Performance Limits and Design Issues in Wireless Networks

Thesis by

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To

my family

Baba, Maman, Sara,

and Helia

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Abstract

The increasing utilization of networks, especially wireless networks, for different applications and in different aspects of modern life, has directed a great deal of attention towards the analysis and optimal design of networks. Distinguishing features of the wireless environment and the distributed nature of the network setup have raised many important challenges in finding the performance limits of different tasks such as communication, control, and computation over networks. There are also many design issues concerning the complexity and the robustness of wireless systems that should be addressed for a thorough understanding and an efficient operation of wireless networked systems. This thesis deals with a few of the challenges associated with the fundamental performance limits and optimal design of wireless networks.

In the first part, we analyze performance limits of two applications for a special class of wireless networks called wireless erasure networks. These networks incorporate some of the essential features of the wireless environment. We look at the performance limits of two applications over these networks. The first application is data transmission with two different traffic patterns, namely multicast and broadcast. The capacity region and the optimal coding scheme for the multicast scenario are found, and outer and inner bounds on the capacity region for the broadcast scenario are provided. The second application considered in this thesis is estimation and control of a dynamical process at a remote location connected through a wireless erasure network to a sensor observing the process. In this case, we characterize the minimum steady-state error and its dependency on the parameters of the network.

The final problem considered in the first part of the thesis concerns power consumption (as a performance measure) in wireless networks. We propose and analyze a simple scheme based on the idea of distributed beamforming that saves us in terms of power consumption for dense sensor and ad-hoc networks. We quantify this gain compared to the case when nodes have isolated communications without participating in the network.

The second part of the thesis deals with two design issues in the downlink of cellular wireless networks. The first issue is related to quality of service provisioning in the downlink scenario. We investigate the problem of differentiated rate scheduling in which different users demand different sets of rates. We obtain explicit and practical scheduling schemes to achieve the rate constraints and at the same time maximize the throughput. These schemes are based on the idea of opportunistic beamforming, are simple, and require little amount of feedback to the transmitter. We further show that the throughput loss due to imposing the rate constraints is negligible for large systems.

The next issue considered in this thesis is the robustness of the capacity region of multiple antenna Gaussian broadcast channels to the channel estimation error at the transmitter and the users. These channels are mathematical models for the downlink of cellular systems. We provide an inner bound on the capacity region of these channels and show that this inner bound is equivalent to the capacity region of a dual multiple access channel with a noise covariance that depends on the transmit powers. This duality is explored to show the effect of the estimation error on the sum-rate for a large number of users and in the large power regime. Finally, a training-based scheme for the block fading multiple antenna broadcast channels is proposed.

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Chapter 1

Introduction

The range of applications of wireless technology is no longer limited to transmission of voice in cellular networks. Numerous types of services, e.g., images, file transfer, and streaming video, with different Quality of Service (QoS) requirements, are available in today's highly heterogeneous cellular networks.

Furthermore, cellular networks are no longer the only type of wireless networks in use. Wireless local and metropolitan area networks (LAN, MAN) offer connectivity in the office, at home, or among buildings. Wireless networks, such as ad-hoc and sensor networks, are also deployed for various purposes such as environmental monitoring, industrial, transportation, and home systems automation, control of distributed embedded systems (such as robots or Unmanned Aerial Vehicles), and even medical services [66, 67, 68]. These networks are interconnected systems of devices that are capable of communication, computation, data storage, and adaptation in a distributed fashion. Unlike cellular networks, these networks typically operate without any predetermined infrastructure.

The increasing utilization of networks, especially wireless networks, in different aspects of modern life has redirected the attention from the point-to-point setup towards the network (or multi-user) setup. There are many features of the wireless environment and the distributed nature of a network setup that make the analysis and design of wireless networks a very challenging problem.

It is already known that the theory and protocols developed for point-to-point communications channels do not necessarily lead to satisfactory performance in a network setup. Therefore, new techniques and methodologies, designed specifically for networks, are required that capture the features of wireless networks and provide optimal performance in terms of rate, delay, and robustness. Also, because of the limited amount of available resources, such as power, bandwidth, antenna, and memory, a new look at the efficient usage of these resources is essential.

In essence, we are seeking the best strategy in the network that combats (and in many cases exploits [102]) the features of the wireless medium, uses the resources available as efficiently as possible, and provides the users with different types of demands. The key point is that in a network setup these tasks should be performed in a *distributed* fashion through the cooperation of the users and based on the *local* information available to each user.

The work presented in this thesis can be thought of as ways to find these optimal strategies or to bound the optimal performance for special classes of wireless networks. Figure 1.1 demonstrates the different sources of challenges in the design and analysis of wireless networks. In the following section we look at each source in more detail.

1.1 Sources of Challenges in Wireless Networks

Features of the Wireless Medium

There are many features of the wireless medium that distinguish it from other media. The wireless medium is a *shared* medium. This means that unlike wireline systems, where there exists dedicated physical connections between users, every user can essentially receive an attenuated version what other users are transmitting. In such a system, the manner of transmission is *broadcast* of the signal and there is *interference* in reception of a signal. Another property of a wireless channel is its random time-

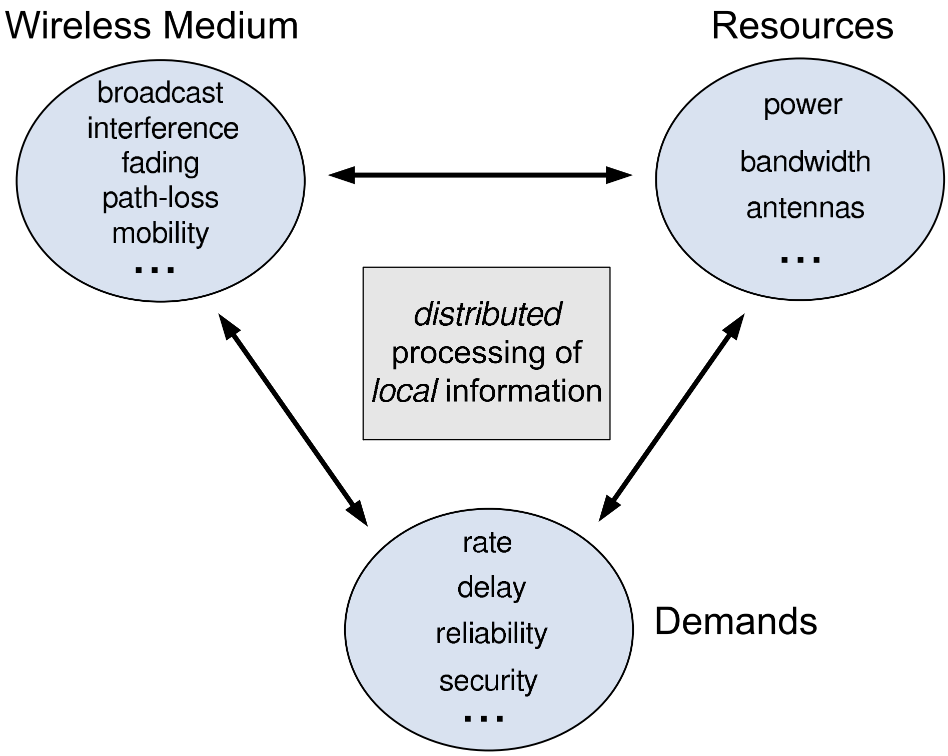


Figure 1.1: Sources of challenges in analysis and design of wireless networks.

varying behavior due to the mobility of users and other objects, as well as obstacles in the environment. More specifically, the channel to a given user might have poor conditions at some times and favorable conditions at other times. This is called the *fading* behavior of the channel [1]. In many situations, multiple copies of the transmitted signal may be received with different delays and different strengths. This is referred to as “multipath fading” and can severely deteriorate the performance when the transmitted signals have shorter duration (e.g., broadband transmission). Conventionally, the goal is to combat the randomness introduced by the environment. However, in recent years, there has been another view and that is to exploit the inherent randomness in the environment to increase the performance [102, 127, 121]. For instance, the multi-user diversity gain in the downlink of cellular systems is based on this idea, i.e., in a system of many users with random quality of reception (fading), there exists one user with good quality of reception with very high probability.

Efficient Use of Resources

Today's wireless systems are faced with an ever-growing demand for higher rates and quality of services. However, the available resources such as bandwidth, power, and number of antennas is limited. Therefore, efficient usage and allocation of these resources is more important than ever. In many scenarios, from mobile users in cellular networks to sensor nodes deployed in a remote area, the components of the network have limited power supplies. In these networks, efficient use of the available energy (power) supply is a critical issue. Bandwidth is another valuable resource in wireless systems, especially in high data rate broadband communications. Also, there has been a great interest in exploiting the spatial degrees of freedom in wireless systems by deploying multiple antennas at the transmitter and the receiver [109, 150, 148, 105]. The possibility of using multiple antennas in a network (multi-user) setup have been recently explored [117, 133].

Demands and Services

As mentioned earlier, wireless systems have become highly heterogenous. Different types of applications such as voice, internet, and video-on-demand, are provided over wireless networks. Depending on the application, the main performance measure will vary. For instance, video-on-demand applications not only require high rates but also are sensitive to delay. For many detection schemes, in ad-hoc networks reliability is the main concern. In many scenarios, resource allocation in wireless networks aims at optimizing over two conflicting performance measures at the same time, such as reliability and rate or delay and rate [154, 155]. Finding strategies that provide these different demands in an efficient manner is a challenging task.

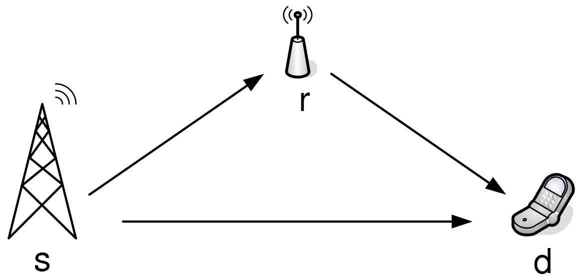


Figure 1.2: A simple network with one relay component, one source and one destination.

Operation In a Network Setup

Finding the optimal strategy for the nodes of a network in order to optimally perform a given task is very much an open problem. Consider the simple network, with only three nodes, shown in Figure 1.2. The desired task is reliable communication from the source to the destination with the aid of the relay node. The relay node is connected to both the source and destination through communications channels. Even for this simple network, finding the optimal operation at each node for maximizing the rate of reliable communication is unsolved [53]. The main difficulty in a network setup is the distributed nature of the information in the network. Each user has only access to local information and has to cooperate with other nodes in a distributed fashion to maximize the performance.

The above sources have raised many important and interesting challenges regarding the performance limits of different tasks such as communications and computation over networks. In addition, there are many design issues concerning the complexity and the robustness of the systems that should be addressed for a thorough understanding and efficient operation of wireless networked systems.

This thesis deals with a few of the challenges associated with wireless networks. In the first part of the thesis, Chapters 2 through 6, we analyze the capabilities of wireless networks for different tasks. More specifically, we look at the limits of rate, power efficiency, and estimation error for the tasks of reliable data-transmission and estimation across wireless networks.

In the second part of the thesis, Chapters 7 and 8, we look at a few design issues for wireless cellular networks, namely differentiated QoS provisioning and the robustness issues in downlink scheduling of cellular networks.

1.2 Performance Limits In Wireless Networks

In his ground breaking paper “A mathematical theory of communication,” [2], Claude E. Shannon revealed the fundamental performance limit of reliable communications, i.e., capacity, between a transmitter and receiver pair. This work started the new field of information theory and laid down a foundation for modern digital communications. Since then, one of the main goals in communications theory has been to devise practical methods to approach the capacity of point-to-point channels.

As mentioned earlier, the increase in utilization of wireless networks that involves multiple users rather than a single pair of users for different applications has spurred a great deal of research in various areas of wireless networks. Multi-terminal information theory is an example of these areas that looks at the fundamental performance limits of reliable communication in a multi-user setup [53]. Furthermore, since tasks like distributed estimation, computation, and control across communication networks (channels) are emerging as (primary) applications for many wireless networks, there has been a great deal of research aiming at finding the fundamental limits of performance for these tasks as well [3, 4, 5].

However, unlike single-user information theory, we do not have a complete understanding of the optimal performance for these applications. This is mainly because of the challenges discussed in Section 1.1. Multi-terminal information theory is still in its early stages. Many problems in multi-terminal information theory are still unsolved. As mentioned in the previous section, even for the simple network of Figure 1.2, the capacity region and the optimal strategy are not known [14]. Finding the performance limits of different applications is a fundamental task for the following reasons:

- It gives us a measure of the capabilities of the network and sets a standard for what we should aim for.
- It provides us with intuition on the optimal solution and guides practical strategies for achieving the limits.
- Identifying the limits of performance and the main parameters of the network affecting those limits gives insights for the optimal design of networks and even convergence of different applications on the same network.

In the first part of this thesis, we mainly look at the performance limits of different applications over wireless networks. In Chapters 2 through 6, we consider a simple model that captures some of the essential features of wireless networks. We look at two applications, namely data transmission and control and estimation, over these networks and find the performance limits and optimal operation for these applications.

1.3 Design Issues in Wireless Cellular Networks

The downlink of cellular systems is known to be a major bottleneck for future broadband wireless communications. The down-link scenario refers to the case in which the base station (possibly equipped with multiple antennas) simultaneously provides service to multiple mobile subscribers in a cell (as shown in Figure 1.3). From an information-theoretic perspective, broadcast channels [13], and in particular the Gaussian broadcast channel (GBC), can be used to model the downlink in cellular systems. There exists an abundance of information-theoretic results describing the limits of the achievable rates to the users in single-input single-output (SISO) Gaussian broadcast channels (see e.g., [115, 116]). More recently, there has been growing interest in the use of multiple antennas (at the transmitter, receivers, or both) for wireless communication networks. This has led to an interest in the multiple-input multiple-output (MIMO) Gaussian broadcast channel, where the transmitter and the various users may be equipped with multiple transmit and receive antennas, respectively. For these channels, the entire capacity region [118] is shown to be achieved by an interference cancelation scheme referred to as *dirty paper coding* (DPC) [119].

While dirty paper coding is the optimal transmission scheme, it is computationally expensive and requires the transmitter to have perfect knowledge of the channel state information (CSI) for all the users. More importantly, unlike point-to-point multi-antenna channels, the multi-user capacity depends heavily on whether the transmitter knows the channel coefficients to each user [121, 117, 118]. For instance, in a Gaussian broadcast channel with M transmit antennas and n single antenna users, the sum rate capacity scales like $M \log\log n$ for large n if perfect channel state information (CSI) is available at the transmitter, yet only logarithmically with M if it is not. Therefore, it seems that the performance of the broadcast channels is very sensitive to the channel knowledge. However, in practice the transmitter and the users do not have access to the exact channel realization and have only an estimate of it.

Moreover, in today's cellular systems, users might request different applications such as voice, internet, or video-on-demand. This implies that the base station has to provide differentiated services to different users, yet at the same time maximize the throughput. Therefore, the best operating point on the capacity region will no longer be the sum-rate capacity point. While achieving the sum-rate capacity point has been studied before, devising simple and practical schemes that come close to other points on the capacity region has not been studied in depth.

The issues mentioned above raise the following questions:

- Do there exist simple schemes that require little amount of feedback to the transmitter and yet maximize the throughput while providing different users

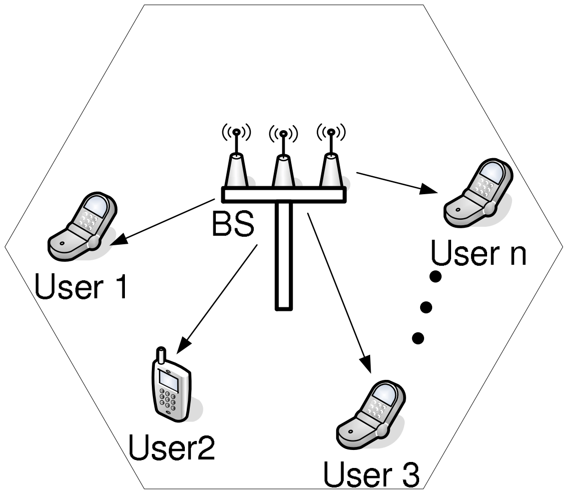


Figure 1.3: A cellular system with multiple antennas in the transmitter (base station).

with different demands (rates)?

- What is the effect of channel estimation errors at the transmitter and/or users on the capacity region of MIMO Gaussian broadcast channels?

These questions are mainly dealt with in Chapters 7 and 8. First, we propose simple scheduling schemes that require little CSI and provide users with different rates. Then, we look at the performance of broadcast channels with channel estimation error at the transmitter and the receivers and propose a training-based scheme for these channels.

1.4 Contributions of the Thesis

In this section, we review the contents of each chapter and mention the main contributions. The thesis is organized in such a way that different chapters can be more or less read independently.

Chapter 2 through Chapter 6 deal with finding performance limits of different applications over wireless networks.

In Chapter 2 we introduce a simple model for wireless networks called *wireless erasure networks*. This model incorporates the broadcast nature of transmission in wireless environment, but assumes that there is no interference. This model is reasonable for wireless networks where information is packetized and some interference-avoidance scheme is in effect. Examples of this scenario can be Software Defined Radios (SDR) operating based on Orthogonal Frequency Division Multiplexing (OFDM) techniques [19]. We look at a specific traffic pattern called the *multicast problem*. In this problem, a subset of users, denoted by the "destination set," are interested in the same message from a user, denoted by the "source node." The main contribution of this chapter is to find the multicast capacity of wireless erasure networks with arbitrary topology and show that it has an interesting interpretation related to the concept of minimum-cut capacity in flow graphs. However, in our case, the definition of the cut-capacity should be modified so that it takes into account the broadcast feature of the network. It is further shown that random linear encoding of received packets at each node suffices to achieve the optimal performance.

In Chapter 3 we look at another traffic pattern, referred to as the *broadcast problem*, over wireless erasure networks. In this problem, a subset of users, denoted by the "destination set," demand different types of information from the same user, called the "source node." The goal is to find the capacity region, i.e., set of achievable rates of communication between the source node and the destinations. First, we find the capacity region of broadcast problems for a subclass of wireless erasure networks called *erasure broadcast channels*. We show that the capacity region of these channels is achieved by time-sharing between the destinations at each node in the network. The proof technique used relies on optimization theory and may be extended to more general networks. We use the result for erasure broadcast channels to find an outer bound for the capacity region of the broadcast problems for general wireless erasure networks.

In Chapter 4 we look back at the multicast problem for wireless networks with *limited* operation allowed at each node. More specifically, unlike Chapter 2, we assume that the nodes can only perform one of two operations. They can either *forward* the signal they have received without any attempt to recover the content of the signal, or they can *decode and re-encode* the received signal, recover its content, and re-encode and send it across the network. The second operation is basically what is known as "routing" (or switching) in today's conventional network protocols. We first show that because of the distinguishing features of the wireless medium, routing data across the network does not necessarily give satisfactory results in a wireless setup. For instance, a multi-hop approach, in which every relay node *decodes* the received message, is not always the best paradigm for wireless networks, and forwarding the signal can increase the end to end throughput in wireless networks. We then propose a greedy algorithm that finds the best operation among forwarding and decoding and re-encoding at each node in order ro maximize the achievable rate in the network.

In Chapter 5 we turn our attention from data communications to estimation and control across wireless networks. We consider the problem of estimation and control of a dynamical process at a remote destination node. The destination node is connected to the sensor that observes the process through a wireless erasure network model introduced in Chapter 2. The performance measure in this problem is the steady-state estimation error covariance at the destination node. We find the condition for stability of the system and the exact performance measure in terms of the parameters of the process and the network. It is shown that the stability condition resembles a max-flow min-cut condition; however, the definition of the cut-value is different from the one introduced in Chapter 2. In addition, we find that the effect of the network on the performance is mainly through the *latency* introduced by the network in receiving recent measurements at the destination. The results of this chapter and Chapter 2 provide guidelines for the design of networks that are optimal for both communication and estimation.

Chapter 6 looks at another measure of performance in wireless networks, namely power consumption. We are interested in the power efficiency of large and random wireless sensor and ad-hoc networks. The fundamental question that we answer is, "Is it beneficial in terms of power consumption for nodes to operate in a network setup?" We show that densely deployed wireless sensors networks can gain in terms of power consumption by operating in a network and using a simple protocol. The protocol is essentially based on performing *distributed beamforming* at different nodes of the network. We quantify this gain by showing that for large random wireless ad-hoc network with n users and $r \leq \sqrt{n}$ simultaneous source/destination pairs located in a domain of fixed area, one can support the same rate as a single point-to-point system, but by expending only $\frac{1}{\sqrt{n}}$ the energy. This result shows the value of cooperation among nodes in terms of power consumption.

Having looked at the performance limits of wireless networks in previous chapters, we look at two design issues in cellular networks in Chapters 7 and 8.

In Chapter 7 we consider the downlink of wireless cellular systems where users have different rate demands. We model this scenario as a multiple input multiple output (MIMO) Gaussian broadcast channel. We assume n homogenous users divided into K groups. Users in the same group require the same rate. We further assume that the ratio of the groups' rates are given. The objective is to design strategies that are simple, require limited amount of information about the channel at the transmitter, and maximize the throughput (sum of the rates to all users) while maintaining the rational rate constraints. In general, this problem appears to be computationally intractable since the ergodic capacity region is described as the convex hull of (an infinite) set of rates. We focus on the asymptotic regime of large n where explicit results can be found. In particular, we propose three scheduling schemes to provide the rational rate constraints, namely the weighted opportunistic beamforming (WO), time division opportunistic beamforming (TO), and superposition coding (SC) for the single antenna case. In WO, we transmit each beam to the user that has the largest "weighted" signal to interference plus noise ratio (SINR). In TO, each group has its own time slot during which the transmitter chooses the user with the best SINR from the corresponding group. Superposition coding is the scheme that achieves the information-theoretic capacity region for the single antenna case. For each scheduling we give an explicit scheme to guarantee the rational rate constraints. We also analyze the throughput loss due to the rate constraints for all three different schemes. In particular, we show that the throughput loss compared to the maximum throughput (i.e., the sum rate capacity without any rate constraints) tends to zero for large n. Thus, there is not much of a penalty in providing different levels of service to different users. We also analyze the convergence rate of all the schemes and provide simulations supporting the theoretical analysis.

In Chapter 8 we look at robustness issues for the downlink scheduling in cellular networks. More specifically, we consider a MIMO Gaussian broadcast channel with channel estimation error at the transmitter and the receivers. We propose an achievable region based on the dirty paper coding scheme. We further show that the achievable region is equivalent to the capacity region of a dual Gaussian multiple access channel with noise covariance that depends on the transmit powers. We look at the achievable sum-rate for large broadcast channels with estimation error and show that as long as the estimation error does not vary with the number of users, the scaling behavior of the sum-rate is similar to the case with no estimation error. At the end of this chapter, we analyze the performance of training-based methods for broadcast channels with estimation error. We find the optimal energy and time that should be allocated for training. Based on the training scheme, it is shown that in order to achieve a linear increase in the sum-rate in terms of the number of transmit antennas in the high SNR regime (also known as the multiplexing gain), a linear fraction of energy should be allocated for the training phase. In other words, to achieve the multiplexing gain one needs a high fidelity description of the channel at the transmitter.

Finally **in Chapter 9**, we discuss a few interesting open problems that have been brought up by the research undertaken in this thesis.

Chapter 2

Capacity of Wireless Erasure Networks

2.1 Introduction

Determining the capacity region for general multi-terminal networks has been a long-standing open problem. An outer bound for the capacity region is proposed in [53]. This outer bound has a nice min-cut interpretation: The rate of flow of information across any cut (a cut is a partition of the network into two parts) is less than the corresponding cut-capacity. The cut-capacity is defined as the maximum rate that can be achieved if the nodes on each side of the cut can fully cooperate and also use their inputs as side-information.

The difficulty in multi-terminal information theory is that this outer bound is not necessarily tight. For instance, for the single relay channels introduced in [8], no known scheme achieves the min-cut outer bound of [53].

However, for a class of network problems called multicast problems in *wireline* networks, it is shown that the max-flow min-cut outer bound can be achieved [9, 10, 54]. A multicast problem comprises one or more source nodes (at which information is generated), several destinations (that demand all information available at the source nodes), relay nodes, and directed communication channels between some nodes. It is assumed that each channel is statistically independent of all other channels. Also, as the name suggests, the communication between different nodes is done through physically separated channels (wires). This means that the communication between two particular nodes does not affect the communication between others. In this setup, the maximum achievable rate is given by the minimum cut-capacity over all cuts separating the source nodes and a destination node. Because of the special structure of wireline networks, the cut-capacity for any cut is equal to the sum of the capacities of the channels crossing the cut.

This remarkable result for wireline networks is proved by performing separate channel and network coding in the network. First, we perform channel coding on each link of the network, so as to make it operate error-free at any rate below its capacity. This way, the problem is transformed into a flow problem in a graph where the capacity of each edge is equal to the information-theoretic capacity of the corresponding channel in the original network. If there is only one destination node, standard routing algorithms for finding the max-flow (min-cut) in graphs [52] achieve the capacity. However, when the number of destinations is more than one, these algorithms can fail. The key idea in [9] is to perform coding at the relay nodes. By [10, 54], linear codes are sufficient to achieve the capacity in multicast problems. These ideas are formulated in an algebraic framework and generalized to some other special network problems in [54]. Since then, there has been a great deal of research on the benefits of coding over traditional routing schemes in networks from different viewpoints, such as network management, security, etc. [11, 12].

In a wireless setup, however, the problem of finding the capacity region is more complicated. The main reason is that unlike wireline networks, in which communication between different nodes is done using separated media, in a wireless system the communication medium is shared. Hence, all transmissions across a wireless network are *broadcast*. Also, any communication between two users can cause *interference* to the communication of other nodes. These two features, broadcast and interference, present new issues and challenges for performance analysis and system design. The capacity regions of many information-theoretic channels that capture these effects are not known. For instance, the capacity region for general broadcast channels is an unsolved problem [13]. The capacity of general relay channels is not known. However, there are some achievable results based on block Markov encoding and random binning[14]. These ideas have been generalized and applied to a multiple relay setup in [15] and [16].

In this chapter we look at a special class of wireless networks which only incorporates the broadcast feature of wireless networks.[1]We model each communication channel in the network as a memoryless erasure channel. We will often assume that the erasure channels are independent; however, we show that the results also hold when the various erasure channels are correlated. We require that each node sends out the same signal on each outgoing link. However, for reception we use a multiple access model without interference, i.e., messages coming into a node from different incoming links do not interfere. In general, this is not true for a wireless system. However, this can be realized through some time, frequency, or code division multiple access scheme. [2] This simplification is important in making solution of the problem tractable. Even the capacity of a single relay channel is not known.

Finally, we assume that complete side-information regarding erasure locations on each link is available to the destination (but not to the relay) nodes. If we assume that the erasure network operates on long packets, i.e., packets are either erased or received exactly on each link, then this assumption can be justified by using headers in the packets to convey erasure locations or by sending a number of extra packets containing this information. By making the packets very long, the overhead of transmitting the

[1] [17] and [18] have considered applications of network coding at the network layer for cost (energy) minimization in lossless wireless ad-hoc networks. In this chapter, we look at wireless features of the network in the physical layer.

[2] Emerging technologies like Software-Defined Radio (SDR) based on OFDM techniques [19] also have these properties.

erasure locations can be made negligible compared to the packet length. We should remark that provided that the side-information is available to the destinations, all results in this work hold for any packet length.

We should mention that our model is appropriate for wireless networks where all information transmission is packetized and where some form of interference-avoidance is already in place. Channel coding within each packet can be used to make each link behave as a packet erasure channel. Although our model does not incorporate interference (primarily because it is not clear what interference means for erasure channels) one way, perhaps, to account for interference is to allow the erasure channels coming into any particular node to be correlated (something that is permitted in our model).

The main result is that a max-flow min-cut type of result holds for multicast problems in wireless erasure networks under the assumptions mentioned above. The definition of cut-capacity in these networks is such that it incorporates the broadcast nature of the network. We further show that similar to the wireline case, for multicast problems over wireless erasure networks, linear encoding at nodes achieves all the points in the capacity region. Working with linear encoding functions reduces the complexity of encoding and decoding. Building on the results of this work and using ideas from LT coding [20], it is shown in [21] that it is possible to reduce the delay incurred in the network. In their scheme, instead of using linear *block* codes, which is what we do here, the nodes send random linear combinations of their previously received signals at each time. This way nodes do not need to wait for receiving a full block before transmitting, which reduces the delay.

We once more need to emphasize the importance of the side-information on the erasure locations (or any other mechanism that provides the destination with the mapping from the source nodes to their incoming signals) for our result to hold. Interestingly, all the cut capacities of the network remain unchanged by making the above described side-information available to the receiver nodes. Thus, in some sense, what is shown here is that with appropriate side information made available to the receivers, the min-cut upper bound on capacity can be made tight. It would therefore be of further interest to see whether for other classes of networks it is possible to come up with the appropriate side-information to make the min-cut bounds tight.

This chapter is organized as follows. Section 2.2 defines notation used in this chapter and reviews some graph theoretic definitions of importance. We introduce the network model in Section 2.3 and the problem setup in Section 2.4. Section 2.5 states the main result for multicast problems over wireless erasure networks with side-information available at destinations. Section 2.6 includes proofs of these results. Section 2.7 demonstrates the optimality of linear encoding. Section 2.8 includes a discussion of our network assumptions. Also, the performance of different coding schemes when side-information is not available is analyzed and compared. We mention future directions of our work and conclude in Section 2.9.

2.2 Preliminaries

2.2.1 Notation

Upper case letters (e.g., X, Y, Z) usually denote random variables, and lower case letters (e.g., x, y, z) denote the values they take. Underlined letters (e.g., $\underline{x}$) are used to denote vectors. Sets are denoted by calligraphic alphabet (e.g., $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$). The complement of a set $\mathcal{A}$ is shown by $\mathcal{A}^c$. The transpose of matrix $\underline{x}$ is shown by $\underline{x}^\dagger$. $\exp(x)$ is used to denote 2^x.

Subscripts specify nodes, edges, inputs, outputs, and time. For instance, v_2 and X_2 could denote node number two and the output of node number two in the network, respectively. Unless otherwise mentioned, commas are used to separate time subscripts from other subscripts. Superscripts are also used to refer to different sources.

$\mathcal{V}$	node set
$\mathcal{E}$	edge set
$\mathcal{S}$	the set of source nodes
$\mathcal{D}$	the set of destination nodes
$[\mathcal{V}_x, \mathcal{V}_y]$	$x - y$ cut-set described by x-set $\mathcal{V}_x$
X_i	symbol transmitted from node i
X_i^n	a transmitted block of n symbols from node i
Y_{ij}	channel output of edge (i, j)
Y_i	symbols received at node i from all incoming channels
$w^{(s)}$	message transmitted from source s
$\mathcal{W}^{(s)}$	message index set at source node s
$\hat{w}_{d_i}^{(s)}$	estimate at destination d_i of the message transmitted from s
$P_{d_i}^{(n)(s)}$	prob. of error in decoding source s at destination d_i

Table 2.1: Some important notation in this chapter

For example, $w^{(s)}$ could denote the message sent by node s.

Consider a sequence of numbers $x_1, x_2, x_3, \ldots$. We use notation x^n to denote the sequence $x_1, x_2, \ldots, x_n$. We also use notation $(x_i, \quad i \in \mathcal{I})$ to denote the ordered tuple specified by index set $\mathcal{I}$. Finally, $|\mathcal{X}|$ is the cardinality of set $\mathcal{X}$, and $2^{\mathcal{X}}$ is the set of all the subsets of $\mathcal{X}$. Table 2.1 summarizes our notation.

2.2.2 Definitions for Directed Graphs

In this Section, we briefly review the concepts and definitions from graph theory used in this work[22].

A directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ has vertex set $\mathcal{V}$ and directed edge set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. Without loss of generality, let

$$\mathcal{V} = \{1, 2, \ldots, |\mathcal{V}|\}.$$

We assume that the graph is finite, i.e., $|\mathcal{V}| < \infty$. For each node, $v \in \mathcal{V}$, $\mathcal{N}_O(v)$ and $\mathcal{N}_I(v)$ are the set of edges leaving from and the set of edges going into v, respectively.

Formally,

$$\begin{aligned}\mathcal{N}_O(v) &= \{(v,u)|(v,u) \in \mathcal{E}\} \quad \text{and} \\ \mathcal{N}_I(v) &= \{(u',v)|(u',v) \in \mathcal{E}\}. \end{aligned} \tag{2.1}$$

The *out-degree* $d_O(v)$ and *in-degree* $d_I(v)$ of v are defined as $d_O(v) = |\mathcal{N}_O(v)|$ and $d_I(v) = |\mathcal{N}_I(v)|$. A sequence of nodes $v_0, v_1, \ldots, v_n$ such that $(v_0, v_1), (v_1, v_2), \ldots, (v_n, v_0)$ are all in $\mathcal{E}$ is called a cycle. An acyclic graph is a directed graph with no cycles.

An $x-y$ *cut* for $x, y \in \mathcal{V}$ is a partition of $\mathcal{V}$ into two subsets, $\mathcal{V}_x$ and $\mathcal{V}_y = \mathcal{V}_x^c$, such that $x \in \mathcal{V}_x$ and $y \in \mathcal{V}_y$. The *x-set* $\mathcal{V}_x$ (or *y-set* $\mathcal{V}_y$) determines the cut uniquely. For the $x-y$ cut given by $\mathcal{V}_x$, the *cut-set* $[\mathcal{V}_x, \mathcal{V}_y]$ is the set of edges going from the *x-set* to *y-set*, i.e.,

$$[\mathcal{V}_x, \mathcal{V}_y] = \{(u,v)|(u,v) \in \mathcal{E}, u \in \mathcal{V}_x, v \in \mathcal{V}_y\}.$$

We also define $\mathcal{V}_x^*$ as

$$\mathcal{V}_x^* = \{v|\exists\, u \text{ s.t. } (v,u) \in [\mathcal{V}_x, \mathcal{V}_y]\}.$$

$\mathcal{V}_x^*$ is the set of nodes in the x-set that has at least one of its outgoing edges in the cut-set.

Example 2.1. Consider the acyclic directed graph shown in Figure 2.1. $\mathcal{V} = \{1, 2, 3, 4\}$ is the set of nodes, and $\mathcal{E} = \{(1,2), (3,2), (1,3), (3,4), (2,4)\}$ is the set of edges. The source and destination nodes are $s = 1$ and $d = 4$, respectively. The *out-degree* of node 3 is 2, i.e., $d_O(3) = 2$. Looking at the $s-d$ cut specified by *s-set* $\mathcal{V}_s = \{1, 3\}$, the cut-set $[\mathcal{V}_s, \mathcal{V}_d]$ is the set $\{(3,4), (3,2), (1,2)\}$, and $\mathcal{V}_s^* = \{1, 3\}$.

At the end of this section we define the notion of partial ordering of the nodes in the graph. Consider two distinct nodes i and j of the network. Exactly one of the following three will occur:

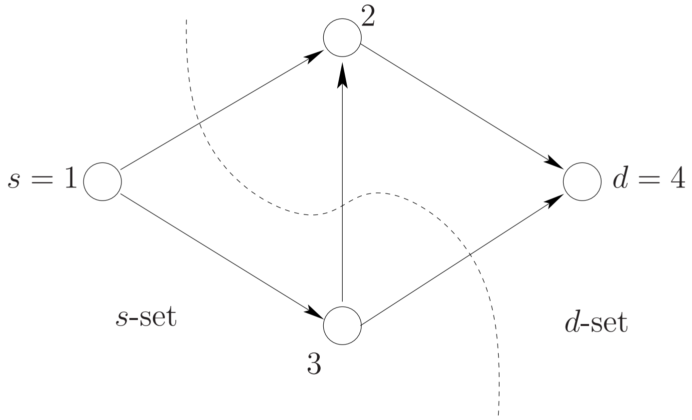


Figure 2.1: A directed acyclic graph with four nodes and five edges. The cut-set $\{(3,4),(3,2),(1,2)\}$ is shown by the dashed line.

1. There is a directed path from i to j. In this case we will say that $i \prec j$.

2. There is a directed path from node j to node i. In this case we will say that $j \prec i$.

3. There is no directed path from node i to node j or from node j to node i. In this case we will say that j and i are incomparable.

Note that since we assume acyclic networks, we cannot have directed paths both from i to j and from j to i. Thus, we have a partial ordering for nodes in the network. For example, in Figure 2.3 we have $3 \prec 4$, but 2 and 5 are incomparable. Note that the partial ordering gives us a (non-unique) sequence of nodes starting with s such that for every node i, all the nodes j that satisfy $j \prec i$ are before it in the sequence [58]. Call such a sequence $\mathcal{T}$. A possible sequence $\mathcal{T}$ for Figure 2.3 is $(s, 3, 2, 5, 4, d)$.

2.3 Network Model

Wireless Packet Erasure Networks

We model the wireless packet[3] erasure network by a directed acyclic graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Each edge $(i, j) \in \mathcal{E}$ represents a memoryless packet erasure channel from node i to node j. For most of this chapter, we assume that erasure events across different links are independent. However, as described later, the results go through for correlated erasure events. For independent erasure events, a packet sent across link (i, j) is either erased with probability of erasure ϵ_{ij} or received without error. We denote the input alphabet (the set of possible packets) of the erasure channel by $\mathcal{X}$.[4]

Let $Z_{ij,t}$ be a random variable indicating erasure occurrence across channel (i, j) at time t. For independent erasure events, $Z_{ij,t}$ has a Bernoulli distribution with parameter $1 - \epsilon_{ij}$. If an erasure occurs on link $(i, j) \in \mathcal{E}$ at time t, the value of $Z_{ij,t}$ will be zero; otherwise $Z_{ij,t}$ will be one. Note that the behavior of the network can be fully determined by the values of $Z_{ij,t}$ for all links and all times and the operation performed at each node.

We assume that transmissions on each channel experience one unit of time delay. The input of all the channels originating from node i is denoted by X_i chosen from input alphabet $\mathcal{X}$. Note that with this definition we have required that each node transmit the same symbol on all its outgoing edges, i.e., all channels corresponding to edges in $\mathcal{N}_O(i)$ have the (same) input X_i (see Figure 2.2.) This constraint incorporates broadcast in our network model. The output of the communication channel corresponding to edge $(i, j) \in \mathcal{E}$ is denoted by Y_{ij}; Y_{ij} lies in output alphabet $\mathcal{Y} = \mathcal{X} \cup \{e\}$, where e denotes the erasure symbol. We also assume that the outputs

[3]Throughout this chapter a packet can be of any length. When the length of packets is one, the channel is a binary erasure channel.

[4]For simplicity and without loss of generality we consider $\mathcal{X} = \{0, 1\}$ in our analysis and proofs. However, we should remark that all the results and analysis hold for input alphabet of arbitrary length.

of all channels corresponding to edges in $\mathcal{N}_I(i)$ are available at node i. This condition is equivalent to having no interference in receptions in the network. Having this, let $Y_i = (Y_{ji},\ (j,i) \in \mathcal{N}_I(i))$ be the symbols that are received at node i from all its incoming channels. We have $Y_i \in \prod_{j:(j,i)\in\mathcal{E}} \mathcal{Y}$. The relation between the Y_is and X_is defines a coding scheme for the network.

Based on the properties of the network mentioned above, if we consider the inputs and outputs up to time t, then the conditional probability function of the outputs of all the channels (edges) up to time t, given all the inputs of all the channels up to time t and all the previous outputs, can be written as follows for all t

$$\begin{aligned}&\Pr\left((y_{ij,t},\ (i,j)\in\mathcal{E})\middle|(x_l^t,\ l\in\mathcal{V}),(y_{ij}^{t-1},\ (i,j)\in\mathcal{E})\right)\\&=\mathrm{P}\left((Y_{ij}=y_{ij,t},\ (i,j)\in\mathcal{E})|(X_l=x_{l,t},\ l\in\mathcal{V})\right).\end{aligned}$$

For independent erasure events, we further have

$$\begin{aligned}&\Pr\left((y_{ij,t},\ (i,j)\in\mathcal{E})\middle|(x_l^t,\ l\in\mathcal{V}),(y_{ij}^{t-1},\ (i,j)\in\mathcal{E})\right)\\&=\prod_{i\in\mathcal{V}}\ \prod_{j:\,(i,j)\in\mathcal{N}_O(i)}\Pr\left(Y_{ij}=y_{ij,t}|X_i=x_{i,t}\right).\end{aligned}\tag{2.2}$$

Network Problems

Any network problem is characterized by a collection of information sources, a collection of source nodes at which one or more information sources are available, and a collection of destination nodes. Each destination node demands a subset of information sources. More specifically, a network problem is a quintuple $\mathcal{P} = (\mathcal{M}, \mathcal{S}, \mathcal{D}, \Upsilon_g, \Upsilon_r)$, where

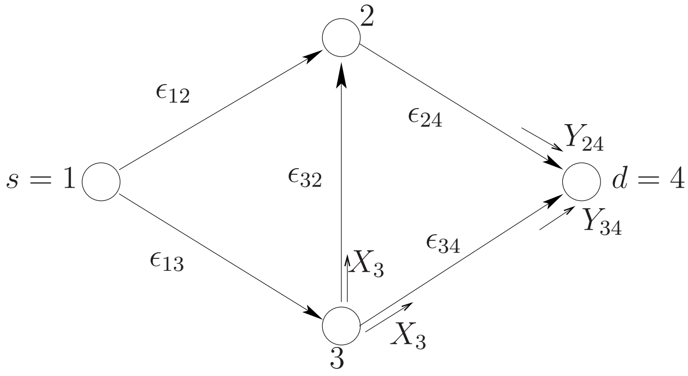


Figure 2.2: (i): An erasure wireless network with the graph representation of example 2.1. Probability of erasure on link (i, j) is ϵ_{ij}. Each node (e.g., node 3) transmits the same signal (X_3) across its outgoing channels. Since the network is interference-free, node 4 receives both signals Y_{24} and Y_{34} completely. (ii): In this network, cut-capacity for s-set $\mathcal{V}_s = \{1, 3\}$ is $C(\mathcal{V}_s) = 1 - \epsilon_{12} + 1 - \epsilon_{32}\epsilon_{34}$.

- $\mathcal{M} = \{m_1, \ldots, m_{|\mathcal{M}|}\}$ denotes the set of information messages (sources).[5] We assume that each of the information messages is an i.i.d uniformly distribution random process. Different information messages are assumed to be independent.

- $\mathcal{S} = \{s_1, s_2, \ldots, s_{|\mathcal{S}|}\} \subset \mathcal{V}$ denotes the source nodes.

- $\Upsilon_g : \mathcal{S} \to 2^{\mathcal{M}}$, where $\Upsilon_g(s_i)$ is the subset of messages generated at source node s_i.

- $\mathcal{D} = \{d_1, d_2, \ldots, d_{|D|}\} \subset \mathcal{V}$ denotes the set of destination nodes.

- $\Upsilon_r : \mathcal{D} \to 2^{\mathcal{M}}$, where $\Upsilon_r(d_j)$ is the subset of message demanded at destination node d_j.

We should remark that in the above definition $\mathcal{S} \cap \mathcal{D}$ may not be empty, i.e., a node can be a destination node for one information source and a source node for another. Also, destination nodes can act as relay nodes for other destination nodes in the network.

[5] Throughout this thesis, $\mathcal{M}$ and $\mathcal{W}$ are used interchangeably to denote the set of information messages.

The class of network problems that we consider in this chapter is the multiple source/multiple destination *multicast* problem, where each of the destinations demands all of the information sources and source nodes generate disjoint messages. Mathematically, a multicast problem, $\mathcal{P}_{mc} = (\mathcal{M}, \mathcal{S}, \mathcal{D}, \Upsilon_g, \Upsilon_r)$, is a network problem where $\Upsilon_g(\cdot)$ is a partition of $\mathcal{M}$ and $\Upsilon_r(d_i) = \mathcal{M}$ for all destination nodes. In Chapter 3, we look at another type of network problems referred to as *broadcast problem.*

Side-information at Destinations

In most parts of the chapter we assume that each destination node $d \in \mathcal{D}$ has complete knowledge of the erasure locations on each link of the network that is on a path from the source set to d. In other words, d knows values of the $z_{ij,t}$, for all $(i,j) \in \mathcal{E}$ and all times t, for which (i,j) is on at least one path from one of the sources to d. This serves as channel side-information provided to the destinations from across the network. In the case when we consider large packets (alphabet), this side-information can be provided using negligible overhead. More discussion of this model appears in Section 2.8.

Cut-capacity Definition

Consider an $s-d$ cut given by *s-set* $\mathcal{V}_s$ as defined in Section 2.2.2. We define $X(\mathcal{V}_s)$ and $Y(\mathcal{V}_s)$ as

$$\begin{aligned} X(\mathcal{V}_s) &= \{X_i | i \in \mathcal{V}_s^*\} \quad \text{and} \\ Y(\mathcal{V}_s) &= \{Y_{ij} | (i,j) \in [\mathcal{V}_s, \mathcal{V}_s^c]\}. \end{aligned} \tag{2.3}$$

At the end of this section, we define the cut-capacity for wireless erasure networks. In wireline networks, the value of the cut-capacity is the sum of the capacities of the edges in the cut-set [54]. Such a definition of cut-capacity in wireline networks makes sense because the nodes can send out different signals across their outgoing edges. However, this is not the case for wireless erasure networks where broadcast transmissions are required. The following definition of cut-capacity is different from that in the wireline network settings, and it incorporates the broadcast nature of transmission in our network.

Definition 2.1. *Consider an erasure wireless network represented by* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ *and probabilities of erasure* ϵ_{ij} *as described in Section 2.3. Let* s *and* d_l *be the source and destination nodes, respectively. The cut-capacity corresponding to any* $s - d_l$ *cut represented by s-set* $\mathcal{V}_s$ *is denoted by* $C(\mathcal{V}_s)$ *and is equal to*

$$C(\mathcal{V}_s) = \sum_{i \in \mathcal{V}_s^*} \left(1 - \prod_{j:\,(i,j) \in [\mathcal{V}_s, \mathcal{V}_{d_l}]} \epsilon_{ij} \right). \tag{2.4}$$

Example 2.2. Consider the network represented by the directed graph of example 2.1. (See Figure 2.2.) For the $s - d$ cut specified by the s-set $\mathcal{V}_s = \{1, 3\}$, the cut-capacity is

$$C(\mathcal{V}_s) = 1 - \epsilon_{12} + 1 - \epsilon_{32}\epsilon_{34}.$$

Looking at this example, we see that all edges in the cut-set that originate from a common node, i.e., edges $(3, 2)$ and $(3, 4)$, together contribute a value of one minus the product of their erasure probabilities, i.e., $1 - \epsilon_{32}\epsilon_{34}$ to the cut-capacity. This observation holds in general for wireless erasure networks.

Example 2.3. As another example, consider the network shown in Figure 2.3 with one source $s = 1$ and one destination $d = 6$. The cut-capacity corresponding to the $s - d$ cut specified by $\mathcal{V}_s = \{1, 3, 4\}$ is $C(\mathcal{V}_s) = 1 - \epsilon_{12} + 1 - \epsilon_{46} + 1 - \epsilon_{35}\epsilon_{32}$.

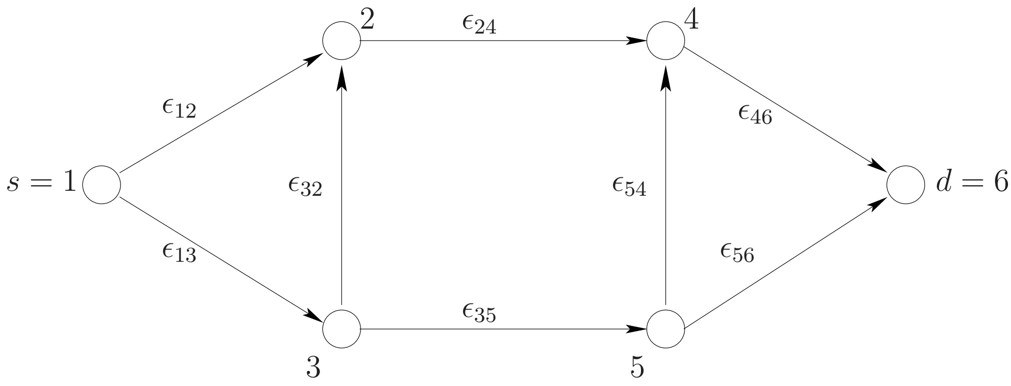


Figure 2.3: For the cut-set specified by the s-set $\mathcal{V}_s = \{1, 3, 4\}$, the cut-capacity is $C(\mathcal{V}_s) = 1 - \epsilon_{12} + 1 - \epsilon_{46} + 1 - \epsilon_{35}\epsilon_{32}$.

2.4 Problem Statement

We next define the class of block codes considered. A $(\lceil 2^{nR_1} \rceil, \ldots, \lceil 2^{nR_{|\mathcal{S}|}} \rceil, n)$ code for the multicast problem in a wireless erasure network described in the previous sections consists of the following components:

- A set of integers $\mathcal{W}^{(s_i)} = \{1, 2, \ldots, \lceil 2^{nR_i} \rceil\}$ for each source node $s_i \in \mathcal{S}$. $\mathcal{W}^{(s_i)}$ represents the set of message indices corresponding to node s_i. $w^{(s)}$ denotes the message of source $s \in \mathcal{S}$. We assume that the messages are equally likely and independent.

- A set of encoding functions $\{f_{i,t}\}_{t=1}^{n}$ for each node $i \in \mathcal{V}$, where

$$x_{i,t} = f_{i,t}(w^{(i)}, y_i^{t-1})$$

  is the signal transmitted by node i at time t. Note that $x_{i,t}$ is a function of the message $w^{(i)}$ that node $i \in \mathcal{V}$ wants to transmit in the current block [6] and all symbols received so far by node i from its incoming channels. If i is not a source node, we set $w^{(i)} = 0$ for all blocks and all times.

[6]The value of $w^{(i)}$ does not change in one block.

- A decoding function g_{d_i} at destination node $d_i \in \mathcal{D}$,

$$g_{d_i} : \mathcal{W}^{(d_i)} \times \mathcal{Y}_{d_i}^n \times \{0,1\}^{n|\mathcal{E}|} \to \prod_{s\in\mathcal{S}} \mathcal{W}^{(s)}$$

  such that

$$\underline{\hat{w}}_{d_i} = (\hat{w}_{d_i}^{(s)}, \quad s \in \mathcal{S}) = g_{d_i}(w^{(d_i)}, y_{d_i}^n, (z_{ij,t}\,, (i,j) \in \mathcal{E},\, 1 \le t \le n)), \tag{2.5}$$

  where $\hat{w}_{d_i}^{(s)}$ is the estimate of the message sent from source $s \in \mathcal{S}$ based on received signals at d_i, information source available at d_i[7], $w^{(d_i)}$, and also the erasure occurrences on all the links of the network in the current block.

Note that X_i, Y_{ij}, and Y_i all depend on the message vector $\underline{w} = (w^{(s)},\ s \in \mathcal{S})$ that is being transmitted. Therefore, we will write them as $X_i(\underline{w})$,$Y_{ij}(\underline{w})$, and $Y_i(\underline{w})$ to specify what specific set of messages is transmitted.

Associated with every destination node $d \in \mathcal{D}$ and every information source $s \in \mathcal{S}$ is a probability that the message will not be decoded correctly:[8]

$$P_d^{(n)(s)} = \Pr(\hat{W}_d^{(s)} \neq W^{(s)}), \tag{2.6}$$

where $P_d^{(n)(s)}$ is defined under the assumption that all the messages are independent and are uniformly distributed over $\mathcal{W}^{(s)}, s \in \mathcal{S}$. The set of rates $(R_s,\ s \in \mathcal{S})$ is said to be achievable if there exist a sequence of $(\lceil 2^{nR_1} \rceil, \ldots, \lceil 2^{nR_{|\mathcal{S}|}} \rceil, n)$ codes such that $P_d^{(n)(s)} \to 0$ as $n \to \infty$ for all $s \in \mathcal{S}$ and $d \in \mathcal{D}$. The capacity region is the closure of the set of achievable rates.

[7]If $d_i \notin \mathcal{S}$ without loss of generality, we set $w^{(d_i)} = 0$ and $\mathcal{W}^{(d_i)} = \{0\}$ for all blocks.

[8]Note that if d is a source node, we assume without loss of generality that $P_d^{(n)(d)} = 0$.

2.5 Main Results

In this section we present the main results of this chapter.

Theorem 2.1. *Consider a single source/ single destination wireless erasure network described by the directed acyclic graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and the assumptions of Section 2.3. Let $s \in \mathcal{V}$ and $d \in \mathcal{V}$ denote the network's source and destination, respectively. Then the capacity of the network with side-information at the destination is given by the value of the minimum value s-d cut. More precisely, we have*

$$C = \min_{\mathcal{V}_s : \mathcal{V}_s \text{an } s-d \text{ cut}} C(\mathcal{V}_s). \tag{2.7}$$

Remark 2.1. *The results presented above are stated for erasure wireless networks with broadcast property (and no interference). However, based on these results, it is possible to derive the capacity of multicast problems over* error-free *networks (with the broadcast property and without interference), with or without capacitated links.*

Remark 2.2. *Although we have assumed that the erasure events across the network are independent, the capacity results also hold for the case when the erasure events are correlated, i.e. $Z_{ij}, (i, j) \in \mathcal{E}$ are dependent on each other. In that case the definition of the cut capacity should be modified to*

$$C(\mathcal{V}_s) = \sum_{i \in \mathcal{V}_s^*} \left(1 - \Pr\left(Z_{ij} = 0,\ j : (i, j) \in [\mathcal{V}_s, \mathcal{V}_s^c]\right) \right). \tag{2.8}$$

Example 2.4. Recall the single source / single destination network of example 2.2. (See Figure 2.2.) By Theorem 2.1, the capacity of this network is

$$C = \min\{1 - \epsilon_{12} + 1 - \epsilon_{32}\epsilon_{34}, 1 - \epsilon_{34} + 1 - \epsilon_{24}, 1 - \epsilon_{12}\epsilon_{13}, 1 - \epsilon_{13} + 1 - \epsilon_{24}\}.$$

The following theorems generalize the single source/ single destination result to general multicast problems.

Theorem 2.2. *Consider a multiple source/single destination wireless erasure network described by directed acyclic graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ *and the assumptions of Section 2.3. Suppose that the destination requests all of the information from all of the sources. Let* $\mathcal{S} \subset \mathcal{V}$ *and* $d \in \mathcal{V}$ *denote the set of source nodes and the destination node, respectively. The capacity region of the network with side-information provided at the destination is given by*

$$C(\mathcal{G}, \mathcal{S}, d) \triangleq \left\{ (R_s,\ s \in \mathcal{S}) \middle| 0 \leq \sum_{s \in \mathcal{V}' \cap \mathcal{S}} R_s \leq C(\mathcal{V}') \qquad \forall\, \mathcal{V}' \subset \mathcal{V} - \{d\} \right\}. \tag{2.9}$$

In other words, the total rate of information transmission to d *across any cut* $[\mathcal{V}', \mathcal{V}_d]$ *should not exceed the cut-capacity of that cut.*

Example 2.5. Consider the network shown in Figure 2.4 with two sources $\{1, 2\}$ and one destination $\{3\}$. Then according to Theorem 2.2, the capacity region is

$$\{(R_1, R_2) \in \mathbb{R}^+ \times \mathbb{R}^+ | R_1 \leq 1 - \epsilon_{12}\epsilon_{13}, R_2 \leq 1 - \epsilon_{23}, R_1 + R_2 \leq 1 - \epsilon_{23} + 1 - \epsilon_{13}\}.$$

Theorem 2.3. *Consider a multicast problem with multiple sources and multiple destinations. Let* $\mathcal{S}, \mathcal{D} \subset \mathcal{V}$ *denote the set of source nodes and destination nodes, respectively. The capacity region of the network with side-information is given by the intersection of the capacity regions of the multicast problem between the sources and each of the destinations, i.e,*

$$C(\mathcal{G}, \mathcal{S}, \mathcal{D}) = \bigcap_{d \in \mathcal{D}} C(\mathcal{G}, \mathcal{S}, d). \tag{2.10}$$

Corollary 2.3. *Consider a multicast problem with one source denoted by* s *and multiple destinations denoted by* $d_1, \ldots, d_{|\mathcal{D}|}$*. The capacity of the network is given by the* *minimum value of the cuts between the source node and any of the destinations, i.e.,*

$$C = \min_{d_i \in \mathcal{D}} \min_{\mathcal{V}_s : s - d_i \, cut} C(\mathcal{V}_s).$$

Example 2.6. Consider the network shown in Figure 2.2. Suppose that we are decoding at node 2 and 4, i.e., $\mathcal{D} = \{2, 4\}$. Based on Corollary 2.3, the capacity of this network is

$$C = \min\{1 - \epsilon_{12} + 1 - \epsilon_{32}, 1 - \epsilon_{34} + 1 - \epsilon_{24}, 1 - \epsilon_{12}\epsilon_{13}, 1 - \epsilon_{13} + 1 - \epsilon_{24}\}.$$

The above results show that the capacity region for multicast problems over wireless erasure networks has a max-flow min-cut interpretation. This result is similar to multicast problems in wireline networks [9]; however the definition of the cut-capacity is different. Recall from [9] that in wireline networks, the cut-capacity is the sum of the capacities of the links in the cut-set. Since wireless erasure networks incorporate broadcast, the cut-capacity is the sum of the capacities of each broadcast system that operates across the cut.

The next theorem states that linear network coding is sufficient for achieving the capacity region.

Theorem 2.4. *Consider a multicast problem with multiple sources and multiple destinations. Then, any rate vector in the capacity region* $C(\mathcal{G}, \mathcal{S}, \mathcal{D})$ *of the network defined in Theorem 2.3 is achievable with linear block coding.*

In the next section, we prove Theorems 2.1, 2.2, and 2.3. In Section 2.7, we look at the performance of the network using random linear coding and prove Theorem 2.4.

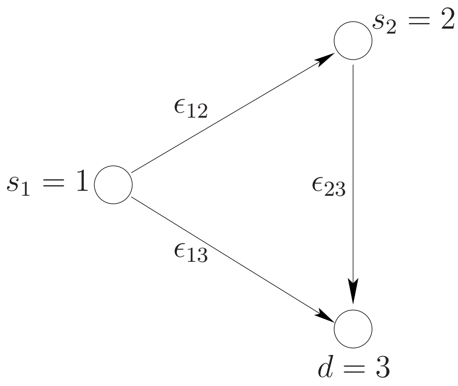


Figure 2.4: A wireless erasure network with two sources, $\mathcal{S} = \{1, 2\}$, and one destination, $\mathcal{D} = \{3\}$.

2.6 Proof of Theorems

2.6.1 Proof of Theorems 1 and 2

In this section we prove the results stated for multi-source/ single destination network problems. We start by proving the converse.

2.6.1.1 Converse

We prove the converse part by considering perfect cooperation among subsets of nodes. Consider the cut specified by d-set $\mathcal{V}_d$. Let all of the nodes in $\mathcal{V}_d$ and all of the nodes in $\mathcal{V}_d^c$ cooperate perfectly, i.e., each node has access to all of the information known to nodes in its set. In this case, we have a multiple input, multiple output point-to-point erasure channel. Consider all source nodes in $\mathcal{V}_d^c$. Then, clearly the sum-rate of these source nodes must be less than the capacity of the multiple input multiple output point-to-point erasure channel. The capacity of this point-to-point communication channel is

$$C_{col} = \max_{P(x_i, \quad i \in \mathcal{V}_d^{c*})} I((X_i,\, i \in \mathcal{V}_d^{c*}); (Y_{ij},\, (i,j) \in [\mathcal{V}_d^c, \mathcal{V}_d])).$$

Since the channels are independent and memoryless, the mutual information is maximized when the different X_is are i.i.d. and uniform on the input alphabet $\mathcal{X}$. In this case, the above mutual information equals the cut-capacity corresponding to the cut-set $[\mathcal{V}_d^c, \mathcal{V}_d]$, i.e.,

$$C_{col} = C(\mathcal{V}_d^c).$$

Therefore, for any cut-set $[\mathcal{V}_d^c, \mathcal{V}_d]$,the sum-rate of the information sources in set $\mathcal{V}_d^c$ satisfies

$$\sum_{s \in \mathcal{S} \cap \mathcal{V}_d^c} R_s \leq C_{col} = C(\mathcal{V}_d^c).$$

The complete analysis appears in [57]. The proof follows the same lines as the max-flow min-cut upper bound of Cover and Thomas for multi-terminal networks [53, Sec. 14.10].

2.6.1.2 Achievability

In this section we prove that all of the rates arbitrarily close to rates in the capacity regions given in Theorems 2.1 and 2.2 are achievable for a multiple sources/ single destination multicast problem. We next use random coding techniques to show this result.

We employ random block codes in the network. Each node transmits the next block of n symbols only after it has received all n symbols corresponding to the present block from each of its incoming channels. Let $L_{\max}$ denote the length of the longest path from a source to the destination in the network. Since each transmission introduces one unit of time delay, the maximal delay between the transmission of a message from one source and to its receipt at the destination using block codes of length n is $nL_{\max}$. We do not use any information from previously decoded blocks to decode the current set of messages. Also note that since our model assumes that the reception is interference-free, there is no confusion up among different blocks at any node. Therefore, if the network operates for nB units of time (i.e., B blocks of length-n symbols), then the destination has received all of the information required for decoding the $B - L_{\max}$ first messages transmitted from each source $s \in \mathcal{S}$, i.e., $w_b^{(s)}$, $b = 1, \ldots, B - L_{\max}$. Since the network size is finite, as $B \to \infty$, for fixed n, the rate $R_s \frac{B-L_{\max}}{B}$ approaches R_s.[9]

The same codebook and encoding and decoding functions are used for all the blocks. We explain the coding scheme for transmitting one set of messages from the sources to the destination. Below we describe the encoding and decoding processes.

- **Codebook Generation and Encoding:** For each node $i \in \mathcal{V}$, the encoding function

$$f_i : \mathcal{W}^{(i)} \times \mathcal{Y}_i^n \to \mathcal{X}^n$$

  is generated randomly as follows. For each $y^n \in \mathcal{Y}_i^n$ and for each $w^{(i)} \in \mathcal{W}^{(i)}$ we draw the symbols of $f_i(w^{(i)}, y^n) \in \mathcal{X}^n$ randomly and independently according to a binary Bernoulli distribution with parameter 1/2 . Thus, the channel input at node $i \in \mathcal{V}$ is $X_i^n = f_i(w^{(i)}, y^n)$ when the message at node i is $w^{(i)}$ and the incoming sequence is $y^n \in \mathcal{Y}_i^n$. The destination has perfect knowledge of all the encoding functions $f_i(.)$, $i \in \mathcal{V}$ thus generated.[10]

- **Decoding:** The destination "simulates" the network to decode the messages. Suppose that message vector $\underline{w}_0 = (w_0^{(s)}, \quad s \in \mathcal{S})$ is transmitted and $y_d^n(\underline{w}_0)$ is received at destination d. By assumption, the receiver knows the erasure locations on all the links of the network, i.e., $(z_{ij}^n, \ (i,j) \in \mathcal{E})$. Having all of the

[9] We could also consider the case when different sources transmit different numbers of messages in B block uses. In that case, if L_s denotes the longest path from $s \in \mathcal{S}$ to the destination, we could transmit $B - L_s$ messages from information source s to the destination. However, for simplicity of notation and analysis we assume that all of the nodes send the same number of messages in a synchronized fashion.

[10] Note that the encoding functions thus constructed satisfy a causality condition that is more strict than what is defined in Section 2.4. Here, each transmitted block is only a function of immediately previous block of received symbols. In Section 2.3, each transmitted symbol could be a function of all previous symbols.

erasure locations and all of the encoding functions applied at different nodes in the network,[11] the destination can compute the values of $X_i^n(\underline{w}), Y_{ij}^n(\underline{w})$ and $Y_i^n(\underline{w})$ for all nodes and edges for any $\underline{w} \in \prod_{s\in\mathcal{S}} \mathcal{W}^{(s)}$. If there exists a unique message vector $\underline{w} \in \prod_{s\in\mathcal{S}} \mathcal{W}^{(s)}$ such that the computed value of $Y_d^n(\underline{w})$ equals the value $y_d^n(\underline{w}_0)$ of the received signal at the destination, then $\underline{w}$ is declared as the decoder output. Otherwise, the decoder declares an error.

Since the computed value of $Y_d^n(\underline{w}_0)$ for transmitted message $\underline{w}_0$ always matches the received signal at the destination, an error occurs if and only if there is another message vector $\underline{w} \neq \underline{w}_0$ for which $Y_d^n(\underline{w}) = Y_d^n(\underline{w}_0) = y_d^n(\underline{w}_0)$. In the next section we compute the probability of this event and show that for large blocks this probability can be made arbitrarily close to zero provided that the rate vector $(R_s,\, s \in \mathcal{S})$ is inside the capacity region described in Theorems 2.1 and 2.2.

2.6.1.3 Probability of Error

Let $\Pr(err)$ be the probability of error averaged over all possible functions f_i. In other words, if $P_e^{(n)}$ is the probability that $\hat{w}_0^{(s)}$, the destination's estimate of the transmitted message $\underline{w}_0$, is not equal to $\underline{w}_0$, then $\Pr(err)$ is the expected value of $P_e^{(n)}$ over all possible encoding functions at all nodes.[12] More precisely,

$$P_e^{(n)} = \Pr(\exists s \in \mathcal{S} \text{ s.t. } \hat{w}_0^{(s)} \neq w_0^{(s)}),$$

and $\Pr(err) = \mathrm{E}\, P_e^{(n)}$. Because of the symmetry of the code construction,

$$\Pr(err) = \Pr(err|\underline{W} = \underline{w}_0 \text{ is transmitted}), \tag{2.11}$$

[11]We also assume that the destinations knows the topology of the network.

[12]Note that if $P_e^{(n)}$ goes to zero as n grows larger, so will $P_d^{(n)(s)}$ of (2.6) for every $s \in \mathcal{S}$.

where $\underline{W} = (W^{(s)}, \quad s \in \mathcal{S})$. Therefore we will find the average probability of error when message vector $\underline{w}_0$ is transmitted from the sources. Recall the notation $X_i^n(\underline{w}_0)$ and $Y_i^n(\underline{w}_0)$ and $Y_{ij}^n(\underline{w}_0)$, $(i,j) \in \mathcal{E}$. For each $\underline{w} \in \underline{\mathcal{W}} \triangleq \prod_{s\in\mathcal{S}} \mathcal{W}^{(s)}$, $\underline{w} \neq \underline{w}_0$, define the following event:

$$E(\underline{w}) = \{Y_d^n(\underline{w}) = Y_d^n(\underline{w}_0)\}. \tag{2.12}$$

Let $\mathcal{A}_\delta^{(n)}(i)$ be the event that the erasure locations on the channels going out of node i are jointly δ-strongly typical, i.e.,

$$\mathcal{A}_\delta^{(n)}(i) = \{(z_{ij}^n,\ j : (i,j) \in \mathcal{E}) \quad \text{are jointly } \delta\text{-strongly typical}\},$$

[1, Equation (13.107)] and define

$$\mathcal{A}_\delta^{(n)} = \bigcap_{i=1}^{|\mathcal{V}|} \mathcal{A}_\delta^{(n)}(i).$$

Note that by the weak law of large numbers [53], $\Pr(\mathcal{A}_\delta^{(n)}(i)) \to 1$ as $n \to \infty$, and hence for all $\delta > 0$,

$$\Pr(\mathcal{A}_\delta^{(n)}) \geq 1 - |\mathcal{V}|\delta,$$

for n sufficiently large. Using the definition of the above events, $\Pr(err)$ can be written as

$$\begin{aligned}
&\Pr(err) \\
&= \Pr(err|\underline{W} = \underline{w}_0) \\
&= \Pr\Bigg(\bigcup_{\underline{w}\in\underline{\mathcal{W}}-\{\underline{w}_0\}} E(\underline{w})\Bigg) \\
&= \Pr\Bigg(\bigcup_{\underline{w}\in\underline{\mathcal{W}}-\{\underline{w}_0\}} E(\underline{w})|\mathcal{A}_\delta^{(n)}\Bigg)\Pr(\mathcal{A}_\delta^{(n)}) + \Pr\Bigg(\bigcup_{\underline{w}\in\underline{\mathcal{W}}-\{\underline{w}_0\}} E(\underline{w})|\mathcal{A}_\delta^{(n)^c}\Bigg)\Pr(\mathcal{A}_\delta^{(n)^c}) \\
&\leq \sum_{\underline{w}\in\underline{\mathcal{W}}-\{\underline{w}_0\}} \Pr(E(\underline{w})|\mathcal{A}_\delta^{(n)}) + |\mathcal{V}|\delta.
\end{aligned} \tag{2.13}$$

Therefore, using strong typicality and the union bound on the probability of events, we only look at network instantiations that are "strongly typical." We next bound the conditional probability of $E(\underline{w})$ given $\mathcal{A}_\delta^{(n)}$.

Corresponding to each cut in the network represented by *d-set* $\mathcal{V}_d \ni d$, define the following event:

$$B[\mathcal{V}_d] = \left(\bigcap_{i \in \mathcal{V}_d} \{Y_i^n(\underline{w}) = Y_i^n(\underline{w}_0)\} \right) \bigcap \left(\bigcap_{i \in \mathcal{V}_d^c} \{Y_i^n(\underline{w}) \neq Y_i^n(\underline{w}_0)\} \right). \tag{2.14}$$

The interpretation of the above event is as follows. By definition of $E(\underline{w})$, we know that the received signal at the destination is the same for $\underline{w}$ and $\underline{w}_0$, but $\underline{w} \neq \underline{w}_0$. Therefore, we can partition the nodes of the network into two sets: the "distinguishable" and the "indistinguishable" set. The "distinguishable" set contains all nodes for which the signal received at those nodes when $\underline{w}$ is transmitted differs from the signal received when $\underline{w}_0$ is transmitted. All the other nodes for which the received signals for $\underline{w}$ and $\underline{w}_0$ are the same are in the "indistinguishable" set. Clearly, these two sets define a cut. Event $B[\mathcal{V}_d]$ corresponds to the case when the "indistinguishable" set (containing d) is equal to $\mathcal{V}_d \subset \mathcal{V}$. Note that these events are all disjoint and also $E(\underline{w}) = \bigcup_{\mathcal{V}_d:\text{d-set}} B[\mathcal{V}_d]$.

