-I_2 = \pi i\left[\text{Res. of } F(z) \text{ at } z = +1\right]$$

Thus,

$$I = \frac{1}{4}\left\{2\pi i\left[\text{Res. of } F(z) \text{ at } z = \frac{e^{i\frac{\pi}{4}}}{a}\right] + 2\pi i\left[\text{Res. of } F(z) \text{ at } z = \frac{e^{i\frac{3\pi}{4}}}{a}\right]\right.$$

$$\left. + \pi i\left[\text{Res. of } F(z) \text{ at } z = -1\right] + \pi i\left[\text{Res. of } F(z) \text{ at } z = +1\right]\right\} \qquad \text{(C-13)}$$

$$\left[\text{Res. of } F(z) \text{ at } z = \mp 1\right] = -\frac{i\pi m}{2(1 + a^4)}$$

$$\left[\text{Res. of } F(z) \text{ at } z = \frac{e^{i\frac{\pi}{4}}}{a}\right] = \frac{a^3}{i}\,\frac{\left[1 - e^{\frac{\sqrt{2}\pi m}{a}(-1+i)}\right]}{2\sqrt{2}\,(1+i)(a^2 - i)^2}$$

$$\left[\text{Res. of } F(z) \text{ at } z = \frac{e^{i\frac{3\pi}{4}}}{a}\right] = \frac{a^3}{i}\,\frac{\left[1 - e^{\frac{\sqrt{2}\pi m}{a}(-1-i)}\right]}{2\sqrt{2}\,(1-i)(a^2 + i)^2}$$

Substitution of the residues into Eq. (C-13), followed by the method outlined previously, results in the following equation for λ_α :

$$\lambda_\alpha = \frac{A}{\left[\left(\frac{\omega_i}{\omega_n}\right)^4+1\right]}\sqrt{\frac{\overline{n^2(t)}}{2\pi m}\,\frac{\omega_i}{\omega_n}}\left\{\sqrt{2}\left[\left(\frac{\omega_i}{\omega_n}\right)^4+1\right]-\frac{1}{\pi m}\left(\frac{\omega_i}{\omega_n}\right)^3\left[\left(\frac{\omega_i}{\omega_n}\right)^4-2\left(\frac{\omega_i}{\omega_n}\right)^2-1\right]\left(e^{-\sqrt{2}\pi m\frac{\omega_n}{\omega_i}}\sin\sqrt{2}\,\pi m\frac{\omega_n}{\omega_i}\right)\right.$$
$$\left.+\frac{1}{\pi m}\left(\frac{\omega_i}{\omega_n}\right)^3\left[\left(\frac{\omega_i}{\omega_n}\right)^4+2\left(\frac{\omega_i}{\omega_n}\right)^2-1\right]\left(1-e^{-\sqrt{2}\pi m\frac{\omega_n}{\omega_i}}\cos\sqrt{2}\,\pi m\frac{\omega_n}{\omega_i}\right)\right\}^{1/2} \qquad \text{(C-14)}$$

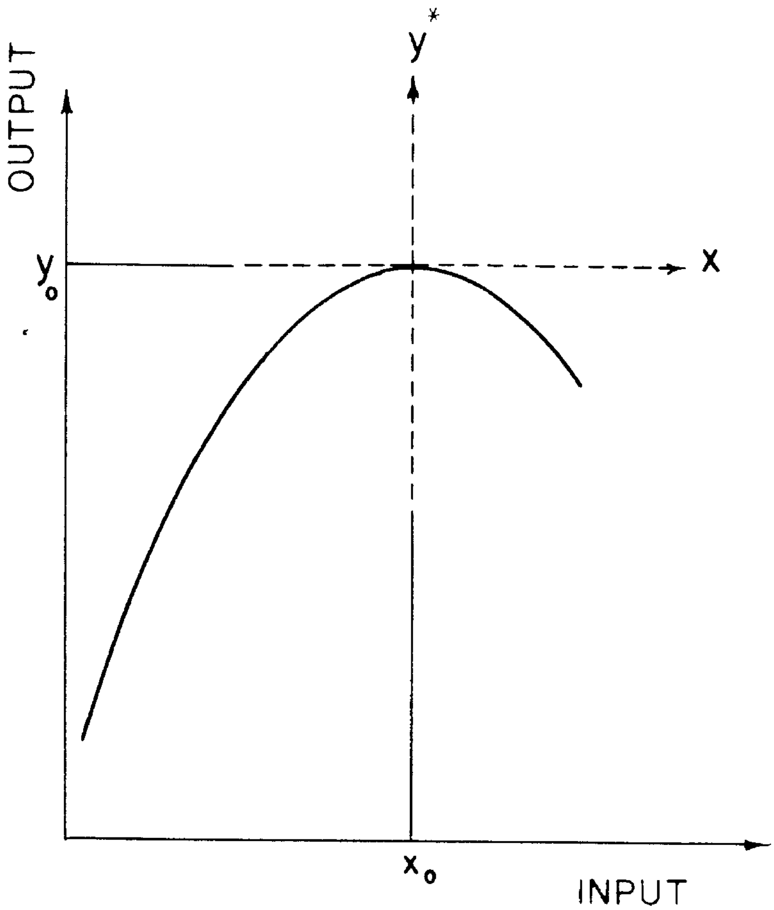

Fig. 2·1 Input-Output Characteristics of Controlled System.

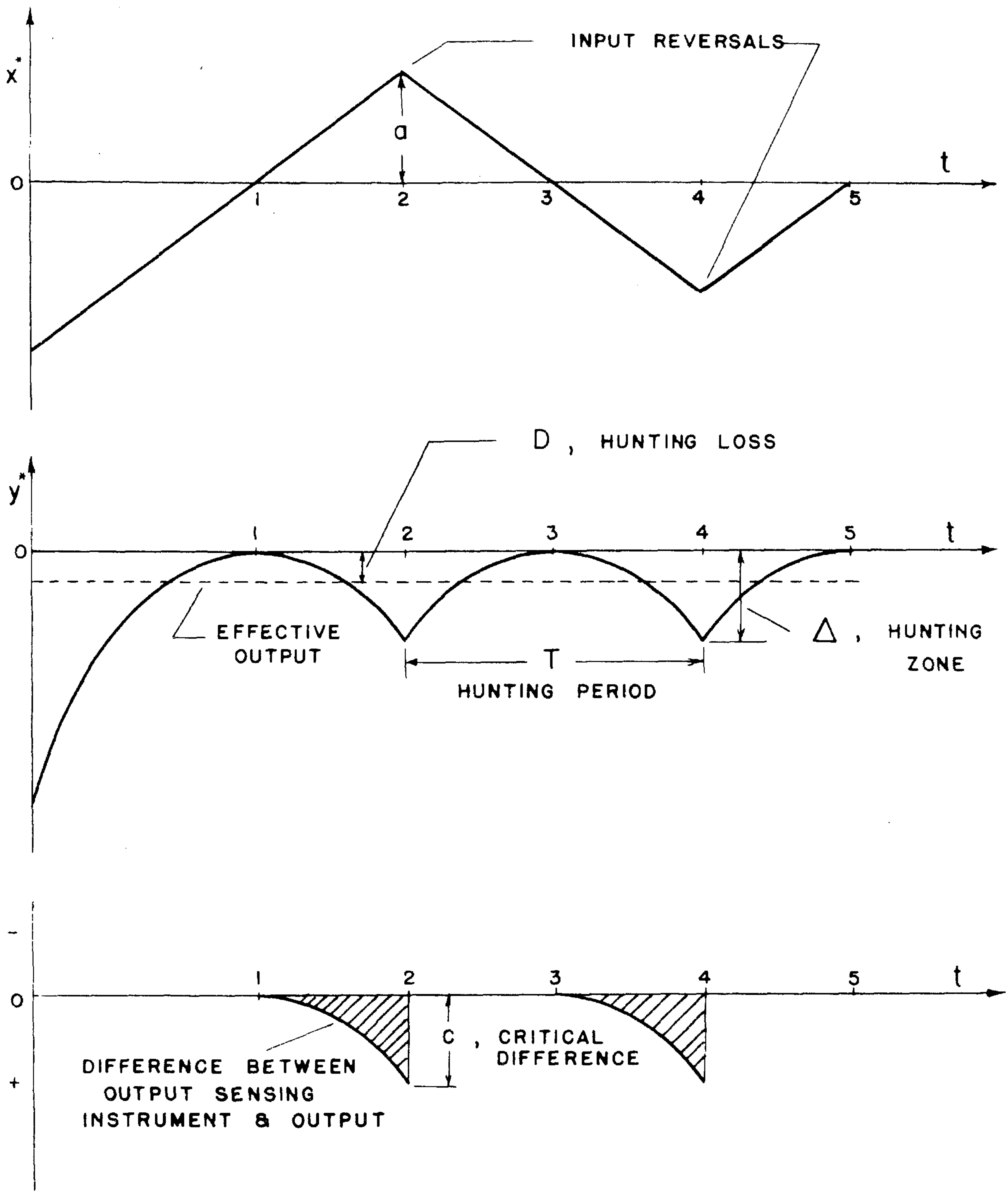

Fig. 2 2 Typical Performance Diagram for an Ideal Peak-holding Optimalizing Control System.

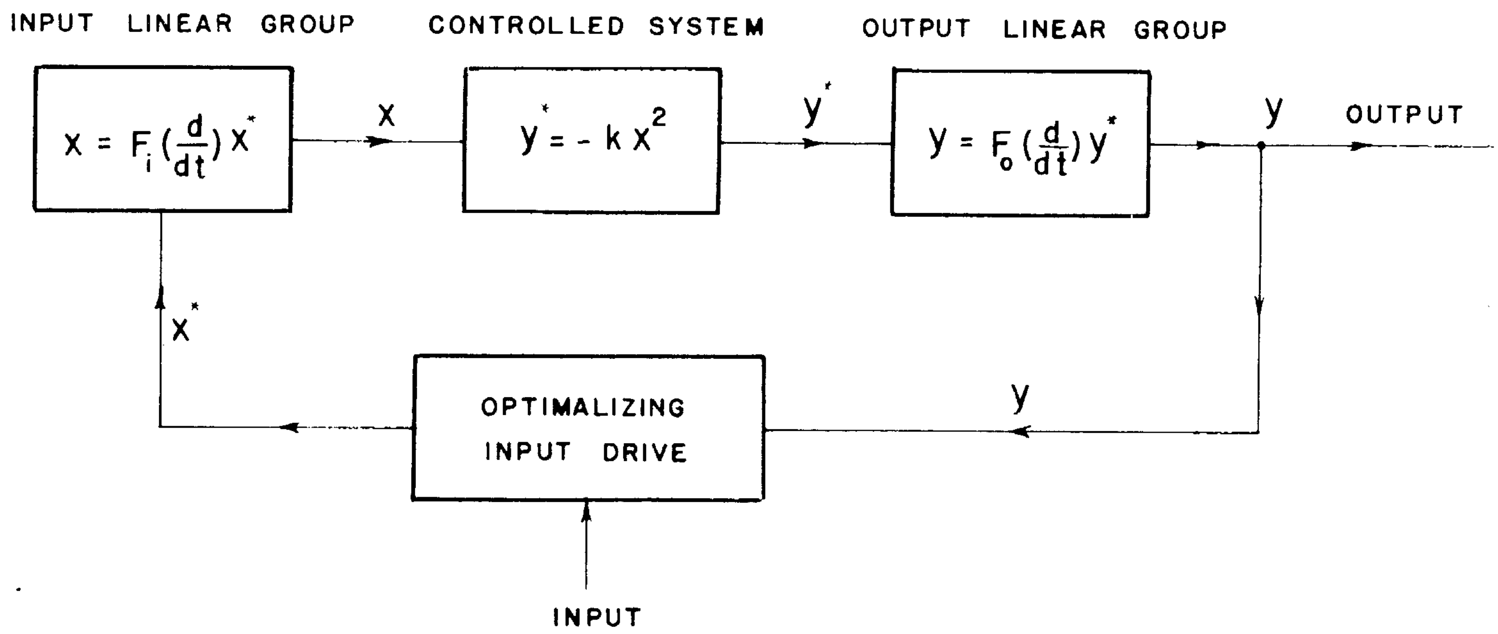

Fig. 2·3 Block Diagram of a Complete Peak-holding Optimalizing Control System.

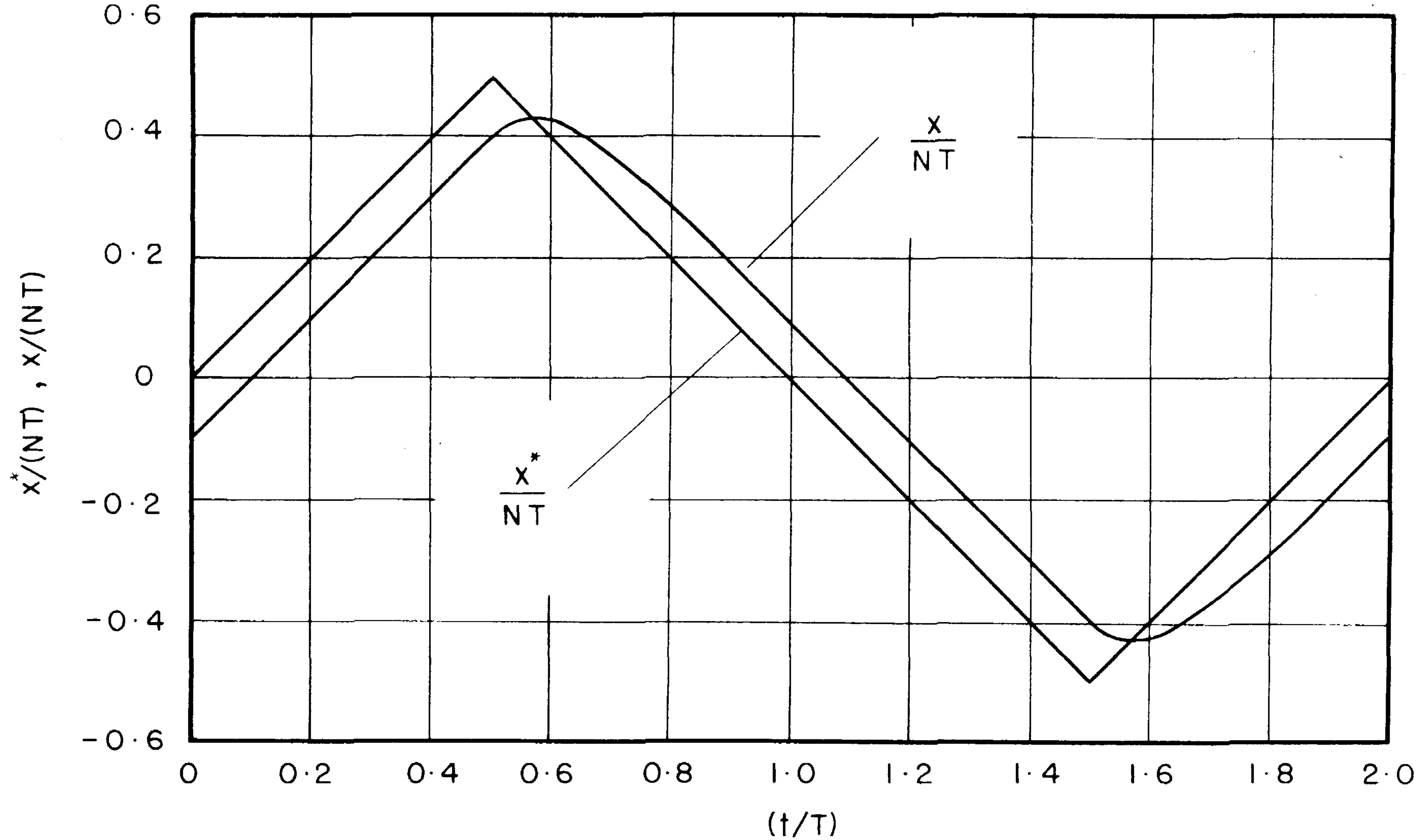

Fig. 2·4 "Potential" Input and Actual Input for $(\tau_i/T) = 0{\cdot}10$.

