STUDIES ON THE PERFORMANCE
OF A ROCKET PROPELLED ORBITING MISSILE

Thesis by

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In Partial Fulfillment of the

Requirements for the

Degree of Aeronautical Engineer

California Institute of Technology
Pasadena, California

June, 1946

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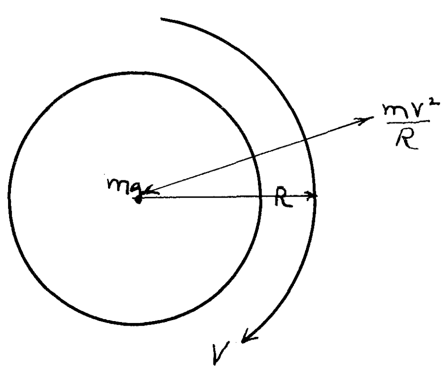
ACKNOWLEDGEMENT

The authors are deeply indebted to Dr. W. Z. Chien for the use of his study of this problem and for his valuable guidance in the formulation of this report. They particularly wish to express their thanks to Dr. H. J. Stewart for the valuable suggestions he contributed. The assistance of members of the Jet Propulsion Laboratory Computing Section, California Institute of Technology, is also deeply appreciated.

INTRODUCTION

This report is a study of the performance requirements necessary to launch a rocket propelled missile into a circular orbit about the earth. Various trajectories by which a missile can be launched into such an orbit are investigated and an estimation is made of the minimum propellant-gross weight ratio required to reach stable orbital conditions at various altitudes.

The velocity of a projectile traveling about the earth in a circular orbit must be such that its weight is exactly balanced by the centrifugal force acting upon it. The expression for the required velocity, relative to a non-rotating earth, can be derived from the following figure.



$$\frac{mV^2}{R} = mg_0 \frac{R_0^2}{R^2}$$

$$V = \sqrt{g_0 \frac{R_0^2}{R^2} R}$$

By the law of Conservation of Energy, such an orbit, once obtained, must remain circular unless influenced by additional forces. The action of atmospheric drag would be to cause a continual loss of energy; this loss would be made up at the expense of potential energy, causing the projectile to move in toward the earth in a contracting spiral. It can be shown (cf. ref. 2) that, for altitudes greater than one hundred and twenty miles, the rate of this contraction is negligible.

The practical applications of an orbiting missile are numerous. A few are as follows:

a) A radio relay station for television and other high frequency communication.

b) To carry instruments for continuous measurement of upper atmospheric phenomena.

c) A military weapon of unlimited range, provided it can be brought back to earth at will.

The possibility of launching a vehicle into a circular orbit by means of rocket propulsion depends primarily upon the conditions of propellant consumption which are required to reach a stable orbital velocity. These conditions are:

a) Propellant characteristics (in particular, specific impulse).

b) Rate of propellant consumption.

c) Propellant-gross weight ratio.

The velocity necessary for a circular orbit is in the neighborhood of 26,000 ft/sec. For an unboosted rocket with an uniform burning rate the terminal velocity (neglecting drag and gravity forces) is given by

$$V_p = -I_{sp}\, g \ln(1-\nu) \tag{1}$$

where I_{sp} is assumed constant. If we take I_{sp} to be equal to

400 sec., a representative value for a rocket fuel consisting of oxygen and hydrogen, the value of ν, the propellant-gross weight ratio, corresponding to $V_p = 26{,}000$ ft/sec., is $\nu = 0.867$. Thus, even with the most powerful rocket fuel, the required propellant-gross weight ratio is very high, and the importance of a more careful estimation of this ratio is evident.

A comprehensive analysis of the important factors which influence the performance of an orbiting rocket has been made by W. Z. Chien of GALCIT. In this analysis, Dr. Chien considered a vertical trajectory as a first approximation in obtaining estimates of the altitude of the circular orbit and of the propellant-gross weight ratio required (cf. ref. 2).

This report extends the considerations of Chien's work to oblique trajectories of various types, all of which terminate at the end of burning with the proper velocity conditions for a circular orbit. The first part of the report is a recapitulation of Chien's analysis. The second part deals with the characteristics of various oblique trajectories by which a rocket may be launched directly into a circular orbit; and the third part analyses a procedure whereby the rocket is launched into an elliptical orbit near its perihelion, is allowed to travel as a free body around the earth to the aphelion of the ellipse and then, by an additional boost of velocity, is projected into a circular orbit.

Explanation of Symbols

a	acceleration
a_p	acceleration at end of burning
α	angle of attack, measured from line of flight to axis of rocket
C	arbitrary constant
e	eccentricity of elliptical orbit
F	thrust of rocket
g	gravitational constant; does not include component due to centripetal acceleration since all quantities are relative to a rotating earth
g_o	apparent value of g at earth's surface at equator (32.08 ft/sec.^{2})
h	altitude
k	trajectory control factor as defined in Eq. (14)
L	control force perpendicular to flight path
m	mass of rocket
p	angular momentum of a free projectile in an orbit around earth $= R^2 \frac{d\theta}{dt}$
θ	inclination of flight path from vertical through launching point
ψ	angle from aphelion of elliptic orbit to displacement vector of projectile
R	distance from earth's center to rocket
R_o	earth's radius at equator (taken as 3963.34 miles)
t	time

t_p burning time of rocket

W weight of rocket

V velocity

V_s stable orbital velocity

V_p velocity at end of burning

ν ratio of propellant weight to initial gross weight of rocket

δ 1) correction
2) small increment

π 3.1416

ϕ inclination of velocity of projection into elliptic orbit with respect to a tangent to a circular orbit through the point of projection.

All units are in the foot-pound-second system.

FIGURES

Figures	Title	Page

PART I - VERTICAL TRAJECTORY

The basis of the solution to the problem of part two of this report is the velocity-time relationship for a rocket in vertical flight. This relationship is worked out in detail in ref. 2, and is plotted in Fig. 4. The calculations involved the following assumptions:

a) The specific impulse of the propellant and its variation with altitude are taken from data prepared by D. A. Young, Aerojet Engineering Corp., November 15, 1945.

b) The gravitation constant is assumed to vary inversely as the square of the distance from the center of the earth.

c) The vehicle is launched at zero velocity from sea level at the equator.

d) The velocity is relative to a rotating earth and the orbital velocity is that required by a projectile traveling in an eastward direction.

e) Air resistance is neglected (but the final required orbital velocity is increased five percent to allow for this loss.)

The variation of the specific impulse with altitude is approximated by the equation

$$I_{sp} = I_{sp_0} + I_{sp_1}\left(1 - e^{-\lambda h}\right). \qquad (2)$$

where

I_{sp_0} = specific impulse at sea level

$I_{sp_0} + I_{sp_1}$ = specific impulse in vacuum

h = altitude, ft.

λ = altitude exponential factor.

It is assumed that the designed expansion ratio at the earth's surface is 23: i.e., the chamber pressure corresponds to 330 p.s.i. For a gaseous oxygen-hydrogen combination, and using the N.A.C.A. standard atmosphere, the parameters are found to be

$I_{sp_0} = 330$ sec. $\quad I_{sp_1} = 111$ sec. $\quad \lambda = 1.707 \times 10^{-5}\ ft.^{-1}$

This empirical formula checks very well with results estimated theoretically by D. A. Young (cf. Fig. 13). With these parameters in Eq. (2), one can see that the specific impulse increases by about thirty percent at high altitude. Thus the velocity correction for variation of specific impulse with altitude is additive and calculations in ref. 2 show it to be of the order of magnitude of 6,000 ft/sec., for the vertically fired rocket at the end of burning.

The following data on the earth are taken from the Handbook of Chemistry and Physics (cf. ref. 3):

R_0 equatorial radius, 3,963.34 miles

V_e mean linear velocity of rotation of the surface at the equator, 0.289 miles/second

g_0 gravitation at equator, 32.08 ft/sec.^{2}

A part of the gravitational force at the equator is counterbalanced by the centrifugal force due to the earth's rotation. It should be noticed that if a missile is fired vertically from a rotating earth's surface, the angular momentum must be conserved. However, it can be shown (cf. ref. 2) that the gravitational constant at various distances from the earth is very closely approximated by

$$g = g_0 \frac{R_0^2}{R^2} \tag{3}$$

where

$$g_0 = 32.08 \ ft/sec^2 .$$

If h denotes the altitude of the orbit, the stable orbital velocity relative to a non-rotating earth is given by

$$V = R_0 \sqrt{\frac{g_0}{R_0 + h}} . \tag{4}$$

If the space vehicle in its orbit is moving in the eastward direction, then the stationary orbital velocity relative to the rotating earth's surface becomes

$$V_1 = V - V_e .$$

In this investigation, V_1 is the required velocity to be reached through the combustion of the propellant.

In ref. 2, air resistance was neglected in calculating the velocity-time relationship. The justification for this assumption can be shown by investigation of the magnitude of the drag force. Assuming drag may be expressed as

$$D = \frac{1}{2} \rho(h) C_D A V^2$$

where

$\rho(h)$ = air density in slugs, based on the thermal variation taken by Whipple (cf. ref. 5)

A = cross-sectional area, ft.^{2}

C_D = drag coefficient

If the mean free path of the air molecules is greater than the length of the missile, the drag coefficient, C_D, may be estimated on the basis of the transfer of impact momentum of the molecules from the surrounding space.

The ratio $\frac{\text{mean free path}}{\text{length of missile}} \cong \frac{\text{Mach number}}{\text{Reynolds number}}$.

At an altitude of 50 miles, a velocity of 20,000 ft/sec., and a missile diameter of 10 ft.,

$$\frac{M}{R_e} \cong \frac{10}{10{,}000} \cong \frac{1}{1000}$$

Therefore, C_D is estimated on the basis of a continuum flow pattern rather than by the momentum transfer theory.

Since the exhaust jet is operating, base drag is approximately zero. Nose drag can be estimated for a 20° (half angle) cone as $C_{D_N} \cong 0.2$. At a Reynolds number of 10,000, a missile of ten calibers length has a skin friction coefficient $\cong 0.2$. Thus the total drag coefficient, $C_D \cong 0.4$.

For example: At an altitude of 50 miles, a 20 ton vehicle with a diameter of 10 ft. would be subject to a drag force of

$$D = 0.4 \times \frac{6.42 \times 10^{-8}}{2} \times (25{,}000)^2 \times 25\pi \cong 630 \text{ lbs.}$$

The decelerating effect of this drag is

$$\frac{D}{m} = \frac{624 \times 32.08}{40{,}000} \cong 0.5 \text{ ft/sec}^2.$$

The acceleration due to thrust during the latter stages of the trajectory is of the order of several hundred ft/sec.^{2}, and thus the effect of air resistance is negligible during that part of its flight. Although the loss due to drag in the early stages of the flight may be appreciable, its proportionate effect on the terminal velocity is small. To make ample allowance for any drag loss, the required terminal velocity is taken five percent greater than that necessary for a stable orbit.

FIRST ORDER SOLUTION, VERTICAL TRAJECTORY.

The equation of motion for a vertically fired rocket is

$$\frac{dV}{dt} = \frac{F-D}{m} - g_0 \frac{R_0^2}{R^2} . \qquad (5)$$

In this equation,

$$F = g_0 \frac{R_0^2}{R^2}\left(-\frac{dm}{dt}\right)\left\{I_{sp_0} + I_{sp_1}(1-e^{-\lambda h})\right\} \cong g_0 \frac{dm}{dt} I_{sp_0}$$

$$D = \frac{1}{2}\rho(h)\, V^2 C_D A \cong 0$$

$$R^2 = (R_0 + h)^2 \cong R_0^2$$

Assuming also that the rate of propellant consumption is uniform, Eq. (5) may then be written as

$$\frac{dV}{dt} = \left\{ \frac{I_{sp_0}\frac{\nu}{t_p}}{\left(1-\frac{\nu}{t_p}t\right)} - 1 \right\} g_0 . \qquad (6)$$

The integration of Eq. (5) for zero initial velocity at the ground is

$$V = g_0 I_{sp_0} \ln\left(1-\frac{\nu}{t_p}t\right) - g_0 t \qquad (7)$$

$$h = \frac{I_{sp_0} g_0}{\nu} t_p \left(1-\frac{\nu}{t_p}t\right) \ln\left(1-\frac{\nu}{t_p}t\right) + I_{sp_0} g_0 t - \frac{1}{2} g_0 t^2 . \qquad (8)$$

Numerical computation of the altitude dependent corrections are carried out in ref. 2 for various combinations of ν and t_p . These corrections are added to the corresponding evaluations of Eqs. (7) and (8) to give the true velocity and altitude vs. time relationships for a vertical trajectory. The proper combinations of ν and t_p to obtain stable orbital velocity are obtained by correlating these empirical functions with the velocity requirements of a stable orbit at various altitudes. This constitutes the first order solution to the problem. Based upon this solution, the minimum propellant-gross weight ratio required is $\nu = 0.89$. Lower values of ν reduce the terminal velocity below that necessary for a stable orbit at any altitude within possible reach. Since the trajectory is vertical, an estimation of the altitude of the stable orbit is uncertain. Ref. 2 sets this value at 100 to 200 miles.

PART II - OBLIQUE TRAJECTORIES

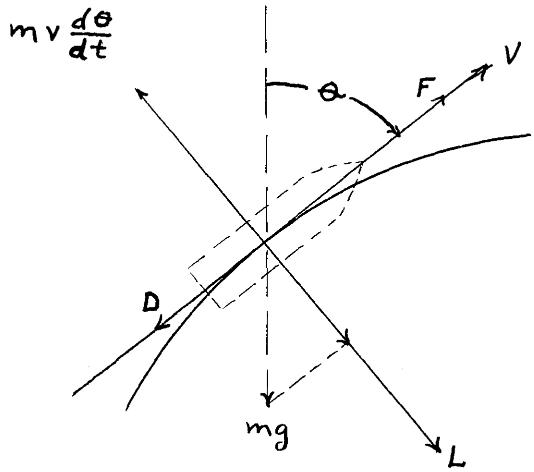


Fig. 1

Resolution of Forces Acting on a Rocket Propelled Missile in an Oblique Trajectory.

The general equations of motion for a rocket propelled missile in an oblique trajectory are:

$$\frac{dV}{dt} = g\left\{\frac{F-D}{W} - \cos\theta\right\} \tag{9}$$

$$\frac{d\theta}{dt} = \frac{g}{V}\left\{\sin\theta + L\right\} \quad . \tag{10}$$

Here the thrust is assumed to be along the flight path and lift forces are obtained from aerodynamic surfaces. For the particular missile under consideration, these equations are modified by the following assumptions:

a) $g \cong g_o$

b) $D \cong 0$

c) The rocket is essentially a wingless missile and control forces are supplied by a component of the thrust perpendicular to the flight path; thus

$L = F \sin \alpha$, where α is the angle of attack of the rocket.

In the velocity-time function used in these computations, g is approximated by $g_0 \frac{R_0^2}{R^2}$. Since Eq. (9) will be used only to obtain a correction to $V(t)$ for a vertical trajectory, assumption (a) introduces only a second order error. Similarly, assumption (b) introduces negligible errors since drag is allowed for in $V(t)$.