Define

$$\mathcal{K}(\underline{w}) = \{s | s \in \mathcal{S}, w_0^{(s)} \neq w^{(s)}\} \tag{2.15}$$

to be the subset of source nodes for which the corresponding messages in $\underline{w}$ and $\underline{w}_0$ are different. Set $\mathcal{K}(\underline{w})$ is not empty since $\underline{w}_0 \neq \underline{w}$ by assumption. In what follows we bound the probability of event $B[\mathcal{V}_d]$ by considering the edges in the cut-set $[\mathcal{V}_x, \mathcal{V}_x^c]$, where $\mathcal{V}_x \triangleq \mathcal{V}_d^c \cup \mathcal{K}(\underline{w})$. Note that $\mathcal{V}_x^c$ is a d-set since if the destination is a source of information, it is aware of the message it has transmitted, and so $d \notin \mathcal{K}(\underline{w})$.

Consider any edge $(i, j) \in [\mathcal{V}_x, \mathcal{V}_x^c]$. We know that the transmitted signal $X_i^n = f_i(W^{(i)}, Y_i^n)$ from node i is a function of the message it wants to transmit , $W^{(i)}$, and the received signal at its incoming edges, Y_i^n. For any node i in $\mathcal{V}_d^c \cup \mathcal{K}(\underline{w})$, either the received signal Y_i^n or message $w^{(i)}$ is different for message vectors $\underline{w}$ and $\underline{w}_0$. Thus, for a randomly designed code, the transmitted signal by node i for message vector $\underline{w}$ is independent of the corresponding X_i^n for message vector $\underline{w}_0$. Using this observation we next bound the probability of the event $E(\underline{w})$ conditioned on $\mathcal{A}_\delta^{(n)}$.

$$\begin{aligned}
&\Pr\left(E(\underline{w})|\mathcal{A}_\delta^{(n)}\right)\\
=\ &\Pr\left(\bigcup_{\mathcal{V}_d:d\text{-}set} B[\mathcal{V}_d]|\mathcal{A}_\delta^{(n)}\right) = \sum_{\mathcal{V}_d:d\text{-}set}\Pr\left(B[\mathcal{V}_d]|\mathcal{A}_\delta^{(n)}\right)\\
=\ &\sum_{\mathcal{V}_d:d\text{-}set}\Pr\left(\left(\bigcap_{j\in\mathcal{V}_d}\{Y_j^n(\underline{w})=Y_j^n(\underline{w}_0)\}\right)\bigcap\left(\bigcap_{i\in\mathcal{V}_d^c}\{Y_i^n(\underline{w})\neq Y_i^n(\underline{w}_0)\}\right)|\mathcal{A}_\delta^{(n)}\right)\\
\overset{(a)}{\leq}\ &\sum_{\mathcal{V}_x:\mathcal{K}(\underline{w})\subset\mathcal{V}_x}\Pr\Big(\bigcap_{i,j:(i,j)\in[\mathcal{V}_x,\mathcal{V}_x^c]}\{(w^{(i)},Y_i^n(\underline{w}))\neq(w_0^{(i)},Y_i^n(\underline{w}_0)),\ Y_{ij}^n(\underline{w})=Y_{ij}^n(\underline{w}_0)\}|\mathcal{A}_\delta^{(n)}\Big)\\
\overset{(b)}{=}\ &\sum_{\mathcal{V}_x:\mathcal{K}(\underline{w})\subset\mathcal{V}_x}\Pr\Big(\bigcap_{i\in\mathcal{V}_x^*}\bigcap_{j:(i,j)\in[\mathcal{V}_x,\mathcal{V}_x^c]}\{(w^{(i)},Y_i^n(\underline{w}))\neq(w_0^{(i)},Y_i^n(\underline{w}_0)),\ Y_{ij}^n(\underline{w})=Y_{ij}^n(\underline{w}_0)\}|\mathcal{A}_\delta^{(n)}\Big)
\end{aligned}$$

Here (a) follows since $\Pr(A,B)\leq\Pr(A)$ for any events A and B. Instead of looking at equalities on every edge and every node of the network, we are looking at the nodes having an edge from $[\mathcal{V}_x,\mathcal{V}_x^c]$ connected to them, where $\mathcal{V}_x=\mathcal{V}_d^c\cup\mathcal{K}(\underline{w})$. (b) is clear from the definition of $\mathcal{V}_x^*$. Simplifying the probability of error further, we have

$$\begin{aligned}
&\Pr\left(E(\underline{w})|\mathcal{A}_\delta^{(n)}\right) \qquad (2.16)\\
\overset{(c)}{=}\ &\sum_{\substack{\mathcal{V}_x\\ \mathcal{K}(\underline{w})\subset\mathcal{V}_x}}\Pr\left(\bigcap_{i\in\mathcal{V}_x^*}\bigcap_{(i,j)\in[\mathcal{V}_x,\mathcal{V}_x^c]}\{Y_{ij}^n(\underline{w})=Y_{ij}^n(\underline{w}_0)\}\middle|\mathcal{A}_\delta^{(n)},\ \bigcap_{i\in\mathcal{V}_x^*}\{(w^{(i)},Y_i^n(\underline{w}))\neq(w_0^{(i)},Y_i^n(\underline{w}_0))\}\right)\\
&\qquad\cdot\Pr\left((w^{(i)},Y_i^n(\underline{w}))\neq(w_0^{(i)},Y_i^n(\underline{w}_0)),\ \forall\, i\in\mathcal{V}_s^*\right)\\
\leq\ &\sum_{\substack{\mathcal{V}_x\\ \mathcal{K}(\underline{w})\subset\mathcal{V}_x}}\Pr\left(\bigcap_{i\in\mathcal{V}_x^*}\bigcap_{(i,j)\in[\mathcal{V}_x,\mathcal{V}_x^c]}\{Y_{ij}^n(\underline{w})=Y_{ij}^n(\underline{w}_0)\}\middle|\mathcal{A}_\delta^{(n)},\ \bigcap_{i\in\mathcal{V}_x^*}\{(w^{(i)},Y_i^n(\underline{w}))\neq(w_0^{(i)},Y_i^n(\underline{w}_0))\}\right)\\
\overset{(d)}{=}\ &\sum_{\substack{\mathcal{V}_x:\mathcal{K}(\underline{w})\subset\mathcal{V}_x\\ d\notin\mathcal{V}_x}}\prod_{i\in\mathcal{V}_x^*}\Pr\left(\bigcap_{j:(i,j)\in[\mathcal{V}_x,\mathcal{V}_x^c]}\{Y_{ij}^n(\underline{w})=Y_{ij}^n(\underline{w}_0)\}\middle|\mathcal{A}_\delta^{(n)},\ \{(w^{(i)},Y_i^n(\underline{w}))\neq(w_0^{(i)},Y_i^n(\underline{w}_0))\}\right),
\end{aligned}$$

where (c) is clear from the definition of conditional probability and (d) follows from fact that averaged over all possible functions f_i, the conditional events shown in the equation are independent for different is in $\mathcal{V}_x^*$.

Now we bound the expression given in (2.16) for any node $i \in \mathcal{V}_x^*$. Note that since $(w^{(i)}, Y_i^n(\underline{w})) \neq (w_0^{(i)}, Y_i^n(\underline{w}_0))$ at node i, $X_i^n(\underline{w}) = f_i(w^{(i)}, Y_i^n(\underline{w}))$ and $X_i^n(\underline{w}_0) = f_i(w_0^{(i)}, Y_i^n(\underline{w}_0))$ are chosen independently and uniformly from $\{0,1\}^n$. Therefore, the probability that they are the same in at least α_i specific locations is at most $2^{-\alpha_i}$. Looking at a fixed node i, $Y_{ij}^n(\underline{w}) = Y_{ij}^n(\underline{w}_0)$ for all j such that $(i,j) \in [\mathcal{V}_x, \mathcal{V}_x^c]$ only if all the locations where $X_i^n(\underline{w})$ and $X_i^n(\underline{w}_0)$ differ get erased on all these edges. Because of the δ-strong typicality of the erasure locations on edges $(i,j) \in [\mathcal{V}_x, \mathcal{V}_x^c]$, the number of locations at which erasure occurs on all the edges of interest, say $\alpha_i(\mathcal{V}_x)$, satisfies

$$\left| \frac{1}{n}\alpha_i(\mathcal{V}_x) - \Pr\left(Z_{ij} = 0,\ j : (i,j) \in [\mathcal{V}_x, \mathcal{V}_x^c]\right) \right| \leq \frac{\delta}{2^{|\{j:(i,j)\in[\mathcal{V}_x,\mathcal{V}_x^c]\}|}} \leq \delta.$$

Therefore, $X_i^n(\underline{w})$ and $X_i^n(\underline{w}_0)$ cannot differ in more than $n(\Pr(Z_{ij} = 0,\ j : (i,j) \in [\mathcal{V}_x, \mathcal{V}_x^c]) + \delta)$ locations, and the probability of this event is no more than

$$\exp\left(-n(1 - \Pr\left(Z_{ij} = 0,\ j : (i,j) \in [\mathcal{V}_x, \mathcal{V}_x^c]\right) - \delta)\right) = \exp\left(-n(1 - \prod_{j:(i,j)\in[\mathcal{V}_x,\mathcal{V}_x^c]} \epsilon_{ij} - \delta)\right). \tag{2.17}$$

Combining this with the last equation of (2.16) gives [13]

$$\begin{aligned}
\Pr\left(E(\underline{w})|\mathcal{A}_\delta^{(n)}\right) &\leq \sum_{\mathcal{V}_x:\mathcal{K}(\underline{w})\subset\mathcal{V}_x} \prod_{i\in\mathcal{V}_x^*} \exp\left(-n(1 - \prod_{j:\,(i,j)\in[\mathcal{V}_x,\mathcal{V}_x^c]} \epsilon_{ij} - \delta)\right) \\
&= \sum_{\mathcal{V}_x:\mathcal{K}(\underline{w})\in\mathcal{V}_x, d\notin\mathcal{V}_x} 2^{n|\mathcal{V}_x^*|\delta} \cdot \exp\left(-n\sum_{i\in\mathcal{V}_x^*}(1 - \prod_{j:\,(i,j)\in[\mathcal{V}_x,\mathcal{V}_x^c]} \epsilon_{ij})\right) \\
&\leq 2^{n|\mathcal{V}|\delta} \sum_{\mathcal{V}_x:\mathcal{K}(\underline{w})\subset\mathcal{V}_x, d\notin\mathcal{V}_x} 2^{-nC(\mathcal{V}_x)}.
\end{aligned} \tag{2.18}$$

[13]Using (2.17) it can be easily verified that the arguments that follow will exactly go through for correlated erasure events with cut-capacity, $C(\mathcal{V}_x)$, defined as in 2.8.

Combining (2.13) and (2.18) together gives

$$
\begin{aligned}
\Pr(err) &\le |\mathcal{V}|\delta + 2^{n|\mathcal{V}|\delta} \sum_{\underline{w}\in\underline{\mathcal{W}}-\{\underline{w}_0\}} \sum_{\substack{\mathcal{V}_x:\mathcal{K}(\underline{w})\subset\mathcal{V}_x \\ d\notin\mathcal{V}_x}} 2^{-nC(\mathcal{V}_x)} \\
&= |\mathcal{V}|\delta + 2^{n|\mathcal{V}|\delta} \sum_{\mathcal{K}\subset\mathcal{S}} \sum_{\substack{\underline{w}\in\underline{\mathcal{W}}-\{\underline{w}_0\} \\ \mathcal{K}(\underline{w})=\mathcal{K}}} \sum_{\substack{\mathcal{V}_x:\mathcal{K}(\underline{w})\subset\mathcal{V}_x \\ d\notin\mathcal{V}_x}} 2^{-nC(\mathcal{V}_x)} \\
&= |\mathcal{V}|\delta + 2^{n|\mathcal{V}|\delta} \sum_{\mathcal{K}\subset\mathcal{S}} \sum_{\substack{\mathcal{V}_x:\mathcal{K}\subset\mathcal{V}_x \\ d\notin\mathcal{V}_x}} \sum_{\substack{\underline{w}\in\underline{\mathcal{W}}-\{\underline{w}_0\} \\ \mathcal{K}(\underline{w})=\mathcal{K}}} 2^{-nC(\mathcal{V}_x)} \\
&= |\mathcal{V}|\delta + 2^{n|\mathcal{V}|\delta} \sum_{\mathcal{K}\subset\mathcal{S}} \sum_{\substack{\mathcal{V}_x:\mathcal{K}\subset\mathcal{V}_x \\ d\notin\mathcal{V}_x}} \prod_{s\in\mathcal{K}} (\lceil 2^{nR_s}\rceil - 1) 2^{-nC(\mathcal{V}_x)} \\
&\overset{(a)}{\le} |\mathcal{V}|\delta + 2^{n|\mathcal{V}|\delta} \sum_{\mathcal{K}\subset\mathcal{S}} \sum_{\substack{\mathcal{V}_x:\mathcal{K}\subset\mathcal{V}_x \\ d\notin\mathcal{V}_x}} 2^{-n(C(\mathcal{V}_x)-\sum_{s\in\mathcal{K}} R_s)} \\
&\overset{(b)}{=} |\mathcal{V}|\delta + 2^{n|\mathcal{V}|\delta} \sum_{\mathcal{V}_x:\mathcal{V}_x\subset\mathcal{V}-\{d\}} \sum_{\mathcal{K}\subset\mathcal{S}\cap\mathcal{V}_x} 2^{-n(C(\mathcal{V}_x)-\sum_{s\in\mathcal{K}} R_s)} \\
&\overset{(c)}{\le} |\mathcal{V}|\delta + 2^{n|\mathcal{V}|\delta} \sum_{\mathcal{V}_x:\mathcal{V}_x\subset\mathcal{V}-\{d\}} 2^{|\mathcal{V}_x\cap\mathcal{S}|} 2^{-n(C(\mathcal{V}_x)-\sum_{s\in\mathcal{S}\cap\mathcal{V}_x} R_s)},
\end{aligned} \tag{2.19}
$$

where we have used the inequality $\lceil 2^{nR_s}\rceil - 1 \le 2^{nR_s}$ in (a). Also, (b) is derived by changing the order of summation, and (c) follows from bounding $\sum_{s\in\mathcal{K}} R_s$ by $\sum_{s\in\mathcal{V}_x\cap\mathcal{S}} R_s$ in (c). Now, by assumption the rate vector $(R_s, \quad s\in\mathcal{S})$ is inside the capacity region given in Theorem 2.2. Therefore, for any partition of the nodes into $\mathcal{V}_x$ and $\mathcal{V}_x^c \ni d$ we have $C(\mathcal{V}_x) - \sum_{s\in\mathcal{S}\cap\mathcal{V}_x} R_s > 0$. Therefore the exponent in the last term of the above summation is negative. The above result holds for any $\delta > 0$ and n sufficiently large. By letting $n\to\infty$ and $\delta\to 0$, we can make the upper bound on the probability of error arbitrarily close to zero. Now, by standard coding arguments we conclude that there exists some deterministic choice of encoding functions that has an arbitrarily small probability of error for the rates in the achievable rate region $C(\mathcal{G}, \mathcal{S}, d)$.

2.6.2 Proof of Theorem 3

In this section we outline the proof of Theorem 2.3. The analysis is very similar to Theorem 2.1. The converse part is straightforward. We know that the sources can be recovered at all the destinations; therefore we have the same argument as the converse part of Theorem 1 for the sources and any of the destinations. In particular, for any destination $d_i, \quad i \in \mathcal{D}$, we have $(R_s, \quad s \in \mathcal{S}) \in C(\mathcal{G}, \mathcal{S}, d_i)$. Therefore, any achievable rate vector should be in the intersection of these capacity regions, i.e,

$$(R_s,\ s \in \mathcal{S}) \in \cap_{d_i \in \mathcal{D}} C(\mathcal{G}, \mathcal{S}, d_i) = C(\mathcal{G}, \mathcal{S}, \mathcal{D}).$$

Hence, the converse part is done.

In order to prove the achievability of the above rates, we can use the random coding argument of Section 1. Note that averaged over all the codebooks and functions, the probability of error for each destination goes to zero. Therefore, using the union bound on probability of events, the probability of having an error in at least one destination (averaged over all the functions and codebooks) goes to zero. Using standard arguments, there exists some deterministic choice of codebooks and functions for which the probability of error in the network become arbitrarily small and that shows the achievability of the rates in $C(\mathcal{G}, \mathcal{S}, \mathcal{D})$ of Theorem 2.3 for the multiple destination case.

2.7 Linear Encoding

In Section 2.6.1.2 we showed the achievability of the capacity region as defined in Theorem 2.2 by using general random coding functions at different nodes of the network. In this section we restrict our attention to linear encoding schemes. The advantage of using linear encoding scheme is that the decoding process becomes much easier. In this case, the equivalent transfer function of the network from any source to any destination, having the erasure locations at that destination, is linear. Hence, decoding at the destination is simply forming and solving a linear system of equations.

In this section we show that linear encoders achieve capacity. Let us first define the linear block coding scheme with block length of n:

Recall that $\mathcal{W}^{(s)} = \{1, 2, \ldots, \lceil 2^{nR_s} \rceil\}$ is the message set for information source $s \in \mathcal{S}$. We assume that different messages are equiprobable and independent of each other. For any $w^{(s)} \in \mathcal{W}^{(s)}$, let $\underline{b(w^{(s)})}$ be the length-nR_s binary expansion of $w^{(s)} - 1$.

The encoding operation is as follows:

Each node $i \in \mathcal{V}$ transmits n linear combinations of the non-erased symbols received from its incoming edges and the binary representation of the message it wants to transmit across the network. More precisely, node i generates a random binary matrix B_i of size $n \times n(d_I(i) + R_i)$, where $d_I(i)$ is the in-degree of node i and R_i is the rate of the codebook used at node i (in the case where i is not a source of information $R_i = 0$). Each element of B_i is drawn i.i.d. Bernoulli(1/2). For a given sequence y, let $\widetilde{y}$ be a sequence derived by replacing every e with 0. Note that $\widetilde{y}$ and y have the same length.[14] If node i receives $Y_i^n = y_i^n$ on its incoming edges and wants to transmit message $w^{(i)}$, then it sends out $x_i = B_i \cdot [\underline{b(w^{(i)})}, \widetilde{y_i^n}]^\dagger$. (Since the input-output relation at each node is linear, setting the erased symbols equal to zero is the same as finding linear combinations of only the non-erased bits.)

Each destination d knows all the matrices B_i and also the erasure locations Z^n on all the links across the network, since each received and transmitted symbol at any node is a linear combination of the elements of vector $b(\underline{w}) \triangleq (\underline{b(w^{(s)})},\ s \in \mathcal{S})$. Therefore, each destination receives a collection of linear combinations of elements of $b(\underline{w})$. Using $\{B_i\}_{i \in \mathcal{V}}$ and Z^n, destination node d can construct the matrix that corresponds to the linear input-output relation of the network. We denote this matrix

[14]The corresponding mapping from alphabet $\mathrm{GF}(q) \cup \{e\}$ to $\mathrm{GF}(q)$ again replaces e with 0. This variation is useful for packet erasure networks.

by $F(\{B_i\}, Z^n)$, giving $\widetilde{Y_d^n}(\underline{w}) = F(\{B_i\}, Z^n) \cdot b(\underline{w})^\dagger$. Note that matrix F is a function of different nodes' encoding matrices $\{B_i\}$ and Z^n.

Now, upon receiving $Y_d^n = y \in \{0, 1, e\}^{nd_I(d)}$, the destination node d looks (solves) for the message vector $\underline{w} \in \underline{\mathcal{W}} \triangleq \prod_{s \in \mathcal{S}} \mathcal{W}^{(s)}$ such that $F(\{M_i\}, Z^n) \cdot b(\underline{w})^\dagger = \widetilde{y}$. If there is a unique $\underline{w}$ with this property, node d declares it as the transmitted message vector; otherwise it declares an error. Note that the actual transmitted message vector, say $\underline{w}_0 \in \underline{\mathcal{W}}$, always satisfies the above property. Therefore, an error occurs only if there is another message vector $\underline{w} \neq \underline{w}_0$ such that $Y_d^n(\underline{w}) = Y_d^n(\underline{w}_0) = y$.

2.7.1 Achievability Result for Linear Encoding

Looking at the achievability proof and probability of error analysis for general random coding in Sections 2.6.1.2 and 2.6.1.3, it can be easily verified that the linear case requires the same error events (2.12). Since the erasure vector Z^n is available at the destination, there is no difference between $\widetilde{Y}_i$ and Y_i, and we can determine one from the other. By expanding the conditional error event $E(\underline{w})$ given $A_\delta^{(n)}$ for different cuts in the network, all of the relations up to step (d) of equation (2.16) go through for the linear case. In fact, the relations up to step (d) only require the independence of encoding functions for different nodes of the network, which holds for the linear case. Now, we look at the following probability in (2.16):

$$P_i \triangleq \Pr\left(\bigcap_{j:(i,j) \in [\mathcal{V}_x, \mathcal{V}_x^c]} \{Y_{ij}^n(\underline{w}) = Y_{ij}^n(\underline{w}_0)\} \middle| \mathcal{A}_\delta^{(n)},\ \{(w^{(i)}, Y_i^n(\underline{w})) \neq (w_0^{(i)}, Y_i^n(\underline{w}_0))\} \right). \tag{2.20}$$

As in the general random coding argument, for a fixed i we have $Y_{ij}^n(\underline{w}) = Y_{ij}^n(\underline{w}_0)$ for all j such that $(i, j) \in [\mathcal{V}_x, \mathcal{V}_x^c]$, only if $X_i^n(\underline{w})$ and $X_i^n(\underline{w}_0)$ differ in locations where an erasure occurs on all the edges of the interest. Because of strong typicality, the number of these location is at most $n(\prod_{j:\ \ (i,j) \in [\mathcal{V}_x, \mathcal{V}_x^c]} \epsilon_{ij} + \delta)$. Therefore $X_i^n(\underline{w})$ and $X_i^n(\underline{w}_0)$ should be the same in at least $n(1 - \prod_{j:\ \ (i,j) \in [\mathcal{V}_s, \mathcal{V}_s^c]} \epsilon_{ij} - \delta)$ locations. But by our encoding scheme this means that

$$B_i \cdot \underbrace{([w^{(i)}, Y_i^n(\underline{w})]^\dagger - [w_0^{(i)}, Y_i^n(\underline{w}_0)]^\dagger)}_{\underline{z}}$$

should be zero in at least $n(1 - \prod_{j:\ (i,j)\in[\mathcal{V}_s,\mathcal{V}_s^c]} \epsilon_{ij} + \delta)$ specific locations. Also note that since $(w^{(i)}, Y_i^n(\underline{w})) \neq (w_0^{(i)}, Y_i^n(\underline{w}_0))$, $\underline{z}$ is a non-zero vector. From the above argument we have

$$\begin{aligned} P_i &\leq \Pr\left(B_i \cdot \underline{z} \text{ be 0 in at least } n\alpha_i \text{ specific locations} \,\middle|\, \underline{z} \neq 0\right) \\ &\overset{(a)}{\leq} 2^{-n\alpha_i} = 2^{-n(1-\prod_{j:\,(i,j)\in[\mathcal{V}_x,\mathcal{V}_x^c]} \epsilon_{ij} - \delta)}, \end{aligned} \tag{2.21}$$

where $\alpha_i = 1 - \prod_{j:\,(i,j)\in[\mathcal{V}_x,\mathcal{V}_x^c]} \epsilon_{ij} - \delta$, and (a) follows from the following Lemma and its Corollary. Proof of this lemma is provided in [57].

Lemma 2.4. *Let X be a non-zero vector of size $n \times 1$ from some finite field $GF(q)$. Suppose that A is a random matrix of size $m \times n$ with i.i.d. components distributed uniformly over $GF(q)$. Then, the coordinates of $Y = A \cdot X$ are i.i.d. uniform random variables over $GF(q)$.*

Corollary 2.5. *The probability that $Y = A \cdot X$ is zero in k specific coordinates equals q^{-k}.*

Now note that by replacing P_i in (2.20) and (2.16) with its bound from (2.21), we get the same bound as (2.18) for random linear codes. Therefore, linear operations are sufficient for achieving the capacity.

2.8 Discussion

2.8.1 Packet Size and Cycles

We considered binary erasure networks in this work. However, as mentioned earlier, the obtained results hold for any packet length (or more generally for any input alphabet size). The theorems stated in Section 2.5 give the maximum achievable rate per packet. Therefore, if one is interested in the maximum achievable rate per bit, assuming that the size of each packet is L bits (resp. the alphabet size is M) across the network, the capacity will be L (respectively $\log_2 M$) times the capacity as stated in the theorems.

Also, we assume that the graph representation of the wireless erasure network is acyclic. However, the upper bound derived in [57] does not rely on this assumption. By an approach similar to [9], [54], [23] (Section 11.5.2), it can be shown that the upper bound is still achievable, and therefore the capacity theorems holds. We do not get into this problem in detail here.

2.8.2 Side-information at Destination Nodes

The results stated up to now are based on the perfect knowledge of the erasure locations for each link of the network to be available at destination nodes.

Erasure channels are usually used in modeling networks for which there exists a mechanism by which the receiver (destination) can be informed of a packet dropping. Usually, this side-information is provided by using sequencing numbers in the packet header to detect lost packets. However, if we do not provide the destination with this side-information, even for the simplest case of point to point communication, the capacity is not known. In this case, the communication system is usually modeled by the *deletion channel*. This channel has been studied by some researchers, and lower and upper bounds on its capacity are found in [24, 25, 26].

Looking back at our network, for each block there are $n|\mathcal{E}|$ transmissions of packets across the network. Therefore, the erasure locations on the links of the network can be represented by $n|\mathcal{E}|$ bits. These $n|\mathcal{E}|$ should be provided to the destination through some mechanism. One approach is to use part of each packet as a header to transmit this information.

If the size of each packet is L bits, then based on our result we are able to send nCL bits across the network in a block of length n, where C is the minimum cut-capacity of the network. If the size of the packet, L, is large compared to the size of the network, or if the network is small, i.e., $|\mathcal{E}|$ is small, the amount of side-information required is negligible compared to the amount of information sent across the network. We should remark that if one is trying to map a real network to our model, the packet length and the probability of erasure are closely related, and there will be some trade-off between them.

A number of techniques can be used to reduce the required overhead for providing side-information at the destination(s). For instance, consider a wireless erasure network with one destination d. Let C denote the minimum cut-capacity for this node. Based on Theorem 2.1, this is the maximum achievable rate at d. Now consider $\mathcal{Q}$ to be the subset of nodes for which the minimum cut-capacity is greater than C. If we decode the messages completely and then re-encode them using random code-books, we can still achieve the capacity at destination d. However, doing this may reduce the amount of overhead required at the destination. As an example, let's look at a line network (Figure 2.5) from this point of view. The source node is the leftmost node, and the destination is the rightmost node. The minimum cut-capacity for the destination is less than or equal to the minimum cut-capacity for every other node. Therefore, the intermediate nodes can decode without degrading the performance at the destination d. Further, this approach decreases the amount of overhead required. In fact, for this special case, by decoding at every node no side-information from

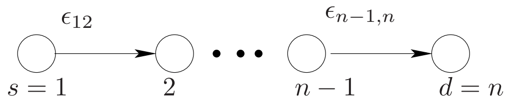


Figure 2.5: A wireless erasure line network.

previous links is required at the destination. Another technique is scheduling among the nodes to minimize the average header size needed for sending across the required side-information. In this scheduling, for any link in the network we determine the nodes that should include the erasure locations on that link as a header in their transmitted packets.

Remark: A closer look at the achievability proof of Section 2.6.1.2 reveals that all that the destination nodes need to know is the mapping from the source nodes to their incoming signals for every instantiation of the network. (In other words, for every instantiation of the network, the destinations should be able to unambiguously compute their output for any given input to the network.) Any mechanism that provides destination nodes with the knowledge of this mapping will work. Providing the erasure locations for each link of the network is one possible mechanism. In the subsequent work of [21], another kind of side-information is considered for the linear encoding scenario. There, the side information is the global encoding vectors, which also allow for the input output mapping to be determined.

2.8.3 Achievable Rates Without Side-information

In this section we look at the achievable rate for single source / single destination wireless erasure networks under a number of coding schemes when the side-information is not available at the destination.

"Forward" and "Decode" Scheme

In this scheme we analyze the performance of wireless erasure networks when limited operations are allowed at each node. Consider a codebook $\mathcal{C}$ of rate R and block size n. This codebook is available at all the nodes and is used for encoding the information message. The source node uses this codebook to "encode" the information. We assume that all the other nodes are allowed to perform one of the following operations:

- **Forward:** A node operating at this mode forwards received strings unchanged.[15]
- **Decode and re-encode:** In this case, the node first decodes the message transmitted from the source node based on what it has received and the codebook $\mathcal{C}$. Then, it sends out the codeword corresponding to that message in $\mathcal{C}$ across its outgoing links. In this way, each relay node acts as a "source" of information for other nodes in the network. The need for successful decoding at intermediate nodes may reduce R the rate of the codebook used.

The main observation is that since the source message is intended to be decoded only at destination nodes, decoding at one relay node may be sub-optimal. In fact, as it will be shown in Chapter 4, the distinguishing features of wireless media imply that decoding at every relay node and operating below the capacity of each link in the network can result in severe degradation in the achievable rate in the network (for more examples See [27, 55, 56]). For instance, for the erasure wireless networks considered here, the maximum rate when all the nodes are decoding the source message using codebook $\mathcal{C}$ is given by the minimum capacity of the links in the network, i.e.,

$$R_{AD} = 1 - \max_{(i,j)\in\mathcal{E}} \epsilon_{ij}, \tag{2.22}$$

[15]In this scheme we consider a modified erasure channel between any two nodes in which the nodes can forward the erasure symbol without error. In other words, although similar to previous sections; however the nodes are not allowed to perform coding on erasure symbol, they can inform their local neighbors of erasure of packets.

where subscript AD refers to the all decoding case.

In the "forward" and "decode" scheme, instead of requiring all the nodes to decode the source message, we allow for another operation: "forward"ing. The objective is to find the optimal operation at every node so as to maximize the achievable rate, i.e., the rate of the codebook, in this network. In Chapter 4 we propose an efficient algorithm that finds the optimal rate for this scheme. We use R_{FD} to refer to the optimal rate using the "forward" and "decode" scheme.

Block Markov Superposition Coding

Cover and El Gamal [14] developed a coding strategy for general single relay channels based on block Markov coding and random partitioning. This strategy is generalized and used in a multiple relay setup in [15],[16].

Let $\pi(.)$ be a permutation on $\mathcal{V}$ that fixes the source and destination nodes. This permutation determines the order in which the nodes decode and encode the information. Using this coding strategy, it is shown in [15] that we can achieve

$$R_{BM} = \max_{\pi(.)} \min_{1 \leq t \leq |\mathcal{V}|-1} I(X_{\pi(1:t)}; Y_{\pi(t+1)} | X_{\pi(t+1:|\mathcal{V}|-1)}),$$

where subscript BM is used to refer to block Markov coding, and $\pi(i:j) = \{\pi(i), \pi(i+1), \ldots, \pi(j)\}$. We should remark that one can choose any distribution on $(X_1, X_2, \ldots, X_{|\mathcal{V}|})$ in the above result. Applying this result to our case, we show in the following lemma that the maximum rate is achieved when the X_is are independent and uniformly distributed.

Lemma 2.6. *The maximum achievable rate using block Markov coding for wireless erasure networks is given by*

$$R_{BM} = \min_{j \in \mathcal{V}} \sum_{i:\,(i,j) \in \mathcal{E}} (1 - \epsilon_{ij}). \tag{2.23}$$

Furthermore, this rate is achieved by applying independent coding at different nodes of the network.

Proof. As mentioned earlier, the achievable rate using block Markov coding is given by

$$R_{BM} = \max_{\pi(.)} \min_{1\leq t\leq |\mathcal{V}|-1} I(X_{\pi(1:t)}; Y_{\pi(t+1)}|X_{\pi(t+1:|\mathcal{V}|-1)}),$$

where the maximization is over the joint distribution of the $(X_i, \quad i \in \mathcal{V})$ and $\pi(.)$ permutations of the nodes that keep source and destination fixed. Now, for any mutual information term in the above equation we have

$$\begin{aligned}
& I(X_{\pi(1:t)}; Y_{\pi(t+1)}|X_{\pi(t+1:|\mathcal{V}|-1)}) \\
&= H(Y_{\pi(t+1)}|X_{\pi(t+1:|\mathcal{V}|-1)}) - H(Y_{\pi(t+1)}|X_{\pi(t+1:|\mathcal{V}|-1)}, X_{\pi(1:t)}) \\
&\leq H(Y_{\pi(t+1)}) - H(Y_{\pi(t+1)}|X_i, \quad i \in \mathcal{V}) \\
&\overset{(a)}{=} H(Y_{\pi(t+1)}) - H(Y_{\pi(t+1)}|X_i, \quad (i, \pi(t+1)) \in \mathcal{E}) \\
&\overset{(b)}{\leq} \sum_{i:\,(i,\pi(t+1))\in\mathcal{E}} H(Y_{i\pi(t+1)}) - H(Y_{i\pi(t+1)}|X_i) \\
&= \sum_{i:\,(i,\pi(t+1))\in\mathcal{E}} I(Y_{i\pi(t+1)}; X_i) \\
&\overset{(c)}{\leq} \sum_{i:\,(i,\pi(t+1))\in\mathcal{E}} 1 - \epsilon_{i\pi(t+1)}
\end{aligned} \tag{2.24}$$

where

- (a),(b) follows from the fact the network is memoryless; therefore given $X_i \quad (i, j) \in \mathcal{E}$, Y_{ij}s are independent from each other and also output of other nodes.
- (c) follows from the fact that the capacity of an erasure channel with probability of erasure ϵ_{ij} is $1 - \epsilon_{ij}$.

Using (2.24), we have

$$R_{BM} \leq \min_{j\in\mathcal{V}} \sum_{i:\,(i,j)\in\mathcal{E}} 1 - \epsilon_{ij}.$$

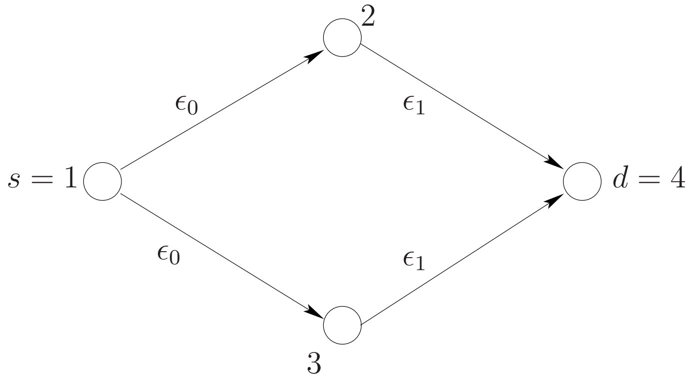


Figure 2.6: A simple network.

Now we can easily verify that by choosing X_is independent and uniformly distributed and by considering a permutation that is faithful to partial ordering of the nodes defined in Section 2.2.2, we can achieve the right-hand side of the above equation. ∎

The above lemma (and in particular (2.23)) suggests that the achievable rate in block Markov scheme is constrained by the minimum of the sum-capacities of incoming edges to any node in the network. This constraint is less severe than the all decoding case in (2.22). Here, instead of requiring all the links to be error-free, we only require that the relay nodes be able to decode the information. However, the achievable rate is still constrained by the capability of each relay node to decode the information. The example that follows demonstrates that block Markov coding is not always efficient in our network.

There are other strategies such as compress-forward and also partial decoding at the relay nodes that can be implemented in the network [14, Theorem 6],[28],[16]. Finding the achievable rates for these schemes explicitly and in terms of parameters of the network is usually intractable since the expressions for these schemes are not simple and involve a number of auxiliary random variables. In the rest of this section we compare the performance of these schemes for a very simple but interesting erasure broadcast network. The capacity of this simple network is not known to our knowledge.

A Simple Example

The network under study here has a graph representation shown in Figure 2.6. It has two relay nodes and one destination. We assume that the relay nodes are identical in the sense that their connections to the source (respectively destination) have the same probability of erasure equal to ϵ_0 (respectively ϵ_1). The capacity of the network given side-information at the destination is $C_{SI} = \min\{1 - \epsilon_0^2, 2 - 2\epsilon_1, 2 - \epsilon_1 - \epsilon_0\}$. Using the "forward" and "decode" strategy, the maximum achievable rate is given by [27]

$$R_{FD} = \max\{1 - \max\{\epsilon_1, \epsilon_0\}, 1 - (1 - (1 - \epsilon_0)(1 - \epsilon_1))^2\}.$$

Using (2.23), the block Markov coding scheme achieves rates up to

$$R_{BM} = \min\{1 - \epsilon_0, 2(1 - \epsilon_1)\}.$$

Another strategy that can be used is for the relay nodes to encode and compress their received signals, Y_i, with rate R_i and send them to the destination node reliably. Since the received signals at the relay nodes are correlated, the Slepian-Wolf encoding scheme can be used [29]. However, this scheme works only if the capacity of the channel between relay node i and the destination is larger than R_i. Combining this with the Slepian-Wolf rate region [29], we should have

$$H(Y_2|Y_3) \leq 1 - \epsilon_1 \quad \text{and} \quad H(Y_2, Y_3) \leq 2(1 - \epsilon_1).$$

If the above conditions are satisfied, the destination will have access to both observations Y_2 and Y_3, and therefore we can achieve a rate of $I(X_1; Y_2, Y_3)$ in the network.

Now suppose that the distribution on the input signal, X_1, is given by vector $\underline{p}$.

It can be verified that $H(Y_2|Y_3) = H(\epsilon_0) + \epsilon_0(1-\epsilon_0)H(\underline{p})$, $H(Y_3, Y_2) = (1-\epsilon_0^2)H(\underline{p}) + 2H(\epsilon_0)$, and $I(X_1; Y_2, Y_3) = (1-\epsilon_0^2)H(\underline{p})$. By choosing the probability distribution appropriately, we can achieve

$$R_{SW} = \min\{1 - \epsilon_0^2, 2\{1 - \epsilon_1 - H(\epsilon_0)\}^+\},$$

where $\{x\}^+ = \max\{0, x\}$, and the subscript SW is used to refer to the Slepian-Wolf coding scheme. Note that this scheme does not work if the quality of the channels from the relay nodes to the destinations is low, i.e., if ϵ_1 is large. In this case the second term in the above formula becomes zero, and therefore R_{SW} equals zero.

We have plotted the performance of the above-mentioned strategies for four different scenarios in Figure 2.7. In Figure 2.7.(a), we plot the performance for $\epsilon_0 = \epsilon_1 \in [0, 1]$. For small values of ϵ_0, the Slepian-Wolf strategy achieves capacity. Unfortunately, this approach performs poorly for large values of ϵ_0 since the quality of the channel between relay nodes and the destination is not good enough to pass the compressed data reliably. Figure 2.7.(b) shows the results for $\epsilon_1 = \epsilon_0^3$. This choice corresponds to the case when the quality of the channels from the relay nodes to the destination is better than the source-to-relay connections. Note that the performance of the block Markov scheme is not good because of the rate constraint introduced by decoding at the relay nodes. The Slepian-Wolf scheme works for a larger range of ϵ_0 compared to the network considered in part (a). This is because the channels between the source and the destination are better than in the former case. In Figure2.7.(c), we look at one extreme case when ϵ_1 is zero, giving a perfect channel between the relays and the destination. As the figure shows, the "forward" and "decode" scheme achieves the capacity in this cases. The rate of the Slepian-Wolf scheme decreases for intermediate values of ϵ_0, but as we increase ϵ_0 again this scheme achieves the capacity. This happens because for values of ϵ around 0.5, the entropy of the re-

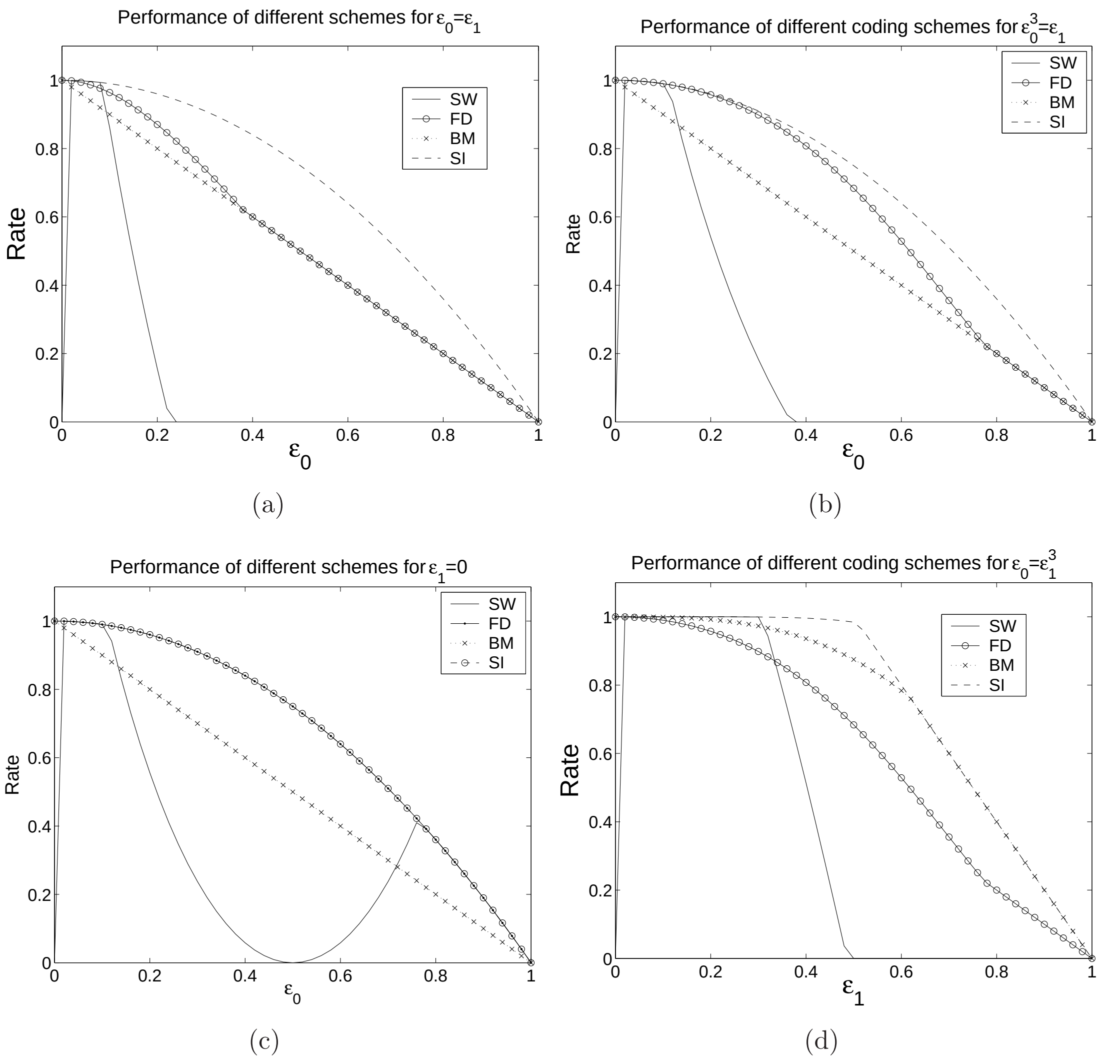


Figure 2.7: Performance of different schemes for four scenarios.

ceived signals at the relay nodes increases. Therefore, the minimum required rate for reliable transmission also increases. However, as we increase the value of ϵ the required rate decreases, and the compressed signals can pass across without error. Hence, we achieve capacity. In Figure 2.7.(d) we look at the case when the quality of channels between the source and relays is better than of those between the relays and destination. For this, we have chosen $\epsilon_0 = \epsilon_1^3$. In this plot, block Markov coding outperforms other schemes and achieves capacity for large values of ϵ_1.

There are other strategies that can be used that are not analyzed here. For instance, the authors of [14] propose a compression-based scheme for general relay channels. The approach is generalized to a multiple relay setup in [16],[28]. This scheme is based on the Wyner-Ziv compression technique with side-information at the receiver [30]. One expects this scheme to be more efficient than the Slepian-Wolf scheme proposed here. However, finding an explicit formula for the achievable rate requires a seemingly intractable optimization over a number of auxiliary random variables. Designing efficient and analytically tractable coding schemes based on these ideas deserves further investigation and can be a subject of further research.

2.9 Conclusion

We have obtained the capacity for a class of wireless erasure networks with broadcast and no interference at reception. We have generalized some of the capacity results that hold for wireline networks [9],[54] to these networks. Furthermore, we have shown that linear encoding suffices to achieve the optimal performance. We see from the proof that it is not necessary to perform channel coding and network coding separately from each other. In fact, as we will see in Chapter 4, decoding at the relay nodes and operating below the capacities of each link can actually significantly reduce the achievable rate (for more examples, see [27],[55]). Therefore, unlike the wireline scenario where each link is made error free by channel coding and network coding is then employed on top of that, our scheme only requires a single encoding function. Only the destination has to decode the received signal.

Many problems related to wireless networks remain open. Generalizing the results in this work for other network problems is one possible extension of this work. It will also be interesting to see if similar results can be obtained for other types of networks, such as erasure wireless networks, in which interference is incorporated in the reception model, networks involving channels other than erasure channels, etc.

Chapter 3

Broadcast Problems over Wireless Networks

3.1 Introduction

Different traffic patterns are present in today's communication networks. In addition to pairwise communications in the network, it is possible that a collection of users are interested in the same type of information generated in the network. This scenario resembles the multicast problem that was considered in Chapter 2 for a specific model of wireless networks. Another possibility is that a collection of users demand different types of information from a specific user referred to as source node. We refer to this problem as the *broadcast problem* in the network. What is apparent from the above two scenarios is that unlike point-to-point communication systems, where the source (transmitter) and the sink (receiver) of the information are specified, in communication networks a user can be a transmitter and receiver (or even relay) for different types of information at the same time, and this gives rise to different *network problems.* At a high level of generality, a network problem over a communication network is specified by the set of information messages, the source nodes, and the destination nodes. Each source node has access to a subset of information messages, and each destination demands some subset of the information messages. In Section 2.3 of Chapter 2 we defined a network problem, $\mathcal{P}$ by a quintuple $\mathcal{P} = (\mathcal{M}, \mathcal{S}, \mathcal{D}, \Upsilon_g, \Upsilon_r)$, where $\mathcal{M}$ is the set of information message, $\mathcal{S}$ and $\mathcal{D}$ are the set of source and destination nodes, and Υ_r (respectively Υ_g) is the function that specifies the set of messages requested (generated) at each destination (source).

Note that the above definition is independent of the communication model defined for the network. A network problem can be defined over wireline and wireless, fixed, or mobile networks. Given a communication model for the network (which specifies the connectivity of the users and their communication capabilities), the main questions regarding a network problem $\mathcal{P}$ are

- What is the set of possible rates (for information messages) that can be supported for $\mathcal{P}$ over the network?
- What is the optimal communication strategy to achieve those rate?

As mentioned in Chapter 2, these questions are answered for multicast problems over wireline networks in [9, 10] and a class of deterministic networks in [32]. We also looked at multicast problems over a class of wireless networks called wireless erasure networks and found the capacity region and the optimal strategy in Chapter 2. However, for a general network problem, the answer to the above questions is unknown. In [33, 23], outer and inner bounds on the capacity region of general network problems over wireline networks are obtained.

In this chapter we consider *broadcast problems*. In a broadcast problem one source has access to multiple information messages. Each of these information messages is requested by a particular destination. In accordance to our definition, a *broadcast problem* $\mathcal{P}_{bc} = (\mathcal{M}, \mathcal{S}, \mathcal{D}, \Upsilon_g, \Upsilon_r)$ is a network problem where $\mathcal{S} = \{s\}$, $\Upsilon_g = \mathcal{M}$, and $\Upsilon_r(.)$ is a partition of $\mathcal{M}$. There are many scenarios that can be modeled by broadcast problems. A well-known example is TV broadcasting stations. Downlink of cellular systems where a base station provides service to different users is another example of such problems that has received a lot of attention in the past few years. It should be noted that the capacity region of a broadcast problem for a general network is still unknown. In fact, even in the multi-user setup where there are no relays present in the network and the destinations are directly connected to the source, the broadcast problem is not completely solved. In this case the problem is referred to as *broadcast channels*, which was first introduced by Cover in [13].

In a network setup the broadcast problem is solved for wireline networks with error-free links in [54]. The capacity region in this case has a min-cut interpretation. Furthermore, it can be shown that in this case the capacity is achieved by routing, and no coding at the intermediate nodes is required.

In this chapter we look at broadcast problems over wireless erasure networks. These networks were introduced in Chapter 2. At the beginning of the chapter we give the problem statement. We find an achievable region for broadcast problems over these networks and find the exact capacity region in the multiuser setup. Using this result we will give an outer bound on the capacity region of broadcast problems over wireless erasure networks which is tighter than the outer bounds that we get from the multicast type of arguments.

3.2 Problem Statement

The communication networks considered in this chapter, namely Wireless Erasure (WE) networks, are modeled by an acyclic directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with node set $\mathcal{V}$ and link (edge) set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. Each link $(i, j) \in \mathcal{E}$ corresponds to a communication channel between the node i and node j. Looking back at Chapter 2, the input of all channels originating from node i is denoted by X_i chosen from input alphabet $\mathcal{X}$. The output of the communication channel corresponding to link (i, j) is denoted by Y_{ij}; Y_{ij} lies in alphabet set $\mathcal{Y}_{ij}$. We assume interference-free property for all the incoming links to a node and denote the collection of received signal at node j by

$Y_j = (Y_{ij},\ (i,j) \in \mathcal{E}) \in \mathcal{Y}_j \triangleq \prod_{i:(i,j)\in\mathcal{E}} \mathcal{Y}_{ij}$. Link (i,j) corresponds to an erasure channel with probability of erasure ϵ_{ij} in WE networks.

Let $\mathcal{D} = \{d_1, \ldots, d_{|\mathcal{D}|}\}$ denote the set of destination nodes and s be the source node for the broadcast problem. Next, we define the class of block codes considered in this chapter.

A $(\lceil 2^{nR_1} \rceil, \ldots, \lceil 2^{nR_{|\mathcal{D}|}} \rceil, n)$ code for the broadcast problem in a wireless erasure network described in previous sections consists of the following components:

- A set of integers $\mathcal{W}^{(d_i)} = \{1, 2, \ldots, \lceil 2^{nR_i} \rceil\}$ that represent the message indices corresponding to information source intended for destination node $d_i \in \mathcal{D}$. We assume that all the messages are equally likely. All the information sources are available at the source node indexed by $s \in \mathcal{V}$.

- An encoding function for the source node s: $f_s : \prod_{d\in\mathcal{D}} \mathcal{W}^{(d)} \to \mathcal{X}_s^n$.

- A set of encoding functions $\{f_{i,t}\}_{t=1}^n$ for each node $i \neq s \in \mathcal{V}$, where $x_{i,t} = f_{i,t}(y_i^{t-1})$ is the signal transmitted by node i at time t. Note that $x_{i,t}$ is a function of all the received symbols from all its incoming channels up to time $t-1$.

- A decoding function g_{d_i} at destination node $d_i \in \mathcal{D}$, $g_{d_i} : \mathcal{Y}_{d_i}^n \times \{0,1\}^{n|\mathcal{E}|} \to \mathcal{W}^{(d_i)}$ such that
$$\hat{w}^{(d_i)} = g_{d_i}(y_{d_i}^n, (s_{ij,t}\,, (i,j) \in \mathcal{E},\ 1 \le t \le n)), \tag{3.1}$$
where $\hat{w}^{(d_i)}$ is the estimate of the message sent from source s based on received signals at d_i and also the erasure occurrences on all the links of the network in the current block.

Note that X_i, Y_{ij} and Y_i, all depend on the message vector $\underline{w} = (w^{(d_i)},\ d_i \in \mathcal{D})$ that is being transmitted. Therefore, we will write them as $X_i(\underline{w})$,$Y_{ij}(\underline{w})$, and $Y_i(\underline{w})$ to specify what specific set of messages is transmitted.

We define the probability of error as the probability that the decoded message at one of the destinations is not equal to the transmitted message, i.e.,

$$P_{err} = \Pr\left(\exists\, d_i \in \mathcal{D} : \hat{W}^{(d_i)} \neq W^{(d_i)}\right). \tag{3.2}$$

The set of rates ($R_i, \quad 1 \leq i \leq |\mathcal{D}|$) is said to be achievable if there exist a sequence of $(\lceil 2^{nR_1} \rceil, \ldots, \lceil 2^{nR_{|\mathcal{D}|}} \rceil, n)$ codes such that $P_{err} \to 0$ as $n \to \infty$. The capacity region, $\mathcal{C}(\mathcal{G}, \{\epsilon_{ij},\ (i,j) \in \mathcal{E}\})$, is the set closure of the set of achievable rates.

In the remainder of this chapter we will look at the capacity region for broadcast problem in WE networks.

3.3 Broadcast Problems Over WE Networks

As we saw in Chapter 2, with an appropriate definition of the cut-capacity, the capacity of multicast problems over wireless erasure networks has a nice min-cut interpretation. In particular, for a unicast scenario (i.e., single destination) the capacity is given by the min-cut capacity over all the source destination cuts (see Theorem 2.1). In the following we propose a time-sharing scheme for broadcast problems based on the result for unicast problems in wireless erasure networks.

3.3.1 Time-sharing Scheme for WE networks

We consider a scheme where each node in the network performs time-sharing between the destinations. In other words, each node $i \in \mathcal{V}$ allocates a fraction $\alpha_{id}, \quad d \in \mathcal{D}$ of its block length to transmit to destination $d \in \mathcal{D}$. These fractions may not be the same for different nodes. In fact, as we will see later, in some cases the optimal fractions are unequal.

In order to analyze the set of achievable rates with the above time-sharing schemes, it is important to note that given time-sharing parameters $\alpha_{id}, \quad d \in \mathcal{D}$, the achievable rate for each destination is given by a min-cut formulation. The only difference to be taken into account is that the length of the block code used at node i for transmitting to destination d is of size $\lceil \alpha_{id} n \rceil$ (rather than n in the proof of Theorem 2.1). However, with a generalized definition of the cut-capacity, a similar min-cut result will still hold even if the block lengths are not equal across the network. We have stated this result without proof as the following lemma.

Lemma 3.1. *Consider an erasure wireless network with single source and single destination d. Furthermore, suppose that node $i \in \mathcal{V}$ uses a block code of length $\lceil \alpha_{id} n \rceil, \quad \alpha_{id} \leq 1$ to perform encoding. Then, the capacity of the network with side-information at the destination and under this coding scheme is given by the minimum of the cut-capacities over all the s-d cuts, where the cut-capacity of s-d cut $\mathcal{V}_s$ is defined as*

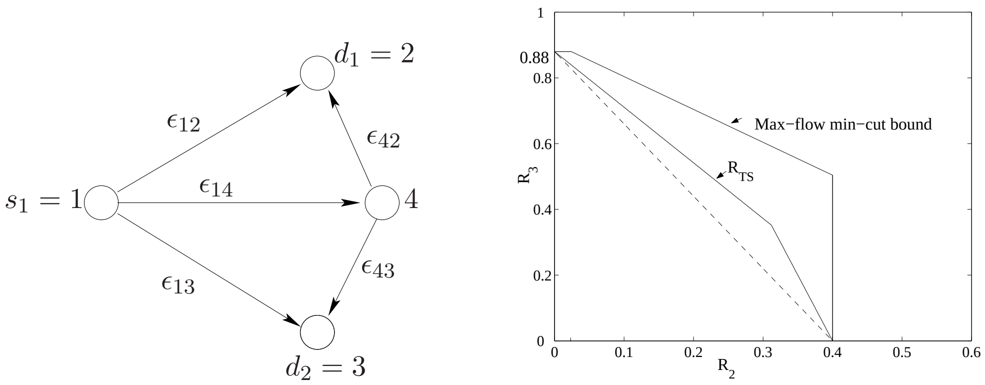
$$C(\mathcal{V}_s, \{\alpha_{id}\}_{i \in \mathcal{V}}) = \sum_{i \in \mathcal{V}_s^*} \alpha_{id} (1 - \prod_{j:(i,j) \in [\mathcal{V}_s, \mathcal{V}_s^c]} \epsilon_{ij}). \tag{3.3}$$

Now consider our broadcast problem in wireless erasure networks. We represent any admissible time-sharing policy by $\underline{\alpha} = (\alpha_{id}, \quad i \in \mathcal{V}, d \in \mathcal{D})$, where, as mentioned earlier, α_{id} specifies the fraction of the block length allocated by node i for transmission to destination d. It is clear that for any admissible time-sharing $\underline{\alpha}$, we should have $\alpha_{id} \geq 0$ and $\sum_{d \in \mathcal{D}} \alpha_{id} = 1$ for any $i \in \mathcal{V}$ and $d \in \mathcal{D}$. According to the previous lemma, the achievable rate region using a fixed time-sharing scheme given by $\underline{\alpha}$ will be

$$\mathcal{R}_{TS}(\underline{\alpha}) = \{(R_d, \quad d \in \mathcal{D}) | \forall d \in \mathcal{D}, \quad 0 \leq R_d \leq \min_{\mathcal{V}_s: \text{s-d cut}} C(\mathcal{V}_s, \{\alpha_{id}\}_{i \in \mathcal{V}})\}, \tag{3.4}$$

and therefore the achievable rate region using time-sharing in the network is

$$\mathcal{R}_{TS} = \bigcup_{\underline{\alpha}} R_{TS}(\underline{\alpha}), \tag{3.5}$$



(a) (b)

Figure 3.1: (a): A wireless erasure network with one source and two destination nodes. (b): Different achievable regions for the network in part (a).

where union is taken over all admissible $\underline{\alpha}$s and subscript TS is used to refer to the time-sharing scheme.

Example 3.1. Consider the wireless erasure network shown in Figure 3.1, with one source, one relay node, and two destination nodes. Based on the above argument, the achievable rate region using time-sharing is

$$\mathcal{R}_{TS} = \cup_{(\alpha,\beta)} \left\{ \begin{pmatrix} R_2 \\ R_3 \end{pmatrix} \middle| \begin{array}{c} R_2 \leq \min\{\alpha(1-\epsilon_{12}\epsilon_{14}), \alpha(1-\epsilon_{12}) + \beta(1-\epsilon_{42})\} \\ R_3 \leq \min\{(1-\alpha)(1-\epsilon_{13}\epsilon_{14}), (1-\alpha)(1-\epsilon_{13}) + (1-\beta)(1-\epsilon_{43})\} \end{array} \right\},$$

where $0 \leq \alpha \leq 1$ (respectively $0 \leq \beta \leq 1$) is the fraction of the block length that node 1 (respectively 4) allocates to transmit to destination 2. In Figure 3.1.(b), we have plotted the achievable rate region using the time-sharing scheme described above for $(\epsilon_{12}, \epsilon_{13}, \epsilon_{14}, \epsilon_{42}, \epsilon_{43}) = (0.8, 0.2, 0.6, 0.8, 0.2)$. As it can be seen from the plot, the time-sharing region is given by

$$\mathcal{R}_{TS} = \left\{ (R_2, R_3) \,\middle|\, \frac{R_3}{0.88} + \frac{R_2}{0.52} \leq 1,\ \frac{R_2}{0.4} + \frac{R_3}{1.6} \leq 1 \right\}. \tag{3.6}$$

The dashed line specifies boundaries of the region achievable by time-sharing between source and one of the destination sub-networks (this corresponds to the case when

$\alpha = \beta$). As we can observe, the optimal time-sharing is not achieved by equal fractions α and β. Figure 3.1.(b) also compares the time-sharing region to the naïve min-cut outer bound derived by multicast type arguments, i.e.,

$$\begin{aligned} R_2 &\leq 0.4, \\ R_3 &\leq 0.88, \text{ and} \\ R_2 + R_3 &\leq 0.904. \end{aligned}$$

Note that the outer bound provided in the example is simply an application of the outer bound developed for the mutlicast problem in Chapter 2. The counterpart of this outer bound in wireline networks is tight for broadcast problems [54]. However, as we will see in the following, this outer bound is not tight for wireless erasure networks. In the following section, we first look at the performance of the time-sharing scheme proposed above for a subclass of WE networks referred to as Erasure Broadcast Channels (EBC). We will show that time-sharing achieves the capacity for this subclass and then use this result to find tighter outer bounds for a general WE network.

3.4 Erasure Broadcast Channel

We consider a subclass of wireless erasure networks called erasure broadcast channels. An (m, n)-erasure broadcast channel (see Figure 3.2.(a)) with channel matrix $\underline{\epsilon}_{m\times n}$ is a broadcast channel with m inputs and n receivers (or destinations). Destination i is connected to node j with a packet erasure channel with probability of erasure ϵ_{ij}. We can think of an (m, n)-erasure broadcast channel as a wireless erasure network with n destinations and m intermediate nodes, each connected to the source node with a link with no erasure and unlimited capacity. We can also consider an (m, n)-

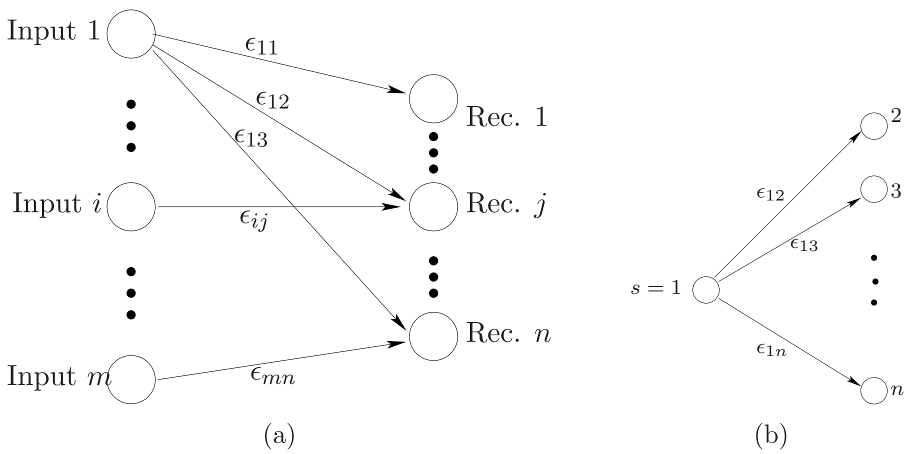


Figure 3.2: (a):An (m, n)-erasure broadcast channel (b): An $(1, n-1)$-erasure broadcast channel

erasure broadcast channel as the product of m scalar (single input) erasure broadcast channels. The capacity of the product of two broadcast channels with degraded components is considered in [37, 38].

The (m, n)-erasure broadcast channel does not in general belong to the classes of broadcast channels for which the capacity is known, e.g., “degraded,” “more capable,” and “less noisy” (see [35] for an overview). However, the single input erasure broadcast channel belongs to the “degraded” class, and its capacity region has been found in [42, 39]. It is shown in [42, 39] that the time-sharing scheme is actually capacity achieving. Here we look at the general case. Note that for the (m, n)-erasure broadcast channel, the time-sharing region of (3.5) simplifies to the following:

$$\mathcal{R}_T(\underline{\epsilon}) = \bigcup_{\underline{\alpha}} \{(R_1, \ldots, R_n) | 0 \leq R_j < \sum_{i=1}^{m} \alpha_{ij}(1-\epsilon_{ij})\}, \tag{3.7}$$

where the union is over all admissible time-sharing matrices $\underline{\alpha}$ that satisfy:

(a) $\alpha_{ij} \geq 0$.

(b) $\sum_{j=1}^{n} \alpha_{ij} = 1$ for all inputs $1 \leq i \leq m$.

3.4.1 Degraded (m,n)-Erasure Broadcast Channels

Here we try to find necessary and sufficient conditions so that an (m,n)-erasure broadcast channel with erasure matrix $\underline{\epsilon}$ belongs to the "degraded" class.

Definition 3.1. *Channel i with probability transition matrix $\Pr_i(Y|X)$ is a statistically "degraded" version of channel j with transition matrix $\Pr_j(Z|X)$ iff there exists a channel $\Pr(Y|Z)$ such that*

$$\Pr_i(Y|X) = \sum_Z \Pr_j(Z|X)\Pr(Y|Z).$$

In this case we write $\Pr_i(\cdot|\cdot) \preceq \Pr_i(\cdot|\cdot)$.

Now note that if channel i is a degraded version of channel j, then for any distribution on X we should have

$$I(X;Y_j) \geq I(X;Y_i).$$

Therefore, for an (m,n)-erasure broadcast channel, if the i-th user is a "degraded" version of j-th user, then we should have $I(X;Y_j) \geq I(X;Y_i)$ for any distribution on X. In particular, if we keep all the inputs except the k-th one constant and let X_k be an i.i.d. Bernoulli($\frac{1}{2}$), the following condition should hold:

$$\epsilon_{kj} \leq \epsilon_{kj} \qquad \forall\, k \in \{1,\ldots,m\}.$$

It can be easily checked that this condition is also sufficient for channel i being a "degraded" version of channel j. Hence,

Lemma 3.2. *An (m,n)-erasure channel with erasure matrix $\underline{\epsilon}$ belongs to "degraded"* *class iff there exists some permutation* $\pi(\cdot)$ *on* $\{1, 2, \dots, n\}$ *such that*

$$\epsilon_{i\pi(j)} \leq \epsilon_{i\pi(j+1)}, \quad \forall\, 1 \leq i \leq m,\, 1 \leq j \leq n.$$

Now, we show that the capacity of the degraded erasure broadcast channels is given by the time-sharing scheme described in Section 3.3.1. In doing so, we use the following lemma, the proof of which we omit.

Lemma 3.3. *Suppose* $U, V, X,$ *and* Y *are random variables with probability distribution of form*

$$Pr(U, V, X, Y) = Pr(U, V, X)Pr_\epsilon(Y|X),$$

where $Pr_\epsilon(.|.)$ *is the transition probability of an erasure channel with probability of erasure* ϵ*. Then*

$$I(U; Y|V) = (1 - \epsilon)I(U; X|V).$$

Theorem 3.1. *The capacity region,* $\mathcal{C}_g(\underline{\epsilon})$*, of a degraded* (m, n)*-erasure channel with erasure matrix* $\underline{\epsilon}$ *is given by time-sharing between the receivers at each input, i.e.,*

$$\mathcal{C}_g(\underline{\epsilon}) = \mathcal{R}_T(\underline{\epsilon}),$$

where $\mathcal{R}_T(\underline{\epsilon})$ *is given in (3.7).*

Proof. Without loss of generality let us assume that for $i < j$, receiver j is a "degraded" version of receiver i. According to [40], the capacity of the degraded broadcast channel is given by the convex hull of the closure of the $(R_1, \dots, R_n)$ satisfying

$$0 \leq R_j \leq I(U_j; Y_j|U_{j+1}, \dots, U_n)$$

for $j = 1, 2, \dots, n$, where $U = (U_1, \dots, U_n)$ and $X = (X_1, \dots, X_m)$ and $U - X - Y_1 - \cdots - Y_n$ forms a Markov chain.

Now consider the following maximization problem:

$$S^\star(\underline{\mu}) = \max_{(R_1,\ldots,R_n)\in\mathcal{C}_g(\underline{\epsilon})} \sum_{j=1}^{n} \mu_j R_j,$$

where $\underline{\mu} = (\mu_1, \ldots, \mu_n) \geq 0$. Note that every point on the boundary of the capacity region is the maximizing solution for some $\underline{\mu} \geq 0$. Also, the maximizing solution of the above optimization problem corresponds to a boundary point of the capacity region.