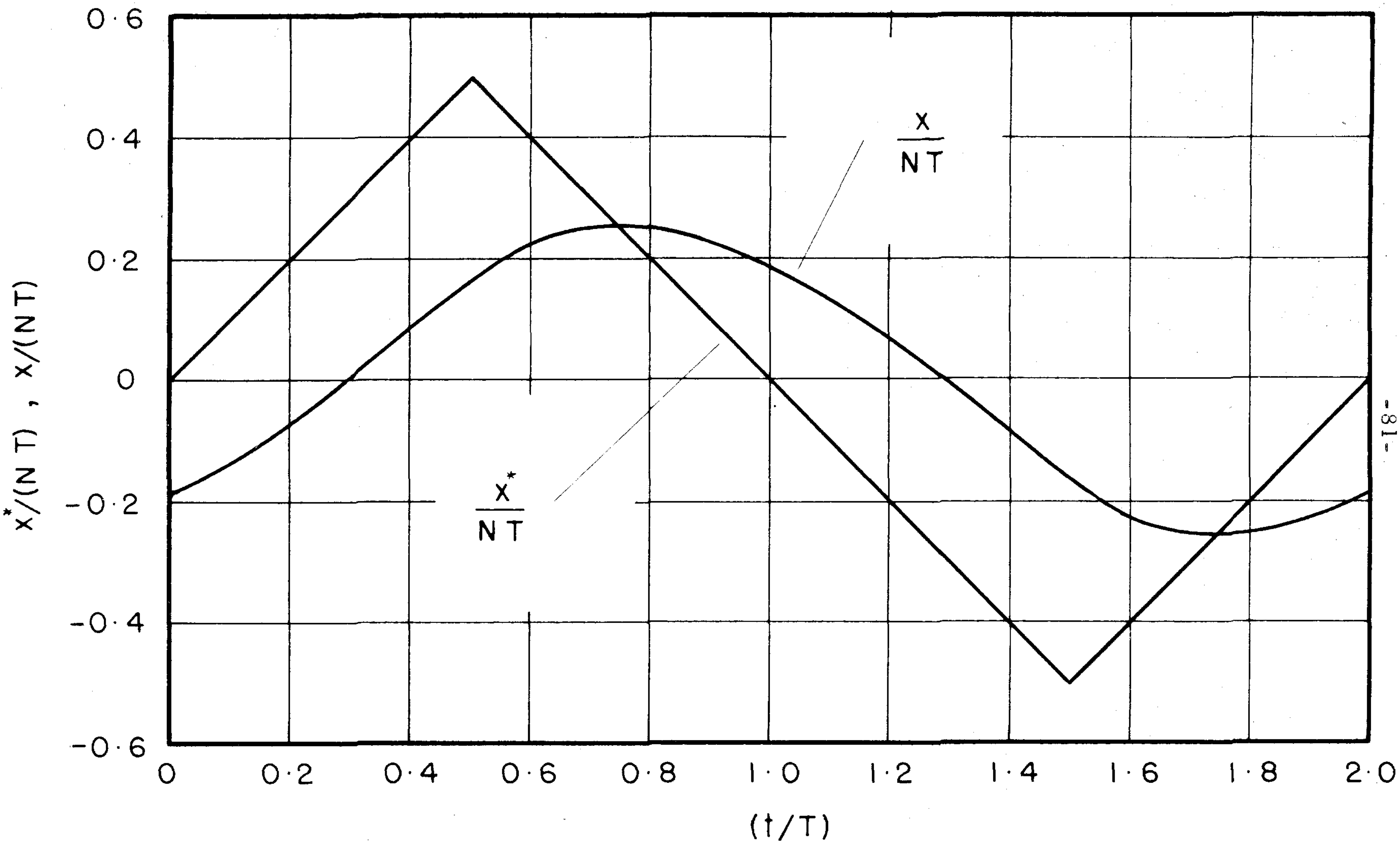

Fig. 2·5 "Potential" Input and Actual Input for $(\tau_i/T) = 0{\cdot}40$.

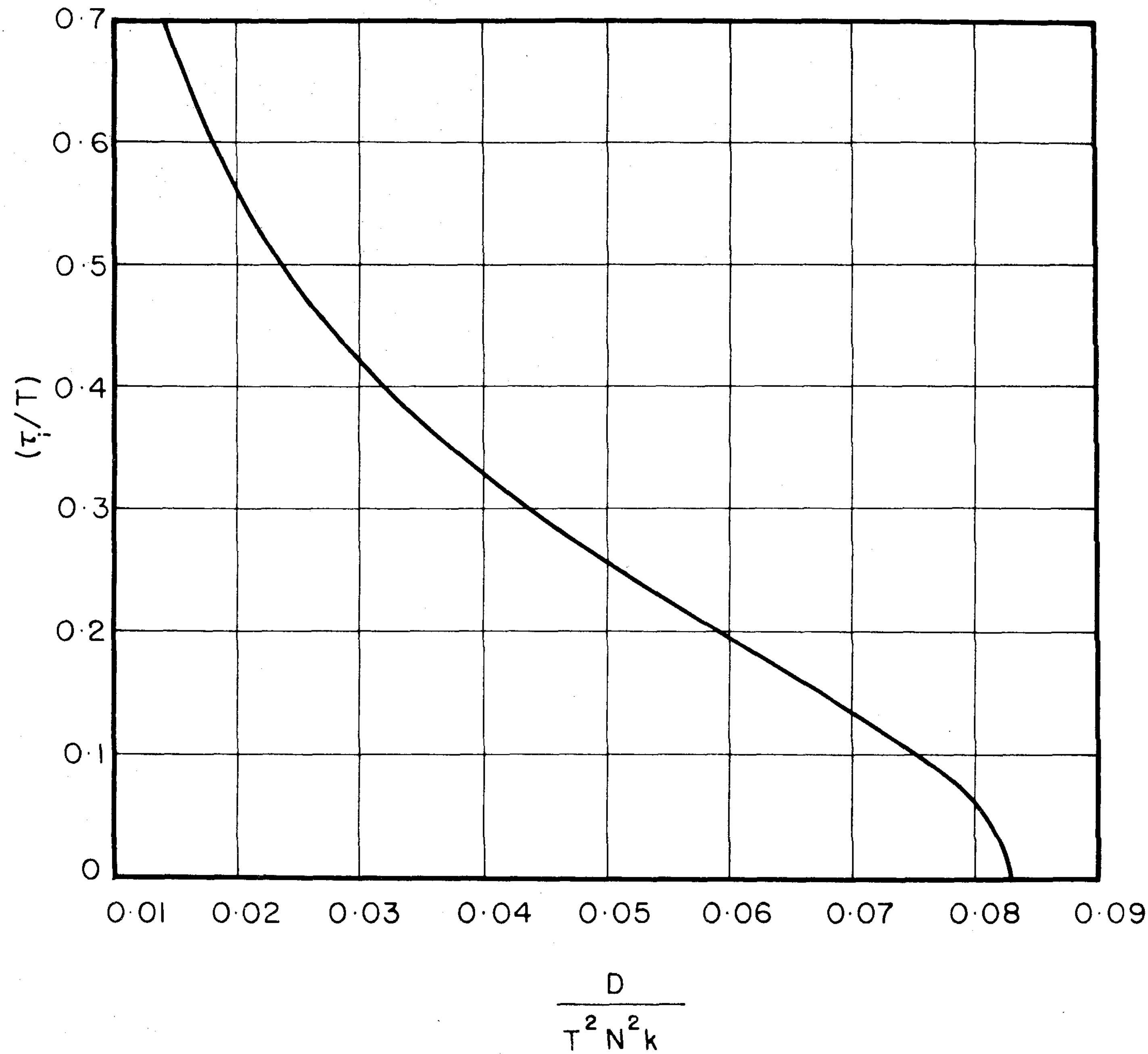

Fig. 2·6 Variation of Dimensionless Hunting Loss with τ_i/T .

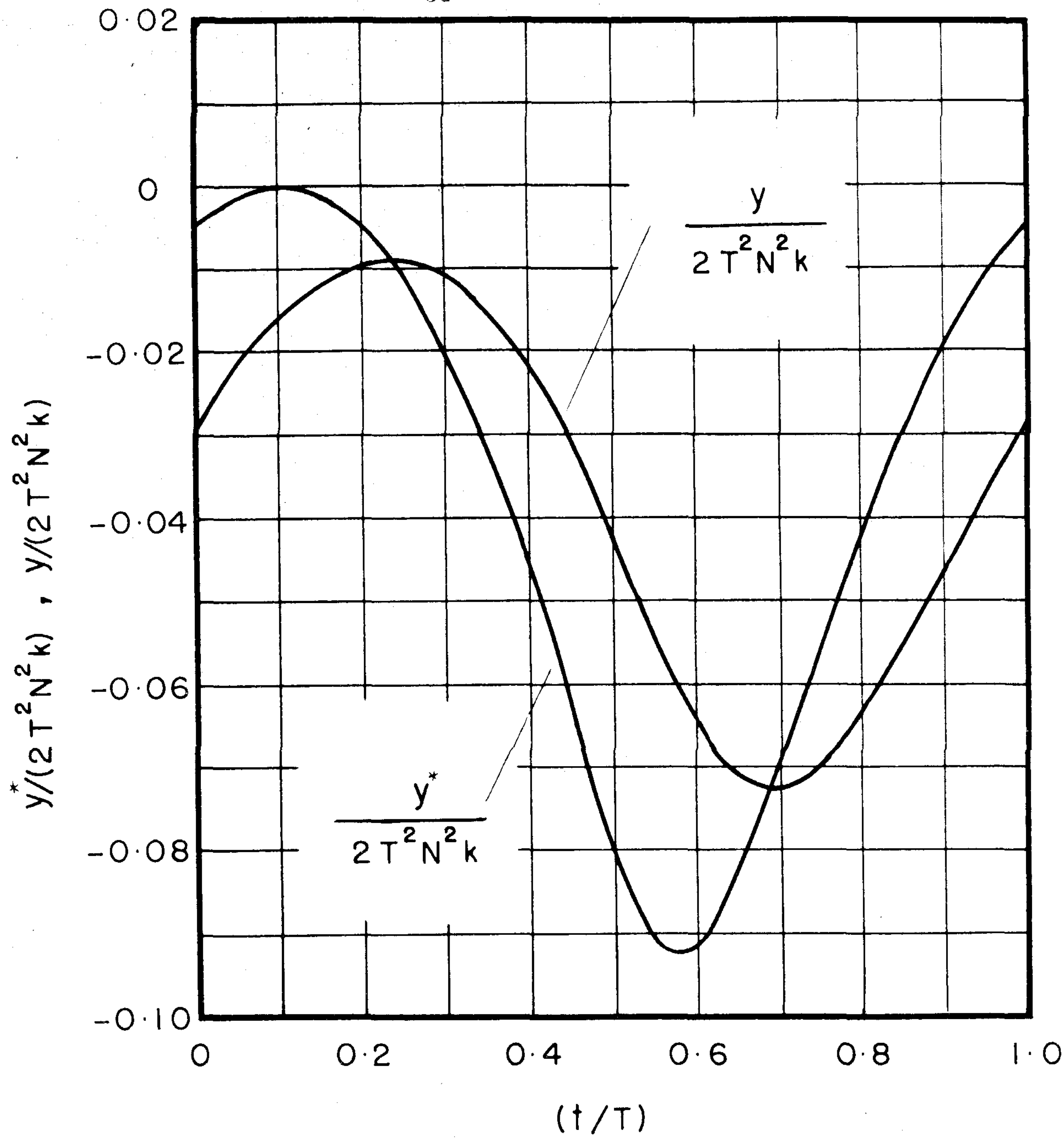

Fig. 2·7 "Potential" Output and Actual Output for $(\tau_i/T)=0{\cdot}10$ and $(\tau_o/T)=0{\cdot}15$.

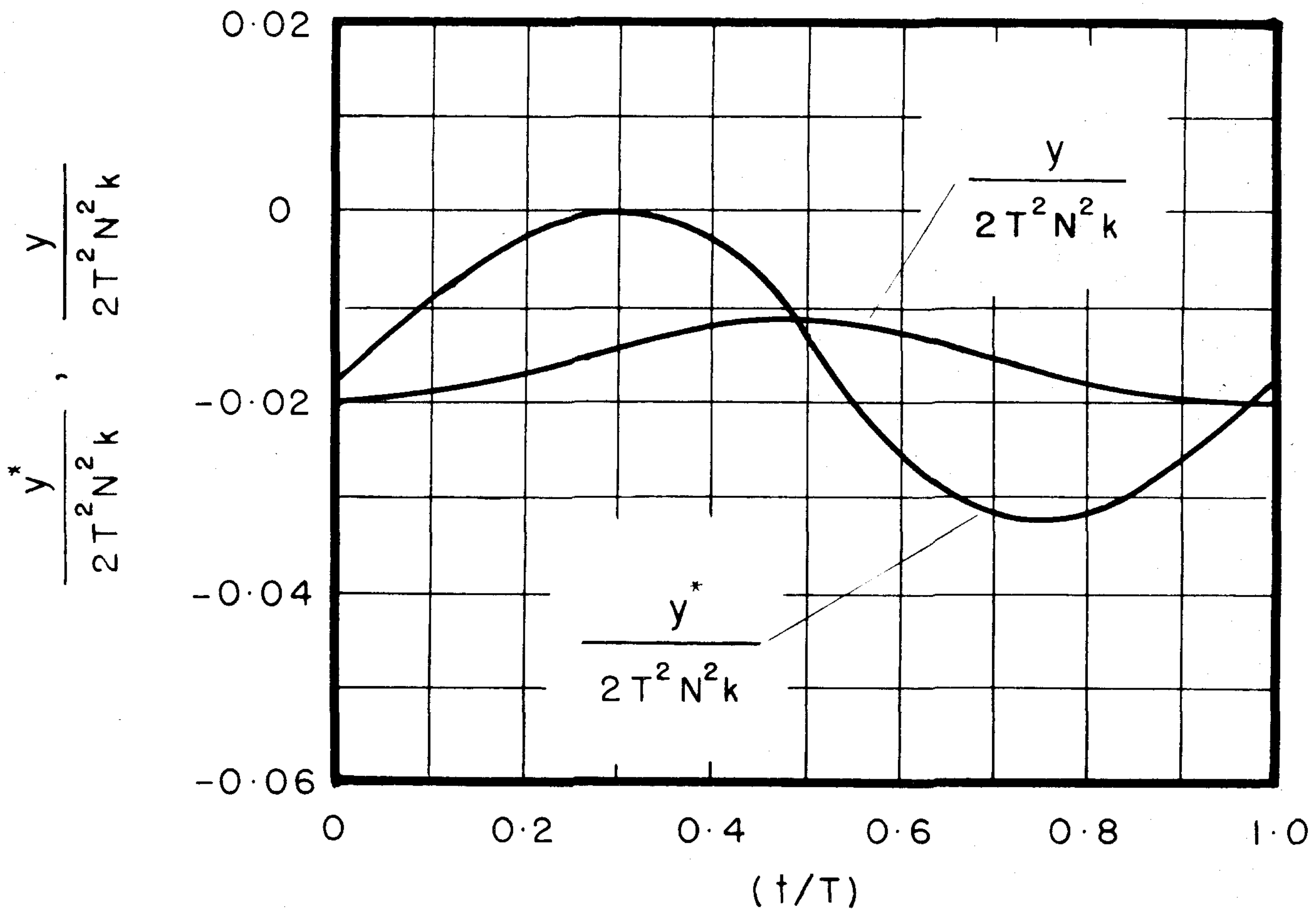

Fig. 2·8 "Potential" Output and Actual Output for $(\tau_i/T)=0{\cdot}40$ and $(\tau_o/T)=0{\cdot}60$.