It will be shown later that the control force, L, necessary to obtain stable orbital velocity in a horizontal direction, corresponds to an acceleration of several g's normal to the flight path. A wing capable of producing this force at high altitudes would be very large, complicating the structure of the missile and causing a much larger drag. By utilizing a component of the thrust of the rocket motor to supply this control force, these complications are eliminated and the equations of motion are greatly simplified.

With these assumptions, and referring to Fig. 2, Eq. (9) and (10) are modified to the following general form.

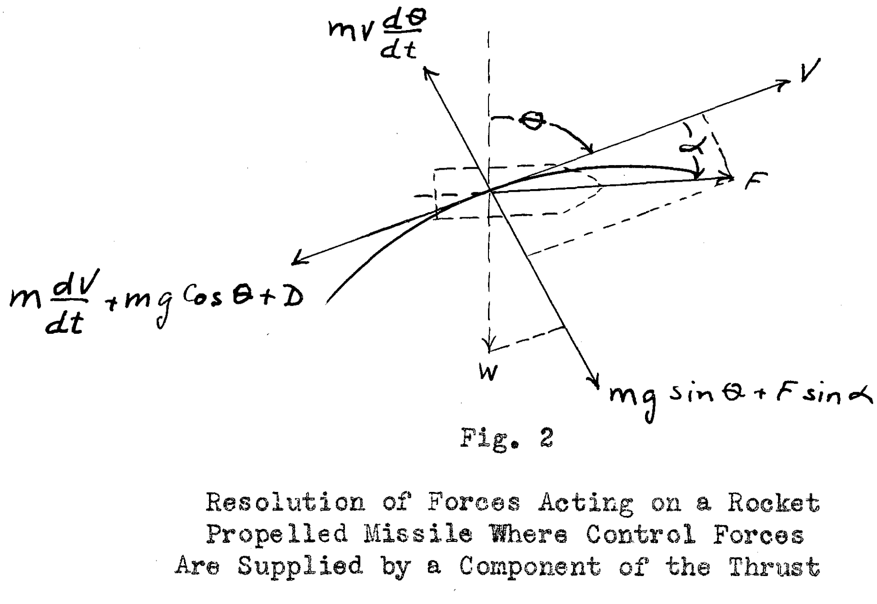


Fig. 2

Resolution of Forces Acting on a Rocket Propelled Missile Where Control Forces Are Supplied by a Component of the Thrust

$$\frac{dV}{dt} = g\left\{\frac{F}{W}\cos\alpha - \cos\theta\right\} \tag{11}$$

$$\frac{d\theta}{dt} = \frac{g}{V}\left\{\sin\theta + \frac{F}{W}\sin\alpha\right\} \tag{12}$$

The above equations will be developed first for a free, i.e. uncontrolled, trajectory and then for a controlled trajectory.

THE FREE TRAJECTORY.

For a free trajectory, $\alpha = 0$, and Eqs. (11) and (12) become

$$\frac{dV}{dt} = g\left\{\frac{F}{W} - \cos\theta\right\} \tag{13}$$

$$\frac{d\theta}{dt} = \frac{g}{V}\sin\theta . \tag{14}$$

From Eq. (14)

$$\int_{\theta}^{\theta_p} \frac{d\theta}{\sin\theta} = \int_{t}^{t_p} \frac{g}{V}\,dt = G(t)$$

$$\log_e\left(\frac{\sin\theta}{1+\cos\theta}\right)\Big]_{\theta}^{\theta_p} = G$$

$$\frac{\sin\theta_p}{1+\cos\theta_p} = \frac{\sin\theta}{1+\cos\theta}\,e^{G} .$$

For the missile to enter the circular orbit at the end of burning, it must be traveling in a horizontal direction, that is, $\theta_p = 90^\circ$. Thus

$$\frac{\sin\theta}{1+\cos\theta}\,e^{G} = 1$$

this can be further reduced to

$$\frac{1+\cos\theta}{1-\cos\theta} = e^{2G}$$

solving for $\cos\theta$

$$\cos\theta = \frac{e^{2G}-1}{e^{2G}+1}$$

since

$$\tanh G = \frac{e^{G} - e^{-G}}{e^{G} + e^{-G}} = \frac{e^{2G} - 1}{e^{2G} + 1}$$

$$\cos\theta = \tanh \int_{t}^{t_p} \frac{g}{V}\, dt. \tag{15}$$

Eq. (15) can be integrated numerically using values of velocity obtained from the computations for a vertical trajectory (cf. ref. 2), giving a first approximation of $\theta(t)$.

For a free inclined trajectory, i.e., with $\alpha = 0$,

$$\frac{dV}{dt} = g\left\{\frac{F}{W} - \cos\theta\right\},$$

and for a vertical trajectory,

$$\frac{dV}{dt} = g\left\{\frac{F}{W} - 1\right\}$$

The correction, δV, to be applied to the velocity of a vertical trajectory, is $v_{inclined} - v_{vertical}$, and differentiating

$$\frac{d\,\delta V}{dt} = \frac{dV_{inclined}}{dt} - \frac{dV_{vertical}}{dt}$$

$$= g(1 - \cos\theta), \quad \text{or}$$

$$\delta V = \int_{t_1}^{t} g(1 - \cos\theta)\, dt \tag{16}$$

where $\theta = \theta(t)$ as obtained from the integration of Eq. (15). This gives a first approximation of $\delta V(t)$ and hence of $V(t)$.

Repeating the integration of Eq. (15) and Eq. (16) using the first approximation of $\theta(t)$ and $V(t)$, a second approximation is obtained. It was found by carrying out this process that the second approximation differed from the first by less than one percent of the velocity, and it may be concluded that first approximations of these quantities are sufficiently accurate.

With $\theta(t)$ and $V(t)$ known, altitude and horizontal distance can be obtained from

$$h = \int_0^t V \cos\theta \, dt \qquad \text{(cf. Figs. 9 and 16)} \qquad (17)$$

$$X = \int_0^t V \sin\theta \, dt \qquad \text{(cf. Figs. 14 and 15)} \qquad (18)$$

THE CONTROLLED TRAJECTORY.

In a free trajectory, the energy in the propellant is expended in accelerating the rocket to a velocity V_p and in lifting it to an altitude h_p . For any other trajectory, a portion of the propellant must be utilized to provide a controlling force. From the propellant consumption standpoint, the free trajectory is the best; however, even with the highest propellant-gross weight ratio considered, the maximum ordinate is only 29.6 miles. It is thus necessary to go to a controlled trajectory in order to obtain sufficient altitude to avoid the dense atmosphere surrounding the earth.

There are, of course, an infinite number of possible controlled trajectories given by

$$\frac{d\theta}{dt} = f(t)$$

and there must be one such function which will require a minimum amount of fuel to reach stable orbital velocity at a given altitude. A rigorous solution of this problem will not be attempted, but it is believed that the following analysis will give results which are sufficiently accurate for the present purpose.

Defining the angle α such that

$$\frac{F}{W}\sin\alpha = (k-1)\sin\theta$$

where k, a constant greater than one, is the control factor, then

$$\frac{d\theta}{dt} = k\frac{g}{V}\sin\theta$$

and, for constant values of k,

$$\cos\theta = \tanh k\int_{t}^{t_p}\frac{g}{V}\,dt. \tag{19}$$

In a manner similar to that used for the free trajectory, the expression for δV in the controlled trajectory is derived as

$$\delta V = g\int_{t_1}^{t}\left\{(1-\cos\theta) - \frac{F}{W}(1-\cos\alpha)\right\}dt \tag{20}$$

where $\cos\theta$ is given by Eq. (19); also

$$F = \frac{I\nu}{t_p}W_0$$

and

$$W = \left(1-\nu\frac{t}{t_p}\right)W_0$$

or

$$\frac{F}{W} = \frac{I\nu}{t_p - \nu t}\;^{*}$$

and from the definition of α,

$$\sin\alpha = (k-1)\frac{W}{F}\sin\theta. \tag{21}$$

Thus all quantities in Eq. (20) are known, and a numerical integration can be carried out.

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*The variation of I with altitude was taken into consideration in the calculation of V for a vertical trajectory. In this equation an average value over the entire trajectory is used. This value is 420 sec. (cf. Fig. 13).

Finally, $V' = V_{vertical} + \delta V$ and we now have first approximations to V and θ as functions of time. Just as for the free trajectory, it was found that these first approximations are sufficiently accurate. V' is plotted vs. time in Fig. 8 for various combinations of ν, t_p, and k. The altitude and horizontal distance traveled can be calculated from Eqs. (17) and (18).

ESTIMATION OF PROPELLANT-GROSS WEIGHT RATIO.

The velocity at end of burning in the inclined trajectory, V_p', is obtained by adding δV_p (tabulated in Table 1), to the corresponding V_p; i.e., the terminal velocity of the vertical trajectory:

$$V_p' = V_p + \delta V_p \quad .$$

Since the altitude reached in the inclined trajectory is less than that in the vertical trajectory, the velocity required for a stable orbit, V_s, is greater. Thus the net δV_p is given by

$$\delta V_{p(net)} = V_p' - V_s$$

If $\delta V_{p(net)}$ is positive, the burning time of the inclined trajectory can be reduced below that of the vertical trajectory, and this reduction in t_p is given by:

$$\delta t_p = \frac{V_p' - V_s}{a_p'}$$

where a'_p, the acceleration of the inclined trajectory at end of burning is

$$a_p' = \frac{I_p \, \nu / t_p}{1 - \nu} \, \frac{R_o^2}{R^2} \, g_o \qquad \text{(cf. Table 1)}$$

and

$$t_p' = t_p - \delta t_p \qquad \text{(cf. Table 1)}$$

and finally,

$$\nu' = \nu \times \frac{t_p'}{t_p} \qquad \text{(cf. Table 1)} \qquad (22)$$

Fig. 6 gives the fuel requirements of various inclined trajectories. The curves of Fig. 6 give the propellant-gross weight ratio and burning time required by a rocket to reach a given altitude with a velocity just sufficient for a stable orbit. For any point on a given altitude curve, there is a corresponding ν, t_p and control factor, k, but the lowest point on the curve corresponds to the most economical trajectory since it requires the smallest value of ν. Thus, for each altitude, there is an ideal control factor, k, and corresponding burning time, t_p. Furthermore, for each value of ν there is a maximum altitude obtainable, with the necessary values of k and t_p uniquely defined. It is seen that, for $\nu = 0.89$, the highest possible altitude is 53 miles.

It must be remembered that the high velocity required for an orbiting missile places severe limitations upon the possibility of its successful launching. Most of the altitude reached by a vertically fired rocket is obtained during the later stages of flight when the velocity is high. Since the final velocity of an orbiting missile must be horizontal, its altitude is, of necessity, very much reduced. It is desirable, therefore, to keep the flight path nearly vertical for as long as possible and still be able to turn the missile into a horizontal direction at the end of burning.

Since the rocket must eventually turn through 90°, the integral of $\frac{d\theta}{dt}$ is fixed, and if θ is to be kept small during the early stages of flight, $\frac{d\theta}{dt}$ must be small at first and large near the end of the trajectory.

The control force necessary to turn the rocket from the vertical towards the horizontal is $L = mV\frac{d\theta}{dt} - mg \sin\theta$. Herein lies the limitation on the function describing $\frac{d\theta}{dt}$. V is large near the end of the trajectory and, if L is to be kept within reasonable limits, $\frac{d\theta}{dt}$ must be relatively small.

As an example of these considerations, let us examine two types of functions which describe the variation of $\frac{d\theta}{dt}$.

The German V-2 followed a path given by $\theta = C\, t^2/t_p^2$ or $\frac{d\theta}{dt} = C't$ where C was given a series of values throughout the powered trajectory. As a simple case, we will let $\theta = \frac{\pi}{2}\frac{t^2}{t_p^2}$, which meets the condition that $\theta = 0$ when $t = 0$ and $\theta = 90^\circ$ when $t = t_p$.

For t_p = 45 sec., the acceleration normal to the flight path at the end of burning is

$$\frac{L}{m} \cong \frac{\pi}{45} \times 25{,}000 - 32.08 \cong 1718 \text{ ft/sec.} \cong 53.5 g\text{'s}.$$

This acceleration could not possibly be obtained by aerodynamic forces and would require over 70 percent of the direct thrust of the rocket motor; i.e., an angle of attack of over 45 degrees.

The free trajectory furnishes an example of the other extreme. Since no control is used, $\frac{d\theta}{dt}$ and θ are large near the beginning of the trajectory and the resulting altitude is low (cf. Fig. 9).

The conclusion is that a compromise must be made such that $\frac{d\theta}{dt}$ is small in the early stages of flight to give a higher altitude, increase to a maximum somewhere near the middle of the trajectory, and then decrease as V becomes very large, keeping down the required control force.

The type of control investigated in this report accomplishes this compromise to a fair degree. We assume that the entire control force is obtained by a component of thrust perpendicular to the velocity as shown in Fig. 2. The variation of $\frac{d\theta}{dt}$ with time for various values of k is shown in Fig. 12. From the standpoint of obtaining maximum altitude, k should be large, since this allows $\frac{d\theta}{dt}$ and hence θ to remain small for a longer period of time. A very high value of k would result in a flight path which would be nearly vertical until shortly before the end of burning and then curve sharply over into the horizontal.

The value of k is limited, however, since it has a direct effect upon the angle α, which, by definition, is given by Eq. (21).

$$\sin \alpha = (k-1)\frac{W}{F}\sin\theta .$$

Thus α increases with k and since the velocity thrust is reduced by $\cos\alpha$, so also is the velocity. This decrease in velocity represents the cost of control. Since the final velocity must be sufficient for a stable orbit, k has a maximum value for each combination of ν and t_{b0}. It also has an optimum value for given trajectory requirements as seen from Fig. 6. This optimum value is approximately two in most cases.

The angle α is plotted in Fig. 11 and it is noted that, for k optimum, α has a maximum value of about 12 degrees, occurring at around the half burning time when V is still relatively low. The five percent loss in velocity originally allowed for is believed ample to cover the increased loss due to drag resulting from the angle of attack α.

PART III - THE ELLIPTICAL TRAJECTORY

It has been shown that the maximum altitude attainable by continued powered flight is about 53 miles for $\nu = 0.89$; however, it is possible to reach considerably greater altitudes along an elliptical orbit.

If the missile is brought into stable conditions at a given altitude by the method of Part II, and is then allowed to burn for a short additional time, it will enter into an elliptical orbit of small eccentricity with the center of the earth at one focus. The aphelion of this orbit will, in general, be at a greater altitude than the point of projection; and if, at the aphelion, the rocket is given another small boost of velocity, it may enter a circular orbit at that altitude.

The following analysis gives, to a first approximation, the requirements and results of such a method for launching a rocket into a stable orbit.