Using chain rule for mutual information, we can write

$$\begin{aligned}
S^\star(\underline{\mu}) &\leq \max_{P(U,X)} \sum_{j=1}^{n} \mu_j I(U_j; Y_j | U_{j+1}, \ldots, U_n) \qquad (3.8)\\
&= \max_{P(U,X)} \sum_{j=1}^{n} \mu_j I(U_j; Y_{1j}, Y_j' | U_{j+1}, \ldots, U_n)\\
&= \max_{P(U,X)} \Bigg(\sum_{j=1}^{n} \mu_j I(U_j; Y_j' | U_{j+1}, \ldots, U_n)\\
&+ \sum_{j=1}^{n} \mu_j I(U_j; Y_{1j} | U_{j+1}, \ldots, U_n, Y_j') \Bigg)\\
&= \max_{P(U,X)} \Bigg(\sum_{j=1}^{n} \mu_j I(U_j; Y_j' | U_{j+1}, \ldots, U_n) \qquad (3.9)\\
&+ \sum_{j=1}^{n} \mu_j (1 - \epsilon_{1j}) I(U_j; X_1 | U_{j+1}, \ldots, U_n, Y_j') \Bigg),
\end{aligned}$$

where $Y_j' = (Y_{2j}, \ldots, Y_{mj})$ for all $1 \leq j \leq n$, and (3.9) follows from Lemma 3.3. Defining $X' = (X_2, \ldots, X_m)$, it can be easily verified that $(U_1, \ldots, U_n, X_1) - X' - Y_1' - \cdots - Y_n'$. This Markov property implies that

$$H(X_1 | U_j, \ldots, U_n, Y_{j-1}') = H(X_1 | U_j, \ldots, U_n, Y_j') - I(Y_{j-1}'; X_1 | U_j, \ldots, U_n, Y_j'). \qquad (3.10)$$

Therefore,

$$H(X_1|U_j,\ldots,U_n,Y'_{j-1}) \leq H(X_1|U_j,\ldots,U_n,Y'_j).$$

Using the above inequality, one can show that

$$\sum_{j=1}^{n} I(U_j;X_1|U_{j+1},\ldots,U_n,Y'_j) \leq H(X_1).$$

Therefore, the second weighted sum of (3.9) is at most $\max_j \mu_j(1-\epsilon_{1j})H(X_1)$, and replacing this in (3.9), we have

$$S^\star(\underline{\mu}) \leq \max_{P(U,X')} \sum_{j=1}^{n} \mu_j I(U_j;Y'_j|U_{j+1},\ldots,U_n) + \max_j \mu_j(1-\epsilon_{1j}). \tag{3.11}$$

The first summation in the right hand side of the inequality corresponds to the maximum weighted sum rate of an $(m-1,n)$-degraded broadcast channel obtained by excluding the connections from the first input of the transmitter to all the receivers. Using similar arguments for the new $(m-1,n)$-degraded broadcast channel, it can be verified that

$$\max_{P(U,X')} \sum_{j=1}^{n} \mu_j I(U_j;Y'_j|U_{j+1},\ldots,U_n) \leq \sum_{i=2}^{m} \max_j \mu_j(1-\epsilon_{ij}). \tag{3.12}$$

Using this in (3.11), we have

$$S^\star(\underline{\mu}) \leq \sum_{i=1}^{m} \max_j \mu_j(1-\epsilon_{ij}). \tag{3.13}$$

Now, the right hand side value can be achieved by time-sharing. For that, input i transmits only to the receiver with maximum $\mu_j(1-\epsilon_{ij})$. Therefore ,each boundary point of the capacity region can be achieved by time-sharing and $\mathcal{C}_g(\underline{\epsilon}) = \mathcal{R}_T(\underline{\epsilon})$. ∎

3.4.2 Non-degraded (m, n)-Erasure Broadcast Channel

In this section we show that the capacity region of the general erasure broadcast channel is given by time-sharing. In [38], the capacity of the product of two reversely degraded broadcast channels is characterized. Using this result it can be easily verified that the capacity region of $(2, 2)$-erasure broadcast channel is given by time-sharing. However, applying and specializing the technique of [38] for the general (m, n)-erasure broadcast channel does not seem plausible. Instead, in this chapter we use another argument to show that every boundary point of the capacity region for general erasure broadcast channel is achieved by time-sharing. The argument used here is very close to the one used in [118] to prove that the capacity of the MIMO Gaussian broadcast channel is given by Dirty Paper coding (DPC) introduced in [119].

Consider an (m, n)-erasure broadcast channel with erasure matrix $\underline{\epsilon} = [\epsilon_{ij}]$. Let $\mathcal{C}_g(\underline{\epsilon})$ denote the capacity region of this general erasure broadcast channel. For every boundary point $\underline{R}'$, there exist positive $\mu_1, \ldots, \mu_n$ such that $\underline{R}'$ is the optimal solution of

$$\max_{(R_1,\ldots,R_n)\in \mathcal{C}_g(\underline{\epsilon})} \sum_{j=1}^{n} \mu_j R_j.$$

The idea is to construct for each value of $\mu_1, \ldots, \mu_n$ a degraded (m, n)-erasure broadcast channel with channel matrix $\underline{\epsilon}^\star$ whose capacity region, $\mathcal{C}_g(\underline{\epsilon}^\star)$, contains the capacity region of the original channel, $\mathcal{C}_g(\underline{\epsilon})$. Moreover, we require that the time-sharing region of both channels meet at some specific point(s) on the boundary.

Let us look at the $(2, 2)$-erasure broadcast channels first. Suppose that the channel is not degraded. Without loss of generality, assume that

$$\epsilon_{11} \leq \epsilon_{12}, \quad \epsilon_{21} \geq \epsilon_{22}.$$

Consider the following maximization problem:

$$\max_{(R_1,R_2)\in\mathcal{C}_g} \mu_1 R_1 + \mu_2 R_2. \tag{3.14}$$

We construct a $(2,2)$-degraded erasure broadcast channel that contains the capacity region $\mathcal{C}_g$. For this, we consider the following two cases separately.

- $\frac{\mu_1}{\mu_2} \geq 1$: In this case, consider the erasure broadcast channel with the erasure matrix constructed from $\underline{\epsilon}$ as follows:

$$\underline{\epsilon}^\star = \begin{bmatrix} \epsilon_{11} & \epsilon_{11} \\ \epsilon_{21} & \epsilon_{22} \end{bmatrix}.$$

First note that the above channel is degraded. Moreover, the erasure probabilities on every edge of this new channel are less than or equal to that of the corresponding edge in the original channel. Therefore, the capacity region of the degraded channel contains $\mathcal{C}_g$. Now let's look back at the maximization problem in (3.14). Based on the above discussions we know that the optimal solution is less than

$$\max_{(R_1,R_2)\in\mathcal{C}_d} \mu_1 R_1 + \mu_2 R_2,$$

where $\mathcal{C}_d$ is the capacity region of the obtained degraded channel with erasure matrix $\underline{\epsilon}^\star$. Based on Theorem 3.1, the maximum of the above problem is achieved by time sharing, and it equals

$$\max\{\mu_1(1-\epsilon_{21}), \mu_2(1-\epsilon_{22})\} + \max\{\mu_1(1-\epsilon_{11}), \mu_2(1-\epsilon_{11})\}.$$

Since $\mu_1 \geq \mu_2$, we can write the above rate as

$$\max\{\mu_1(1-\epsilon_{21}), \mu_2(1-\epsilon_{22})\} + \mu_1(1-\epsilon_{11}).$$

Looking closely at the above rate, we observe that we can achieve the above rate in the original erasure channel by time-sharing as well. Input 1 transmits only to the first receiver, and the second input sends to the maximum of the to terms appearing in the above formula. Therefore, the boundary point corresponding to μ_1, μ_2 is achieved by time-sharing.

- $\frac{\mu_1}{\mu_2} < 1$: In this case, consider the $(2,2)$-erasure broadcast channel with the following erasure matrix:

$$\underline{\epsilon}^{\star} = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} \\ \epsilon_{22} & \epsilon_{22} \end{bmatrix}.$$

  Similar to the first case, the broadcast channel represented by $\underline{\epsilon}^{\star}$ is degraded, and its capacity region contains $\mathcal{C}_g$. Also, the solution of (3.14) coincides with the solution of the same cost function over region $\mathcal{C}_d$. Thus, the boundary point corresponding to μ_1, μ_2 is achieved by time-sharing.

Based on the above results and Theorem 3.1, we have the following result.

Theorem 3.2. *The capacity region of any* $(2,2)$*-erasure broadcast channel is given by time-sharing.*

We can generalize the preceding arguments to general (m, n)-erasure broadcast channels by finding a degraded erasure broadcast channel that contains the capacity region of the original channel and its time-sharing region coincides with that of the original one at a specific point. We have stated the above result as a theorem:

Theorem 3.3. *Consider an* (m, n)*-erasure broadcast channel with channel matrix* $\underline{\epsilon}$*. The capacity region,* $\mathcal{C}_g(\underline{\epsilon})$*, of this broadcast channel is given by time-sharing between the receivers at each input, i.e.,*

$$\mathcal{C}_g(\underline{\epsilon}) = \mathcal{R}_T(\underline{\epsilon}),$$

where $\mathcal{R}_T(\underline{\epsilon})$ is defined in (3.7).

Proof. Similar to previous discussions, we need to show that every boundary point of the capacity region is achieved by time-sharing. For any $\underline{\mu} \geq 0$[1], consider the following maximization problem:

$$f_g(\underline{\mu}, \underline{\epsilon}) = \max_{(R_1,\ldots,R_n)\in \mathcal{C}_g(\underline{\epsilon})} \sum_{j=1}^{n} \mu_j R_j, \tag{3.15}$$

where the maximization is over the capacity region of the erasure broadcast channel with channel matrix $\underline{\epsilon}$. We construct a *degraded* (m, n)-erasure broadcast channel with channel matrix denoted by $\underline{\epsilon}^\star$ from the original channel such that

- its capacity region $\mathcal{C}_g(\underline{\epsilon}^\star)$ contains that of the original channel and, therefore,

$$f_g(\underline{\mu}, \underline{\epsilon}) \leq f_g(\underline{\mu}, \underline{\epsilon}^\star), \text{ and} \tag{3.16}$$

- the following maximization problem over the time-sharing region

$$f_T(\underline{\mu}, \underline{\epsilon}) = \max_{(R_1,\ldots,R_n)\in \mathcal{R}_T(\underline{\epsilon})} \sum_{j=1}^{n} \mu_j R_, \tag{3.17}$$

  gives the same values for both channels, i.e.,

$$f_T(\underline{\mu}, \underline{\epsilon}) = f_T(\underline{\mu}, \underline{\epsilon}^\star). \tag{3.18}$$

Combining (3.16) and (3.18) we get

$$f_g(\underline{\mu}, \underline{\epsilon}) \leq f_g(\underline{\mu}, \underline{\epsilon}) \overset{(a)}{=} f_T(\underline{\mu}, \underline{\epsilon}^\star) = f_T(\underline{\mu}, \underline{\epsilon}),$$

where (a) follows from Theorem 3.1 and the fact that $\underline{\epsilon}^\star$ corresponds to a degraded

[1] Here, by $\underline{a} \geq 0$ we mean every component of $\underline{a}$ should be greater than equal zero.

channel, and this proves that time-sharing is capacity-achieving.

Before we start constructing the degraded channel, we should remark that the solution to the optimization problem in (3.17) can be written explicitly as

$$f_T(\underline{\mu}, \underline{\epsilon}) = \sum_{i=1}^{m} \max_{1 \le j \le n} \mu_j (1 - \epsilon_{ij}). \tag{3.19}$$

Next, we claim that there exists at least one input i and two receivers j, k such that

$$\mu_j \le \mu_k, \quad \epsilon_{ik} < \epsilon_{ij}.$$

If not, then it can be easily verified that for all is, ϵ_{ij}s are ordered in the reverse order that μ_js are ordered and, therefore, the erasure matrix $\underline{\epsilon}$ satisfies the constraint of Lemma 3.2 with the permutation that sorts μ_js in decreasing order; hence our (m, n)-erasure channel is "degraded." But, in that case we already know from Theorem 3.1 that the capacity region is achieved by time-sharing. Therefore, let i^*, j^*, and k^* be such numbers. Consider a new (m, n)-erasure channel with erasure matrix $\underline{\epsilon}^{(1)}$ derived from $\underline{\epsilon}$ by replacing $\epsilon_{i^*j^*}$ with $\epsilon_{i^*k^*}$ in the i^*j^* coordinate of $\underline{\epsilon}$. In other words, each coordinate of the new matrix is as follows:

$$\epsilon_{ij}^{(1)} = \epsilon_{ij} + (\epsilon_{i^*k^*} - \epsilon_{i^*j^*})\delta(i - i^*)\delta(j - j^*),$$

where $\delta(\cdot)$ denotes the Dirac's delta function. This new channel has the following properties:

- Its capacity region, $\mathcal{C}_g(\underline{\epsilon}^{(1)})$, contains $\mathcal{C}_g(\underline{\epsilon})$, since a link is replaced by a link with lower probability of erasure; therefore,

$$f_g(\underline{\mu}, \underline{\epsilon}) \le f_{g)}(\underline{\mu}, \underline{\epsilon}^{(1)}).$$

- The value of (3.19) remains unchanged for the new channel, i.e., $f_T(\underline{\mu}, \underline{\epsilon}) = f_T(\underline{\mu}, \underline{\epsilon}^{(1)})$. To see this, first notice that

$$\begin{aligned}
\max_{1\le j\le n} \mu_j(1-\epsilon^{(1)}_{i^*j}) &= \max\{\mu_{j^*}(1-\epsilon^{(1)}_{i^*j^*}), \max_{1\le j\neq j^*\le n} \mu_j(1-\epsilon^{(1)}_{i^*j})\} \\
&= \max\{\mu_{j^*}(1-\epsilon_{i^*k^*}), \max_{1\le j\neq j^*\le n} \mu_j(1-\epsilon_{i^*j})\} \\
&= \max\{\mu_{j^*}(1-\epsilon_{i^*k^*}), \mu_{k^*}(1-\epsilon_{i^*k^*}) \max_{1\le j\neq j^*,k^*\le n} \mu_j(1-\epsilon_{i^*j})\} \\
&= \max\{\mu_{k^*}(1-\epsilon_{i^*k^*}) \max_{1\le j\neq j^*,k^*\le n} \mu_j(1-\epsilon_{i^*j})\} \\
&= \max_{1\le j\le n} \mu_j(1-\epsilon_{i^*j}),
\end{aligned}$$

where we have used our initial assumption that $\mu_{j^*} \le \mu_{k^*}$.

Now we repeat the above process and obtain erasure channels $\underline{\epsilon}^{(l)}$ for $l \ge 1$, until we cannot find any input i, receivers j, k with $(\mu_j \le \mu_k)$ and $(\epsilon_{ik} < \epsilon_{ij})$ at round $l = l_0$.[2] In that case, we know that channel $\underline{\epsilon}^\star = \underline{\epsilon}^{(l_0)}$ is degraded. Furthermore, its capacity region $\mathcal{C}_g(\underline{\epsilon}^\star)$ contains the capacity region of all previously derived channels (in particular the original channel), and the value of (3.19) for it remains unchanged, i.e.,

$$f_T(\underline{\mu}, \underline{\epsilon}^\star) = f_T(\underline{\mu}, \underline{\epsilon}).$$

Based on Theorem 3.1, for the derived degraded channel we have

$$f_g(\underline{\mu}, \underline{\epsilon}^\star) = f_T(\underline{\mu}, \underline{\epsilon}^\star).$$

Putting these together, we get $f_T(\underline{\mu}, \underline{\epsilon}) = f_g(\underline{\mu}, \underline{\epsilon})$. This completes the proof. ∎

Remark 3.4. *Theorem 3.3 can be stated in another form. That is, the capacity region of a general (m, n)-erasure broadcast channel is the Minkowski sum of the capacity regions of the scalar (single input) erasure broadcast channels between each input and*

[2]It is clear that this process comes to an end since at each step of the process the number of triplets (i, j, k) that $\mu_j \le \mu_k, \epsilon_{ij} \ge \epsilon_{ik}$ is reduced.

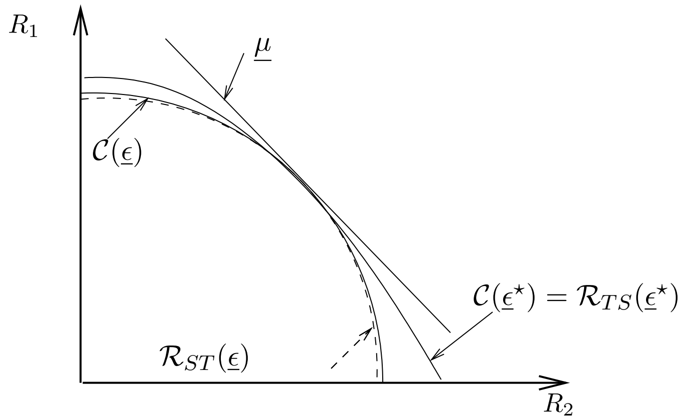


Figure 3.3: Illustrating the proof technique.

the receivers, where The Minkowski sum of two set $\mathcal{A}$ *and* $\mathcal{B}$ *in* $n-$*dimensional vector space is the result of adding every element of* $\mathcal{A}$ *to every element of* $\mathcal{B}$*, i.e. the set*

$$\mathcal{A} \oplus \mathcal{B} = \{a + b | a \in \mathcal{A}, b \in \mathcal{B}\}.$$

In mathematical terms, $\mathcal{C}_g(\underline{\epsilon})$ *can be written as*

$$\mathcal{C}_g(\underline{\epsilon}) = \oplus_{i=1}^{m} \mathcal{C}_g(\underline{\epsilon}_i),$$

where $\underline{\epsilon}_i$ *is the i-th row of* $\underline{\epsilon}$ *and corresponds to a single input erasure channel with* n *receivers.*

Remark 3.5. *The above statement of Theorem 3.3 can be generalized to a broader class of channels. Using very similar techniques, it is possible to show that the capacity region of broadcast problems for the product of a number of degraded broadcast channels with arbitrary degradation order is given by the Minkowski sum of the capacity regions of the degraded broadcast channels between each input and the receivers. This is a generalization of the result reported previously in [37].*

3.4.3 Discussion

In the previous section, we showed that the capacity region of erasure broadcast channels is achieved by the time-sharing scheme introduced in Section 3.3.1. This result will enable us to provide tighter outer bounds on the capacity region of broadcast problems over general wireless erasure networks. However, before we proceed to provide the outer bounds, we point out some important remarks regarding the previous result and the proof method.

3.4.3.1 Correlated Erasures

The result of Theorem 3.3 is for independent and memoryless erasure events. One might think that the result goes through for correlated erasure events across the links. However, this is not true. In fact, we can show that time-sharing is not capacity-achieving even for "degraded" channels.

Consider an (m, n)-erasure channels with correlated erasure events.[3] We model the correlated event as follows. For each set $\mathcal{A} \subseteq \{1, \ldots, m\}$, $\Pr_j(\mathcal{A})$ denotes the probability that the transmitted signals from inputs $i \in \mathcal{A}^c$ are erased and signals from $i \in \mathcal{A}$ are received successfully at receiver j. In the following Proposition, we look at the necessary and sufficient conditions for "degraded"-ness.

Proposition 3.1. *Considering the previous notation, receiver j is a degraded version of receiver k iff for any subset $\mathcal{U}$ of the power set of $\{1, \ldots, m\}$, with the property that*

$$\mathcal{B} \in \mathcal{U},\, \mathcal{A} \subseteq \mathcal{B} \Longrightarrow \mathcal{A} \in \mathcal{U},$$

we have

$$\sum_{\mathcal{A} \in \mathcal{U}} \Pr{}_j(\mathcal{A}) \leq \sum_{\mathcal{A} \in \mathcal{U}} \Pr{}_k(\mathcal{A}). \tag{3.20}$$

[3] Since the capacity region of broadcast channels with average probability of error constraint only depends on the marginals [35], we assume that different users' channels are independent from each other.

Proof. It is not hard to check that $\Pr_j(\cdot)$ is a degraded version of $\Pr_k(\cdot)$ if and only if there exists a stochastic matrix $\gamma(\cdot,\cdot)$ of dimension of $2^m \times 2^m$ indexed by the subsets of $\{1,\ldots,m\}$ with $\gamma(\mathcal{S},\mathcal{T}) = 0$ for $\mathcal{S},\mathcal{T},\ \mathcal{S} \nsubseteq \mathcal{T}$ such that

$$\Pr_j(\mathcal{A}) = \sum_{\mathcal{B}\subseteq\mathcal{A}} \gamma(\mathcal{A},\mathcal{B}) \Pr_k(\mathcal{B})$$

for all $\mathcal{A},\mathcal{B}$ subsets of $\{1,\ldots,m\}$.

With this, the necessity of (3.20) is clear. To prove the sufficiency, we use an argument used in [44]. The idea is based on constructing a max-flow network. Apart from source s and destination t, there are two nodes $l(\mathcal{A})$ and $r(\mathcal{A})$ for each $\mathcal{A} \subseteq \{1,\ldots,m\}$. s is connected to $l(\mathcal{A})$ with a link with capacity $\Pr_k(\mathcal{A})$. $r(\mathcal{A})$ is connected to t with a link with capacity $\Pr_j(\mathcal{A})$. Finally, $l(\mathcal{A})$ and $r(\mathcal{B})$ are connected with a link of unlimited capacity if and only if $\mathcal{B} \subseteq \mathcal{A}$. Consider a cut $[\mathcal{V}_s, \mathcal{V}_t]$ of finite capacity in this network. Let $\mathcal{V}_t = \mathcal{V}_t^l \cup \mathcal{V}_t^r$ be the partition of $\mathcal{V}_t$ to the nodes in the l side and the r side. Now, because of the finiteness of the capacity of the cut, we should have

$$r(\mathcal{B}) \in \mathcal{V}_t^r,\ \mathcal{B} \subseteq \mathcal{A} \Longrightarrow l(\mathcal{A}) \in \mathcal{V}_r^l.$$

Therefore, if we define $\mathcal{V}^* = \{\mathcal{A} | \exists\, r(\mathcal{B}) \in \mathcal{V}_t^r \text{ s.t. } \mathcal{B} \subseteq \mathcal{A}\}$, we have

$$\mathcal{V}^r \subseteq r(\mathcal{V}^*) \quad \text{and} \quad l(\mathcal{V}^*) \subseteq \mathcal{V}_t^l. \tag{3.21}$$

The cut-capacity $C(\mathcal{V}_s)$ can be written as

$$\begin{aligned} C(\mathcal{V}_s) &= \sum_{l(\mathcal{A})\in\mathcal{V}_t^l} \Pr_k(\mathcal{A}) + 1 - \sum_{r(\mathcal{B})\in\mathcal{V}_t^r} \Pr_j(\mathcal{B}) \\ &\geq 1 - \sum_{\mathcal{B}\in\mathcal{V}^*} \Pr_j(\mathcal{B}) + \sum_{\mathcal{A}\in\mathcal{V}^*} \Pr_k(\mathcal{A}) \\ &\geq 1, \end{aligned}$$

where the second line follows from (3.21), and the last line follows from (3.20). This suggests that the value of min-cut is equal to one, and the minimum cut is the one that isolates either s or t from the rest of nodes. In this case, using the max-flow min-cut theorem, the max-flow is also one. Therefore, the flow in each of the links between s (respectively t) and the intermediate nodes is equal to the capacity of the corresponding link. Defining the flow between $l(\mathcal{A})$ and $r(\mathcal{B})$ as $\gamma(\mathcal{B}, \mathcal{A}) \cdot \Pr_k(\mathcal{A})$, we can easily see that $\gamma(\cdot, \cdot)$ is the desired stochastic matrix. ∎

Remark 3.6. *As mentioned earlier, for the independent erasure case, different notions of "degraded," "more capable," and "less noisy" are equivalent. However, it is not clear whether the same is true for the case when erasures are correlated. In Proposition 3.1 we identified the class of "degraded" channels. However, characterizing the class of "more capable" channels is not an easy task. In order to see it, note that channel j is "more capable" than channel k if and only if for any distribution on the input $X = (X_1, \ldots, X_m)$ we have*

$$I(X; Y_k) \leq I(X; Y_j).$$

For (possibly correlated) erasure channels, the above condition can be written as a linear inequality involving the joint entropies of subsets of inputs, i.e.,

$$0 \leq \sum_{\mathcal{A}} \underbrace{(\Pr_j(\mathcal{A}) - \Pr_k(\mathcal{A}))}_{\alpha(\mathcal{A})} H(X_i,\, i \in \mathcal{A}),$$

where $\sum_{\mathcal{A}} \alpha(\mathcal{A}) = 0$. Therefore, to identify the class of "more capable" channels, one should be able to characterize the set of all valid linear inequalities over the entropies of subsets of m random variables. However, the later problem has proved to be a very challenging problem. Identifying the set of linear inequalities over the space of entropy vectors is still an unsolved problem for $m \geq 4$ [46, 47, 23].

Now consider a 2 by 2 degraded erasure broadcast channel with probability of erasure events as shown in table . Consider the supporting hyperplane $\underline{\mu} = (\mu_1, \mu_2)$, and let us look at the boundary point corresponding to this point.

It is not hard to check that the boundary point of the time-sharing region corresponding to this hyperplane is

$$(R_1, R_2) = \left(\max_{i=1,2} \left\{ \mu_i (\Pr{}_i(\{1\}) + \Pr{}_i(\{1,2\})) \right\}, \max_{i=1,2} \left\{ \mu_i (\Pr{}_i(\{2\}) + \Pr{}_i(\{1,2\})) \right\} \right).$$

On the other hand, using the Bergman's formula [40], we know that the rate vectors in the capacity region of this degraded channel should satisfy

$$R_1 \leq I(Y_1; X|U) = \Pr{}_1(\{1,2\})H(X_1, X_2|U) + \Pr{}_1(\{1\})H(X_1|U) + \Pr{}_1(\{2\})H(X_2|U)$$
$$R_2 \leq I(Y_2; U) = \Pr{}_2(\{1,2\})I(X_1, X_2; U) + \Pr{}_2(\{1\})I(X_1; U) + \Pr{}_2(\{2\})I(X_2; U),$$

for some joint distribution $\Pr(U, X = (X_1, X_2))$.

To find the boundary point of the capacity region corresponding to hyperplane $\underline{\mu} = (\mu_1, \mu_2)$, we have to solve the following optimization problem

$$f(\underline{\mu}) = \max_{\Pr(U,X)} \mu_1 R_1 + \mu_2 R_2.$$

One can further write

$$\begin{aligned} f(\underline{\mu}) &= \max_{\Pr(U,X)} \mu_2 \Pr{}_2(\{1,2\})H(X_1, X_2) + \mu_2 \Pr{}_2(\{1\})H(X_1) + \mu_2 \Pr{}_2(\{2\})H(X_2) + \\ &\quad aH(X_1, X_2|U) + bH(X_1|U) + cH(X_2|U), \end{aligned}$$

where $a = (\mu_1 \Pr{}_1(\{1,2\}) - \mu_2 \Pr{}_2(\{1,2\}))$, $b = (\mu_1 \Pr{}_1(\{1\}) - \mu_2 \Pr{}_2(\{1\}))$, and $c = (\mu_1 \Pr{}_1(\{2\}) - \mu_2 \Pr{}_2(\{2\}))$. We can upper bound $f(\underline{\mu})$ by substituting $H(X_1)$,

$H(X_2)$, and $H(X_1, X_2)$ with 1, 1, and 2 respectively. This gives

$$\begin{aligned} f(\underline{\mu}) \;&\geq\; \max_{\Pr(U,X)} aH(X_1, X_2|U) + bH(X_1|U) + cH(X_2|U) \\ &+\; 2\mu_2 \Pr{}_2(\{1,2\}) + \mu_2 \Pr{}_2(\{1\}) + \mu_2 \Pr{}_2(\{2\}). \end{aligned} \tag{3.22}$$

The optimization problem in (3.22) can be equivalently viewed as a Linear Program (LP) over the set of entropy vectors $h = (H(X_1, X_2|U)$, and $H(X_1|U), H(X_2|U))$, subject to the following constraints

$$\begin{aligned} H(X_1, X_2|U) &\leq H(X_1|U) + H(X_2|U), \\ H(X_i|U) &\leq H(X_1, X_2|U), \qquad i = 1, 2, \\ 0 &\leq H(X_i|U) \leq 1, \qquad i = 1, 2. \end{aligned}$$

Solving this LP gives the following upper bound for $f(\underline{\mu})$:

$$f(\underline{\mu}) \geq \max\{0, a+b, a+c, a+b+c, 2a+b+c\} + 2\mu_2 \Pr{}_2(\{1,2\}) + \mu_2 \Pr{}_2(\{1\}) + \mu_2 \Pr{}_2(\{2\}).$$

Furthermore, the following upper bound can be always achieved by considering X_1, X_2 to be i.i.d. uniform and letting U be a member of $\{\emptyset, X_1, X_2, (X_1, X_2), X_1 \oplus X_2\}$. It can be checked that the first four possibilities for U correspond to performing time-sharing. However $U = X_1 \oplus X_2$ does not correspond to time-sharing. Based on the above argument we have the following result.

Proposition 3.2. *The capacity region, $\mathcal{C}$, of a* degraded $(2,2)$*-erasure broadcast channel with correlated erasure events is*

$$\mathcal{C} = \text{conv}\left(\mathcal{R}_{TS} \cup \{(1 - \Pr{}_1(\emptyset), \Pr{}_2(\{1,2\}))\}\right),$$

where $\mathcal{R}_{TS}$ denotes the time-sharing region and conv $(\mathcal{A})$ *represents the convex hull*

$\mathcal{A}$	$\emptyset$	$\{1\}$	$\{2\}$	$\{1,2\}$
$\Pr_1(\cdot)$	0	0.35	0.35	0.3
$\Pr_2(\cdot)$	0.5	0.15	0.15	0.2

Table 3.1: Probability of erasure events for Example 3.2.

of set $\mathcal{A}$.

Example 3.2. Consider a degraded $(2,2)$-erasure channel with probability of erasures shown in Table 3.1. Using Proposition 3.2, we have plotted the capacity region of this channel in Figure 3.4. The time-sharing region is also shown in the same figure. Note that for this channel the capacity region is strictly greater than the time-sharing region.

3.4.3.2 Generalizing the Proof Technique

The main idea behind the proof of Theorem 3.3 is in understanding and exploiting the structure of the achievable region (here given by the time-sharing scheme). First, it is shown that for the "degraded" case, time-sharing is capacity achieving. Next, it is shown that for any point, $\underline{R}$, on the boundary of the time-sharing region one can find another (erasure) broadcast channel that is "degraded," its capacity region contains the capacity region of the original channel and shares the same boundary point $\underline{R}$ with the time-sharing region of the original channel (see Figure 3.3).

Now, turning our attention to general broadcast channels, the capacity region of these channels is still unknown. The best known inner bound is given by Marton [49]. The Marton inner bound is tight for all the cases that the capacity region is known. It is possible to generalize the approach taken above is a systematic way as follows. Consider a general broadcast channel g with n users and transition probability matrix $\Pr_g(Y_1, \ldots, Y_n|X)$, and let $\mathcal{R}_{MT}(g)$ denote the Marton region introduced in [49].

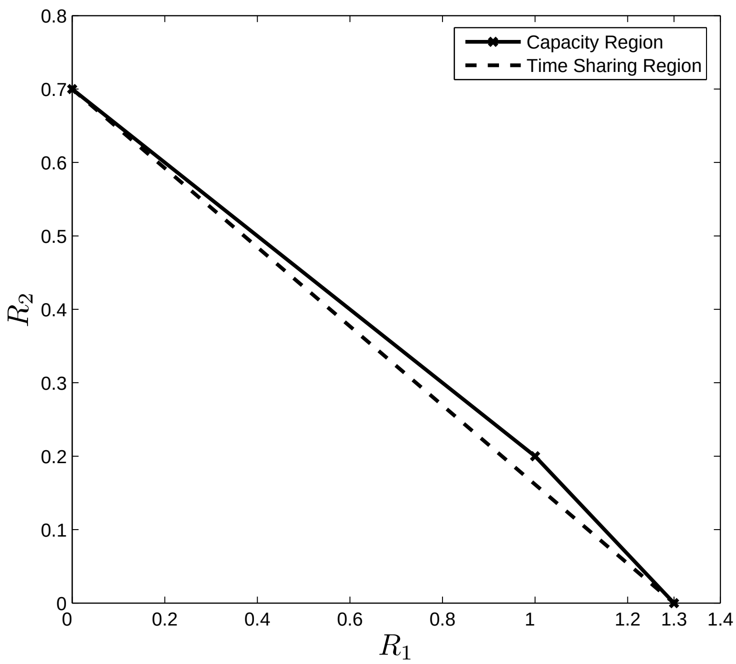


Figure 3.4: Capacity region of the degraded (2, 2)-erasure broadcast channel of Example 3.2.

Consider the following optimization problem for a given $\underline{\mu} \geq 0$ vector:

$$f_{MT}(\underline{\mu}, \Pr{}_g(\cdot|\cdot)) = \max_{(R_1,\ldots,R_n)\in\mathcal{R}_{MT}(g)} \sum_{i=1}^{n} \mu_i R_i. \tag{3.23}$$

Let $\mathcal{A}_g$ be the set of all channels (defined over the same input and output alphabet as g) for which g is a degraded version of, i.e.,

$$\mathcal{A}_g = \{\Pr{}_h(Y_1, \ldots, Y_n|X) | \Pr{}_g(\cdot|\cdot) \preceq \Pr{}_h(\cdot|\cdot)\}.$$

Furthermore, let $\mathcal{B}_g(\underline{\mu})$ be a subset of $\mathcal{A}_g$ such that for any channel h in $\mathcal{B}_g(\underline{\mu})$ we have

$$f_{MT}(\underline{\mu}, \Pr{}_g(\cdot|\cdot)) = f_{MT}(\underline{\mu}, \Pr{}_h(\cdot|\cdot)).$$

Now, if one can prove that $\mathcal{B}_g(\underline{\mu})$ is not empty, then the boundary point of the capacity region of g corresponding to the supporting plan $\underline{\mu}$ is achievable with the Marton coding scheme [49].

However, we should remark that the challenge in the aforementioned approach is in characterizing the Marton inner bound region. This is not an easy task since the cost function appearing in (3.23) should be maximized over the distribution of the input and a number of auxiliary random variables present in the Marton region formula. In general, this optimization problem is not convex. As a matter of fact, both in the erasure broadcast channel analyzed here and the MIMO Gaussian broadcast channel considered in [118], the optimization problem of (3.23) can not be solved explicitly. Rather, in both cases one constrains the joint probability distribution of the transmitted signal and the auxiliary random variables to a particular form and perform that optimization over the smaller space of probability distributions, e.g., i.i.d. Bernoulli random variables in erasure broadcast channels and jointly Gaussian random variables in the MIMO Gaussian broadcast channel. In both cases, one can find channels in $\mathcal{A}_g$ that share the same boundary point. The hope is that one can take the approach proposed here for other classes of broadcast channels and use the well-developed tools in optimization theory to find their capacity region.

3.5 Tighter Outer bounds for WE networks

In this section we will give tighter outer bounds for the capacity region of a general wireless erasure network. We will use the result derived in the previous section regarding the capacity region of erasure broadcast channels. The bound presented in this section is similar to the edge-cut bounds proposed in [45].

Consider a wireless erasure network represented by the acyclic directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Let $\mathcal{E}' \subseteq \mathcal{E}$ be a collection of links in the network. Let $\mathcal{D}(\mathcal{E}') \subseteq \mathcal{D}$ denote the subset of destination nodes that will be disconnected from s by removing the edges in $\mathcal{E}'$ from the network. For each destination node d_l in $d_l \in \mathcal{D}(\mathcal{E}')$, $\mathcal{J}(d_l, \mathcal{E}') \subseteq \mathcal{E}'$ is a subset of links from $\mathcal{E}'$ with the following properties:

- Removing $\mathcal{J}(d_l, \mathcal{E}')$ from the network disconnects d_l from s.
- No two edges in $\mathcal{J}(d_l, \mathcal{E}')$ lie on the same path from s to d_l.

In a sense, $\mathcal{J}(d_l, \mathcal{E}')$ is a minimal edge-cut for d_l. We should remark that $\mathcal{J}(d_l, \mathcal{E}')$ may not be unique. It is possible that there are more that two minimal edge-cuts. In this case, considering any of them will lead to an outer bound on the capacity region of the wireless erasure network.

Now, it is possible to construct a multiple input, multiple output erasure broadcast channel corresponding to $\mathcal{E}'$ and find a bound for the simultaneous rate of destinations in $\mathcal{D}(\mathcal{E}')$.

We construct the corresponding erasure broadcast channel as follows. Let $\mathcal{V}(\mathcal{E}') \subseteq \mathcal{V}$ denote the set of vertices that are the originating node of at least one edge in $\mathcal{E}'$. The corresponding erasure broadcast channel has one input for each node in $\mathcal{V}(\mathcal{E}')$ and one receiver (destination) for each node in $\mathcal{D}(\mathcal{E}')$. Consider a (the) minimal edge-cut for each of the destinations in $\mathcal{D}(\mathcal{E}')$. For each edge (i, v) in $\mathcal{J}(d_l, \mathcal{E}')$, we connect input i to receiver d_l through a packet erasure channel with probability of erasure ϵ_{iv}. Note that if more than one link originating from the same node is in $\mathcal{J}(d, \mathcal{E}')$, without loss of generality, we can put one link in the corresponding erasure broadcast channel with the probability of erasure equal to the product of the probabilities of erasure of those links. This way we will get an erasure broadcast channel. We refer to the channel matrix of this broadcast channel as $\underline{\epsilon}(\mathcal{J}(d, \mathcal{E}'),\ d \in \mathcal{D}(\mathcal{E}'))$ (or $\underline{\epsilon}(\mathcal{E}')$ whenever the edge-cuts are uniquely defined.)

Example 3.3. Let us revisit the wireless erasure network shown in Figure 3.1.(a). Let $\mathcal{E}' = \{(1, 2), (1, 3), (1, 4)\}$. In this case, $\mathcal{D}(\mathcal{E}') = \{2, 3\}$, $\mathcal{J}(2, \mathcal{E}') = \{(1, 2), (1, 4)\}$, and $\mathcal{J}(3, \mathcal{E}') = \{(1, 3), (1, 4)\}$. The broadcast channel corresponding to this choice of $\mathcal{E}'$ is

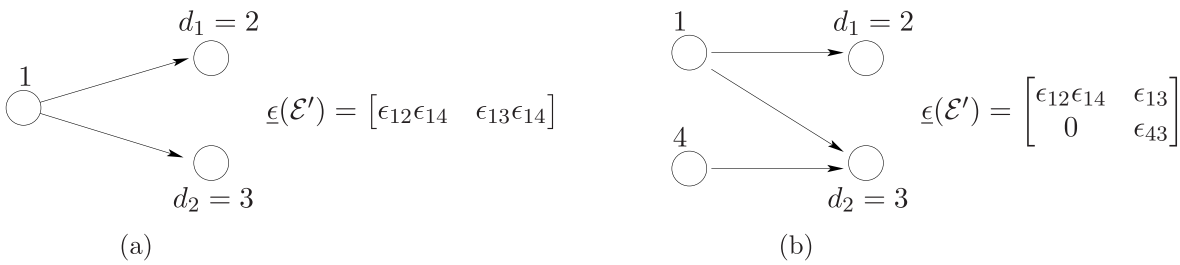


Figure 3.5: (a): Erasure broadcast channel derived from $\mathcal{E}' = \{(1,2),(1,3),(1,4)\}$ for the network in Figure 3.1. (b): Erasure broadcast channel constructed from $\mathcal{E}' = \{(1,2),(1,4),(4,3)\}$ for the same network as in (a).

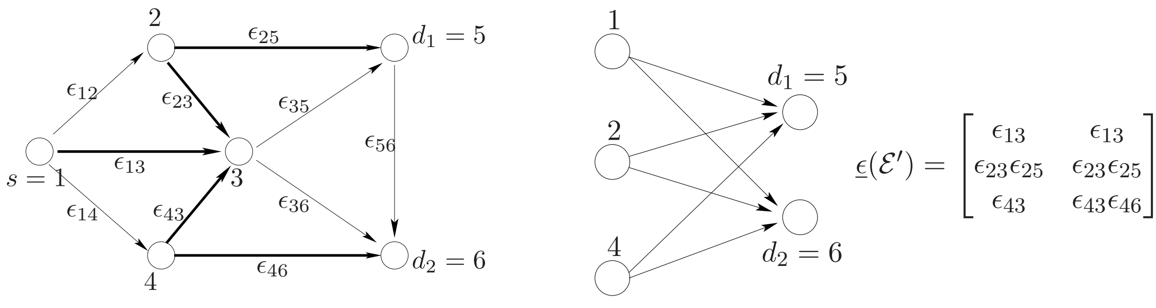
shown in Figure 3.5.(a). For $\mathcal{E}' = \{(1,2),(1,4),(1,3),(4,3)\}$, we have $\mathcal{D}(\mathcal{E}') = \{2,3\}$. A possible edge-cut for each destination is $\mathcal{J}(2,\mathcal{E}') = \{(1,2),(1,4)\}$ and $\mathcal{J}(3,\mathcal{E}') = \{(1,3),(4,3)\}$. The broadcast channel corresponding to this choice of $\mathcal{E}'$ is shown in Figure 3.5.(b).

Example 3.4. Consider the wireless erasure network shown in Figure 3.6.(a). In this network, node 1 is the source node, and the destination set is given as $\mathcal{D} = \{5,6\}$. Let $\mathcal{E}' = \{(1,3),(2,3),(2,5),(4,3),(4,6)\}$. In this case, $\mathcal{D}(\mathcal{E}') = \{5,6\}$ and $\mathcal{V}(\mathcal{E}') = \{1,2,3,4\}$. Furthermore, the edge-cut sets for destinations are $\mathcal{J}(6,\mathcal{E}') = \{(4,3),(4,6),(1,3),(2,3),(4,6)\}$ and $\mathcal{J}(5,\mathcal{E}') = \{(2,5),(1,3),(4,3),(2,3)\}$. The erasure broadcast channel corresponding to this choice of edge-cuts is shown in Figure 3.6.(b).

The following theorem provides an outer bound on the capacity region of broadcast problems for WE networks based on the previous construction.

Theorem 3.7. *Consider a* broadcast problem *over wireless erasure network $\mathcal{G} = (\mathcal{V},\mathcal{E})$ with source node s and destination set $\mathcal{D} = \{d_1,\ldots,d_{|\mathcal{D}|}\}$, and let $\mathcal{E}'$ be any subset of the edges. If $(R_1,R_2,\ldots,R_{|\mathcal{D}|})$ is a set of achievable rates for the broadcast problem, we should have*

$$(R_d,\, d \in \mathcal{D}(\mathcal{E}')) \in \mathcal{C}(\underline{\epsilon}\,(\mathcal{J}(d,\mathcal{E}'),\; d \in \mathcal{D}(\mathcal{E}'))), \tag{3.24}$$



(a) (b)

Figure 3.6: (a): A wireless erasure network with two destinations. (b): Erasure broadcast channel derived form $\mathcal{E}' = \{(1,3),(2,3),(2,5),(4,3),(4,6)\}$ for the network of part (a).

where $\mathcal{C}(\underline{\epsilon}\,(\mathcal{J}(d,\mathcal{E}'),\ d \in \mathcal{D}(\mathcal{E}')))$ *denotes the capacity region of the erasure broadcast channel corresponding to edge-cuts* $\mathcal{J}(d,\mathcal{E}')$ *for all* $d \in \mathcal{D}(\mathcal{E}')$.

Proof. For each destination $d \in \mathcal{D}(\mathcal{E}')$, consider the received signals on the links in $\mathcal{J}(d,\mathcal{E}')$, i.e., $Y(\mathcal{J}(d,\mathcal{E}')) = (Y_{ij},\ (i,j) \in \mathcal{J}(d,\mathcal{E}'))$. Now consider any sequence of block codes defined in Section 3.2 whose average probability of error goes to zero as the block length, n, increases. It can be shown that $W_d - Y^n(\mathcal{J}(d,\mathcal{E}')) - Y_d^n$ forms a Markov chain. This is true because the network is acyclic and $\mathcal{J}(d,\mathcal{E}')$ disconnects s and d. Hence, using Data Processing Inequality [53], we have

$$I(W_d;Y_d^n) \leq I(W_d;Y^n(\mathcal{J}(d,\mathcal{E}'))).$$

Now we have

$$\begin{aligned} nR_d &= H(W_d) \\ &= I(W_d;Y_d^n) + H(W_d|Y_d^n) \\ &\leq I(W_d;Y_d^n) + n\zeta_n \\ &\leq I(W_d;Y^n(\mathcal{J}(d,\mathcal{E}'))) + n\zeta_n, \end{aligned} \tag{3.25}$$

where we have used the Fano's inequality in the third line, and ζ_n will go to zero as the block length n goes to infinity. By noting that the capacity region of a broadcast channel depends on the marginals only, it becomes clear that the last inequality in (3.25) suggests that $(R_d,\, d \in \mathcal{D}(\mathcal{E}'))$ is an achievable rate vector for the broadcast channel constructed from edge-cuts $\mathcal{J}(d, \mathcal{E}')$ for all $d \in \mathcal{D}(\mathcal{E}')$, and this proves the theorem. ■

Remark 3.8. *As we can see from the above theorem, the outer bound is in fact intersection of some time-sharing regions, each corresponding to a subset of nodes and edges in the original network. However, unlike the time-sharing scheme of Section 3.3.1, our outer bounding technique does not take into account the fact that operation of a node across different subset of nodes and edges should be the same in all these time-sharing regions. In other words, the time-sharing parameters used for each node across different subsets of nodes and edges is different from each other and, therefore, these parameters loose their* operational *meaning in the original network. This is the main difference between our outer bound and the achievable region given in Section 3.3.1.*

The above theorem enables us to provide tighter outer bounds for the capacity region of general wireless erasure networks. We illustrate this fact in the following example.

Example 3.5. Let us consider the wireless erasure network shown in Figure 3.1.(a) again. As in Example 3.1, we set the erasure probabilities across the network to $(\epsilon_{12}, \epsilon_{13}, \epsilon_{14}, \epsilon_{42}, \epsilon_{43}) = (0.8, 0.2, 0.6, 0.8, 0.2)$. Using Theorem 3.7, we can derive the following four new bounds for any point in the capacity region of the network shown in Figure 3.1.(a):

$$(R_2, R_3) \in \mathcal{C}_1 \triangleq \mathcal{C}(\underline{\epsilon}(\{(1,2),(1,4)\},\{(1,3),(1,4)\})),$$
$$(R_2, R_3) \in \mathcal{C}_2 \triangleq \mathcal{C}(\underline{\epsilon}(\{(1,2),(1,4)\},\{(1,3),(4,3)\})),$$
$$(R_2, R_3) \in \mathcal{C}_3 \triangleq \mathcal{C}(\underline{\epsilon}(\{(1,2),(4,2)\},\{(1,3),(4,3)\})), \text{ and}$$
$$(R_2, R_3) \in \mathcal{C}_4 \triangleq \mathcal{C}(\underline{\epsilon}(\{(1,2),(4,2)\},\{(1,3),(1,4)\})).$$

Any achievable (R_2, R_3) should be in the intersection of the capacity region of the above four broadcast channels. We can describe each of the capacity regions $\mathcal{C}_i$, $1 \leq i \leq 4$, using Theorem 3.3 as follows:

$$\begin{aligned}
\mathcal{C}_1 &= \left\{(R_2,R_3)\middle| \frac{R_3}{0.88} + \frac{R_2}{0.52} \leq 1\right\} \\
\mathcal{C}_2 &= \left\{(R_2,R_3)\middle| \{\frac{R_3 - 0.8}{0.8}\}^+ + \frac{R_2}{0.52} \leq 1\right\} \\
\mathcal{C}_3 &= \left\{(R_2,R_3)\middle| \frac{R_2}{0.4} + \frac{R_3}{1.6} \leq 1\right\} \\
\mathcal{C}_4 &= \left\{(R_2,R_3)\middle| \{\frac{R_2 - 0.2}{0.2}\}^+ + \frac{R_3}{0.88} \leq 1\right\},
\end{aligned}$$

where $\{x\}^+ = \max\{0, x\}$. Hence, the capacity region of broadcast problem for this wireless erasure network is bounded as

$$\mathcal{C}(\mathcal{G}, \{\epsilon_{ij},\ (i,j) \in \mathcal{E}\}) \subseteq \cap_{i=1}^{4} \mathcal{C}_i = \left\{(R_2,R_3)\ \middle|\ \frac{R_3}{0.88} + \frac{R_2}{0.52} \leq 1,\ \frac{R_2}{0.4} + \frac{R_3}{1.6} \leq 1\right\}.$$

However, this outer bound and the time-sharing achievable region of (3.6) coincide for the particular choice of ϵ_{ij}, $(i,j) \in \mathcal{E}$ considered in this example. Therefore, the capacity region of the wireless network shown in Figure 3.1.(a) is given by time-sharing scheme of Section 3.3.1.

Example 3.6. Consider the network shown in Figure 3.1.(a) again. Assume the

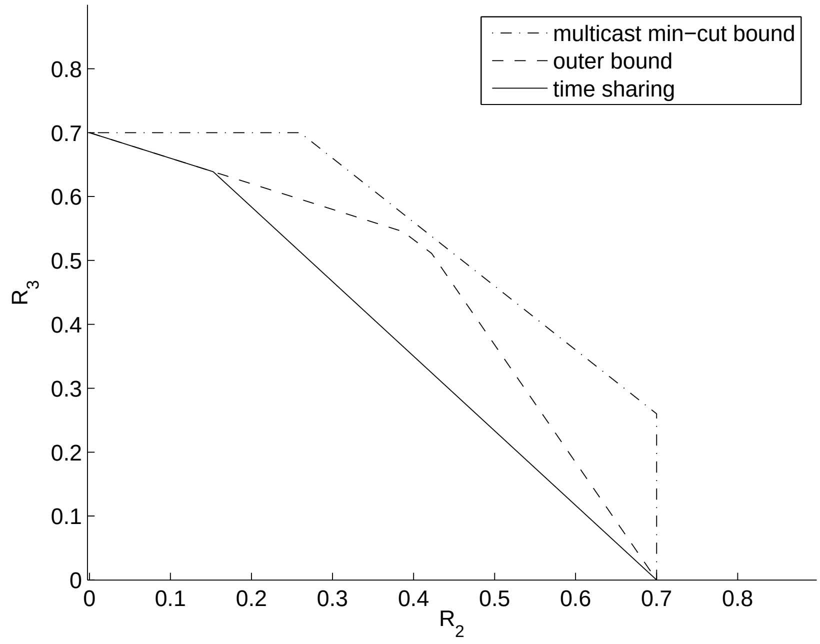


Figure 3.7: Time-sharing region and different outer bounds for Example 3.6.

erasure probabilities are as follows: $(\epsilon_{12}, \epsilon_{13}, \epsilon_{14}, \epsilon_{42}, \epsilon_{43}) = (0.5, 0.8, 0.1, 0.8, 0.5)$. In Figure 3.7 we have plotted the time-sharing region of Section 3.3.1, the outer bound given by Theorem 3.7, and the multicast type min-cut outer bound. As we see from the figure, the outer bound of Theorem 3.7 is tight for some regions; however, the min-cut outer bound is not tight anywhere.

3.6 Conclusion

In this chapter we considered a special class of network problems defined as *broadcast problem* over wireless erasure networks. In a broadcast problem one source has access to multiple information messages. Each of these information messages is requested by a particular destination. Downlink of cellular systems and TV broadcasting stations are examples of these problems. We proposed an achievable region based on time-sharing for these problems. We further showed that the time-sharing scheme is capacity-achieving in the multiuser setup, where the destinations are connected directly to the source node. The proof technique in this case requires understanding the structure of the achievable (time-sharing) region and borrows insights from optimization theory.

Using the capacity result for the multiuser setup, we gave an outer bound on the capacity region of broadcast problems over wireless erasure networks. This bound is tighter than the outer bounds that we get from the multicast type of arguments.

Finding the capacity region of broadcast problems over general wireless erasure networks remains an open problem. In order to find the capacity region, we require tighter outer bounds. Towards this end, an optimization theory viewpoint of the problem seems promising.

Chapter 4

A Practical Scheme for Wireless Networks

4.1 Introduction

In a wireline network having a single source and a single destination, we can think of information flow in the same terms as fluid flow and obtain a max-flow min-cut result to get capacity. This treatment closely follows that of the Ford-Fulkerson [52] algorithm to give us a neat capacity result. This has been well understood for many years. However, until recently, similar min-cut capacity results were not known for any other class of network problems. The remarkable results of [9, 10] say that in a wireline network setting, we can indeed achieve the min-cut upper bounds for multicast problems. These problems were defined in Chapter 2. Furthermore, it is shown that *separation* of channel coding and network coding does not degrade performance. This means that one can first perform channel coding for each link in the systems so as to make it error-free. By employing network coding on the resulting error-free network we achieve the min-cut capacity.

In Chapter 2 we also looked at the capacity of multicast problems for a class of wireless networks called wireless erasure networks. We showed that the capacity has a nice min-cut interpretation. Furthermore, it was shown that linear combining of data at intermediate nodes of the network achieves capacity. We further saw that separate channel coding and network coding is not necessary for achieving the capacity.

In this chapter we will look into the effect of *separation* of channel and network coding for *wireless* networks. In the early sections of this chapter, we will present simple *wireless* networks where this principle of separation fails. Thus we will show that operating wireless networks in a multihop manner, where each relay node decodes the message it receives, is not necessarily the right approach. This observation was first made in [55, 56]. We will also suggest some schemes of operation that will outperform those that require the ability of relay nodes to decode.

We will focus attention on two specific wireless network models. The important features that characterize a wireless network are broadcast and interference. The first model has Gaussian channels as links and incorporates broadcast as well as interference. The second model is the wireless erasure network introduced in Chapter 2 and has erasure channels as links and incorporates broadcast, but not interference. For these models, we will show that making links error-free can sometimes degrade the performance. In fact, asking nodes to simply forward their data rather than decoding it is sometimes more advantageous. This tells us that wireless networks need to be understood differently from wireline networks. We will see some explanations as to why this is the case later in the chapter.

In our study of wireless networks, we propose a scheme of network operation that permits only two operations at nodes . One is decoding to get the original data and then resending the same message as the source. The other is forwarding the data as is received. Since each node has two options, we have an exponential-sized set of possible operations. We will present an algorithm that goes over each node at most once to find the optimal operation among this set of restricted operations. This will be a greedy algorithm that avoids searching over the exponential-sized set of possible operation allocations. We also present an algorithm that can approach the best rate arbitrarily closely in an iterative manner. This will be a "decentralized" algorithm in the sense that each node needs only one bit of information from the destination in every iteration and no knowledge of the rest of the network in order to determine its own operation.

The organization of this chapter is as follows. In Section 4.2 we present two wireless network models. These will be the Gaussian wireless network and the wireless erasure networks studied in Chapter 2. In Section 4.3 we show that with these wireless models, making links or sub-networks error-free can be sub-optimal. In Section 4.4 we will formally state the two operations that nodes will be permitted to perform. With this setup, we will state our problem of allocating appropriate operations in Section 4.5. In Section 4.6 we will see how rates are calculated for all nodes in the network and how asking certain nodes to decode and others to forward can affect the rate of the network. In Section 4.7 we will state our algorithm to find the optimal policy. In Section 4.8 we will prove optimality of the algorithm. We will see some examples in Section 4.9 that will show that the gap between the "all nodes decode" strategy and our method can be significant. In Section 4.10 we will discuss the decentralized algorithm. We present upperbounds on the rate achievable by our scheme in Section 4.11. Conclusions and further questions are presented in Section 4.12.

4.2 Two Wireless Network Models

In this section we formalize two wireless network models considered in this chapter. In both cases the network consists of a directed, acyclic graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V} = \{1, \ldots, |\mathcal{V}|\}$ is the set of vertices, and $\mathcal{E}$ is the set of directed edges where each edge is a communication channel.

In this chapter we assume that there is only one source and one destination, i.e., we have a *unicast problem.* Without loss of generality, let $s = 1$ be the source node and $d = |\mathcal{V}|$ be the destination.[1] The remaining nodes are the relay nodes which must aid communication between s and d. We will assume that every edge is on some directed path from s to d. If we have edges other than these, we remove them, and what remains is our graph $\mathcal{G}$. We will denote the message transmitted by node i by X_i and that received by node j by Y_j. Figure 4.1 represents a network with 6 vertices

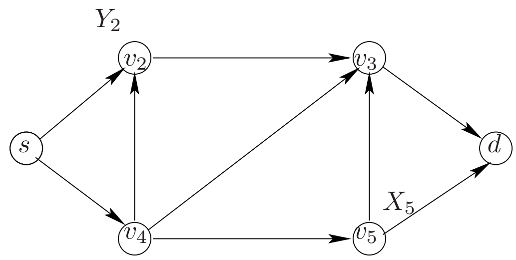


Figure 4.1: Example of a network.

and 9 edges where 1 is the source s and 6 is the destination d. X_5 is the message transmitted by node 5 and Y_2 is that received by node 2.

(a) **Gaussian Wireless Networks** In these networks each edge (i, j) of the network is a Gaussian channel with some fixed attenuation factor $h_{i,j}$ associated with it. In a practical system, this may be some pathloss that depends on the physical distances between the nodes. We will assume $h_{i,j}$ to be a non-negative constant. We will assume that nodes broadcast messages, i.e., a node transmits the same message on all outgoing edges. Assuming that Figure 4.1 represents a Gaussian wireless network, X_5 is the message transmitted on edges $(5, 3)$ and $(5, 6)$. We will also assume interference, i.e., the received signal at node i is the sum of all the signals transmitted on edges coming in to it and additive white Gaussian noise n_i of variance σ_i^2. Therefore, in general, we have

$$Y_i = n_i + \sum_{j:(j,i)\in\mathcal{E}} h_{j,i} X_j.$$

[1]The results of this chapter can be generalized easily for a multicast problem.

All n_is are assumed independent of each other as well as the messages. For Figure 4.1 this implies that $Y_2 = h_{1,2}X_1 + h_{4,2}X_4 + n_2$. We will assume that all transmitting nodes have a power constraint of P.

(b) **Wireless Erasure Networks**

These networks were introduced in Chapter 2. In Chapter 2, the multicast capacity of these networks were found. To remind ourselves, in these networks each edge (i, j) of the network is a binary erasure channel with erasure probability ϵ_{ij}. In this chapter we further assume that nodes (other than the source node) can transmit erasures e. If an edge takes e as input, the received signal on that edge is always e. In short, the channel for edge (i, j) (for $i \neq s$) is modified as in Figure 4.2 compared to the one in Chapter 2. We incorporate broadcast in the model, i.e., each transmitting node must send out the same signal on each outgoing edge. However, we assume that the is no interference.

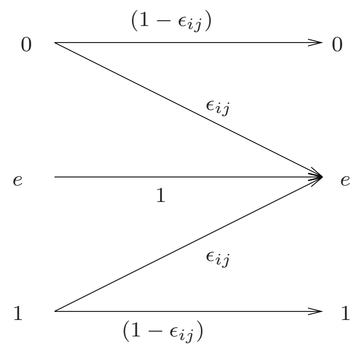


Figure 4.2: Modified Erasure Channel.

4.3 Optimizing Over Sub-networks Does Not Work

Theorem 4.1. *For the wireless networks described in Section 4.2, making sub-networks error-free can be suboptimal.*

Proof. We give some examples to demonstrate this.

- **Gaussian Relay Networks**: Consider a Gaussian parallel relay network consisting of two relay nodes and one source-destination pair. See Figure 4.3(a). All four channel coefficients are assumed to be 1. The relay nodes 2 and 3 are solely to aid communication from source to destination. We assume that the noise power at each receiver is σ^2, and the transmit power at each node is P. Let $\rho \triangleq \frac{P}{\sigma^2}$ be the Signal to Noise Ratio (SNR).

  One way to view the network is as a cascade of a broadcast channel (from s to $\{2,3\}$) and a multiple access channel (from $\{2,3\}$ to d). This is equivalent to assuming that the relays decode their messages correctly and code them again and transmit. If the relays are receiving independent information at rates R_1 and R_2, we have $R_1 + R_2 \leq \log(1+\rho)$ as the capacity region. These rate pairs (R_1, R_2) can be supported by the multiple access channel, and hence the maximum rate from s to d is no greater than $\log(1+\rho)$. If the relays are receiving exactly the same information from the source, the maximum rate of this is $\log(1+\rho)$. In this case, the multiple access channel is used for correlated information and can support rates up to $\log(1+4\rho)$. In either case, asking the relay nodes to decode limits the rate from s to d to $\log(1+\rho)$. (We note also that the broadcast sub-network is the bottleneck in both cases.)

  Now consider another strategy in which the relay nodes do not decode but only normalize their received signal to meet the power constraint and transmit it to the destination. In this case the received signal at the destination is

$$Y_4 = \sqrt{\frac{P}{P+\sigma^2}}(2X_1 + n_2 + n_3) + n_4,$$

  where X_1, Y_4, n_2, n_3, n_4 are, respectively, the transmitted signal from the source, the received signal at the destination, and the noises introduced at nodes 2, 3 and d. Thus, the signal received by d is a scaled version of X_1 with additive

Gaussian noise. The maximum achievable rate, denoted by R_f, is

$$R_f = \log\left(1 + \frac{\frac{4P^2}{P+\sigma^2}}{\sigma^2 + \frac{2P\sigma^2}{P+\sigma^2}}\right) = \log\left(1 + \frac{4\rho^2}{3\rho+1}\right),$$

where ρ is as before. Here, the subscript f stands for *forwarding*.

Comparing R_d and R_f, we can see $\rho = 1$ is a critical value in the following sense. For $\rho > 1$, we have superior performance in the forwarding scheme, and for $\rho < 1$ we have better rate with relay nodes decoding and re-encoding. This implies that making a sub-network error-free (in this case the broadcast section, or the links $(1, 2)$ and $(1, 3)$) can sometimes be sub-optimal.

We note that decoding at one of the relay nodes and forwarding at the other is always sub-optimal.

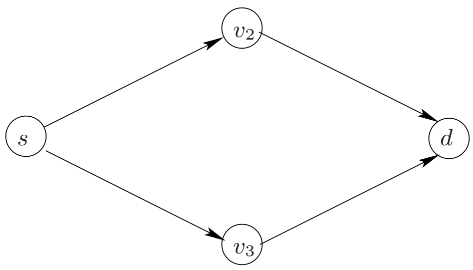
In general, if we have $k(\geq 2)$ relay nodes in parallel rather than two, it can be easily checked that

$$R_d = \log(1+\rho) \quad \text{and} \quad R_f = \log\left(1 + \frac{k^2\rho^2}{(k+1)\rho+1}\right).$$

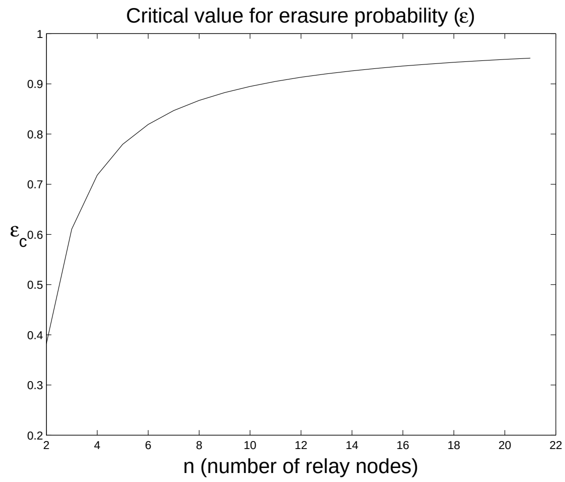
With this we get a critical value of $\rho = \frac{1}{k^2-k-1}$, below which decoding is better and above which forwarding is better. Clearly, this goes to zero for large k. Therefore, in the limit of $k \to \infty$, it is always favorable to forward.

It turns out that this fact is also true for Gaussian relay networks in the presence of fading. The work of [55] shows that for fading Gaussian relay networks with n nodes, the asymptotic capacity achievable with the relay nodes decoding (and re-encoding) scales like $O(\log\log n)$, whereas with the forward scheme it scales like $O(\log n)$.

Similar problems are considered in [60] and [91]. The former considers bounds and achievable rates for the Gaussian network with two parallel links, and the latter considers a network with a single source and destination and the other nodes acting as relays. The second result shows that the maximum rate achievable is $O(\log n)$. This is the same as that achieved by forwarding in our scheme.



(a) Graph representation of a relay network with two relay nodes.



(b) Critical value of erasure probability for k relay nodes.

Figure 4.3: Proof of Theorem 4.1.

- **Erasure Relay Network**: Consider, once again, the network of Figure 4.3(a), where, now, each link represents an erasure channel with erasure probability $\epsilon_{ij} = \epsilon$. Since we have broadcast, node s transmits the same messages to relay nodes 2 and 3. If the relay nodes decode and re-encode, the rate is bounded by the sum-rate capacity of the broadcast system, which gives

$$R_d = 1 - \epsilon.$$

If the relay nodes simply forward what they receive, it is easy to see that the destination sees an effective erasure probability of $(1 - (1 - \epsilon)^2)$. (We will spell out how to do this calculation for a general network in Section 4.6.) Forwarding erasures is possible since we are assuming the modified erasure channel of Figure 4.2. With this we have $R_f = 1 - (1 - (1 - \epsilon)^2)^2$. Comparing R_f and R_d, we can see that $\epsilon = \frac{3-\sqrt{5}}{2}$ is a critical value, above which decoding and re-encoding is better and below which forwarding is better.

Thus we see that for this network also, making the broadcast sub-network error-free is not always optimal.

In general, if we have k relay nodes in parallel rather than two, we have

$$R_d = 1 - \epsilon \quad \text{and} \quad R_f = 1 - (1 - (1 - \epsilon)^2)^k,$$

and the critical value of e is as plotted in Figure 4.3(b). Below this, forwarding is better, and above this, decoding is better. In the limit of large k, it is always better to forward.

From this we see that making links or sub-networks error-free does not ensure optimal network operation. It can sometimes be provably sub-optimal. ■

In this proof a simple operation like forwarding the received data proved to be better than decoding it. We understand this as follows. Because of the broadcast present in wireless networks, the same data naturally gets passed on to the destination along many different paths. Therefore, some nodes receive better versions of the data on incoming links than other nodes and are automatically in a better position to decode. Forcing all the nodes to decode and be error-free only imposes additional bottlenecks on the rate. Therefore, it is beneficial to carefully check the quality of the effective signal that various nodes get to see and then decide whether to ask them to decode or not.

4.4 A Possible Set of Network Operations

It follows from the previous discussions that to obtain the optimum rate over wireless networks, the nodes must perform operations other than just decoding. Determining what the optimum operation at each node should be, especially for a general wireless network, appears to be a daunting task. We shall therefore simplify the problem by allowing one of only two operations at every node. One will be the *decode and re-encode* operation as before. The other is the far simpler operation of *forwarding* the received data as is. The first operation, viz., decode and re-encode is typically the only operation used in multihop networks and many wireline networks. In effect, we are attempting to attain higher rates by introducing the additional operation of forwarding.

We will assume that the network operates in blocks of length n. We assume that the source s has a set of message indices

$$\Omega = \{1, 2, \ldots, 2^{\lfloor nR \rfloor}\}$$

and an encoding function

$$f : \Omega \to \mathcal{X}^n,$$

where $\mathcal{X}$ is $\mathbb{R}$ for the Gaussian wireless network and $\{0, 1\}$ for the erasure wireless network. To transmit message $i \in \Omega$, the source transmits $f(i)$. With this the source operates at rate R. $\{f(1), f(2), \ldots, f(2^{\lfloor nR \rfloor})\}$ is the set of codewords or possible transmitted messages. This set is called the codebook and is denoted by $\mathcal{C}$. We assume that all nodes have the codebook. For the Gaussian network we will assume that the codebook meets the power constraint, i.e., $E\|f(i)\|^2 \leq P$.

Here we restrict the relay nodes to two operations. These have been introduced in the examples of Section 4.3, viz., "forward" and "decode and re-encode." We now state them formally.

(a) **Decode and Re-encode**: This operation implies that when node i receives message Y_i it performs ML decoding of Y_i to determine which message index was transmitted by s. Since it has the codebook, it re-encodes the message using the same codeword that the source s would have used and transmits the same codeword. In short, it should act like a copy of the source.

However, for this to happen, we need that the decoding be error-free. This implies that the rate R at which the source operates should be no greater that the maximum rate at which node i can decode. We will see the relevance of this constraint in Section 4.5.

(b) **Forward**: We will describe this operation separately for the two network models. In the Gaussian network, node i receives message Y_i given by

$$Y_i = n_i + \sum_{j:(j,i)\in\mathcal{E}} h_{j,i} X_j. \tag{4.1}$$

"Forwarding" implies that the node normalizes this signal to meet the power constraint and then transmits the message. Therefore, it transmits X_i given by

$$X_i = \sqrt{\frac{P}{E\|Y_i\|^2}}\; Y_i.$$

We will assume that $E\|Y_i\|^2$ is known to i.

For the erasure network, nodes either decode without error and transmit the original codeword or "forward" the received data. Consider node i which sees data coming in on several edges in the form of n-length blocks of bits and erasures. For the b-th bit of such a block, it either sees erasures on every edge (and this sees an *effective* erasure) or gets to see the bit on at least one incoming edge. (It cannot happen that the node sees 1 on a particular edge and 0 on another edge for the b-th position. This is because of our assumption that whenever an earlier node decodes it does so without error.) Therefore, in our interference-free model, every relay node sees an effective erasure channel from the source, i.e., it sees the codeword transmitted by the source with some bits erased. "Forwarding" means broadcasting this sequence of bits and erasures.

Note that the effective erasure probability seen by node i is a function of the network topology and parameters, ϵ_{ij}. We will see in Section 4.6.2 how this effective erasure probability can be calculated.

By restricting ourselves to only two operations, we have ensured that all nodes in the network see a Gaussian channel (with some effective SNR) or erasure channel (with some effective erasure probability) with respect to the transmitted codeword. Therefore, they can do ML decoding or typical set decoding if R is no greater than the rate that they can support. We will always ensure that R satisfies this constraint.

We can think of both operations as specific forms of network coding. In both networks and with both operations, all the information coming in at a node on different edges gets pooled together – this happens automatically in the Gaussian network and is done by the node itself in the erasure network. But the node has the choice of trying to decode, thus imposing a rate constraint, or can simply forward the information, hoping that some other node would have a better chance of decoding.

Having described the two operations permitted to the relay nodes in the two networks, we are now ready to formally state the problem.

4.5 Problem Statement

Since we allow only two operations to nodes, viz., "decode and re-encode" and "forward," and every relay node must perform one of these, it is enough to specify the set of relay nodes that "decode and re-encode" in order to completely specify the working of the network. The source and destination will always be excluded from this set.

If a set $D \subseteq \mathcal{V} - \{s, d\}$ is the set of nodes [2] that "decode and re-encode," we will call D a **policy** for network operation.