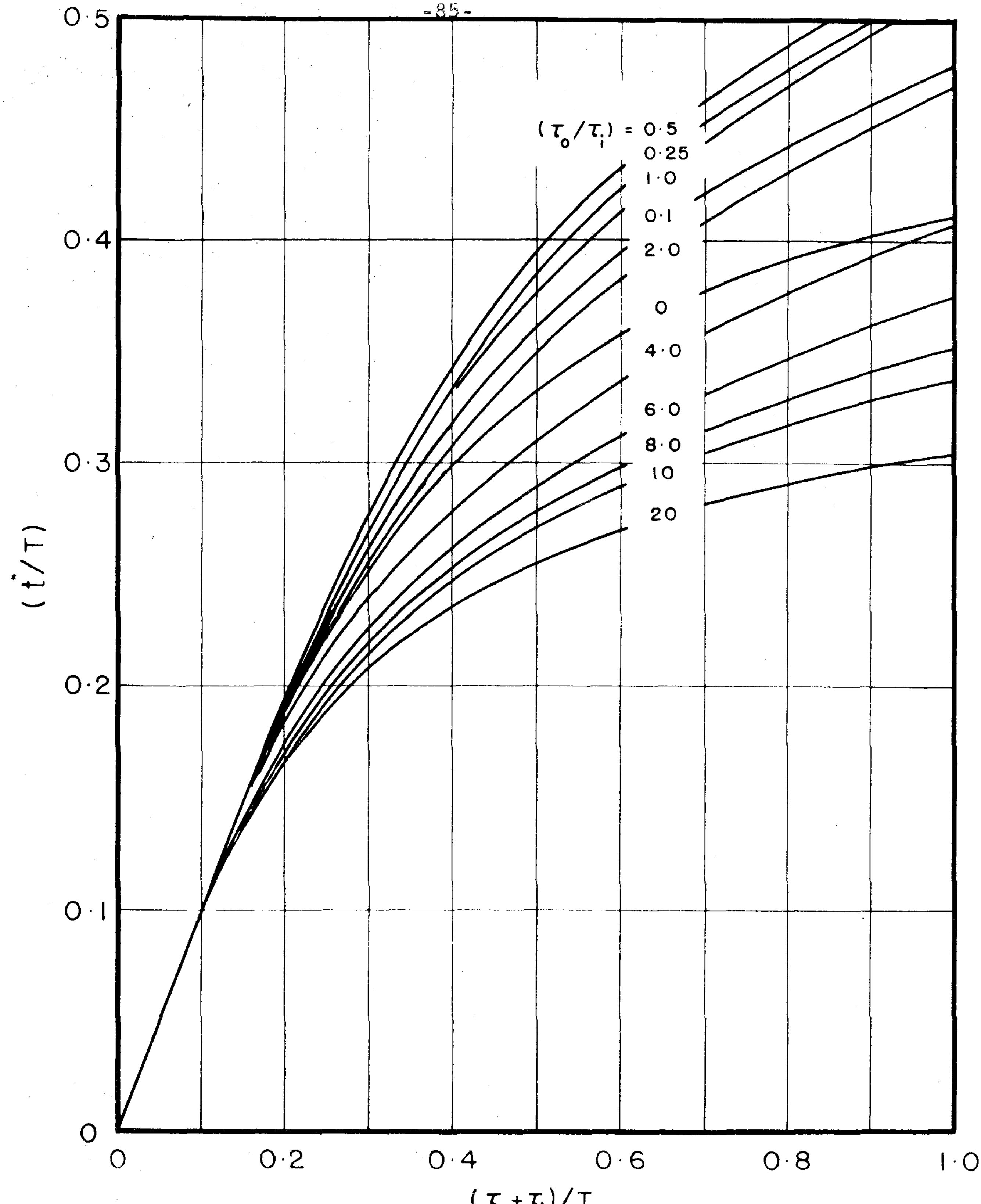

Fig. 2.9 Maximum Output Occurrence Instant, t^*/T versus $(\tau_o + \tau_i)/T$ with τ_o/τ_i as Parameter, in Interval $(0 \leq \frac{t}{T} \leq \frac{1}{2})$.

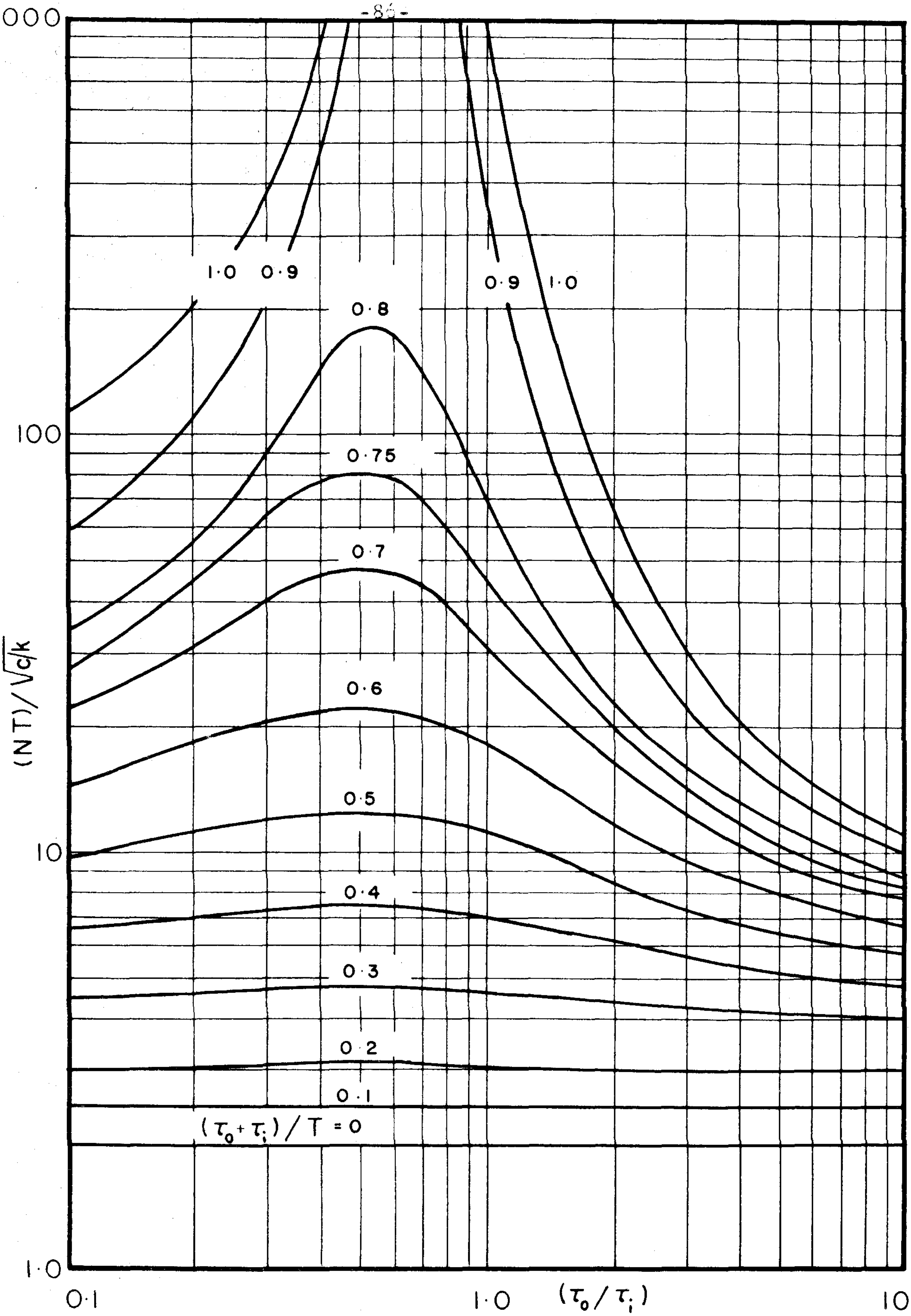

Fig. 2·10 Critical Indicated Difference Parameter $\frac{TN}{\sqrt{c/k}}$ versus $\frac{\tau_o}{\tau_i}$ with $\frac{(\tau_o + \tau_i)}{T}$ as Parameter.

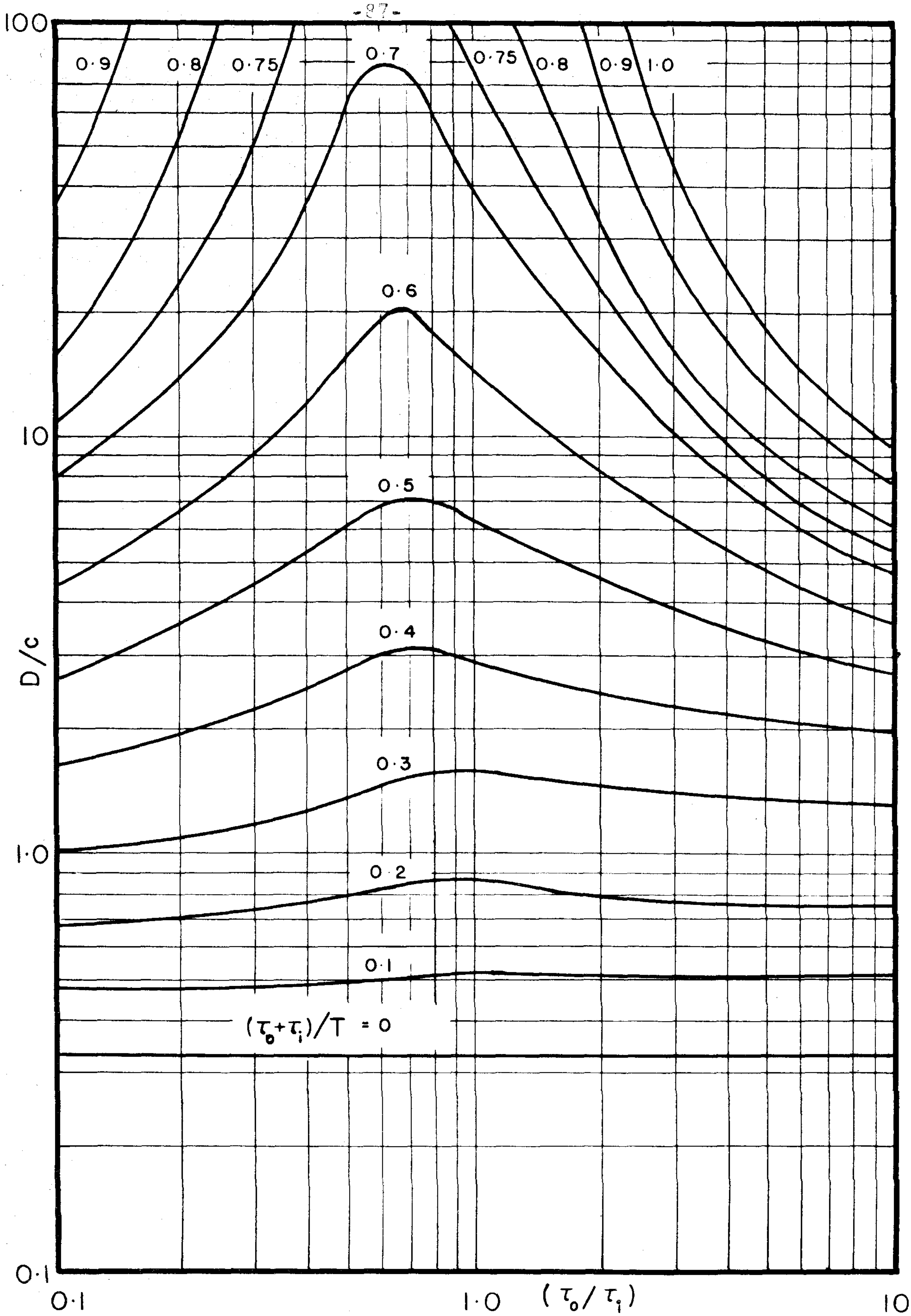

Fig. 2·11 Relative Hunting Loss, $\frac{D}{c}$ versus $\frac{\tau_o}{\tau_i}$ with $(\tau_o+\tau_i)/T$ as Parameter.

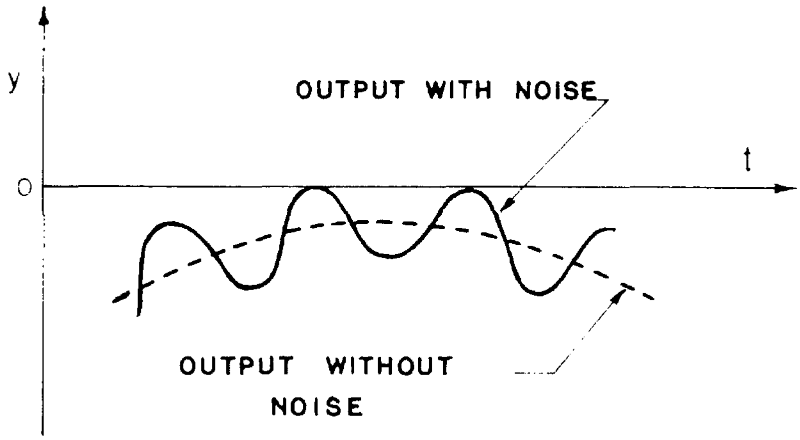

Fig. 3.1 Output Signal with Superimposed High Frequency Sinusoidal Noise.