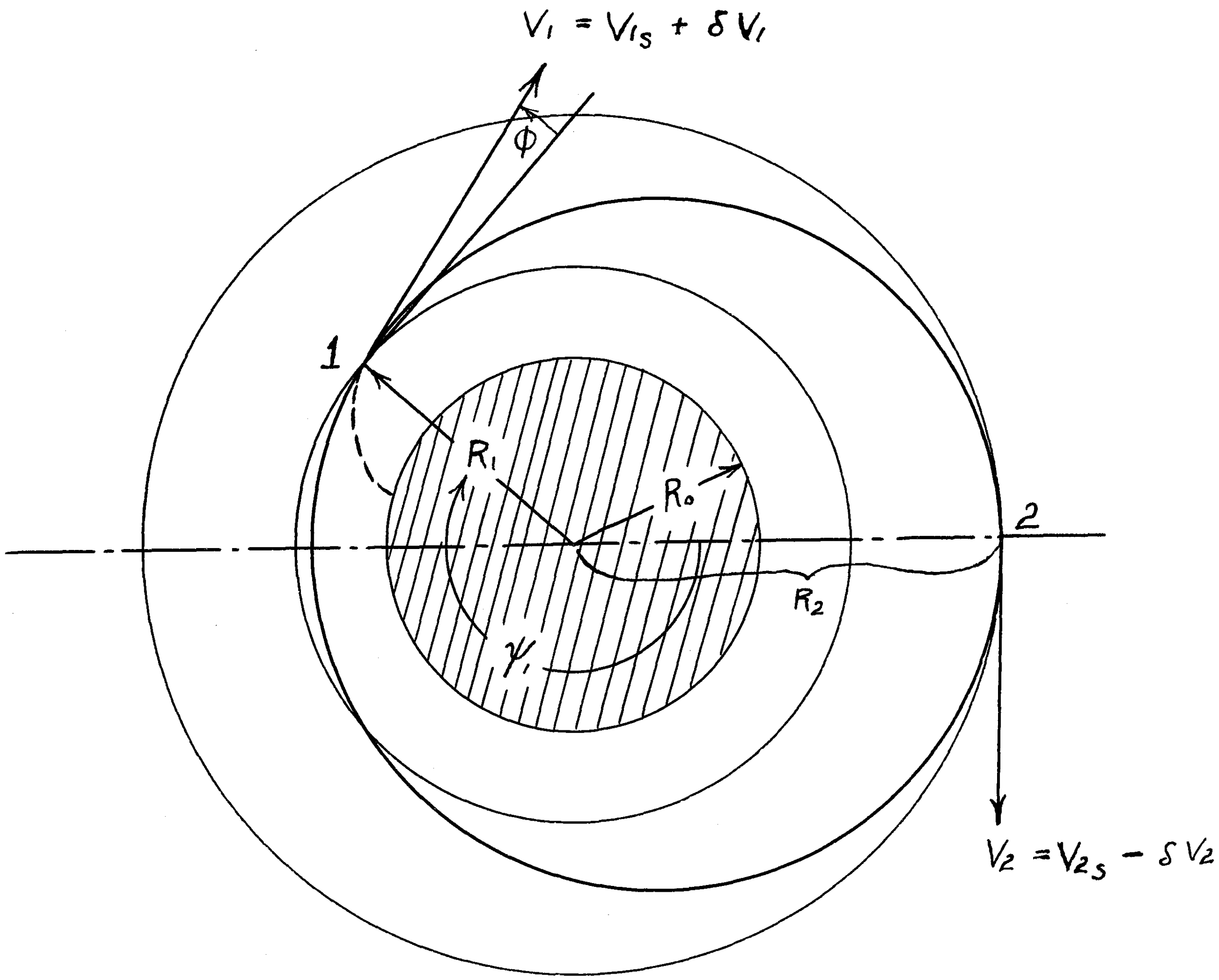


Fig. 3

Diagram of the Elliptical Trajectory for Launching a Rocket Propelled Missile into the Stable Orbit.

The path of a free body traveling in an elliptical orbit around the earth is given by

$$R = \frac{p^2}{g_0 R_0^2 (1 - e \cos \psi)} \tag{23}$$

where the center of the earth is at one focus of the orbit.

The notations are:

R is the distance of the projectile from the earth's center

R_0 is the mean radius of the earth, 3963.34 miles at the equator

g_0 is the gravitation constant at the earth's surface

p is the angular momentum of the projectile $= R^2 \frac{d\psi}{dt}$ and is constant

e is the eccentricity of the orbit

ψ is measured from the aphelion (point 2)

The shape and orientation of the orbit are dependent only upon the direction and magnitude of the velocity with which the body is projected into free flight. The approximations and assumptions introduced are

a) The point of projection (point 1) is at an altitude h_1, great enough that atmospheric drag may be neglected.

b) The eccentricity of the orbit is small. This is necessary in order that powers of e may be neglected in the approximate solution. Small eccentricity results in the velocity at aphelion, V_2, being only slightly less than that required for a circular orbit.

c) Small eccentricity requires that the projection velocity, V_1, be

1) very nearly tangent to the circular orbit through point 1.

2) very nearly equal in magnitude to stable orbital velocity. In order to gain the most altitude, point 1 must be near the perhelion; hence V_1 should be slightly greater than stable orbital velocity at point 1.

From the general expression, Eq. (23), one obtains

$$\left(\frac{dR}{dt}\right)^2 = \frac{R^2 e^2 \sin^2\psi \left(\frac{d\psi}{dt}\right)^2}{(1 - e\cos\psi)^2}$$

also

$$V^2 = \left(\frac{dR}{dt}\right)^2 + R^2\left(\frac{d\psi}{dt}\right)^2$$

or

$$V^2 = \frac{p^2(e^2 + 1 - 2e\cos\psi)}{R^2(1 - e\cos\psi)^2}$$

and neglecting e^2

$$V^2 = \frac{p^2(1 - 2e\cos\psi)}{R^2(1 - e\cos\psi)^2} = \frac{g_0^2 R_0^4(1 - 2e\cos\psi)}{p^2}$$

therefore

$$V = \frac{g_0 R_0^2}{p}\sqrt{1 - 2e\cos\psi}$$

Since e is small, the radical may be expanded, and again neglecting second order terms,

$$V = \frac{g_0 R_0^2}{p}(1 - e\cos\psi) \qquad (24)$$

This is the approximate expression for V at various positions in the elliptical orbit provided p and e are known. The values of p and e can be determined as follows:

At point 2, $R = R_2$, $\psi = 0$, thus

$$R_2 = \frac{p^2}{g_0 R_0^2 (1 - e)}$$

and

$$R_1 = \frac{p^2}{g_0 R_0^2 (1 - e\cos\psi_1)}$$

so

$$\frac{R_2}{R_1} = \frac{1 - e\cos\psi_1}{1 - e}$$

or

$$e = \frac{R_2 - R_1}{(R_2 - R_1) + R_1(1 - \cos\psi_1)} = \frac{\delta R}{\delta R + R_1(1 - \cos\psi_1)}$$

Since ψ_1 lies between 90° and 180°, δR is small compared to $R_1(1 - \cos\psi_1)$; hence

$$e = \frac{\delta R}{R_1(1 - \cos\psi_1)} \qquad (25)$$

Since p is constant,

$$p = \sqrt{g_0 R_0^2 (1 - e) R_2} = \sqrt{g_0 R_0^2 R_1 \left(1 + \frac{\delta R}{R_1}\right)(1 - e)}$$

Here again $\frac{\delta R}{R_1}$ and e are small, hence the radical may be expanded. Neglecting second order terms,

$$p = \sqrt{g_0 R_0^2 R_1}\left(1 + \frac{1}{2}\frac{\delta R}{R_1} - \frac{1}{2} e\right)$$

and putting in Eq. (25) for e,

$$p = \sqrt{g_0 R_0^2 R_1}\left(1 - \frac{\delta R \cos\psi_1}{2R_1(1 - \cos\psi_1)}\right) \qquad (26)$$

Eqs. (25) and (26) give the values of e and p for given R_1 and Ψ_1. In order to bring this vehicle into such an orbit, velocity increments δV_1 and δV_2 must be imparted at points 1 and 2, respectively, of the elliptical orbit. These increments can be determined by using Eqs. (24), (25) and (26).

Let

$$\delta V_1 = V_1 - V_{1_S} \tag{27}$$

$$= \frac{g_0 R_0^2}{\sqrt{g_0 R_0^2 R_1}} \left\{ \frac{1 - \dfrac{\delta R \cos \Psi_1}{R_1(1-\cos\Psi_1)}}{1 - \dfrac{\delta R \cos\Psi_1}{2R_1(1-\cos\Psi_1)}} \right\} - \sqrt{\frac{g_0 R_0^2}{R_1}}$$

Neglecting second order terms, this reduces to

$$\delta V_1 = \frac{\delta R}{2R_1}\sqrt{\frac{g_0 R_0^2}{R_1}}\left(\frac{-\cos\Psi_1}{1-\cos\Psi_1}\right)$$

or

$$\delta V_1 = \frac{\delta R}{2R_1} V_{1_S}\left(\frac{-\cos\Psi_1}{1-\cos\Psi_1}\right) . \tag{28}$$

Similarly,

$$\delta V_2 = V_2 - V_{2_S}$$ and since $\Psi = 0$ at point 2 (29)

$$\delta V_2 = \sqrt{\frac{g_0 R_0^2}{R_2}} - \frac{g_0 R_0^2}{p}(1-e)$$

which reduces to:

$$\delta V_2 = \frac{\delta R}{2R_1} V_{1_S}\left(\frac{1}{1-\cos\Psi_1}\right) \tag{30}$$

It is interesting to note that the total velocity increment, $\delta V_1 + \delta V_2$, is independent of ψ_1:

$$\delta V = \delta V_1 + \delta V_2 = \frac{\delta R}{2R_1} V_{1s} \left(\frac{1}{1-\cos\psi_1} + \frac{-\cos\psi_1}{1-\cos\psi_1} \right)$$

or

$$\delta V = \frac{\delta R}{2R_1} V_{1s} . \qquad (31)$$

Thus the altitude reached by this method depends only upon the total velocity increment, $\delta V_1 + \delta V_2$; or, more basically, upon the amount of propellant remaining after the missile enters the elliptical orbit.

We can closely approximate the final altitude as follows:

a) Let ν correspond to total amount of fuel.

b) Let ν_1 correspond to minimum fuel required to reach an altitude h_1 with stable orbital velocity (obtain ν_1 from Fig. 6).

c) Obtain t_1 = burning time to h_1 from Fig. 6.

d) Since δR is independent of ψ_1 we need consider only the total velocity increment $\delta V = \delta V_1 + \delta V_2$.

Thus

$$\delta V = a \times \delta t$$

where

$$\delta t = t_1 \frac{\nu - \nu_1}{\nu_1} = t_1 \frac{\delta \nu}{\nu_1}$$

$$a = \frac{I_{h_1} (\nu_1 + \frac{1}{2}\delta\nu) \times g_0 R_0^2}{(t_1 + \frac{1}{2}\delta t)(1 - \nu_1 - \frac{1}{2}\delta\nu) R_1^2} \; * \qquad (32)$$

---

*In this computation, an average value of I_{h_1} was used corresponding to the altitude h_1 . See Fig. 13.

e) $\delta R = 2 R_1 \frac{\delta V}{V_{1_S}}$

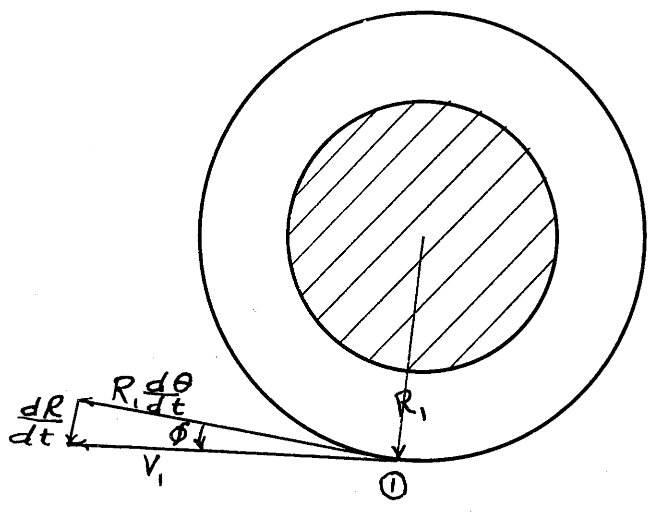
where V_{1_S} is obtained from Eq. (1)

f) Finally, $h_2 = h_1 + \delta R$.

If, at point 1, the velocity is tangent to a circular orbit through point 1, then the projection point is at the perhelion of the elliptical orbit. For this case:

$$\delta V_1 = \delta V_2 = \frac{\delta R}{4 R_1} V_{1_S} . \qquad (33)$$

If the velocity of projection is inclined by a small angle, ϕ, from the tangent to the circular orbit at point 1, the values of γ_1 and eccentricity, e, of the orbit can be determined for any required value of δR. By observing the component of velocity tangent to the circular orbit at point 1 in the following figure, one may write



$$\cos \phi = \frac{R_1 \frac{d\theta}{dt}}{V_1} = \frac{p}{R_1 V_1} \qquad (34)$$

The first order approximations of the preceding paragraphs give $p = R_1 V_1$, hence $\cos \phi$ must be evaluated to a second order of approximation. This gives:

$$\cos^2\phi = \frac{(1-e\cos\psi_1)^2}{e^2\sin^2\psi_1 + (1-e\cos\psi_1)^2}$$

$$= (1-e\cos\psi_1)^2(1+2e\cos\psi_1 - e^2 + 4e^2\cos^2\psi_1)$$

Neglecting terms of the third order and higher,

$$\cos^2\phi = 1 - e^2(1-\cos^2\psi_1)$$

or

$$\sin\psi_1 = \frac{1}{e}\sin\phi \qquad (35)$$

Solving Eqs. (25) and (35) for ψ_1 in terms of the known quantities ϕ, R_1, and the required δR:

$$\tan\frac{\psi_1}{2} = \frac{\delta R}{R_1}\csc\phi .$$

The values of e, δV_1, and δV_2 can be obtained by substituting ψ_1 from Eq. (36) into Eqs. (25), (28) and (30).

This approximate solution of the elliptic orbit is based primarily on the assumption that the eccentricity is small. On this basis, it was found that the gain in altitude does not depend upon the eccentricity, but only upon the velocity increments δV_1 and δV_2.

If δV_1 should be negative, i.e., V_1 is less than V_{1s} and if ϕ is fairly large, the eccentricity of the orbit is no longer small. The errors introduced by assuming δR independent of e under these conditions, though perhaps not negligible, are not large. This consideration leads to at least a partial solution to the drag problem during the elliptic part of the trajectory.

Heretofore, it has been assumed that the missile is brought into stable orbital conditions just before projection into the elliptical orbit. If, instead, the rocket motor is cut off some time before stable orbital velocity is reached, the missile may have a vertical component of velocity sufficient to carry it quickly above the region where drag is important. The sum of the velocity increments, $\delta V_1 + \delta V_2$, is unchanged in this case and hence the gain in altitude is still given by Eq. (31), subject only to the errors introduced by assuming δR independent of e.

The calculations for the elliptical trajectory are summarized by Fig. 7. It is noted that an appreciable gain in altitude is possible by reserving a small portion of the fuel for accomplishing the elliptical orbit. Since atmospheric drag was neglected, the curves of Fig. 7 represent an optimum. The point of projection must be high enough that drag does not seriously reduce the velocity during the free flight in the elliptic orbit, since a small reduction in velocity during this stage may greatly reduce the final altitude.

CONCLUSIONS.