Under policy D, each node of the network sees an effective (Gaussian or erasure) channel from the source. Let the effective SNR that node i sees under policy D be denoted by $\rho_D(i)$ for Gaussian networks. For erasure networks we denote the effective erasure probability seen by node i under policy D by $\epsilon_D(i)$. Therefore, the rate that node i can support under policy D is $\log(1 + \rho_D(i))$ or $(1 - \epsilon_D(i))$ for Gaussian or erasure networks, respectively. In general we will call this $R_D(i)$. Nodes in D as well as the destination must be able to perform error-free decoding. This means that the rate at which the source transmits must be no greater than the rates at which these nodes can decode. This tells us that under policy D, the rate R at which we can operate the network is constrained by

$$R \leq \min_{i \in D \cup \{d\}} R_D(i). \tag{4.2}$$

We denote this minimum by R_D. So,

$$R_D = \min_{i \in D \cup \{d\}} R_D(i). \tag{4.3}$$

Intuitively, asking some nodes to decode means that there are more copies of the source in the network, and hence the rate which the destination can support increases. On the other hand, asking a node to decode introduces a constraint on the rate R. This is the tradeoff for any policy D. For instance, in Figure 4.1 consider nodes 2 and 4. If 4 forwards, node 2 sees an effective erasure probability of $\epsilon_{42}\epsilon_{12} + \epsilon_{14}\epsilon_{12}(1 - \epsilon_{42})$. (We will see how this has been calculated in Section 4.6.2.) On the other hand, if 4

[2] The set D should not be confused with the set of destinations defined in Chapter 2. Here, since we are looking at unicast problems, there exists only one destination d.

decodes, node 2 is at an advantage since it sees a lower effective erasure probability, viz., $\epsilon_{12}\epsilon_{42}$. However, asking 4 to decode puts a constraint on the rate as seen by (4.2) since the rate that 4 can support is only $(1 - \epsilon_{14})$. This constraint is $R_D \leq 1 - \epsilon_{14}$.

Our problem is to find the policy that gives the best rate, i.e., to find D such that R_D is maximized, viz.,

$$\max_{D} \min_{i \in D \cup \{d\}} R_D(i).$$

First we need to address the question of finding $R_D(i)$, i.e., of finding the rate at node i under policy D. Recall that X_i and Y_i are the transmitted and received messages at node i. If we are using policy D, we will denote these by $X_D(i)$ and $Y_D(i)$. We may drop the subscript D if it is clear which policy we are referring to. Note that for the source, the transmitted message is X_1 irrespective of the policy.

4.6 Determining the Rate at a Node – $R_D(i)$

In this section we describe a method to find the rate at an arbitrary node i when the set of decoding nodes is given by D. Therefore, we need to find the effective SNR or erasure probability of the received signal $Y_D(i)$. In order to do that, we need the concept of a partial ordering on the nodes introduced in Section 2.2.2.

Next we address the issue of determining the rate under a particular policy. We discuss this separately for Gaussian wireless networks and Erasure wireless networks.

4.6.1 Finding the Rate in Gaussian Wireless Networks

Recall that $Y_D(j)$ is the received signal at v_j under policy D. Once we know $Y_D(j)$, we can determine the signal power and the noise power in it. Denote these by $P_D(j)$ and $N_D(j)$, respectively. Consider node j. If it is decoding, $X_D(j) = X_1$. If it is forwarding,

$$X_D(j) = \sqrt{\frac{P}{E\|Y_D(j)\|^2}}\ Y_D(j) = \sqrt{\frac{P}{P_D(j)+N_D(j)}}\ Y_D(j).$$

We now outline a method for finding the rate for all the nodes by proceeding in the order given by $\mathcal{T}$ defined in Section 2.2.2. Without loss of generality, assume that the nodes are already numbered according to a partial ordering. Therefore $\mathcal{T} = (s, 2, \ldots, d)$. Then, for 2 we only have an edge coming in from s, and hence

$$Y_D(2) = h_{1,2}X_1 + n_2.$$

Let our induction hypothesis be that we know $Y_D(j)$ for $j = 1, \ldots, i-1$. For $Y_D(i)$ we now have

$$\begin{aligned} Y_D(i) &= n_i + \sum_{j:(j,i)\in\mathcal{E}} h_{j,i}X_D(j) \qquad (4.4) \\ &= n_i + \sum_{j:(j,i)\in\mathcal{E}, j\in D\cup\{s\}} h_{j,i}X_1 + \sum_{j:(j,i)\in\mathcal{E}, j\notin D\cup\{s\}} h_{j,i}X_D(j) \\ &= n_i + \sum_{j:(j,i)\in\mathcal{E}, j\in D\cup\{s\}} h_{j,i}X_1 + \sum_{\substack{j:(j,i)\in\mathcal{E} \\ j\notin D\cup\{s\}}} h_{j,i}\sqrt{\frac{P}{P_D(j)+N_D(j)}}\ Y_D(j). \end{aligned}$$

By our hypothesis, we know all the $Y_D(j)$ that occur in the last summation; Substituting for these, we get $Y_D(i)$. Careful observation indicates that this will be a linear combination of X_1 and the noise terms $n_2, \ldots, n_i$.

In general, if this linear combination is given by

$$Y_D(i) = a_D X_1 + \sum_{j=2}^{i} a_{D,j}(i)n_j,$$

we have $P_D(i) = a_D^2 P$ and $N_D(i) = \sum_{j=2}^{i} a_{D,j}^2(i)\sigma_j^2$. Once these are known, the SNR is simply $\rho_D(i) = \frac{P_D(i)}{N_D(i)}$, and the rate can be calculated as $R_D(i) = \log(1 + \rho_D(i))$. Clearly, the complexity of this procedure is $O(|\mathcal{V}|)$.

4.6.2 Finding Rate in Erasure Wireless Networks

We first put this problem in a graph theoretic setting. We are given a directed, acyclic graph where certain nodes act as sources. For us, the set $D \cup \{s\}$ is the set of source nodes. All the edges of the graph have certain probabilities of failing, i.e., of being absent. For us, these are the erasure probabilities of the channel. With this setup, for every node v in the network (excluding s, but including those in D) we need to find the probability that there exists at least one directed path from some source node to this node. This is the Network Reliability problem in one of its most general formulations [59, 61]. This is a well-studied problem and is known to be $\#P$-hard [61]. Although no polynomial-time algorithms to solve the problem are known, efficient algorithms for special graphs are known. An overview of the network reliability problem can be found in [62]. In the rest of this section we propose two straightforward methods to compute the probabilities of connectivity that we are interested in. We will also mention some techniques that can reduce the computation involved in these methods.

Assume we have a policy D. Consider a node i of the network. To find $R_D(i)$ we need to find $\epsilon_D(i)$. A bit is erased at node i if it is erased on all incoming links. Remember from Chapter 2 that the erasure event of link (i, j) is represented by Z_{ij}. This random variable takes the value 0 when a bit is erased and the value 1 when a bit is not erased. Thus, it is a Bernoulli random variable with probability $(1 - \epsilon_{ij})$.

Consider all the directed paths from s to i. Let there be k_i paths. Denote the paths by $B_1, \ldots, B_{k_i}$. Let path B_j consist of l_j edges. We specify path B_j by writing in order the edges it traverses, i.e., with the sequence $((v_{j_1}, v_{j_2}), (v_{j_2}, v_{j_3}), \ldots, (v_{j_{l_j}}, v_{j_{l_j+1}}))$. We know that $s = v_{j_1}$ and $i = v_{j_{l_j+1}}$. Consider the set of vertices excluding i that are on path j, i.e., $\{v_{j_i} : i = 1, \ldots, l_j\}$. Some nodes in this set may belong to D, i.e., they are decoding nodes. In this case we know that they transmit the original codeword exactly. Let t be the largest index in this set such that v_{j_t} decodes. Therefore, i will not receive bit b along path B_j only if an erasure occurs on an edge that comes after v_{j_t} in the path. We associate with path B_j the product of the random variables that affect this, viz.,

$$Z_j^{\star} = Z_{j_t,j_{t+1}} \cdot Z_{j_{t+1},j_{t+2}} \cdot \cdots \cdot Z_{j_{l_j},j_{l_j+1}}.$$

This product is zero if one of the Z random variables takes value zero, which, in turn, means that an erasure occurred on that edge.

Now, i sees an erasure only when none of the paths from s to itself manage to transmit the bit to it. Therefore, i sees an erasure when $Z_j^{\star} = 0$ for *all* the paths $B_j, j = 1, \ldots, k_i$. Therefore we have

$$\begin{aligned} R_D(i) &= 1 - \epsilon_D(i) \\ &= 1 - \Pr\,(\bigcap_{j=1}^{k_i}(Z_j^{\star} = 0)) \\ &= \Pr\,(\bigcup_{j=1}^{k_i}(Z_j^{\star} \neq 0)) \end{aligned}$$

One way to evaluate this is by checking all possible combinations of values that the Z variables can take and finding the total probability of those combinations that satisfy $\bigcup_{j=1}^{k_i}(Z_j^{\star} \neq 0)$. This procedure has complexity $O(2^{|\mathcal{E}|})$. One observation that can make this procedure more efficient is the following – if we know that setting a certain subset of the Z variables to 1 is enough to make the event $\bigcup_{j=1}^{k_i}(Z_j^{\star} \neq 0)$ happen, then for every superset of this subset, setting all the Z variables in that superset to 1 is also enough to make the event $\bigcup_{j=1}^{k_i}(Z_j^{\star} \neq 0)$ happen. With this, we may have to check out fewer than the $2^{|\mathcal{E}|}$ possible combinations of values for the Z variables and reduce the complexity.

Another way to evaluate this is by using the Inclusion Exclusion Principle [58].

This gives us

$$\Pr(\bigcup_{j=1}^{k_i} Z_j^\star \neq 0) = \sum_{r=1}^{k_i} \sum_{1\leq j_1<\cdots<j_r\leq k_i} (-1)^{r+1}\Pr(Z_{j_1}^\star \neq 0, \ldots, Z_{j_r}^\star \neq 0).$$

Since we have k_i paths, the above expression has $2^{k_i} - 1$ terms. A general term of the form $\Pr(Z_{j_1}^\star \neq 0, \ldots, Z_{j_r}^\star \neq 0)$ can be evaluated by first listing all the Z variables that occur in at least one of the r terms. Say these are $Z_{i_1 j_1}, \cdots, Z_{i_q j_q}$. Now $\Pr(Z_{j_1}^\star \neq 0, \ldots, Z_{j_r}^\star \neq 0)$ is given by the product $(1 - \epsilon_{i_1 j_1}) \times \cdots \times (1 - \epsilon_{i_q j_q})$. This procedure has complexity $O(|\mathcal{E}|2^k)$ where k is the $\max_i k_i$. In this procedure, the complexity of listing all the variables in a certain set of r terms can be reduced by storing the lists that one makes for sets of $(r-1)$ terms and simply adding on the z terms from the r-th term to the appropriate list.

4.7 Algorithm to Find Optimum Policy

In general, since we have $|\mathcal{V}| - 2$ relay nodes and each node has two options, viz., "forwarding" and "decoding and re-encoding," we have $2^{|\mathcal{V}|-2}$ policies. To find the optimum policy we can analyze the rate for each of these policies and determine the one that gives us the best rate. This strategy of exhaustive search requires us to analyze $2^{|\mathcal{V}|-2}$ policies.

Here, we propose a greedy algorithm that finds the optimum policy D which maximizes the rate. This algorithm requires us to analyze at most $|\mathcal{V}| - 2$ policies. In the next section we will give a proof of correctness for this algorithm.

(a) Set $D = \emptyset$.

(b) Compute $R_D(i)$ for all $v_i \in \mathcal{V}$. (Use techniques of Section 4.6.)
Find $R_D = \min_{i\in D\cup\{d\}} R_D(i)$.

(c) Find $M = \{i | i \notin \{s,d\} \cup D, R_D \leq R_D(i)\}$.

(d) If $M = \emptyset$, terminate. D is the optimal strategy.

(e) If $M \neq \emptyset$, find the largest $D' \subseteq M$ such that $\forall v \in D'$, $R_D(v) = \max_{i \in M} R_D(i)$.
Let $D = D \cup D'$.
Return to 2.

At each stage of the algorithm we look for nodes that are seeing a rate as good as or better than the current rate of network operation. If there are no such nodes, the algorithm terminates. If there are such nodes, we choose the best from among them. Thus, in every iteration, the nodes we add are such that they do not put additional constraints on the rate of the network. Therefore, the rate of the network can only increase in successive iterations.

Note that since we assume a finite network, this algorithm is certain to terminate. Also, since D cannot have more than $(|\mathcal{V}| - 2)$ nodes, the algorithm cycles between steps 2 to 5 at most $(|\mathcal{V}| - 2)$ times. This is significantly faster than the strategy of exhaustive search that requires us to analyze $2^{|\mathcal{V}|-2}$ policies.

The complexity of the algorithm depends on how fast the computation of $R_D(i)$ can be done. We have seen techniques for this computation in Section 4.6.

4.8 Analysis of the Algorithm

We first prove a Lemma regarding the effect of decoding at a particular node on the rates supportable at other nodes.

Lemma 4.1. *When node v is added to the decoding set D, the only nodes i that may see a change in rate are $v \prec i$. This change can only be an increase in rate, i.e., $\forall i$ such that $v \prec j$ we have $R_D(i) \leq R_{D \cup \{v\}}(i)$. Every other node j is unaffected, i.e., $R_D(j) = R_{D \cup \{v\}}(j)$.*

Proof. We give a proof for the Gaussian Network. We omit the proof for erasure networks since it uses the same ideas.

Gaussian Network : Recall the computation of $\rho_D(i)$ described in Section 4.6.1. The computation for $Y_D(i)$ depends only on (some of) the $Y_D(j)$ where (j, i) is an edge. Therefore, inductively, it is clear that $Y_D(v_i)$ (and hence $\rho_D(i)$) depends only on the nodes v where $v \prec i$. Therefore, the only nodes that are affected when v changes its operation (from "forwarding" to "decoding and re-encoding") are $v \prec i$. The rest are unaffected.

Consider one of the $X_D(j)$ terms in (4.4). Note that each of these are of power P of which some power is the signal power and the rest is the noise power. If node j changes its operation from forwarding to decoding, $X_D(j) = X_1$, i.e., the signal power increases to P and the noise power goes to 0. If node j is forwarding, $X_D(j)$ is only a scaled version of $Y_D(j)$. Since it is always of power P, if the SNR at node j increases, the signal power in $X_D(j)$ increases while the noise power decreases. From (4.4) we see that in both these cases, there is an increase in the signal power of $Y_D(i)$ and a decrease in the noise power. This implies an increase in the SNR.

Therefore, when v is added to D, by induction, for all nodes $v \prec i$, the SNR, if affected, can only undergo an increase. Naturally, we have the same conclusion for the rate. ■

This Lemma tells us that adding nodes to the set of decoding nodes can only increase the rate to other nodes. While this sounds like a good thing, it also puts a constraint on the rate as indicated by (4.2). It is this tradeoff that our algorithm seeks to resolve by finding the optimal set of decoding nodes.

4.8.1 Proof of Optimality

Theorem 4.2. *The algorithm of Section 4.7 gives us an optimal set of decoding nodes.*

Proof. Let S be an optimal set of decoding nodes. Let D be the set returned by the algorithm. We will prove that $R_D \geq R_S$. Then, since S is optimal, we will have

$R_D = R_S$.

We prove $R_D \geq R_S$ in two steps. First we show that $R_{S\cup D} \geq R_S$. Then we show that $S \cup D - D = \emptyset$, i.e., $S \cup D = D$. This will complete the proof.

Step 1: In every iteration, the algorithm finds subsets D' and adds them to D. Denote by D_i the subset that is added to D in the i-th iteration. Assuming the algorithm goes through m iterations, we have $D = D_1 \cup \cdots \cup D_m$ where the union is over disjoint sets. In the algorithm, when D_i is added to D, all the nodes in it are decoding at the same rate, which is $R_{D_1\cup\cdots\cup D_{i-1}}(v)$ for $v \in D_i$. We will call this rate $R_{\text{algo},i}$. Consider the smallest i such that $D_i \nsubseteq S$, i.e., D_i is not already entirely in S.

Claim: Adding D_i to S does not decrease the rate, i.e., $R_{S\cup D_i} \geq R_S$.

Proof. Because of the acyclic assumption on the graph, we will have some nodes $v \in S$ such that $\forall\, u(\neq v) \in S$, we either have $v \prec u$, or v and u are incomparable. Let L be the set of all such nodes v. Note that by Lemma 4.1, node v supports a rate $R_S(v) = R_\emptyset(v)$. By (4.3), for every $v \in L$ we have the necessary condition

$$R_S \leq R_S(v) = R_\emptyset(v). \tag{4.5}$$

Also note that $D_1, \ldots, D_{i-1}$ are all in S, and by the definition of L and Lemma 4.1 we have

$$R_\emptyset(v) = R_{D_1\cup\cdots\cup D_{i-1}}(v). \tag{4.6}$$

We now consider two cases.

- If for some $w \in L$ we also have $w \in D_i$, then from (4.5) and (4.6) we have $R_S \leq R_S(w) = R_\emptyset(w) = R_{D_1\cup\cdots\cup D_{i-1}}(w) = R_{\text{algo},i}$.

- On the other hand, if none of the nodes in L are in D_i, pick any node $v \in L$. We have $v \notin D_i$. We now consider two subcases.

  - Let $v \notin D_1, \ldots, D_{i-1}$. We note from Steps 3 and 5 of the algorithm that it picks out from the set of nodes not in D, all nodes with the best rate. Since v does not get picked, we have $R_{\text{algo},i} > R_{D_1\cup\cdots\cup D_{i-1}}(v)$. This along with (4.5) and (4.6) gives us $R_S \le R_{D_1\cup\cdots\cup D_{i-1}}(v) < R_{\text{algo},i}$.

– The other possibility is that $v \in D_1 \cup \cdots \cup D_{i-1}$. Since the D_is are disjoint, there is a unique j such that $v \in D_j$. Since $v \in L$, by Lemma 4.1, $R_{\text{algo},j} = R_{D_1\cup\cdots\cup D_{j-1}}(v)$. With the same argument as that for (4.6), we have $R_\emptyset(v) = R_{D_1\cup\cdots\cup D_{j-1}}(v)$. But since the algorithm never decreases rate from one iteration to the next, we have $R_{\text{algo},i} \ge R_{\text{algo},j}$. Putting these together we get $R_{\text{algo},i} \ge R_{\text{algo},j} = R_{D_1\cup\cdots\cup D_{j-1}}(v) = R_\emptyset(v)$. With (4.5) this gives us $R_S \le R_S(v) = R_\emptyset(v) \le R_{\text{algo},i}$.

Therefore, in every case, we have shown that $R_S \le R_{\text{algo},i}$. This implies that adding the rest of the nodes from D_i to S will not put additional constraints on R_S and, hence, cannot decrease the rate. Therefore, we have $R_{S\cup D_i} \ge R_S$. ■

Since S is optimal, this proves that $S \cup D_i$ also achieves optimal rate. We can now call this set S, and for the next value of i such that $D_i \not\subseteq S$, we can prove that $S \cup D_i$ has optimal rate. Continuing like this we have that $S \cup D$ is optimal, or, in other words, $R_{S\cup D} \ge R_S$.

Step 2: Next we wish to show that $S \subseteq D$, i.e., $S \cup D - D = \emptyset$. Let us assume the contrary. Let $T = S\cup D - D$. Therefore, $T \cap D = \emptyset$, but $T \subseteq S$. Thus, $D\cup S = D\cup T$, where D and T are disjoint. Consider $v \in T$ such that $\forall u (\ne v) \in T$, we either have $v \preceq u$, or v and u are incomparable. We have $R_{D\cup T}(v) = R_{D\cup S}(v)$. By Lemma 4.1, $R_{D\cup T}(v) = R_D(v)$. Also, the constraint of (4.2) tells us that $R_{D\cup S} \le R_{D\cup S}(v)$. Finally, note that since the algorithm terminates without adding v to D, we have $R_D > R_D(v)$. Putting these inequalities together we have $R_D > R_D(v) = R_{D\cup T}(v) = R_{D\cup S}(v) \ge R_{D\cup S}$. But this contradicts the fact that $S \cup D$ is optimal. Thus we have $S \subseteq D$, i.e., $S \cup D = D$.

From Steps 1 and 2 we have $R_D \geq R_S$. But since S was an optimal policy, D is also an optimal policy. This proves that the algorithm does indeed return an optimal set of decoding nodes.

The only case in which this proof does not go through is when the algorithm returns $D = \emptyset$ and $S \neq \emptyset$. In this case, consider node $v \in L \subseteq S$, where L is as defined earlier. Since the algorithm does not pick up v, we have $R_\emptyset > R_\emptyset(v)$. But, $R_S \leq R_S(v) = R_\emptyset(v)$ from (4.5). Thus, $R_S < R_\emptyset$; however, this contradicts the optimality of S. Therefore, if there exists an optimal, non-empty S, the algorithm cannot return an empty D. ■

Corollary 4.2. *The algorithm of Section 4.7 returns the largest optimal policy D.*

Proof. In the proof above, we have shown that for any optimal policy S, we have $S \subseteq D$. This implies that D is the largest optimal policy. ■

Remark 4.3. *In this chapter we consider a unicast problem. However, it is possible to easily generalize the algorithm of Section 4.7 for single source multicast problems where there is more than one destination. The only difference in the algorithm would be in the definition of R_D defined in (4.3) . For a multicast problem with destination nodes $d_1, \ldots, d_k$, R_D should be defined as $R_D = \min_{i \in D \cup \{d_1, \ldots, d_k\}} R_D(i)$.*

4.9 Examples

In this section we present some examples of networks and show how the algorithm runs on them.

4.9.1 Multistage Erasure Relay Networks

In Figure 4.4(a) we have depicted a multistage relay network. In this we have a single source and destination and k layers of relay nodes. The i-th layer consists of l_i nodes.

Between the i-th and the $(i+1)$-th layer we have a complete bipartite graph where all the edges are directed from the i-th layer to the $(i+1)$-th. We assume that each of these edges has erasure probability ϵ_i. The source is connected to all the nodes in the first layer by erasure channels with erasure probability ϵ_0, and all the nodes in the k-th layer are connected to the destination by erasure channels with erasure probability ϵ_k. We will also call d the $(k+1)$-th layer and $l_{k+1} = 1$.

Because of the structure of this network, finding the rate under a particular policy is easier than indicated in Section 4.6.2. Denote by $Q_{i,j}$ the probability that in layer i there are j nodes that do not see an erasure. This defines $Q_{i,j}$ for $i = 1, 2, \dots, (k+1)$, and $j = 0, 1, \dots, l_i$. With this, for $i = 1$ we obtain

$$Q_{1,k} = \binom{l_1}{k} \epsilon_0^{l_1-k} (1-\epsilon_0)^k. \tag{4.7}$$

For $i > 1$, we can show the recursion below:

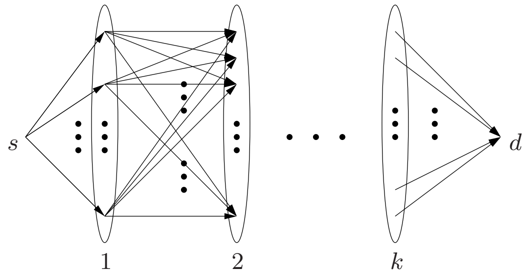
$$Q_{i,k} = \binom{l_i}{k} \sum_{t=0}^{l_{i-1}} \epsilon_{i-1}^{t(l_i-k)} (1-\epsilon_{i-1}^t)^k Q_{i-1,t}. \tag{4.8}$$

Denote by e_i the probability that the at least one node in the i-th layer does not see an erasure. We can show that

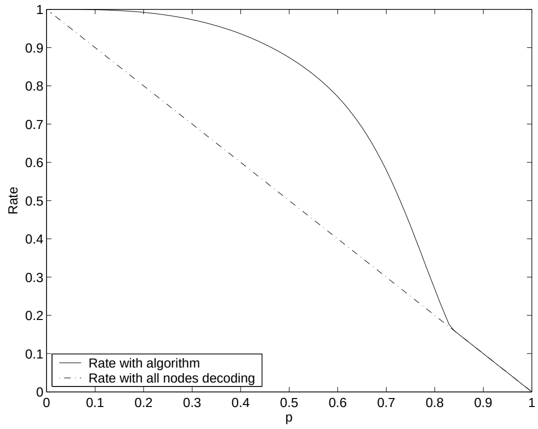
$$e_i = \sum_{k=0}^{l_i} Q_{i,k} \left(1 - \frac{k}{l_i}\right).$$

Note that, by symmetry, whenever a node decides to decode, all the nodes in that layer decode. When layer i decides to decode, we set $Q_{i,l_i} = 1$ and $Q_{i,j} = 0$ for $j \neq l_i$ and continue with the recursion of (4.8) for the other layers. This also extends to the case when more than one layer decodes.

Now, our algorithm proceeds as before, but operates on layers rather than nodes, and the effective erasure probability at layer i is e_i. As an explicit example, consider



(a) Model of a multistage relay network.



(b) Maximum rate as given by the algorithm.

Figure 4.4: Multistage relay network.

a multistage relay network with four layers between the source and destination. Let $l_1 = 3, l_2 = 6, l_3 = 4, l_4 = 5$ and $\epsilon_0 = p, \epsilon_1 = p^2, \epsilon_2 = p, \epsilon_3 = p^3, \epsilon_4 = p$, where p is any number in the interval $[0, 1]$. For a fixed value of p, we can find the optimum policy for the network, and this will give us the optimal rate. Figure 4.4(b) shows this optimal rate for the parameter p going from 0 to 1 (solid curve). This is not a smooth curve. The point where the right and left derivatives do not match is where either the optimum policy or the rate-determining layer changes. The rate with all nodes decoding has also been plotted (dashed curve). This rate is $1 - p$, and we see that the algorithm gives us dramatically higher rates.

4.9.2 Multistage Gaussian Relay Networks

We consider a multistage network similar to the one of the previous section, but in which the links represent Gaussian channels with fading coefficients h_i and with additive noise σ_i^2 at layer i. The indexing is identical to that in the erasure network.

Because of the structure of the network, it is easy to compute SNRs. Let $\rho(i)$ denote the SNR at layer i. Then, in the situation where all the nodes are forwarding, the following recursion gives us the SNR. We initialize the recursion as follows:

$$a(1) = h_0^2 P \quad b(1) = \sigma_1^2 \quad \rho(1) = \frac{a(1)}{b(1)}.$$

For the rest of the layers, i.e., $i \geq 2$ we have

$$\begin{aligned} a(i) &= a(i-1)\frac{h_i^2 l_i^2}{1+\frac{1}{\rho(i-1)}} \\ b(i) &= b(i-1)\frac{h_i^2 l_i}{1+\frac{1}{\rho(i-1)}} + \sigma_i^2 \\ \rho(i) &= \frac{a(i)}{b(i)} \end{aligned}$$

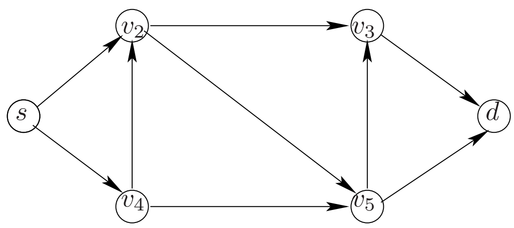
As with the erasure relay network, whenever a node decides to decode, all the nodes in that layer decode. If some layers decide to decode, a simple modification of the above recursion gives us the new rates. If i is the smallest number such that the i-th layer decodes, then, clearly, the above recursion gives us rates for layers l_1 to l_i. For l_{i+1}, we set $a(i+1) = h_i^2 l_i^2 P$ and $b(i+1) = \sigma_{i+1}^2$. We have $\rho(i+1) = a(i+1)/b(i+1)$ as before, and we can continue with the recursion above for layers $(i+2)$, etc. We repeat this modification for each layer that decodes.

Once the SNR at a layer is known, the rate is given by $\log(1+\rho)$ as usual. With this procedure for calculating rates, we use the algorithm of Section 4.7. It now operates on layers rather than nodes.

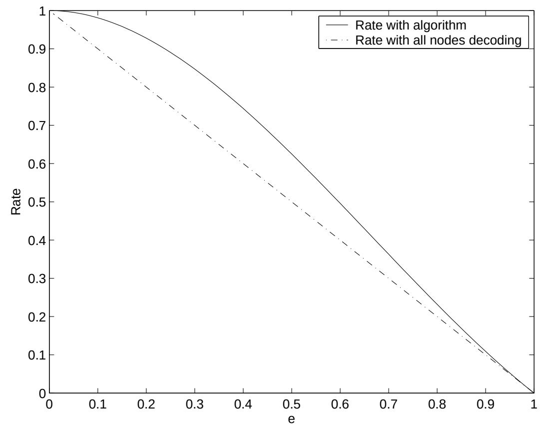
As an explicit example, consider a multistage relay network with three layers between the source and destination. Each node is restricted to using power $P = 1$. Let $l_1 = 2, l_2 = 5$, and $l_3 = 3$ and $h_0 = 0.7, h_1 = 10, h_2 = 0.1$, and $h_3 = 1$. We will have $\sigma_1^2 = m^2, \sigma_2^2 = m, \sigma_3^2 = m^3$, and $\sigma_4^2 = m^2$, where m can be any positive real number. For a fixed value of m, we can find the optimum policy for the network, and this will give us the optimal rate. Figure 4.4(b) shows this optimal rate for the parameter m going from 0.5 to 1.5 (solid curve). As with the multistage erasure network, the curve is not smooth at points where the optimum policy or the rate-determining layer changes. We also see the advantage compared to the case when all nodes decode (dashed curve).

4.9.3 Erasure Network with Four Relay Nodes

Consider the relay network of Figure 4.5(a). All the links have the same erasure probability p, where p is any number between 0 and 1. For this range of p, the algorithm has been used to find the optimum rates and policies. The rate is plotted in Figure 4.5(b) (solid curve). Throughout, the optimal policy is $D = \{2, 3, 5\}$. The rate with all nodes decoding is $1 - p$ and is also plotted (dashed curve). As expected, the algorithm outperforms the all-decoding scheme.



(a) Erasure network with four relay nodes.

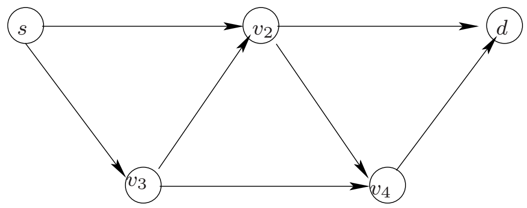


(b) Maximum rate as given by the algorithm.

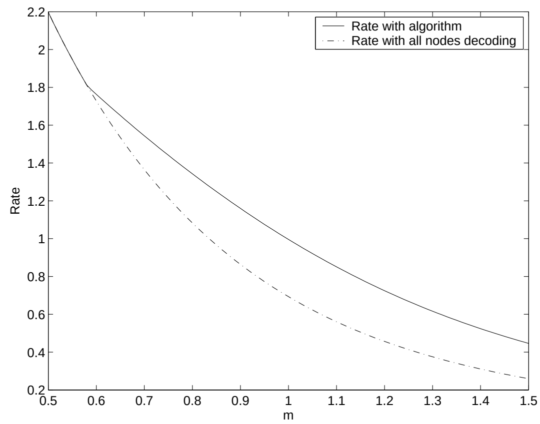
Figure 4.5: Erasure network with four relay nodes.

4.9.4 Gaussian Network with Three Relay Nodes

In Figure 4.6(a) we see a Gaussian network with three relay nodes. We assume that each node is restricted to use power $P = 1$. Let the additive noise variances be $\sigma_2^2 = m, \sigma_3^2 = m^3, \sigma_4^2 = m^2$, and $\sigma_5^2 = m^1$ where m can be an arbitrarily chosen real number. In Figure 4.6(b) we see the rate returned by the algorithm for the optimal



(a) Gaussian network with three relay nodes.



(b) Maximum rate as given by the algorithm.

Figure 4.6: Gaussian network with three relay nodes.

policy for $m \in [0.5, 1.5]$ (solid curve). The rate with all nodes decoding is also plotted (dashed curve). In the region $m \in [0.5, 0.58]$ we see that the optimal policy is in fact that of decoding at all nodes and the two curves match. After that, the optimal policy changes, and hence we see that the optimal rate curve is not smooth.

4.9.5 Gaussian Network with Four Relay Nodes

In Figure 4.7 we see a Gaussian network with four relay nodes. Each node, including the source, is restricted to using power $P = 1$. The attenuation factors associated with the edges are $h_{1,2} = 1, h_{1,4} = 2, h_{4,2} = 3, h_{2,3} = 4, h_{4,3} = 5, h_{4,5} = 1, h_{3,6} = 3, h_{5,3} = 2$, and $h_{5,6} = 4$. The additive noise variances associated with the nodes are $\sigma_2^2 = m, \sigma_3^2 = m^3, \sigma_4^2 = m^2, \sigma_5^2 = m$, and $\sigma_6^2 = m^3$, where m can be any positive real number. In Figure 4.10 we see the rate returned by the algorithm for the optimal policy for $m \in [1, 3]$ (solid curve). The rate with all nodes decoding is also plotted (dashed curve). We see that the forward/decode scheme gives us significant improvements in the rate.

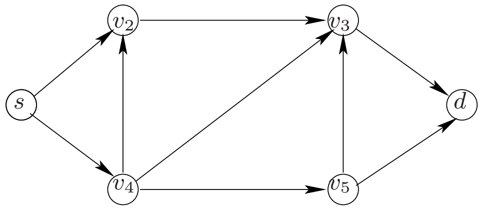


Figure 4.7: Gaussian network with four relay nodes.

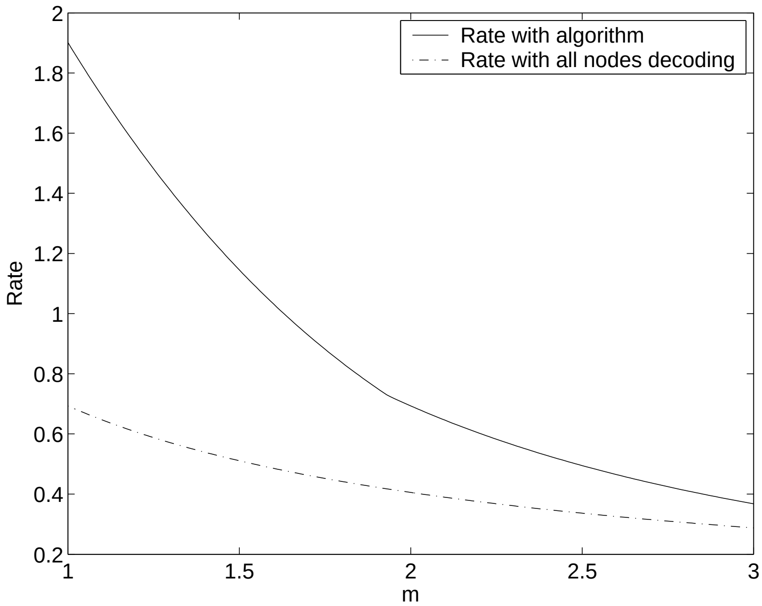


Figure 4.8: Rates for the Gaussian network of Figure 4.7.

4.10 A Distributed Algorithm for the Optimal Policy

The algorithm as proposed in Section 4.7 requires that the network parameters (viz., noise variances or erasure probabilities) be known before the network operation begins so that the optimum policy is known beforehand. With the algorithm in its current form the nodes cannot determine for themselves if they should decode or forward. In this section we propose a scheme that can permit nodes to determine their own operation.

The algorithm works iteratively to converge to a rate. In each iteration, the rate of operation of the network is incremented or decremented depending on whether the previous transmission was successful or not. In every iteration, all the nodes get to decide their operation for themselves.

Let R^* be the maximum rate of the network. This is not known beforehand. We assume that parameters R, δ, and N are known to all the nodes beforehand. The blocklength n is also predetermined and known to all the nodes. In addition, we require that the nodes have a common source of randomness so that they can generate the *same* random codebook individually. With this, consider the following algorithm:

(a) All nodes generate the (same) codebook for rate R. They all set $k = 0$.

(b) s transmits a randomly chosen codeword X_1.

(c) Every relay node i attempts to decode the received message Y_i.
If it can decode without error, it transmits the decoded codeword. [3]
Otherwise, it forwards the received message (with appropriate scaling, for the Gaussian network).

(d) The destination attempts to decode the received message.
If it decodes without error, it sends back bit 1 to all the other nodes to indicate successful decoding.
Otherwise, it sends back bit 0 to all other nodes.

(e) All nodes increment k. $k = k + 1$.
If transmitted bit was zero, all nodes set $R = R - \delta/2^k$.
If transmitted bit was one, all nodes set $R = R + \delta/2^k$.

(f) While $k \leq N$, go to Step 1.

Theorem 4.3. *If the maximum rate of the network, viz. R^* is in the range $[R - \delta, R + \delta]$, the algorithm above converges to it with an accuracy of $\frac{\delta}{2^N}$.*

[3]One method of error detection is for a node to perform typical set decoding and assume an error if it finds more than one codeword that is jointly typical with the received message. Other methods of error detection are the introduction of Cyclic Redundancy Checks (CRCs) or an ARQ protocol, e.g., [63].

Proof. The source starts by transmitting at rate R. Each relay node receives messages on all incoming links and decodes the message if it can. If it cannot, it simply forwards what it has received. With this procedure, nodes decide their own operation. (The order in which they decide this is a partial order in the sense defined in Section 2.2.2.) After the destination receives all its incoming messages, it tries to decode. If $R > R^*$, the destination will definitely not be able to decode. If $R \leq R^*$, we claim that the destination will be able to decode. This is because when a node decodes, it only improves the rates for other nodes. Also, note that an arbitrary node v decides whether to decode or not only after all the nodes before it in the partial order have already determined if the rate they can support is greater or smaller than R. Since, by Lemma 4.1, these are the only nodes that affect the rate for v and they decode whenever they can, node v always gets to see the best situation it can as far as rate R is concerned. This is true for the destination also.

Therefore, depending on whether the destination can decode or not, we can say if R^* is greater or smaller than R. If this bit of information is transmitted back to the source and other nodes, they can accordingly decide whether to increase or decrease the rate for the next transmission. Thus, we have a decision tree of rates such that the ability or inability of the decoder tells us which path to traverse in that tree we can finally converge on a rate sufficiently close to the actual rate R. ■

This algorithm provides a very natural mode of network operation that obviates the need for a central agent to know the entire network and decide the optimum policy. Although some communication from the destination to the source and other nodes is required, this is minimal and should be easily possible in a practical network setting.

We mention that the algorithm we present can be made more sophisticated such that it works for all values of R^*, rather than just those in the interval $[R-\delta, R+\delta]$. We omit the details in the interests of brevity.

4.11 Upperbounds On the Maximum Rate

The algorithms of Section 4.7 as well as Section 4.10 converge to the maximum rate possible with the decode/forward scheme, but we have no way of simply looking at the network and saying what this maximum rate will be. In this section we present upperbounds on the rate achievable with the limited operations that we use in this chapter.

Before we derive upperbounds, let us remind ourselves of the notion of a cut-set defined in Section 2.2.2. An $s - d$ *cut* is defined as a partition of the vertex set $\mathcal{V}$ into two subsets, $\mathcal{V}_s$ and $\mathcal{V}_d = \mathcal{V} - \mathcal{V}_s$, such that $s \in \mathcal{V}_s$ and $d \in \mathcal{V}_d$. The cutset $[\mathcal{V}_s, \mathcal{V}_d]$ is the set of edges passing across the cut. Also, $\mathcal{V}_s^*$ is the subset of nodes in the source set that have an edge in the cutset. Similarly, $\mathcal{V}_d^*$ is the subset of nodes in the destination set that have an edge in the cutset.

4.11.1 Upperbound for Gaussian Networks

For Gaussian networks, it is evident that making the additive noise zero at certain nodes can only increase the maximum rate available at d. In particular, let us make the additive noise zero at all nodes except $\mathcal{V}_d^*$. Therefore, the received messages (and the transmitted messages) at all nodes in $\mathcal{V}_s$ are exactly the same as that transmitted by the source. Now, if we permit the nodes in the destination set to decode co-operatively, the rate at which they can decode will give us an upperbound on the rate that the destination can decode.

Note that the SNR at node $v_j \in \mathcal{V}_d^*$ is

$$\frac{P}{\sigma_j^2}\left(\sum_{i:(i,j)\in[\mathcal{V}_s,\mathcal{V}_d]} h_{i,j}\right)^2.$$

Since our codebook and noise are Gaussian distributed, the optimum scheme for decoding co-operatively is taking a suitable linear combination of received messages and then decoding that. For optimal decoding, we find the linear combination that gives us the best SNR. It is easy to show that the best SNR possible is the sum of the SNRs seen by each node in $\mathcal{V}_d^*$.

Therefore, an upperbound on the rate is

$$R \leq \log\left(1 + \sum_{j \in Y(\mathcal{V}_s)} \frac{P}{\sigma_j^2}\left(\sum_{i:(i,j)\in[\mathcal{V}_s,\mathcal{V}_d])} h_{i,j}\right)^2\right)$$

for every cut $\mathcal{V}_s$.

4.11.2 Upperbound for Erasure Networks

As in the above section, we can obtain an upperbound on the rate for erasure networks by making certain links perfect, or free of erasures. Therefore, we can obtain an upperbound on the rate by making all edges other than those in $[\mathcal{V}_s, \mathcal{V}_d]$ perfect. With this, all the received (and transmitted) messages in $\mathcal{V}_s$ are exactly the same as the codeword transmitted by the source. Now, it is clear that the rate at which the nodes in $\mathcal{V}_d^*$ can decode co-operatively is an upperbound on the rate available at the destination.

Clearly, the effective erasure probability seen by the set of nodes $\mathcal{V}_s^*$ is $\prod_{(i,j)\in[\mathcal{V}_s,\mathcal{V}_d]} \epsilon_{ij}$. This gives us an upperbound on the rate. We have

$$R \leq 1 - \prod_{(i,j)\in[\mathcal{V}_s,\mathcal{V}_d]} \epsilon_{ij}$$

for every cut $\mathcal{V}_s$.

Note that in Chapter 2 a different min-cut upperbound was proposed and was shown to be achievable. This gives the capacity of the network under the assumption that the destination has perfect side-information regarding erasure locations from across the network. This is very different from the setup of this chapter.

4.12 Conclusions

To summarize, we have shown that making each link error-free in a wireless network is sub-optimal. Thus, a multihop approach, in which every relay node decodes the received message, is not necessarily the correct approach for all wireless networks. We have proposed a scheme for network operation that is of use in practical networks and in which operations performed by a node are restricted to decoding and forwarding – both of which are common operations performed in a network setting. We have suggested an algorithm that finds the optimum policy without exhaustive search over an exponential number of policies and also proposed a method to converge to the correct policy without having a central decision-making agent.

The algorithm of Section 4.7 can find the maximum rate and optimum policy for any Gaussian or wireless erasure network. In addition, the bounds presented in Section 4.11 give us some idea of what sort of optimal rates to expect. However, we still do not know what sort of policies are optimal in what ranges of erasure probabilities or SNR. The examples of Section 4.3 suggest that when the links are poor (high erasure probabilities or low SNR), it is better to decode. It would be interesting to know if this is true for general networks and what thresholds exist below which a certain operation is always preferred.

Also, Corollary 4.2 tells us that the algorithm returns the largest decoding set. Since decoding is the more costly of the two operations considered here, an algorithm that finds the smallest decoding set such that the maximum rate is obtained is of interest.

Finally, we note that in this chapter we considered only two types of operations. However, it is possible to imagine a larger set of operations and the optimal choice of operation from among these. Finding practical schemes that improve upon the present algorithm is an interesting avenue for future work.

Chapter 5

Estimation over Wireless Erasure Networks

5.1 Introduction

Recent advances in Micro-Electro-Mechanical Systems (MEMS) technology have provided us with cheap, low power, customizable sensors capable of sensing, signal processing, and communication in wireless media (for university and industrial prototype of these sensors, see [64, 65]). These advances have given rise to an increasing number of applications for networks of sensors in different aspects of our life. As mentioned earlier in Chapter 1, examples of these applications appear in environmental monitoring, industrial, transportation, and home systems automation, control of distributed embedded systems (such as robots or UAVs), and even medical services [66, 67].

One important feature of these applications is that not in all of them the main objective is high-data *rate* communication between components of the network. Different tasks such as distributed computation, detection, and control can be the main purpose for deploying these networks.

Given the increasing use of wireless sensor networks for different tasks other than data communication, a theoretical framework for analysis of the ultimate performance and the optimal schemes of operation for each of these tasks is required.

Towards this end, recently a great deal of attention has been directed towards networked control systems in which components communicate over wireless links or communication networks that may also be used for transmitting other unrelated data (see, e.g., [68, 69] and the references therein). The estimation and control performance in such systems is severely affected by the properties of the communication channels. Communication links introduce many potentially detrimental phenomena, such as quantization error, random delays, data loss, and data corruption to name a few, that lead to performance degradation or even stability loss. As emergent applications in distributed control mature, these issues have gained a lot of focus from the community.

In the previous chapters of this thesis, we looked at the performance of different classes of wireless networks for different network problems. In these problems the main objective is maximizing the reliable rate of *communication* between the nodes of the network. In this chapter, we look at another task, namely *control and estimation*, over these networks. We are interested in the problem of estimation and control of a dynamical process across the wireless erasure network model introduced in Section 2.3. We consider dynamical process evolving in time that is being observed by a sensor. The sensor needs to transmit the data over a network to a sink (destination) node, which can either be an estimator or a controller. However, the links in the network stochastically erase packets.

Prior work in this area has focused on studying the effect of packet erasures by a single link in an estimation or control problem. Assuming certain statistical models for the packet erasure process, stability and control performance of such systems were analyzed in [70]-[73]. To counteract the degradation in performance, some approaches have been proposed in the literature [74]-[79]. In particular, in [76], a sub-optimal estimator and regulator is proposed that minimizes a quadratic cost. This approach was later extended by [77, 78]. [79] also considered the related problem of optimal estimation across a packet erasure link that erases packets in an independent and identically distributed (i.i.d.) fashion, and obtained bounds on the expected error covariance.

Most of the above designs aimed at designing a packet-loss compensator. The compensator accepts those packets that the link successfully transmits and comes up with an estimate for the time steps when data is lost. If the estimator is used inside a control loop, the estimate is then used by the controller. A more general approach is to design both an encoder and a decoder for the communication link to counteract the effect of stochastic packet erasures. This was considered for the case of a single communication link in [80] and [81]. It was demonstrated that using encoders and decoders can improve both the stability margin and the performance of the system.

In this chapter, we consider the design of encoders and decoders for wireless erasure network model introduced in Chapter 2. The optimal transmission strategy over general networks for the purpose of estimation and control is largely an open problem. In [82], Tatikonda studied some issues related to quantization rates required for stability when data was being transmitted over a network of digital memoryless channels. Also relevant is the work of Robinson and Kumar [83], who considered the problem of optimal placement of the controller when the sensor and the actuator are connected via a series of communication links. They ignore the issue of delays over paths of different lengths (consisting of different number of links), and under a *Long Packet Assumption* come up with the optimal controller structure. There are two main reasons why the problem of encoding data for transmission is much more complicated in the case of transmission over a network:

- If the intermediate nodes are allowed to process data, it introduces an element of memory. The network is not equivalent to an erasure channel with probability of successful transmission as the reliability of the network.

- There are potentially many paths from the source to any node that offer data with varying amounts of delay.

We begin by proving a separation principle that allows us to separate the control problem into one of designing a state-feedback optimal controller, and another of transmitting information across unreliable links. This also allows us to identify the information that needs to be made available to the controller for achieving optimal performance. We then propose a simple recursive algorithm that ensures that this information is available to the controller. Even though the algorithm requires a constant amount of memory, transmission, and processing at any node, it is optimal for *any* packet erasure pattern and has many additional desirable properties that we illustrate. The analysis of the algorithm identifies a property of the network called the max-cut probability that is relevant for the purpose of stability of the control loop. We also provide a framework to analyze the performance of our algorithm. The main contributions of this chapter are as follows:

(a) We identify the optimal information processing strategy that should be followed by the nodes of the network to allow the sink to calculate the optimal estimate at every time step. This algorithm is optimal for *any* packet erasure process, yet requires a constant amount of memory, processing, and transmission by any node per time step. Due to a separation principle, the algorithm also solves the optimal control problem.

(b) We analyze the stability of the expected error covariance for this strategy when the packet erasure events are independent from one time step to the next and across channels. For any other scheme (e.g., transmitting measurements without any processing), our conditions are necessary for stability. For channels with correlated erasures, we show how to extend this analysis.

(c) We calculate the performance for our algorithm for channels that erase packets independently. We provide a mathematical framework for evaluating the performance for a general network and provide expressions for networks containing links in series and parallel. We also provide lower and upper bounds for the performance over general networks. For any other strategy, these provide lower bounds for achievable performance.

Our results can also be used for *synthesis* of networks to improve estimation performance. We consider a simple example in which the optimal number of relay nodes to be placed is identified for estimation performance. We also consider optimal routing of data in unicast networks.[1] Simulation results are provided to illustrate the results.

The chapter is organized as follows. In the next section, we set up the problem and state the various assumptions. Then, we state a separation principle that allows us to focus on the optimal estimation problem. In Section 5.4 we identify a recursive yet optimal processing and transmission algorithm. We then specialize to the case of packet erasure events occurring in a memoryless fashion and independently across different links. We first do a stability analysis of the algorithm in Section 5.5 to obtain conditions on the packet erasure probabilities under which the estimate error at the sink retains a bounded covariance. Following that, in Section 5.6 we analyze the performance of the algorithm. We derive an expression for general networks and evaluate it explicitly for specific classes of networks. We also provide bounds for general networks. We then illustrate the results using some examples. Finally, we consider some extensions of the analysis by considering correlated erasures and using the results already derived for optimal routing in unicast networks and for network synthesis.

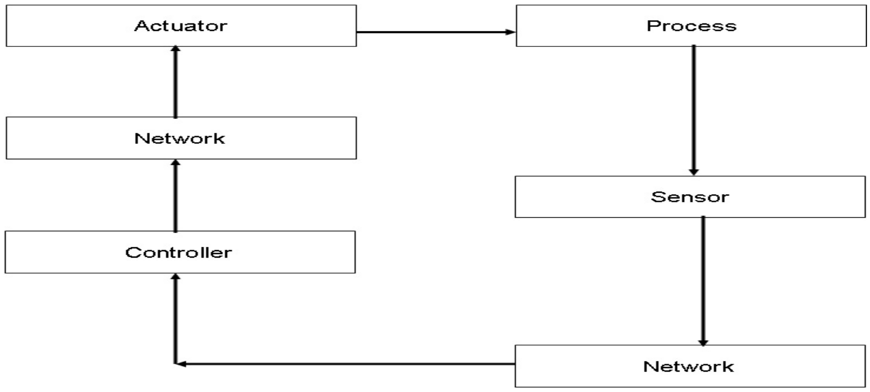


Figure 5.1: The set-up of the control across communication networks problem. For most of the discussion in the chapter, we will ignore the network between the controller and the actuator. See, however, Section 5.4.2.

5.2 Problem Setup

Consider the arrangement in Figure 5.1. Let a discrete-time linear process evolve according to the equation

$$x(k+1) = Ax(k) + Bu(k) + w(k), \tag{5.1}$$

where $x(k) \in \mathbf{R}^n$ is the process state, $u(k) \in \mathbf{R}^m$ is the control input, and $w(k)$ is the process noise assumed to be white, Gaussian, and zero mean with covariance matrix R_w.[2] The initial condition $x(0)$ is assumed to be independent of $w(k)$ and to have mean zero and covariance matrix $R(0)$. The state of the plant is measured by a sensor that generates measurements according to the equation

$$y(k) = Cx(k) + v(k). \tag{5.2}$$

The measurement noise $v(k)$ is white, zero-mean, Gaussian (with covariance matrix R_v), and independent of the plant noise $w(k)$. We assume that the pairs (A, B) and

[1] Unicast network in this chapter refers to routing the data over *only* one particular path in the network.

[2] The results we present continue to hold for time-varying systems, but we consider the time-invariant case to simplify the presentation.

$\{A, R_w^{\frac{1}{2}}\}$ are stabilizable and the pair (A, C) is observable.

The sensor communicates with a controller across a wireless erasure network that was introduced in Chapter 2. The sensor constitutes the source node and is denoted by s. The controller is designated as the sink, or the destination node d. As in Chapter 2, the network is represented by a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. The edges of the graph represents packet erasure channels. We do not make any assumptions a priori regarding the erasure process across the network. The following remarks regarding the model are in order:

- We assume that the links take in as input a finite vector of real numbers. Therefore, the assumption is that sufficient bits per data packet and a high enough data rate are present so that quantization error is negligible.

- We will nominally consider the delays introduced by the channel to be less than one time step according to which the discrete-time dynamical process evolves. Most of the results in the chapter can, however, be extended to the case when delays are present. In particular, the algorithm in the case of delays is provided in Section 5.4.1.

- In this chapter we also assume a global clock so that each node is synchronized.

If the packet erasure process is independent for different links and also from one time step to the next (or, in other words, memoryless), the probability of dropping a packet on link $(i, j) \in \mathcal{E}$ is given by ϵ_{ij} independent of time. More sophisticated models of erasure process such as erasures occurring according to a Markov chain to capture the *bursty* nature of them can readily be thought of and will be considered towards the end of this chapter.

We refer to individual realizations of the erasure process as a packet drop (erasure) sequence. The operation of the different nodes in the network at every time-step k can be described as follows:

(a) Every node computes a function of all the information it has access to at that time.

(b) It transmits the function on all the out-going edges. We allow some additional information in the message that tells us the time step j such that the function that the node transmits corresponds to the state $x(j)$. The sink node calculates the control input $u(k)$ based on the information it possesses.

(c) Every node observes the messages from all the incoming links and updates its information set for the next time step. For the source node, the message it receives at time step k corresponds to the observation $y(k)$.

Note that the time line we have proposed ensures a strictly causal operation. Thus, at time step k, the function that the source node transmits depends on measurements $y(0)$, $y(1)$, $\cdots$, $y(k-1)$. Similarly, even if there was no packet erasure, if the sink node is d hops away from the source node (i.e., the shortest path from the source node to the sink node involves d edges), its control input $u(k)$ at time k can only depend on measurements $y(0)$, $y(1)$, $\cdots$, $y(k-d-1)$. Thus, unlike the model in [83], every communication edge consumes one hop, or in other words, one time step, as data is transmitted over it. We can easily adapt the discussion presented below to the causal case.

The controller at every time step calculates a control input $u(k)$ and transmits it to the actuator. For the time being, we ignore any communication channel between the controller and the actuator. We will revisit the presence of a controller - actuator channel later in the chapter and show how simple modifications to our design can take care of them. The controller aims at minimizing a quadratic cost function

$$J_T = E\left[\sum_{k=0}^{T}\left(x^T(k)Qx(k) + u^T(k)Ru(k)\right) + x^T(T+1)P^c_{T+1}x(T+1)\right], \qquad (5.3)$$

where the expectation is taken over the uncorrelated variables $x(0)$, $\{w(k)\}$ and $\{v(k)\}$. Note that the cost functional J_T above also depends on the random packet drop sequences in each link. However, we do *not* average across packet-drop processes; *the solution we will present is optimal for arbitrary realizations of the packet dropping processes.* For considering the stability of the system, we will consider the infinite horizon cost

$$J_\infty = \lim_{T\to\infty} \frac{J_T}{T}.$$

Without the communication network, the problem is thus the same as the classical LQG control synthesis problem. The presence of the network, however, alters the problem drastically. It is unclear *a priori* what the structure of the optimal control algorithm should be and in what way the links in the network should be used to transmit information. We aim to solve the following problems:

(a) Identify the optimal processing and transmission algorithm at the nodes that allow the controller to minimize the cost J_T. Clearly, sending measurements alone might not be the optimal thing to do, since in such a scheme, erasing a packet would mean loss of information that cannot be compensated for in the future. We are particularly interested in strategies that do not entail an increasing amount of transmission and memory at the nodes.

(b) Identify the conditions on the network that would lead to a stable system.

(c) Identify the best possible performance of the system in terms of the quadratic cost that can be achieved.

For future reference, we will denote this problem set-up as problem $\mathcal{P}_1$.

5.3 Preliminary Results

We wish to construct the optimal control input and the optimal information processing algorithm at each node that minimizes the cost function J_T. If the packet dropping links were not present, the optimal performance would have been achieved if every node in the network transmitted the latest measurement it received and the controller calculated the LQ optimal control input based on the estimate obtained by using a Kalman filter. However, the presence of packet erasures disrupts this operation since the Kalman filter requires continuous access to measurements, which is denied by the packet erasures. To solve for the optimal controller design and the optimal information processing algorithm at each node, we begin by introducing some notation.

For the node i, denote by $\mathcal{I}^i(k)$ the information set that it can use to generate the message that it transmits at time step k. This set contains the aggregate of the information the node has received on the incoming edges at time steps $t = 0, 1, \cdots,$ $k-1$. As an example, for the source node s,

$$\mathcal{I}^s(k) = \{y(0), y(1), \cdots, y(k-1)\}.$$

Without loss of generality, we can restrict our attention to information-set feedback controllers, i.e., controllers of the form $u(k) = u(\mathcal{I}^d(k), k)$. For a given information set at the destination $\mathcal{I}^d(.)$, let us denote the minimal value of J_T by $J_T^\star(\mathcal{I}^d)$. The packet drops occur according to a random process. Let $\lambda_{pq}(k)$ be the binary random variable describing the packet erasure event on link $(p, q) \in \mathcal{E}$ at time k. $\lambda_{pq}(k)$ assumes the value "*dropped*" if the packet is dropped on link (p, q) at time k, and "*received*" otherwise. For a network with independent and memoryless packet erasures, $\lambda_{pq}(k)$ is distributed according to Bernoulli with parameter p_{pq}. We define $\lambda_{pp}(k) =$ "*received*". Given the packet drop sequences in each link, at time step k we can define a time stamp $t^i(k)$ for node i such that the packet erasures did not allow any information transmitted by the source after $t^i(k)$ to reach the i-th node in time for it to be a part of $\mathcal{I}^i(k)$.

Now consider an algorithm $\mathcal{A}_1$ that proceeds as follows. At time step k, every node takes the following actions:

(a) Calculate the estimate of state $x(k)$ based on the information set at the node.

(b) Transmit its entire information set on the outgoing edges.

(c) Receive any data successfully transmitted along the incoming edges.

(d) Update its information set and affix a time stamp corresponding to the time of the latest measurement in it.

When this algorithm is executed for a particular drop sequence, the information set at node i will be of the form

$$\mathcal{I}^i(k) = \{y(0), y(1), \cdots, y\left(t^i(k)\right)\},$$

where $t^i(k) < k$ is the time stamp as defined above. This is the maximal information set $\mathcal{I}^{i,\max}(k)$ that the node i can possibly have access to with any algorithm. For any other algorithm, the information set will be smaller than this since earlier packets, and hence measurements, might have been dropped.

Note that for two information sets $\mathcal{I}^d(k,1)$ and $\mathcal{I}^d(k,2)$ related by $\mathcal{I}^d(k,1) \subseteq \mathcal{I}^d(k,2)$, we have $J_T^\star(\mathcal{I}^d(k,1)) \leq J_T^\star(\mathcal{I}^d(k,2))$. Thus, in particular, one way to achieve the optimal value of J_T is through the combination of an information processing algorithm that makes the information set $\mathcal{I}^{d,\max}(k)$ available to the controller and a controller that optimally utilizes the information set. Further, one such information processing algorithm is the algorithm $\mathcal{A}_1$ described above. However, this algorithm requires increasing data transmission as time evolves. Surprisingly, in a lot of cases, we can achieve performance equivalent to this naïve solution using a constant amount of transmission and memory.

To see this, we first state the following separation principle. For any random variable $\alpha(k)$, denote by $\hat{\alpha}\left(k|\beta(k)\right)$ the minimum mean squared error (mmse) estimate of $\alpha(k)$ given the information $\beta(k)$.

Proposition 5.1. *[Separation Principle] Consider the packet-based optimal control problem $\mathcal{P}_1$ defined in section 5.2. Suppose that each node transmits all the measurements it has access to at every time step, so that the decoder has access to the maximal information set $\mathcal{I}^{d,\max}(k)$ at every time step k. Then, for an optimizing choice of the control, the control and estimation costs decouple. Specifically, the optimal control input at time k is calculated by using the relation*

$$u(k) = \hat{u}_{LQ}\left(k|\mathcal{I}^{d,\max}(k), \{u(t)\}_{t=0}^{k-1}\right),$$

where $u_{LQ}(k)$ is the optimal LQ control law and $\hat{u}_{LQ}\left(k|\mathcal{I}^{d,\max}(k), \{u(t)\}_{t=0}^{k-1}\right)$ denotes the minimum mean squared error (mmse) estimate of $u_{LQ}(k)$ given the information set $\mathcal{I}^{d,\max}(k)$ and the previous control inputs $u(0)$, $\cdots$, $u(k-1)$.

Proof. The proof is along the lines of the standard separation principle (see, e.g., [84, Chapter 9]; see also [77, 80]) and is omitted for space constraints. ∎

Informally, the separation principle states that if every node transmits all previous measurements at every time step, then the controller design consists of an estimator (e.g., a Kalman filter) that calculates the minimum mean squared error estimate of the current state value using all available measurements and the previous control inputs together with an LQ optimal controller. There are two reasons this principle is useful to us:

(a) We recognize that the optimal controller does not need to have access to the information set $\mathcal{I}^{d,\max}(k)$ at every time step k. The encoders and the decoder only need to ensure that the controller receives the quantity $\hat{u}_{LQ}\left(k|\mathcal{I}^{d,\max}(k), \{u(t)\}_{t=0}^{k-1}\right)$, or equivalently, $\hat{x}\left(k|\mathcal{I}^{d,\max}(k), \{u(t)\}_{t=0}^{k-1}\right)$.

(b) If we can ensure that the controller has access to this quantity, the controller design part of the problem is solved. The optimal controller is the solution to the LQ control problem.

Thus, the controller needs access to the estimate $\hat{x}\left(k|\mathcal{I}^{d,\max}(k), \{u(t)\}_{t=0}^{k-1}\right)$ based on the information set $\mathcal{I}^{d,\max}(k)$ (that is of the form $\{y(0), y(1), \cdots, y(j)\}$ for some $j < k$) and the previous control inputs. We can make another simplification in the problem by separating the dependence of the estimate on measurements from the effect of the control inputs. In the context of our problem, this is useful since the nodes in the network do not then need access to the control inputs and can concentrate solely on the effect of measurements. The effect of the control inputs can be taken care of by the decoder or the controller that has access to all previous control inputs. To this end, we state the following Theorem.

Theorem 5.1. *Consider the problem $\mathcal{P}_1$ defined in section 5.2. The quantity $\hat{x}\left(k|\mathcal{I}^{d,\max}(k), \{u(t)\}_{t=0}^{k-1}\right)$, where $\mathcal{I}^{d,\max}(k)$ is of the form $\{y(0), y(1), \cdots, y(j)\}$ for some $j < k$, can be calculated as the sum of two quantities:*

$$\hat{x}\left(k|\mathcal{I}^{d,\max}(k), \{u(t)\}_{t=0}^{k-1}\right) = \bar{x}(k) + \psi(k),$$

where $\bar{x}(k)$ depends only on $\mathcal{I}^{d,\max}(k)$ and should be provided to the controller by the network and $\psi(k)$ depends only on the control inputs and can be computed by a recursive linear filter at the controller. Further, term $\bar{x}(k)$ that the network needs to deliver is, in fact, the mmse estimate of the state $x(k)$ of a process evolving as

$$x(k+1) = Ax(k) + w(k), \tag{5.4}$$

given the measurements $y(0)$, $y(1)$, $\cdots$, $y(j)$ that are assumed to originate from a sensor of the form (5.2).

Proof. The proof of the above theorem is provided in [90]. ∎

As mentioned above, the advantage of separating the effects of measurements and the control inputs is that the nodes in the network can concentrate on delivering $\bar{x}(k)$ to the controller, the controller (which has access to all the control inputs) can then calculate $\psi(k)$ and, in turn, $\hat{x}\left(k|\{y(t)\}_{t=0}^{j}, \{u(t)\}_{t=0}^{k-1}\right)$. The nodes in the network do not need access to the control inputs.

Thus, consider an alternative estimation problem $\mathcal{P}_2$. A process of the form (5.4) is observed by a sensor of the form (5.2). There is an estimator across the network that needs to estimate the state $x(k)$ of the process in the mmse sense at every time step k. The network is modeled in the same manner as in the original problem $\mathcal{P}_1$. We can once again define the information set $\mathcal{I}^i(k)$ that the node i has access to at time k and the corresponding maximal information set $\mathcal{I}^{i,\max}(k)$. What is the optimal information processing algorithm to be followed by each node that allows the estimator to calculate the estimate of $x(k)$ based on the information set $\mathcal{I}^{d,\max}(k)$? By the arguments above, the optimal information processing algorithm for the nodes in the network in the problems $\mathcal{P}_1$ and $\mathcal{P}_2$ is identical. For the presentation of the algorithm and analysis of its properties, we will consider this equivalent problem $\mathcal{P}_2$ while keeping in mind that, to solve problem $\mathcal{P}_1$, the controller can then calculate $\psi(k)$ to include the effect of the previous control inputs and, finally, the new control input $u(k)$ by utilizing the separation principle. We now move on to present the algorithm.

5.4 Optimal Encoding at Each Node

We saw that one algorithm that ensures that the estimator (or the sink node) has access to $\mathcal{I}^{d,\max}(k)$ and hence can calculate the mmse estimate of $x(k)$ is $\mathcal{A}_1$. However, that algorithm involves an increasing amount of memory and transmission at each node. We will now describe an algorithm $\mathcal{A}_2$ that achieves the same performance at the expense of constant memory, processing, and transmission (modulo the transmission of the time stamp). The algorithm proceeds as follows. At each time step k, every node i takes the following actions:

(a) Calculate its estimate $\hat{x}^i(k)$ of the state $x(k)$ based on any data received at the previous time step $k-1$ and its previous estimate. The estimate can be computed using a switched linear filter, as shown later.

(b) Affix a time stamp corresponding to the latest measurement used in the calculation of the estimate in step 1, and transmit the estimate on the outgoing edges.

(c) Receive data on the incoming edges, if any, and store it for the next time step.

To prove that algorithm $\mathcal{A}_2$ is indeed optimal, we need the following intermediate result.

Lemma 5.2. *Consider any edge (i,j) and any packet drop pattern. At time step k, let the node i transmit the measurement set*

$$S^{ij} = \{y(0), y(1), \cdots, y(l)\}$$

on the edge (i,j) if algorithm $\mathcal{A}_1$ is executed. If, instead, algorithm $\mathcal{A}_2$ is executed, the node i transmits the estimate

$$\hat{x}\left(k|S^{ij}\right) = \hat{x}(k|\{y(0), y(1), \cdots, y(l)\})$$

along the edge (i, j) at time step k.

Proof. The proof readily follows by induction on the time step k. For time $k = 1$, the source node s transmits $\{y(0)\}$ along all edges of the form $(s, .)$ while following algorithm $\mathcal{A}_1$ and the estimate $\hat{x}(1|y(0))$ while executing algorithm $\mathcal{A}_2$. If any edge is not of the form $(s, .)$, there is no information transmitted along that edge in either algorithm. Thus, the statement is true for $k = 1$. Now assume that the statement is true for $k = n$. Consider the node i at time $k = n + 1$. If the node i is the source node, the statement is true by an argument similar to that at $k = 1$. Let us assume that node i is not the source node. Consider all edges that transmitted data at time step $k = n$ to node i. Let each of the edges $(j, i) \in \mathcal{N}_I(i)$ transmits the measurement set

$$S^{ji} = \{y(0), y(1), \cdots, y(t(j))\}$$

if algorithm $\mathcal{A}_1$ is being executed. Also, denote the measurement set that the node i has access to from time step $k = n - 1$ as

$$S^{ii} = \{y(0), y(1), \cdots, y(t(i))\}.$$

Note that at time step $k = n$, the node i transmitted the set S^{ii} along all outgoing edges in $\mathcal{N}_O(i)$. Let v be the node for which

$$t(v) = \max\{t(i) \cup \{t(j)|(j, i) \in \mathcal{N}_I(i)\}\}.$$

Then at time $k = n+1$, the node i transmits along all outgoing edges the measurement set

$$\mathcal{S}^1 = \{y(0), y(1), \cdots, y(t(v))\}.$$

Now consider the case when algorithm $\mathcal{A}_2$ is being executed. By the assumption of the statement being true at time step $k = n$, the edges (j, i) transmit the estimate

$$\hat{x}(n|S^{ji}) = \hat{x}(n|\{y(0), y(1), \cdots, y(t(j))\})$$

for all node js such that $(j, i) \in \mathcal{N}_I(i)$. Also, since at time $k = n$ the node transmitted S^{ii} on any edge $(i, .)$ in algorithm $\mathcal{A}_1$, it has access to the estimate $\hat{x}(n|S^{ii})$ when algorithm $\mathcal{A}_2$ is executed. Clearly, the set S^{vi} is the superset of all sets S^{ii} and S^{ji} where $(j, i) \in \mathcal{N}_I(i)$ and v have been defined above. Thus, the estimate that the node i calculates at time $k = n + 1$ is $\hat{x}(n + 1|S^{vi})$. But the measurement set S^{vi} is simply the set $\mathcal{S}^1$. Hence, at time step $k = n + 1$, the node i transmits along all outgoing edges the estimate $\hat{x}(n + 1|\mathcal{S}^1)$. Thus, the statement is true at time step $k = n + 1$ along all edges of the form $(i, .)$. Since the node i was arbitrary, the statement is true for all edges in the graph. Thus, we have proven that if the statement is true at time $k = n$, it is true at time $k = n + 1$. But it is also true at time $k = 1$. Thus, by the principle of mathematical induction, it is true at all time steps. ∎

Note that we have also shown that if at time step k, the node has access to the measurement set S^{ii} from time step $k-1$ when algorithm $\mathcal{A}_1$ is executed, it has access to the estimate $\hat{x}(k-1|S^{ii})$ from time step $k-1$ when algorithm $\mathcal{A}_2$ is executed. We can now state the following result.