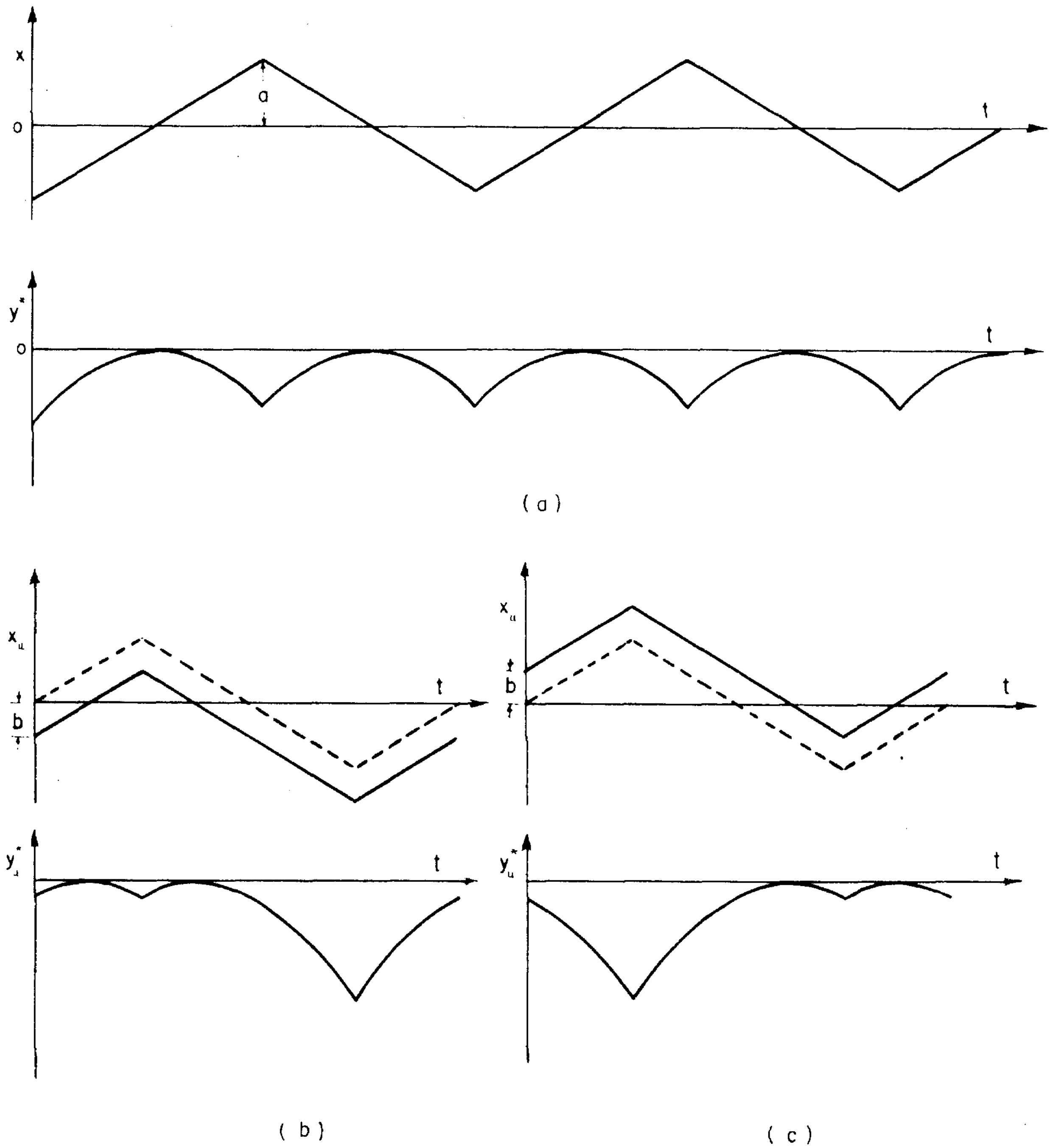

Fig. 3·2 Performance of an Ideal Peak-holding Optimalizing Control System for Various Amounts of Incorrect Input.

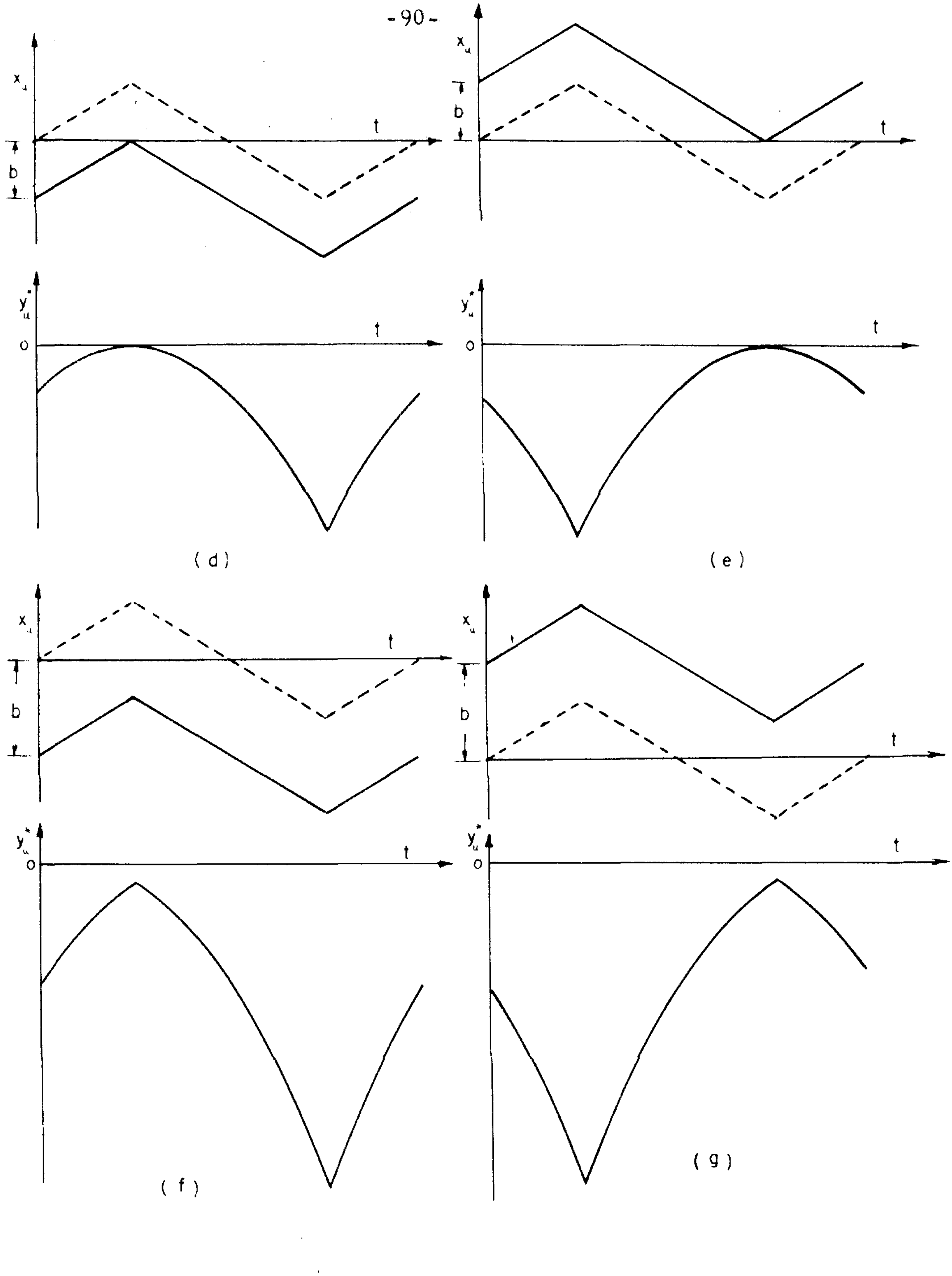

Fig. 3·2 (Continued)

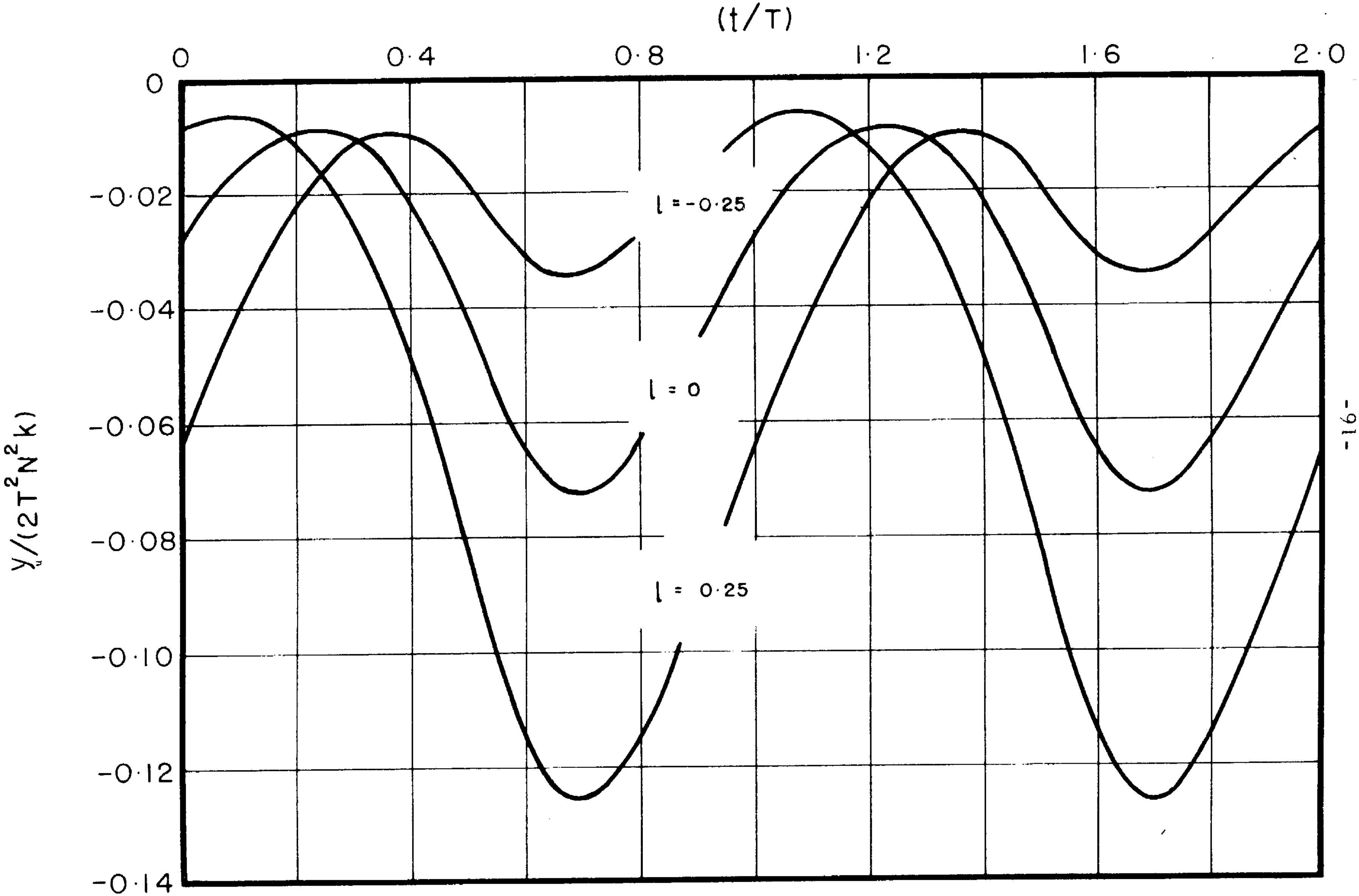

Fig. 3·3 Actual Output for $\frac{\tau_i}{T} = 0{\cdot}10$ and $\frac{\tau_o}{T} = 0{\cdot}15$ with l Parameter.

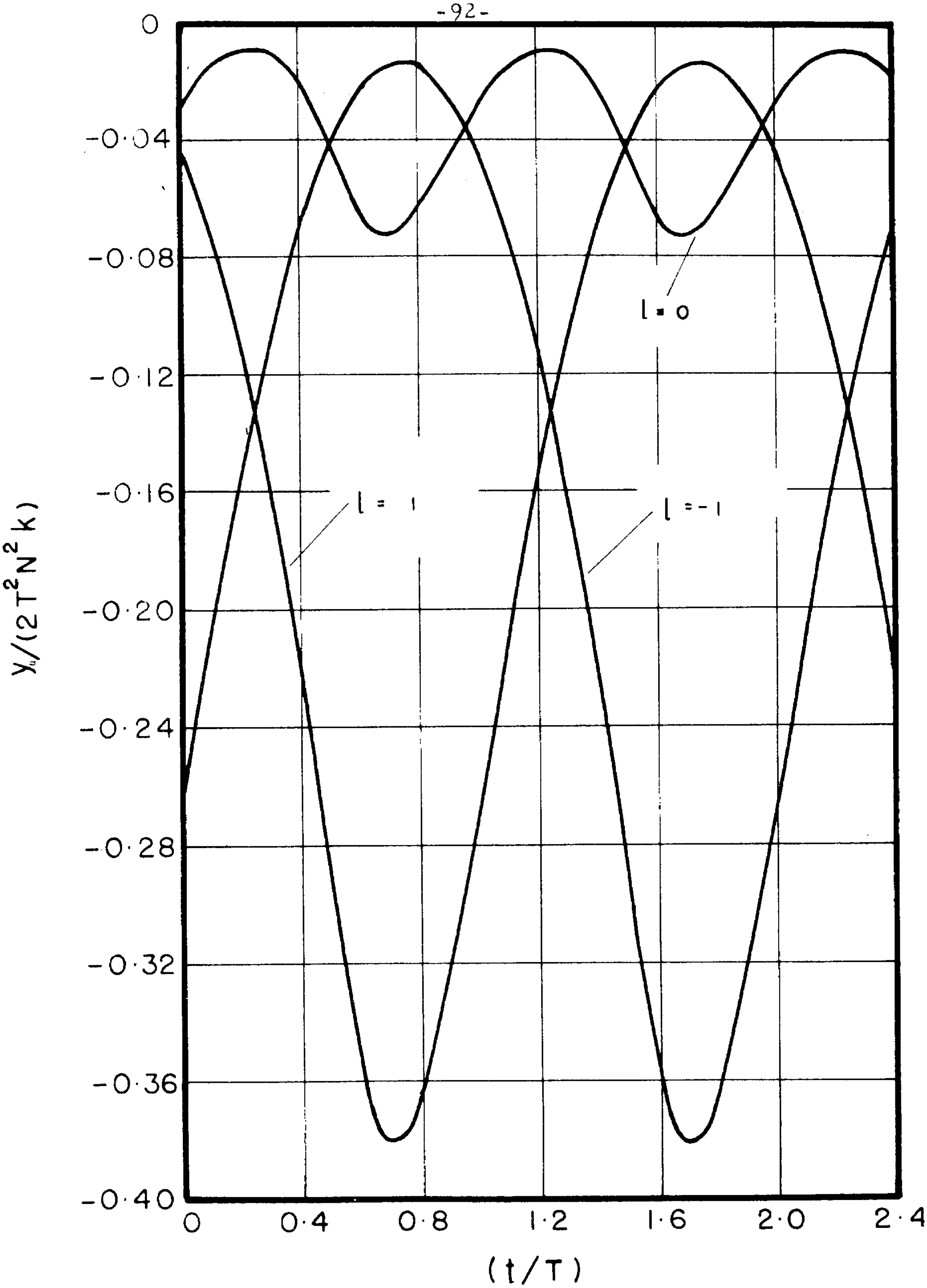

Fig. 3 4 Actual Output for $\frac{\tau_1}{T} = 0{\cdot}10$ and $\frac{\tau_2}{T} = 0{\cdot}15$ with l as Parameter.