The possibility of launching an unboosted rocket missile into a circular orbit around the earth depends very critically upon the propellant-gross weight ratio. The curves of Fig. 7 give approximately the maximum altitude of the stable orbit for various values of ν , regardless of the method of launching. The value of $\nu = 0.89$ seems to be the minimum value of propellant gross weight ratio with which a single stage missile can be successfully launched into a stable orbit. If ν can be raised to .90, the possibility becomes very much more certain. In view of the difficulty in obtaining these high propellant-gross weight ratios, it is somewhat doubtful that a single stage rocket can be launched into the circular orbit with the present fuels.

The analysis in Part II of this report can be extended directly to trajectories for launching a missile into an elliptic orbit of larger eccentricity by simply setting the boundary condition on θ at some angle less than 90°. This will allow choice of a larger control factor, k , and result in higher altitudes at the end of the first launching stage. By utilizing an elliptic orbit of larger eccentricity, the effect of drag will be minimized and the altitudes given by Fig. 7 can be more closely approached.

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TABLE I

	k	X MILES	h MILES	δV_p ft/sec	ν'	t'_p	a'_p g's	$a_{t=0}$ g's
	1	123.42	7.65	1995	.8643	76.61	36.53	3.72
$\nu = .88$	2	117.67	13.87	1572	.8687	77.00	38.28	"
$t_p = 78$	3	112.09	18.91	1125	.8722	77.36	39.20	"
	4	106.48	23.00	629	.8772	77.76	39.77	"
	5	101.57	26.57	136	.8815	78.13	40.03	"
	1	106.22	14.13	2434	.8679	101.01	30.42	2.84
$\nu = .885$	2	148.19	23.95	1755	.8739	101.71	31.53	"
$t_p = 103$	3	138.41	31.37	1148	.8790	102.31	31.89	"
	4	130.61	37.26	557	.8838	102.87	32.05	"
	5	122.72	42.12	-15	.8885	103.40	32.05	"
	1	192.79	18.81	2819	.8715	123.38	26.72	2.33
$\nu = .890$	2	176.58	31.53	2017	.8782	124.33	27.56	"
$t_p = 126$	3	162.81	40.88	1237	.8842	125.18	27.65	"
	4	150.72	48.15	430	.8904	126.06	27.52	"
	5	140.38	54.06	-335	.8965	126.92	27.50	"
	1	218.50	24.68	3065	.8762	143.92	24.57	2.01
$\nu = .895$	2	197.37	40.53	2138	.8831	145.05	24.86	"
$t_p = 147$	3	180.15	51.08	1254	.8896	146.11	24.79	"
	4	166.04	59.62	416	.8959	147.15	24.77	"
	5	152.90	66.16	-525	.9029	148.29	24.69	"
	1	236.13	29.58	3288	.8815	164.54	23.36	1.77
$\nu = .900$	2	215.90	48.67	2243	.8882	165.80	22.93	"
$t_p = 168$	3	194.42	61.20	1268	.8951	167.08	22.83	"
	4	176.84	70.47	272	.9061	169.13	22.76	"
	5	166.15	78.90	-459	.9073	169.36	22.67	"

TABLE II $\quad V = .880 \quad t_p = 78$ sec.

t sec.	k=1	k=2	k=3	k=4	k=5	k=1	k=2	k=3	k=4	k=5
	Altitude, h, (Miles)					Distance, X, (Miles)				
0	0	0	0	0	0	0	0	0	0	0
15.6	1.08	1.7	2.00	2.12	2.17	2.05	1.44	.92	.56	.38
46.8	5.36	9.5	12.50	14.8	16.70	29.60	26.40	23.40	20.67	18.38
78.0	7.65	13.87	18.91	23.0	26.57	123.42	117.67	112.09	106.48	101.57

t sec.	k=1	k=2	k=3	k=4	k=5	k=1	k=2	k=3	k=4	k=5
	θ (Degrees)					α (Degrees)				
0	0	0	0	0	0		0	0	0	0
7.8	57.6	33.6	18.9	10.4	5.7		6.1	7.1	6.0	4.4
15.6	69.2	51.1	36.5	25.7	17.9		7.8	11.9	13.1	12.3
23.4	75.9	62.7	50.7	40.6	32.2		7.9	13.9	17.6	19.3
31.2	80.3	70.9	62.0	53.8	46.2		7.4	14.0	19.3	23.3
39.0	83.3	76.7	70.2	63.5	58.2		6.6	12.9	18.6	23.7
46.8	85.5	81.0	76.6	72.2	68.1		5.7	11.2	16.5	21.7
54.6	87.2	84.3	81.3	78.7	76.5		4.6	9.2	13.8	18.3
62.4	88.4	86.9	85.2	83.5	82.1		3.6	7.2	10.8	14.3
70.2	89.3	88.7	87.9	87.3	86.7		2.5	5.0	7.6	10.1
78.0	90.0	90.0	90.0	90.0	90.0		1.4	2.9	4.4	5.8

t sec.	k=1	k=2	k=3	k=4	k=5	k=1	k=2	k=3	k=4	k=5
	Velocity, $V + \delta V$, (ft/sec)					Control Acceleration (g's)				
0	0	0	0	0	0		0	0	0	0
7.8	734	734	734	734	734		.5534	.6466	.543	.399
15.6	1,798	1,661	1,599	1,581	1,577		.7776	1.190	1.300	1.229
31.2	4,506	4,232	4,032	3,889	3,809		.9451	1.766	2.421	2.895
46.8	8,547	8,187	7,865	7,493	7,298		.9878	1.946	2.858	3.709
62.4	14,865	14,460	14,054	13,618	13,208		.9984	1.993	2.982	3.962
78.0	27,319	26,806	26,360	25,863	25,307		1.000	2.000	3.000	4,000

TABLE II (Continued)

$\nu = .885$ $t_p = 103$ sec.

t (sec)	Altitude, h, (Miles) k=1	k=2	k=3	k=4	k=5	Distance, x, (Miles) k=1	k=2	k=3	k=4	k=5
0	0	0	0	0	0	0	0	0	0	0
20.6	1.78	2.4	2.58	2.63	2.75	2.14	1.13	.57	.29	.15
61.8	10.10	16.3	20.5	23.4	25.5	36.51	29.81	24.62	20.49	17.18
103.0	14.13	23.95	31.37	37.26	42.12	160.22	148.19	138.41	130.61	122.72

t (sec)	θ (Degrees) k=1	k=2	k=3	k=4	k=5	α (Degrees) k=1	k=2	k=3	k=4	k=5
0	0	0	0	0	0		0	0	0	0
10.3	44.5	19.0	7.8	3.2	1.4		4.7	3.9	2.4	1.3
20.6	59.5	36.2	21.2	12.2	7.0		7.7	9.5	8.3	6.3
30.9	68.4	49.6	34.8	24.1	16.5		8.9	13.4	14.4	13.4
41.2	74.4	59.8	47.2	36.7	28.2		8.8	15.2	18.7	21.6
51.5	78.4	67.4	57.1	47.9	39.9		8.2	15.1	20.3	23.3
61.8	83.9	77.8	71.9	66.2	60.7		7.3	14.3	20.9	26.9
72.1	86.2	82.4	78.5	74.8	71.1		6.0	11.9	17.8	23.6
82.4	87.9	85.7	83.6	81.5	79.4		4.6	9.2	14.1	18.6
92.7	89.1	88.2	87.4	86.4	85.6		3.2	6.4	9.7	13.0
103.0	90.0	90.0	90.0	90.0	90.0		1.8	3.6	5.5	7.3

t (sec)	Velocity, $V + \delta V$ (ft/sec) k=1	k=2	k=3	k=4	k=5	Control Acceleration (g's) k=1	k=2	k=3	k=4	k=5
0	0	0	0	0	0		0	0	0	0
10.3	660	660	660	660	660		.325	.272	.167	.091
20.6	1,674	1,485	1,451	1,447	1,448		.590	.722	.633	.507
41.2	4,321	3,911	3,718	3,636	3,614		.863	1.464	1.790	1.890
61.8	8,392	7,821	7,425	7,143	6,965		.978	1.903	2.753	3.780
82.4	14,823	14,173	13,634	13,148	12,717		.997	1.987	2.970	3.985
103.0	27,525	26,843	26,239	25,648	25,076		1.000	2.000	3.000	4.000

TABLE II (Continued)

$\nu = .890$ $t_p = 126$ sec.

t(sec)	Altitude, h, (Miles) k=1	k=2	k=3	k=4	k=5	Distance, x, (Miles) k=1	k=2	k=3	k=4	k=5
0	0	0	0	0	0	0	0	0	0	0
25.2	2.14	2.73	2.87	2.91	2.92	2.14	1.00	.48	.22	.11
75.6	12.72	20.15	24.89	28.15	30.48	41.90	33.49	26.98	21.91	17.93
126.0	18.81	31.53	40.88	48.15	54.06	192.79	176.58	162.81	150.72	140.38

t(sec)	θ (Degrees) k=1	k=2	k=3	k=4	k=5	α (Degrees) k=1	k=2	k=3	k=4	k=5
0	0	0	0	0	0		0	0	0	0
12.6	37.7	13.2	4.5	1.6	.7		4.0	2.7	1.4	2.1
25.2	56.4	32.1	17.6	9.5	5.1		8.5	9.6	7.9	5.6
37.8	65.4	44.9	29.7	19.4	12.6		10.1	14.2	14.3	12.4
50.4	73.2	57.8	44.6	33.9	25.5		10.6	17.7	21.3	21.9
63.0	78.5	67.4	57.1	48.0	39.9		9.9	18.3	24.7	28.8
75.6	82.3	74.8	67.4	60.6	54.0		8.8	16.9	24.2	30.6
88.2	85.3	80.4	75.7	71.1	66.6		7.2	13.7	21.1	27.8
100.8	87.3	84.7	82.1	79.4	76.8		5.6	11.1	16.6	22.3
113.4	88.9	87.7	86.6	85.6	84.4		3.8	7.7	11.6	15.5
126.0	90.0	90.0	90.0	90.0	90.0		2.2	2.1	6.4	8.6

t(sec)	Velocity, $V + \delta V$ (ft/sec) k=1	k=2	k=3	k=4	k=5	Control Acceleration (g's) k=1	k=2	k=3	k=4	k=5
0	0	0	0	0	0		0	0	0	0
12.6	591	591	591	591	591		.221	.158	.080	.116
25.2	1,491	1,344	1,319	1,316	1,316		.532	.604	.494	.354
50.4	4,093	3,666	3,463	3,388	3,374		.887	1.403	1.67	1.72
75.6	8,204	7,563	7,084	6,734	6,517		.965	1.847	2.61	3.24
100.8	14,730	13,971	13,282	12,636	12,077		.990	1.985	2.95	3.89
126.0	27,805	27,003	26,223	25,416	24,651		1.000	2.000	3.00	4.00

TABLE II (Continued)

$\gamma = .895$ $t_p = 147$ sec.

t (sec.)	k=1	k=2	k=3	k=4	k=5	k=1	k=2	k=3	k=4	k=5
	Altitude, h, (Miles)					Distance, x, (Miles)				
0	0	0	0	0	0	0	0	0	0	0
29.4	2.59	3.14	3.06	3.07	3.08	1.78	.63	.23	.09	.04
88.2	16.40	25.21	29.85	32.90	35.30	44.50	33.80	26.10	20.45	16.16
147.0	24.68	40.53	51.08	59.62	66.16	218.50	197.37	180.15	166.04	152.90
	θ (Degrees)					α (Degrees)				
0	0	0	0	0	0		0	0	0	0
14.7	27.1	6.7	1.7	.7	.6		2.4	1.1	.4	.6
29.8	46.8	21.2	9.2	4.1	1.9		6.7	5.9	3.8	2.4
44.1	60.4	37.4	22.2	13.0	7.7		10.0	12.5	11.2	8.7
58.8	69.7	51.8	37.4	26.6	18.7		11.4	17.7	19.7	18.8
73.5	76.1	62.8	51.1	40.9	32.5		11.1	19.6	25.1	27.7
88.2	80.5	71.6	62.9	54.9	47.6		9.8	18.9	26.4	32.4
102.9	84.2	78.6	73.1	67.8	62.4		8.3	16.3	22.0	31.3
117.6	86.9	83.8	80.6	77.5	74.5		6.3	12.7	18.9	25.3
132.3	88.7	87.4	86.1	84.8	83.5		4.3	8.7	13.1	17.6
147.0	90.0	90.0	90.0	90.0	90.0		2.4	2.4	7.1	9.4
	Velocity, $V + \delta V$ (ft/sec)					Control Acceleration (g's)				
0	0	0	0	0	0		0	0	0	0
14.7	529	529	529	529	529		.115	.055	.189	.006
29.4	1,318	1,210	1,191	1,199	1,200		.362	.322	.210	.130
58.8	3,793	3,366	3,201	3,158	3,160		.786	1.215	1.341	1.281
88.2	7,912	7,203	6,715	6,401	6,225		.949	1.781	2.456	2.95
117.6	14,516	13,648	12,883	12,257	11,593		.994	1.973	2.929	3.85
147.0	27,981	27,054	26,170	25,332	24,391		1.000	2.000	3.000	4.00

TABLE II (Continued)

$\nu = .900$ $t_0 = 168$ sec.

t (sec)	Altitude, h, (Miles) k=1	k=2	k=3	k=4	k=5	Distance, x, (Miles) k=1	k=2	k=3	k=4	k=5
0	0	0	0	0	0	0	0	0	0	0
33.6	2.77	3.10	3.12	3.13	3.14	1.38	.40	.11	.05	.02
100.8	19.21	29.10	34.15	37.21	39.60	43.96	33.25	23.75	18.50	14.35
168.0	29.58	48.67	61.20	70.47	78.90	236.13	215.90	194.39	176.84	166.15

t (sec)	θ (Degrees) k=1	k=2	k=3	k=4	k=5	α (Degrees) k=1	k=2	k=3	k=4	k=5
0	0	0	0	0	0		0	0	0	0
16.8	19.0	3.3	.9	.8	.7		1.3	.2	.3	.4
33.6	39.2	14.4	5.1	1.8	.9		5.3	3.8	2.0	1.0
50.4	54.7	30.0	15.8	8.2	4.3		9.3	10.1	7.9	5.5
67.2	65.9	45.5	30.4	20.0	13.0		11.7	15.5	16.9	14.8
84.0	73.5	58.3	45.3	34.6	26.2		12.1	20.3	24.6	25.6
100.8	79.1	68.6	58.8	49.9	42.0		11.0	20.5	28.0	33.2
117.6	83.3	74.4	70.2	64.1	58.1		9.2	18.0	26.3	33.9
134.4	86.4	82.7	79.1	75.5	72.0		7.1	14.1	21.2	28.3
151.2	88.5	87.0	85.5	84.0	82.6		4.8	9.7	14.6	19.6
168.0	90.0	90.0	90.0	90.0	90.0		2.6	5.1	7.7	10.3

t (sec)	Velocity, $V + \delta V$ (ft/sec) k=1	k=2	k=3	k=4	k=5	Control Acceleration (g's) k=1	k=2	k=3	k=4	k=5
0	0	0	0	0	0		0	0	0	0
16.8	469	469	469	469	469		.056	.009	.013	.018
33.6	1,090	1,082	1,078	1,078	1,078		.249	.181	.097	.050
67.2	3,400	3,105	2,983	2,959	2,981		.714	.938	1.025	.9
100.8	7,100	6,817	6,342	6,060	6,071		.932	1.705	2.285	2.68
134.4	13,900	13,279	12,458	11,731	11,331		.991	1.965	2.90	3.80
168.0	28,133	27,088	26,101	24,573	24,386		1.000	2.000	3.00	4.00