Proposition 5.2. *The algorithm $\mathcal{A}_2$ is optimal in the sense that it leads to the minimum possible error covariance at any node at any time step.*

Proof. Consider a node i. At time k, let $j \in \{i\} \cup \{q|(q, i) \in \mathcal{N}_I(i)\}$ such that $\lambda_{ji}(k-1) =$ "*received*". Denote the measurement set that is transmitted from node j to node i at time step k under algorithm $\mathcal{A}_1$ by S^{ji}. As in the proof of Lemma 5.2, there is a node v such that S^{vi} is the superset of all the sets S^{ji}. Thus, the estimate of node i at time k under algorithm $\mathcal{A}_1$ is

$$\hat{x}^{\mathcal{A}_1}(k) = \hat{x}(k|S^{vi}).$$

From Lemma 5.2, when algorithm $\mathcal{A}_2$ is executed at time step k, the node i has access to the estimates $\hat{x}(k-1|S^{ji})$. Once again, since S^{vi} is the superset of all the sets S^{ji}, the estimate of node i at time step k is simply

$$\hat{x}^{\mathcal{A}_2}(k) = A\hat{x}(k-1|S^{vi}) = \hat{x}(k|S^{vi}).$$

Thus we see that for any node i, the estimates $\hat{x}^{\mathcal{A}_1}(k)$ and $\hat{x}^{\mathcal{A}_2}(k)$ are identical for any time step k for any packet drop pattern. But algorithm $\mathcal{A}_1$ leads to the minimum possible error covariance at each node. Thus, algorithm $\mathcal{A}_2$ is optimal. ■

The following remarks regarding the above algorithm are in order:

(a) The step of calculating the estimate at each node in the algorithm $\mathcal{A}_2$ can be implemented as follows. The source node implements a Kalman filter and updates its estimate at every time step with the new measurement received. Every other node i checks the time-stamps on the data coming on the incoming edges. The time-stamps correspond to the latest measurement used in the calculation of the estimate being transmitted. Then, node i updates its time-stamp using the relation

$$t^i(k) = \max_{j\in\{q|(q,i)\in\mathcal{N}_I(i)\}\cup\{i\}} \lambda_{ji}(k-1)t^j(k-1). \tag{5.5}$$

Suppose the maximum of (5.5) is achieved by node $n \in \{q|(q,i) \in \mathcal{N}_I(i)\} \cup \{i\}$. Then the node i updates its estimate as

$$\hat{x}^i(k) = A\hat{x}^n(k-1),$$

where $\hat{x}^t(k)$ denotes the estimate of the state $x(k)$ maintained by the node t. Thus, the processing can be done as a switched linear filter.

(b) We have made no assumptions on the packet drop pattern. The algorithm provides the optimal estimate based on $\mathcal{I}^{d,\max}(k)$ for an arbitrary packet drop sequence, irrespective of whether the packet drop can be modeled as an i.i.d. process or a more sophisticated model like a Markov chain or even adversarial. The algorithm results in the optimal estimate at every time step for any instantiation of the packet drop sequence, not merely in the optimal average performance. We also do not assume any knowledge of the statistics of the packet drops at any of the nodes.

(c) We have proved that the algorithm is optimal for any node. Thus we do not need to assume only one sink. The algorithm is also optimal for multiple sources if all sources have access to measurements from the same sensor. For multiple sources with each source obtaining measurements from a different sensor, the problem remains open.

(d) Any received data vector $\hat{x}(k|j)$ "washes away" the effect of all previous packet drops. It ensures that the estimate at the receiving node is identical to the case when all measurements $y(0)$, $y(1)$, $\cdots$, $y(j)$ were available, irrespective of which previous data packets had been dropped.

(e) A priori we had not made any assumption about a node transmitting the same message along all the out-going edges. It turned out that in this optimal algorithm, the messages are the same along all the edges. This property is especially useful in the context of wireless communication, which is inherently broadcast in nature.

(f) The communication requirements can be reduced by adopting an event-based protocol in which a node transmits only if it updated its estimate based on data arriving on an incoming edge. This will not degrade the performance but reduce the number of transmissions, especially if packet drop probabilities are high.

In a sense our algorithm corresponds to communication of information over a digital channel, while the strategy of using no encoding is an analog communication scheme. Our algorithm allows the intermediate nodes to play the role of repeaters that help to limit the effect of the channel by decoding and re-encoding the data along the way. In analog channels, repeaters make no difference in the received SNR and hence the signal quality. Similarly, in our setting, if raw measurements are being transmitted, presence of intermediate nodes does not help in improving the estimation performance.

5.4.1 Presence of Delays

If the links introduce random delays, the algorithm remains optimal irrespective of the possibility of packet rearrangements. Each node, at every time step, still calculates the estimate of the state $x(k)$ based on any information received at that time step and the previous estimate from its memory, affixes the correct time stamp, and transmits it along out-going edges. Further, if the graph is finite, the stability conditions of the algorithm we present in the next section also do not change.

5.4.2 Channel Between the Controller and the Actuator

If we look at the proof of the separation principle given above, the crucial assumption was that the controller knows what control input is applied at the plant. Thus, if we have a channel between the controller and the plant, the separation principle would still hold, provided there is a provision for acknowledgment from the receiver to the transmitter for any packet successfully received over that channel.[3] The optimal information processing algorithm presented above carries over to this case as well. We can also ask the question of the optimal encoder-decoder design for the controller-actuator channel. The optimal decoding at the actuator end will depend on the information that is assumed to be known to the actuator (e.g., the cost matrices Q and R and the measurements from the sensor). Design of the decoder for various information sets is an interesting open problem.

For Sections 5.5 and 5.6, we will analyze the stability and performance of the above algorithm by assuming that packets are erased independently from one time step to the next and are uncorrelated in space. We will return to more general packet dropping processes in Section 5.8.

5.5 Stability Analysis

We are interested in stability in the bounded second moment, or the mean squared sense. Thus, for the problem $\mathcal{P}_1$, we say that the system is stable if $E\left[J_\infty\right]$ is bounded, where the expectation is taken over the packet dropping processes in the network. For the problem $\mathcal{P}_2$, denote the error at time step k as

$$e^d(k) = x(k) - \hat{x}^d(k),$$

where $\hat{x}^d(k)$ is the estimate of the destination node. We can compute the covariance of the error $e(k)$ at time k as

$$R^d(k) = E\left[e^d(k)(e^d(k))^T\right],$$

[3]Note that we do not require acknowledgements for the sensor-controller channel.

where the expectation is taken over the initial condition $x(0)$, the process noise $w(j)$, and the measurement noise $v(j)$. We can further take the expectation with respect to the packet dropping process in the network and denote

$$P^d(k) = E\left[R^d(k)\right].$$

We consider the steady-state error covariance in the limit as k goes to infinity, i.e.,

$$P^d(\infty) = \lim_{k\to\infty} P^d(k). \tag{5.6}$$

If the limit exists and is bounded, we will say that the estimate error is stable; otherwise it is unstable.[4] Note that because of the separation principle, the cost J_T and the estimation error covariance $R^d(k)$ are related through the equation

$$\begin{aligned} J_T = \quad & E\left[x^T(0)S(0)x(0)\right] + \text{tr}\,\left(S(0)R^d(0)\right) + \\ & \sum_{k=0}^{T-1} \text{tr}\,\left(S(k+1)R_w + \left(A^T S(k+1)A + Q - S(k)\right) R^d(k)\right), \end{aligned} \tag{5.7}$$

where $S(k)$ is the Riccati variable that arises because of the LQ optimal control being calculated and evolves as

$$S(k) = A^T S(k+1)A + Q - A^T S(k+1)B\left(B^T S(k)B + R\right)^{-1} B^T S(k+1)A.$$

Because of the stabilizability assumptions, $S(k)$ would tend to a constant value S as the horizon T becomes longer. Thus, in the limit as $T \to \infty$, we obtain

$$J_\infty = \text{tr}\,(SR_w) + \text{tr}\,\left(\left(A^T SA + Q - S\right) R^d(\infty)\right).$$

[4]Our definition of stability requires the average estimation error or quadratic control cost to be bounded. Other metrics that require the probability density function of the cost to decay at a fast enough rate are also possible.

If we now take the expectation with respect to the packet dropping processes in the network, we obtain

$$E\left[J_{\infty}\right] = \text{tr}\ \left(SR_w\right) + \text{tr}\ \left(\left(A^TSA + Q - S\right)P^d(\infty)\right). \tag{5.8}$$

Thus, the stability conditions for problems $\mathcal{P}_1$ and $\mathcal{P}_2$ are identical. We now proceed to evaluate these conditions.

For node d and time k, let $t^d(k)$ denote the time-stamp of the most recent observation used in estimating $x(k)$ at the destination node d. This time-stamp evolves according to (5.5). The expected estimation error covariance at time k at node d can thus be written as

$$\begin{aligned} P^d(k) &= E\left[e^d(k)(e^d(k))^T\right] \\ &= E\left[\left(x(k) - \hat{x}^d(k)\right)\left(x(k) - \hat{x}^d(k)\right)^T\right] \\ &= \sum_{l=0}^{k} E\left[\left(x(k) - \hat{x}^d(k|t^d(k) = l)\right)\left(x(k) - \hat{x}^d(k|t^d(k) = l)\right)^T\right]\text{Pr}\ \left(t^d(k) = l\right), \end{aligned}$$

where in the final equation we have evaluated the expectation with respect to the packet dropping process, and $\hat{x}^d(k|t^d(k) = l)$ denotes the estimate of $x(k)$ at the destination node given all the measurements $\{y(0), y(1), \cdots, y(l)\}$. We see that the effect of the packet dropping process shows up in the distribution of the time-stamp of the most recent observation used in estimating $x(k)$. For future use, we denote the *latency* for the node d at time k as

$$l^d(k) = k - 1 - t^d(k).$$

Also, denote the mmse estimate of $x(k)$ given all the measurements $\{y(0)$, $y(1)$, $\cdots$, $y(k-1)\}$ by $P(k)$. It is well-known that $P(k)$ evolves according to the Riccati recursion

$$P(k+1) = AP(k)A^T + R_w - AP(k)C^T\left(CP(k)C^T + R_v\right)^{-1}CP(k)A^T.$$

We can now rewrite the error covariance $P^d(k)$ as

$$\begin{aligned} P^d(k) &= \sum_{l=0}^{k-1} E\left[\left(x(k) - \hat{x}^d(k|l^d(k)=l)\right)\left(x(k) - \hat{x}^d(k|l^d(k)=l)\right)^T\right] \times \Pr\left(l^d(k)=l\right) \\ &= \sum_{l=0}^{k-1}\left[A^lP(k-l)\left(A^l\right)^T + \sum_{j=0}^{l-1}A^jQ\left(A^j\right)^T\right] \times \Pr\left(l^d(k)=l\right). \end{aligned} \quad (5.9)$$

The above equation gives the expected estimation error covariance for a general network with any packet dropping process. The effect of the packet dropping process appears in the distribution of the latency $l^d(k)$. As we can see from (5.9), the stability of the system depends on how fast the probability distribution of the latency decreases.

To analyze the stability, we use the following result from [80] restated here for independent packet drops.

Proposition 5.3. *Consider a process of the form (5.4) being estimated using measurements from a sensor of the form (5.2) over a packet-dropping link that drops packets in an i.i.d. fashion with probability* ϵ*. Suppose that the sensor calculates the mmse estimate of the measurements at every time step and transmits it over the channel. Then, the estimate error at the receiver is stable in the bounded second moment sense if and only if*

$$\epsilon|\rho(A)|^2 < 1,$$

where $\rho(A)$ *is the spectral radius of the matrix* A *appearing in (5.4).*

5.5.1 Network with Links in Parallel

We begin by considering a network consisting only of links in parallel. Consider the source and the sink node being connected by a network with m links in parallel with the probability of packet drop in the i-th link being ϵ_i. Since the same data is being transmitted over all the links, the distribution of the latency in (5.9) remains the same if the network is replaced by a single link that drops packets when all the links in the original network drop packets and transmits the information if even one link in the original network allows transmission. Thus, the packet drop probability of this equivalent link is $\epsilon_1\epsilon_2\cdots\epsilon_m$. The necessary and sufficient condition for the error covariance to diverge thus becomes

$$\epsilon^\star|\rho(A)|^2 < 1,$$

where

$$\epsilon^\star = \epsilon_1\epsilon_2\cdots\epsilon_m.$$

5.5.2 Necessary Condition for Stability in Arbitrary Networks

Using the result for parallel networks, we can obtain a necessary condition for stability for general networks as follows.

Proposition 5.4. *Consider a process of the form (5.4) being estimated using measurements from a sensor of the form (5.2) through a wireless erasure network. Now, for each s-d cut-set* $[\mathcal{V}_s, \mathcal{V}_s^c]$ *define the equivalent erasure probability as*

$$\epsilon(\mathcal{V}_s) = \prod_{(i,j)\in[\mathcal{V}_s,\mathcal{V}_s^c]} \epsilon_{ij}. \tag{5.10}$$

Then, a necessary condition for the error covariance to converge is

$$\epsilon_{\text{net}}|\rho(A)|^2 < 1,$$

where ϵ_{net} *is the network erasure probability defined as*

$$\epsilon_{\text{net}} = \max_{\mathcal{V}_s : s-d \,\text{cut}} \epsilon(\mathcal{V}_s). \tag{5.11}$$

Proof. Denote the given network by $\mathcal{G}_1$. Consider a cut-set of this network with source set $\mathcal{V}_s$. Form another network $\mathcal{G}_2$ by replacing all links in $\mathcal{E} - [\mathcal{V}_s, \mathcal{V}_d]$ by perfect links, i.e., links that do not drop packets and additionally do not consume one time step to transmit data across. Now, for any packet drop pattern, denote the information sets that the destination node has access to at any time step k over networks $\mathcal{G}_2$ and $\mathcal{G}_1$ by $\mathcal{I}^{d,\mathcal{G}_2}(k)$ and $\mathcal{I}^{d,\mathcal{G}_1}(k)$, respectively. It is obvious that

$$\mathcal{I}^{d,\mathcal{G}_1}(k) \subseteq \mathcal{I}^{d,\mathcal{G}_2}(k).$$

Thus, the estimate error covariances at the destination node for the two networks are related by

$$P^d(k|\mathcal{I}^{d,\mathcal{G}_1}(k)) \succeq P^d(k|\mathcal{I}^{d,\mathcal{G}_2}(k)).$$

Hence, by considering the stability of error covariance over network $\mathcal{G}_2$, we can obtain a necessary condition for the stability of error covariance over network $\mathcal{G}_1$. Since the edges within each source and the destination sets do not introduce any delay or error, $\mathcal{G}_2$ consists of the source and the sink joined by parallel edges in the cut-set only, i.e., $[\mathcal{V}_s, \mathcal{V}_d]$. The condition for the error covariance across $\mathcal{G}_2$ to converge is thus

$$\epsilon(\mathcal{V}_s)|\rho(A)|^2 < 1,$$

where $\epsilon(\mathcal{V}_s)$ is defined in (5.10). This is thus a necessary condition for error covariance across $\mathcal{G}_1$ to be stable. One such condition is obtained by considering each cut-set. Thus, a necessary condition for the error covariance to converge is

$$\epsilon_{\text{net}}|\rho(A)|^2 < 1,$$

where ϵ_{net} is defined in (5.11). ∎

5.5.3 Network with Links in Series

Consider a network consisting of two links in series with probability of packet drops ϵ_{sr} and ϵ_{rd}, where the relay node is denoted by r, and s and d denote the source and destination nodes. Denote the estimate at node i at time k by $\hat{x}^i(k)$. Also, let $e^s(k)$ be the error between $x(k)$ and $\hat{x}^r(k)$. Similarly, let $e^r(k)$ be the error between $\hat{x}^r(k)$ and $\hat{x}^d(k)$. We are interested in the second moment stability of $e^s(k)+e^r(k)$. Clearly, a sufficient condition is that both $e^s(k)$ and $e^r(k)$ individually be second moment stable. Applying Theorem 5.3, this translates to the condition

$$\begin{aligned}\epsilon_{sr}|\rho(A)|^2 &< 1\\ \epsilon_{rd}|\rho(A)|^2 &< 1.\end{aligned}$$

If $\epsilon^\star$ is the greater of the probabilities ϵ_{sr} and ϵ_{rd}, the sufficient condition thus is

$$\epsilon^\star|\rho(A)|^2 < 1.$$

But this is identical to the necessary condition stated in Theorem 5.4. Thus, the condition above is both necessary and sufficient. Clearly this argument can be extended to any finite number of links in series. If there are m links in series with the probability of drop of the i-th link being ϵ_i, then a necessary and sufficient condition for the estimate error to converge at the sink node is

$$\epsilon^{\star}|\rho(A)|^2 < 1,$$

where

$$\epsilon^{\star} = \max(\epsilon_1, \epsilon_2, \cdots, \epsilon_m).$$

5.5.4 Sufficient Condition for Arbitrary Networks

We now proceed to prove that the condition stated in Theorem 5.4 is sufficient as well for stability.

Theorem 5.3. *Consider the assumptions of Theorem 5.4 on the erasure process and the network. Then the estimation error covariance under algorithm $\mathcal{A}_2$ will be stable if*

$$\epsilon_{\text{net}}|\rho(A)|^2 < 1,$$

where ϵ_{net} is defined in (5.11).

Proof. First note that if a packet erasure link between two nodes u and v with probability of erasure ϵ_{uv} is replaced by two parallel links with erasure probabilities $p_{uv}^{(1)}$ and $p_{uv}^{(2)}$ such that $p_{uv} = p_{uv}^{(1)} p_{uv}^{(2)}$, then the average error covariance of the estimation under algorithm $\mathcal{A}_2$ will not change at any node. This is true simply because the probability distribution of the latency in (5.9) will not change with this replacement.

Next consider the set $\Gamma = \{\gamma_1, \gamma_2, \cdots \gamma_m\}$ of all simple directed paths from the source to the destination in the network graph. Note that these paths may share edges. Consider the following optimization problem

$$\min_{\beta_j} \prod_{j=1}^{m} \beta_j, \tag{5.12}$$

subject to the following constraints

$$\prod_{j:(u,v)\in\gamma_j} \beta_j \geq \epsilon_{uv} \quad \forall\,(u,v) \in \mathcal{E} \tag{5.13}$$
$$1 \geq \beta_j \geq 0 \quad \forall j = 1, 2, \cdots, m.$$

A simple change of variables,

$$\psi_j = -\log \beta_j, \tag{5.14}$$

transforms the above optimization problem into the following linear program in the variables ψ_j's:

$$\max_{\psi_j} \sum_{j=1}^{m} \psi_j \tag{5.15}$$

subject to

$$\sum_{j:(u,v)\in\gamma_j} \psi_j \leq -\log \epsilon_{uv} \quad \forall\,(u,v) \in \mathcal{E}$$
$$\psi_j \geq 0 \quad \forall j = 1, 2, \cdots, m.$$

The solutions of the optimization problems (5.12) and (5.15), denoted by $\{\beta_j^\star\}$ and $\{\psi_j^\star\}$, are related through the relation

$$\psi_j^\star = -\log \beta_j^\star.$$

The structure of the linear program (5.15) is the same as the one used for finding the maximum flow possible in a fluid network [85, Page 59], which has the same topology as our packet dropping network with the capacity of the link (u, v) equal to $-\log \epsilon_{uv}$. The solution to the problem of finding the maximum flow through a fluid network is well-known to be given by the max-flow min-cut theorem. Using this fact, we see that the solution to the optimization problem (5.15) is given by

$$\psi_j^\star = \min_{\mathcal{V}_s : s-d\,\text{cut}} \sum_{(u,v)\in[\mathcal{V}_s,\mathcal{V}_s^c]} -\log \epsilon_{uv}.$$

Thus, for the optimization problem (5.12), the solution is given by

$$\begin{aligned}\beta_j^\star &= \max_{\mathcal{V}_s : s-d\,\text{cut}} \prod_{(u,v)\in[\mathcal{V}_s,\mathcal{V}_s^c]} -\log \epsilon_{uv} \\ &= \max_{\mathcal{V}_s : s-d\,\text{cut}} \epsilon(\mathcal{V}_s) \\ &= \epsilon_{\text{net}},\end{aligned} \tag{5.16}$$

where $\epsilon(\mathcal{V}_s)$ and ϵ_{net} have been defined in (5.10) and (5.11), respectively.

Consider the paths in the set Γ. Form a new set $\mathcal{B}$ of all those paths γ_js for which the associated optimal variable $\beta_j^\star$ is strictly less than one. The remaining paths in Γ have equivalent erasure probability as unity and can thus be ignored. Now form a new network $\mathcal{G}'$ as follows. The node set of $\mathcal{G}'$ is the union of those nodes of the original network $\mathcal{G}$ that are present on any path in $\mathcal{B}$. Each pair of nodes (u, v) in the node set of $\mathcal{G}'$ is connected by (possibly) multiple links. If an edge (u, v) between two nodes u and v is present in a path $\gamma_j \in \mathcal{B}$, we add an edge between nodes u and v in $\mathcal{G}'$ and associate with it an erasure probability $\beta_j^\star$. By considering all the edges in $\mathcal{G}$ and following this procedure, we construct the edge set of $\mathcal{G}'$. The following properties of $\mathcal{G}'$ are easily verified.

- By construction, $\mathcal{G}'$ can be presented as a union of edge-disjoint paths. Each path in $\mathcal{G}'$ corresponds to one path in $\mathcal{B}$. Furthermore, for each path, the probabilities of packet erasure on all the links of that path are equal.

- By virtue of (5.16) and the procedure followed to construct $\mathcal{G}'$, the product of the probabilities of packet erasure of the different paths is equal to the equivalent probability of the network, ϵ_{net}, for the network $\mathcal{G}$.

- For any pair of nodes that were connected by a link in $\mathcal{G}$, the product of the probabilities of packet erasure of the links in $\mathcal{G}'$ connecting these two nodes is greater than or equal to the erasure probability of the link between the same pair of nodes in $\mathcal{G}$. This can be seen from the first inequality constraint of (5.13).

Therefore, the estimate error covariance at the sink by following algorithm $\mathcal{A}_2$ in the original network $\mathcal{G}$ is less than or equal to the error covariance by following $\mathcal{A}_2$ in the new network $\mathcal{G}'$. Thus, to obtain a sufficient condition on stability, we can analyze the performance of $\mathcal{A}_2$ in the network $\mathcal{G}'$. For this we consider another algorithm, which we denote as $\mathcal{A}_3$. In this algorithm we consider the disjoint paths given in $\mathcal{G}'$ and assume that estimates on different paths are routed separately. Thus, if a node lies on many paths, on each path it forwards the packets it received on that path only. Clearly, the performance $\mathcal{A}_3$ cannot be better than $\mathcal{A}_2$ since in $\mathcal{A}_2$ we send the most recent estimate received from different paths at any node compared to forwarding the estimates on different paths separately from each other.

Therefore, to prove the theorem we only need to show the stability of estimation using protocol $\mathcal{A}_3$, assuming that the condition of Theorem 5.4 holds. Since we do not mix the estimates obtained from different paths in $\mathcal{A}_3$, the network can be considered as a collection of parallel paths, with each path consisting of links with equal drop probability. Therefore, using the stability analysis of serial networks presented earlier, each path (from a stability point of view) can be viewed as an erasure channel with drop probability equal to the drop probability of one link in that path. Using the stability analysis of parallel networks, we see that the stability of the new network under algorithm $\mathcal{A}_3$ is equivalent to the stability of a packet erasure link with probability of erasure equal to the product of the drop probabilities of different paths, which as mentioned earlier is ϵ_{net}. Thus, if

$$\epsilon_{\text{net}}|\rho(A)|^2 < 1, \tag{5.17}$$

the network $\mathcal{G}'$ is stable under protocol $\mathcal{A}_3$. But the performance of $\mathcal{A}_3$ cannot be better than of $\mathcal{A}_2$. Thus $\mathcal{G}'$ is stable under $\mathcal{A}_2$. Therefore, the original network $\mathcal{G}$ is stable under protocol $\mathcal{A}_2$ assuming (5.17) is satisfied. ∎

The following remarks are in order:

(a) We have provided a necessary and sufficient condition for the expected error covariance to remain bounded for a network of arbitrary topology. For any other causal data processing algorithm, it provides a necessary condition for stability.

(b) Let us, in particular, compare the stability conditions for the algorithm $\mathcal{A}_2$ to those for a simpler algorithm $\bar{\mathcal{A}}$ in which the intermediate nodes do not have *any* memory. At each time step k, the source node forwards the measurement $y(k-1)$. The intermediate nodes compare the time stamps of the measurements they received at the previous time step along different incoming edges and forward the most recent one. If they did not receive any measurement on the last time step, they do not transmit anything. It is clear that the probability that the destination node receives any particular measurement $y(k)$ from the source over the network is upper-bounded by the reliability of the network (see, e.g., [86]). Let us consider a simple example of a line network in which n edges each with drop probability ϵ are combined in series. With our optimal algorithm, the necessary and sufficient condition for expected estimate error covariance to be stable is $\epsilon|\rho(A)|^2 < 1$. On the other hand, in algorithm $\bar{\mathcal{A}}$, the probability that any measurement is received by the destination node is

$q = 1 - (1-\epsilon)^n$. By a method similar to the one used in [79], it can be proven that a *necessary* condition for stability is $q|\rho(A)|^2 < 1$. As an example, for $n = 5$ links and drop probability $p = 0.2$, $q = 0.67$. Thus, our algorithm yields a huge improvement for the stability margin, from $\rho(A) \leq 1.22$ for the simple algorithm to $\rho(A) \leq \sqrt{5}$ for our algorithm. This is an instance of the effect of having "repeaters" in the path as mentioned above.

5.6 Performance Analysis

In this section we calculate the performance of the algorithm $\mathcal{A}_2$ for drops independent in time and uncorrelated in space. Once again, from (5.8) we realize that the cost function for the problem $\mathcal{P}_1$ can be calculated easily as long as we are able to calculate the steady-state expected estimate error covariance defined in (5.6). We now provide a framework to calculate the expected error covariance at *any* node.

Let $L_{uv}(k)$ be the difference between k and the time at which the last successful transmission before time k occurred on link (u, v):

$$L_{uv}(k) = \min\{j \geq 1 | \lambda_{uv}(k+1-j) = \text{``}received\text{''}\}.$$

By convention, we adopt $L_{uu}(k) = 1$. Thus, the last time that any message is received at node v from link (u, v) is $k - L_{uv}(k) + 1$, and that message has time-stamp $t^u(k - L_{uv}(k))$. Then, (5.5) can be rewritten as

$$t^v(k) = \max_{u \in \{j | (j,v) \in \mathcal{N}_I(v)\} \cup \{v\}} t^v(k - L_{uv}(k)).$$

Now $L_{uv}(k)$ is distributed as a truncated geometric random variable with the density function

$$\Pr\left(L_{uv}(k)=i\right)=\begin{cases}(1-\epsilon_{uv})\epsilon_{uv}^{i-1} & \forall\, k>i\geq 1\\ 1-\sum_{i=1}^{k-1}\Pr\left(L_{uv}(k)=i\right), & k=i.\end{cases}$$

As an example, for the source node s, without extending the definition we have $t^s(k)=k-1$ for $k\geq 1$. We can get rid of the truncation by extending the definition of $t^u(k)$. For all $k<0$, we define $t^u(k)=0$. Thus, using the extended definition, $t^s(k)=(k-1)^+$for all k, where $x^+=\max\{0,x\}$. In general, using the extended definition of $t^u(k)$ for all k and for any node u, we can easily verify that

$$t^v(k)=\max_{u\in\{j|(j,v)\in\mathcal{N}_I(v)\}\cup\{v\}} t^u(k-L_{uv}(k)), \tag{5.18}$$

where $L_{uv}(k)$'s are now independent random variables distributed according to a geometric distribution with parameters ϵ_{uv}s. Thus

$$\Pr\left(L_{uv}(k)=i\right)=(1-\epsilon_{uv})\epsilon_{uv}^{i-1}\quad \forall i\geq 1, \forall k. \tag{5.19}$$

Since $L_{uv}(k)$s do not depend anymore on k from now on, we will omit the argument k. From (5.18) we can write $t^v(k)$ in terms of the time-stamp at the source node $(k-1)^+$ as

$$t^v(k)=\max_{P\text{:an s-v path}}(k-1-\sum_{(u,v)\in P} L_{uv})^+, \tag{5.20}$$

where the maximum is taken over all paths P in the graph $\mathcal{G}$ from source s to the node v. Therefore, the latency at node v can be written as

$$l^v(k)=k-1-t^v(k)=\min\{k-1,\min_{P\text{:an s-v path}}(\sum_{(u,v)\in P} L_{uv})\}.$$

Since we are interested in the steady-state expected error covariance, we consider the steady-state behavior of the latency $l^v(k)$. As $k \to \infty$, the distribution of $l^v(k)$ approaches that of the variable l^v defined as

$$l^v = \min_{P:\text{an s-v path}} \big(\sum_{(u,v)\in P} L_{uv} \big). \tag{5.21}$$

Let us now concentrate on the destination node.[5] For the destination node d, we refer to l^d as the *steady-state latency* of the network. From (5.9), the steady-state error covariance can now be rewritten as

$$P(\infty) = \sum_{l=0}^{\infty} \Pr\left(l^d = l\right) \left[A^l P^\star A^l + \sum_{j=0}^{l-1} A^j Q A^j \right], \tag{5.22}$$

where $P^\star$ is the steady-state estimation error covariance of $x(k)$ based on $\{y(0), y(1), \cdots, y(k-1)\}$ and is the solution to the Discrete Algebraic Riccati Equation (DARE)

$$P^\star = AP^\star A^T + R_w - AP^\star C^T (CP^\star C^T + R_v)^{-1} CP^\star A^T.$$

Since the pair $\{A, R_w^{\frac{1}{2}}\}$ is stabilizable, the rate of convergence of $P(0)$ to $P^\star$ is exponential [87], and the substitution of $P^\star$ for $P(k-1)$ in (5.9) does not change the steady-state error covariance.

Let us define the generating function of the complementary density function $G(X)$ and the moment generating function $F(X)$ of the steady state latency l_d

$$G(X) = \sum_{l=0}^{\infty} \Pr\left(l^d \geq l+1\right) X^l, \quad \text{and} \quad F(X) = \sum_{l=0}^{\infty} \Pr\left(l^d = l\right) X^l, \tag{5.23}$$

where X is an arbitrary matrix. Thus

$$F(X) = (X - I)G(X) + I. \tag{5.24}$$

[5] If we are interested in the error covariance at some other node v, simply denote v as the destination node.

On vectorizing (5.22), we obtain

$$\begin{aligned}\mathrm{vec}\,(P(\infty)) &= F\,(A \otimes A)\,\mathrm{vec}\,(P^{\star}) + G\,(A \otimes A)\,\mathrm{vec}\,(Q) \\ &= ((A \otimes A - I)G(A \otimes A) + I)\,\mathrm{vec}\,(P^{\star}) + G(A \otimes A)\mathrm{vec}\,(Q)\end{aligned} \quad (5.25)$$

where $A \otimes B$ is the Kronecker product of matrices A and B. Thus, the performance of the system depends on the value of $G(X)$ evaluated at $X = A \otimes A$. In particular, the system is stable if and only if $G(X)$ is bounded at $A \otimes A$. Since $G(X)$ is a power series, boundedness of $G(x)$ at $A \otimes A$ is equivalent to the boundedness of $G(x)$ (evaluated for a scalar x) at the square of the spectral radius of A. We thus have the following result.

Theorem 5.4. *Consider a process of the form (5.4) being observed using a sensor of the form (5.2) through a wireless erasure network. Let the packet erasures be independent from one time step to the next and across links. Then, the minimum expected steady-state estimation error covariance at the receiver is given by (5.25). Furthermore, the error covariance is stable in the sense of bounded expected steady-state error iff* $|\rho(A)|^2$ *lies in the region of convergence of* $G(x)$*, where* $\rho(A)$ *is the spectral radius of* A*.*

The above theorem allows us to calculate the steady state expected error covariance for any network as long as we can evaluate the function $G(X)$ for that network. We now consider some special networks and evaluate the performance explicitly. We start with a network consisting of links in series, or a line network.

5.6.1 Networks with Links in Series

In this case, the network consists of only one path from the source to the destination. Thus, we have

$$F(X) = E\left[X^{l^d}\right] = E\left[X^{\sum_{(u,v)} L_{uv}}\right],$$

where the summation is taken over all the edges in the path. Since the drops across different links are uncorrelated, the variables L_{uv}s are independent. Thus, we have

$$F(X) = E\left[X^{\sum_{(u,v)} L_{uv}}\right] = \prod_{(u,v)} E\left[X^{L_{uv}}\right],$$

where we have used the independence of L_{uv}s. Since L_{uv} is a geometric random variable (5.19),

$$E\left[X^{L_{uv}}\right] = (1-\epsilon_{uv})X\left(I-\epsilon_{uv}X\right)^{-1},$$

provided that $\rho(X)\epsilon_{uv} < 1$, where $\rho(X)$ is the spectral radius of matrix X. Therefore,

$$F(X) = E\left[X^{l_d}\right] = \prod_{(u,v)} \left[(1-\epsilon_{uv})X\left(I-\epsilon_{uv}X\right)^{-1}\right].$$

Using partial fractions and the relation in (5.24), we then obtain

$$G(X) = \sum_{i=0}^{n-1} X^i + X^n \sum_{(u,v)} c_{uv}\frac{\epsilon_{uv}}{1-\epsilon_{uv}}(I-\epsilon_{uv}X)^{-1},$$

where

$$c_{uv} = \Big(\prod_{(r,s)\neq(u,v)} (1-\frac{\epsilon_{uv}}{\epsilon_{rs}})\Big)^{-1}.$$

Therefore, the cost can be written as

$$\mathrm{vec}\left(P(\infty)\right) = \prod_{(u,v)} \left[(A\otimes A)\left(\frac{I-\epsilon_{uv}A\otimes A}{1-\epsilon_{uv}}\right)^{-1}\right] \mathrm{vec}\left(P^{\star}\right) + G\left(A\otimes A\right)\mathrm{vec}\left(Q\right). \tag{5.26}$$

We can also see from the above argument that the system is stable if for every link (u,v) we have $\epsilon_{uv}|\rho(A)|^2 < 1$ or, equivalently, $\max_{(u,v)} \epsilon_{uv}|\rho(A)|^2 < 1$. This matches with the condition in section 5.5 Also note that for the case that some of ϵ_{uv}s are equal, a different partial fraction expansion applies. In particular, for the case when there are n links all with the erasure probability ϵ, we obtain

$$\text{vec}\left(P(\infty)\right) = (A \otimes A)^n \left(\frac{I - \epsilon A \otimes A}{1-\epsilon}\right)^{-n} \text{vec}\left(P^\star\right)$$
$$+ \sum_{i=0}^{n-1} \left[\frac{\epsilon}{1-\epsilon}(A \otimes A)^n \left(\frac{I - \epsilon A \otimes A}{1-\epsilon}\right)^{-i-1}\right] \text{vec}(Q) + \sum_{i=0}^{n-1} (A \otimes A)^i \text{vec}(Q). \quad (5.27)$$

Finally, when there is only one link between the source and the destination, the cost reduces to a particularly simple form. It is easily seen that, in that case, the steady state error covariance will be the solution to the Lyapunov equation

$$P(\infty) = \sqrt{\epsilon} A P(\infty) \sqrt{\epsilon} A + \left(Q + (1-\epsilon) A P^\star A\right).$$

This expression can alternately be derived using Markov jump linear system theory as in [80].

5.6.2 Network of Parallel Links

Consider a network with one sensor connected to a destination node through n links with probabilities of erasure $\epsilon_1, \ldots, \epsilon_n$. Since the same data is being transmitted over all the links, using (5.21) the steady state latency can be written as

$$l_d = \min_{1 \le i \le n} (L_i).$$

Since L_is are all independent geometrically distributed variables with parameters ϵ_is respectively, their minimum is itself geometrically distributed with parameter $\epsilon_{eq} = \prod_i \epsilon_i$. Thus, $F(X)$ can be evaluated as

$$F(X) = (1 - \epsilon_{eq}) X (I - \epsilon_{eq} X)^{-1},$$

and $G(X)$ can, in turn, be written as

$$G(X) = (I - \prod_i \epsilon_i X)^{-1}.$$

Thus, the steady-state error can be evaluated using (5.25). Note that the region of convergence of $G(X)$ enforces for stability $\prod_i \epsilon_i |\rho(A)|^2 < 1$, which again matches with the condition in Section 5.5.

5.6.3 Arbitrary Network of Parallel and Serial Links

We can similarly find the steady-state error covariance of any network derived from the parallel and serial concatenations of sub-networks using the following two simple rules. Let $l_d(\mathcal{G})$ denote the steady-state latency function of network $\mathcal{G}$. Also, given two subnetworks $\mathcal{G}_1$ and $\mathcal{G}_2$, denote their series combination by $\mathcal{G}_1 \oplus \mathcal{G}_2$ and their parallel combination by $\mathcal{G}_1 \| \mathcal{G}_2$.

(a) Suppose the network $\mathcal{G}$ can be decomposed as a series of two subnetworks $\mathcal{G}_1$ and $\mathcal{G}_2$. Since packet erasures in the two subnetworks are independent of each other, we have

$$l^d(\mathcal{G}_1 \oplus \mathcal{G}_2) = l^d(\mathcal{G}_1) + l^d(\mathcal{G}_2).$$

Thus, we obtain

$$\begin{aligned} F_{\mathcal{G}_1 \oplus \mathcal{G}_2}(X) &= E\left[X^{l^d(\mathcal{G}_1 \oplus \mathcal{G}_2)}\right] \\ &= E\left[X^{l^d(\mathcal{G}_1) + l^d(\mathcal{G}_2)}\right] \\ &= E\left[X^{l^d(\mathcal{G}_1)}\right] E\left[X^{l^d(\mathcal{G}_2)}\right] \\ &= F_{\mathcal{G}_1}(X)\, F_{\mathcal{G}_2}(X). \end{aligned}$$

Finally, using (5.24), the complementary density function of the network $\mathcal{G}$ is given by

$$\begin{aligned} G_{\mathcal{G}_1 \oplus \mathcal{G}_2}(X) &= (X - I)^{-1} \left(F_{\mathcal{G}_1 \oplus \mathcal{G}_2}(X) - I\right) \\ &= G_{\mathcal{G}_1}(X)\,(X - I) G_{\mathcal{G}_2}(X) + G_{\mathcal{G}_1}(X) + G_{\mathcal{G}_2}(X) \\ &= (X - I) G_{\mathcal{G}_1}(X)\, G_{\mathcal{G}_2}(X) + G_{\mathcal{G}_1}(X) + G_{\mathcal{G}_2}(X), \end{aligned}$$

where in the last line we have used the fact that

$$G(X)(X - I) = (X - I)G(X).$$

(b) If the network $\mathcal{G}$ can be decomposed as parallel combination of two sub-networks $\mathcal{G}_1$ and $\mathcal{G}_2$, we have

$$l^d(\mathcal{G}_1 \| \mathcal{G}_2) = \min\{l^d(\mathcal{G}_1), l^d(\mathcal{G}_2)\}.$$

Once again, the erasures in the two subnetworks are independent of each other. Thus

$$\Pr\left(l^d(\mathcal{G}_1 \| \mathcal{G}_2) \ge l\right) = \Pr\left(l^d(\mathcal{G}_1) \ge l\right) \Pr\left(l^d(\mathcal{G}_2) \ge l\right).$$

Thus, we see that if

$$G_{\mathcal{G}_1}(X) = \sum_{i=0}^{\infty} a_i X^i \qquad \text{and} \qquad G_{\mathcal{G}_2}(X) = \sum_{i=0}^{\infty} b_i X^i,$$

then

$$G_{\mathcal{G}_1 \| \mathcal{G}_2}(X) = \sum_{i=0}^{\infty} a_i b_i X^i.$$

Thus, we can use (5.25) in conjunction with the above two rules to derive the steady state error of any network consisting of links in series and in parallel with each other. As an example, consider the network depicted in Figure 5.2. In this case

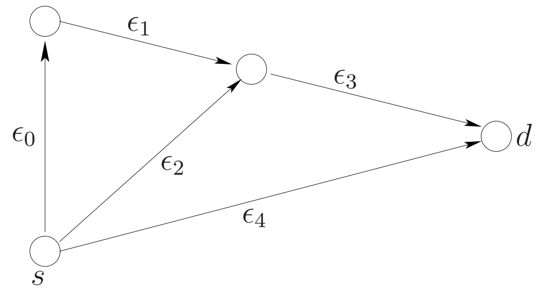


Figure 5.2: Example of a network of combination of parallel and serial links.

$$\mathcal{G} = (((\mathcal{G}_0 \oplus \mathcal{G}_1)\|\mathcal{G}_2) \oplus \mathcal{G}_3)\|\mathcal{G}_4) \,,$$

where each of the sub-networks $\mathcal{G}_i$ is just a link with probability of packet drop ϵ. The generating function of a link with erasure probability ϵ is given by $G(X) = (I-\epsilon X)^{-1}$. Moreover, for a subnetwork with generating function $G(X)$ in parallel with a link with erasure probability ϵ, the generating function of the entire network is given by $\mathcal{L}_\epsilon(G)(X)$, where $\mathcal{L}_\epsilon$ is an operator such that $\mathcal{L}_\epsilon(G)(X) = G(\epsilon X)$. Thus, the generating function of the network can be written as

$$G(X) = \mathcal{L}_\epsilon(\mathcal{L}_\epsilon(G_0 * G_1) * G_3)(X),$$

where

$$G_i(X) = (I - \epsilon X)^{-1}, \; i = 0, 1, 3$$

is the generating function for the i-th link and $G_i * G_j$ denotes the generating function of the series combination of link i and j. The steady state error covariance can thus be evaluated.

5.6.4 Networks with Arbitrary Topology

In this section, we consider the performance of general networks. Finding the distribution of the steady-state latency l_d of a general network is not an easy task because different paths may overlap. This can introduce dependency in the delays incurred along different paths and the calculation of the minimum delay and, hence, the steady-state latency becomes involved. However, using a method similar to the one used in Section 5.5, we can provide upper and lower bounds on the performance. We first mention the following intuitive lemma without proof.

Lemma 5.5. *Let* $P^{\infty}(\mathcal{G}, \{\epsilon_{uv},\, (u,v) \in \mathcal{E}\})$ *denote the expected steady-state error of a system with a communication network represented by graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ *and probabilities of packet drop* ϵ_{uv}*,* $(u,v) \in \mathcal{E}$*. Then, the expected steady-state error is non-increasing in* ϵ_{uv}*s, i.e., if* $\epsilon_{uv} \leq q_{uv} \quad \forall\, (u,v) \in \mathcal{E}$*:*

$$P^{\infty}(\mathcal{G}, \{\epsilon_{uv},\, (u,v) \in \mathcal{E}\}) \preceq P^{\infty}(\mathcal{G}, \{q_{uv},\, (u,v) \in \mathcal{E}\}),$$

where $A \preceq B$ *means that* $B - A$ *is positive semi-definite.*

5.6.4.1 Lower Bound

Using the above lemma we can lower-bound the steady-state error by making a subset of links erasure free. This is similar to the method we used to obtain a necessary condition for stability in Section 5.5. Thus, once again consider any s-d cut $\mathcal{V}_s$ of the network. Setting the probability of erasure equal to zero for every link except those crossing the cut (i.e., of the form $(u,v) \in [\mathcal{V}_s, \mathcal{V}_s^c]$) gives a lower bound on the error. Therefore,

$$P^{\infty}(\mathcal{G}, \{\epsilon_{uv},\, (u,v) \in \mathcal{E}\}) \succeq P^{\infty}(\mathcal{G}, \{q_{uv},\, (u,v) \in \mathcal{E}\}),$$

where

$$q_{uv} = \begin{cases} \epsilon_{uv} & (u,v) \in [\mathcal{V}_s, \mathcal{V}_s^c] \\ 0 & \text{otherwise.} \end{cases}$$

Now $P^{\infty}(\mathcal{G}, \{q_{uv},\ (u,v) \in \mathcal{E}\})$ can be evaluated using the results given above for a network of parallel links. By considering the maximum along all possible cut-sets, we obtain the closest lower bound.

5.6.4.2 Upper Bound

We use a method similar to the one used to obtain the sufficient condition for stability in Section 5.5. In the proof of Theorem 5.3, it is shown that the performance of the network $\mathcal{G}$ is lower-bounded by the performance of another network $\mathcal{G}'$ that has series and parallel links only and has the following properties:

- $\mathcal{G}$ and $\mathcal{G}'$ have the same node set.
- $\mathcal{G}'$ is the combination of edge-disjoint paths from the source to destination.
- The value of the max-cut in $\mathcal{G}'$ is the same as in the original network $\mathcal{G}$.

The performance of $\mathcal{G}'$ can be computed based on the results given above for arbitrary networks composed of subnetworks in series and parallel. This provides an upper bound on the performance of the original network.

5.7 Examples

We now illustrate the above results using some simple examples. Consider a scalar process evolving as

$$x(k+1) = 0.8x(k) + w(k)$$

that is being observed through a sensor of the form

$$y(k) = x(k) + v(k).$$

The noises $w(k)$ and $v(k)$ are assumed zero-mean, white, independent, and Gaussian with unit variances.

Suppose that the source and the destination nodes are connected using two links in series, each with a probability of packet erasure p. Figure 5.3 shows the performance of our strategy for the estimation problem $\mathcal{P}_2$ as the probability p is varied. The simulation results refer to data generated by a random run averaged over 100000 time steps, while the theoretical values refer to the value predicted by (5.25). We can see that the two sets of values match quite closely.

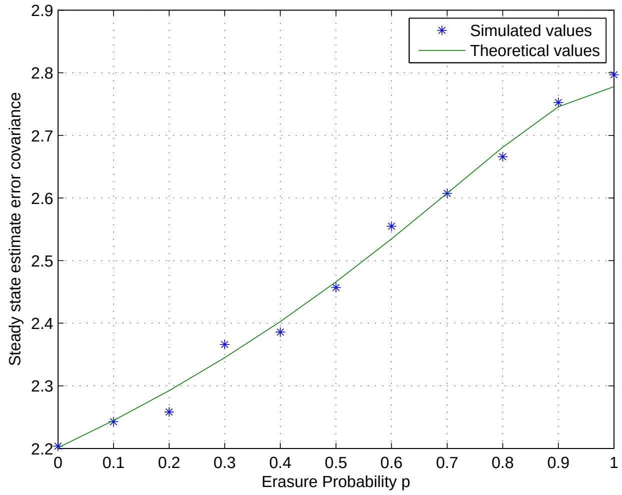


Figure 5.3: Simulated and theoretical results for a line network.

We also carried out a similar exercise for the source and destination nodes connected by two links in parallel, each with packet erasure probability p. The results are plotted in Figure 5.4. We can once again see that the simulated values match quite closely with the theoretical values.

Finally, we consider the source and destination nodes connected by a bridge network shown in Figure 5.5. We assume all the links in the network to have probability of erasure p. This network cannot be reduced to a series of series and parallel sub-networks. We can, however, calculate the performance analytically in this particular case and compare it to the upper and lower bounds presented earlier. The networks

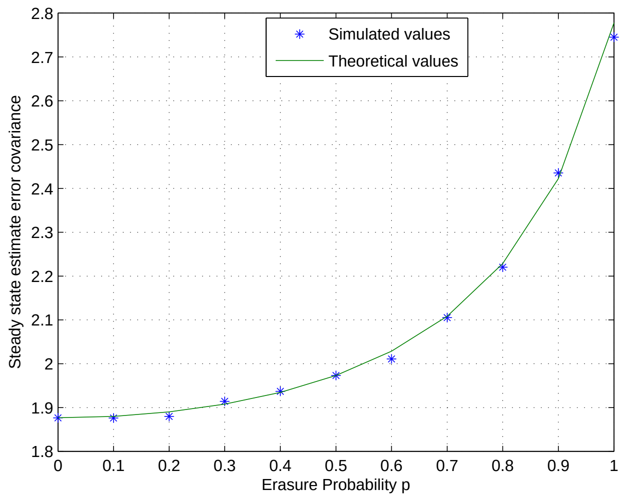


Figure 5.4: Simulated and theoretical results for a parallel network.

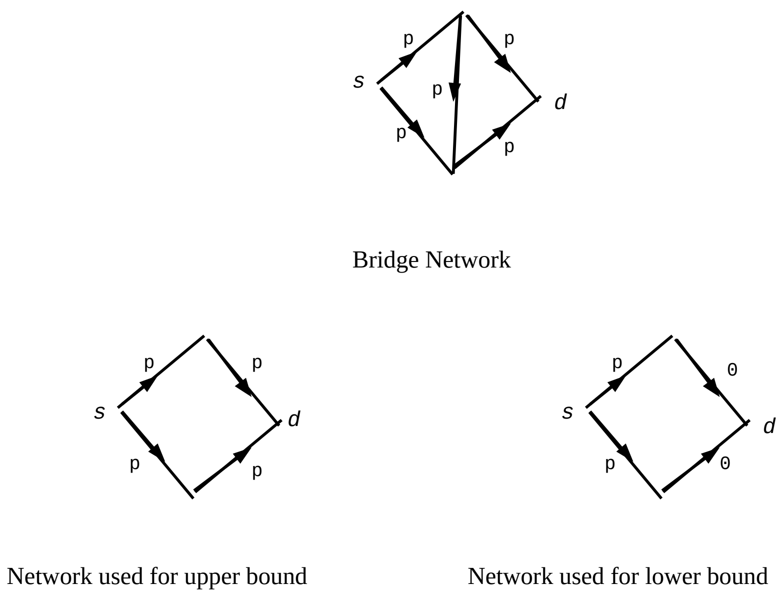


Figure 5.5: Bridge network and the networks used for calculating lower and upper bounds.

used for calculating the bounds are also shown in Figure 5.5. The bounds can be computed analytically from series and parallel network results. Figure 5.6 shows a comparison of the analytical and simulated values with the lower and upper bounds. We can see that the bounds are quite tight.

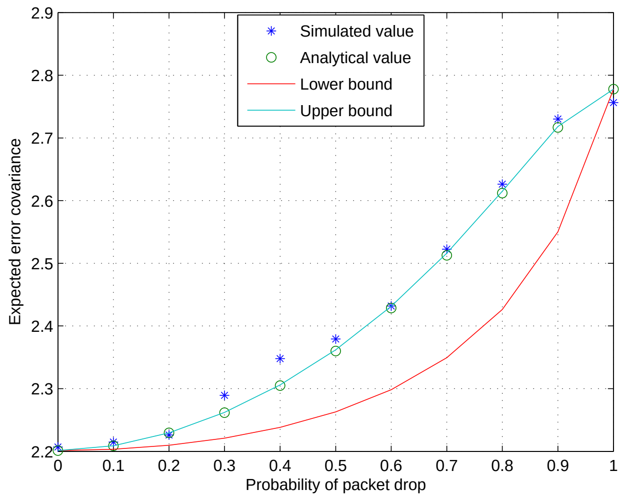


Figure 5.6: Simulated values and theoretical bounds for the bridge network.

We can also compare the performance of our algorithm with that obtained if no encoding were done and only measurements were transmitted. Consider the process

$$x(k+1) = \begin{bmatrix} 1.25 & 0 \\ 1 & 1.1 \end{bmatrix} x(k) + w(k)$$

being observed by a sensor of the form

$$y(k) = x(k) + v(k),$$

where $w(k)$ and $v(k)$ are white independent Gaussian noises with means zero and covariance identity. We consider transmission of data across a series of n channels. Figure 5.7 shows the difference in simulated performance of the algorithm in which no encoding is done and for our algorithm for various values of n. It can be seen that the error covariance is much higher if no encoding is being done, even for a few number of links and moderate values of drop probability. For each point we did 50000 simulations, with each simulation being 1000 time steps long. As a final example, we

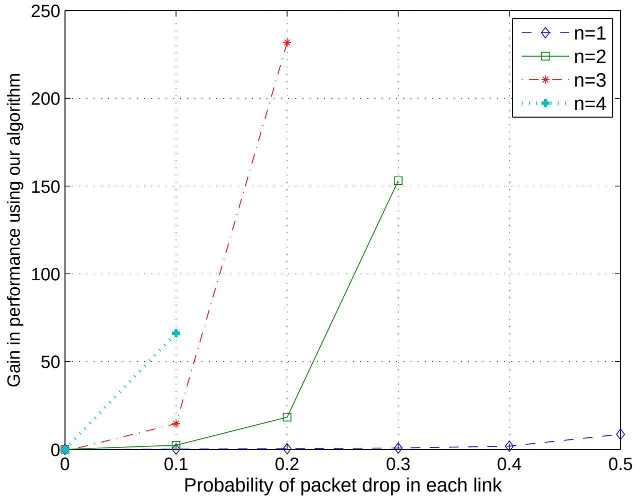


Figure 5.7: Simulated difference in performance between algorithm in which no encoding is done and our optimal algorithm for series connection of n links.

consider the process

$$x(k+1) = 1.2x(k) + u(k) + w(k)$$

being observed through a sensor of the form

$$y(k) = x(k) + v(k),$$

which communicates with the controller over a series network of n links each with packet drop probability p. The controller is interested in minimizing the quadratic cost

$$J = \lim_{K\to\infty} \frac{1}{K} E\left[\sum_{k=0}^{K} x^T(k)Qx(k) + u^T(k)Ru(k)\right].$$

The cost matrices Q and R as well as the noise variances R_w and R_v are assumed to be unity. Figure 5.8 shows the variation of the cost with the probability for different number of links n. The loss in performance is very rapid with the increase in number of links.

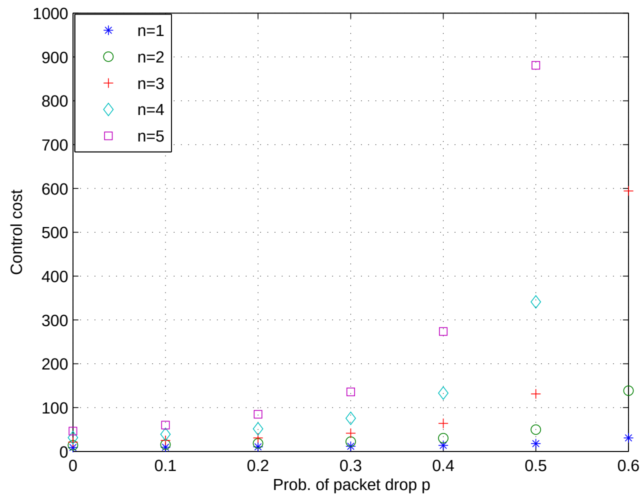


Figure 5.8: Loss in performance as a function of packet drop probability for n links in series.

5.8 Generalizations

In this section we consider problems that are related to the problem set-up we considered above.

5.8.1 Correlated Erasure Events

Even though the algorithm $\mathcal{A}_2$ is optimal for any packet dropping pattern, the stability and performance analysis so far assumed that the erasure events are memoryless and independent across different links in the network. We could thus formulate the performance in terms of a generating function of the steady-state latency distribution as defined in (5.21). We now look at the effect of dropping these assumptions.

5.8.1.1 Markov Events

If we assume that the erasure events on each link are governed by a Markov chain (but are still independent of other links), we can obtain the performance as follows. Suppose that the packet drop event on link (u, v), denoted by $\lambda_{uv}(k) =$ "*dropped*" evolves according to a Markov chain with transition matrix M_{uv}. We further assume that M_{uv} is irreducible and reversible. Let us first consider the case where the initial distribution of packet erasure on each link is the stationary distribution of the Markov chain on that link. Then we can rewrite (5.5) in a similar fashion as (5.20), with L_l being a geometric random variable with distribution

$$\Pr(L_{uv} = l) = \begin{cases} \alpha_{uv} M_{uv}(1,2) M_{uv}(1,1)^{l-2} & \forall\, l \geq 2 \\ 1 - \alpha_{uv} & l = 1 \end{cases},$$

where α_{uv} is the probability of packet drop based on the stationary distribution of link $e = (u, v)$, and $M_{uv}(i, j)$ as the (i, j)-th element of M_{uv}. Thus, all the previous analysis goes through. In particular, the stability condition is

$$\max_{\mathcal{V}_s : s-d\text{cut}} \prod_{(u,v)\in \text{c}} M_{uv}(1,1)|\rho(A)|^2 < 1.$$

Now, if the initial distribution is not the stationary distribution, the variables $L_{uv}(k)$ will not be time-independent, and the analysis does not goes through. However, since for large k the Markov chains will approach their stationary distribution, the stability condition remains unchanged.

5.8.1.2 Spatially Correlated Events

Suppose that the erasure events are correlated across the network but memoryless over time. In other words, at each time step k, the packet erasure occurs according to distribution $\Pr(\lambda_{uv}, (u,v) \in \mathcal{E})$. Now, $L_{uv}(k)$s are not independent across the network, and hence finding the steady-state error covariance does not seem to be tractable. However, we can find the condition for stability. For this we define a generalized notion of equivalent probability of packet drop for correlated events. Consider a cut-set with source set $\mathcal{V}_s$. Then, the equivalent probability of packet drop for this cut is

$$\epsilon_{eq}(\mathcal{V}_s) = \Pr\left(\lambda_{uv} = \text{``}dropped\text{''}, \quad \forall\, (u,v) \in [\mathcal{V}_s, \mathcal{V}_s^c]\right).$$

The value of the max-cut for the network is the maximum of $\epsilon_{eq}(\mathcal{V}_s)$ over all the cuts, $\epsilon_{\text{net}}(\mathcal{G}) = \max_{\mathcal{V}_s : s-d\text{cut}} \epsilon_{eq}(\mathcal{V}_s)$. We can then show that the condition for stability of the system is $\epsilon_{\text{net}}(\mathcal{G})|\rho(A)|^2 < 1$. To see this, consider the scenario when only one packet is to be routed from the source to destination starting at time t_0. For each time-step $t \geq t_0$, let $\mathcal{V}_r(t)$ denote the set of nodes that have received the packet at time t. Clearly $\mathcal{V}_r(t_0) = \{s\}$. We want to bound the probability that at time $t_0 + T$, the destination node has not yet received the packet. Note that for every time-step between t_0 and $t_0 + T$, $\mathcal{V}_r(t)$ defines a cut-set since it contains s and not d. Now, the size of $\mathcal{V}_r(t+1)$ does not increase with respect to time-step t iff all the links that cross the cut generated by $\mathcal{V}_r(t)$ drop packets. However, by the definition of $\epsilon_{\text{net}}(\mathcal{G})$, the probability of this event is at most $\epsilon_{\text{net}}(\mathcal{G})$. Therefore, we have

$$|\mathcal{V}_r(t+1)| \begin{cases} \geq |\mathcal{V}_r(t)| + 1 & \text{with prob. at most}\, \epsilon_{\text{net}}(\mathcal{G}) \\ = |\mathcal{V}_r(t)| & \text{with prob. at least}\, 1 - \epsilon_{\text{net}}(\mathcal{G}) \end{cases}$$

Thus, for large T the probability that at time $t_0 + T$ the destination node has not received the packet is upper-bounded by $n(1 - \epsilon_{\text{net}}(\mathcal{G}))^n T^n \epsilon_{\text{net}}(\mathcal{G})^{T-n}$, where n is the number of nodes in the network. Although a new packet is generated at the source at each time step, the importance of the packets is increasing with time; hence the error can be upper-bounded by considering that the network is only routing the packet generated at time $k-l$. The probability that the latency is larger than l grows like $f(l)\epsilon_{\text{net}}(\mathcal{G})^l$, where $f(l)$ is polynomial in l with bounded degree and thus the sufficiency of the stability condition follows. The necessity part involves similar ideas and is omitted.

5.8.2 Synthesis of a Network

One can use the performance results above to design networks for the purpose of estimation and control. To consider a simple example, consider a scalar system observed by sensor s. Assume that the destination is located at distance d_0 from the sensor. The probability of dropping a packet on a link depends on its physical length. A reasonable model for probability of dropping packets is given by[6] $\epsilon(d) = 1 - \exp(-\beta d^\alpha)$, where β and α are positive constants. α denotes the exponent of power decay in the wireless environment. We are interested in the optimal number n of relay nodes that we should place between the sensor and the destination so as to minimize the expected steady-state error covariance (or in turn to minimize the cost J_∞). Clearly, there is a trade-off involved since more nodes will reduce the probability of erasure but at the same time lead to a higher delay before the destination receives a packet. Assuming that n sensors are uniformly placed, there are $n+1$ links each with erasure probability q. Thus, from (5.27), $P(\infty)$ can be written as

$$P(\infty) = \left(\frac{a^2(1-q)}{1-qa^2}\right)^{n+1}(P^\star + \frac{R_w}{a^2-1}) - \frac{R_w}{a^2-1}.$$

Thus, the optimal number of relay nodes, (assuming that $a^2 > 1$) is the solution to the problem

$$\min_n \left(\frac{a^2(1-\epsilon(\frac{d_0}{n+1}))}{1-\epsilon(\frac{d_0}{n+1})a^2}\right)^{n+1}.$$

[6]This expression can be derived by considering the probability of outage in a Rayleigh fading environment.

If $a^2 < 1$, then the minimization in the above problem is replaced with maximization.

5.8.3 Unicast Networks

So far, we have assumed that the topology of the network was fixed and that a node could transmit a message on all the out-going edges. We can consider networks that are unicast in the sense that each node should choose one out of a set of possible edges to transmit the message on. There are two parts of the problem:

(a) Choose the optimal path for data to flow from the source node to the sink node.

(b) Find the optimal information processing scheme to be followed by each node.

The two parts can clearly be solved separately in the sense that given any path, the optimal processing strategy is the algorithm described in Section 5.4.

To choose the optimal path, we need to define a metric for the cost of a path. We can consider two choices:

(a) If the metric is the condition for stability, then the problem can be recast as choosing the shortest path in a graph with the length of a path being given by its equivalent probability of packet drop. Thus, we need to find the path that has the minimum ϵ_{path}. Since each path is just a line network, this reduces to the problem $\min_{P:s-d\,\text{path}} \max_{(u,v)\in P} \epsilon_{uv}$. The above problem is well studied in the computer science community and can be solved as a shortest-path problem over a min-max semi-ring in a distributed fashion [88].

(b) If the metric is performance, the problem is more complicated in general. We can consider the special case of a scalar system and no process noise. In this case, from (5.26), we have for path q

$$\log P_q(\infty) = \sum_{e\in q} \log(\frac{(1-\epsilon_e)a^2}{1-\epsilon_e a^2})$$

Now the problem is equivalent to

$$\min_{q\text{:s-d path}} \sum_{e\in q} \log(\frac{(1-\epsilon_e)a^2}{1-\epsilon_e a^2}).$$

This problem can also be solved in a distributed way [89].

5.9 Conclusions

In this chapter we considered the problem of estimation and control of a dynamical process across a wireless erasure network. We identified the optimal information processing strategy to be followed by each node in the network that allows the estimator to calculate the best possible estimate in the minimum mean square sense and the controller to minimize a quadratic cost. The recursive algorithm that we propose requires a constant amount of memory, processing, and transmission at every node in the network per time step, yet is optimal for any erasure process and at every time step. It has numerous other important properties as well, such as being able to take care of delays and packet reordering. For the case when the erasure probabilities are memoryless and independent across links, we also carried out the stability and performance analysis for this algorithm. The results can also be used for the problems of routing of data through a network and synthesis of networks for the purpose of estimation and control.

The work in this chapter can be extended in many ways. We have ignored issues of quantization so far. Finding stability condition and performance bounds for a limited bit rate erasure channels is a possible avenue. Another important extension is analysis of stability conditions for the case that more than one sensor is providing measurements in the network. This seems like a very challenging and interesting problem that merits further investigation.

And finally, at a more general level, given the variety of tasks, e.g. communication, computation, estimation, and control that wireless networks are deployed for today, a mathematical framework capable of analyzing their ultimate performance is necessary. Such a framework enables us to find the important features of the network affecting the performance of each of these tasks and design networks that are efficient for different applications. The work presents in the present and previous chapters is a step toward this goal.

Chapter 6

Power Efficiency of Sensor and Ad-Hoc Networks

6.1 Introduction

In recent years, there has been great interest in the analyses of capabilities of wireless networks from different aspects. Most of the analysis have dealt with the analysis of the following two types of networks [91], [92]:

(a) **Sensory networks:** A sensory network consists of $n + 1$ fixed nodes with a single receiver that collects data/information from the sensor nodes. At any given time, there can be at most one sensory transmitter. All other nodes in the network can be thought of as relay nodes. (See Figure 6.1.(a).)

(b) **Ad-hoc networks:** At any time, an ad-hoc network consists of n fixed relay nodes and r fixed simultaneous transmitter/receiver pairs, where $r \leq n$. In this network relay nodes cooperate for transmission of information from one transmit node to the corresponding receiver node. (See Figure 6.1.(b).)

Unfortunately, finding the exact ultimate performance of a general wireless network for different applications, e.g., communications, estimation, and control, is very much open. For instance, finding the maximum reliable rate of communication for a network with only one relay is still open [53]. Therefore, most of the research

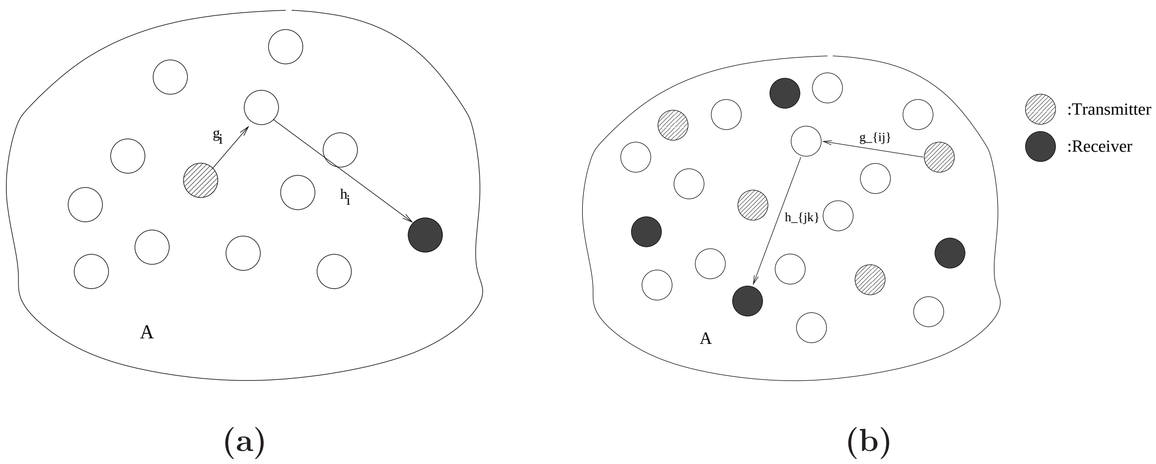


Figure 6.1: Sensory and Ad-hoc wireless networks.

for wireless sensory and ad-hoc networks involves asymptotically large networks and gives scaling laws for performance measures such as capacity or distortion rather than exact results for general networks. For instance, it is shown in [91] that for a sensory network, the capacity scales as $\Theta(\log n)$.[1] For ad-hoc networks, the problem is much more challenging. The groundbreaking work of [92] shows that the capacity grows at least as $\Theta(\sqrt{n})$. Using information theoretic tools, it is shown in [93, 94, 95] that under some mild assumptions on the channel model, $\Theta(\sqrt{n})$ is an upperbound on the sum-capacity in the extended wireless networks, i.e., networks where the density of the nodes per area does not increase with the number of nodes. In both sensory and ad-hoc wireless networks, these results are discouraging from a practical point of view because they suggest that for sensory and ad-hoc wireless networks, the per-user capacity scales as $\Theta(\frac{\log n}{n})$ and $\Theta(\frac{\sqrt{n}}{n})$, respectively. This represents rewards that rapidly diminish to zero as the number of nodes (users) in the network increases.

Therefore, one interesting problem is to see whether there exists any favorable scalings in ad-hoc and sensory wireless networks. In other words, are there any

[1]The following notation will be used in this thesis. For two functions f, g defined on natural numbers, we have $f(n) = O(g(n))$ if $\lim_{n\to\infty} \inf f(n)/g(n) < \infty$; we have $f(n) = \Omega(g(n))$ if $\lim_{n\to\infty} \inf f(n)/g(n) > 0$; and we have $f(n) = o(g(n))$ if $\lim_{n\to\infty} \inf f(n)/g(n) = 0$. Finally, we have $f(n) = \Theta(g(n))$ if $f(n) = \Omega(g(n))$ and $f(n) = O(g(n))$.

scenarios in which it is actually beneficial to form a sensory or ad-hoc network and obtain increasing gains as the network size grows? Several researchers have looked at this problem from different points of view. In [96] the authors look at a wireless network in which users are mobile (not fixed), and they show that the total capacity of such a network scales like $\Theta(n)$. The work in [97],[98] and [99] also consider the feasibility of wireless networks from a distributed source coding point of view.

In previous chapters we were concerned with performance analysis of special classes of wireless networks (mainly wireless erasure networks) with *arbitrary* size. We found exact ultimate performance measures for applications like communications and estimation across the network. In this chapter we look at a more general class of wireless networks, but instead of finding the exact performance measure for arbitrary size networks, we analyze the performance for random and asymptotically large size networks. The performance measure considered in this chapter, unlike the references mentioned above, will be the power consumption in the network.

Power consumption is one of the main concerns in wireless networks, especially in sensory networks [100]. Since the source of energy for each user is limited (usually a battery)[65], users in these networks need to use power efficiently. Two major sources of power consumption at each node are the computation power and the transmit power. We only consider the power consumption due to transmission and not due to computation. However, we should mention that it is not clear whether at low SNR (where many wireless networks usually operate at) the computation power is negligible compared to transmit power. Incorporating computation power as well is very interesting and can be a subject of research in itself.

We will show that it is beneficial to form large networks of users in terms of power consumption. We consider sensory and ad-hoc wireless networks where the users are placed randomly in a domain of fixed area A. We show that users in these networks can support the same rate as a single user system, but by expending less power.