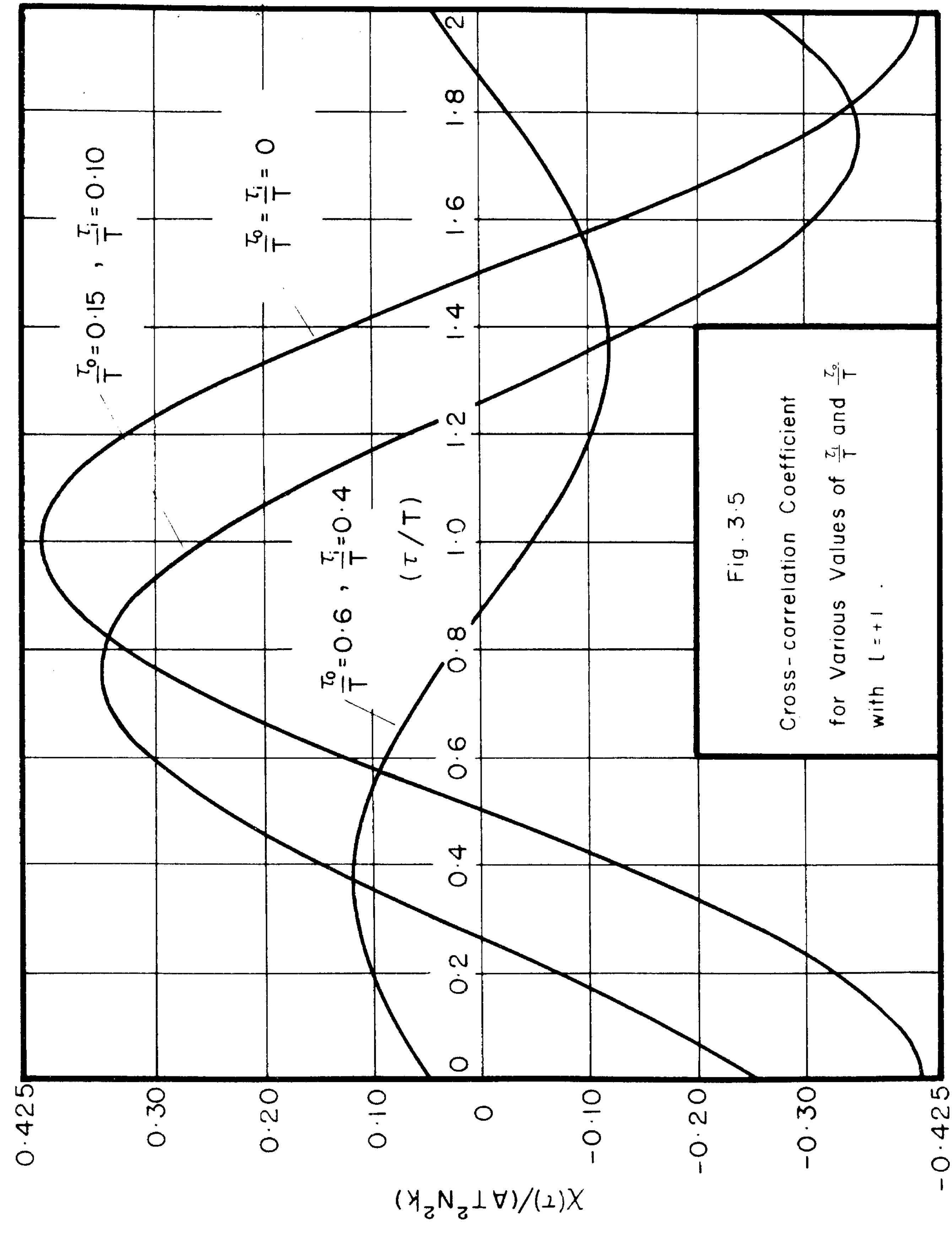

Fig. 3.5

Cross-correlation Coefficient for Various Values of $\frac{\tau_i}{T}$ and $\frac{\tau_o}{T}$ with $l = +1$.

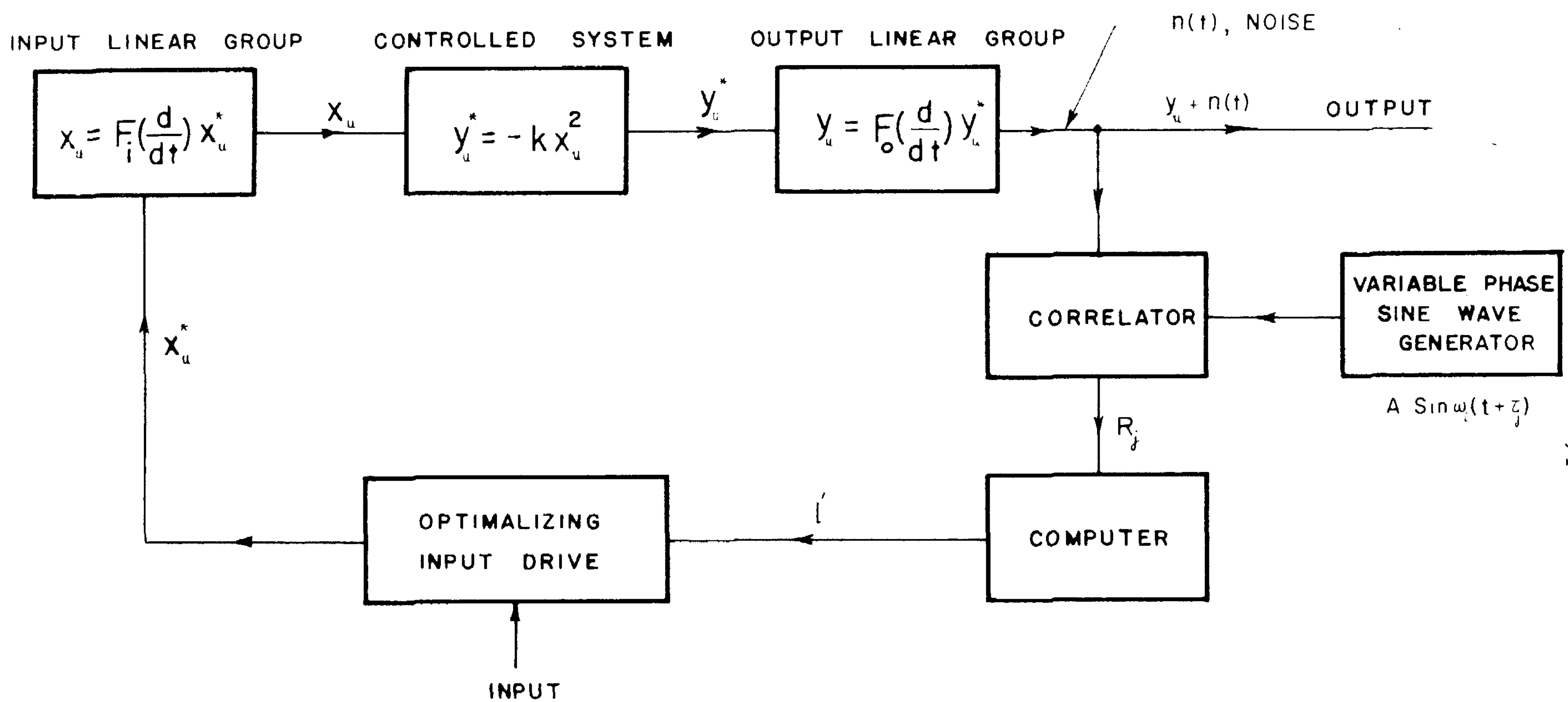

Fig. 3·6 Block Diagram of a Modified Optimalizing Control System with Filtering by means of Cross - correlation .

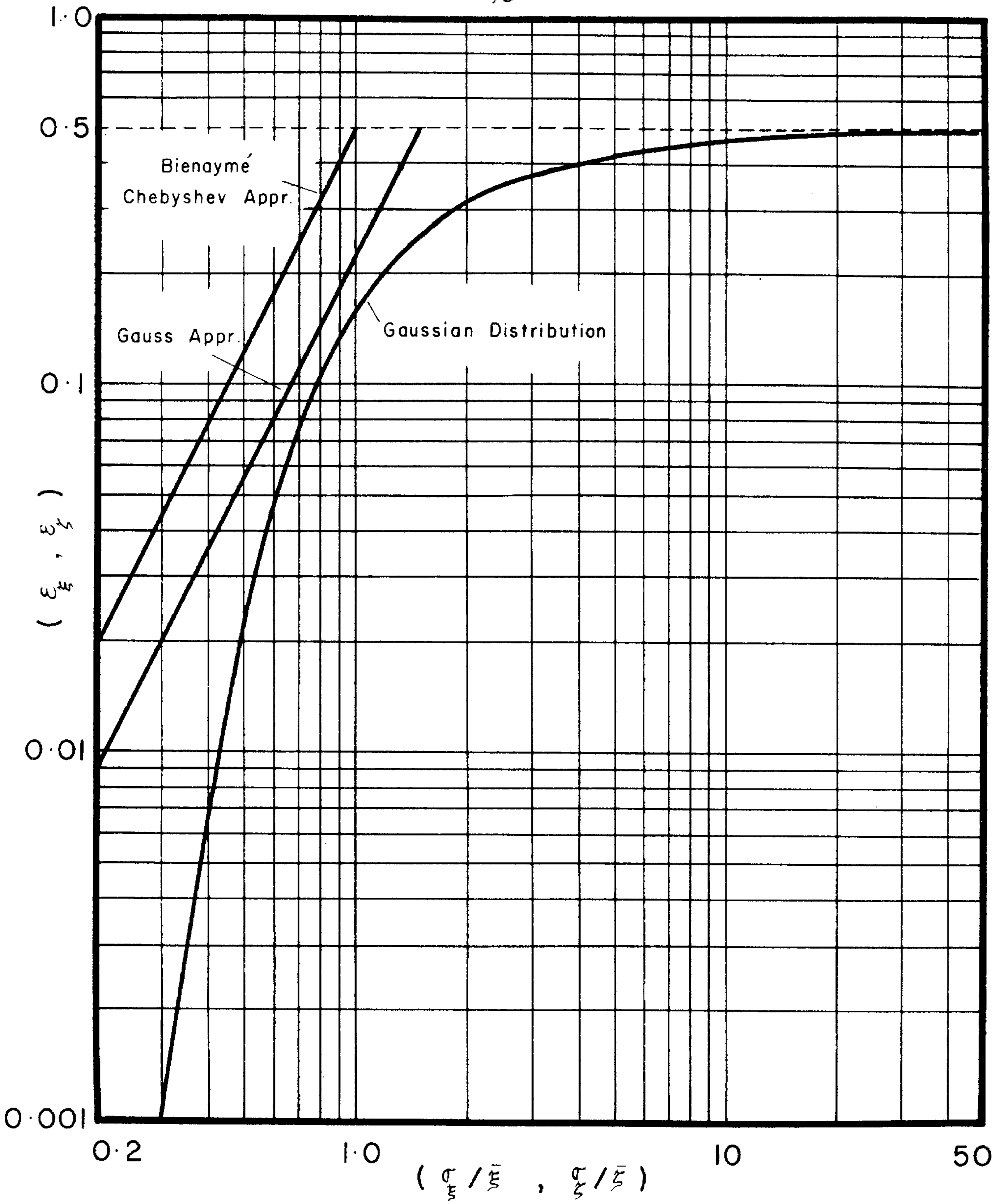

Fig. 3·7 Probable Error for Decision on Incorrect Input Level.

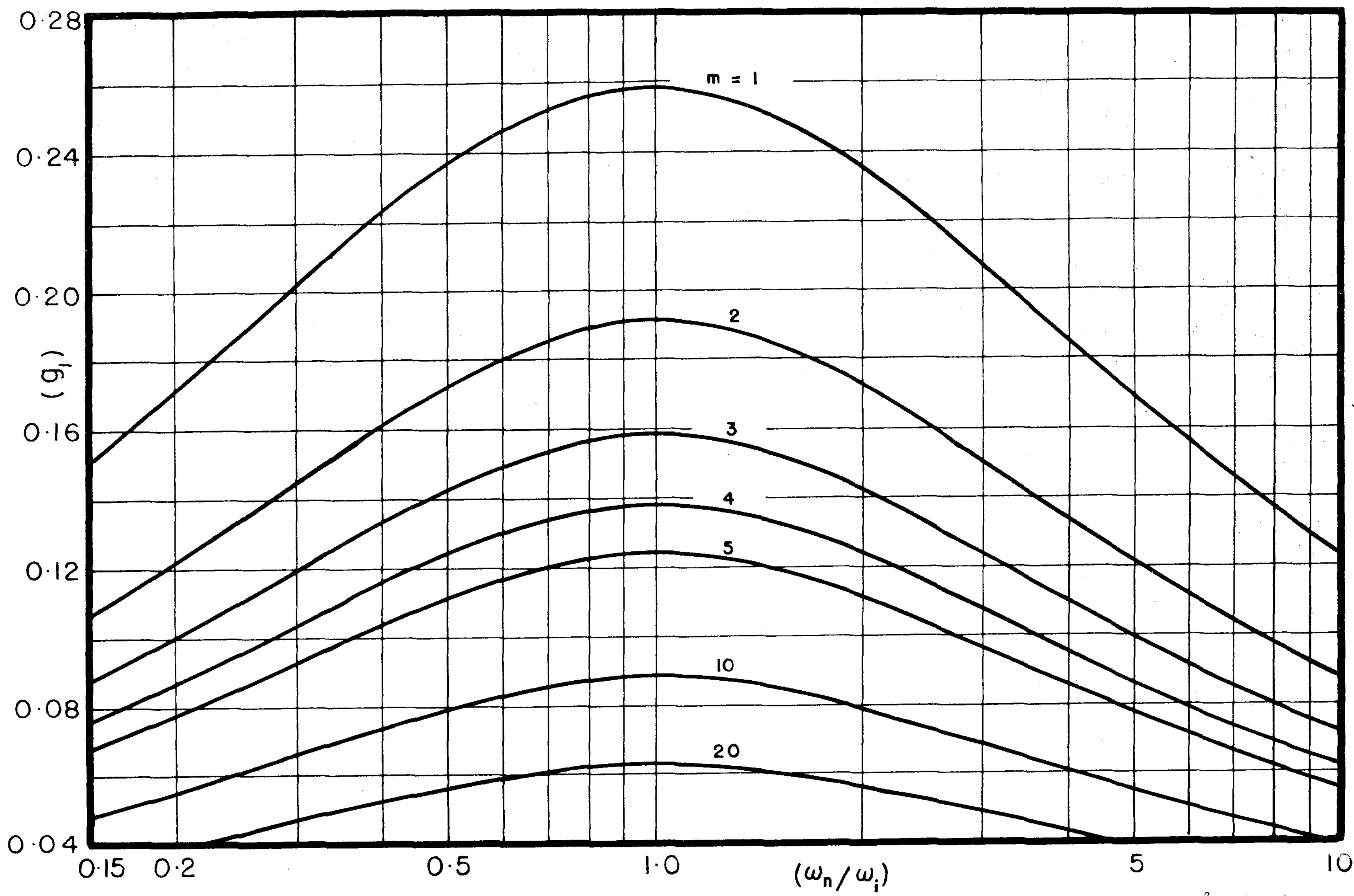

Fig. 3·8 Characteristic Noise Function g_1 versus ω_n/ω_i with m as Parameter for $\Phi_{nn}(\omega) = \nu^2/(\omega^2+\omega_n^2)$.