TABLE III

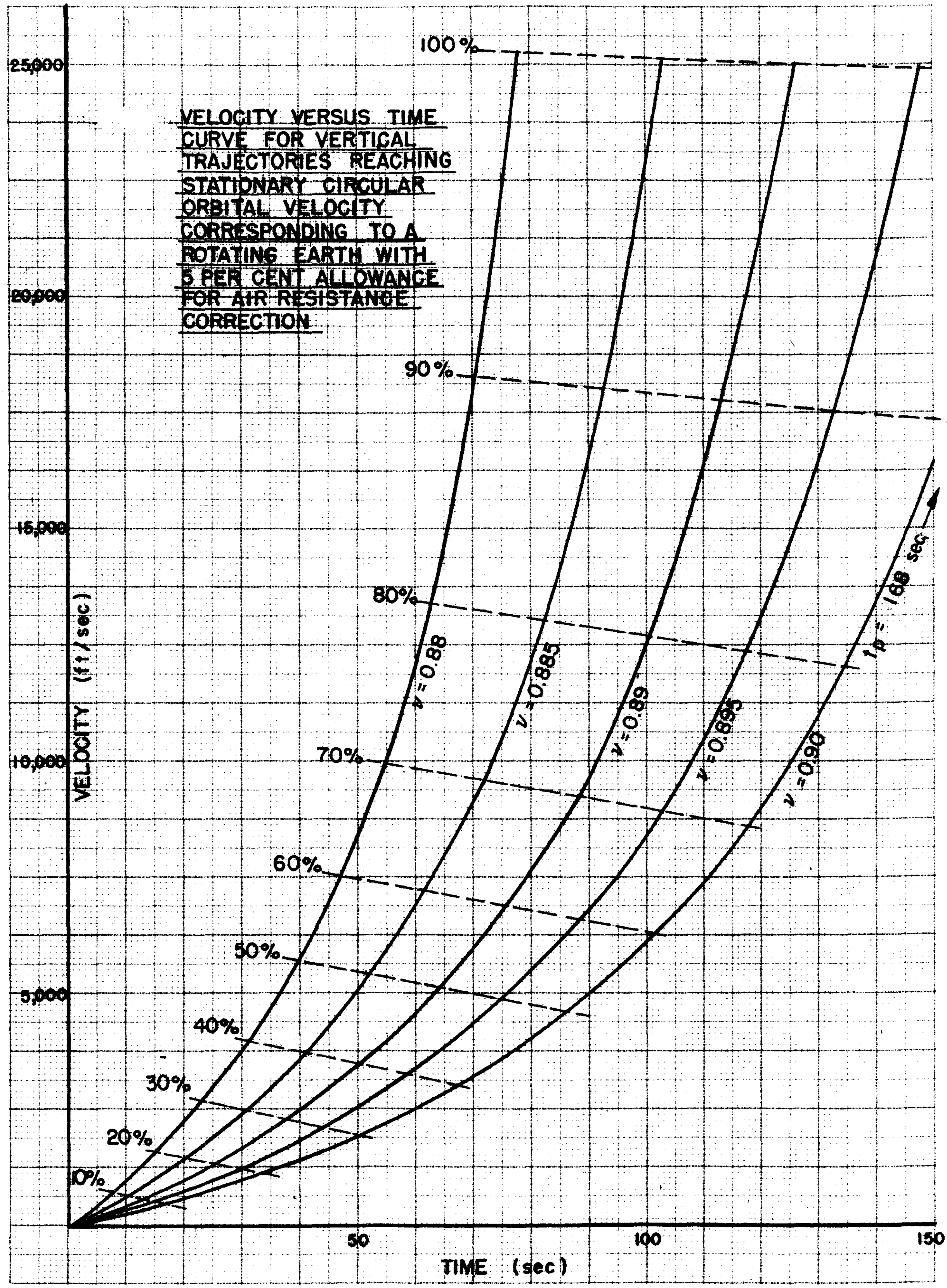
Altitude in Feet Lost Per Revolution Due to Air Resistance[1]

Altitude (miles)	R (miles)	ρ (slug/ft^3)		ΔR (ft/revolution)	
		Whipple[2]	Gutenberg[3]	Whipple[2]	Gutenberg[3]
50	4013	6.42×10^{-8}	1.81×10^{-7}	4.7×10^{7}	1.33×10^{8}
60	4023	8.8×10^{-9}	3.33×10^{-8}	6.5×10^{6}	2.46×10^{6}
70	4033	1.16×10^{-9}	5.23×10^{-9}	8.6×10^{5}	3.88×10^{6}
80	4043	1.66×10^{-10}	(9.3×10^{-10})	1.24×10^{5}	6.94×10^{5}
90	4053	(2.16×10^{-11})	(1.64×10^{-10})	1.62×10^{4}	1.23×10^{5}
100	4063	(309×10^{-12})	(2.92×10^{-11})	2.33×10^{3}	2.2×10^{4}
110	4073	(4.51×10^{-16})	(8.56×10^{-12})	3.41×10^{1}	6.48×10^{3}
120	4083	(6.89×10^{-15})	(9.3×10^{-13})	5.2	7.07×10^{2}

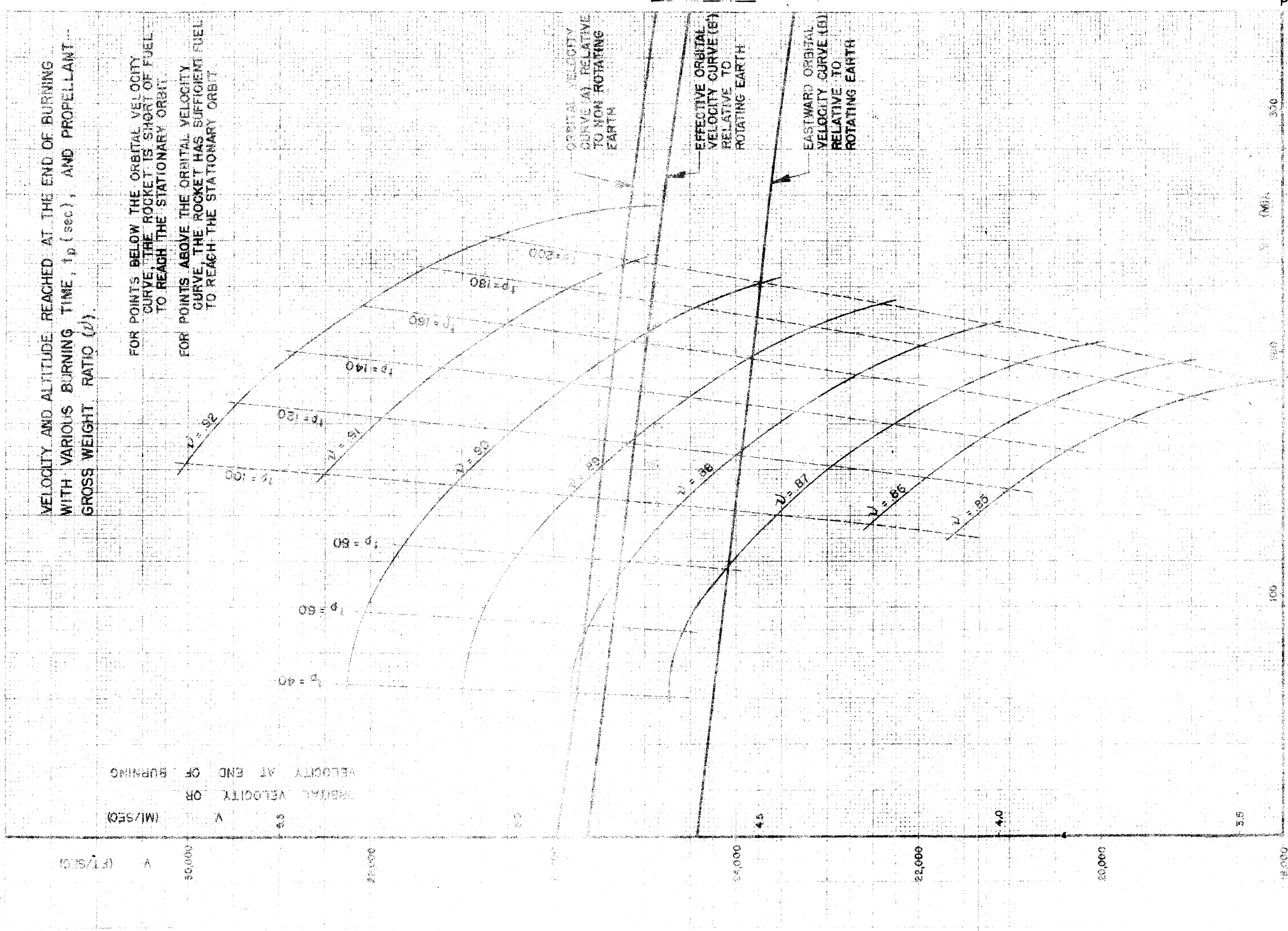
[1]The values in parentheses in the density columns are extrapolated from the values from the lower altitudes.

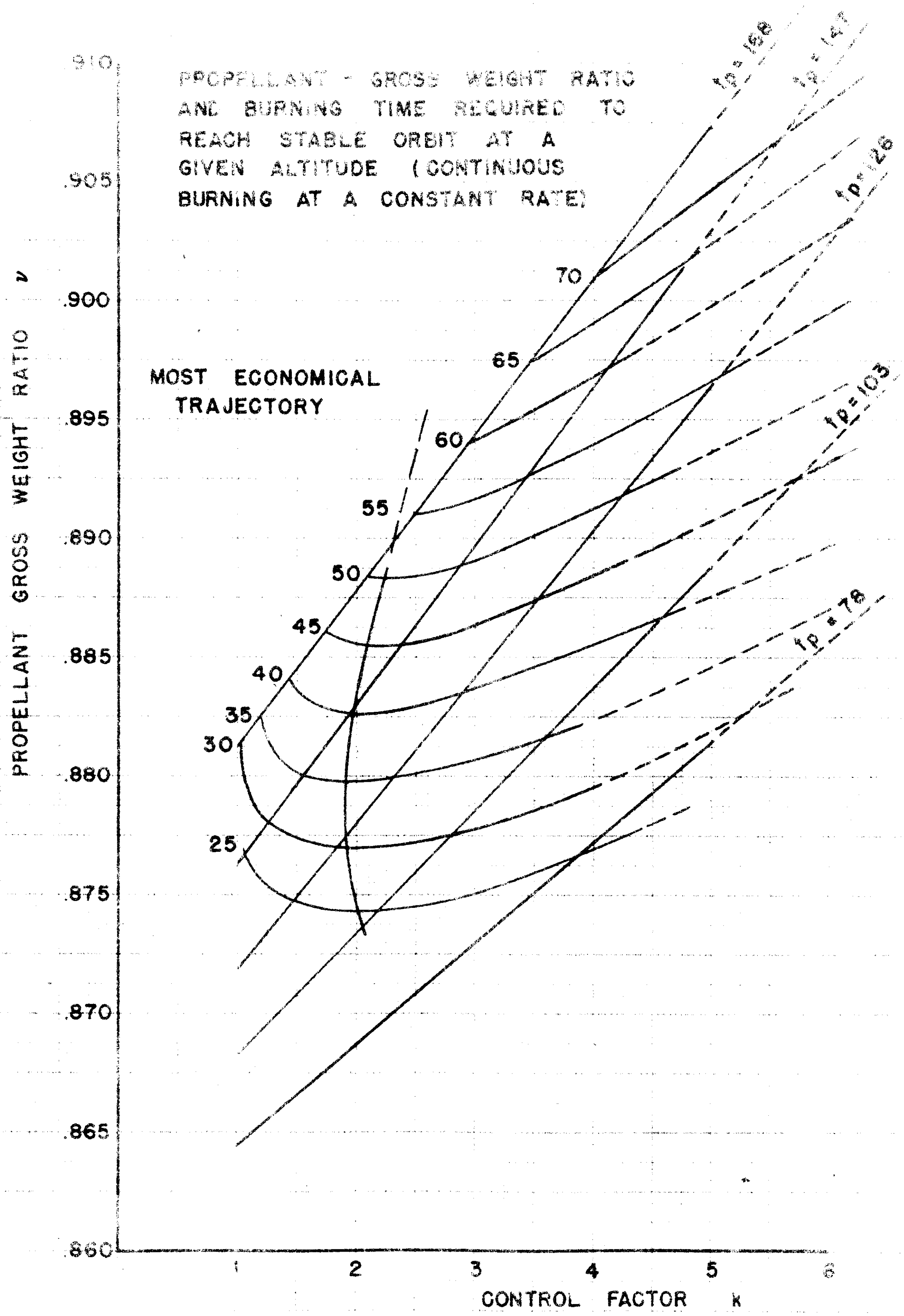
[2]Whipple, Review of Modern Physics, 1943.

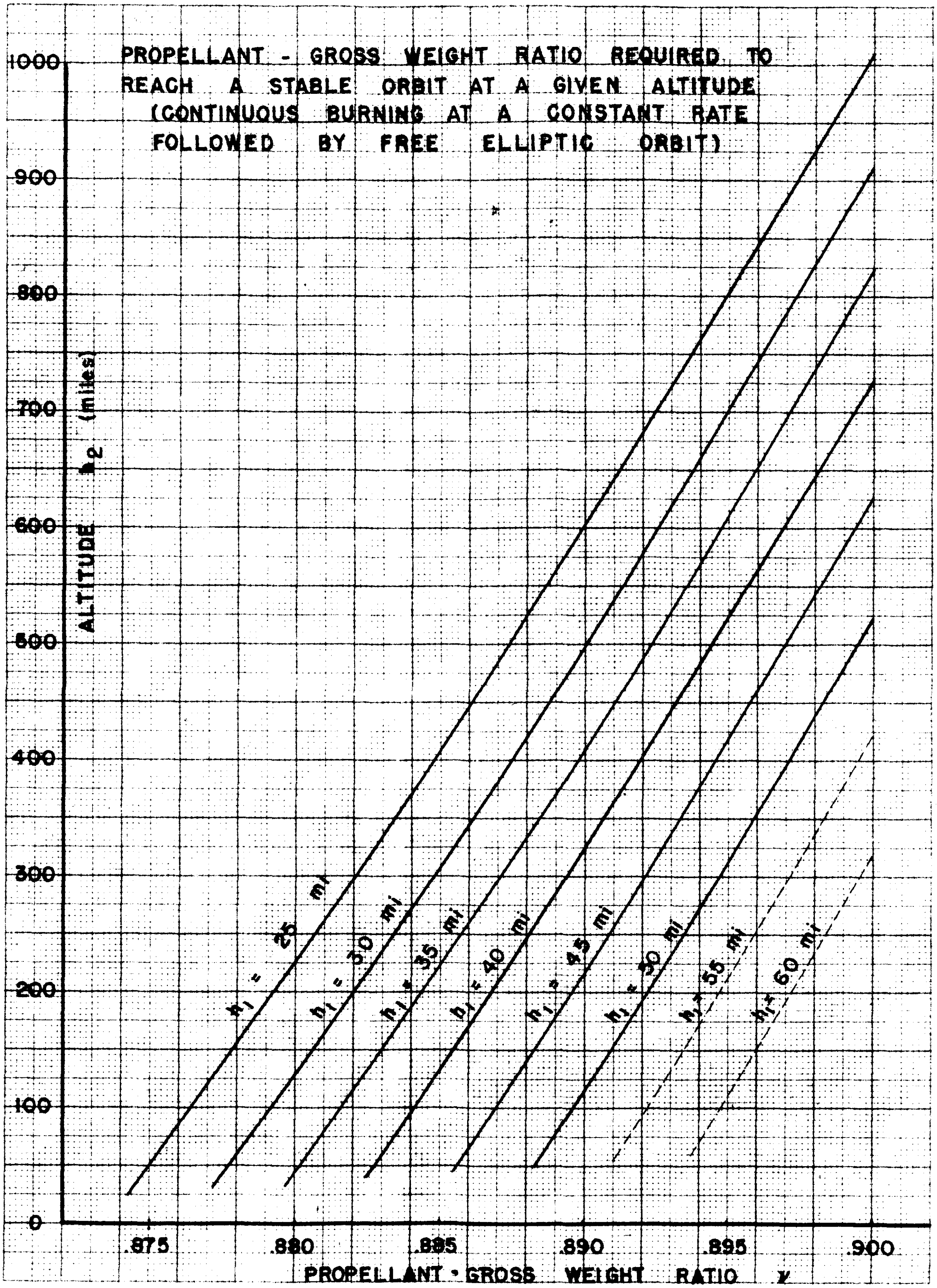
[3]Gutenberg, B., Abstracts of Papers Presented at the Guided Missile and Upper Atmosphere Symposium Held March 13-16, 1946, Jet Propulsion Laboratory, GALCIT, California Institute of Technology (To be published) Publication No. 3.