Furthermore, the power that each user needs to expend decreases as we increase the size of the network where the rate of communication is kept fixed. To look at the power efficiency of these networks we will follow the same approach and concept as in [101]. The power efficiency η of a communications channel is defined as the ratio between the capacity (data rate) of the channel and the transmit power (energy rate). For AWGN channels, this is given by

$$\eta = \frac{\log(1 + \sigma_s^2/\sigma_n^2)}{\sigma_s^2}, \tag{6.1}$$

where σ_s^2 represents the transmit power and σ_n^2 represents the noise power. Clearly, for fixed σ_n^2 as $\sigma_s^2 \to \infty$, the value of η approaches zero, meaning that we are highly power inefficient at high SNR. On the other hand, we are power efficient at low SNR and, in fact,

$$\max_{\sigma_s^2} \eta = \lim_{\sigma_s^2 \to 0} \eta = \frac{\log e}{\sigma_n^2}. \tag{6.2}$$

This implies that, at low SNR, capacity is proportional to the transmit power. In [101], the power efficiency (or capacity per unit cost as the author defines it) of several other communication systems is computed.

In this chapter we will find a lower bound for the power efficiency of sensory and ad-hoc wireless networks formed in a domain of fixed area. For this, we will propose a protocol for communication among the nodes. The key idea used in the protocol is to exploit features of wireless networks and operate the network at low SNR (thereby avoiding the logarithmic scaling of the capacity). The main features that distinguish wireless networks from wireline networks are path-loss, fading, and interference. Path-loss has been exploited in cellular networks. Fading also is exploited in multi-user systems by scheduling transmissions when a user has favorable channel conditions [102, 103]. However, most current approaches avoid interference in the network. For instance, in [92] most of the emphasis is on interference-avoidance and the construction of a multi-hop network. In our protocol we will exploit the interference and fading inherent in any wireless network for achieving good power efficiency. Also, the protocol proposed in this chapter is a double-hop protocol. Although it is thought that the power efficiency of multi-hop networks is better than that of double-hop, networks it can be shown that if the nodes are placed in a domain of fixed area, this is not true. A similar observation is made in [104]. The authors have observed that the most energy-efficient protocol to use depends on the network topology and the radio parameters of the system.

We have shown in [105] that for sensory and ad-hoc wireless networks for which the channel coefficients between users can be modeled by independent zero-mean, unit variance, and bounded fourth order moment random variables, the power efficiency scales at least as $\Theta(\sqrt{n})$. However, the model used for channel coefficients in this chapter is more general. We will see that even with this general model we are still able to achieve a power efficiency that scales favorably as the size of the network grows. The net result is that under some mild assumptions on the channel coefficients that will be mentioned in Section 6.2.2, with high probability the power efficiency of a random network, i.e., the data rate per energy rate, scales as $\Theta(\sqrt{n})$ for each user.

This chapter is organized as follows. Section 6.2 describes the system model and assumptions and presents the statement of the problem considered in this chapter. In Section 6.3 we will compute the power efficiency of a multi-antenna communication system for comparison. In Section 6.4 we consider the power efficiency of sensory wireless networks. We describe the proposed “Listen and Transmit” protocol for achieving scalable power efficiency for sensory networks. In Section 6.5 power efficiency of ad-hoc wireless networks is considered and analyzed. We first present a generalization of the "Listen and Transmit" protocol for ad-hoc networks and then optimally allocate powers to achieve a scalable power efficiency for the network. At the end of this section we will compare the performance of our protocol with an interference suppression scheme that requires complete knowledge of the channel. Conclusions and proposals of further work are provided in Section 6.6.

6.2 Notation and System Model

6.2.1 Notation and Definitions

Throughout this chapter matrices and vectors are denoted by boldface characters. $\mathrm{tr}\,(\mathbf{A})$, $\lambda_{max}(\mathbf{A})$, and $\lambda_{min}(\mathbf{A})$ denote the trace, the maximum eigenvalue, and the minimum eigenvalue of a square hermitian matrix $\mathbf{A}$. The superscript $*$ denotes conjugate transposition for matrices and complex conjugate for scalars. Complex conjugation for matrices is shown by using bar. Transposition is also denoted by superscript T. $\mathbf{A}^*$, $\mathbf{A}^T$, and $\bar{\mathbf{A}}$ are the conjugate transpose, transpose, and conjugate of the matrix $\mathbf{A}$, and α^* is the complex conjugate of the scalar α. $\mathbf{I}_r$ is the $r \times r$ identity matrix. For a matrix $\mathbf{A}$, $\mathrm{vec}\,(\mathbf{A})$ denotes the vector obtained from stacking all the columns of $\mathbf{A}$, one on top of another. For a vector $\mathbf{g} = (g_1, \dots, g_n)$, $\mathrm{diag}\,(\mathbf{g})$ denotes the $n \times n$ diagonal matrix with i-th diagonal element equal to g_i. We may also write $\mathrm{diag}\,(\mathbf{g})$ as $\mathrm{diag}\,(g_1, \dots, g_n)$. Finally, $\|\mathbf{g}\|$ denotes the Euclidean norm of vector $\mathbf{g}$.

We consider random wireless networks over a fixed area A. We randomly select points in A to form the nodes of the network (either as transmitters, receivers, or relay nodes). Since the network is wireless, the connections between any two nodes will be subject to fading. Thus, the randomness in the network will be due to two sources: the random choice of points in A, and the random fading between the connections. When we fix the position of the nodes, we denote the expectation over channel fading by $\mathrm{E}_{\mathrm{f}}\,[.]$. The expectation over the location of some set of the nodes, say P, in a random network is denoted by $\mathrm{E}_{\,P}[.]$, which from now on we shall call the spatial average . For instance, the spatial average of the mean value of the channel coefficient between node i and node j, h_{ij}, over the position of node i while node j is fixed, can be written as $\mathrm{E}_{\{i|j\}}[\mathrm{E}_{\mathrm{f}}[h_{ij}]]$. The expectation over the location of all the nodes is denoted by E_{loc}, and whenever $\mathrm{E}[.]$ is used without any subscript, expectation over both fading and the location of the nodes is implied. Channel coefficients are denoted by h, g, or c depending on the context. Usually c is used as a generic channel between two arbitrary points in the domain.

6.2.2 System Model and Problem Statement

Sensory Networks

As mentioned earlier, by a sensory wireless network we mean one with n relay nodes and a single transmitter/receiver pair (see Figure 6.1). We assume that the nodes are placed randomly and independently according to some distribution function (not necessarily uniform) in a domain of fixed area, say A. We denote the channel coefficient from the transmitter to the relay node i by g_i, and the channel coefficient from the relay node i to the receiver by h_i. We assume that, averaged over the fading, different channels are independent. Furthermore, we assume that each node i knows only its local connections $h_i,\ g_i$, but not the other connections in the network.

Ad-hoc Networks

As mentioned earlier, for ad-hoc networks we assume that at any time there are n relay nodes and at most r simultaneous transmit/receive pairs in the network. The nodes are placed randomly and independently according to some distribution function (not necessarily uniform) in a domain of fixed area, say A. The channel coefficient from transmitter i ($i = 1, \ldots, r$) to relay node j ($j = 1, \ldots, n$) is denoted by g_{ij}, and from relay node j to receive node k ($k = 1, \ldots, r$) is denoted by h_{jk}. Similar to the sensory case, we assume that, averaged over the fading, distinct channels are independent. Furthermore, if we fix the location of the transmitters and the receivers and randomly choose relay nodes j and j', the channel coefficients $\{g_{ij}, h_{jk}\}$ and $\{g_{ij'}, h_{j'k}\}$ are independent for all i and k. As with the sensory case, we assume that all the relay nodes know their local connections, but not the remaining connections in the network. In other words, node j knows all the connections $\{g_{ij}, i = 1, \ldots, r\}$ and $\{h_{jk}, k = 1, \ldots, r\}$.

Additional Assumptions for Ad-hoc Networks

For ad-hoc networks, we have a few more assumptions. Thus, denote the channel coefficient between two points x_i and x_j by c_{ij}. With this notation, we have the following additional assumptions:

A. $\mathrm{E}_{\{x_5|x_2\}}[\mathrm{E}_{\mathrm{f}}[c_{52}]] = 0, \quad \forall x_2 \in A$

B. $\mathrm{E}_{\{x_5|x_1,x_2\}}[\mathrm{E}_{\mathrm{f}}[c_{51}c_{52}^*]] = 0 \quad \forall x_1 \neq x_2 \in A$

C. $\mathrm{E}_{\{x_5|x_1,x_2,x_3,x_4\}}[\mathrm{E}_{\mathrm{f}}[c_{51}^*c_{52}^*c_{53}c_{54}]] = 0 \quad \forall x_1, x_2, x_3, x_4 \in A$ at least 3 of x_i are distinct.

Note that the above conditions are clearly met if the fading is zero mean. In general, however, there may be line-of-sight between different nodes in the network, and the fading may be nonzero mean. The above conditions are more general and do not require zero mean fading. The first assumption says that the spatial average of the mean of a channel coefficient between a random point and a fixed point is zero. The second assumption is that the channel coefficients between one random point, x_5, and two different points, x_1, x_2, are uncorrelated when averaged over both the fading and the point placement of x_5. In other words, although the channels c_{51} and c_{52}, given that x_1 and x_2 are fixed, are not independent and may be correlated, the spatial average of the correlation between these two channels is zero. The last condition also says that the expectation of the product of the channel coefficients between one random point and four fixed points averaged over the location of the random point

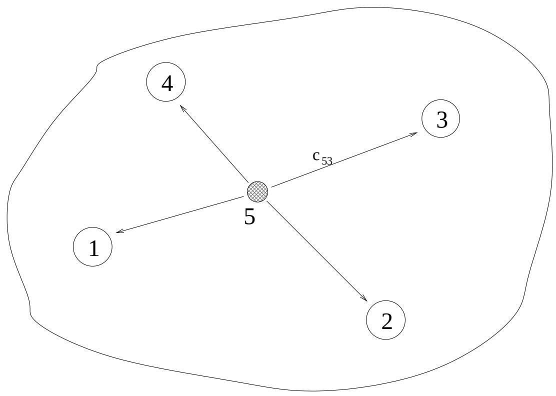


Figure 6.2: Condition on the channel coefficients.

is zero (See Figure 6.2). These assumptions appear to be reasonable, especially if we assume that the environment is rich in scattering. We obtain two achievable bounds for the power efficiency of ad-hoc networks. The first bound (Theorem 6.9) relies only on the first and second assumptions, and the second bound (Theorem 6.12) requires the last assumption as well.

Power Assumptions

In the sensory network we assume that the transmit power is p. For ad-hoc networks we assume that *all* the transmitters transmit with the same power p. In both cases, we will assume that the relay nodes transmit with identical power σ_r^2. The noise introduced in every reception is an additive white circularly-symmetric Gaussian noise with zero mean and variance σ_n^2, which is denoted by $\mathcal{CN}(0, \sigma_n^2)$.

Path Loss

In this work we will not be concerned with explicit path loss models. The main reason is that since we consider a fixed domain A, the only characteristics of the path-loss that enter our analysis are the second and fourth order moments of the channel. In fact, a strength of our results is that the asymptotics are not sensitive to the path-loss

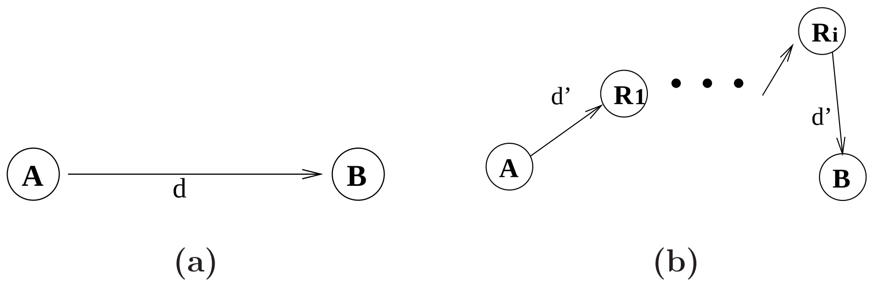


Figure 6.3: Single-hop versus Multi-hop.

model (the model and the geometry of the domain A affect the constants but not the scaling behavior). We further discuss path-loss models when comparing single-hop and multi-hop systems below.

Single-hop vs. Multi-hop Communication

The proposed protocol in this work achieves a power efficiency that scales with the number of nodes, n. The communication model that we are using is a double hop (transmit and relay) communication protocol. In this protocol, which will be explained in detail later on, the communication is done in two intervals. In the first interval, the transmit node(s) sends their data signal. In the second interval, relay nodes send a signal based on what they have received in the first interval.

Typically, in order to increase power efficiency in wireless networks, one must move toward a multi-hop system so as to avoid long hops (which are subject to severe path loss) [106], [107]. While this is certainly true for networks that grow in physical size as the number of nodes increases (thereby increasing the size of the hops), it is not true for networks in which the physical domain is *fixed* while the number of nodes increases.[2] In this case, there is nothing to be gained by using multi-hop schemes in which the number of hops scales with the number of nodes in the network and the

[2] If the area of the network increases with the number of nodes, a combination of multi-hop routing and the "Listen and Transmit" protocol described here is necessary to achieve a power efficiency that scales with the number of nodes.

length of the hops becomes shorter and shorter as the number of nodes increases. To make this more explicit, we use the following qualitative argument. Suppose that n nodes are located in a domain of fixed area A. Consider two nodes (users) of distance d which want to communicate with each other (Figure 6.3.(a)). Assume that the channel is AWGN and that the power loss between any two points is a decreasing function of their distance d and is denoted by $f(d)$. In this case, the relation between the transmitted signal s from A and the received signal y at B is

$$y = \sqrt{f(d)}s + v.$$

The capacity is clearly $\log(1 + f(d)\frac{p}{\sigma_n^2})$, and the power efficiency achieved at low SNR is

$$\eta = \frac{f(d)}{\sigma_n^2}.$$

Assume now that we employ a multi-hop scheme to communicate between A and B where each node relays to its nearest neighbor (Figure 6.3.(b)). Since we have n nodes, the distance to a nearest neighbor will be $d' = O(\frac{1}{\sqrt{n}})d$, and the number of hops will be of $\Omega(\sqrt{n})$. Here each relay will communicate at rate $\log(1 + f(d')\frac{\sigma_r^2}{\sigma_n^2})$, and since the total transmit power is $\Omega(\sqrt{n})\sigma_r^2$, the power efficiency achieved at low SNR will be

$$\eta^{'} = \frac{f(d')}{\sigma_n^2\Omega(\sqrt{n})} = \frac{f(dO(\frac{1}{\sqrt{n}}))}{\sigma_n^2\Omega(\sqrt{n})}.$$

For any reasonable path-loss model $\lim_{d\to 0} f(d) = constant$.[3] Therefore, the power efficiency of the multi-hop system scales like $O(\frac{1}{\sqrt{n}})$ as n increases. This says that for a fixed-size network, increasing the number of hops in fact reduces the power efficiency.

Remark 6.1. *Note that if the size of the domain also increases with n, then d will*

[3]Note that the common power law function used in literature, $f(d) = \frac{1}{d^n} \quad n \geq 2$, does not satisfy this property since this model is only valid for far field approximation.

also increase. In this case, since

$$\frac{\eta'}{\eta} = \frac{f(dO(\frac{1}{\sqrt{n}}))}{f(d)\Omega(\sqrt{n})}$$

depending on the path-loss model $f(.)$ and how *d scales with n, it may be more power efficient to use multi-hop.*

Channel Knowledge and Synchronicity

As mentioned earlier, we have assumed that the nodes have knowledge of their local connections. This is a much more reasonable assumption than the nodes knowing the entire network. However, it does require that the network remain relatively stationary in time so that the local connections can be learned via the transmission of pilot symbols, etc. Furthermore, we assume a synchronous system. In other words, all the transmissions and receptions are synchronized. Later on we will argue that the system performance is not very sensitive to timing errors and lack of perfect synchronicity.

6.3 An Example: Multi-antenna Systems

In order to obtain some insight into how the power efficiency of a sensory or ad-hoc wireless network might scale, it is useful to look at the example of a multi-antenna system. For more details see [108].

Consider an n transmit single receive multi-antenna channel, described by the channel vector

$$H = \left[\begin{array}{cccc} h_1 & h_2 & \ldots & h_n \end{array} \right],$$

where h_i denotes the channel coefficient from the i-th transmitter to the receiver. (Assume that the channel coefficients are zero-mean and unit variance and have fourth order moment κ.) Two cases can be envisioned.

- The channel matrix is *known* to the transmitter: In this case the optimal scheme is for beam-forming. Thus, if each antenna transmits with power p, the power efficiency becomes

$$\eta = \frac{\mathrm{E}\,\log\left(1 + \frac{p}{\sigma_n^2}(\sum_{i=1}^n |h_i|^2)^2\right)}{np}.$$

  This is maximized when $p \to 0$, which yields

$$\begin{aligned} \eta &= \frac{\log e}{n\sigma_n^2}\mathrm{E}\,(\sum_{i=1}^n |h_i|^2)^2 \\ &= \frac{\log e}{n\sigma_n^2}\,(n\kappa + n(n-1)) = \Theta(n). \end{aligned} \tag{6.3}$$

- The channel matrix is *unknown* to the transmitter. In this case, beam-forming cannot be done. However, the capacity is known from [109], and so the power efficiency becomes

$$\eta = \frac{\mathrm{E}\,\log\left(1 + \frac{p}{\sigma_n^2}\sum_{i=1}^n |h_i|^2\right)}{np}.$$

  Looking at the low SNR we have

$$\eta = \frac{\log e}{n\sigma_n^2}\mathrm{E}\,(\sum_{i=1}^n |h_i|^2) = \frac{\log e}{n\sigma_n^2}n = \Theta(1). \tag{6.4}$$

What distinguishes an $n \times 1$ multi-antenna system from an n node sensory network is that the n antenna elements are allowed to cooperate, but the n nodes in a sensory network are not. What the above result says is that when the nodes are allowed to cooperate *and* the nodes know the channel coefficients, the power efficiency scales as $\Theta(n)$. However, even if the nodes are allowed to cooperate, as long as they do not know the channel coefficients, the power efficiency does not improve over $\Theta(1)$. But what about a sensory network where the nodes are not allowed to cooperate but know the local channel coefficients? Moreover, what about ad-hoc networks? These are the questions we shall address.

6.4 Sensory networks

We begin by describing a simple protocol that achieves a power efficiency of $\Theta(\sqrt{n})$ for random sensory and, as we shall see in the next section, with some modification for ad-hoc wireless networks. As mentioned earlier, the protocol assumes synchronous transmission and receptions, as well as local channel knowledge at the nodes.

6.4.1 "Listen and Transmit" Protocol

Consider a random sensory network with n relay nodes and one transmitter/receiver pair. We are interested in a probabilistic bound for the achievable power efficiency in this network, i.e., a bound that with high probability is achievable for a random network in the domain. We begin by explaining the protocol that achieves power efficiency of $\Theta(\sqrt{n})$ for sensory wireless networks. In this, so called "Listen and Transmit," protocol communication is done in two intervals:

(a) **Listen interval:** In this interval the transmitter sends the data and the relay nodes only listen. Relay node i receives:

$$r_i = g_i s + v_i \qquad i = 1, 2, \ldots, n, \tag{6.5}$$

where v_i is $\mathcal{CN}(0, \sigma_n^2)$.

(b) **Transmit interval:** Each node, using its knowledge of the local connections, transmits a scaled version of the signal it has received in the first interval:

$$t_i = \frac{\sigma_r g_i^* h_i^*}{|g_i||h_i|\sqrt{|g_i|^2 p + \sigma_n^2}} r_i \qquad i = 1, 2, \ldots, n. \tag{6.6}$$

The scalar is chosen so that the relay node power is σ_r^2 and so that the signal parts coherently add at the receiver.

This protocol is similar to the protocol proposed in [110]. In [110], the relay nodes transmit the exact signals they have received, scaled to meet the power constraint. In the "Listen and Transmit" protocol, the channel coefficients can be complex. Therefore, the relay nodes change the phase of their received signal appropriately so that the signal parts of the received signal (at the receiver) add up coherently. The received signal at the receiver is:

$$\begin{aligned} y &= \sum_{i=1}^{n} h_i t_i + w \\ &= \underbrace{\sum_{i=1}^{n} \frac{\sigma_r |g_i||h_i|}{\sqrt{|g_i|^2 p + \sigma_n^2}}}_{\alpha} s + \underbrace{\sum_{i=1}^{n} \frac{\sigma_r |h_i| g_i^*}{|g_i|\sqrt{|g_i|^2 p + \sigma_n^2}} v_i + w}_{w'}, \end{aligned} \tag{6.7}$$

where w is $\mathcal{CN}(0, \sigma_n^2)$. From (6.7) it is clear that the signal part from different relay nodes adds up coherently, but the noise part does not. In this sense, the "Listen and Transmit" protocol can be regarded as performing *distributed beamforming.*

6.4.2 Finding a Lower Bound

We break α, defined in (6.7), into $\bar{\alpha}$ and $\tilde{\alpha} = \alpha - \bar{\alpha}$, where $\bar{\alpha} = \mathrm{E}_{\mathrm{f}}[\alpha]$ and $\mathrm{E}_{\mathrm{f}}[\tilde{\alpha}] = 0$. Now, if we rewrite (6.7) as $y = \bar{\alpha}s + \tilde{\alpha}s + w'$, then as it is shown in [111], the capacity of this system can be lower-bounded by the capacity of the AWGN channel $y = \bar{\alpha}s + w''$, where w'' is a Gaussian noise with variance equal to the variance of $\tilde{\alpha}s + w'$ (in this analysis we assume that the receiver is provided with the mean of α). Therefore, the capacity of the system in (6.7) may be lower-bounded by

$$C \geq \frac{1}{2} \log \left(1 + \frac{|\bar{\alpha}|^2 p}{\mathrm{Var}_{\mathrm{f}}[w'] + \mathrm{Var}_{\mathrm{f}}[\tilde{\alpha}] p} \right). \tag{6.8}$$

Note that the $\frac{1}{2}$ in front of the logarithmic term comes from the fact that the transmitter transmits half of the time. By substituting $\bar{\alpha}$, $\mathrm{Var_f}\,[w']$, and $\mathrm{Var_f}\,[\tilde{\alpha}]$ in (6.8) with

$$\begin{aligned}
\bar{\alpha} = \mathrm{E_f}\,[\alpha] &= \sigma_r \sum_{i=1}^{n} \mathrm{E_f}\, \frac{|g_i||h_i|}{\sqrt{|g_i|^2 p + \sigma_n^2}}, \\
\mathrm{Var_f}\,[\tilde{\alpha}] = \mathrm{Var_f}\,[\alpha] &= \sigma_r^2 \sum_{i=1}^{n} \mathrm{E_f}\, \frac{|g_i|^2|h_i|^2}{|g_i|^2 p + \sigma_n^2} - \sigma_r^2 \sum_{i=1}^{n} \big(\mathrm{E_f}\, \frac{|g_i||h_i|}{\sqrt{|g_i|^2 p + \sigma_n^2}}\big)^2, \quad \text{and} \\
\mathrm{Var_f}\,[w'] &= \sigma_n^2 \big(1 + \sigma_r^2 \sum_{i=1}^{n} \mathrm{E_f}\, \frac{|h_i|^2}{|g_i|^2 p + \sigma_n^2}\big),
\end{aligned}$$

and rearranging the terms, we get

$$C \geq \frac{1}{2} \log \left(1 + \frac{\sigma_r^2 p \big(\sum_{i=1}^{n} \mathrm{E_f}\, \frac{|g_i||h_i|}{\sqrt{|g_i|^2 p + \sigma_n^2}} \big)^2}{\sigma_n^2 + \sigma_r^2 \big(\sum_{i=1}^{n} \mathrm{E_f}\, |h_i|^2 - \sum_{i=1}^{n} p \big(\mathrm{E_f}\, \frac{|g_i||h_i|}{\sqrt{|g_i|^2 p + \sigma_n^2}} \big)^2 \big)} \right). \tag{6.9}$$

Define $c_i = \mathrm{E_f}\, \frac{|g_i||h_i|}{\sqrt{|g_i|^2 p + \sigma_n^2}}$ and $b_i = \mathrm{E_f}\, |h_i|^2$. By ignoring the negative term in the denominator we can rewrite (6.9) as

$$C \geq \frac{1}{2} \log \left(1 + \frac{n^2 \sigma_r^2 p \big(\frac{\sum_{i=1}^{n} c_i}{n} \big)^2}{\sigma_n^2 + n \sigma_r^2 \big(\frac{\sum_{i=1}^{n} b_i}{n} \big)} \right). \tag{6.10}$$

The above lower bound holds for every fixed network. For a random network, the capacity, the power efficiency, and the above lower bound are random variables depending on the placement of the nodes in the network. Since the nodes are placed independently and according to the same distribution on the available area, the b_is and c_is are i.i.d random variables for different relay nodes (i.e., different is). Therefore, denoting the transmitter and receiver location by $\{t, r\}$, we have

$$\begin{aligned}
\mathrm{E}_{loc}(\sum_{i=1}^{n} b_i) &= \mathrm{E}_{\{t,r\}} \sum_{i=1}^{n} \mathrm{E}_{\{i|t,r\}}[b_i] = n\mathrm{E}_{loc}[b] \\
\mathrm{Var}_{loc}(\sum_{i=1}^{n} b_i) &= \mathrm{E}_{\{t,r\}} \sum_{i=1}^{n} \mathrm{E}_{\{i|t,r\}}([b_i^2] - [b_i]^2) = n\mathrm{Var}_{loc}[b] \\
\mathrm{E}_{loc}(\sum_{i=1}^{n} c_i) &= \mathrm{E}_{\{t,r\}} \sum_{i=1}^{n} \mathrm{E}_{\{i|t,r\}}[c_i] = n\mathrm{E}_{loc}[c] \\
\mathrm{Var}_{loc}(\sum_{i=1}^{n} c_i) &= \mathrm{E}_{\{t,r\}} \sum_{i=1}^{n} \mathrm{E}_{\{i|t,r\}}([c_i^2] - [c_i]^2) = n\mathrm{Var}_{loc}[c],
\end{aligned}$$

where $c = \mathrm{E}_{\mathrm{f}} \frac{|g||h|}{\sqrt{|g|^2 p + \sigma_n^2}}$ and $b = \mathrm{E}_{\mathrm{f}} |h|^2$ are the random variables depending on the channel coefficients between one random point and two other random points. Now, for any $\epsilon > 0$, using Chebyshev's inequality and the union bound on the probability of the events, we have

$$\Pr\left\{\left|\frac{\sum_{i=1}^{n} c_i}{n} - \mathrm{E}_{loc} c\right| \le \epsilon, \left|\frac{\sum_{i=1}^{n} b_i}{n} - \mathrm{E}_{loc} b\right| \le \epsilon\right\} \ge 1 - \frac{\mathrm{Var}_{loc}[b] + \mathrm{E}_{loc}[c^2]}{n\epsilon^2}. \quad (6.11)$$

The inequality of (6.11) shows that as $n \to \infty$, the quantities $\frac{\sum_{i=1}^{n} c_i}{n}$ and $\frac{\sum_{i=1}^{n} b_i}{n}$ behave like their spatial averages. This implies that with high probability, C in (6.10) is bounded by

$$\frac{1}{2} \log\left(1 + \frac{n^2 \sigma_r^2 p \mathrm{E}_{loc}^2[c]}{\sigma_n^2 + n\sigma_r^2 \mathrm{E}_{loc}[b]}\right).$$

Remark 6.2. *Note that* $\mathrm{E}_{loc}[b]$ *and* $\mathrm{E}_{loc}[c]$ *depend only on the domain* A *on the fading characteristics and on the distribution of the points, and they do not depend on* n*. Thus, for fixed* p *and* σ_r^2*, as* $n \to \infty$ *the lower bound on capacity with high probability behaves like*

$$\frac{1}{2} \log n + O(1).$$

Note that this is the same asymptotic growth obtained for Gaussian relay channels in [91]. Thus, we conclude that the "Listen and Transmit" protocol (i.e., distributed *beamforming) achieves the optimal asymptotic capacity growth. We, of course, are not primarily interested in capacity, but rather in power efficiency.*

Now we will focus on how to optimally allocate the powers (p and σ_r^2 as a function of n) to maximize the power efficiency. As mentioned earlier, $\mathrm{E}_{loc}[b]$ and $\mathrm{Var}_{loc}[b]$ do not depend n. Using the Taylor series expansion of $\mathrm{E}_{loc}[c]$ and $\mathrm{E}_{loc}[c^2]$ in p, we have

$$\begin{aligned} \mathrm{E}_{loc}[c] &= \frac{\mathrm{E}_{loc}\mathrm{E}\,|h||g|}{\sigma_n} - \frac{\mathrm{E}_{loc}\mathrm{E}\,|h||g|^3}{\sigma_n^3}p + o(p) \\ \mathrm{E}_{loc}[c^2] &= \frac{\mathrm{E}_{loc}\mathrm{E}\,|h|^2|g|^2}{\sigma_n^2} - \frac{\mathrm{E}_{loc}\mathrm{E}\,|h|^2|g|^4}{\sigma_n^4}p + o(p). \end{aligned} \tag{6.12}$$

Note that $|h|$ and $|g|$ do not depend on n, so the only dependence of $\mathrm{E}_{loc}[c]$ and $\mathrm{E}_{loc}[c^2]$ on n can be through p. Since the total power consumed in the network is $\frac{1}{2}(p + n\sigma_r^2)$ ($\frac{1}{2}$ comes from the fact that each node is sending only half of the time), the power efficiency is

$$\eta = \frac{2C}{p + n\sigma_r^2}.$$

From (6.11) and (6.10) we can find a probabilistic lower bound for the power efficiency of the network. In other words, for a random placement of the nodes in the domain we have

$$\Pr\left\{\eta = \frac{2C}{n\sigma_r^2 + p} \geq \frac{1}{p + n\sigma_r^2}\log\left(1+\frac{n^2\sigma_r^2 p\big(\mathrm{E}_{loc}[c] - \epsilon\big)^2}{\sigma_n^2 + n\sigma_r^2\big(\mathrm{E}_{loc}[b] + \epsilon\big)}\right)\right\} \geq 1-\frac{\mathrm{Var}_{loc}[b] + \mathrm{E}_{loc}[c^2]}{n\epsilon^2}.$$

By choosing the transmit power, $p = \frac{1}{\sqrt{n}}$, and the relay node power, $\sigma_r^2 = \frac{1}{n\sqrt{n}}$, we have from (6.12) and the above equation that

$$\Pr\left\{\eta \geq \frac{\sqrt{n}}{2}\log\left(1 + \frac{(\beta_1 + o(\frac{1}{\sqrt{n}}) - \epsilon)^2}{\sigma_n^2 + \frac{\mathrm{E}_{loc}[b]+\epsilon}{\sqrt{n}}}\right)\right\} \geq 1 - \frac{\mathrm{Var}_{loc}[b] + \beta_2 + o(\frac{1}{\sqrt{n}})}{n\epsilon^2},$$

where $\beta_1 = \frac{\mathrm{E}_{loc}\mathrm{E}\,|h||g|}{\sigma_n}$ and $\beta_2 = \frac{\mathrm{E}_{loc}\mathrm{E}\,|h|^2|g|^2}{\sigma_n^2}$ are some constants independent of n. Close inspection of the above inequality reveals that the term in the logarithm is of order one. This says that the capacity achieved with the "Listen and Transmit" protocol is of $\Theta(1)$. Moreover, there exists a constant $K_1 \geq 0$ such that

$$\Pr\{\eta \geq K_1\sqrt{n}\} \geq 1 - \frac{\mathrm{Var}_{\,loc}[b] + \beta_2 + o(\frac{1}{\sqrt{n}})}{n\epsilon^2}. \tag{6.13}$$

From the above inequality we can see that for a random placement of the nodes in A, with a high probability that approaches one as the number of nodes increases , we can achieve a power efficiency that grows like $\sqrt{n}$. Also, the rate achieved is of $\Theta(1)$. The choice of transmit and relay node power in this case is $p = n\sigma_r^2 = \frac{1}{\sqrt{n}}$.

6.4.3 Finding an Upper Bound

We can also find an upper bound on the achievable rates using the "Listen and Transmit" protocol. For this, we consider the case where the receiver in (6.7) knows $\{h_i, g_i, v_i\}$ for $i = 1, 2, \ldots, n$. In this case we have

$$C = \sup_{p(s)} \mathrm{I}\,(y; s) \leq \sup_{p(s)} \mathrm{I}\,(s; y|\{h_i, g_i, v_i\}), \tag{6.14}$$

where $\mathrm{I}\,(y; s)$ is the mutual information between y and s. Now, if the receiver knows the channel coefficients and the v_is, then the system in (6.7) becomes an AWGN channel and therefore

$$\sup_{p(s)} \mathrm{I}\,(s; y|\{h_i, g_i\}) = \frac{1}{2}\mathrm{E_f}_{\,\{h_i,g_i\}}\left[\log\left(1 + \frac{|\alpha|^2 p}{\sigma_n^2}\right)\right]. \tag{6.15}$$

The v_is do not contribute to the noise power in the denominator of (6.15) since the receiver has complete knowledge of them and can cancel out their effect. Combining (6.14) and (6.15) and using the convexity of the log function, we may write

$$C \leq \frac{1}{2}\log\left(1 + \frac{\mathrm{E_f}\,|\alpha|^2 p}{\sigma_n^2}\right). \tag{6.16}$$

Substituting the value of α from (6.7) in the above equation gives

$$C \leq \frac{1}{2}\log\left(1+\frac{\sigma_r^2 p}{\sigma_n^2}\mathrm{E_f}\,(\sum_{i=1}^{n}\frac{|g_i||h_i|}{\sqrt{|g_i|^2 p+\sigma_n^2}})^2\right),$$

where again the expectation is taken over the fading of the channels for a fixed placement of the nodes. Without loss of generality we can ignore the $|g_i|^2 p$ term in denominator and rewrite the above equation as

$$C \leq \frac{1}{2}\log\left(1+\frac{\sigma_r^2 p}{\sigma_n^4}\mathrm{E_f}\,(\sum_{i=1}^{n}|g_i||h_i|)^2\right). \tag{6.17}$$

Since, averaged over the fading, the $|h_i||g_i|$s are independent for different is, we have $\mathrm{E_f}\,(\sum_{i=1}^{n}|g_i||h_i|)^2 \leq (\sum_{i=1}^{n}\mathrm{E_f}\,|g_i||h_i|)^2+\sum_{i=1}^{n}\mathrm{E_f}\,|h_i|^2|g_i|^2$, and therefore

$$C \leq \frac{1}{2}\log\left(1+\frac{n^2\sigma_r^2 p}{\sigma_n^4}\left(\frac{\sum_{i=1}^{n}\mathrm{E_f}\,|g_i||h_i|}{n}\right)^2+\frac{n\sigma_r^2 p}{\sigma_n^4}\frac{\sum_{i=1}^{n}\mathrm{E_f}\,|h_i|^2|g_i|^2}{n}\right). \tag{6.18}$$

Given that the location of the transmitter and the receiver is fixed, $\mathrm{E_f}\,|h_i||g_i|$ and $\mathrm{E_f}\,|h_i|^2|g_i|^2$ for all i depend only on the placement of the relay nodes and are i.i.d random variables. Thus, according to the law of large numbers, their average converges to their statistical mean. More specifically, for any $\epsilon > 0$, using Chebyshev's inequality and the union bound on the probability of the events we have,

$$\begin{aligned}&\Pr\left\{\left|\frac{\sum_i \mathrm{E_f}\,|h_i||g_i|}{n}-\kappa\right|\geq\epsilon,\left|\frac{\sum_i \mathrm{E_f}\,|h_i|^2|g_i|^2}{n}-\mu\right|\geq\epsilon\right\}\\&\geq 1-\frac{\mathrm{Var}_{\,loc}\mathrm{E_f}\,|h||g|+\mathrm{Var}_{\,loc}\mathrm{E_f}\,|h|^2|g|^2}{n\epsilon^2},\end{aligned} \tag{6.19}$$

where $\kappa = \mathrm{E}\,|h||g|$ and $\mu = \mathrm{E}\,|h|^2|g|^2$. Combining (6.19) and (6.18) gives

$$\begin{aligned}&\Pr\left\{\eta\leq\frac{1}{p+n\sigma_r^2}\log\left(1+\frac{n\sigma_r^2 p}{\sigma_n^4}\big(n(\kappa+\epsilon)^2+\mu+\epsilon\big)\right)\right\}\\&\geq 1-\frac{\mathrm{Var}_{\,loc}\mathrm{E_f}\,|h||g|+\mathrm{Var}_{\,loc}\mathrm{E_f}\,|h|^2|g|^2}{n\epsilon^2}.\end{aligned} \tag{6.20}$$

It can be easily verified that for the extreme point of the above upper bound (with respect to p and $n\sigma_r^2$), we have $p = n\sigma_r^2$. Therefore,

$$\eta \leq \max_p \left\{ \frac{1}{4p} \log \left(1 + \frac{(n(\kappa+\epsilon)^2 + \mu + \epsilon)p^2}{\sigma_n^4} \right) \right\}. \tag{6.21}$$

By defining $x = \frac{\sqrt{n(\kappa+\epsilon)^2+\mu+\epsilon}}{\sigma_n^2} p$, (6.21) may be written as

$$\eta \leq \frac{\sqrt{n(\kappa+\epsilon)^2 + \mu + \epsilon}}{4\sigma_n^2} \underbrace{\max_x \frac{\log(1+x^2)}{x}}_{<2}. \tag{6.22}$$

Since κ, μ, ϵ, and σ_n^2 do not depend on n, it is clear from the above equation that $\eta \leq O(\sqrt{n})$. Also the maximization over x is uniquely achieved by some constant x in the interval $[0, 2]$. Therefore, from the definition of x, the optimal value of p, and hence $n\sigma_r^2$, is $\Theta(\frac{1}{\sqrt{n}})$. Thus, we have shown that for a random placement of nodes in the domain A, the "Listen and Transmit" protocol with high probability achieves a power efficiency of at most of order $\sqrt{n}$.

6.4.4 Main Result: Sensory Case

In previous sections we found a lower bound on the power efficiency of sensory networks. Combining these bounds together, we have the following theorem.

Theorem 6.3. *Consider a random sensory network with a transmitter/receiver pair and n relay nodes where all the nodes are placed randomly and independently on a domain of fixed area A. Assume that averaged over the fading, the various channels are independent, i.e., for every two different channels c_1, c_2 we have $\mathrm{E_f}[c_1c_2] = \mathrm{E_f}[c_1]\mathrm{E_f}[c_2]$, and the measurement noises are all i.i.d $\mathcal{CN}(0, \sigma_n^2)$. Furthermore, assume that the relay nodes have knowledge of their channels to and from the receiver and the transmitter and that the receiver knows the mean of α in* (6.7)*. Then, with* *high probability the power efficiency of the network is at least* $\Theta(\sqrt{n})$*, i.e., there exists a scheme such that*

$$\Pr\{K_2\sqrt{n} \geq \eta \geq K_1\sqrt{n}\} \geq 1 - \frac{K}{n}, \tag{6.23}$$

where K_1, K_2*, and* K *are constants depending on the domain and the fading characteristics, but not on* n*. Moreover, the listen transmit protocol achieves* $\eta = \Theta(\sqrt{n})$ *with the power allocation* $p = n\sigma_r^2 = \Theta(\frac{1}{\sqrt{n}})$*.*

Remark 6.4. *It was shown that in the "Listen and Transmit" protocol the rate of communication is of order constant. Therefore we are getting the same rate of communication as the case when the transmitter and the receiver communicate in isolation. The difference is that in the former protocol, the total power consumption is of order* $\Theta(\frac{1}{\sqrt{n}})$ *which is* $\sqrt{n}$ *time less than the power consumption in the later case. Thus, we are getting a fixed rate with less power consumption.*

Remark 6.5. *Implicit in the "Listen and Transmit" protocol there is a notion of fairness: nodes in relay mode consume,* n *times less power that the node in transmit mode.*

Remark 6.6. *Comparing the power efficiency achieved in the sensory networks with the power efficiency of multi-antenna systems, we observe that it is better than the power efficiency of a* $n \times 1$ *multi-antenna system with no channel knowledge at the transmit antennas where unlike the sensory case, cooperation between different antennas is allowed. However, as we expected it is worse than the power efficiency of a* $n \times 1$ *multi-antenna system with perfect knowledge at the transmit antennas.*

6.4.5 Discussion on Synchronicity

The key idea in the "Listen and Transmit" protocol is to scale the received signals at the relay stage in such a way that the information-bearing signal parts add up coherently at the receiver. Therefore, the protocol is sensitive to any error in the phase, and hence to synchronicity. In this section we try to make a qualitative analysis of the effect of asynchronicity on the "Listen and Transmit" protocol.

Instead of considering an asynchronous system, we consider the lack of synchronicity by introducing a phase error in the channel knowledge used by the relay nodes. More precisely, we assume that instead of knowing the channel h_i perfectly, the i-th relay node uses $h_i e^{j\delta_i}$ for processing its received signa, where δ_i is the phase error that models the time lag corresponding to the transmission from i-th relay node to the receiver. We assume that the phase errors are i.i.d random variables and independent from the channel coefficients. Furthermore, we assume that $\mathrm{E}\left[e^{j\delta_i}\right]$ is not zero and is equal to some constant $|\lambda| \neq 0$. In other words, we assume that by the aid of the receiver and by using a training sequence, the relay nodes have some estimate of their time lag and therefore the phase error is not distributed uniformly over the unit circle.

In this case, the received signal at the receiver is

$$y = \sum_{i=1}^{n} h_i t_i e^{j\delta_i} + w,$$

where t_i, defined in (6.6), is the transmitted signal in the case of perfect synchronicity. By plugging in t_i from (6.6) and using the same approach as before, we have

$$C \geq \frac{1}{2}\log\left(1 + \frac{n^2\sigma_r^2 p\left|\frac{1}{n}\sum_{i=1}^{n}\mathrm{E}_{\mathrm{f}}\,\frac{|g_i||h_i|}{\sqrt{|g_i|^2 p+\sigma_n^2}}e^{j\delta_i}\right|^2}{\sigma_n^2 + \sigma_r^2\sum_{i=1}^{n}\mathrm{E}_{\mathrm{f}}\,|h_i|^2}\right) \tag{6.24}$$

Note that the lack of synchronicity appears as the phase errors $e^{j\delta_i}$ in the lower bound. Looking at the numerator of the lower bound, since the phase errors are independent of the channels, we can see that as we increase n

$$\frac{1}{n}\sum_{i=1}^{n} \mathrm{E_f}\, \frac{|g_i||h_i|}{\sqrt{|g_i|^2 p+\sigma_n^2}} e^{j\delta_i} \to \lambda \mathrm{E}_{\,loc}[\mathrm{E_f}\, \frac{|g_i||h_i|}{\sqrt{|g_i|^2 p+\sigma_n^2}}] \quad a.s. \quad .$$

In other words, as the number of nodes increases, for any random network, the term in the numerator of (6.24) with high probability is close to its average over the phase error and the location of the points. Therefore, using the same approach as in the previous section, with high probability, the power efficiency is lower-bounded by

$$\sqrt{n}\log(1+K'|\lambda|^2),$$

where K' depends only on the geometry of the domain and the fading characteristics. From the above discussion, we have the following observations:

- As $|\lambda|$ decreases, i.e., as we become more and more uncertain about the phase of the channel, the power efficiency also decreases. For the case where $|\lambda| = 0$, i.e., the case where we have no estimate of the phase, the lower bound on the achieved power efficiency become zero.

- In terms of n, we see that, as long as $|\lambda| \neq 0$, the asymptotics of the lower bound do not change, and we can still achieve a power efficiency of $\Theta(\sqrt{n})$ with the "Listen and Transmit" protocol.

6.5 Ad-hoc Networks

We now turn our attention to ad-hoc networks. The key difference, compared to the sensory networks, is that we now have r simultaneous transmitter/receiver pairs. Therefore, we are interested in the following question. Assume that in isolation, to maintain some fixed communication rate, each transmitter/receiver pair needs to operate at some power. Now, if the r transmitter/receiver pairs are required to communicate simultaneously and are members of a random wireless network with n nodes, how much can the total power consumption in the network be reduced (from the power consumption required in isolation) to maintain the same communication rate between the transmitters and receivers? We remark that since the capacity of an ad-hoc wireless network scales as $\Theta(\sqrt{n})$ [92, 93, 94, 95], to maintain a fixed rate for each transmitter/receiver pair we need to assume that $r \leq \sqrt{n}$. This will be our standing assumption throughout. To answer the question above, we will construct an extension of "Listen and Transmit" protocol developed for sensory networks. As in the sensory case, the main idea is to exploit interference in the network. For ad-hoc networks the power efficiency is defined as the ratio between the sum of the mutual information of different transmitter/receiver pairs and the total power consumption of the network:

$$\eta = \frac{C_{total}}{P_0}.$$

Note that both C_{total} and P_0 are random variables that depend on the placement of the points. We first consider an alternative form of power efficiency, namely $\eta' = \frac{C_{total}}{\mathrm{E}_{loc}[P_0]}$, where the denominator is averaged over all point placements in the network. The reason is that it is easier to establish scaling laws for η'. We then show that similar scaling laws apply to η.

6.5.1 "Listen and Transmit" Protocol

As in the sensory case, the communication in the "Listen and Transmit" protocol is divided into two intervals.

(a) **Listen interval:** Each of the r transmit users transmit the signal s_i, where $\{s_i\}$ are independent random variables. All other nodes listen. Relay node j

receives:

$$r_j = \sum_{i=1}^{r} g_{ij} s_i + v_j \qquad i = 1, 2, \ldots, r \quad j = 1, 2, \ldots, n. \tag{6.25}$$

(b) **Transmit interval:** Each relay node j transmits t_j, a scaled version of what it has previously received:

$$t_j = \sigma_r d_j r_j \qquad j = 1, 2, \ldots, n,$$

where the scalar d_j can depend only on the local knowledge of the channel coefficients at relay node j. Before describing the particular choice of $\{d_j\}$, it is instructive to consider what can be accomplished by having the relay nodes just scale their received signals. To this end, if, for a particular choice of the $\{d_j\}$, we focus on the $r \times r$ channel matrix relating the r transmit signals to the r receive signals, it is clear that the entries of this matrix are linear combinations of the $\{d_j\}$ (see Section 6.5.7 and (6.64) below). Since the channel matrix has r^2 entries, if $n \geq r^2$, then we have enough free parameters in the $\{d_j\}$ to "generically" make the channel matrix diagonal. This totally suppresses the interference and yields r independent channels. Therefore, in principle, a sum-rate of order r is achievable.[4]

The problem with this approach is that it requires complete knowledge of all the channel coefficients at every node of the network (so that each node can solve the system of linear equations required to diagonalize the channel). Since this is not acceptable, we need to introduce a method that only uses local channel

[4]Of course, one should worry about satisfying the power constraints. As we will see later, this scheme is not power efficient.

knowledge, and so we propose the following choice for d_j

$$d_j = \frac{\left(\sum_{i=1}^{r} g_{ij}^* h_{ji}^*\right)}{\sqrt{r\kappa_1\sigma_n^2 + r((r-1)\kappa_2 + \kappa_3)p}}, \tag{6.26}$$

where κ_1, κ_2, and κ_3 are defined as

$$\begin{aligned} \kappa_1 &= \mathrm{E}_{\{j\}}(\mathrm{E}_{\{\gamma|j\}}\mathrm{E}_{\mathrm{f}}\,|g_{\gamma j}|^2)^2 \\ \kappa_2 &= \mathrm{E}_{\{j\}}(\mathrm{E}_{\{\gamma|j\}}\mathrm{E}_{\mathrm{f}}\,|g_{\gamma j}|^2)^3 \\ \kappa_3 &= \mathrm{E}_{\{j\}}(\mathrm{E}_{\{\gamma|j\}}\mathrm{E}_{\mathrm{f}}\,|g_{\gamma j}|^2\mathrm{E}_{\{\gamma|j\}}\mathrm{E}_{\mathrm{f}}\,|g_{\gamma j}|^4), \end{aligned} \tag{6.27}$$

where γ is a location of a random point in the domain. Note that these parameters do not depend on n and r and depend only on the geometry of the domain and the fading characteristics.

With the above choice of d_js, the operation of the relay nodes can be regarded as performing *distributed performing.* It can be further shown that with this choice of d_j, the average transmit power for the relay nodes of the random network is σ_r^2, where the averaging is over both placement of the network as well as channel fading [113]. Since d_j depends only on the local knowledge of the channel coefficients at relay node j, the d_js are identical and independent random variables when the location of the relay nodes is random and the transmitters and the receivers are fixed. We will use this fact later on in our results. Finally, we remark that the above mentioned scheme may be interpreted as follows:

- Each relay node estimates each of the r transmitted signals as

$$\hat{s}_i^j = g_{ij}^* r_j \quad i = 1, 2, \ldots, r \quad j = 1, 2, \ldots, n.$$

  Of course, these are very inaccurate estimates.

- Each node attempts to coherently add its estimate of signal s_i for i-th receiver via multiplication by h_{ji}^* and normalize the sum to power σ_r^2:

$$t_j = \frac{1}{\sqrt{r\kappa_1\sigma_n^2 + r((r-1)\kappa_2 + \kappa_3)p}} \sum_{i=1}^{r} \hat{s}_i^j h_{ij}^*.$$

Note that in both steps of the protocol, we are exploiting interference. Since the wireless medium is a shared medium, each relay node can estimate each of the r transmitted signals. Also, because of the interference, each receiver will receive a summation of the scaled versions of the signals that the relay nodes have sent. So there are n indirect paths for the signal transmitted from transmitter i to receiver i, each passing through one relay node. Each of the relay nodes has transmitted a signal that has a part that adds coherently for receiver i. Therefore, there are n signal parts that add up coherently at receiver i.

The received signal at receiver k is

$$\begin{aligned} y_k &= \sum_{j=1}^{n} h_{jk}t_j + w_k = \sum_{j=1}^{n} \sigma_r h_{jk} d_j r_j + w_k = \sum_{j=1}^{n} \sigma_r h_{jk} d_j (\sum_{i=1}^{r} g_{ij} s_i + v_j) + w_k \\ &= \sigma_r \sum_{i=1}^{r} \underbrace{(\sum_{j=1}^{n} g_{ij} h_{jk} d_j)}_{\alpha_i^k} s_i + \sigma_r \sum_{j=1}^{n} \underbrace{h_{jk} d_j}_{\beta_j^k} v_j + w_k. \end{aligned} \tag{6.28}$$

We should remark that α_i^k for all i is a sum of n independent random variables. Also notice that since the relay nodes are placed independently and the β_j^ks depend only on the channel coefficients between relay node j and the transmitters and the receivers, they are independent for different j's.

6.5.2 Finding Upper and Lower Bounds

By using the same technique as Section 6.4.1, we can find a lower and upper bound for the mutual information between y_k and s_k. Using the results of [111] again, the maximum value of the mutual information $C_k = \sup_{p(s_k)} \mathrm{I}(y_k; s_k)$ can be lower-bounded by the capacity of the AWGN channel with input/output equation:

$$y = \sigma_r \mathrm{E_f}\,[\alpha_k^k] s_k + w_k', \tag{6.29}$$

where w' is a zero mean complex Gaussian noise with variance

$$\begin{aligned} \mathrm{Var_f}\,[w_k'] &= \mathrm{Var_f}\,[\sigma_r(\alpha_k^k - \mathrm{E_f}\,[\alpha_k^k]) s_k + \sigma_r \sum_{i=1, i\neq k}^{r} \alpha_i^k s_i + \sigma_r \sum_{j=1}^{n} \beta_j^k v_j + w_k] \\ &= \mathrm{Var_f}\,[\alpha_k^k]\sigma_r^2 p + (\sum_{i=1, i\neq k}^{r} \mathrm{E_f}\,|\alpha_i^k|^2)\sigma_r^2 p + (\sigma_r^2 \sum_{j=1}^{n} \mathrm{E_f}\,|\beta_j^k|^2 + 1)\sigma_n^2. \end{aligned} \tag{6.30}$$

Therefore, we have

$$C_k = \sup_{p(s_k)} \mathrm{I}(y_k; s_k) \geq \sup_{p(s_k)} \mathrm{I}(y; s_k) = \frac{1}{2}\log\left(1 + \frac{\sigma_r^2 p |\mathrm{E_f}\,[\alpha_k^k]|^2}{\mathrm{Var_f}\,[w_k']}\right). \tag{6.31}$$

Note that the $\frac{1}{2}$ in front of the logarithmic term comes from the fact that the transmitters transmit half of the time. We can obtain an upper bound on C_k, considering the case that the receiver k knows $\{v_j,\, s_i,\, i = 1, \ldots, r,\, i \neq k,\, j = 1, \ldots, n\}$ and all the channel coefficients. For this case we have

$$C_k = \sup_{p(s_k)} \mathrm{I}(y_k; s_k) \leq \sup_{p(s_k)} \mathrm{I}(y_k; s_k | \{v_j, s_i,\, i \neq k, g_{ij}, h_{jk}\}) = \frac{1}{2}\mathrm{E_f}_{\,\{g_{ij}, h_{jk}\}}\left[\log\left(1 + \frac{|\alpha_k^k|^2 \sigma_r^2 p}{\sigma_n^2}\right)\right]. \tag{6.32}$$

Using the convexity of the log function, we may rewrite the above equation as

$$C_k = \sup_{p(s_k)} \mathrm{I}(y_k; s_k) \leq \frac{1}{2}\log\left(1 + \frac{\mathrm{E_f}\,|\alpha_k^k|^2 \sigma_r^2 p}{\sigma_n^2}\right). \tag{6.33}$$

In order to compute the lower and the upper bound in (6.31) and the above equation, first and second order moments of $\{\alpha_i^k\}$ and $\{\beta_j^k\}$ are required. In the following lemma we give probabilistic bounds on $\mathrm{Var_f}\,[w_k']$ and $\mathrm{E_f}\,[\alpha_k^k]$. The proof of this lemma can be found in [113].

Lemma 6.7. *For every domain A of fixed area and every placement of the nodes of the network, there exist constants B_i, $i = 1, \ldots, 8$, κ, κ_1, and κ' ($\kappa' \leq \kappa$) that depend only on the domain A and the fading characteristics such that for every positive ϵ_i, $i = 1, \ldots, 5$ that $\epsilon_4 < \frac{B_7}{\sqrt{r(\kappa_1\sigma_n^2 + rp\kappa)}}$ and every positive ρ and ζ, we have the following relations:*

$$\Pr\left\{|\mathrm{E_f}\left[\alpha_k^k\right]| > n(\frac{B_7}{\sqrt{r(\kappa_1\sigma_n^2 + rp\kappa)}} - \epsilon_4)\right\} \geq 1 - \frac{B_8}{n\epsilon_4^2(\kappa_1\sigma_n^2 + rp\kappa')}$$

$$\Pr\left\{\mathrm{E_f}\left|\alpha_k^k\right|^2 < n(\frac{B_2}{r\kappa' p + \kappa_1\sigma_n^2} + \epsilon_2) + n^2(\frac{B_7}{\sqrt{r(\kappa_1\sigma_n^2 + r\kappa p)}} + \epsilon_4)^2\right\}$$
$$\geq 1 - (\frac{B_8}{n\epsilon_4^2(\kappa_1\sigma_n^2 + rp\kappa')} + \frac{3B_3}{n\epsilon_2^2(\kappa' rp + \kappa_1\sigma_n^2)^2})$$
$$\Pr\left\{\mathrm{Var}\left[w_k^{'}\right] < \sigma_r^2 p(n\epsilon_2 + nr\epsilon_3 + \frac{nr(B_2 + B_5)}{\kappa' rp + \kappa_1\sigma_n^2} + n^\rho r^\zeta \epsilon_5) + \sigma_n^2(1 + n\sigma_r^2(\epsilon_1 + \frac{\kappa}{\kappa rp + \kappa_1\sigma_n^2}))\right\}$$
$$\geq 1 - \frac{1}{(\kappa' rp + \kappa_1\sigma_n^2)^2}\left(\frac{3B_1}{n\epsilon_1^2} + \frac{3B_3}{n\epsilon_2^2} + \frac{3(r-1)B_4}{n\epsilon_3^2} + h(n, r, \epsilon_5)\right), \tag{6.34}$$

where

$$h(n, r, \epsilon_5) = \begin{cases} \frac{rn(\kappa' rp + \kappa_1\sigma_n^2)}{n^\rho r^\zeta \epsilon_5} & \textit{Using Assumptions A,B} \\ \frac{rn(r+n)B_6}{n^{2\rho} r^{2\zeta} \epsilon_5^2} & \textit{Using Assumptions A,B,C.} \end{cases}$$

Using Lemma 1, we can combine (6.31) and (6.33) to get upper and lower bounds for C_k. For this, define

$$C_{lower} = \frac{1}{2}\log\left(1 + \frac{n^2\sigma_r^2 p(\frac{B_7}{\sqrt{r(\kappa_1\sigma_n^2 + rp\kappa)}} - \epsilon_4)^2}{p\sigma_r^2(n\epsilon_2 + nr\epsilon_3 + \frac{nr(B_2+B_5)}{\kappa' rp + \kappa_1\sigma_n^2} + n^\rho r^\zeta\epsilon_5) + \sigma_n^2(1 + n\sigma_r^2(\epsilon_1 + \frac{\kappa}{\kappa rp + \kappa_1\sigma_n^2}))}\right)$$
$$C_{upper} = \frac{1}{2}\log\left(1 + n\frac{\sigma_r^2 p}{\sigma_n^2}(\frac{B_2}{r\kappa' p + \kappa_1\sigma_n^2} + \epsilon_2) + n^2\frac{\sigma_r^2 p}{\sigma_n^2}(\frac{B_7}{\sqrt{r(\kappa_1\sigma_n^2 + r\kappa p)}} + \epsilon_4)^2\right). \tag{6.35}$$

then we have the following relations as probabilistic lower and upper bounds for C_k:

$$\begin{aligned}\Pr\{C_k \geq C_{lower}\} &\geq 1 - \frac{\left(\frac{3B_1}{n\epsilon_1^2} + \frac{3B_3}{n\epsilon_2^2} + \frac{3(r-1)B_4}{n\epsilon_3^2} + \frac{B_8(\kappa' rp+\kappa_1\sigma_n^2)}{n\epsilon_4^2} + h(n,r,\epsilon_5)\right)}{(\kappa' rp + \kappa_1\sigma_n^2)^2} \\ \Pr\{C_k \leq C_{upper}\} &\geq 1 - (\frac{B_8}{n\epsilon_4^2(\kappa_1\sigma_n^2 + rp\kappa')} + \frac{3B_3}{n\epsilon_2^2(\kappa' rp + \kappa_1\sigma_n^2)^2}).\end{aligned} \quad (6.36)$$

In the "Listen and Transmit" protocol, since there are r transmitters and n relay nodes and all the nodes are transmitting half of the time, the average total power consumption is $\frac{1}{2}(rp + n\sigma_r^2)$. The total capacity of the network is $C_{total} = \sum_{k=1}^{r} C_k$. Therefore, the power efficiency of the network is

$$\eta' = \frac{2C_{total}}{rp + n\sigma_r^2} = \frac{2\sum_{k=1}^{n} C_k}{rp + n\sigma_r^2}. \quad (6.37)$$

Remark 6.8. *As mentioned earlier in Lemma 1, the constants B_i and κ, κ_1, and κ' do not depend on n and r. Now, if we fix σ_r^2, p, and $\epsilon_i \quad i = 1, 2, 5$, and set $\epsilon_3 = \frac{1}{r}$, $\epsilon_4 = \frac{B_7}{2\sqrt{r(\kappa_1\sigma_n^2 + rp\kappa)}}$ and $\zeta = 0$ in (6.35) and (6.36), then the total capacity achieved by the "Listen and Transmit" protocol is bounded probabilistically as*

$$\Pr\{C_{total} \geq K_1' r \log(1 + \frac{n^2}{r^2(n + n^\rho)})\} > 1 - \frac{rK_2'}{r^2}(\frac{r^3}{n} + \frac{r}{n^{2(\rho-1)}}),$$

where K_1' and K_2' are some constants and we have considered assumptions A, B, and C in Section 6.2.2. Therefore, by setting $\rho = 1 + \theta$, $\theta > 0$, we have

$$\Pr\{C_{total} \geq K_1' r \log(1 + \frac{n^{1-\theta}}{r^2})\} \geq 1 - K_2'(\frac{r^2}{n} + \frac{1}{n^{2\theta}}).$$

Now note that the maximum of the bound is achieved for $r = \Theta(n^{\frac{1-\theta}{2}})$, and in that case we have

$$\Pr\{C_{total} \geq K_1' n^{\frac{1-\theta}{2}}\} \geq 1 - \frac{K_2'}{n^\theta} \quad \forall\, \theta > 0. \quad (6.38)$$

From (6.38) we see that, with high probability, by using the "Listen and Transmit" protocol we can get arbitrarily close to the $\Theta(\sqrt{n})$ *result of [92]. This result is interesting since we are only using the local knowledge of the channel coefficients at the relay nodes, and the protocol is very simple (it is double-hop and requires no routing).*

6.5.3 Power Allocation

We will now focus on how to optimally allocate the transmit and relay node powers (i.e., p and σ_r^2 as functions of r and n) to maximize the power efficiency. Define $\eta'_{lower} = \frac{2rC_{lower}}{rp+n\sigma_r^2}$ and $\eta'_{upper} = \frac{2rC_{upper}}{rp+n\sigma_r^2}$. By using union bound on the probability of events, we get the following probabilistic lower and upper bounds for the power efficiency of the network using the "Listen and Transmit" protocol:

$$\begin{aligned}
&\Pr\{\eta'_{lower} \leq \eta\} \geq \Pr\{C_{lower} \leq C_k \quad k=1,\ldots,r\} \geq 1 - r\Pr\{C_k < C_{lower}\} \\
&\Pr\{\eta'_{upper} \geq \eta\} \geq \Pr\{C_{upper} \geq C_k \quad k=1,\ldots,r\} \geq 1 - r\Pr\{C_k > C_{upper}\}.
\end{aligned} \tag{6.39}$$

We will consider the lower bound first. We try to choose the values for p, σ_r^2, ζ, and ρ so that with high probability we can achieve a power efficiency that scales with the number of nodes in the network. For this goal we take ϵ_1,ϵ_2,ϵ_3, and ϵ_5 all to be equal to a positive constant denoted by ϵ. We also choose $\epsilon_4 = \frac{B_7}{2\sqrt{r(\kappa_1\sigma_n^2+r\kappa p)}}$. We further consider the network operating in the low SNR regime so that rp is at most constant (in terms of how it scales with n). Later on, when we are looking at the upper bound, we will show that the optimal operating point for this protocol is indeed when rp is of $O(1)$. Using these assumptions in (6.35) and adding $K_3 rp$ to the denominator, we have the following lower bound for the power efficiency η'

$$\begin{aligned}
&\eta'_{lower} = \frac{r}{rp+n\sigma_r^2}\log\left(1 + \frac{K_1\frac{n^2}{r}\sigma_r^2 p}{p\sigma_r^2(n+nrK_2+n^\rho r^\zeta)\epsilon + \sigma_n^2 + K_3(n\sigma_r^2+rp)}\right). \\
&\Pr\{\eta'_{lower} \leq \eta\} \geq 1 - r\left(\frac{rK_4}{n\epsilon^2} + g(n,r,\epsilon)\right).
\end{aligned} \tag{6.40}$$

K_i, $i = 1, \ldots, 5$ are constants and do not depend on r or n. $g(n, r, \epsilon)$ is also derived from $h(n, r, \epsilon)$ after applying the simplifications:

$$g(n, r, \epsilon_5) = \begin{cases} \frac{K_5 rn}{n^\rho r^\zeta \epsilon} & \text{Using Assumptions A,B} \\ \frac{rn(r+n)B_6}{n^{2\rho} r^{2\zeta} \epsilon^2} & \text{Using Assumptions A,B,C.} \end{cases} \tag{6.41}$$

Looking at (6.40), the following conditions are necessary in order to have $\Pr\{\eta'_{lower} \leq \eta'\} \to 1$ for large n and r

$$\frac{r^2}{n} \to 0 \qquad g(n, r, \epsilon) r \to 0. \tag{6.42}$$

Therefore, from the above equation it is clear that this analysis is valid for the case where $r = O(\sqrt{n})$. ζ and ρ should be chosen so that the second condition in (6.42) is satisfied. These two parameters determine the rate of convergence in probability. By looking at (6.41) we observe that the second condition in (6.42) implies that $\frac{1}{n^{\rho-1} r^{\zeta-1}} \to 0$ an n grows. In this stage we maximize the power efficiency with respect to the total transmit power, rp, and total relay power, $n\sigma_r^2$, and subject to the constraints in (6.42). It can be easily verified by taking partial derivatives with respect to rp and $n\sigma_r^2$ that the expression is maximized for $rp = n\sigma_r^2 = x$. Hence, we can write the maximization problem as

$$\eta^\star_{lower} = \max_{rp, n\sigma_r^2} \eta'_{lower} = \max_{x \geq 0} \frac{r}{2x} \log\left(1 + \frac{K_1 \frac{n}{r^2} x^2}{\sigma_n^2 + 2K_3 x + (n + nrK_2 + n^\rho r^\zeta)\epsilon x^2}\right). \tag{6.43}$$

Let $x = \Theta(n^\alpha)$, $\alpha \leq 0$, and $r = \Theta(n^\nu)$, $0 \leq \nu \leq 1/2$. Using the fact that $\alpha \leq 0$ and $\frac{1}{n^{\rho-1} r^{\zeta-1}} \to 0$ in (6.43), we can write

$$\eta^\star_{lower} = \Theta\left(\max_\alpha n^{\nu-\alpha} \log\left(1 + \frac{n^{1+2\alpha}}{n^{2\nu}(\sigma_n^2 + n^{\rho+2\alpha+\nu(\zeta-1)-1})}\right)\right). \tag{6.44}$$

With the following constraints:

$$\alpha \leq 0, \nu \leq \frac{1}{2},$$
$$\begin{cases} \rho - 1 + \nu(\zeta - 2) > 0 & \text{If using assumptions } \mathsf{A},\ \mathsf{B} \\ \rho - 1 + \nu(\zeta - 1) > 0 & \text{If using assumptions } \mathsf{A},\ \mathsf{B},\ \mathsf{C}, \end{cases} \tag{6.45}$$

where the last two constraints are consequences of (6.42).

Consider that we use assumptions A, B, and C. Later on, we analyze the performance of the protocol when only assumptions A and B are used. Set $\rho - 1 + \nu(\zeta - 1)$ fixed and equal to μ. In this case, the rate of convergence in the probability expression of (6.40) is $\min\{1 - 2\nu, \mu\}$. Now we are interested in the maximum achievable power efficiency for a fixed μ. We consider the following cases:

(a) $\mu + 2\alpha \leq 0$: In this case we can see that $\rho + 2\alpha + \nu(\zeta - 1) - 1 \leq 0$. Therefore, the noise power is dominant to the interference in (6.44), and we can simplify the expression as

$$\eta^{\star}_{lower} = \Theta\left(n^{\nu-\alpha} \log(1 + n^{1-2(\nu-\alpha)})\right) = \Theta\left(n^{\beta} \log(1 + n^{1-2\beta})\right), \tag{6.46}$$

where β is defined as $\beta = \nu - \alpha$. Note that we have $\mu + 2\nu \leq 2\beta$. Now we consider the following cases:

- $\beta \geq \frac{1}{2}$: In this case the power of n in the log function in (6.46) is negative, so it is of $\Theta(n^{1-2\beta})$. Therefore we have

$$\eta^{\star}_{lower} = \Theta(n^{1-\beta}).$$

The total rate of transmission R_{sum} is of $\Theta(n^{\nu+1-2\beta})$ in this case. The maximum order of power efficiency is achieved when β takes its smallest possible value, i.e., $\beta = \max\{\frac{1}{2}, \frac{\mu+2\nu}{2}\}$. For this case, the maximum achievable power efficiency and the total rate of transmission R_{sum} are respectively

$$\eta^{\star}_{lower} = \Theta(n^{1-\max\{\frac{1}{2},\frac{\mu+2\nu}{2}\}}) \quad \text{and}$$
$$R_{sum} = \Theta(n^{\min\{\nu,1-(\nu+\mu)\}}).$$

The transmit power and the relay node power for achieving the maximum power efficiency are

$$rp = n\sigma_r^2 = \Theta(n^{\alpha}) = \Theta(n^{\nu-\beta}) = \Theta(n^{\min\{\frac{2\nu-1}{2},\frac{-\mu}{2}\}}) = \Theta(\min\{\frac{r}{\sqrt{n}},\frac{1}{\sqrt{n^{\mu}}}\}). \tag{6.47}$$

Therefore, with the choice of the transmit and relay node power as above, we have

$$\Pr\{\eta' = \Theta(n^{1-\max\{\frac{1}{2},\frac{\mu+2\nu}{2}\}})\} \geq 1 - \Theta(\frac{1}{n^{\min\{\mu,1-2\nu\}}}). \tag{6.48}$$

- $\beta < \frac{1}{2}$: In this case, (6.46) can be rewritten as

$$\eta^{\star}_{lower} = \Theta(n^{\beta}\log n^{1-2\beta}) = (1-2\beta)\Theta(n^{\beta}\log n).$$

Now, since $\log n$ grows slower than any polynomial function in n, and β is strictly less than $\frac{1}{2}$, the maximum achievable power efficiency in this case cannot be better than the previous case, and thus operating the network in this region is not favorable.

(b) $\mu+2\alpha \geq 0$: For this regime the interference will be the dominant term in (6.44), and therefore we have

$$\eta^{\star}_{lower} = \Theta\left(n^{\nu-\alpha}\log(1+n^{1-(2\nu+\mu)})\right)$$

As we can see from the above equation, the power efficiency is maximized when α takes is greatest possible value $\alpha = \frac{-\mu}{2}$. In this case, the power efficiency of the network is

$$\eta^{\star}_{lower} = \Theta\left(n^{\frac{2\nu+\mu}{2}} \log(1 + n^{1-(3\nu+\mu)})\right). \tag{6.49}$$

It can be easily checked that the maximum power efficiency achieved in this region is always less than or equal to the case where $\mu + 2\alpha \leq 0$.