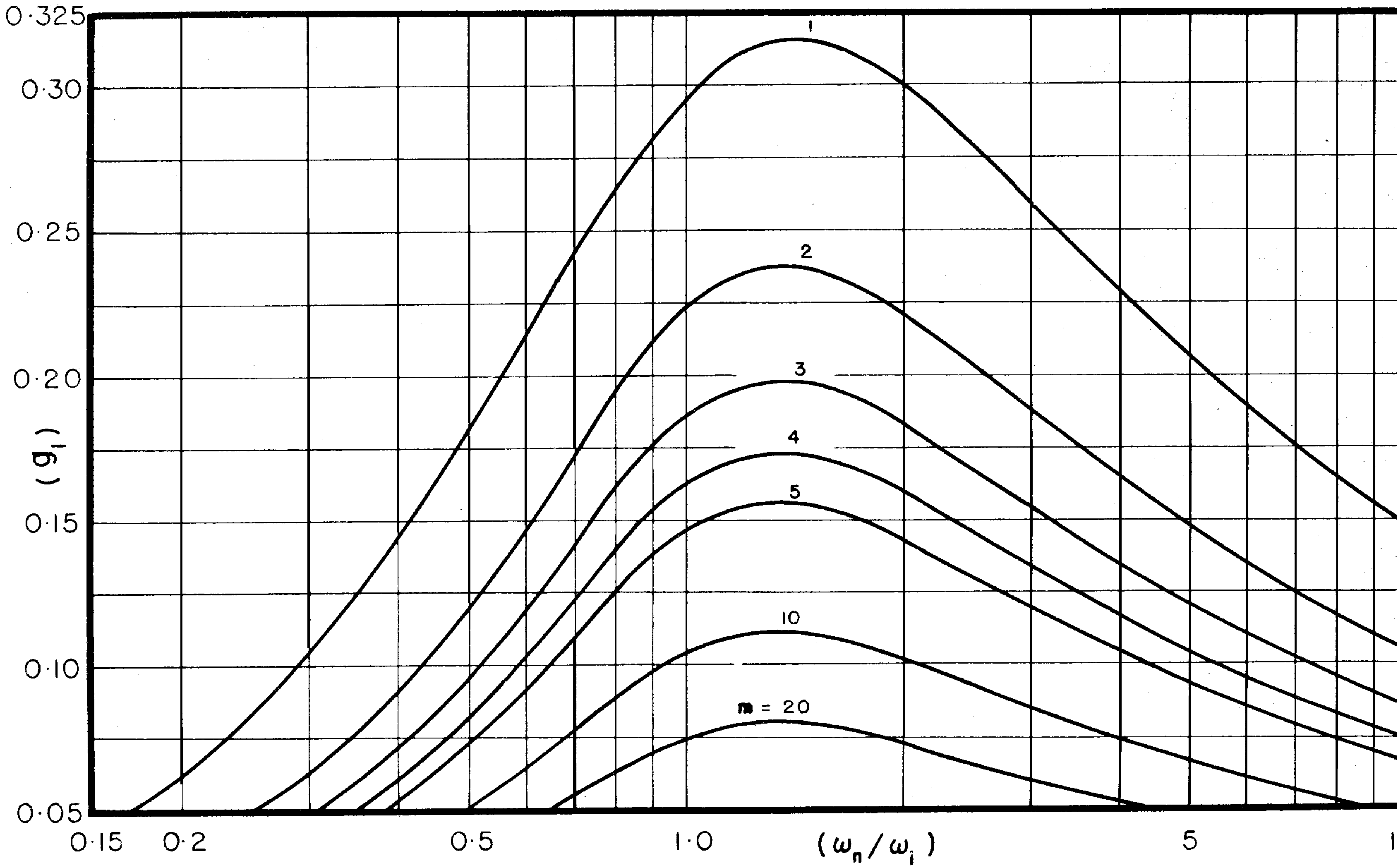

Fig. 3·9 Characteristic Noise Function g_1 versus ω_n/ω_i with m as Parameter for $\Phi_{nn}(\omega) = \gamma^4/(\omega^4+\omega_n^4)$.

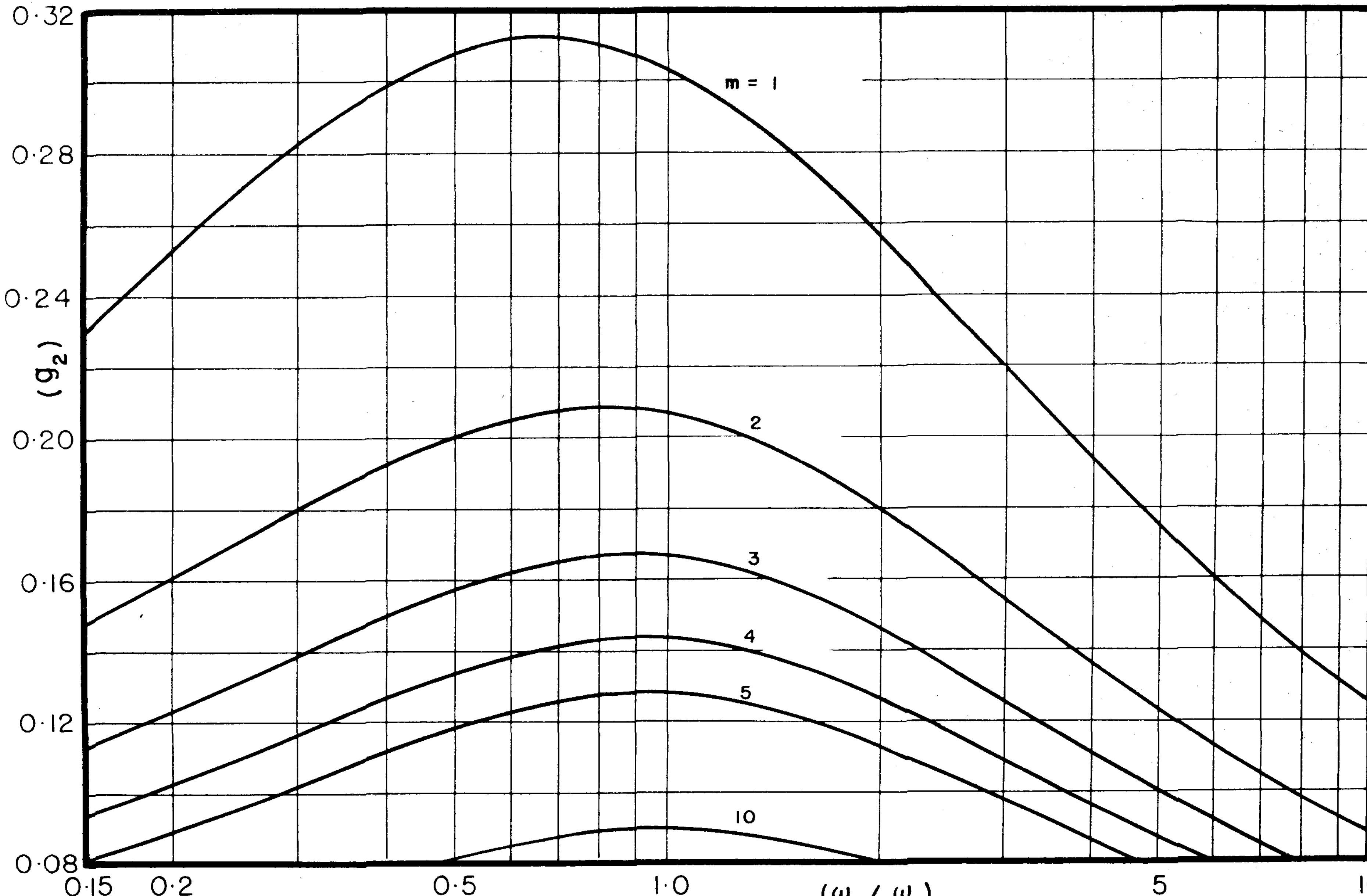

Fig. 3·10 Characteristic Noise Function g_2 versus ω_n/ω_i with m as Parameter for $\Phi_{nn}(\omega) = \vartheta^2/(\omega^2 + \omega_n^2)$.

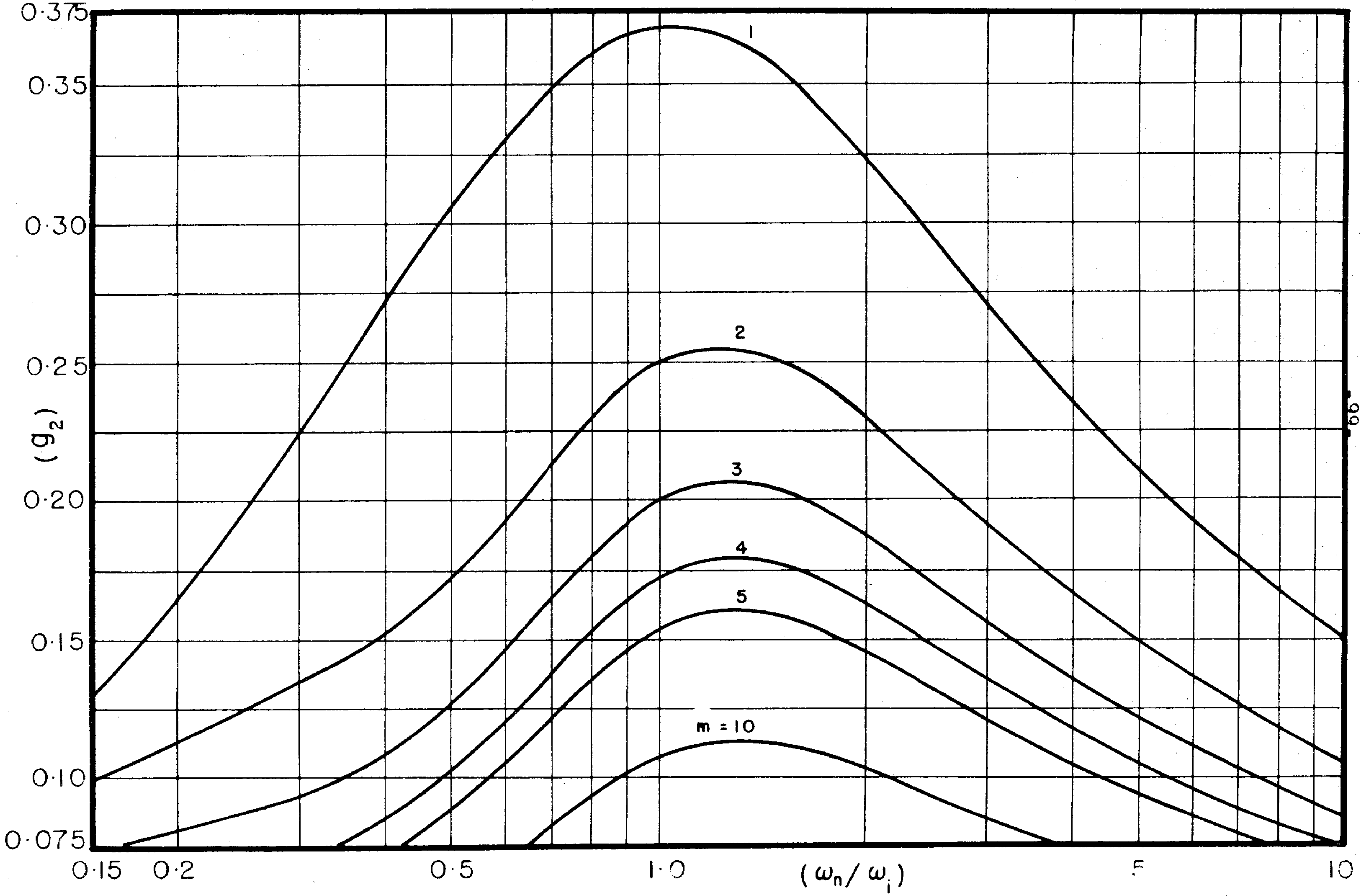

Fig. 3·11 Characteristic Noise Function g_2 versus ω_n/ω_i with m as Parameter for $\phi_{nn}(\omega) = \gamma^4/(\omega^4 + \omega_n^4)$.

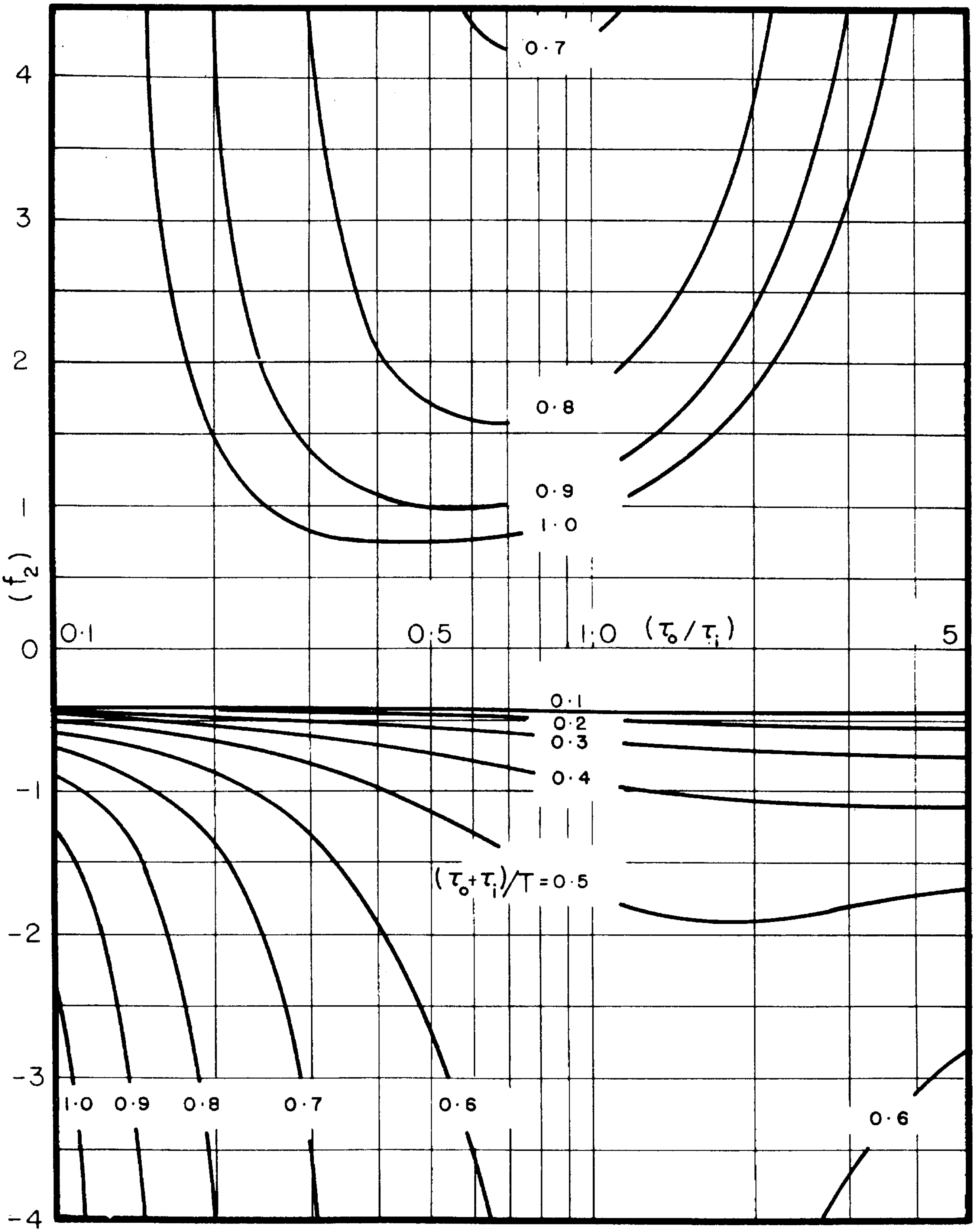

Fig. 3·12 Controlled System Dynamic Characteristics Function f_2 versus τ_o/τ_i with $(\tau_o+\tau_i)/T$ as Parameter.