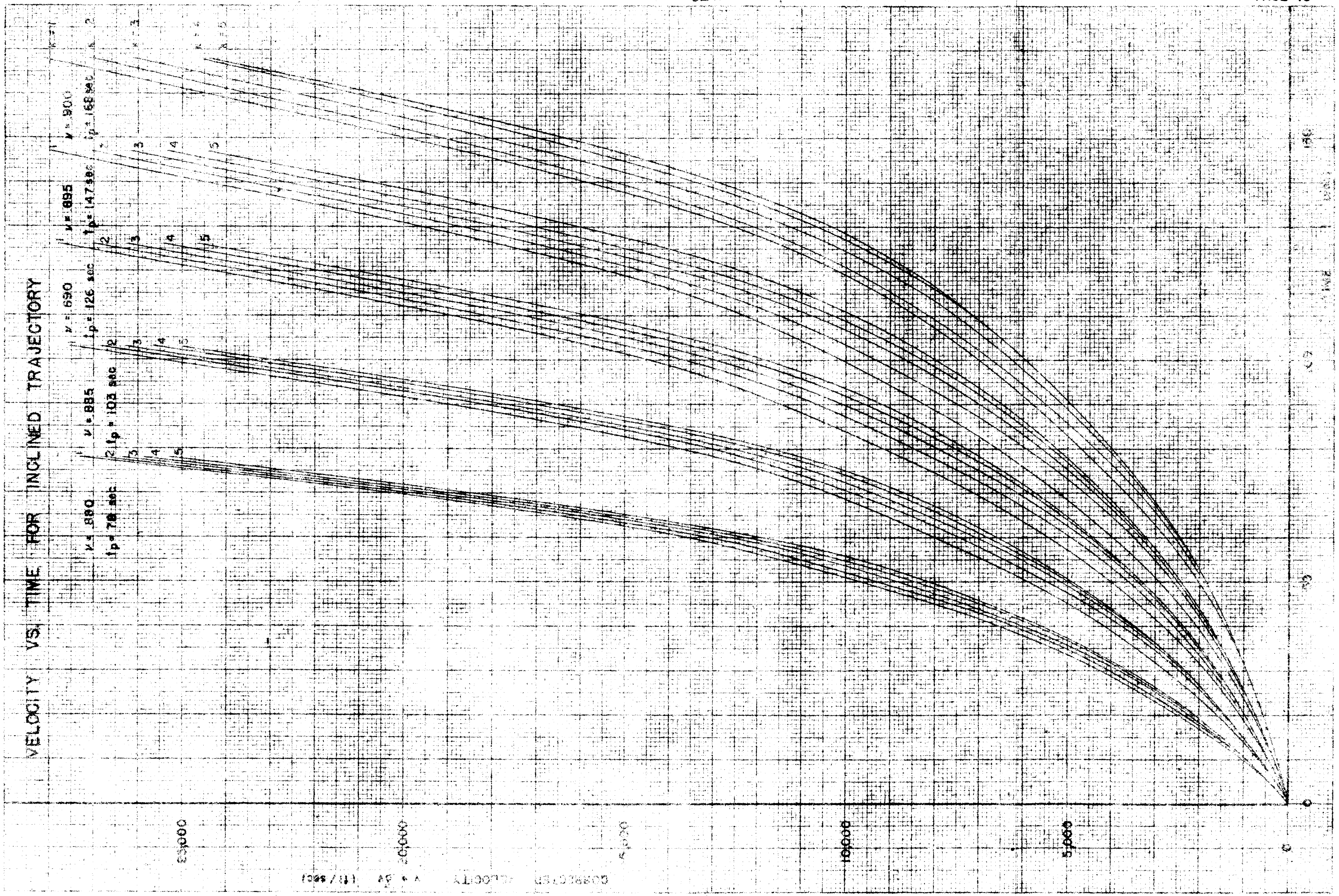


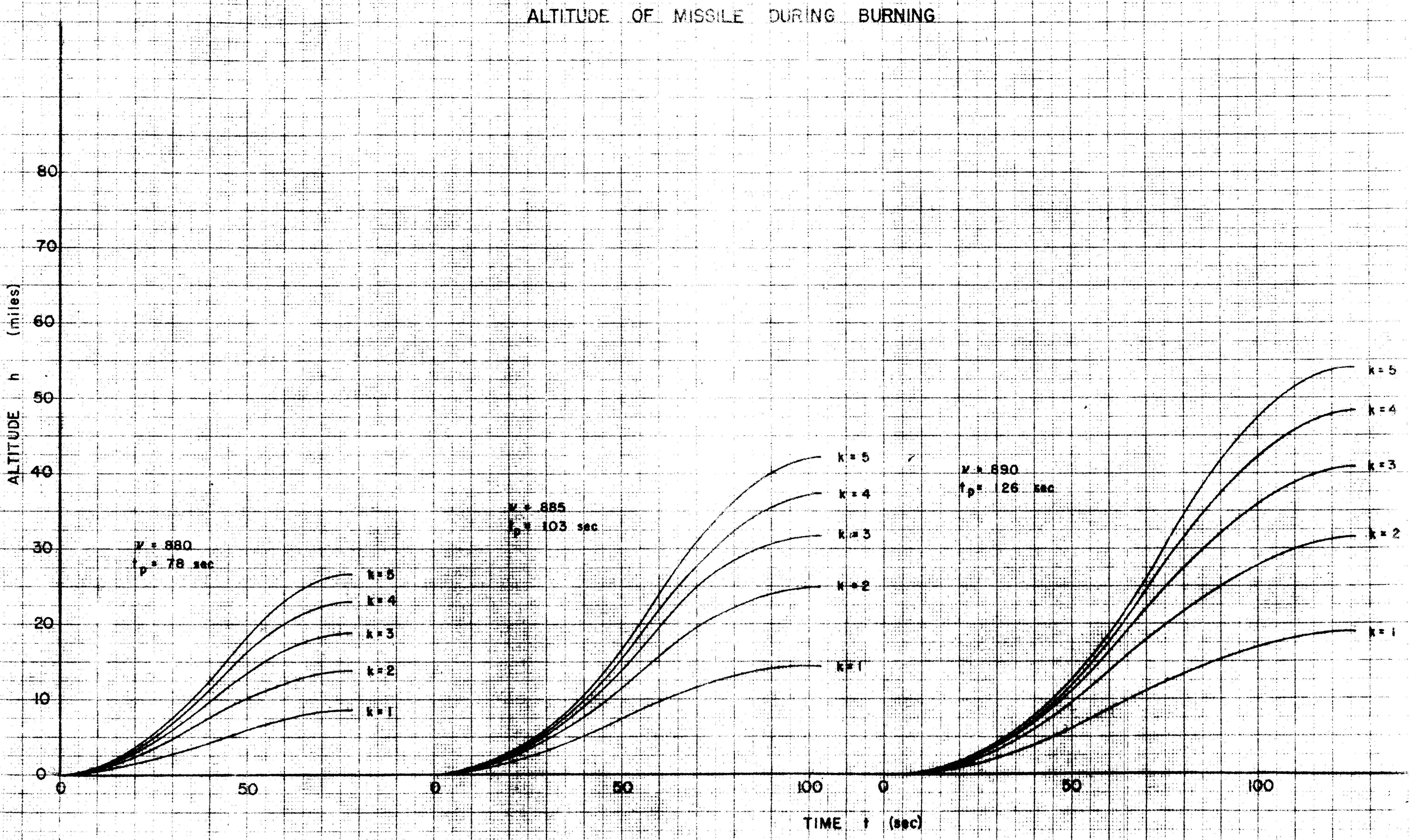
CLASSIFICATION cancelled


VELOCITY AND ALTITUDE REACHED AT THE END OF BURNING WITH VARIOUS BURNING TIME, t_p (sec), AND PROPELLANT-GROSS WEIGHT RATIO (ν).
FOR POINTS BELOW THE ORBITAL VELOCITY CURVE, THE ROCKET IS SHORT OF FUEL TO REACH THE STATIONARY ORBIT.
FOR POINTS ABOVE THE ORBITAL VELOCITY CURVE, THE ROCKET HAS SUFFICIENT FUEL TO REACH THE STATIONARY ORBIT.
ORBITAL VELOCITY CURVE (A) RELATIVE TO NON ROTATING EARTH
EFFECTIVE ORBITAL VELOCITY CURVE (B') RELATIVE TO ROTATING EARTH
EASTWARD ORBITAL VELOCITY CURVE (B) RELATIVE TO ROTATING EARTH
t_p = 200
t_p = 180
t_p = 160
t_p = 140
t_p = 120
t_p = 100
t_p = 80
t_p = 60
t_p = 40
ν = .92
ν = .91
ν = .90
ν = .89
ν = .88
ν = .87
ν = .86
ν = .85
ALTITUDE OF CIRCULAR ORBIT OR ALTITUDE REACHED AT END OF BURNING Y (FT) (MI)
500,000
1,000,000
1,500,000
100
200
300
ORBITAL VELOCITY OR VELOCITY AT END OF BURNING
V (MI/SEC)
5.5
5.0
4.5
4.0
3.5
V (FT/SEC)
30,000
28,000
26,000
24,000
22,000
20,000
18,000

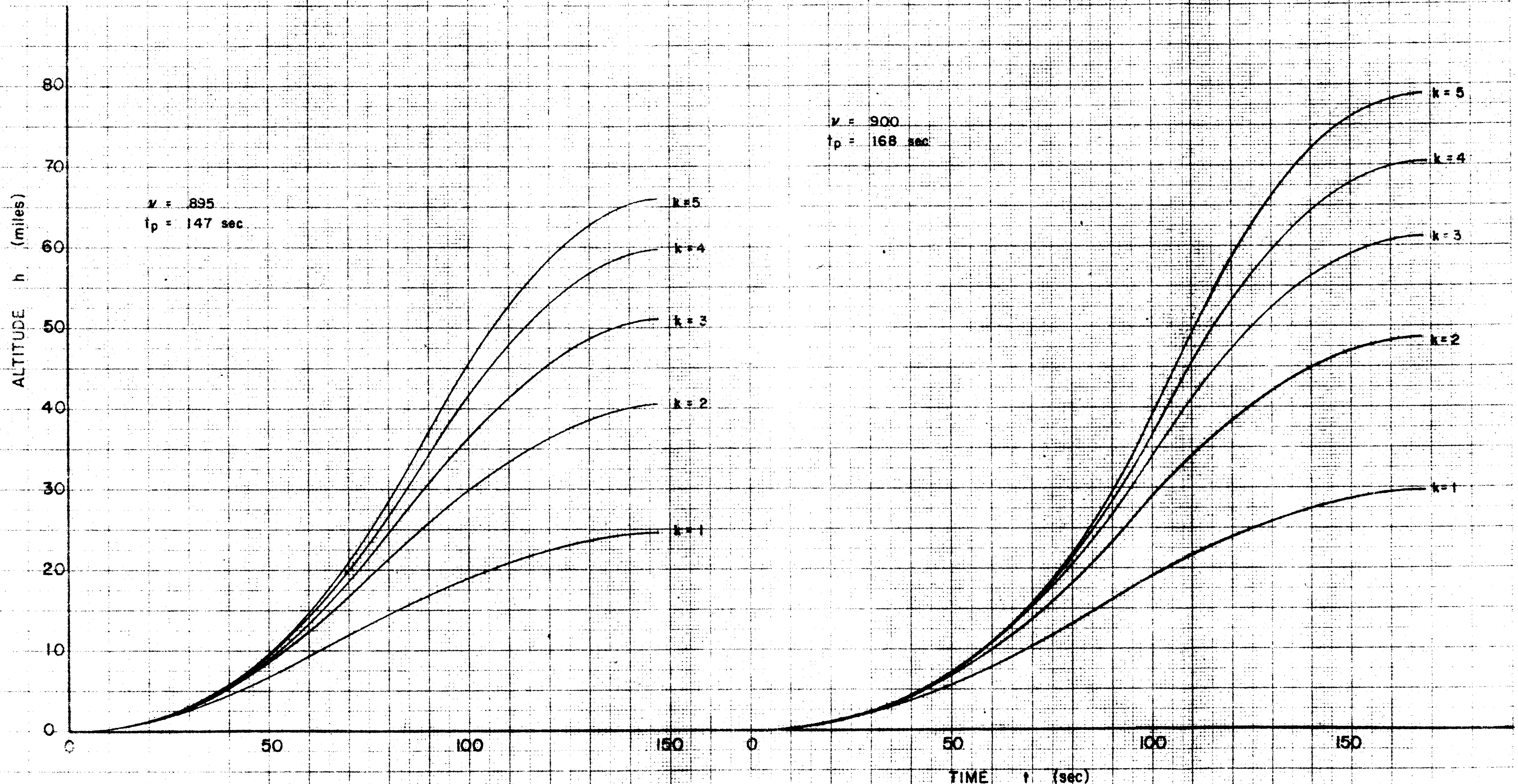

PROPELLANT - GROSS WEIGHT RATIO AND BURNING TIME REQUIRED TO REACH STABLE ORBIT AT A GIVEN ALTITUDE (CONTINUOUS BURNING AT A CONSTANT RATE)
MOST ECONOMICAL TRAJECTORY
PROPELLANT GROSS WEIGHT RATIO ν
CONTROL FACTOR k
.910
.905
.900
.895
.890
.885
.880
.875
.870
.865
.860
1
2
3
4
5
6
25
30
35
40
45
50
55
60
65
70
tp = 78
tp = 103
tp = 126
tp = 168