The discussion above gives the probabilistic lower bound of (6.48) for power efficiency when assumptions A,B, and Cfrom Section 6.2.2 can be used. If only assumptions Aand Bcan be used, then applying the same technique as above, we can easily check that the power efficiency is maximized when the network operates in the noise-dominant regime (i.e., $\mu+\nu+2\alpha \leq 0$) rather than interference-dominant regime (i.e., $\mu + \nu + 2\alpha \geq 0$). Also, similar to the previous discussion, $\beta \triangleq \nu - \alpha$ should be greater than equal to $\frac{1}{2}$.

In this region $\rho + 2\alpha + \nu(\zeta - 1) - 1 \leq 0$, and we are in the noise dominant regime. Hence, we can simplify (6.44) to

$$\eta^{\star}_{lower} = \Theta\left(n^{\nu-\alpha} \log(1 + n^{1-2(\nu-\alpha)})\right). \tag{6.50}$$

Therefore, we have $\mu + 3\nu \leq 2\beta$. Also, since $\beta \geq \frac{1}{2}$, the power of n in the log(.) function in (6.50) is negative, so it is of $\Theta(n^{1-2\beta})$. Therefore, we have

$$\eta^{\star}_{lower} = \Theta(n^{1-\beta}) \quad \text{and}$$
$$R_{sum} = \Theta(n^{\nu+1-2\beta})$$

The best achievable power efficiency is for the case when β takes its smallest possible value. In this case, $\beta = \max\{\frac{1}{2}, \frac{\mu+3\nu}{2}\}$ and the maximum achievable power efficiency and the total rate of transmission R_{sum} are respectively

$$\eta^\star_{lower} = \Theta(n^{1-\max\{\frac{1}{2},\frac{\mu+3\nu}{2}\}})$$
$$R_{sum} = \Theta(n^{\min\{\nu,1-(2\nu+\mu)\}}).$$

The transmit power and the relay node power for achieving the maximum power efficiency are

$$rp = n\sigma_r^2 = \Theta(n^{\min\{\frac{2\nu-1}{2},\frac{-\nu-\mu}{2}\}}) = \Theta(\min\{\frac{r}{\sqrt{n}},\frac{1}{\sqrt{n^\mu r}}\}). \quad (6.51)$$

With this choice of transmit and relay node power we have the following probabilistic lower bound on the power efficiency (using assumptions A and B only):

$$\Pr\{\eta = \Theta(n^{1-\max\{\frac{1}{2},\frac{\mu+3\nu}{2}\}})\} \geq 1 - \Theta(\frac{1}{n^{\min\{\mu,1-2\nu\}}}). \quad (6.52)$$

6.5.4 Main Result: Ad-hoc Case

The analysis in the previous section shows the following result:

Theorem 6.9. *Consider an n node random ad-hoc network where the nodes are placed randomly and independently on a domain of fixed area where, averaged over the fading, the various channels are independent, i.e., for every two different channels c_1, c_2 we have $\mathrm{E_f}[c_1c_2] = \mathrm{E_f}[c_1].\mathrm{E_f}[c_2]$. Furthermore, assume conditions A, B , and C given in Section 6.2.2 and that at any given time there are $r = O(n^\nu)$ $\nu \leq \frac{1}{2}$ transmit/receive pairs. Also, the measurement noises are all i.i.d $\mathcal{CN}(0,\sigma_n^2)$. If we denote the power efficiency of the network by η' (i.e, $\eta' = \frac{C_{total}}{\mathrm{E}_{loc}[P_0]}$), then for every $\mu > 0$*

$$\Pr\{\eta' \geq K_1(n^{1-\max\{\frac{1}{2},\frac{\mu+2\nu}{2}\}})\} \geq 1 - \frac{K_2}{n^{\min\{1-2\nu,\mu\}}}, \quad (6.53)$$

where K_1 and K_2 are independent of n and r but depend on the domain and the fading *characteristics. Moreover, the listen transmit protocol achieves this lower bound. The transmit and the relay node powers that achieve this power efficiency are given in (6.47).*

The following corollary is an immediate consequence of Theorem 6.9 by setting $\mu = 1 - 2\nu$.

Corollary 6.10. *Consider the network model described in Theorem 6.9. If the number of transmitter/receiver pairs in the network is of* $O(n^{\frac{1-\epsilon}{2}})$, *where* $\epsilon > 0$, *then we have*

$$\Pr\{\eta \geq K_3\sqrt{n}\} \geq 1 - \frac{K_4}{n^{\epsilon}}, \tag{6.54}$$

where K_3 *and* K_4 *are independent of* n *and* r *but depend on the domain and the fading characteristics. Moreover, by choosing the transmit and relay node powers as* $rp = n\sigma_r^2 = \Theta(\frac{r}{\sqrt{n}})$, *the listen and transmit protocol achieves this lower bound.*

Remark 6.11. *Note that from (6.54), we can see that there is a trade-off between the number of transmitter/receiver pairs* r *and the rate of convergence. As we increase* r *from* $\Theta(1)$ *to* $\Theta(\sqrt{n})$, *the convergence slows down.*

For the case, when only assumptions A and B are used, we have the following result from (6.51) and (6.52):

Theorem 6.12. *Consider an* n *node random ad-hoc network in a domain of fixed area where, averaged over the fading, the various channels are independent. Furthermore, assume conditions A, B given in Section 6.2.2 and that at any given time there are* $r = O(n^{\nu})$ $\nu \leq \frac{1}{2}$ *transmit/receive pairs. Also, the measurement noises are all i.i.d* $\mathcal{CN}(0, \sigma_n^2)$. *If we denote the power efficiency of the network by* η' *(i.e.,* $\eta' = \frac{C_{total}}{\mathrm{E}_{loc}[P_0]}$*), then for every* $\mu > 0$

$$\Pr\{\eta' \geq K_5\, n^{1-\max\{\frac{1}{2},\frac{\mu+3\nu}{2}\}}\} \geq 1 - \frac{K_6}{n^{\min\{1-2\nu,\mu\}}}$$

where K_5 and K_6 are independent of n and r and depend on the domain and fading characteristics. Moreover, the listen transmit protocol achieves this lower bound. The transmit and relay node powers achieving this power efficiency are given in (6.51).

By considering the case where $\mu + 3\nu = 1$, we have the following corollary:

Corollary 6.13. *Consider the network described in Theorem 6.12. If the number of transmit/receive pairs in the network is of $O(n^{\frac{1-\epsilon}{3}})$, where $\epsilon > 0$, then we have*

$$\Pr\{\eta' \geq K_7\sqrt{n}\} \geq 1 - \frac{K_8}{n^\epsilon},$$

where K_7 and K_8 only depend on the domain and fading characteristics. Moreover, by choosing the transmit and relay node powers as $rp = n\sigma_r^2 = \Theta(\frac{r}{\sqrt{n}})$, the listen and transmit protocol achieves this lower bound.

Corollary 6.13 implies that the maximum number of transmit/receive pairs that the network can support with power efficiency of $\Theta(\sqrt{n})$ is $\Theta(n^{\frac{1}{3}})$. On the other hand, considering the maximum power efficiency of the network with the number of transmit/receive pairs up to $n^{\frac{1}{2}}$, one can write the following corollary by setting μ equal to $1 - 2\nu$ in Theorem 6.9. We should also remark that comparing Corollaries 6.10 and 6.13, we see that the effect of assumption C is on the number of simultaneous transmitter/receiver pairs that can be in the network.

Corollary 6.14. *Consider again the network described in Theorem 6.12. If the number of transmit/receive pairs in the network is of $\Theta(n^{\frac{1-\epsilon}{2}})$, where $0 < \epsilon < 1$, then we have*

$$\Pr\{\eta' \geq Kn^{(\frac{1+\epsilon}{4})}\} \geq 1 - \frac{K'}{n^\epsilon},$$

where K, K' are some constants. Therefore, in this case if the number of transmit/receive pairs is near to $\sqrt{n}$, we can achieve a power efficiency that scales like $n^{\frac{1}{4}}$.

The previous discussion shows that:

- If the number of the transmitter/receiver pairs is less than $\sqrt{n}$, it was shown that a power efficiency that scales with the number of nodes, n, is achievable. The rate per transmitter/receiver pair in this case is of order constant. If we increase the number of simultaneous transmissions to more than $\sqrt{n}$, we can still achieve power efficiency of $\sqrt{n}$ using time-sharing and "Listen and Transmit" protocol together. In this case, at each time instant $\sqrt{n}$ of the transmitters transmit and all the others act as relay nodes. However, in this case, the rate per transmitter/receiver will not be of order constant, but of order $\frac{1}{\sqrt{n}}$. This is in agreement with the result of [92, 93, 94] in that achieving a constant rate per node in this case would require a total sum-capacity larger than $\Theta(\sqrt{n})$, which is not possible.

- There is a notion of fairness implicit in the protocol in the sense that nodes in the relay mode consume $\frac{r}{n}$-th the power of the nodes in the transmit mode.

- For the case where $r = \Theta(n^{\frac{1-\epsilon}{2}})$ (or $r = \Theta(n^{\frac{1-\epsilon}{3}})$, if we do not have assumption C in Section 6.2.2, the optimal choice of the transmit power and relay power is $p = \Theta(\frac{1}{\sqrt{n}})$, $\sigma_r^2 = \Theta(\frac{1}{n})$. The total power consumption is $\Theta(\frac{r}{\sqrt{n}})$, and the total rate is $\Theta(r)$.

- In the case of ad-hoc networks, by using the "Listen and Transmit" protocol, we can see that the we are keeping the rate of transmission for each transmitter/receiver fixed and of order $\Theta(1)$, but the total power consumption decreases as the number of nodes grows larger, as long as $r = O(n^{\frac{1-\epsilon}{2}})$ for some positive ϵ. If we do not have assumption C on the channel coefficients, we still have this property for $r = O(n^{\frac{1-\epsilon}{3}})$ for some positive ϵ.

We should mention that with high probability we cannot get a better power efficiency for ad-hoc networks with this protocol. We can show this by using (6.35), (6.36), and (6.39) to find a probabilistic upper bound. With an argument like the one for the lower bound or the one in [105], we can show that with high probability the maximum achievable power efficiency with this protocol is $O(\sqrt{n})$.

6.5.5 A Further Result

As mentioned earlier, the power efficiency that was considered up to now was defined as the ratio between the sum rate capacity for a specific placement of the nodes of the network and the average of the power consumption over all possible point placements of the network. In other words, for a specific placement of the nodes with sum rate capacity of C_{total} and power consumption of P_0 we defined power efficiency as $\eta' = \frac{C_{total}}{\mathrm{E}_{loc}[P_0]}$. An alternative to this definition is to consider the ratio between the rate and power consumption for a specific network, i.e., $\eta = \frac{C_{total}}{P_0}$ as the power efficiency. In this case, the power efficiency η is a random variable depending on the placement of the nodes. However, because of the law of large numbers, as the size of the network increases, η will be close to its average, and we observe the same behavior as η' for the power efficiency. In order to state this formally, we will need the following lemma. We should remark that in proving this lemma one only needs assumptions A and B of Section 6.2.2. This lemma gives a bound on the power consumption at the relay stage.

Lemma 6.15. *Consider an n node ad-hoc network with assumptions provided in Theorem 6.9; then for any specific placement of the nodes in the network, the total power consumption at relay nodes P_r can be bounded as*

$$\Pr\left\{|P_r - n\sigma_r^2| < \frac{n^{1+\gamma}\sigma_r^2}{\sqrt{r}}(\sigma_n^2 + rp)\right\} \geq 1 - \frac{K'}{n^{2\gamma}(\kappa_1\sigma_n^2 + r\kappa p)^2}, \tag{6.55}$$

where γ is any positive number, and K', κ, κ_1 are constant independent of n and r.

Using this lemma and the fact that the network operates at low SNR regime, i.e.,

$rp << 1$, we can bound the total power consumption of the network $P_0 = rp + P_r$ as follows

$$\Pr\{P_0 \leq (1 + \frac{n^\gamma}{\sqrt{r}})(rp + n\sigma_r^2)\} \geq 1 - \frac{K}{n^{2\gamma}}.$$

Therefore, for $\eta = \frac{R_{sum}}{P_0}$ we have

$$\Pr\{(1 + \frac{n^\gamma}{\sqrt{r}})^{-1}\eta' \leq \eta\} \geq 1 - \frac{K}{n^{2\gamma}}. \tag{6.56}$$

One can combine this relation with the results on the power efficiency η' to get new bounds on η. The following theorems are immediate consequences of (6.56), Theorem 6.9 and Theorem 6.12.

Theorem 6.16. *Consider an n node random ad-hoc network where the nodes are placed randomly and independently on a domain of fixed area where, averaged over the fading, the various channels are independent, i.e., for every two different channels c_1, c_2 we have $\mathrm{E_f}[c_1 c_2] = \mathrm{E_f}[c_1].\mathrm{E_f}[c_2]$. Furthermore, assume conditions A, B and, C given in Section 6.2.2 and that at any given time there are $r = O(n^\nu)$ $\nu \leq \frac{1}{2}$ transmit/receive pairs. Also, the measurement noises are all i.i.d $\mathcal{CN}(0, \sigma_n^2)$. For a specific placement of the nodes of the random network, let C_{total} be the total rate of communication and P_0 be the total power consumption in the network. Then, for every $\mu, \gamma > 0$, the power efficiency of the network defined as $\eta = \frac{C_{total}}{P_0}$ satisfies*

$$\Pr\{\eta \geq K_1 \frac{(n^{1-\max\{\frac{1}{2}, \frac{\mu+2\nu}{2}\}})}{1 + \frac{n^\gamma}{\sqrt{r}}}\} \geq 1 - \frac{K_2}{n^{\min\{1-2\nu, \mu, 2\gamma\}}}, \tag{6.57}$$

where K_1 and K_2 are independent of n and r but depend on the domain and the fading characteristics. Moreover, the listen transmit protocol achieves this lower bound.

By setting $\gamma = \frac{\nu}{2}$ and $\mu = 1 - 2\nu$, we have the following corollary.

Corollary 6.17. *For the network described in Theorem 6.16, we have*

$$\Pr\{\eta \geq K_1\sqrt{n}\} \geq 1 - \frac{K_2}{n^{\min\{1-2\nu,\nu\}}}. \tag{6.58}$$

Therefore, it becomes clear that by considering the power efficiency as the ratio between the sum-rate of transmission and the total power consumption for specific placement of the nodes in the network, we still have similar scaling behavior.

Remark 6.18. *We should remark that the rate of convergence obtained for the probability of the event in (6.58) is not tight (for small r). One expects that as r, the number of pairs requesting service from the network, decreases, the rate of convergence improves (for instance, we can see from Theorem 6.3 that for sensory networks in which $r = 1$, the rate of convergence is proportional to $\frac{1}{n}$). Looking at (6.58), we observe that as r decreases to constant, the convergence slows down. This is an artifact of our approach in bounding η.*

If we use only assumptions A and B in Section 6.2.2, then we can write the following theorem using (6.56) and Theorem 6.12.

Theorem 6.19. *Consider an n node random ad-hoc network where the nodes are placed randomly and independently on a domain of fixed area where, averaged over the fading, the various channels are independent, i.e., for every two different channels c_1, c_2 we have $\mathrm{E_f}[c_1c_2] = \mathrm{E_f}[c_1].\mathrm{E_f}[c_2]$. Furthermore, assume conditions A, B given in Section 6.2.2 and that at any given time there are $r = O(n^\nu) \quad \nu \leq \frac{1}{2}$ transmit/receive pairs. Also, the measurement noises are all i.i.d $\mathcal{CN}(0, \sigma_n^2)$. For a specific placement of the nodes of the random network, let C_{total} be the total rate of communication and P_0 be the total power consumption in the network. Then, for every $\mu, \gamma > 0$, the power efficiency of the network defined as $\eta = \frac{C_{total}}{P_0}$ satisfies*

$$\Pr\{\eta \geq K_5\frac{(n^{1-\max\{\frac{1}{2},\frac{\mu+3\nu}{2}\}})}{1+\frac{n^\gamma}{\sqrt{r}}}\} \geq 1 - \frac{K_6}{n^{\min\{1-2\nu,\mu,2\gamma\}}}, \tag{6.59}$$

where K_5 and K_6 are independent of n and r but depend on the domain and the fading characteristics. Moreover, the listen transmit protocol achieves this lower bound.

By setting $\gamma = \frac{\nu}{2}$ and $\mu = 1 - 3\nu$, we have the following corollary:

Corollary 6.20. *Consider the network model described in Theorem 6.19. Then, for $r = O(n^{\nu}) \quad \nu \leq \frac{1}{3}$ we have*

$$\Pr\left\{\eta \geq K_1\sqrt{n}\right\} \geq 1 - \frac{K_2}{n^{\min\{1-3\nu,\nu\}}}. \tag{6.60}$$

6.5.6 Discussion on Synchronicity

Similar to the sensory case, the key idea of the "Listen and Transmit" protocol used for ad-hoc networks is the coherent and synchronous reception of the signals. Therefore, the protocol is sensitive to synchronicity. In this section we discuss the effect of asynchronocity on our protocol.

Like sensory networks, instead of considering an asynchronous system, we consider the lack of synchronicity by introducing a phase error in the channel knowledge used by the relay nodes. More precisely, we assume that instead of knowing the channel h_{ji} perfectly, the j-th relay node uses $h_{ji}e^{j\delta_{ji}}$ for processing its received signal. δ_{ji} is the phase error that models the time lag, corresponding to the transmission from j-th relay node to the i receiver. We assume that the phase errors are i.i.d random variables and are independent from channel coefficients. Furthermore, we assume that $\mathrm{E}\left[e^{j\delta_{li}}\right]$ is not zero and is equal to some constant λ. In other words, we assume that by the aid of the receivers and by a using training sequence, the relay nodes have some estimate of their time lag and therefore the phase error is not distributed uniformly over the unit circle.

In this case, the scalar $\hat{d}_k$ used by the k-th relay node is proportional to

$$\sum_{l=1}^{r} g_{lk}^{*} h_{kl}^{*} e^{-j\delta_{lk}}.$$

Using these $\hat{d}_k$s, we can find the new α_k^is and β_k^is (6.28) in terms of δ_{li}. Following the lines of Section 6.5.1 and Section 6.5.2, it can be verified that we will still have the same asymptotic behavior for power efficiency in terms of n (i.e., the asymptotic behavior of the achieved power efficiency is still like $\sqrt{n}$), but the constants appearing in the relations will now depend on λ as well. Therefore, similar to the sensory case as $|\lambda|$ decreases, the power efficiency will also decrease and, finally,for the limiting case of $|\lambda| = 0$, the lower bound on the achieved power efficiency also becomes zero.

6.5.7 Complete Knowledge of the Channel

In the "Listen and Transmit" protocol we assumed that relay nodes have only local knowledge of the channels, i.e., they only know their connections to the transmitter and receiver nodes. We addressed another scenario in the previous sections, where the nodes have complete knowledge of all the channel coefficients and try to diagonalize the channel matrix between the transmitters and their corresponding receivers. In this section we analyze the effect of perfect knowledge of the channel on the power efficiency achieved by diagonalizing the channel matrix.

In this section we make an additional assumption that the channel coefficients are independent complex random variables with zero mean and unit variance. Using

(6.25) and (6.28), we can describe our protocol by the following matrix relations

$$\begin{aligned}
\mathbf{r} &= \mathbf{sG} + \mathbf{v} \\
\mathbf{t} &= \mathbf{rD} \\
\mathbf{y} &= \mathbf{sGDH} + \mathbf{vDH} + \mathbf{w} \\
p\,\mathrm{tr}\left(\mathbf{D}^*\mathbf{G}^*\mathbf{GD}\right) + \sigma_n^2 \mathrm{tr}\left(\mathbf{D}^*\mathbf{D}\right) &= n\sigma_r^2
\end{aligned} \tag{6.61}$$

where $\mathbf{s} \in \mathbb{C}^{1\times r}$ is the transmitted vector, $\mathbf{y} \in \mathbb{C}^{1\times r}$ is the received vector, and $\mathbf{r}, \mathbf{t} \in \mathbb{C}^{1\times n}$ are the respective received and transmitted vectors at the relay stage. $\mathbf{w} \sim \mathcal{CN}(\mathbf{0}, \sigma_n^2\mathbf{I}_r)$ and $\mathbf{v} \sim \mathcal{CN}(\mathbf{0}, \sigma_n^2\mathbf{I}_n)$ are the corresponding vectors of noise introduced at the receivers and at the relay stage, respectively. $\mathbf{G} \in \mathbb{C}^{r\times n}$ is the channel matrix between the transmitters and the relay nodes, and $\mathbf{H} \in \mathbb{C}^{n\times r}$ is the channel matrix between the relay nodes and the receivers. Finally, $\mathbf{D} = \mathrm{diag}\left(d_1, \ldots, d_n\right) \in \mathbb{C}^{n\times n}$ is a diagonal matrix with diagonal entries corresponding to the scalars chosen by the relay nodes. Notice that the d_js depend on the channel gains. The last equation in (6.61) is a consequence of the power constraint for the relay nodes. We remark that the power constraint considered here is more general than what was assumed in previous sections.

From (6.61), the equivalent channel matrix between the transmitters and the receivers is $\mathbf{GDH}$. Therefore, diagonalizing the channel matrix amounts to finding diagonal matrix $\mathbf{D}$ such that $\mathbf{GDH} = \alpha\mathbf{I}_r$ for some complex scalar α. The number of complex equations is r^2, and the number of variables is n. Therefore, generically, this equation has a solution for $r^2 \leq n$. In this case, by looking at (6.61) we can write the received signal at receive node k as

$$y_k = \alpha s_k + w_k + \sum_{j=1}^{n} h_{jk} d_j v_j.$$

We can find an upper bound on the achievable rates using the scheme described above by considering that the receiver node has knowledge of the different noises introduced in the relay stage. Hence, we can bound the capacity of the channel between the transmit/receive pair k as

$$C_k \leq \frac{1}{2}\mathrm{E_f}\ \log(1+\frac{|\alpha|^2 p}{\sigma_n^2}) \leq \frac{1}{2}\log(1+\frac{p\mathrm{E_f}\ |\alpha|^2}{\sigma_n^2}).$$

The power efficiency can be bounded as follows:

$$\eta' = \frac{2\sum_{i=1}^{r} C_i}{rp+n\sigma^2} \leq \frac{2r}{rp+n\sigma_r^2}\log(1+\frac{\mathrm{E_f}\ |\alpha|^2 p}{\sigma_n^2}). \tag{6.62}$$

Thus, we only need to find the mean of the maximum possible value of $|\alpha|^2$ subject to the following constraints

$$\begin{aligned} \mathbf{GDH} &= \alpha\mathbf{I}_r \quad \text{and} \\ p\,\mathrm{tr}\,(\mathbf{D}^*\mathbf{G}^*\mathbf{GD}) + \sigma_n^2\mathrm{tr}\,(\mathbf{D}^*\mathbf{D}) &= n\sigma_r^2. \end{aligned} \tag{6.63}$$

First, we try to solve the first equation in (6.63). Define $\mathbf{d} = [d_1, \ldots, d_n]^t$, $\mathbf{b} = \mathrm{vec}\,(\mathbf{I}_r)$. (6.63) can be written in terms of $\mathbf{d}$ and $\mathbf{b}$ as

$$\begin{aligned} \mathbf{Ad} &= \alpha\mathbf{b} \quad \text{and} \\ \mathbf{d}^*\mathbf{\Lambda d} &= n\sigma_r^2, \end{aligned} \tag{6.64}$$

where it can be easily verified that $\mathbf{A} \in \mathbb{C}^{r^2\times n}$ and $\mathbf{\Lambda} \in \mathbb{C}^{n\times n}$ are

$$\mathbf{A}_{r^2\times n} = \begin{bmatrix} \mathbf{H}^T & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \mathbf{H}^T & \ldots & \mathbf{0} \\ \vdots & \ldots & \ldots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & \mathbf{H}^T \end{bmatrix}_{r^2\times rn} \cdot \begin{bmatrix} \mathrm{diag}\,(\mathbf{G}_1) \\ \mathrm{diag}\,(\mathbf{G}_2) \\ \vdots \\ \mathrm{diag}\,(\mathbf{G}_r) \end{bmatrix}_{rn\times n},$$

$$\mathbf{\Lambda}_{n\times n} = \mathrm{diag}\,(p\|\mathbf{g}_1\|^2 + \sigma_n^2, \ldots, p\|\mathbf{g}_n\|^2 + \sigma_n^2). \tag{6.65}$$

$\mathbf{G}_i, \quad i = 1, \ldots, r$, and $\mathbf{g}_j, \quad j = 1, \ldots, n$, denote the i-th row and j-th column of $\mathbf{G}$, respectively. If we define $\mathbf{B}_{r^2 \times n} = \mathbf{A}\mathbf{\Lambda}^{-\frac{1}{2}}$ by using QR-type decomposition [112] for $\mathbf{B}^*$, we can write $\mathbf{B}^* = \mathbf{Q}\begin{pmatrix} \mathbf{R} \\ \mathbf{0} \end{pmatrix}$, where $\mathbf{Q}_{n \times n}$ is unitary matrix (i.e., $\mathbf{Q}\mathbf{Q}^* = \mathbf{Q}^*\mathbf{Q} = \mathbf{I}_n$), and $\mathbf{R}_{r^2 \times r^2}$ is a lower triangular matrix with diagonal elements equal to unity. By writing $\mathbf{Q}^*\mathbf{\Lambda}^{\frac{1}{2}}\mathbf{d} = \begin{pmatrix} \mathbf{d}^{(1)}_{r^2 \times 1} \\ \mathbf{d}^{(2)}_{(n-r^2) \times 1} \end{pmatrix}$, we can rewrite (6.64) as

$$\begin{aligned} \mathbf{R}^*\mathbf{d}^{(1)} &= \alpha \mathbf{b} \quad \text{and} \\ \|\mathbf{d}^{(1)}\|^2 + \|\mathbf{d}^{(2)}\|^2 &= n\sigma_r^2. \end{aligned} \tag{6.66}$$

Now notice that $\mathbf{R}$ is invertible and therefore we can find $\mathbf{d}^{(1)}$ from (6.66), and by substituting its value in the second relation of (6.66) we get

$$|\alpha|^2 \mathbf{b}^*(\mathbf{R}^*\mathbf{R})^{-1}\mathbf{b} + \|\mathbf{d}^{(2)}\|^2 = n\sigma_r^2.$$

It can be easily verified that $\mathbf{R}^*\mathbf{R} = \mathbf{B}\mathbf{B}^* = \mathbf{A}\mathbf{\Lambda}^{-1}\mathbf{A}^*$. Also, the maximum of $|\alpha|^2$ is when $\mathbf{d}^{(2)} = \mathbf{0}_{n-r^2 \times 1}$. Therefore,

$$|\alpha|^2_{max} = \frac{n\sigma_r^2}{\mathbf{b}^*(\mathbf{A}\mathbf{\Lambda}^{-1}\mathbf{A}^*)^{-1}\mathbf{b}}. \tag{6.67}$$

Now, using the following inequality for positive definite matrix B and any vector x ([112], page 452)

$$(\mathbf{x}^*\mathbf{B}\mathbf{x})(\mathbf{x}^*\mathbf{B}^{-1}\mathbf{x}) \geq (\mathbf{x}^*\mathbf{x})^2,$$

we have

$$|\alpha|^2_{max} \leq \frac{n\sigma_r^2(\mathbf{b}^*(\mathbf{A}\mathbf{\Lambda}^{-1}\mathbf{A}^*)\mathbf{b})}{(\mathbf{b}^*\mathbf{b})^2}.$$

$\mathbf{b}^*\mathbf{b} = r$ and $\frac{1}{\sigma_n^2}\mathbf{I}_n - \mathbf{\Lambda}^{-1}$ is positive semi-definite. Therefore, we have

$$\mathrm{E_f}\,|\alpha|^2_{max} \leq \frac{n\sigma_r^2}{r^2\sigma_n^2}\mathrm{E_f}\,[\mathbf{b}^*(\mathbf{A}\mathbf{A}^*)\mathbf{b}]. \tag{6.68}$$

Using the structure of matrix $\mathbf{A}$ in (6.65), it can be verified that

$$\mathrm{E_f}\,[\mathbf{b}^*(\mathbf{A}\mathbf{A}^*)\mathbf{b}] = \sum_{i,j=1}^{r} \mathrm{E_f}\,[\mathbf{b}_i^*\mathbf{H}^T\mathrm{diag}\,(\mathbf{G}_i)\mathrm{diag}^*(\mathbf{G}_j)\bar{\mathbf{H}}\mathbf{b}_j],$$

where b_i is the i-th column of $\mathbf{I}_r$. Now since the entries of $\mathbf{H}$ and $\mathbf{G}$ are independent from each other, the expectation inside the above summation is zero for $i \neq j$. Therefore we have

$$\begin{aligned}
\mathrm{E_f}\,[\mathbf{b}^*(\mathbf{A}\mathbf{A}^*)\mathbf{b}] &= \sum_{i}^{r} \mathrm{E_f}\,[\mathbf{b}_i^*(\mathbf{H}^*\mathbf{H})^T\mathbf{b}_i] \\
&= \mathrm{E_f}\,\mathrm{tr}\,((\mathbf{H}^*\mathbf{H})^T \cdot \sum_{i=1}^{r}\mathbf{b}_i\mathbf{b}_i^*) \\
&= \mathrm{E_f}\,\mathrm{tr}\,((\mathbf{H}^*\mathbf{H})^T \cdot \mathbf{I}_r) \\
&= nr,
\end{aligned} \tag{6.69}$$

where we have used the fact that $\mathrm{tr}\,(\mathbf{AB}) = \mathrm{tr}\,(\mathbf{BA})$. Using the above result in (6.68), we have

$$\mathrm{E_f}\,|\alpha|^2_{max} \leq \frac{n^2\sigma_r^2}{r\sigma_n^2}. \tag{6.70}$$

Combining (6.70) and (6.62) we have

$$\eta' \leq \frac{r}{rp + n\sigma_r^2}\log(1 + \frac{n^2\sigma_r^2 p}{r\sigma_n^2}). \tag{6.71}$$

Using an argument to previous sections (e.g., Section 4.3), we know that the maximum of the right hand side expression is less than $\kappa\sqrt{n}$ for some constant κ dependent on

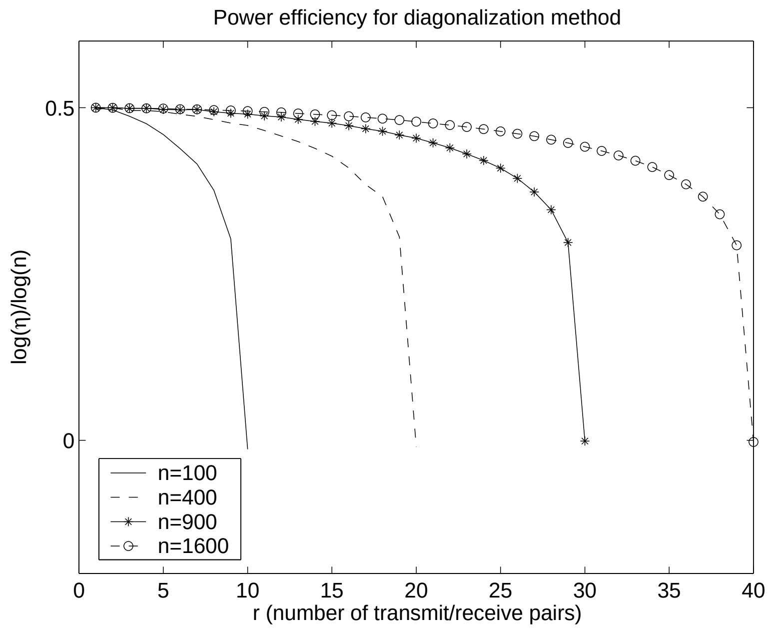


Figure 6.4: Power efficiency for interference suppression case.

σ_n^2. Therefore,

$$\eta' \leq \kappa\sqrt{n},$$

which is the same as the case when we have only local knowledge of the channel and "Listen and Transmit" protocol is used. We have found the upper bound for the maximum power efficiency, η', using the actual value for $\mathrm{E_f}\,|\alpha|^2$ from (6.67) and Matlab simulation. We have plotted the ratio $\frac{\log(\eta')}{\log n}$ for different values of $r \leq \sqrt{n}$ for $n = 100$, 400, 900, and 1600. As we can see from the plots, the upper bound suggests that we cannot do better than $\sqrt{n}$ (or equivalently $\frac{1}{2}$ in the Figure 6.4). Also, as the number of the simultaneous transmitter/receiver pairs, r, increases, the upper bound on the power efficiency of the interference cancelation method becomes smaller. This suggests that this method is not power efficient.

Based on the above argument we have the following theorem.

Theorem 6.21. *Consider a wireless ad-hoc network with n relay nodes and r transmit/receive pair in which $r^2 \leq n$. Moreover, assume the channel coefficients can be* *modeled by independent zero mean unit variance complex random variables. Also assume that the relay nodes have* complete knowledge *of the channel coefficients; then, if the relay nodes cancel out the interference at the receiver nodes, the power efficiency scales as* $O(\sqrt{n})$.

6.6 Conclusion

In this chapter we address the power efficiency of random sensory and ad-hoc wireless networks formed in a domain of fixed area. Under some assumptions on the moments of the channel coefficients, we show that asymptotically, as the number of nodes in the network, n, grows larger, with high probability we can achieve a power efficiency of $\Theta(\sqrt{n})$ for sensory networks. For ad-hoc networks, if the number of transmitter/receiver pairs is of $O(\sqrt{n})$, we can achieve the same result. We also described the protocol used to achieve this power efficiency. Although the best results for capacity per node in sensory and ad-hoc wireless networks decrease as the size of networks grows larger [91], [92], we can see that it pays off to consider these networks in terms of power efficiency.

We can think of the protocol used in this chapter as a simple yet powerful memoryless linear coding scheme for the relay nodes, i.e., the relay nodes simply relay a scaled version of what they have heard. One can generalize this protocol by using other coding schemes for relay nodes. Another interesting problem is to look at the spectral efficiency of sensory and ad-hoc networks and its trade-off with the power efficiency. Partial results about the optimal trade-off is reported in [114].

Chapter 7

Differentiated Rate Scheduling for Cellular Systems

7.1 Introduction

The downlink scheduling in cellular systems is known to be one major bottleneck for future broadband communication systems. From an information-theoretic perspective, broadcast channels [53], and in particular the Gaussian broadcast channels (GBC), can be used to model the downlink in a cellular system. There exist an abundance of information-theoretic results describing the limits of the achievable rates[1] to the users in single-input single-output (SISO) Gaussian broadcast channels (see e.g., [115, 116]). For example, in a homogeneous network, i.e., a network where the fading and noise distributions of all the users are identical, if the transmitter wants to maximize the throughput (or the sum of the rates to all the receivers[2]), it is well known that the optimal strategy is to transmit to the user with the best channel condition at each channel use. This is often referred to as the *opportunistic* transmission strategy [102].

More recently, there has been growing interest in the use of multiple antennas (at the transmitter, receivers, or both) for wireless communication systems. The initial

[1] In the remainder of this work, we assume that the channel is ergodic and rate refers to the average rate over all channel realizations.

[2] We use users and receivers interchangeably.

focus has been on point-to-point communications, where it has been shown that the use of multiple antennas can significantly increase the rate and reliability of a wireless communication link. Given this, both the research and industrial communities have begun to study the use of multiple antenna systems in wireless networks. A most obvious application is in cellular systems, where the use of multiple antennas at the base station can potentially increase the capacity of each cell. This has led to an interest in the multiple-input multiple-output (MIMO) Gaussian broadcast channel, where the transmitter and the various users may be equipped with multiple transmit and receive antennas, respectively. First, the sum-rate of the MIMO broadcast channel, i.e., the maximum possible sum of the rates to all users [117], and then the entire capacity region [118], were shown to be achieved by an interference cancelation scheme referred to as *dirty paper coding* (DPC) [119].

Thus, from a theoretical point of view, the limits of reliable communication in MIMO Gaussian broadcast channels is well understood. Fortunately, the same is true if one takes a *computational* point of view. In other words, it is well known how to computationally obtain any point on the boundary of the capacity region. In a nutshell, the methodology can be explained as follows. Each point on the boundary of the capacity region is characterized by a set of covariance matrices (corresponding to how DPC is used at that particular boundary point). To obtain the desired covariance matrices, one may construct a *dual* multiple-access (MAC) system [120] where, due to the polymatroid structure of the problem, the solution can be found via standard convex optimization techniques [103].

While the aforementioned results gives a good mathematical understanding of the scheduling for downlink of cellular systems, there are many design issues and practical limitations that are not taken into account. For an efficient design of the downlink of cellular systems, these issues and limitations should be addressed. Examples of these issues follow:

- **Channel state information:** A crucial assumption in all the aforementioned results is that the channel coefficients to all the users is known–an assumption referred to as full channel state information (CSI)–at the transmitter. In fact, it is easy to show that with no CSI at the transmitter there is no capacity gain to be had by employing multiple antennas at the transmitter (provided all the users have single antennas) [121]. However, in practice, obtaining full CSI at the transmitter may not be feasible, especially for systems where the number of users is large and/or the users are mobile so that the channel coefficients vary rapidly with time.

- **Computational complexity:** When the number of users is large, the computational complexity of DPC, and even the convex optimization steps required to determine the optimal covariance matrices from the dual MAC, may become prohibitively large. Therefore there is interest in developing *simple* schemes that require *little* CSI at the transmitter, yet deliver on most of the capacity offered by the MIMO broadcast channel. One such scheme that achieves most of the throughput (sum-capacity) in MIMO broadcast channels in certain regimes is described in [121].

- **Differentiated Quality of Service:** In homogenous networks, the sum-rate point is a symmetrical point on the boundary of the capacity region, and so treats all the users equally. In systems which are provisioned to provide differentiated services to different users, the transmitter has to give different services (or rates) to different subsets of receivers, and yet at the same time maximize the throughput (see e.g., [122] for a discussion of the SISO case). Giving differentiated rates to users clearly means operating at non-symmetrical boundary points of the capacity region.

In the present and the following chapters, we will look at two of these issues. In this chapter we consider different quality of service provisioning in the downlink of cellular systems. More specifically, we look at a MIMO broadcast channel where the transmitter has to provide different rates to different subset of receivers, and yet at the same time maximizes the throughput.

As mentioned earlier, this problem can, in principle, be solved since the duality to the MAC allows one to attain any point on the capacity region. However, since this solution requires full CSI at the transmitter and potentially prohibitive computations when the number of users is large, the main goal of this chapter is to develop *simple* schemes that require very *little* CSI, give *differentiated rates* to the users, and that operate *close* to the boundary of the capacity region.

We should also mention that in this chapter we will only be dealing with homogenous networks, in the sense that the SNR of the different users are assumed to have the same probability distribution. Of course many networks are, in fact, heterogenous, with different users having different distributions for their SNRs. The methodology of this paper (and many of the results, we suspect) can be straightforwardly carried over to the heterogenous case, with the caveat that the development will be much more involved and cumbersome. For this reason, and for reasons of space, although quite important in practice, we deem the heterogenous case beyond the scope of the current work. We only remark that a common practice to make a heterogenous network appear homogenous is to use an appropriate power control (after which all our results will directly apply).

In the first part of the chapter we consider channels with a small number of users (i.e., $n = 2, 3$). It turns out that the problem of determining a schedule that satisfies the rate constraints becomes analytically intractable as the number of users grows beyond 3. Therefore, in the second part of the chapter, we assume that the number of users is large. This is, of course, of practical interest since many systems operate in such a regime. Furthermore, it allows us to obtain explicit results in the asymptote of large number of users, i.e., n.

7.2 Problem Formulation

We consider a fading Gaussian broadcast channel with M antennas at the transmitter and n users, each with $N = 1$ receive antennas.[3] The channels to each user are assumed to be block fading with a coherence interval of T; in other words, the channels remain constant for T channel uses, after which they change to different independent values.[4] Furthermore, over different users the fading is assumed to be independent. Thus, during any coherence interval, the signal to the i-th user, $i = 1, 2, \ldots, n$, can be written as

$$x_i(t) = \sqrt{\rho} H_i s(t) + w_i(t), \quad t = 1, \ldots, T, \tag{7.1}$$

where $H_i \in \mathcal{C}^{1\times M}$ is constant during the coherence interval and has i.i.d $\mathcal{CN}(0,1)$ entries. For the purposes of this chapter it will not matter *how* the channels change from block to block, other than the fact that they vary in some stationary and ergodic way.[5]$w_i(t)$ is additive white noise with distribution $\mathcal{CN}(0,1)$, and $s(t) \in \mathcal{C}^{M\times 1}$ is the transmit symbol satisfying $E\|s(t)\|^2 = M$. In other words, the transmit power is assumed to be M. Therefore, the received signal to noise ratio (SNR) of the i-th user will be $\mathrm{E}\,\rho|H_i s(t)|^2 = P = M\rho$; however, to simplify the notation we refer to ρ as the SNR of the users.

In terms of channel knowledge, we assume that H_i is known perfectly at the receiver. Also, we assume that the transmitter is provided with error-free information about the channel states. We denote the (average) rate of the i-th user, $i = 1, \ldots, n$,

[3]It is possible to extend our results to $N \neq 1$ in a straightforward fashion. However, for simplicity, we shall not do so here. From a practical point of view $N = 1$ is also very reasonable.

[4]We should remark that, although the assumption of a constant channel for T channel uses is critical, the requirement that the channels vary independently from one coherence interval to the next is not.

[5]This is because our focus is on the rate. If we had focused on other performance measures, such as delay, then how the channels vary with time would have been important.

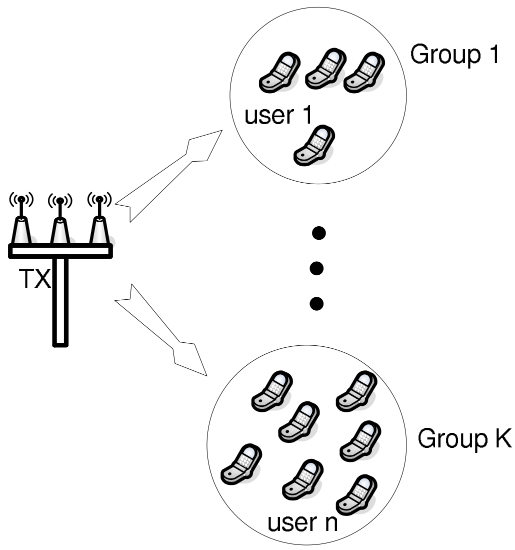


Figure 7.1: A MISO broadcast channels with differentiated rate users.

over the different channel realizations by R_i.

In this chapter we are interested in analyzing differentiated rate scheduling schemes for broadcast systems with n users. We consider a partitioning of the users into K groups $\mathcal{G}_1, \ldots, \mathcal{G}_K$, as shown in Figure 7.1, where different groups require different rates from the transmitter. We also assume that the sizes of the groups are all of the same order and, hence, the cardinality of $\mathcal{G}_k$ is $\alpha_k n$ where K and α_k's are fixed numbers such that $\sum_{i=1}^{K} \alpha_i = 1$.

Assuming that the average rate of a user in the k-th group is denoted by R^k,[6] without loss of generality we may assume $R^1 < \ldots < R^k$. We further impose the constraint that the average rate of a user in the k-th group is β_k times the average rate of a user in the K-th group. $\beta_1 < \ldots < \beta_{K-1} < \beta_K = 1$ are fixed numbers independent of n. In general we are interested in the following problem.

Problem 7.1. *Consider the fading MIMO Gaussian broadcast channel described above. Let R_i denote the rate to i-th user and R^k denote the rate to a user in group*

[6]Throughout the chapter we use superscript k to refer to any user in $\mathcal{G}_k$.

k. Then construct a transmission scheme such that

$$\max_{(R_1,\ldots,R_n)\in\mathcal{C}_{BC}} \sum_{i=1}^{n} R_i$$

$$\textit{subject to } \frac{R^k}{R^K} = \beta_k, \quad k = 1,\ldots,K, \tag{7.2}$$

where $\mathcal{C}_{BC}$ is the capacity region of the broadcast channel given by Dirty Paper coding (or superposition coding in the single antenna case) [118, 40].

Clearly, the solution to Problem 7.1 is given by the intersection of the line $R^k/R^K = \beta_k$, $k = 1,\ldots,K-1$ with the boundary of the capacity region of the broadcast channel. Using the duality between the capacity region of Gaussian broadcast channel and multiple access channel shown in [120, 133, 137], Problem 7.1 can be solved using bisection method in the following way.

(a) Choose a set of rates R'^k satisfying the rate constraints of (7.2).

(b) By appealing to the dual MAC, solve the problem, $\min\sum p_i$, subject to the rates R'^k.

(c) If the minimum sum of powers, $\min\sum p_i$, is less than total transmit power, ρM, then the rate vector is achievable. Increase the rate proportionately (according to vector $\underline{\beta} = (\beta_1,\ldots,\beta_K)$) and go to (a).

(d) Otherwise decrease the rates proportionately (according to vector $\underline{\beta} = (\beta_1,\ldots,\beta_K)$) and go to (a).

While this is all fine, the algorithm is computationally-intensive (even though the problem in step 2 is convex, it is time-consuming if n is large), requires full CSI, and finally requires implementation of DPC. Furthermore, the solution to Problem 7.1 does not give us insight into how much throughput loss we would incur by imposing the rate constraints.

In this chapter we look into devising simpler scheduling schemes such as time-sharing and opportunistic transmission. We also compare the performance of different scheduling schemes in terms of their sum-rate.

It is clear that there is a price to pay in terms of throughput (sum-rate) to maintain the rate constraints. This is due to the fact that we are not working on the sum-capacity point and, therefore, the throughput will be reduced compared to the case where we had no rate constraint. In other words, we are interested in

$$\text{Throughput Loss} = \sum_{\substack{i=1 \\ (R_1,\ldots,R_n)\in\mathcal{C}_{BC}}}^{n} R_i \quad - \sum_{\substack{i=1 \\ (R_1,\ldots,R_n)\in\mathcal{C}_{BC}\text{ and }(7.2)}}^{n} R_i \tag{7.3}$$

for different scheduling schemes.

In Section 7.4 we consider a channel with a small number of users, namely $n = 2$ and $n = 3$ and single antenna at the transmitter, i.e., $M = 1$. We also focus on the rate region achieved by weighted-opportunistic (WO) scheduling in which we transmit to the user that has the maximum "weighted" signal to noise ratio (SNR). We obtain the relationship between the weights for WO scheduling and the ratio of the rates. It turns out that finding an explicit relationship between the weights as a function of the given ratios is analytically intractable for $n > 3$, even if we allow for simplifying assumptions such as considering the low SNR regime. We further look into the throughput loss due to the rate constraints in (7.2).

In order to obtain explicit solutions, in the second part of the chapter we consider a system with many users and, rather than attempting to solve Problem 7.1 and optimization problem (7.2) directly, we propose three scheduling schemes to provide the rational rate constraints, namely weighted opportunistic beamforming (WO), time division opportunistic beamforming (TO), and superposition coding (SC) for single antenna systems. In WO, a generalization of opportunistic random beamforming scheduling, a beam is assigned to the user that has the largest "weighted" signal to noise and interference ratio (SINR) corresponding to that beam. In TO, each group has its own time slot in which the transmitter chooses the user with the best SINR from the corresponding group. Superposition coding (SC) is the scheme that achieves the information-theoretic capacity region for a single antenna broadcast channel. For each scheduling we give an explicit scheme to guarantee the rate constraints. We also analyze the throughput loss due to the rate constraints for all three different schemes. At the end of the chapter, we will provide simulation results showing the performance of the developed scheduling schemes.

7.3 Preliminary Results for MIMO GBC

In this section we will review some of the results for MIMO Gaussian broadcast channels (GBC). In particular, we will look at the sum-capacity of these channels. We will review the concept of "opportunistic" beamforming and analyze the difference between the achievable sum-rate using "opportunistic" beamforming and the sum-capacity point for large systems, i.e., large n.

7.3.1 The Capacity Region of MIMO GBC

The capacity region of MIMO Gaussian broadcast channels was recently found in [118]. It is shown that the capacity region of these channels is given by a precoding scheme at the transmitter, called Dirty Paper Coding (DPC)[119]. Furthermore, it is shown in [120, 133, 137] that the capacity region of the MIMO GBC is equal to the capacity region of a *dual* multiple access channel (MAC) with sum-power constraint equal to the GBC power constraint. Using this duality result we can write the capacity region of the broadcast channel of (7.1) as

$$\mathcal{C}_{BC} = \bigcup_{\{\underline{P}:\sum P_i \leq M\rho\}} \left\{ \underline{R} | \sum_{j\in\mathcal{A}} R_j \leq \log\det(I + \sum_{j\in\mathcal{A}} H_j^* P_j H_j) \quad \forall\, \mathcal{A} \subseteq \{1,\ldots,n\} \right\}, \tag{7.4}$$

where $\underline{R} = (R_1,\ldots,R_n)$ and $\underline{P} = (P_1,\ldots,P_n)$. In particular, the ergodic sum-capacity of the channel is

$$C_{sum} = \mathrm{E} \max_{0\leq P_i:\sum_{i=1}^n P_i \leq M\rho} \log\det(I + \sum_{i=1}^n H_i^* P_i H_i), \tag{7.5}$$

where the expectation is over all the channel realizations.

Sum-Capacity Scaling Laws

In point-to-point multi-antenna systems, the throughput scaling is often equivalent to the "multiplexing gain," defined as $\lim_{P\to\infty} \frac{C_{sum}}{\log P}$ where C_{sum} denotes the ergodic sum-rate capacity (or throughput) of the channel achieved by coding over several coherence intervals.

In broadcast channels, as the number of users can also be large, two different throughput scaling laws can be envisioned with respect to $P = M\rho$ (or equivalently SNR) and with respect to the number of users n.

Theorem 7.1. *(Large Power Regime)[124] Consider the fading MIMO Gaussian broadcast channel of Section 7.2 with received SNR of $P = M\rho$. Then, for a fixed M and n, we have*

$$\lim_{P\to\infty} \frac{C_{sum}}{\log P} = M. \tag{7.6}$$

Theorem 7.2. *(Large Number of Users Regime)[125] Consider the fading MIMO*

Gaussian broadcast channel of Section 7.2. Then, for fixed M and ρ, we have

$$\lim_{n\to\infty} \frac{C_{sum}}{\log\log n} = M, \tag{7.7}$$

where C_{sum} refers to the maximum possible sum of the rates to all n users.

These are clearly two very different regimes and both confirm that the sum-rate is linearly scaled with the number of transmit antennas M. We argue that, from a practical perspective, the latter regime may be more interesting. There are three reasons that come to mind.

(a) Many practical systems operate with a large number of per-cell users (n could be in the hundreds, whereas M may be no more than two, three, or four).

(b) Significant rates can be obtained even at low to moderate SNR, $P = M\rho$.

(c) The first gain requires channel knowledge with very high fidelity at the transmitter (indeed a fidelity that grows with the transmit power) [126], whereas the latter requires very little CSI (see, e.g., [121] and Chapter 8).

In view of the above, in this paper we will focus on the *large* n regime.

Note that in order to perform DPC to achieve the sum-capacity (or any other point in the capacity region), the transmitter requires exact knowledge of the channels. In the following, we briefly describe "opportunistic" beamforming, a simple scheme that achieves most of the sum-capacity in some regimes and yet requires very little CSI at the transmitter. For a complete study of this scheme, see [121, 127].

7.3.2 Opportunistic Beamforming

The main idea behind opportunistic beamforming is to exploit the multi-user diversity available in the network and transmit to the users with the best quality of reception. In this beamforming, during any coherence interval the transmitter constructs M random beams and transmits each beam to the user with the highest signal to interference plus noise ratio (SINR). Let $\phi_m(M \times 1)$, for $m = 1, \ldots, M$, be M random orthonormal vectors generated according to an isotropic distribution [149]. The transmitted signal is

$$s(t) = \sum_{m=1}^{M} \phi_m s_m(t), \qquad t = 1, \ldots, T,$$

where each $s_m(t)$ is a scalar signal (with average unit power, i.e., $\mathrm{E}\,|s_m|^2 = 1$) intended for one of the users. Assuming the users know their own channel coefficients (a much more reasonable assumption than the transmitter knowing *all* the channel gains to the different users), each user can compute its signal-to-interference-plus-noise-ratio (SINR) for every beam as

$$\mathrm{SINR}_{i,m} = \frac{|H_i \phi_m|^2}{\frac{1}{\rho} + \sum_{l \neq m} |H_i \phi_l|^2}, \qquad m = 1, \ldots, M. \tag{7.8}$$

If each user (or, in fact, only those users who have favorable SINRs) feeds back its best SINR and corresponding beam index to the transmitter, the transmitter can assign each beam to the user that has the best SINR for that beam (see [121] for more details.).

For this scheme the following result can be shown.

Theorem 7.3. *[121] Consider the fading MIMO Gaussian broadcast channel of Section 7.2 and let C_{ob} denote the sum-rate obtained by the opportunistic beamforming technique described above. Then, for fixed P and M*

$$\lim_{n \to \infty} \frac{C_{ob}}{\log \log n} = M. \tag{7.9}$$

However, for fixed n and M

$$\lim_{P\to\infty} \frac{C_{ob}}{\log P} = 0. \tag{7.10}$$

In other words, opportunistic beamforming is order-optimal in the large n regime, but not in the large P regime. The reason is that opportunistic beamforming is interference-dominated, and so the sum-rate does not scale with the logarithm of the power. (In fact, to obtain the multiplexing gain of M at high power requires essentially eliminating the interference, such as is done by a zero-forcing solution.)

Before we proceed further, it is useful to mention that the probability distribution function (PDF) of $\text{SINR}_{i,m}$, denoted by $f_s(x)$, can be written as [121]

$$f_s(x) = \frac{e^{-x/\rho}}{(1+x)^M}(\frac{1}{\rho}(1+x)+M-1). \tag{7.11}$$

We can also calculate the cumulative distribution function (CDF) of $\text{SINR}_{i,m}$, $F_s(x)$, as

$$F_s(x) = 1 - \frac{e^{-x/\rho}}{(1+x)^{M-1}}, \qquad x \geq 0. \tag{7.12}$$

7.3.3 Tighter Scaling Laws

In this section we give a tighter result regarding the convergence of the sum-rate of opportunistic beamforming scheme to the sum-capacity.

Theorem 7.4. *Consider the fading MIMO Gaussian broadcast channel of Section 7.2. For fixed ρ and M*

$$C_{sum} = M \log\log n + M \log \rho + o(1), \tag{7.13}$$

where C_{sum} refers to the sum-capacity of the broadcast channel with n users and $o(1)$ is with respect to growing n.

Proof. Using (7.5), we can write the sum-capacity as capacity as

$$\begin{aligned} C_{sum} &= \mathrm{E} \max_{\sum_{i=1}^n P_i = M\rho, P_i \geq 0} \log\det\left(I_M + \sum_{i=1}^n H_i^* P_i H_i\right) \\ &\leq \mathrm{E} \max_{\sum_{i=1}^n P_i = M\rho, P_i \geq 0} M \log\left(1 + \frac{1}{M}\mathrm{tr}\,(\sum_{i=1}^n H_i^* P_i H_i)\right), \\ &= \mathrm{E} \max_{\sum_{i=1}^n P_i = M\rho, P_i \geq 0} M \log\left(1 + \frac{1}{M}\sum_{i=1}^n P_i \|H_i\|^2\right), \\ &= \mathrm{E}\ M \log\left(1 + \rho \max_i \|H_i\|^2\right), \end{aligned} \tag{7.14}$$

where we have used the Hadamard's inequality for $A \succeq 0$ that states $\det(A) \leq (\frac{\mathrm{tr}\,(A)}{M})^M$ [129]. It is worth mentioning that $\|H_i\|^2$s have $\chi^2(2M)$ distribution and they are independent. Therefore, using order statistics results [128], we can show that the random variable $\max_i \|H_i\|^2$ with high probability behaves as $\log n$. More precisely,

$$\Pr\left(\log n + 2(M-2)\log\log n \leq \max_i \|H_i\|^2 \leq \log n + 2M \log\log n\right) = 1 - O\left(\frac{1}{(\log n)^2}\right). \tag{7.15}$$

We can split the expectation in the right hand side of (7.14) in two parts; one is the expectation conditioned on $\max_i \|H_i\|^2$ being less than or equal to $\log n + 2M \log\log n$, and the other is the expectation conditioned on $\max_i \|H_i\|^2$ being greater than $\log n + 2M \log\log n$. This way, we can upper bound the expectation as

$$\begin{aligned} \mathrm{E}\ \log\left(1 + \rho \max_i \|H_i\|^2\right) &\leq \log\left(1 + \rho(\log n + 2M \log\log n)\right) \\ &\quad + \int_{\log n + 2M \log\log n}^{\infty} \log\left(1 + \rho x\right) f_m(x) dx, \end{aligned} \tag{7.16}$$

where $f_m(x)$ is the distribution of $x = \max_i \|H_i\|^2$. It is quite straightforward to show that the second term in the right hand side behaves like $o(1)$ for large n, and the first term scales like $\log\log n + \log\rho + o(1)$. This shows that

$$C_{sum} \le M \log\log n + M\log\rho + o(1).$$

To complete the proof we need a lower bound that has the same behavior. It turns out that the desired lower bound can be obtained by employing opportunistic beamforming. The required result is the next theorem. ∎

Theorem 7.5. *Consider the setting of Theorem 7.4. Then, if we use opportunistic beamforming*

$$C_{ob} = M \log\log n + M\log\rho + o(1). \tag{7.17}$$

Proof. In order to write the sum-rate achieved by random beamforming, we have to take into account the probability that there can be a user that has the best SINR for two different beams. One can lower bound the sum-rate by sending to one of the M best SINRs for each beam such that the user has not been chosen before. Using this selection, we make sure there is no user with the best SINRs for two beams; obviously, there is a rate hit, as we may transmit to the user that has the M-th best SINR as opposed to the best SINR. This, however, has little effect on the sum-rate, as the asymptotic behavior of the M-th best and the best of the SINRs is quite similar (when M is fixed and n grows). In particular,

$$\begin{aligned}
C_{ob} &\ge ME\log\left(1+\rho \max_{1\le i\le n}{}^{M}\,\mathrm{SINR}_{1,i}\right)\\
&\ge M\log\left(1+\rho(\log n - 2M\log\log n)\right)\underbrace{\Pr\left(\max_{1\le i\le n}{}^{M}\,\mathrm{SINR}_{1,i} > \log n - 2M\log\log n\right)}_{=1-O(\frac{1}{\log n})}\\
&= M\log\log n + M\log\rho + o(1),
\end{aligned}$$

where $\max^M x_i$ denotes the M'th maximum of x_i's. Using theorem 7.4, C_{ob} matches the upper bound, and that completes the proof. ■

Theorems 7.4 and 7.5 imply that the difference of the sum-rate achieved by beamforming and DPC tends to zero, i.e.,

$$\lim_{n \to \infty} (C_{sum} - C_{ob}) = 0, \tag{7.18}$$

which is a much stronger result than being simply order optimal. In fact, a careful analysis of the different in the sum-rates shows that

$$(C_{sum} - C_{ob}) = O(\frac{\log \log n}{\log n}). \tag{7.19}$$

For the rest of this chapter we look at the differentiated rate scheduling problem.

7.4 Channels with a Small Number of Users

In this section we start with characterizing the achievable rate region using weighted-opportunistic (WO) scheduling for single antenna (i.e., $M = 1$) systems. In WO, at each channel use we send to only the user that has the maximum weighted signal to noise ratio, i.e., the user for which

$$\max_{1 \le i \le n} \mu_i |h_i|^2. \tag{7.20}$$

Let us look at the rate transmitted to one of the users, say the first user (denoted by R_1^w). Clearly,

$$R_1^w = \int_0^\infty \log(1+x) f_s(x) \Pr(\mu_1 x_1 \ge \mu_i x_i,\, i = 2, \ldots, n | x_1 = x) \mathrm{d}x,$$

where $x_i = \rho|h_i|^2$, $i = 1, \ldots, n$ and $f_s(\cdot)$ is the PDF of the SNR defined in (7.11). The probability of the event inside the above integral can be written as

$$\Pr\left(\mu_1 x_1 \geq \mu_i x_i,\, i = 2, \ldots, n | x_1 = x\right) = \prod_{i=2}^{n}(1 - e^{-\frac{\mu_1 x}{\mu_i \rho}}).$$

Replacing this expression into the formula for R_1^w yields

$$R_1^w = \int_0^\infty \log(1+x)\frac{e^{-x/\rho}}{\rho}\prod_{i=2}^{n}\left(1 - e^{-\frac{\mu_1 x}{\mu_i \rho}}\right)\mathrm{d}x. \tag{7.21}$$

The rates to the other users can be found in a similar fashion. Let us now focus on (7.21) for the case of two and three users.

7.4.1 Case 1: Two-User Channels

In this case (7.21) simply reduces to

$$R_1^w = \int_0^\infty \log(1+x)\left(1 - e^{-\frac{\mu_1 x}{\mu_2 \rho}}\right)\frac{e^{-x/\rho}}{\rho}dx. \tag{7.22}$$

Similarly, the rate for the second user is as in (7.22), with the only difference that μ_1 should be exchanged by μ_2. We would like to find μ_1 and μ_2[7] such that the rate constraint, i.e., $\frac{R_1^w}{R_2^w} = \beta$, is satisfied.

We can simplify (7.22) as

$$\begin{aligned} R_1^w &= \int_0^\infty \log(1+x)\frac{e^{-x/\rho}}{\rho}\mathrm{d}\,x - \int_0^\infty \frac{\mu_2}{\mu_1+\mu_2}\log\left(1 + \frac{\mu_2}{\mu_1+\mu_2}x\right)\frac{e^{-x/\rho}}{\rho}\mathrm{d}\,x \\ &= -e^{\frac{1}{\rho}}\mathrm{Ei}\,(-\frac{1}{\rho}) + \frac{\mu_2}{\mu_1+\mu_2}e^{\frac{\mu_1+\mu_2}{\rho\mu_2}}\,\mathrm{Ei}\left(-\frac{\mu_1+\mu_2}{\rho\mu_2}\right), \end{aligned} \tag{7.23}$$

where we used the definition of the exponential integral function defined as $-\mathrm{Ei}\,(-x) =$

[7]It is worth mentioning that without loss of generality we can set the sum of μ_is equal to 1.

$\int_1^\infty \frac{e^{-tx}}{t} \mathrm{d}\, t$. We can similarly write the rate for the second user as

$$R_2^w = -e^{\frac{1}{\rho}} \mathrm{Ei}\,(-\frac{1}{P}) + \frac{\mu_1}{\mu_1 + \mu_2} e^{\frac{\mu_1+\mu_2}{\rho\mu_1}} \,\mathrm{Ei}\left(-\frac{\mu_1 + \mu_2}{\rho\mu_1}\right). \tag{7.24}$$

In order to find the μ_1 that satisfies the rate constraint of (7.2), we need to solve the following non-polynomial equation:

$$\frac{-e^{\frac{1}{\rho}} \mathrm{Ei}\,(-\frac{1}{\rho}) + (1-\mu_1) e^{\frac{1}{\rho(1-\mu_1)}} \,\mathrm{Ei}\left(-\frac{1}{\rho(1-\mu_1)}\right)}{-e^{\frac{1}{\rho}} \mathrm{Ei}\,(-\frac{1}{\rho}) + \mu_1 e^{\frac{1}{\rho\mu_1}} \,\mathrm{Ei}\left(-\frac{1}{\rho\mu_1}\right)} = \beta. \tag{7.25}$$

It does not seem that (7.25) has a closed form solution because it involves the exponential integral. We can numerically evaluate β versus μ_1, as shown in Figure 7.2. Since $\mu_1 + \mu_2 = 1$, here we assume that μ_1 is varying between zero and one.

The generalization to a system with $n > 2$ users is straightforward: We simply need to expand the products in (7.21) into a summation of exponentials and then repeatedly use the exponential integral. Applying the rate constraints will lead to a non-polynomial system of equations with $n-1$ equalities and $n-1$ variables. Although it may be possible to solve numerically such a system of equations, it gives us little insight into the problem.

In order to find more explicit results, in the next section we simplify the system (7.25) by assuming that ρ is small (low SNR regime).

7.4.2 Case 2: Low SNR Regime

Assuming that the system is working in the regime of small ρ, the instantaneous rate can be approximated to first order as $\rho|h_i|^2$ (instead of $\log(1 + \rho|h_i|^2)$). It turns out that this leads to a system of polynomial equations, which can be theoretically dealt with using Groebner bases.[8] Given a finite set of multivariate polynomials over a

[8]Groebner bases method was introduced by Bruno Buchberger in 1965 [130].

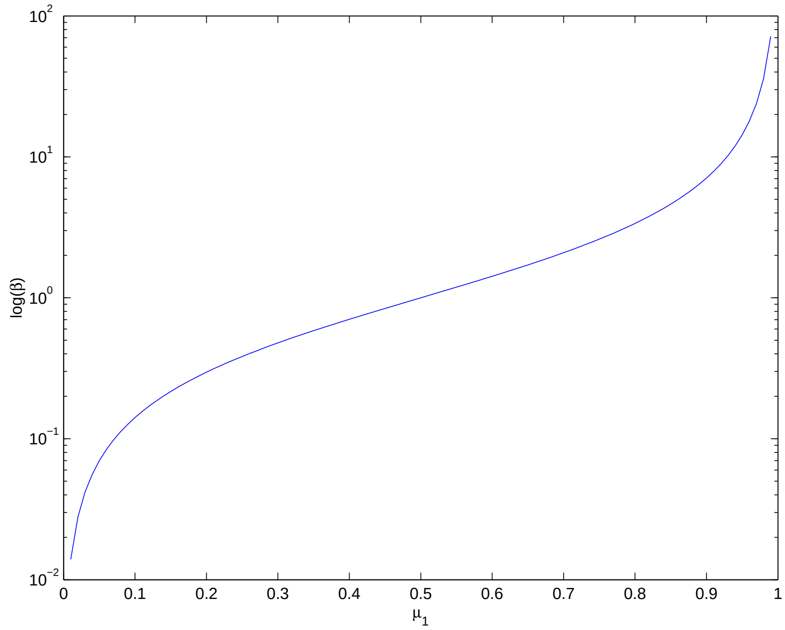


Figure 7.2: β versus μ_1 for a channel with $n = 2$.

field, a new set of polynomials with good properties can be found by an algorithm of Buchberger, called the Groebner basis, which can be used to find the solutions of the polynomial system. This method has been extensively studied, developed, and has been implemented on all major computer algebra systems.

In a channel with two users, the rates can be written as

$$R_1^w = \int_0^\infty x(1 - e^{-\frac{\mu_1 x}{\mu_2 \rho}})\frac{e^{-x/\rho}}{\rho}\mathrm{d}\, x = \rho(2 - \mu_1)\mu_1. \tag{7.26}$$

Similarly,

$$R_2^w = \rho(1 - \mu_1^2). \tag{7.27}$$

Therefore, the boundary of the rate region is characterized by (7.26) and (7.27). This parametric characterization can be made explicit by eliminating μ_1 from (7.26) and (7.27) as

$$\frac{1}{4}\left(1 - \frac{R_2^w}{\rho} + \frac{R_1^w}{\rho}\right)^2 = 1 - \frac{R_2^w}{\rho}. \tag{7.28}$$

Now, given the ratio of the rates and (7.28), we can easily obtain μ_1 such that the ratio of the rates will be equal to β.

This framework can be easily generalized to the case of more than two users. We omit the details and simply state that for $n = 3$, we may write the rates as

$$\begin{aligned}
\frac{R_1^w}{\rho} &= 1 - \frac{\mu_2}{\mu_1 + \mu_3} - \frac{\mu_3}{\mu_1 + \mu_2} + \frac{\mu_2\mu_3}{\mu_2\mu_3 + \mu_1\mu_3 + \mu_1\mu_2}, \\
\frac{R_2^w}{\rho} &= 1 - \frac{\mu_1}{\mu_2 + \mu_3} - \frac{\mu_3}{\mu_2 + \mu_1} + \frac{\mu_1\mu_3}{\mu_2\mu_3 + \mu_1\mu_3 + \mu_1\mu_2}, \\
\frac{R_3^w}{\rho} &= 1 - \frac{\mu_1}{\mu_2 + \mu_3} - \frac{\mu_2}{\mu_3 + \mu_2} + \frac{\mu_1\mu_2}{\mu_2\mu_3 + \mu_1\mu_3 + \mu_1\mu_2}, \\
1 &= \mu_1 + \mu_2 + \mu_3,
\end{aligned}$$

To find the explicit characterization of the rate region, we have to eliminate the μ_is from the above set of polynomial equations. This can be done with the aid of Groebner bases using, say, Mathematica. However, the complexity of the algorithm becomes formidable, even for the case of $n = 3$.

On the other hand, it is possible to attempt to solve the above system of equations numerically. This, in principle, will allow us to map a set of rate constraints to a set of weights for the schedule. However, as mentioned earlier, this gives little insight and, moreover, it too can be quite complex for large n.

7.4.3 Throughput Loss

As mentioned earlier, it is clear that there is a price to pay in terms of throughput (sum-rate) to maintain the rate constraints. In this part, we numerically evaluate the throughput degradation due to imposing the rate constraint of β for a channel with two users. Assuming that the rate of the first user is β times the rate of the second user, Figure 7.3 shows the ratio of the throughput of the WO scheduling over the sum-rate capacity versus β.