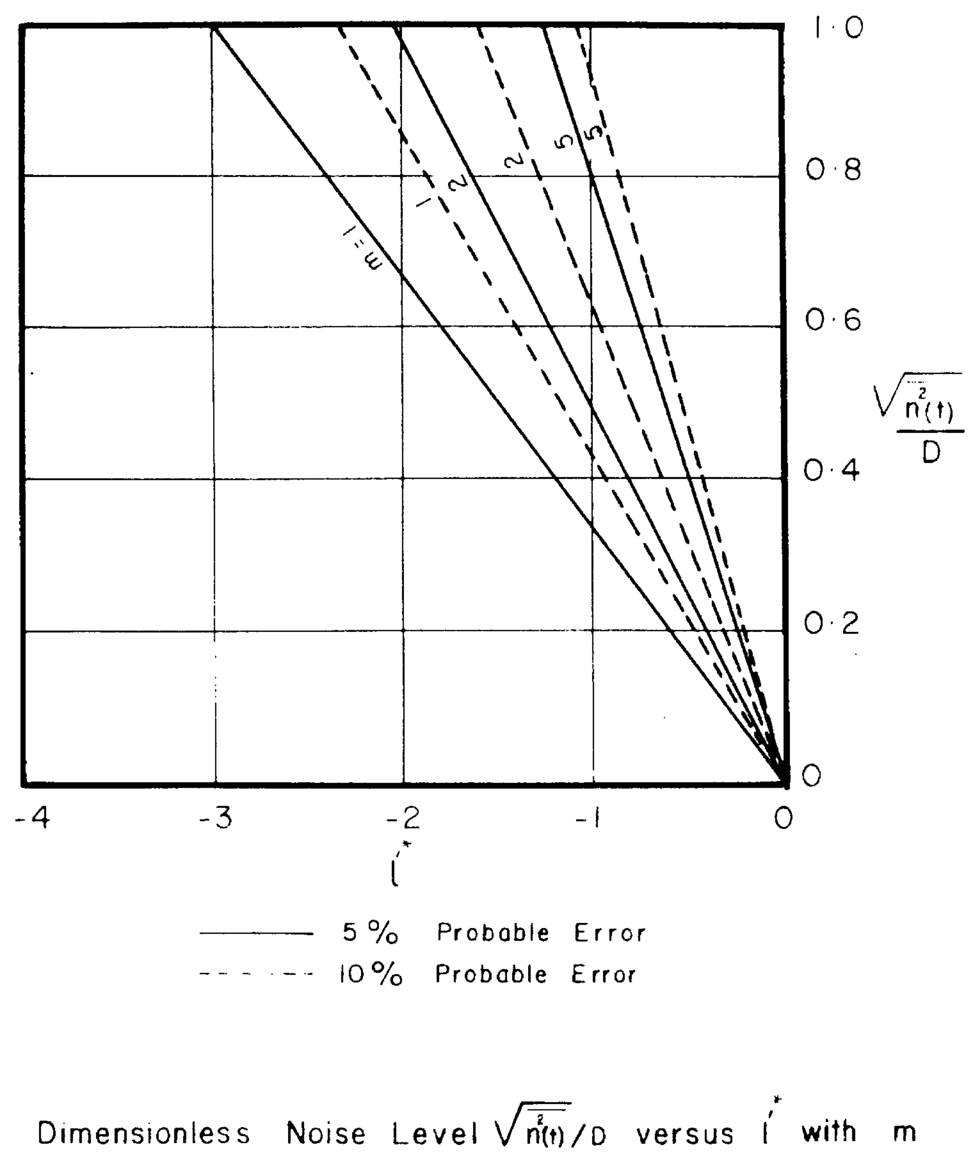

Fig. 3·13 Dimensionless Noise Level $\sqrt{\overline{n^2(t)}}/D$ versus l^* with m as Parameter for $\tau_i/T = 0{\cdot}10$, $\tau_o/T = 0{\cdot}15$, $\omega_n/\omega_i = 1$ and $\Phi_{nn}(\omega) = \gamma^2/(\omega^2 + \omega_n^2)$.

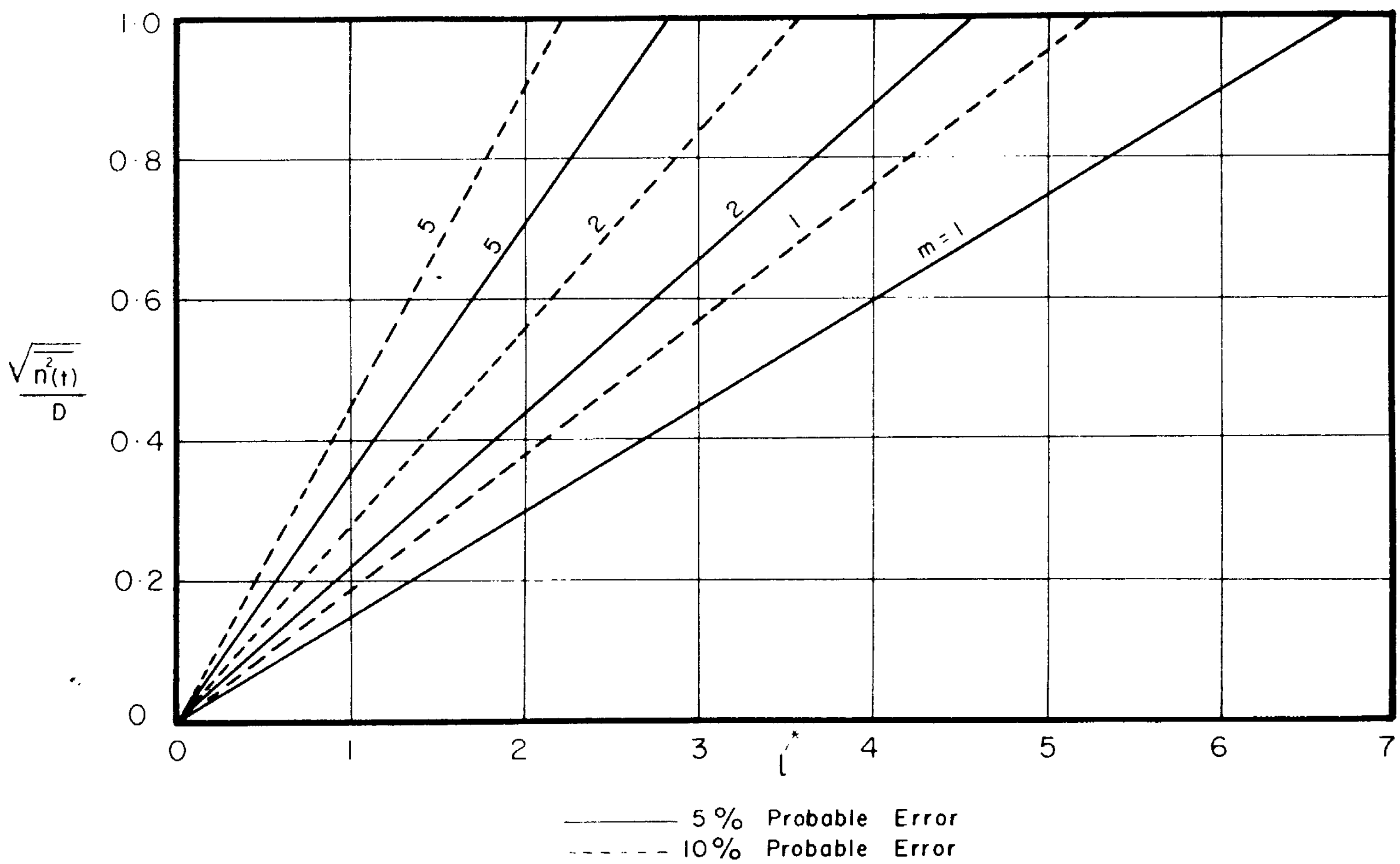

Fig. 3·14 Dimensionless Noise Level $\sqrt{\overline{n^2(t)}}/D$ versus l^* with m as Parameter for values of $\tau_i / T = 0{\cdot}40$, $\tau_o / T = 0{\cdot}60$, $\omega_n/\omega_i = 1$, and $\Phi_{nn}(\omega) = \gamma^2/(\omega^2 + \omega_n^2)$.

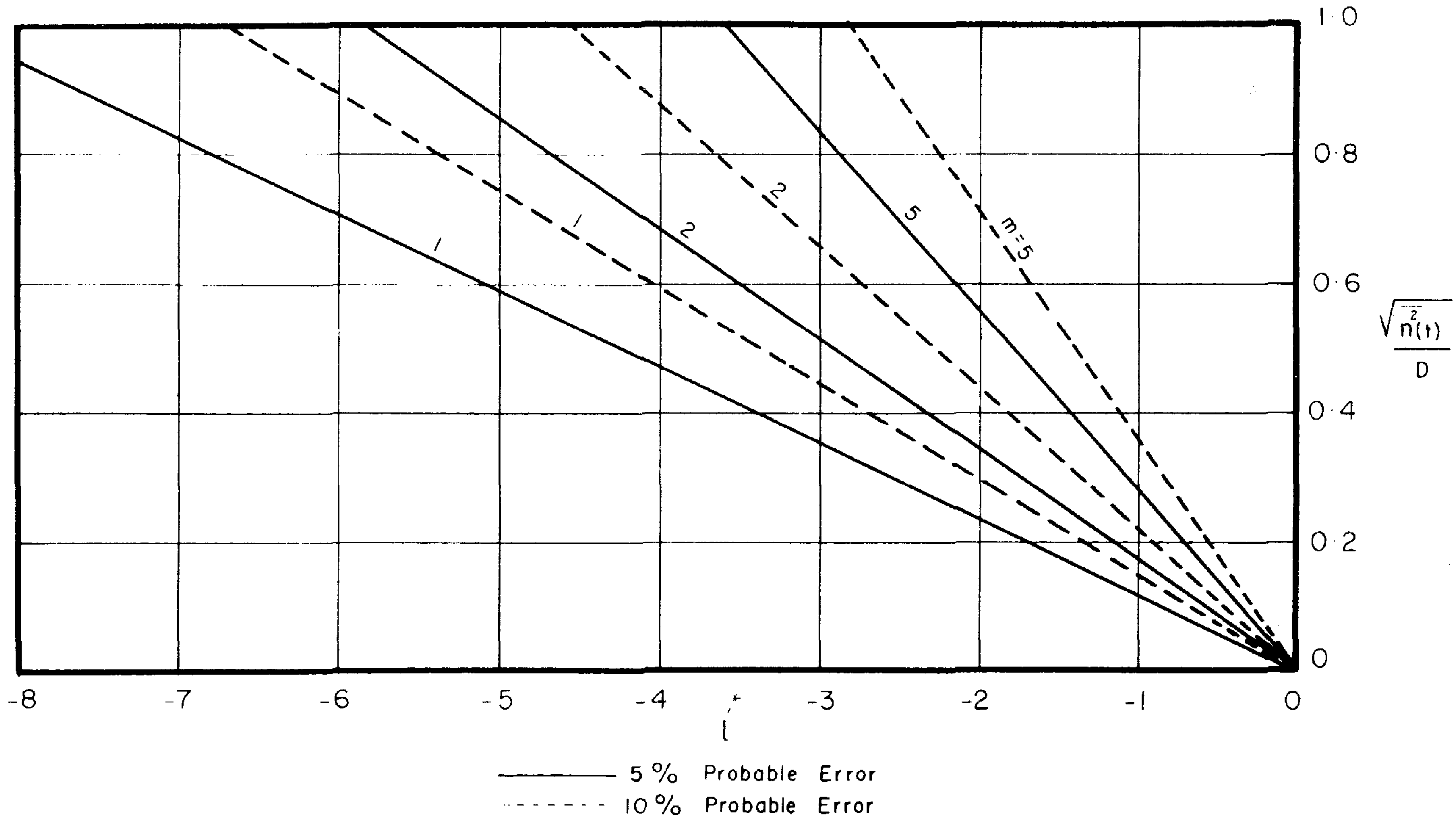

Fig. 3·15 Dimensionless Noise Level $\sqrt{\overline{n^2(t)}}/D$ versus l^{*} with m as Parameter for $\tau_i/T = \tau_o/T = 0 \cdot 25$, $\omega_n/\omega_i = 1$, and $\Phi_{nn}(\omega) = \vartheta^2/(\omega^2 + \omega_n^2)$.

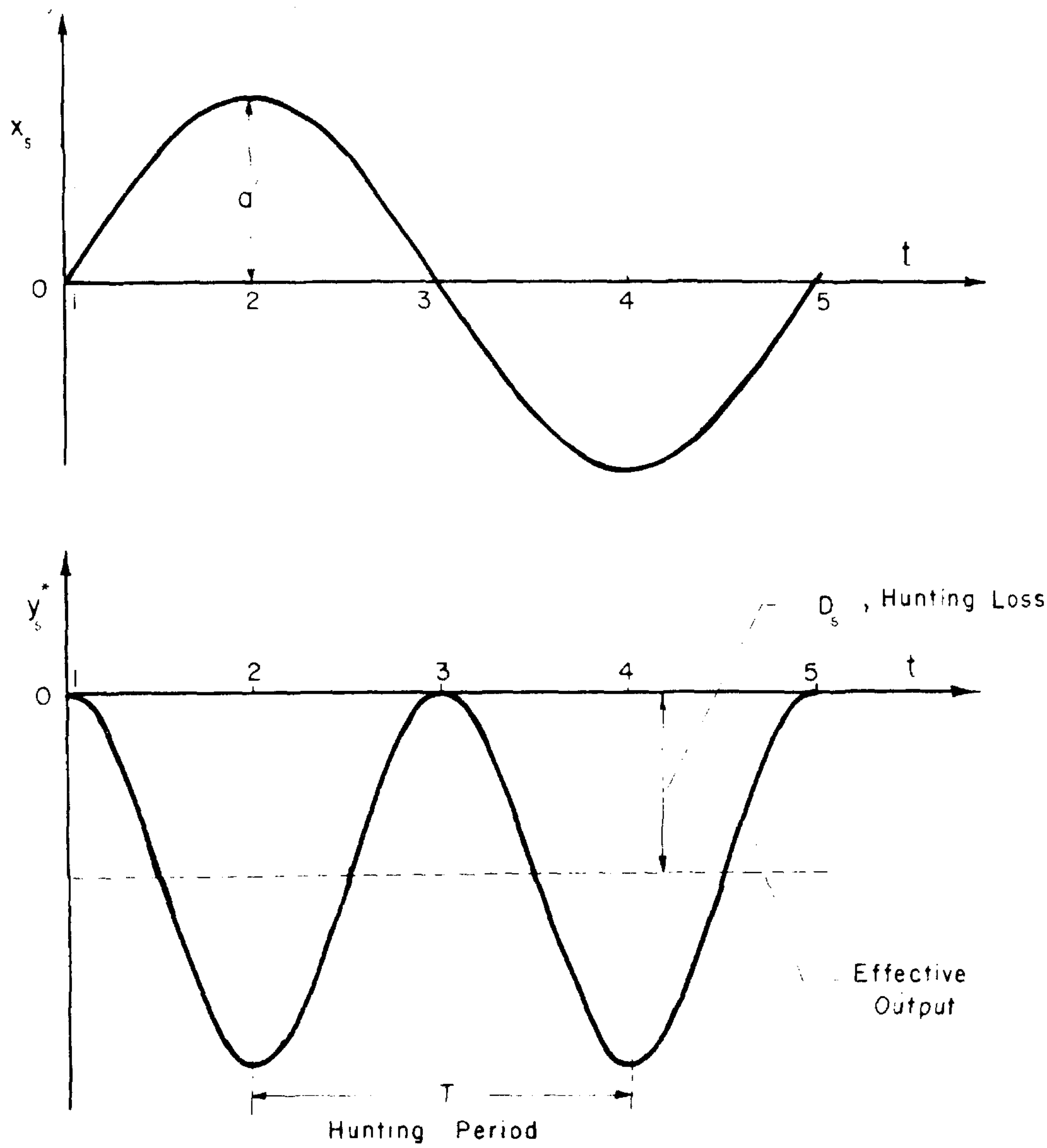

Fig. 4·1 Typical Performance Diagram for a Sinusoidal Input Optimalizing Control System.