PROPELLANT - GROSS WEIGHT RATIO REQUIRED TO REACH A STABLE ORBIT AT A GIVEN ALTITUDE (CONTINUOUS BURNING AT A CONSTANT RATE FOLLOWED BY FREE ELLIPTIC ORBIT)
ALTITUDE h2 (miles)
1000
900
800
700
600
500
400
300
200
100
0
h1 = 25 mi
h1 = 30 mi
h1 = 35 mi
h1 = 40 mi
h1 = 45 mi
h1 = 50 mi
h1 = 55 mi
h1 = 60 mi
.875
.880
.885
.890
.895
.900
PROPELLANT - GROSS WEIGHT RATIO ν

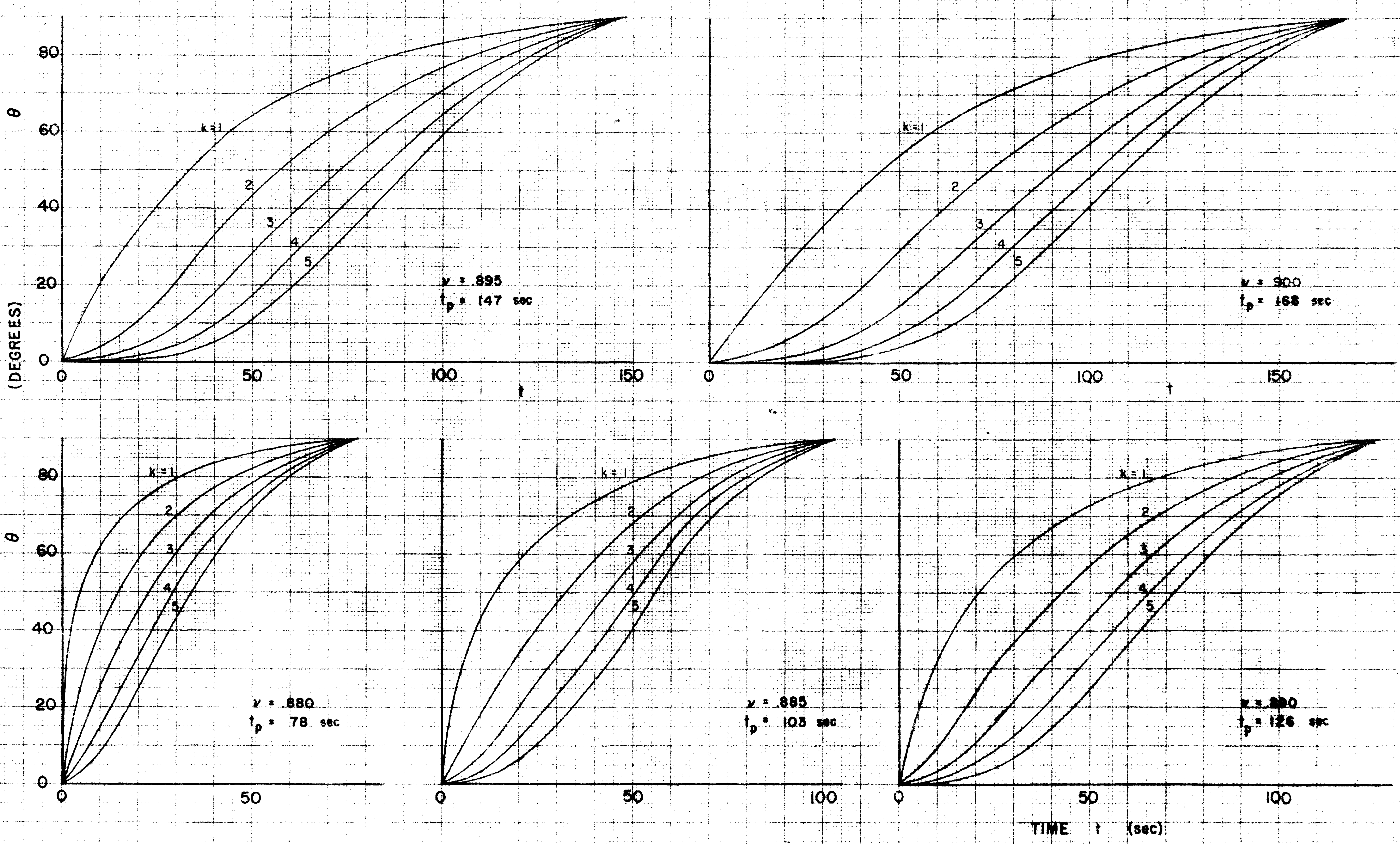

VELOCITY VS. TIME FOR INCLINED TRAJECTORY
ν = .880
t_p = 78 sec
ν = .885
t_p = 103 sec
ν = .890
t_p = 126 sec
ν = .895
t_p = 147 sec
ν = .900
t_p = 168 sec
25,000
20,000
15,000
10,000
5,000
0


ALTITUDE OF MISSILE DURING BURNING
ALTITUDE h (miles)
80
70
60
50
40
30
20
10
0
ν = 880
t_p = 78 sec
k = 5
k = 4
k = 3
k = 2
k = 1
ν = 885
t_p = 103 sec
k = 5
k = 4
k = 3
k = 2
k = 1
ν = 890
t_p = 126 sec
k = 5
k = 4
k = 3
k = 2
k = 1
0
50
0
50
100
0
50
100
TIME t (sec)

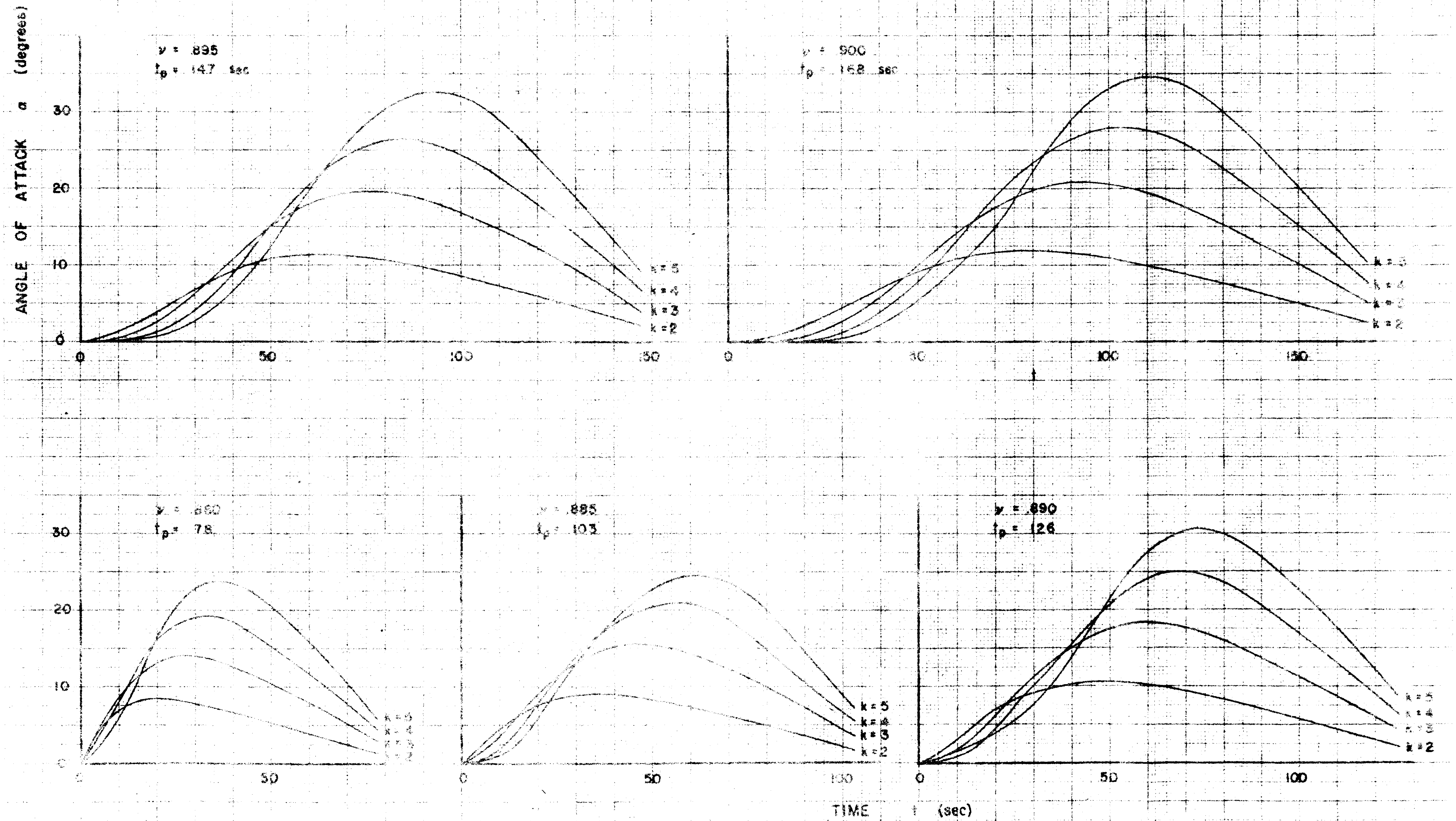
ALTITUDE VS TIME (CONT'D)

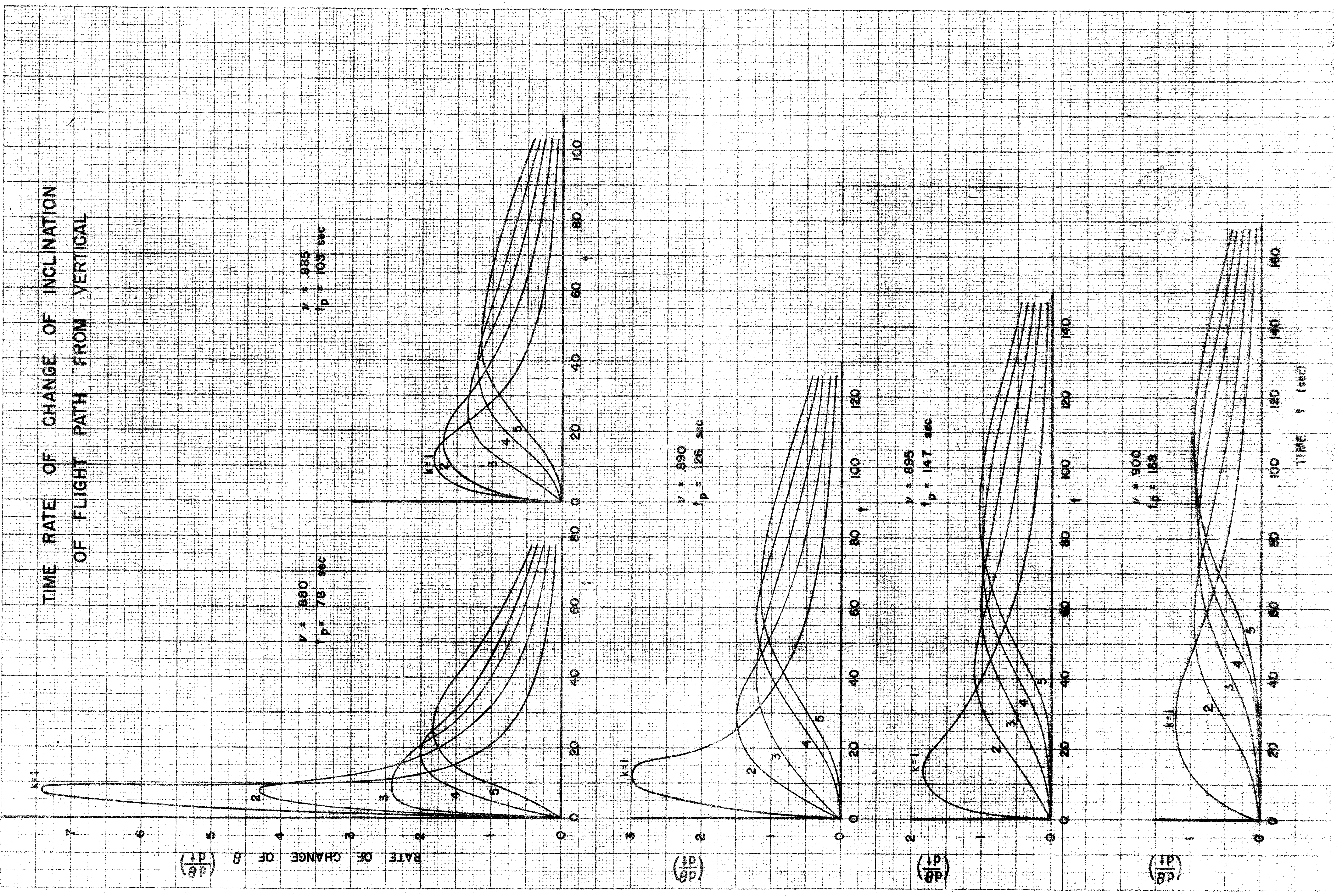


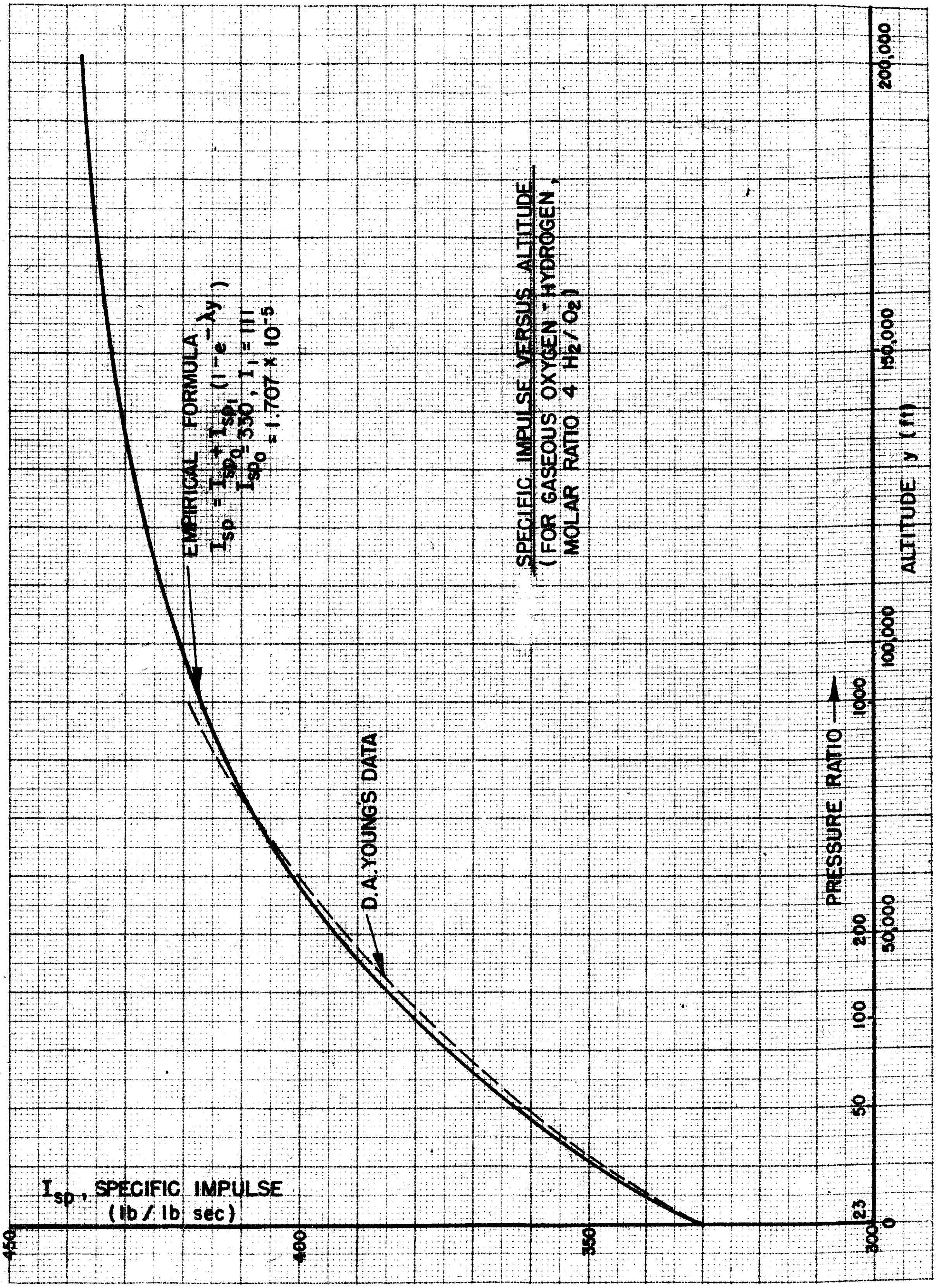
INCLINATION OF FLIGHT PATH FROM VERTICAL DURING BURNING

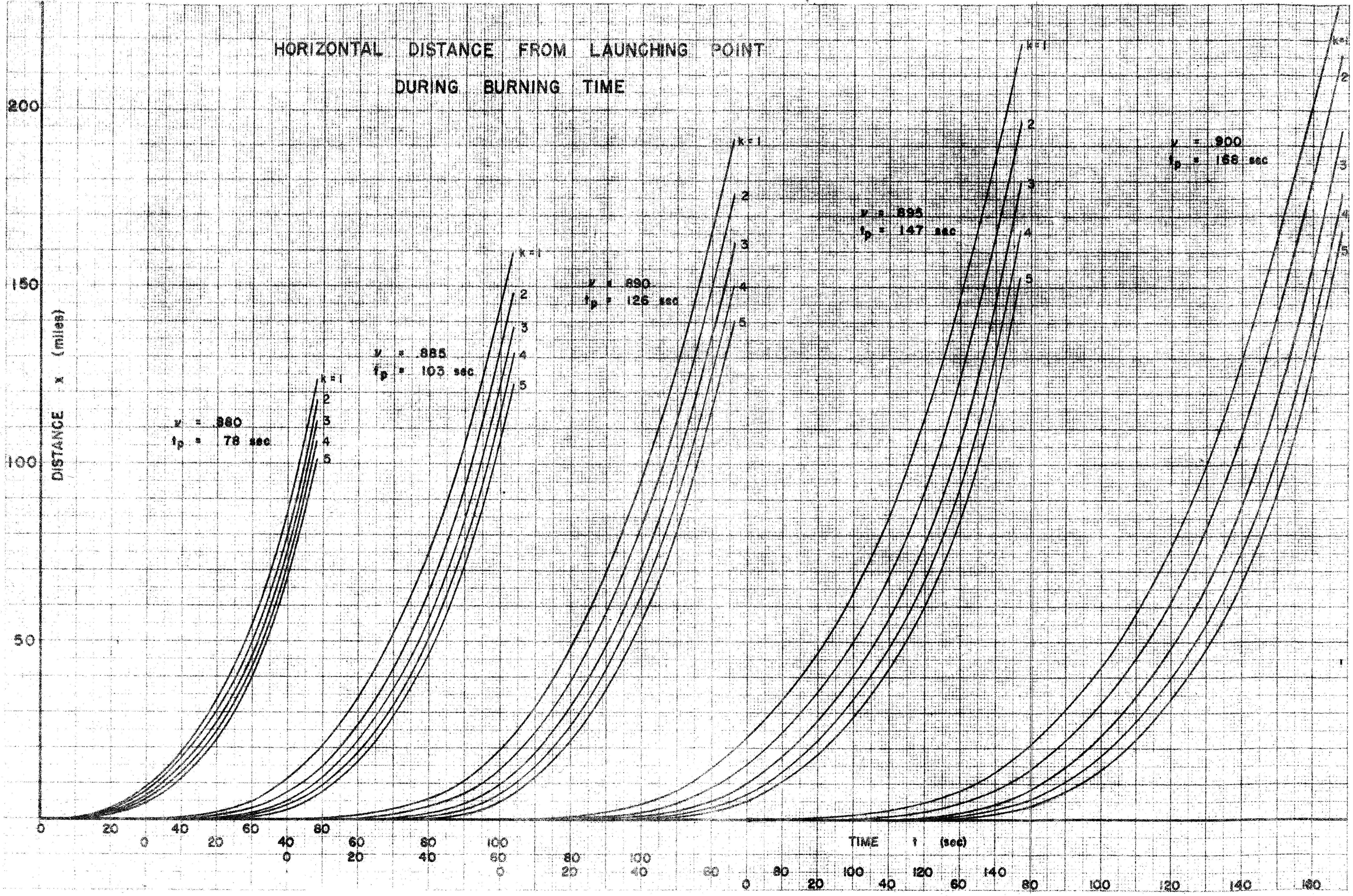


VARIATION DURING BURNING TIME OF ANGLE OF ATTACK OF MISSILE FROM LINE OF FLIGHT

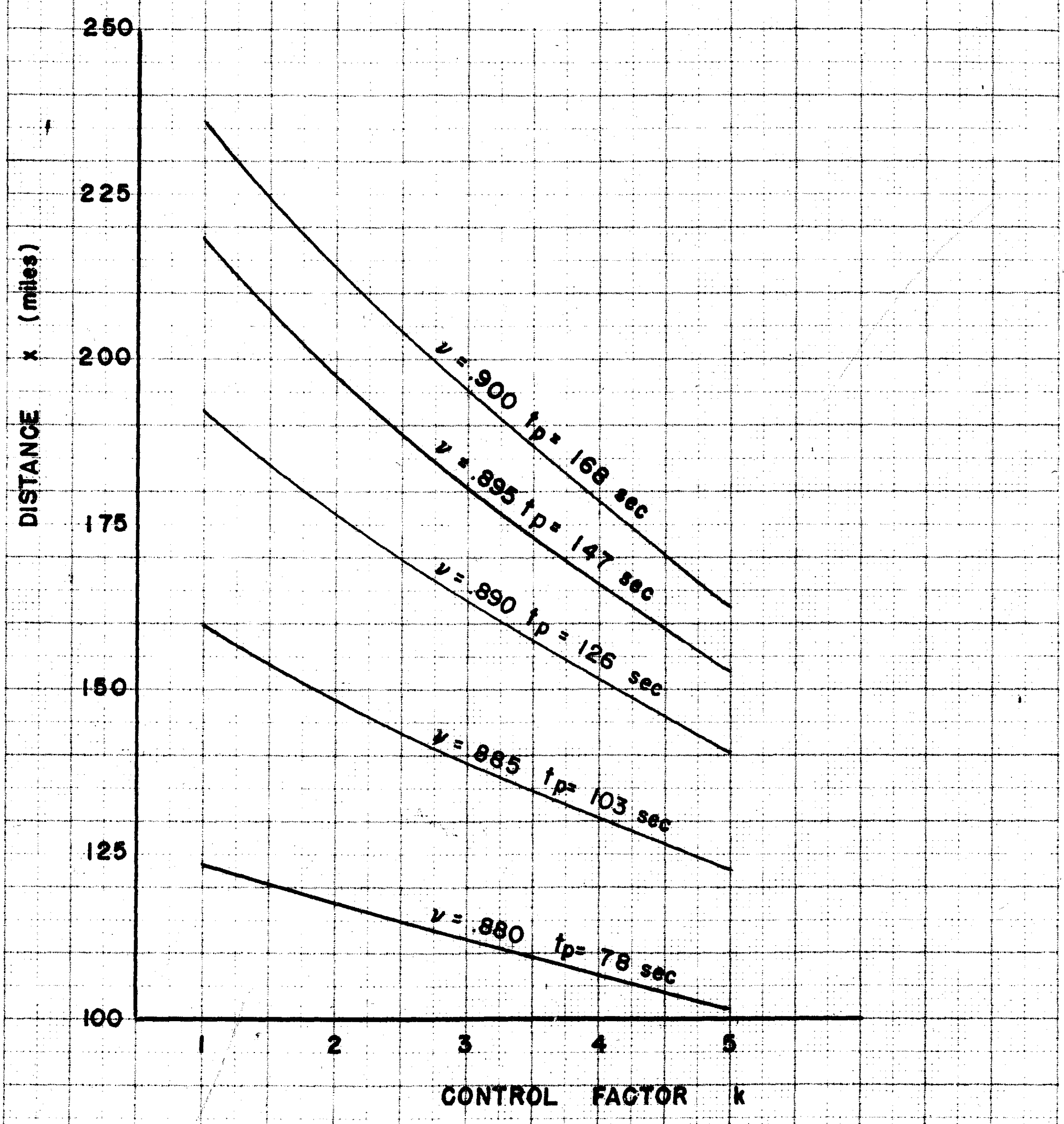


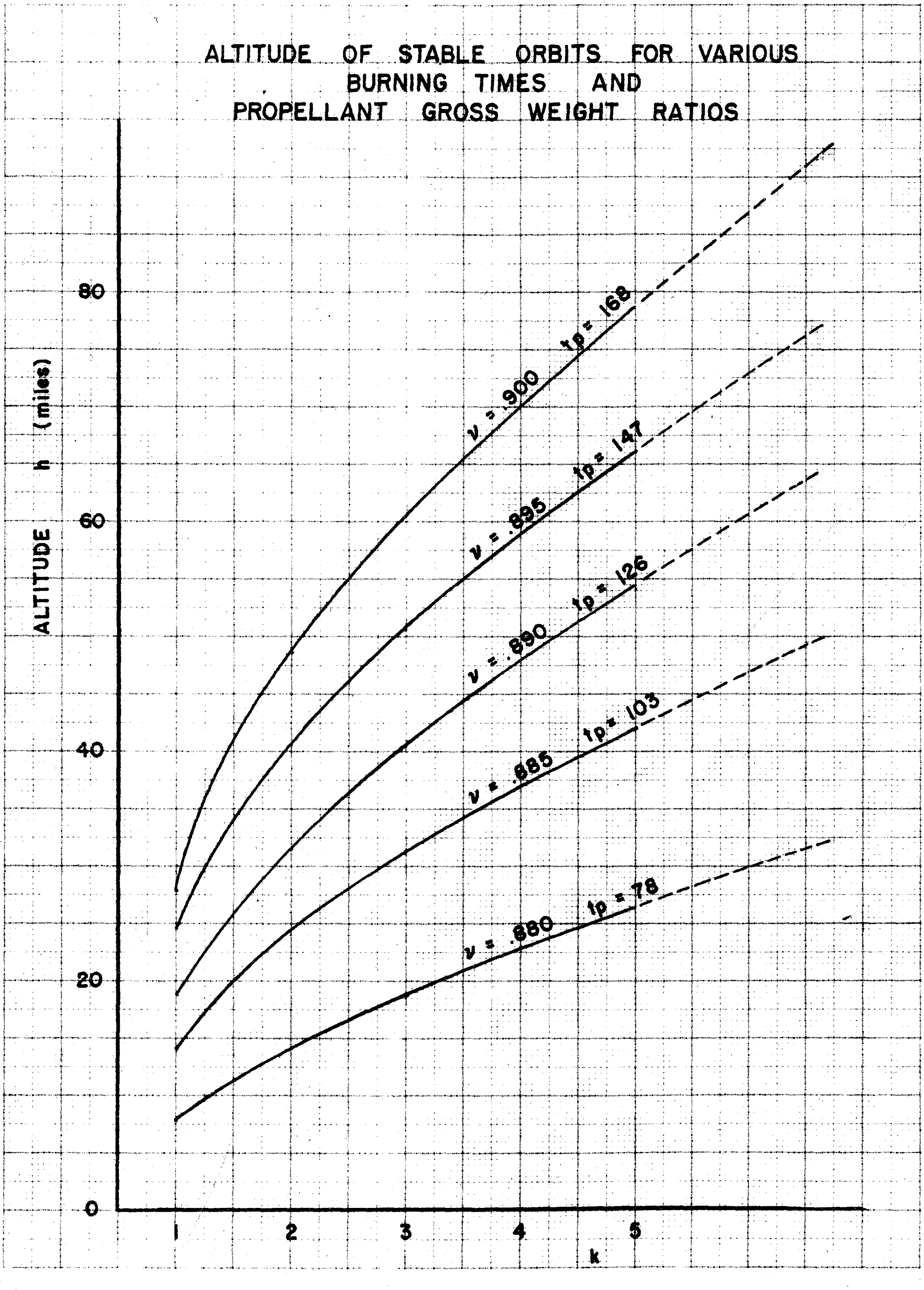

TIME RATE OF CHANGE OF INCLINATION
OF FLIGHT PATH FROM VERTICAL
ν = .880
t_p = 78 sec
ν = .885
t_p = 103 sec
ν = .890
t_p = 126 sec
ν = .895
t_p = 147 sec
ν = .900
t_p = 158
RATE OF CHANGE OF θ (dθ/dt)
TIME t (sec)
k=1
2
3
4
5


SPECIFIC IMPULSE VERSUS ALTITUDE
(FOR GASEOUS OXYGEN - HYDROGEN,
MOLAR RATIO 4 H2/O2)
EMPIRICAL FORMULA
Isp = Isp0 + Isp1 (1 - e^-λy)
Isp0 = 330, I1 = 111
= 1.707 x 10^-5
D.A. YOUNG'S DATA
Isp, SPECIFIC IMPULSE
(lb / lb sec)
450
400
350
300
PRESSURE RATIO
23
50
100
200
1000
ALTITUDE y (ft)
0
50,000
100,000
150,000
200,000


HORIZONTAL DISTANCE FROM LAUNCHING POINT
DURING BURNING TIME
DISTANCE x (miles)
TIME t (sec)
v = 880
t_p = 78 sec
v = 885
t_p = 103 sec
v = 890
t_p = 126 sec
v = 895
t_p = 147 sec
v = 900
t_p = 168 sec
k = 1
2
3
4
5

HORIZONTAL DISTANCE TRAVERSED BY MISSILE AT END OF BURNING TIME FOR VARIOUS PROPELLANT - GROSS WEIGHT RATIOS




ALTITUDE OF STABLE ORBITS FOR VARIOUS BURNING TIMES AND PROPELLANT GROSS WEIGHT RATIOS
ALTITUDE h (miles)
0
20
40
60
80
1
2
3
4
5
k
ν = .900
tp = 168
ν = .895
tp = 147
ν = .890
tp = 126
ν = .885
tp = 103
ν = .880
tp = 78