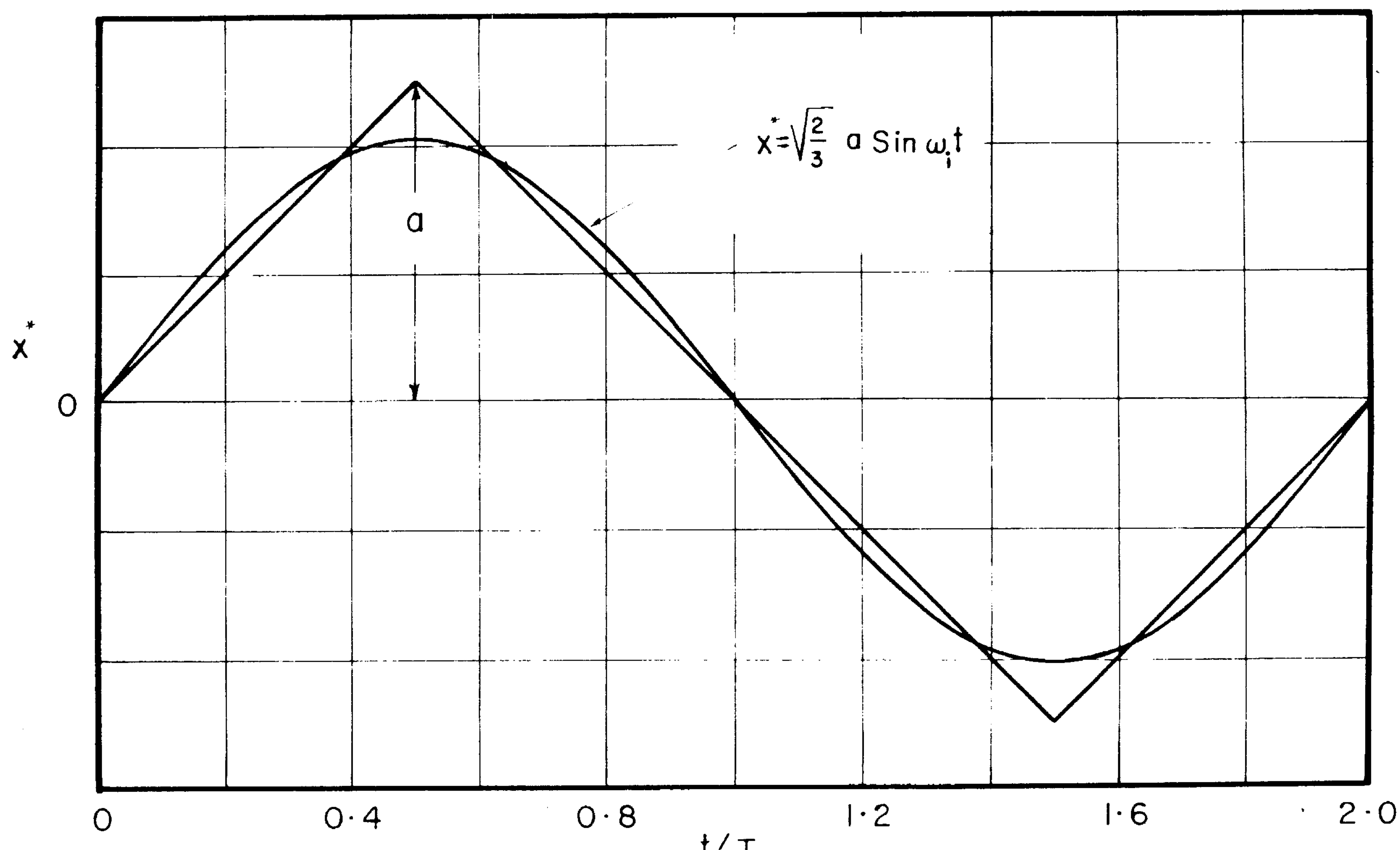

Fig. 4·2 Saw-tooth and Sinusoidal "Potential" Input for Equal R·M·S· Values of "Potential" Inputs.

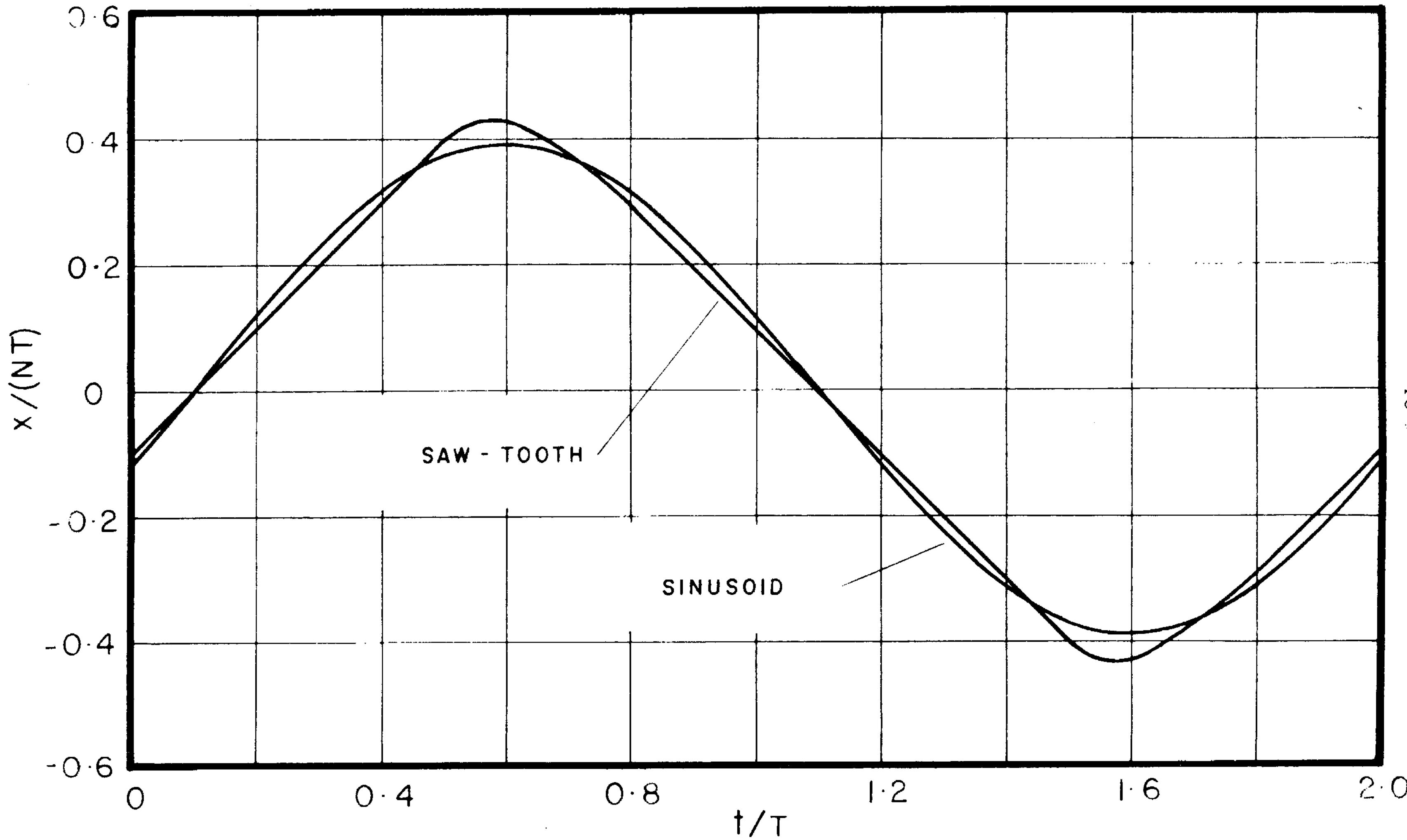

Fig. 4·3 Comparison of Actual Inputs for Equal R·M·S· Values of "Potential" Inputs ($\frac{\tau_i}{T} = 0{\cdot}10$).

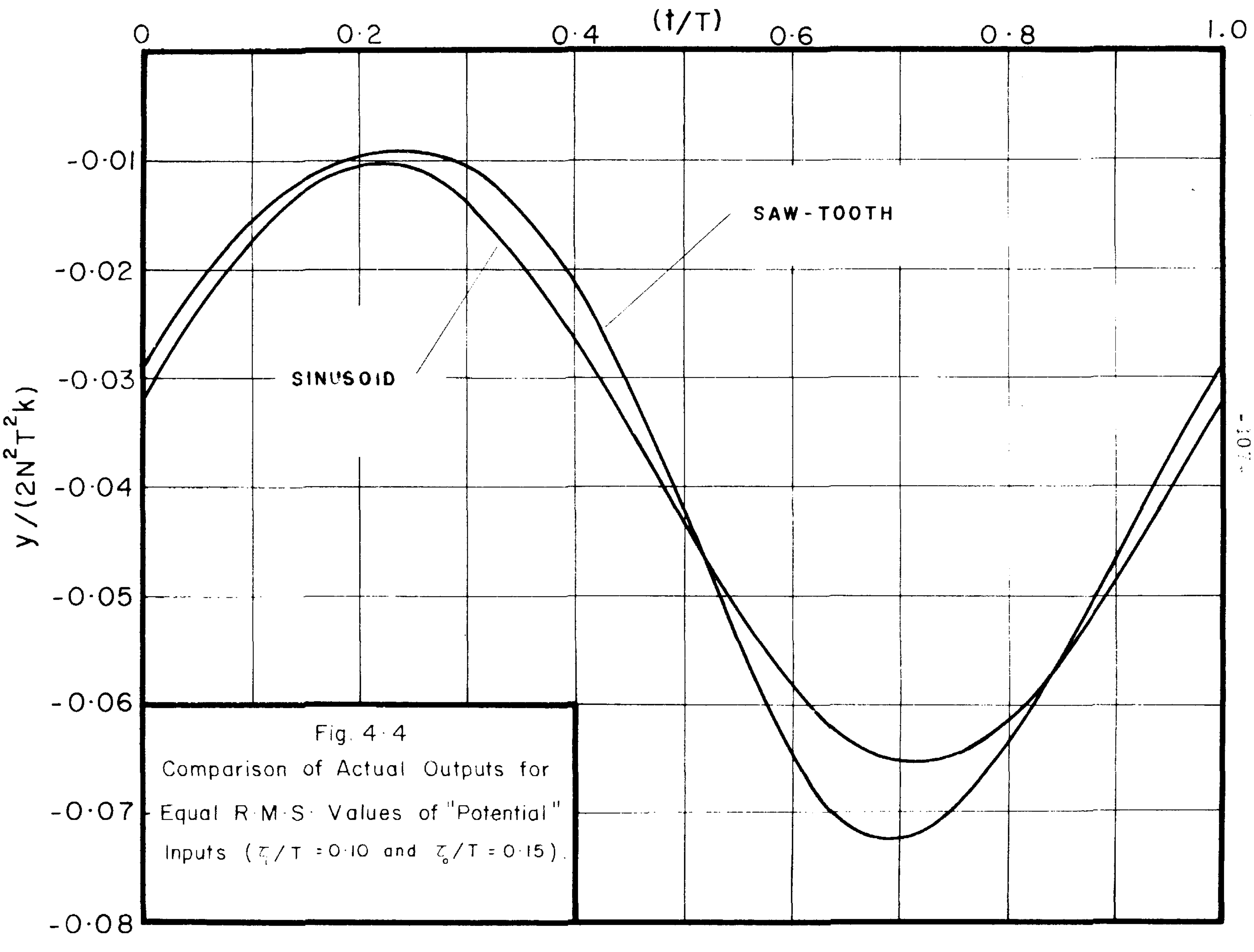

Fig. 4·4
Comparison of Actual Outputs for Equal R·M·S· Values of "Potential" Inputs ($\tau_i/T = 0{\cdot}10$ and $\tau_o/T = 0{\cdot}15$).

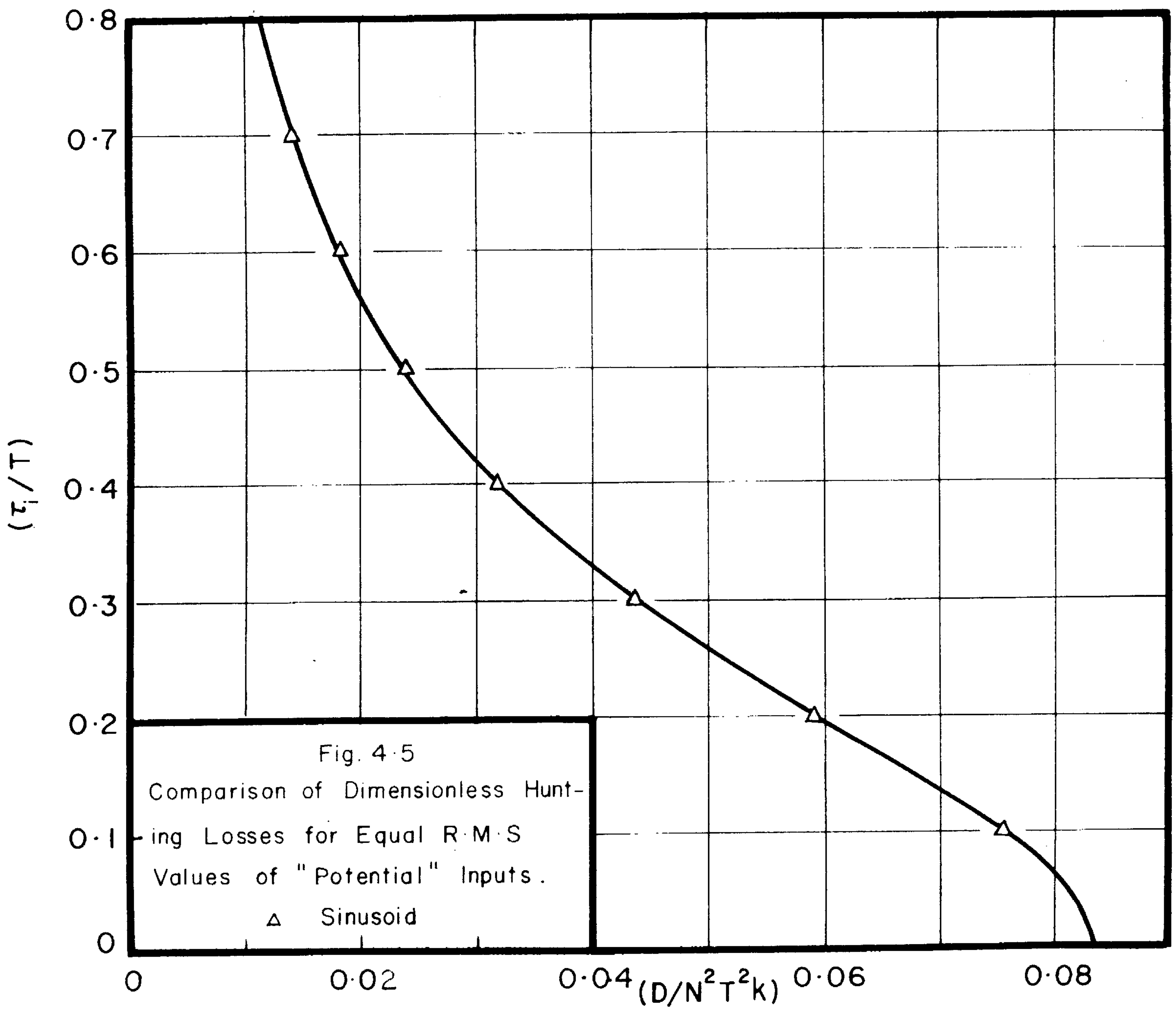

Fig. 4·5
Comparison of Dimensionless Hunting Losses for Equal R·M·S Values of "Potential" Inputs.