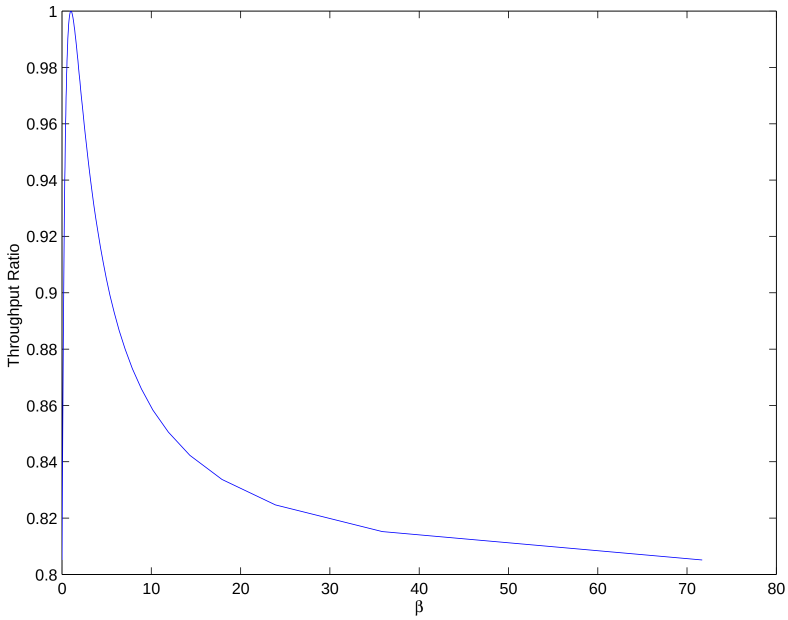


Figure 7.3: Ratio of the throughput with rate constraints over the sum-rate capacity versus β for a channel with $n = 2$.

Clearly, when β equals one, the WO scheduling achieves the sum-rate capacity, and, therefore, throughput will be equal to the sum-rate capacity. As β increases, the throughput loss will be more. It is quite interesting to observe that even for very large β (e.g., close to 70), the throughput is above 80 percent of the sum-rate capacity. Therefore, the throughput does not seem to be too much affected by the differentiated rate scheduling. In the next section, we look into this throughput loss in the regime of large number of users.

7.5 Channels with Many Users

In Section 7.4 we observed that finding an explicit relationship between β_is and the μ_is in WO scheduling becomes very complicated, even for the case of $n = 3$.

Therefore, for the remainder of the chapter, we look into the regime of large number of users. We first characterize the optimal differentiated opportunistic beamforming scheme for maximizing the sum-rate and at the same time satisfy the rational rate constraints.

Then, we consider WO and TO scheduling schemes for multiple antenna GBCs and analyze the performance of superposition coding for single antenna systems for large number of users. It will turn out that having a large number of users will simplify the derivations and lead to explicit results.

7.5.1 Optimal Differentiated Opportunistic Beamforming Scheme

In this section we characterize the *optimal random opportunistic* beamforming scheme for providing different rates to different groups of users. By opportunistic, we mean that the transmitter based on its available knowledge about the channel realization, allocates a fraction of time to transmit each beam to each user. The goal is to maximize the sum-rate and at the same time satisfy the rational rate constraints of (7.2).

Let $\underline{X}^{(m)} = (\mathrm{SINR}_{i,m},\ 1 \le i \le n)$ be the vector of instantaneous SINRs for transmitted beam m of different users. We define a general opportunistic schedule with parameters $\tau_i(\underline{X}^{(m)})$ for $i = 1, \ldots, n$, where $\tau_i(\underline{X}^{(m)})$ is the fraction of the coherence interval that the transmitter assigns beam m to user i, given that the instantaneous SINRs for different receivers correspond to entries of $\underline{X}^{(m)}$. Clearly,

$$\sum_{i=1}^{n} \tau_i(\underline{X}^{(m)}) = 1, \qquad \tau_i(\underline{X}^{(m)}) \ge 0$$

As an example, for the opportunistic beamforming scheme of Section 7.3.2, we have that for each beam m

$$\tau_i(\underline{X}^{(m)}) = \mathbb{1}\left(i = \arg\max_j \underline{X}_j^{(m)}\right),$$

where $\mathbb{1}(a = b)$ is one if $a = b$ and zero otherwise. It is not hard to persuade oneself that WO and TO beamforming are also special cases of the schedule defined here.

With this definition, the average rate assigned to user i is

$$R_i = M\mathrm{E}\left(\tau_i(\underline{X}^{(m)})\log(1+\underline{X}_i^{(m)})\right),$$

where the expectation is over the distribution of SINRs for a particular beam, say, m at different users, i.e., $\underline{X}^{(m)}$. Therefore, we can write Problem 7.1 in terms of $\tau_i(\cdot)$'s as follows

$$\max_{\tau_i(\underline{X}^{(m)})} M\sum_{i=1}^{n}\mathrm{E}\left(\tau_i(\underline{X}^{(m)})\log(1+\underline{X}_i^{(m)})\right), \tag{7.29}$$

subject to

$$\begin{aligned}
\mathrm{E}\left(\tau_i(\underline{X}^{(m)})\log(1+\underline{X}_i^{(m)})\right) &= \beta_i\mathrm{E}\left(\tau_n(\underline{X}^{(m)})\log(1+\underline{X}_n^{(m)})\right) \qquad i = 1,\ldots,n-1\\
\tau_i(\underline{X}^{(m)}) &\geq 0 \qquad \forall\,\underline{X}^{(m)}\in\mathbb{S}^n,\ i = 1,\ldots,n\\
\sum_{i=1}^{n}\tau_i(\underline{X}^{(m)}) &= 1 \qquad \forall\,\underline{X}^{(m)}\in\mathbb{S}^n,
\end{aligned} \tag{7.30}$$

where without of loss of generality we have assumed that user n is in group $\mathcal{G}_K$ and for user i in group k, β_i refers to the β_k defined in Section 7.2.

Looking at (7.29) and (7.30), we can see that the maximization problem is a linear program in $\tau_i(\cdot)$. The following theorem describes the optimal choice of functions $\tau_i(\cdot)$.

Theorem 7.6. *Consider a MIMO Gaussian broadcast channel with the assumptions stated in Section 7.2. Then, the optimal differentiated opportunistic scheduling in the sense defined in (7.29) and (7.30) is given by*

$$\tau_i(\underline{X}^{(m)}) = \mathbb{1}\left(i = \arg\max_j \gamma_j\log(1+\underline{X}_j^{(m)})\right),$$

where $\gamma_j s$ are scalar chosen to satisfy the rational rate constraints of (7.30). In other *words, the optimal differentiated opportunistic scheduling is to assign beam m to the user with the highest "weighted" instantaneous rate in decoding information symbol carried on that beam.*

Proof. We use dual form of the linear program in (7.29) and (7.30) to prove the theorem. The dual linear program can be written as

$$\min_{\eta_i(\underline{X}^{(m)}),\lambda_i,\,\zeta(\underline{X}^{(m)})} M\mathrm{E}\left(\zeta(\underline{X}^{(m)})\right)$$

subject to

$$\begin{aligned}
\eta_i(\underline{X}^{(m)}) &\leq 0 \qquad \forall\, \underline{X}^{(m)} \in \mathbb{S}^n,\ i = 1, \ldots, n \\
(1+\lambda_i)\log(1+\underline{X}_i^{(m)}) &= \eta_i(\underline{X}^{(m)}) + \zeta(\underline{X}^{(m)}) \qquad \forall\, \underline{X}^{(m)} \in \mathbb{S}^n,\ i = 1, \ldots, n-1 \\
(1-\sum_{i=1}^{n-1}\lambda_i\beta_i)\log(1+\underline{X}_n^{(m)}) &= \eta_n(\underline{X}^{(m)}) + \zeta(\underline{X}^{(m)}),
\end{aligned}$$

where λ_i, $\eta_i(\underline{X}^{(m)})$, and $\zeta(\underline{X}^{(m)})$ represent dual variables for the first, second, and third constraint given in (7.30). We can further simplify the above minimization problem to

$$\min_{\lambda_i,\,\zeta(\underline{X}^{(m)})} M\mathrm{E}\left(\zeta(\underline{X}^{(m)})\right) \tag{7.31}$$

subject to

$$\begin{aligned}
(1+\lambda_i)\log(1+\underline{X}_i^{(m)}) &\leq \zeta(\underline{X}^{(m)}) \qquad \forall\, \underline{X}^{(m)} \in \mathbb{S}^n,\ i = 1, \ldots, n-1 \\
(1-\sum_{i=1}^{n-1}\beta_i\lambda_i)\log(1+\underline{X}_n^{(m)}) &\leq \zeta(\underline{X}^{(m)}) \qquad \forall\, \underline{X}^{(m)} \in \mathbb{S}^n. \qquad (7.32)
\end{aligned}$$

Now, for a fixed choice of λ_is, the minimizing $\zeta(\underline{X}^{(m)})$ is $\max_i \gamma_i \log(1+\underline{X}_i^{(m)})$, where we have defined

$$\gamma_i = \begin{cases} 1+\lambda_i & 1 \le i \le n-1, \\ 1-\sum_{i=1}^{n-1}\beta_i\lambda_i & i=n. \end{cases} \tag{7.33}$$

Looking at the KKT conditions [131] at the optimal point $(\eta_i^\star(\cdot), \zeta^\star(\cdot), \tau_i^\star(\cdot), \lambda_i^\star)$, we should have

$$\begin{aligned} \eta_i^\star(\underline{X}^{(m)})\tau_i^\star(\underline{X}^{(m)}) &= 0 \qquad \forall\, \underline{X}^{(m)} \in \mathbb{S}^n,\ 1 \le i \le n \\ \zeta^\star(\underline{X}^{(m)}) &= \max_j(\gamma_j^\star \log(1+\underline{X}_j^{(m)})) \qquad \forall\, \underline{X}^{(m)} \in \mathbb{S}^n \\ \tau_i^\star(\underline{X}^{(m)}) &= \mathbb{1}\left(i = \arg\max_j(\gamma_j^\star \log(1+\underline{X}_j^{(m)}))\right), \qquad \forall\, \underline{X}^{(m)} \in \mathbb{S}^n,\ 1 \le i \le n, \end{aligned}$$

where $\gamma_j^\star$ is defined from $\lambda_j^\star$ according to (7.33). Now, based on the above conditions, it is clear that the maximizing solution is the one that assigns each beam to the user with maximum $\gamma_j^\star \log(1+\underline{X}_j^{(m)})$, and this completes the proof. ∎

Remark 7.7. *It is interesting to note that since* $\log(1+x) \approx x$ *for* $x \ll 1$*, in the low SNR regime, the optimal schedule becomes equivalent to WO beamforming.*

It turned out that analyzing the above optimal scheme does not lead to explicit solution for $\gamma - j$'s. Therefore, we look at the performance of TO and WO beamforming next and devise explicit schemes for these beamforming schedules.

7.5.2 Time-Division Opportunistic (TO) Beamforming

The simplest scheme to give differentiated rates to different users is to assign different lengths of channels uses to different users, i.e., time-sharing. This should be done opportunistically to maximize the sum-rate. In particular, we divide each coherence interval into K slots of duration t_k each, $k = 1, \ldots, K$. During the k-th subinterval, the transmitter performs opportunistic beamforming to *only* the $\alpha_k n$ users in the k-th group. If α_ks are fixed and n grows, it is not hard to convince oneself that to satisfy the rational rate constraints, we must have

$$\frac{t_k}{T} = \frac{\alpha_k \beta_k}{\sum_{l=1}^{K} \alpha_l \beta_l}, \quad k = 1, \ldots, K. \tag{7.34}$$

Intuitively, since each group has a size of order n, the sum-rate for each group scales like $M \log\log n$. Therefore, in order to satisfy the rate constraints, we need to only take into account the size of the group, i.e., α_i, and the rate ratio β_i. Therefore, using (7.34), we can easily show that

$$\lim_{n\to\infty} \frac{R^k}{R^K} = \beta_k, \qquad k = 1, \ldots, K-1. \tag{7.35}$$

We can state the following result that quantifies the sum-rate loss due to the rate constraints and also the sum-optimality of the scheduling.

Theorem 7.8. *Consider the fading MIMO Gaussian broadcast channel of Section 7.2. Let M, ρ, α_k, and β_k be fixed, and let the subintervals be chosen as (7.34). Then, the rational rate constraints are met and*

$$\lim_{n\to\infty} \left(C_{sum} - C_{tdob}\right) = \Theta\left(\frac{1}{\log n}\right), \tag{7.36}$$

where C_{tdob} represents the sum-rate for the time-division opportunistic scheme.

Proof. That the rational rate constraints are met is fairly straight forward. As for the sum-rate we have,

$$\begin{aligned} C_{tdob} &= \sum_{k=1}^{K} \frac{\alpha_k \beta_k}{\sum_{l=1}^{K} \alpha_l \beta_l} \left(M \log\log n\alpha_k + M \log \rho + o(1)\right) \\ &= \sum_{k=1}^{K} \frac{\alpha_k \beta_k}{\sum_{l=1}^{K} \alpha_l \beta_l} \left(M \log\log n + \underbrace{\log\left(1 + \frac{\alpha_k}{\log n}\right)}_{=o(1)} + M \log \rho + o(1) \right) \\ &= M \log\log n + M \log \rho + o(1). \end{aligned}$$

Now, in order to find the difference between the throughputs in the two cases we have

$$C_{ob} - C_{tdob} = \int_0^\infty \underbrace{M \log(1+x) n f_s(x) \left(F_s(x)^{n-1} - \sum_{k=1}^{K} \frac{\alpha_k t_k}{T} F_s(x)^{\alpha_k n - 1} \right)}_{h(x)} \mathrm{d}\, x.$$

Define

$$l^- = \rho(\log n - (M+3)\log\log n), \qquad l^+ = \rho(\log n + (M+3)\log\log n). \qquad (7.37)$$

We break the integral into the following three regions: $\mathcal{I}_1 = [0, l^-]$, $\mathcal{I}_2 = [l^-, l^+]$ and $\mathcal{I}_3 = [l^+, \infty]$. It can be easily checked that

$$\begin{aligned} \left| \int_{\mathcal{I}_1} h(x) \mathrm{d}\, x \right| &= O(\log\log n \int_{\mathcal{I}_1} M n f_s(x) F_s(x)^{\min\{\alpha_k\} n - 1} \mathrm{d}\, x) \qquad (7.38) \\ &= O(\log\log n \, F_s(x)^{\min\{\alpha_k\} n} |_0^{l^-}) \\ &= O\left(\log\log n \left(1 - \frac{(\log n)^{M+3}}{n(1+\rho\log n)^{M-1}} \right)^{\min\{\alpha_k\} n} \right) \\ &= O(\log\log n \cdot e^{-(\log n)^4}). \end{aligned}$$

As for the third region, we have

$$\begin{aligned} \left| \int_{\mathcal{I}_3} h(x) \mathrm{d}\, x \right| &\leq \int_{\mathcal{I}_3} M n \log(1+x) f_s(x) \mathrm{d}\, x \qquad (7.39) \\ &\leq \frac{1}{(\rho \log n)^{M-1}} \int_{\mathcal{I}_3} M n \log(1+x) \frac{e^{-x/\rho}}{\rho} \mathrm{d}\, x \\ &= O\left(\frac{n}{(\rho\log n)^{M-1}} \left(-\log(1+x) e^{-\frac{x}{\rho}} |_{l^+}^{\infty} \right) - e^{\frac{1}{\rho}} \mathrm{Ei}\,(-\frac{l^+}{\rho}) \right), \end{aligned}$$

where $\mathrm{Ei}\,(-x) = -\int_x^\infty \frac{e^{-y}}{y} \mathrm{d}\, y$ is the exponential integral. For large x we have

$\mathrm{Ei}\,(-x) = \Theta(-\frac{e^{-x}}{x})$. Using this identity in the previous inequality gives

$$\left| \int_{\mathcal{I}_3} h(x) \mathrm{d}\, x \right| = O(\frac{\log\log n}{(\log n)^{2M+2}}).$$

Now we move on to the integral over $\mathcal{I}_2$. Using integration by parts we have

$$\begin{aligned} \int_{\mathcal{I}_2} h(x) \mathrm{d}\, x &= \log(1+x)(F(x)^n - \sum_{k=1}^{K} \frac{t_k}{T} F(x)^{\alpha_k n})|_{l^-}^{l^+} \\ & \quad - M \int_{\mathcal{I}_2} \frac{F(x)^n - \sum_{k=1}^{K} \frac{t_k}{T} F(x)^{\alpha_k n}}{1+x} \mathrm{d}\, x. \end{aligned}$$

We can further simplify the integral to

$$\begin{aligned} \int_{\mathcal{I}_2} h(x) \mathrm{d}\, x &= \Theta(\frac{\log\log n}{(\log n)^{2M+2}}) \\ & \quad - \frac{M}{\rho \log n} \left(1 - O(\frac{\log\log n}{\log n})\right) \int_{l^-}^{l^+} \left(F(x)^n - \sum_{k=1}^{K} \frac{t_k}{T} F(x)^{\alpha_k n} \right) \mathrm{d}\, x. \end{aligned}$$

Next, we show that the last integral in the above equation is of order constant. Suppose $i = \arg\min_k \{\alpha_k\}$. Note that with this condition $\alpha_i \leq \frac{1}{2}$. It can be verified that

$$\begin{aligned} \int_{l^-}^{l^+} \left(F(x)^n - \sum_{k=1}^{K} \frac{t_k}{T} F(x)^{\alpha_k n} \right) \mathrm{d}\, x &= \Theta\left(\int_{l^-}^{l^+} \left(F(x)^n - F(x)^{\alpha_i n} \right) \mathrm{d}\, x \right) \qquad (7.40) \\ \overset{(a)}{=} \Theta\left(\int_{-4\log\log n}^{(2M+2)\log\log n} (1 - \frac{e^{-y}}{n(1+O(\frac{\log\log n}{\log n}))^{M-1}})^n - (1 - \frac{e^{-y}}{n(1+O(\frac{\log\log n}{\log n}))^{M-1}})^{\alpha_i n} \mathrm{d}\, y \right) \\ \overset{(b)}{=} \Theta\left(\int_{-4\log\log n}^{(2M+2)\log\log n} \exp\left(-\nu(n) e^{-y}\right) - \exp\left(-\nu(n)\alpha_i e^{-y}\right) \mathrm{d}\, y \right) \\ \overset{(c)}{=} \Theta\left(\int_{\frac{1}{(\log n)^{2M+2}}\nu(n)}^{(\log n)^4 \nu(n)} \frac{e^{-z} - e^{-\alpha_i z}}{z} \mathrm{d}\, z \right) \\ \overset{(d)}{=} \Theta(\ln \alpha_i + \frac{\log\log n}{\log n}) = \Theta(1), \end{aligned}$$

where (a) follows by a change of variable $y = \frac{x}{\rho} - \log n + (M-1)\log\log n$, and (b) holds because

$$(1-\frac{e^{-y}}{n(1+\Theta(\frac{\log\log n}{\log n}))^{M-1}})^n = \Theta\left((1-O(\frac{(\log n)^4}{n}))\exp\left(-\nu(n)e^{-y}\right)\right),$$

for large n, $\nu(n) = (1+O(\frac{\log\log n}{\log n}))^{M-1}$, and for y in the range defined in the integral. (c) follows after change of variable of $z = \nu(n)e^{-y}$, and (d) is a direct consequence of the following asymptotic expansions for $-\mathrm{Ei}\,(-x) = \int_x^\infty \frac{e^{-t}}{t}\mathrm{d}\,t$ for small and large x, respectively [129]:

$$\mathrm{Ei}\,(-x) = \ln x + \gamma_0 + O(x), \qquad x \ll 1, \quad \mathrm{Ei}\,(-x) = -\frac{e^{-x}}{x}(1+O(\frac{1}{x})) \qquad x \gg 1.$$

Putting Equations (7.38) to (7.40) together we get

$$(C_{ob} - C_{tdob}) = \Theta(\frac{1}{\log n}).$$

■

This Theorem implies that rate constraint does not drastically degrade the sum-rate of the systems as long as the group sizes are order n and n is large. It also raises the question whether the difference between the sum-rate of TO beamforming and sum-rate capacity can be improved or not.

7.5.3 Weighted Opportunistic (WO) Beamforming

In the weighted opportunistic scheme we weigh the SINR of each user according to its group by μ_k, $k = 1, \ldots, K$. Then, during each coherence interval, the transmitter assigns the M random beams to the M users that have the largest *weighted* SINR. More specifically, let the SINR corresponding to beam m and user i be denoted by $x_{i,m}$. The distribution of $x_{i,m}$ was given in (7.11). With this notation, beam m is assigned to the user l_0 from group k_0 such that

$$(k_0, l_0) = \arg \max_{(k,i):i \in \mathcal{G}_k} \mu_k x_{i,m}.$$

Equivalently, one can see that beam m assigned to group k_0 such that

$$k_0 = \arg \max_{1 \le k \le K} \mu_k \max_{i \in \mathcal{G}_k} x_{i,m}. \tag{7.41}$$

In the WO beamforming scheme there are two questions to be answered: first, how to determine the weights such that the rational rate constraints are met (Here, unlike the TO case, the answer is not trivial), and second, what is the rate loss compared to the unconstrained sum-rate capacity of the broadcast channel itself?

The following theorems settle the aforementioned questions.

Theorem 7.9. *Consider the fading MIMO Gaussian broadcast channel of Section 7.2. Consider the WO beamforming scheme with*

$$\mu_k = 1 + \frac{\log \beta_k}{\log n - (M-1)\log\log n}. \tag{7.42}$$

Assuming, M, ρ, α_ks and β_ks are fixed, we have

$$\lim_{n\to\infty} \frac{R^k}{R^K} = \beta_k, \quad k = 1, \ldots, K. \tag{7.43}$$

Proof. Without loss of generality let us look at the average transmitted rate to the first user in the first group (denoted by R^1). Clearly,

$$R^1 = M \int_0^\infty \log(1+x) f_s(x) \Pr\left(\mu_1 x_{1,m} \ge \mu_k x_{i,m}, \quad \forall\, k, i \text{ s.t. } 1 \le k \le K, i \in \mathcal{G}_k \middle| x_{1,m} = x\right) \mathrm{d}\, x.$$

Using the independence of the SINRs for different users, the probability of the event defined inside the above integral can be written as

$$\prod_{k=1}^{K} \prod_{i\in\mathcal{G}_k, i\neq 1} \Pr\left(x_{i,m} \leq \frac{\mu_1}{\mu_k} x\right) = F_s(x)^{\alpha_1 n-1} \prod_{k=2}^{K} F(\frac{\mu_1}{\mu_k} x)^{\alpha_k n}.$$

Accordingly, we have

$$R^1 = M \int_0^\infty \underbrace{\log(1+x) f_s(x) F_s(x)^{\alpha_1 n-1} \prod_{k=2}^{K} F_s(\frac{\mu_1}{\mu_k} x)^{\alpha_k n}}_{h_1(x)} \,\mathrm{d}\, x.$$

We further split the above integral to three integrals over the intervals $\mathcal{I}_1 = [0, l^-]$, $\mathcal{I}_2 = [l^-, l^+]$, and $\mathcal{I}_3 = [l^+, \infty]$, where l^- and l^+ are defined in (7.37). The integral over the first region can be written as

$$\begin{aligned}
\int_{\mathcal{I}_1} h_1(x) \mathrm{d}\, x &\overset{(a)}{=} O(\log\log n \int_{\mathcal{I}_1} M f_s(x) F_s(x)^{\alpha_1 n-1} \mathrm{d}\, x) \\
&= O(\frac{\log\log n}{n} F_s(x)^{\alpha_1 n}|_0^{l^-}) \\
&= O(\log\log n \cdot e^{-(\log n)^4}),
\end{aligned}$$

where (a) follows from the fact that $0 \leq F_s(x) \leq 1$. Similarly, it can be shown that for the integral over the third region we have

$$\int_{\mathcal{I}_3} h_1(x) \mathrm{d}\, x = O(\frac{\log\log n}{(\log n)^{2M+2}}).$$

Hence, the main contribution is due to the integral over the second interval. Looking at the behavior of $F_s(\frac{\mu_1}{\mu_k}x)^{\alpha_k n}$ for $x \in \mathcal{I}_2$ we have

$$
\begin{aligned}
F_s(\frac{\mu_1}{\mu_k}x) &= F_s\left(\frac{\mu_1}{\mu_k}(y+\rho(\log n-(M-1)\log\log n))\right) \qquad (7.44)\\
&\overset{(a)}{=} F_s\left(\rho\left(y(1+O(\frac{1}{\log n}))+\log(\frac{\beta_1}{\beta_k})+\log n-(M-1)\log\log n\right)+O(\frac{1}{\log n})\right),
\end{aligned}
$$

where (a) follows by defining $y = \frac{x}{\rho} - \log n + (M-1)\log\log n$, where $y \in \mathcal{I}_2' = [(2M+2)\log\log n, -4\log\log n]$, and using the definition of μ_k's in the theorem statement. Further simplification gives

$$
\begin{aligned}
\left(F_s\left(\frac{\mu_1}{\mu_k}x\right)\right)^{\alpha_k n} &\overset{(b)}{=} \left(1-\frac{\beta_k}{\beta_1}\frac{e^{-y(1+O(\frac{1}{\log n}))}(1-O(\frac{\log\log n}{\log n}))}{n\rho^{M-1}}\right)^{\alpha_k n}\\
&= \exp\left(\alpha_k n\log\left(1-\frac{\beta_k}{\beta_1}\frac{e^{-y(1+O(\frac{1}{\log n}))}(1-O(\frac{\log\log n}{\log n}))}{n\rho^{M-1}}\right)\right)\\
&= (1-O(\frac{\log\log n}{\log n}))\exp(-\frac{\alpha_k\beta_k}{\beta_1\rho^{M-1}}e^{-y}),
\end{aligned}
$$

where (b) follows by substituting $F_s(x)$ in (7.44) with its expression from (7.12) and noting that for $y \in \mathcal{I}_2'$ we have $(1+\frac{y}{\log n})^{1-M} = 1 - O(\frac{\log\log n}{\log n})$. The last equality is obtained by expanding the logarithm. Using similar arguments, it can be shown that for $y \in \mathcal{I}_2'$,

$$
f_s(\rho y+\rho(\log n-(M-1)\log\log n)) = (\frac{1}{\rho^M n} - O(\frac{\log\log n}{n\log n}))e^{-y}.
$$

Using the above asymptotic expressions for $F_s(\frac{\mu_1}{\mu_k}x)^{\alpha_k n}$ and $f_s(x)$, the integral over

$\mathcal{I}_2$ can be written as

$$\begin{aligned}
\int_{\mathcal{I}_2} h_1(x)\mathrm{d}\,x &= M\frac{\log\log\rho n}{n\rho^{M-1}}\left(1+O(\frac{\log\log n}{\log n})\right)\int_{\mathcal{I}_2'} e^{-y}\exp\left(-\frac{\sum_{k=1}^{k}\alpha_k\beta_k}{\beta_1\rho^{M-1}}e^{-y}\right)\mathrm{d}\,y \\
&= M\frac{\log\log\rho n}{n\rho^{M-1}}\left(1+O(\frac{\log\log n}{\log n})\right)\int_{(\log n)^{-2M-2}}^{(\log n)^4}\exp\left(-\frac{\sum_{k=1}^{k}\alpha_k\beta_k}{\beta_1\rho^{M-1}}z\right)\mathrm{d}\,z \\
&\overset{(a)}{=} M\frac{\log\log\rho n}{n\rho^{M-1}}\left(1+O(\frac{\log\log n}{\log n})\right)\frac{e^{-A_0 z}}{A_0}\bigg|_{(\log n)^4}^{(\log n)^{-2M-2}} \\
&= M\frac{\log\log\rho n}{\rho^{M-1}A_0 n}(1-O(\frac{\log\log n}{n\log n})) \\
&= M\frac{\beta_1}{\sum_{k=1}^{K}\beta_k\alpha_k}\frac{\log\log\rho n}{n}+O(\frac{\log\log n}{n\log n}),
\end{aligned}$$

where $A_0 = -\frac{\sum_{k=1}^{k}\alpha_k\beta_k}{\beta_1\rho^{M-1}}$ in (a) and $z = e^{-y}$.

Putting these integrals together we have

$$R^1 = M\frac{\beta_1}{\sum_{k=1}^{K}\beta_k\alpha_k}\frac{\log\log\rho n}{n}+O(\frac{\log\log n}{n\log n}).$$

Similarly, we can check that for a user in group k the average transmitted rate is

$$R^k = M\frac{\beta_k}{\sum_{k=1}^{K}\beta_k\alpha_k}\frac{\log\log\rho n}{n}+O(\frac{\log\log n}{n\log n}),$$

and hence the rational constraints are readily obtained. Also notice that achievable sum-rate using WO beamforming is

$$C_{wob} = \sum_{k=1}^{K}\alpha_k n R^k = Mn\frac{\sum_{k=1}^{K}\beta_k\alpha_k}{\sum_{j=1}^{K}\beta_j\alpha_j}\frac{\log\log\rho n}{n}+O(\frac{\log\log n}{\log n}) = M\log\log\rho n+O(\frac{\log\log n}{\log n}).$$

Comparing this with the sum-capacity we see that

$$\lim_{n\to\infty}(C_{wob}-C_{ob}) = 0.$$

■

Remark 7.10. *Theorem 7.9 asserts that the average rates of users are quite sensitive to the change of μ_is. In order to further understand the impact of a change in μ_is on the rates, we consider a two-group system. Following the methodology in the proof of Theorem 7.9, we can prove the following results. If*

$$\frac{\mu_1}{\mu_2} = 1 - o\left(\frac{1}{\log n}\right)$$

then,

$$\lim_{n\to\infty} \frac{R^1}{R^2} = 1,$$

where R^k is the average rate provided to a user in group k. Moreover, if

$$\frac{\mu_1}{\mu_2} = c < 1$$

then,

$$\lim_{n\to\infty} \frac{R^1}{R^2} = 0,$$

where c is a constant independent of n.

The above Theorem also shows that, as in the case of TO beamforming, WO beamforming achieves the sum-rate of the unconstrained broadcast channel as $n \to \infty$. As a matter of fact, the simulation results of Section 7.6 suggest that throughput of the WO beamforming is much closer to the sum-capacity than the throughput of the TO beamforming.

In the next lemma we show that for a two group case, the difference between the throughput of WO and the sum-rate of opportunistic beamforming with no rational rate constraints scales at most inversely proportional to $\log n$. Carrying the proof over to more than two groups is a possible, however a cumbersome, task.

Lemma 7.11. *Consider the setting of Theorem 7.9 and assume that there are two* *groups present in the network. Let C_{wob} denote the sum of the rates obtained by the weighted opportunistic beamforming scheme. Then, for any group size $\alpha_i n$, and any β_i's for $i = 1, 2$, the sum-rate loss of WO tends to zero with a convergence rate at least as fast as $\frac{1}{\log n}$, i.e.,*

$$\lim_{n\to\infty} (C_{sum} - C_{wob}) = O\left(\frac{1}{\log n}\right). \tag{7.45}$$

Proof. For each beam m, let x_m and y_m denote the maximum SINR in groups $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively. Furthermore, assume that $\beta_1 \geq \beta_2$. We can write the loss in the throughput as

$$C_{ob} - C_{wob} = \sum_{m=1}^{M} \mathrm{E} \log \left(\frac{1 + x_m \mathbb{1}(x_m \geq y_m) + y_m \mathbb{1}(x_m < y_m)}{1 + x_m \mathbb{1}(x_m \geq \frac{\mu_2}{\mu_1} y_m) + y_m \mathbb{1}(x_m < \frac{\mu_2}{\mu_1} y_m)} \right).$$

The difference can be further simplified to get

$$C_{ob} - C_{wob} = \sum_{m=1}^{M} \mathrm{E} \left(\log(\frac{1 + y_m}{1 + x_m}) | y_m \geq x_m \geq \frac{\mu_2}{\mu_1} y_m \right).$$

Note that for $y_m \geq x_m \geq \frac{\mu_2}{\mu_1} y_m$ we have

$$\frac{1 + y_m}{1 + x_m} \leq \frac{1 + y_m}{1 + \frac{\mu_2}{\mu_1} y_m} \leq \frac{\mu 1}{\mu_2}.$$

Therefore, the sum-rate loss is upper-bounded as

$$C_{ob} - C_{wob} \leq \sum_{m=1}^{M} \log(\frac{\mu_1}{\mu_2}) \cdot \Pr\left(y_m \geq x_m \geq \frac{\mu_2}{\mu_1} y_m\right).$$

Using the fact that $\Pr(y_m \geq x_m \geq \frac{\mu_2}{\mu_1} y_m) = \Theta(1)$ and $\frac{\mu_1}{\mu_2} = 1 + \Theta(\frac{1}{\log n})$, we get

$$C_{ob} - C_{wob} = O(\frac{1}{\log n}),$$

and this completes the proof. ∎

In the next subsection we look into a scheme that employs superposition coding and clearly leads to the best throughput for single antenna systems as we actually work on the boundary of the capacity region. As the analysis becomes complicated, we just consider two groups and obtain a scheduling that maximizes the throughput while maintaining the rational rate constraints of (7.2). It should be mentioned that the ergodic capacity region of a broadcast channel with two users has been studied in [115]; here we look at a generalization of the result of [115] in which we have n users divided into two groups with different rate demands.

7.5.4 Superposition Coding for Single Antenna Broadcast Channels

In this section we analyze the performance of superposition coding for the case when there are only two groups of users $\mathcal{G}_1, \mathcal{G}_2$ with sizes $\alpha_1 n$ and $\alpha_2 n$. We assume that the average rate provided to a user in the first group is required to be $\beta > 1$ times the rate provided to a user in the second group.

In order to maximize the rate (sum-rate) while keeping the ratio of different group rates fixed and equal to β, we need to find the point on the boundary of the capacity region of the Gaussian broadcast channel with short-term power constraint ρ that satisfies the differentiated rate constraint. We know that every boundary point is the solution to a maximization problem of form

$$\max_{(R_1,\ldots,R_n)\in\mathcal{C}_{BC}} \sum_{i=1}^{n} \mu_i R_i$$

for some positive values of $\mu_1, \ldots, \mu_n$. In our case, because of the symmetry among the users in each group, the values of μ_is will be the same for the users in the same group. Therefore, we only need to characterize the boundary points that are the maximizing solution to the problem

$$\max_{(R_1,\ldots,R_n)\in\mathcal{C}_{BC}} \mu_1(\sum_{i\in\mathcal{G}_1} R_i) + \mu_2(\sum_{i\in\mathcal{G}_2} R_i)$$

for $\mu_1, \mu_2 > 0$. The following lemma characterizes such boundary points. The proof of this lemma uses the duality of the broadcast channel and the multi-access channel for scalar channels explained in Section 7.3.

Lemma 7.12. *Consider a scalar Gaussian broadcast system with the model described in Section 7.2. Consider the following optimization problem:*

$$\max_{(R_1,\ldots,R_n)\in\mathcal{C}_{BC}} \mu_1(\sum_{i\in\mathcal{G}_1} R_i) + \mu_2(\sum_{i\in\mathcal{G}_2} R_i), \tag{7.46}$$

where $\mathcal{C}_{BC}$ is the ergodic capacity region of broadcast channel with short-term power constraint ρ, and $\mu_1 \geq \mu_2$ are two positive numbers. Then, the solution of the above optimization problem is

$$\begin{aligned} \alpha_1 n R_i \;&=\; E(\log(1+\rho x)|\mu_1 x \geq \mu_2 y) + \\ &\quad\; E(\log(\frac{(\mu_1-\mu_2)y(1+\rho x)}{\mu_1(y-x)})|(x,y)\in\mathcal{R}) \end{aligned} \tag{7.47}$$

for $i \in \mathcal{G}_1$. Similarly, for $i \in \mathcal{G}_2$ we have

$$\begin{aligned}\alpha_2 n R_i &= E(\log(1+\rho y)|\mu_1 x \leq \mu_2 y) - \\ &\quad E(\log(\frac{(\mu_1-\mu_2)x(1+\rho y)}{\mu_2(y-x)})|(x,y) \in \mathcal{R}), \end{aligned} \tag{7.48}$$

where $x = \max_{i \in \mathcal{G}_1} |h_i|^2$, $y = \max_{i \in \mathcal{G}_2} |h_i|^2$, and region $\mathcal{R}$ is defined as

$$\mathcal{R} = \{(x,y) \in R^+ \times R^+ | 0 \leq \frac{\mu_2}{(\mu_1-\mu_2)x} - \frac{\mu_1}{(\mu_1-\mu_2)y} \leq \rho\}. \tag{7.49}$$

Proof. The duality between the broadcast channel and the multi-access channel for the scalar case in [120] (and in (7.4)) states that

$$\mathcal{C}_{BC} = \bigcup_{\sum_i P_i(\cdot)=\rho} \mathcal{C}_{MAC}(P_1(\underline{h}), \ldots, P_n(\underline{h})),$$

where $\underline{h} = (h_1, \ldots, h_n)$, $P_i(\underline{h})$ is the power allocation function of user i, and the union is over all the permissible power allocation functions. Furthermore,

$$\begin{aligned}&\mathcal{C}_{MAC}(P_1(\underline{h}), \ldots, P_n(\underline{h})) = \\ &\big\{\underline{R} : \sum_{i \in \mathcal{S}} R_i \leq \log(1 + \sum_{i \in \mathcal{S}} P_i(\underline{h})|h_i|^2), \forall \mathcal{S} \subset \{1, \ldots, n\}\big\}.\end{aligned}$$

Using the above region, we can rewrite (7.46) as a maximization problem over all the power allocation functions and all the corresponding rate vectors in the capacity region of the dual multi-access channel. Based on this, it can be verified that the maximum of (7.46) occurs when we send only to the users with the best channel in each group. Therefore,

$$V_o = \mathrm{E} \max_{\substack{P_x, P_y \\ P_x + P_y = \rho}} (\mu_1 - \mu_2)\log(1 + P_x x) + \mu_2 \log(1 + P_x x + P_y y),$$

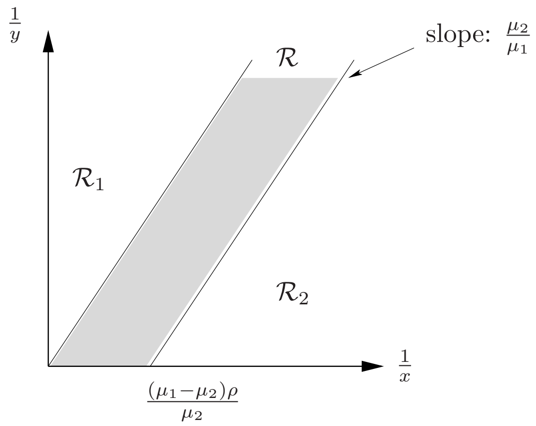


Figure 7.4: The decision region for power allocation in the superposition coding in two group case: If $(x, y) \in \mathcal{R}_1$, all the power is allocated to the best user of group one. If $(x, y) \in \mathcal{R}_2$, all the power is allocated to the best user of group two. If $(x, y) \in \mathcal{R}$, then power is split between the best users of the two groups as in (7.50).

where $x = \max_{i \in \mathcal{G}_1} |h_i|^2$ and $y = \max_{i \in \mathcal{G}_2} |h_i|^2$. Performing the maximization over P_x, P_y, we have one of the following possibilities:

(a) If $\mu_1 x \geq \mu_2 y$, we assign all the power to the best user of the first group.

(b) If $0 \leq \frac{\mu_2}{(\mu_1-\mu_2)x} - \frac{\mu_1}{(\mu_1-\mu_2)y} \leq \rho$, then we split the power between the two best users in the two groups as

$$P_x = \frac{(\mu_1 - \mu_2)Pxy + \mu_1 x - \mu_2 y}{\mu_1(y - x)x}, \quad P_y = \rho - P_x. \tag{7.50}$$

(c) If $\frac{\mu_2}{(\mu_1-\mu_2)x} - \frac{\mu_1}{(\mu_1-\mu_2)y} > \rho$, all the power is assigned to the best user in $\mathcal{G}_2$.

We have plotted the decision region for power allocation in the $(\frac{1}{x}, \frac{1}{y})$ region in Figure 7.5.4. In the weighted opportunistic scheduling, the power allocation policy would be to send to the best user in the first group if (x, y) is in $\mathcal{R}_1$, and to send to the best user in the second group if (x, y) is in $\mathcal{R} \cup \mathcal{R}_2$. ■

The question that remains to be answered is to figure out how to choose μ_1 and μ_2 such that the rate constraint in (7.2) is satisfied. This is answered in the following theorem.

Theorem 7.13. *Suppose $\beta > 1$ is fixed, $\mu_1 = 1$, and $\mu_2 = 1 - \frac{1}{(\log n)^\gamma}$ where $\gamma = 1 + \frac{1}{\alpha_1\beta+\alpha_2}$ (i.e., $1 \le \gamma \le 2$); then*

$$\lim_{n\to\infty} \frac{R^1}{R^2} = \beta. \tag{7.51}$$

Proof. Consider the first terms in the rate expressions for groups one and two in (7.47) and (7.48), respectively. Based on Remark 7.10, since $\frac{\mu_2}{\mu_1} = 1 - o(\frac{1}{\log n})$, it can be easily checked that these two terms are of order $\alpha_1 \log\log n$ and $\alpha_2 \log\log n$, respectively. Now we look at the second expectation in (7.47):

$$A^\star = E\left(\log(\frac{y(1+\rho x)}{((\log n)^\gamma - 1)(y-x)})|(x,y)\in\mathcal{R}\right).$$

It can be verified that for $(x,y)\in\mathcal{R}$ the term inside the logarithm is greater than one. Hence, $A^\star$ is positive. Furthermore, similar to previous cases, the main contribution of the expectation comes from values of x and y around $\log n$. Therefore, we can simplify write $A^\star$ as

$$\begin{aligned} A^\star &= E\left(\log(\frac{y(1+\rho x)}{((\log n)^\gamma - 1)})|(x,y)\in\mathcal{R}\right) - \underbrace{E(\log(y-x)|(x,y)\in\mathcal{R})}_{B^\star} \\ &= (2-\gamma)\log\log n(1 - O(\frac{\log\log n}{\log n}))\cdot \Pr\left((x,y)\in\mathcal{R}\right) - B^\star \end{aligned}$$

It can be shown that $\Pr\left((x,y)\in\mathcal{R}\right) = \alpha_2 - o(1)$. In the following we show that the expectation term in the above equation is $O(1)$. We have

$$B^\star = \Theta\left(\int_{l_1}^{l_2} \alpha_2 n e^{-y}(1-e^{-y})^{\alpha_2 n - 1} \int_{\frac{y(1-\frac{1}{(\log n)^\gamma})}{1+\frac{\rho y}{(\log n)^\gamma}}}^{y(1-\frac{1}{(\log n)^\gamma})} \log(y-x)\alpha_1 n e^{-x}(1-e^{-x})^{\alpha_1 n-1} \mathrm{d}\,x \mathrm{d}\,y\right),$$

where $l_1 = \log n - 4\log\log n$, and $l_2 = \log n + 4\log\log n$. Defining $y - x = z$ and

$w = y - \log n$, $B^\star$ can be simplified to

$$\begin{aligned} B^\star &= \Theta\left(\int_{-4\log\log n}^{4\log\log n} \alpha_2 e^{-w}(1-\frac{e^{-w}}{n})^{\alpha_2 n-1}\int_{(\log n)^{1-\gamma}}^{(\log n)^{2-\gamma}} \log(z)\alpha_1 e^z e^{-w}(1-\frac{e^z e^{-w}}{n})^{\alpha_1 n-1}\mathrm{d}\, z\mathrm{d}\, w\right) \\ &= \Theta\left(\int_{-4\log\log n}^{4\log\log n} \alpha_2 e^{-w}\exp\left(-\alpha_2 e^{-w}\right)\int_{(\log n)^{1-\gamma}}^{(\log n)^{2-\gamma}} \log(z)\alpha_1 e^z e^{-w}\exp\left(-\alpha_1 e^z e^{-w}\right)\mathrm{d}\, z\mathrm{d}\, w\right). \end{aligned}$$

Integration by part of the inner integral and further simplification gives

$$\begin{aligned} B^\star &= \Theta\left(\alpha_2(\gamma-1)\log\log n - \int_{-4\log\log n}^{4\log\log n}\alpha_2\frac{\exp\left(-\alpha_2 e^{-w}\right)}{e^w}\int_{(\log n)^{1-\gamma}}^{(\log n)^{2-\gamma}}\frac{\exp\left(-\alpha_1 e^z e^{-w}\right)}{z}\mathrm{d}\, z\mathrm{d}\, w\right) \\ &\overset{(a)}{=} \Theta\left(\alpha_2(\gamma-1)\log\log n - \alpha_2\int_{(\log n)^{1-\gamma}}^{(\log n)^{2-\gamma}}\frac{\exp\left(-\alpha_1 e^z\frac{1}{(\log n)^4}\right)}{(\alpha_2+\alpha_1 e^z)z}\mathrm{d}\, z\right) \\ &\overset{(b)}{=} \Theta\left(\alpha_2(\gamma-1)\log\log n - \alpha_2\int_{(\log n)^{1-\gamma}}^{1}\frac{1}{(\alpha_2+(1-\alpha_2)e^z)z}\mathrm{d}\, z\right) \\ &\overset{(c)}{=} \Theta\left(\alpha_2(\gamma-1)\log\log n - \alpha_2\int_{(\log n)^{1-\gamma}}^{1}\frac{1}{z}\mathrm{d}\, z\right) \\ &= O(1), \end{aligned}$$

where (a) follows by integrating over w first. Equality (b) follows because in the interval $[1, (\log n)^{2-\gamma}]$, the integrand is upper-bounded by e^{-z}, and therefore the contribution of the integral in this interval is of order constant. Also, for $[(\log n)^{1-\gamma}, 1]$, the numerator of the integrand is $1 - o(1)$. Finally, (c) follows by verifying that the integral is increasing in α_2 and in both extremes, i.e. $\alpha_2 = 1$ and $\alpha_2 = 0$ the integral is $(\gamma - 1)\log\log n + O(1)$.

Putting the evaluated values of $A^\star$ and $B^\star$ together we get the following rate for a user in group $\mathcal{G}_1$:

$$R^1 = (1 + (2-\gamma)\frac{\alpha_2}{\alpha_1})\frac{\log\log n}{n}(1 - O(\frac{\log\log n}{n\log n})). \tag{7.52}$$

Similarly, for a user in group $\mathcal{G}_2$ we have

$$R^2 = (\gamma - 1)\frac{\log\log n}{n}(1 - O(\frac{\log\log n}{n\log n})). \tag{7.53}$$

Therefore, to meet the ratio constraints between the rates of users in different groups, we should have

$$\frac{(\gamma - 1)}{(1 + (2-\gamma)\frac{\alpha_2}{\alpha_1})} = \frac{1}{\beta},$$

or accordingly

$$\gamma = 1 + \frac{1}{\alpha_1\beta + \alpha_2}.$$

This completes the proof. ■

Finally, we look into the throughput loss due to the constraint of (7.2) using superposition coding. It is clear that the convergence rate for the superposition coding should be faster than or equal to TO and WO beamforming. In the next lemma, we provide a bound on the difference between the sum-capacity and the sum-rate obtained by the scheduling discussed in this section.

Lemma 7.14. *Suppose $\beta > 1$ is fixed and μ_1, μ_2 are chosen as in Theorem 7.13. Then*

$$\int_0^\infty n\log(1+\rho x)e^{-x}(1-e^{-x})^{n-1} - \sum_{i=1}^{n} R_i = O\left(\frac{1}{(\log n)^{2\gamma-1}}\right)$$

Proof. Here is the outline of the proof. Using (7.47), we can write the throughput under constraints of (7.2) as [9]

$$\mathrm{E}\,\log\left(1 + \rho x \mathbb{1}(x \geq \mu y) + \rho y \mathbb{1}(x < \mu y)\right) + \mathrm{E}\,\log\left(\frac{(1+\rho x)y}{(1+\rho y)x} | (x,y) \in \mathcal{R}\right),$$

where x and y are defined in Lemma 7.12. Therefore, $\Delta(n)$, the difference of the

[9] In this proof we refer to μ_2 as μ.

sum-capacity and the throughput given above, can be written as

$$0 \leq \Delta(n) = \mathrm{E}\ \log\left(\frac{1+\rho x \mathbb{1}(x \geq y) + \rho y \mathbb{1}(x < y)}{1+\rho x \mathbb{1}(x \geq \mu y) + \rho y \mathbb{1}(x < \mu y)}\right) - \mathrm{E}\ \log\left(\frac{(1+\rho x)y}{(1+\rho y)x} | (x,y) \in \mathcal{R}\right).$$

We can simplify the right hand side of the above equation to get

$$\Delta(n) = \mathrm{E}\ \log\left(\frac{1+\rho y}{1+\rho x}\middle| y \geq x \geq \mu y\right) + \mathrm{E}\ \log\left(\frac{(1+\rho y)x}{(1+\rho x)y} | (x,y) \in \mathcal{R}\right).$$

It can be easily checked that the second term is positive over region $\mathcal{R}$ defined in (7.49). Therefore, we have

$$\begin{aligned} \Delta(n) &\leq \mathrm{E}\ \log\left(\frac{1+\rho y}{1+\rho x}\middle| y \geq x \geq \mu y\right) \\ &\leq \mathrm{E}\ \log\left(\frac{1+\rho y}{1+\rho \mu y}\middle| y \geq x \geq \mu y\right) \\ &\leq \mathrm{E}\ \log\left(1 + \frac{\rho(1-\mu)y}{1+\rho\mu y}\middle| y \geq x \geq \mu y\right) \\ &\leq \log(1+\frac{1-\mu}{\mu}) \cdot \Pr\left(y \geq x \geq \mu y\right) \\ &= -\log\mu \cdot \Pr\left(y \geq x \geq \mu y\right). \end{aligned} \tag{7.54}$$

Now, using the techniques developed so far, it can be shown that

$$\Pr\left(y \geq x \geq \mu y\right) = O\left(\frac{1}{(\log n)^{\gamma-1}}\right).$$

Therefore, substituting μ with its value in (7.54), we have

$$\Delta(n) = O\left(-\log\left(1 - \frac{1}{(\log n)^{\gamma}}\right)\frac{1}{(\log n)^{\gamma-1}}\right) = O\left(\frac{1}{(\log n)^{2\gamma-1}}\right),$$

and this completes the proof. ∎

7.6 Simulation Results

In this section we present simulation results for the three scheduling schemes studied in this chapter. We will show their performance in terms of satisfying the rational rate constraints and also the difference of their throughput with the sum-rate with no constraints present.

The first set of simulations are for $M = 1$, $K = 2$, and $\beta = 2$, i.e., one group requires twice the rate of the second group. We consider the groups to be of equal size. Figure 7.5 shows the sum of the transmitted rate for WO, TO, and SC as a function of the number of users. As expected, all show a $\log \log n$ growth rate. In fact, the sum of the transmitted rates of WO and SC are quite close to the actual sum-rate capacity, signifying that the rate constraints do not lead to much of a rate hit on the throughput.

Figure 7.6 shows the ratio of the rates transmitted to the two groups as a function of the number of users for the WO and SC schedules. As we see, the rate of convergence of the SC to the desired ratio is slower than the WO beamforming. TO is not shown, as it clearly gives the correct ratio of $\beta = 2$.

The second set of simulations are for the same broadcast channel but with $\beta = 4$. The results are shown in Figure 7.7 and Figure 7.8.

Next we consider the performance for multiple antenna broadcast channels. For this set of simulations we consider a broadcast channel with two antennas at the transmitter, $M = 2$. We consider two groups of different size with $\alpha_2 = 2\alpha_1$, and different rate requirements as $\beta_2 = 2\beta_1$. Figure 7.9 shows the achievable throughput for TO and WO and compares it with the sum rate of opportunistic beamforming with no rate constraints. We have shown the ratio between the rates of users in different groups in Figure 7.10. As we see, WO schedule converges fast to the desired ratio. In Figure 7.11, we have shown the ratio of the rate for smaller size networks in

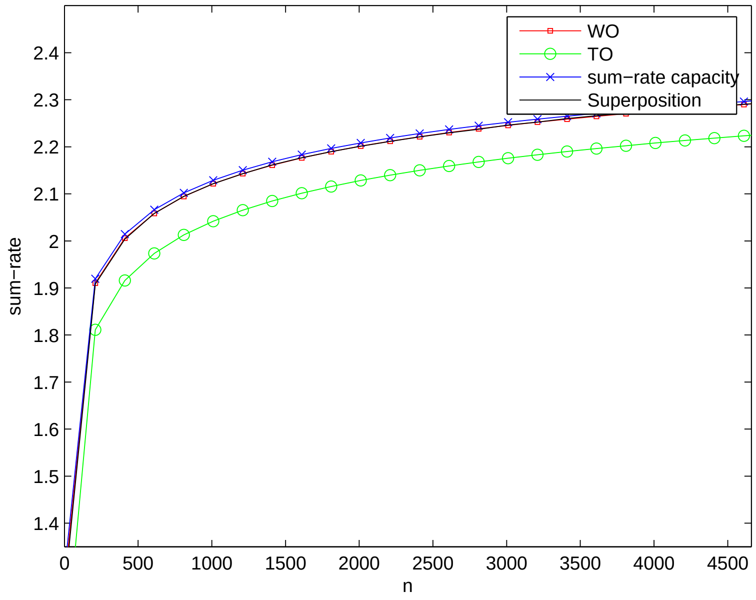


Figure 7.5: The sum of the transmitted rates for WO, TO, and SC, as well as the sum-rate capacity of the single antenna broadcast channel as a function of the number of users for a system with $K = 2$ and $\beta = 2$.

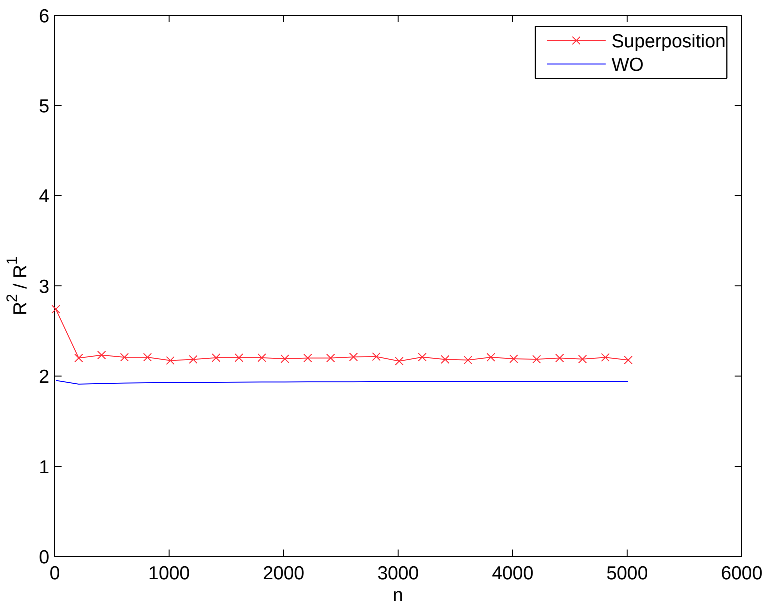


Figure 7.6: The ratio of the rates transmitted to the two groups of users as a function of the number of users for WO and SC for a system with $\beta = 2$.

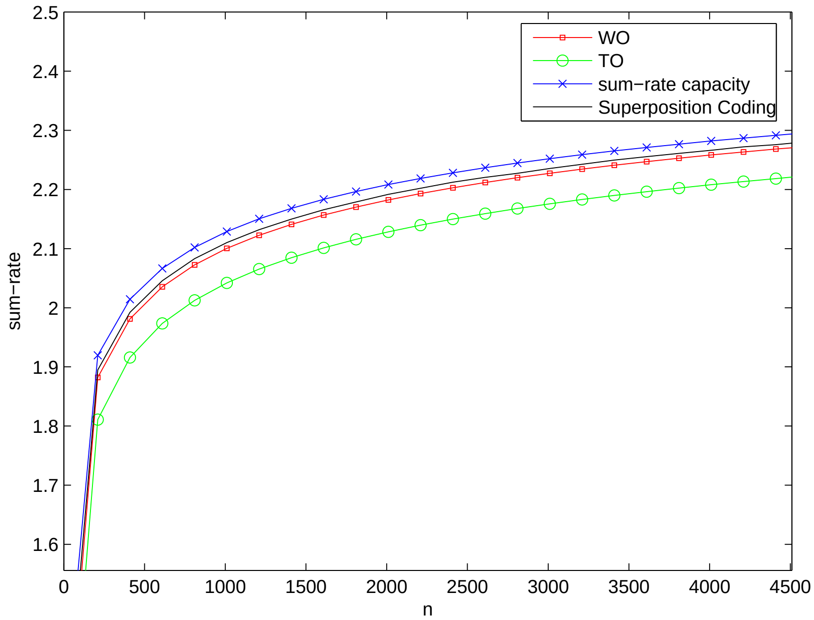


Figure 7.7: The sum of the transmitted rates for WO, TO, and SC, as well as the sum-rate capacity of the single antenna broadcast channel as a function of the number of users for a system with $K = 2$ and $\beta = 4$.

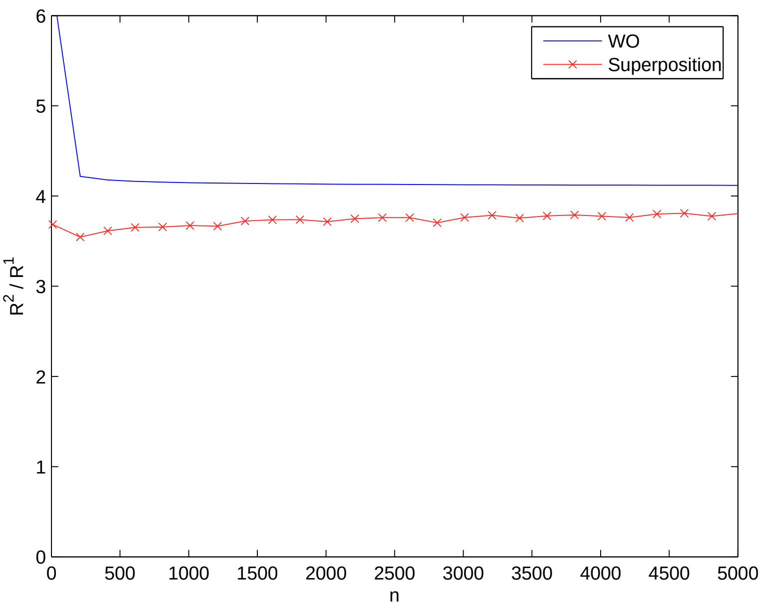


Figure 7.8: The ratio of the rates transmitted to the two groups of users as a function of the number of users for WO and SC for a system with $\beta = 4$.

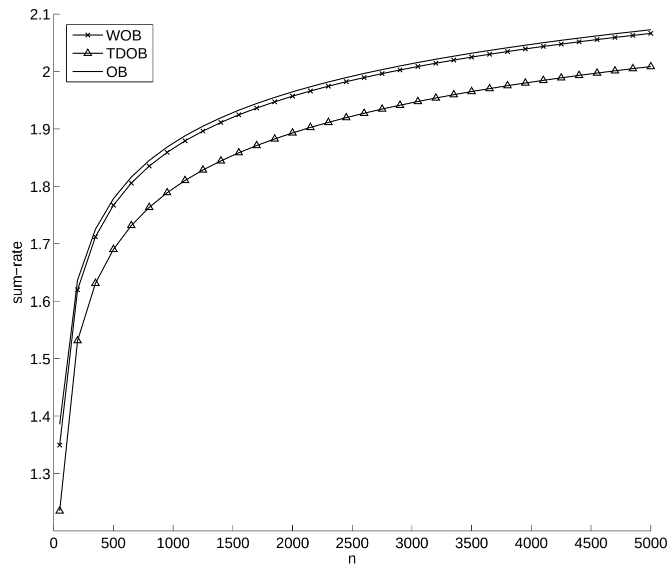


Figure 7.9: The sum of the transmitted rates for WO, TO, as well as the opportunistic for a broadcast channel with $M = 2$, $K = 2$, $\frac{\alpha_2}{\alpha_1} = 2$, and $\frac{\beta_1}{\beta_2} = \frac{1}{2}$ as a function of the number of users.

more detail. As we see, the ratio of the rates converges to the desired ratio even for small to moderate size networks. For instance, for a network of size $n = 50$, the ratio of the rates is 1.95 (only 0.05 off from the desired value).

Finally, the last set of simulation is done for a broadcast channel with two antennas at transmitter and three groups of users, i.e., $K = 3$ with equal size. The desired ratio is given as $\beta_1 = 1, \beta_2 = 2$, and $\beta_3 = 3$. In Figure 7.12 we have plotted the achievable sum-rate for TO and WO scheduling. In Figure 7.13 we have shown the achieved ratio of rates for different groups.

7.7 Conclusion

In this chapter we consider one of the design issues in the downlink of wireless cellular systems, namely differentiated quality of service provisioning. We consider a MIMO broadcast channel with fading, where users have different rate demands. In partic-

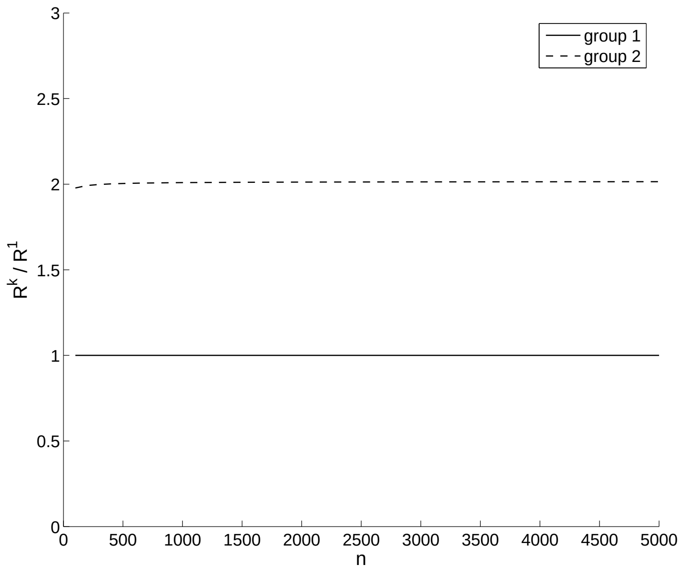


Figure 7.10: The ratio of the rates transmitted to the two groups of users as a function of the number of users for WO and TO for a system with $M = 2$, $K = 2$, $\frac{\alpha_2}{\alpha_1} = 2$, and $\frac{\beta_1}{\beta_2} = \frac{1}{2}$.

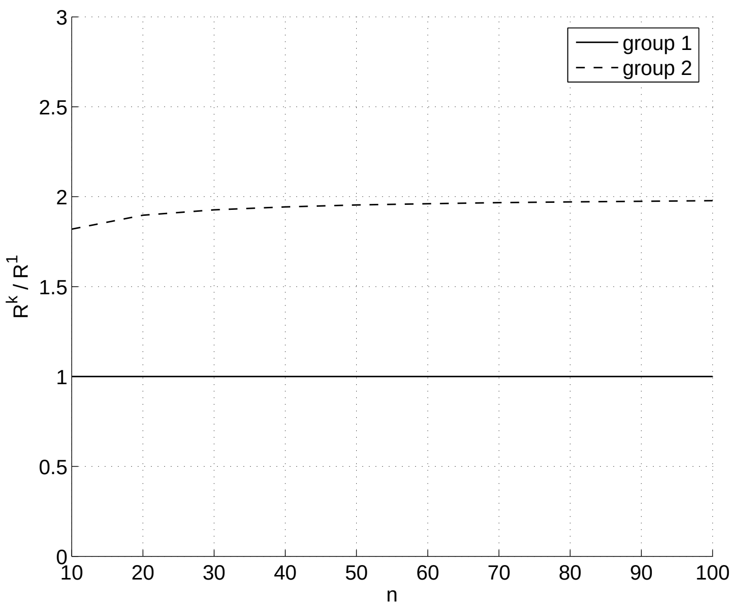


Figure 7.11: A closer look at the ratio of the rates transmitted to users in different groups for the example in Figure 7.10.

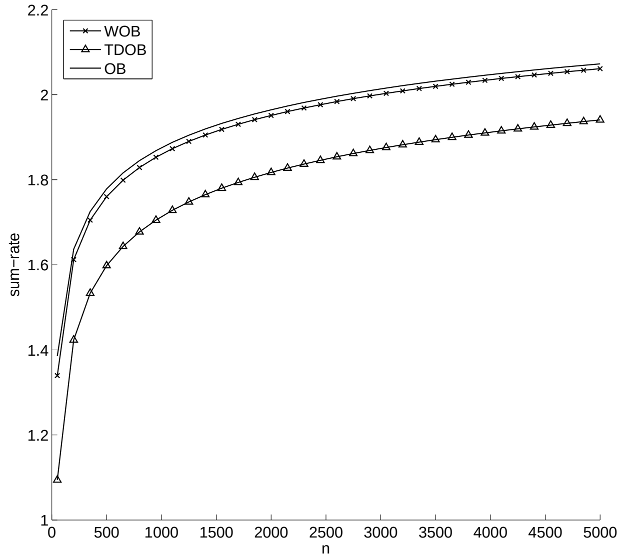


Figure 7.12: The sum of the transmitted rates for WO, TO, as well as the opportunistic for a broadcast channel with $M = 2$, $K = 3$, and $\beta_1 = 1, \beta_2 = 2$, and $\beta_3 = 3$ as a function of the number of users.

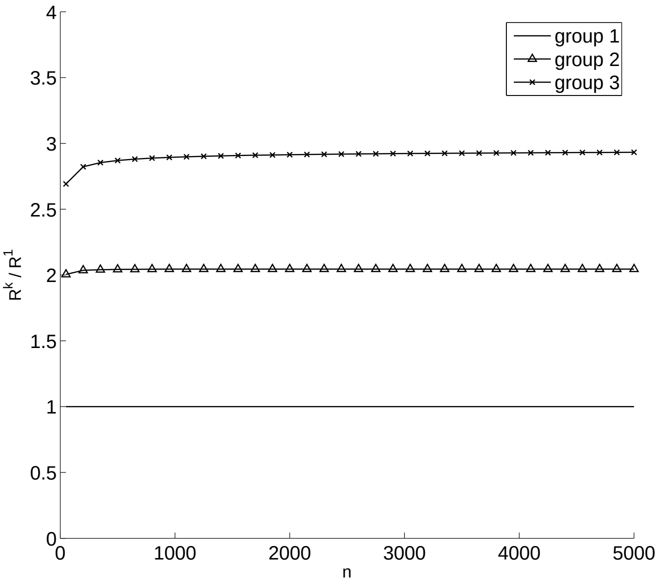


Figure 7.13: The ratio of the rates transmitted to the two groups of users as a function of the number of users for WO and TO for a system with $M = 2$, $K = 3$, and $\beta_1 = 1, \beta_2 = 2$, and $\beta_3 = 3$.

ular, we assume n homogenous users that are divided into K groups, each group of which requires the same rate, and where the ratio of the groups' rates are given. The transmitter would like to maximize the throughput (sum of the rates to all users) while maintaining the rational rate constraints. In general, this problem appears to be computationally intractable since the ergodic capacity region is described as the convex hull of (an infinite) set of rates. Furthermore, finding the exact capacity region requires complete knowledge of the channel states at the transmitter. Therefore, we are interested in *simple* schemes that require a *little* amount of information about the channels at the transmitter and operate close to the optimal capacity-achieving scheme. In particular, we propose three scheduling schemes to provide the rational rate constraints, namely weighted-opportunistic scheduling (WO), time-division opportunistic (TO), and superposition coding (SC) for single antenna systems. WO is a generalization of the opportunistic scheduling in which we transmit to only the user that has the largest "weighted" signal to noise plus interference ratio (SNIR). In TO, each group has its own time slot in which the transmitter chooses the user with the best SNIR from the corresponding group. Superposition coding is the scheme that achieves the information-theoretic capacity region. We first consider systems with $n = 2$ and $n = 3$, where each user requires a different rate. We focus on the achievable region by using the aforementioned WO. It turns out that determining the explicit relationship between the appropriate weights of the schedule and the desired ratios of the rates is analytically intractable even for the case of $n = 3$. For this reason, and also because most practical systems have many users, much of the chapter focuses on the asymptotic regime of large n, where explicit results can be found. For each scheduling we give an explicit scheme to guarantee the rational rate constraints. We also analyze the throughput loss due to the rate constraints for all three different schemes. In particular, we show that the throughput loss compared to the maximum throughput (i.e., the sum rate capacity without any rate constraints) tends to zero for large n. Thus, there is not much of a penalty in providing different levels of service to different users. We also analyze the convergence rate of all the schemes and provide simulations supporting the theoretical analysis.

There are many directions in which this work can be extended. In this chapter we considered homogenous networks, i.e., networks where different users have the same quality of reception statistics. However, many practical network are heterogenous. Generalizing the schedules considered in this chapter to a heterogenous case is an important problem. Looking at other Quality of Service (QoS) requirements such as delay and reliability and also studying the existing trade-off between these measure are also very challenging and interesting problems.

Chapter 8

MIMO Gaussian Broadcast Channel with Estimation Error

8.1 Introduction

There has recently been a great deal of research on the capacity region of the MIMO Gaussian broadcast channels (GBC) (e.g., see [117],[132]-[137]). These channels are of practical importance since they can be used as a model for the downlink of cellular systems equipped with multiple antennas at the transmitter and the receiver side. In [118], the authors show that the entire capacity region is achieved by the dirty paper coding scheme first introduced in [119].

While dirty paper coding is the optimal transmission scheme, it is computationally expensive (although suboptimal schemes such as channel inversion or Tomlinson-Harashima precoding [151, 152, 153] give relatively close performance to the optimal scheme) and also requires the transmitter to have perfect knowledge of the channel state information (CSI) for all the users. As mentioned in the previous chapter, for a complete and optimal design of cellular systems, there are many practical issues that still need to be answered. Identifying the critical CSI needed at the transmitter and the amount feedback required for providing this CSI, finding computationally-efficient scheduling schemes, and providing different quality of service to different users in the network, are examples of these issues.

In Chapter 7 we looked at one of these issues, namely differentiated quality of service provisioning in cellular systems. We proposed *simple* scheduling schemes based on random beamforming that requires *little* amount of side-information at the transmitter and provides different users with different rates.

In this chapter we look at another design issue mentioned above, namely the issue of the robustness of the capacity result with respect to error in channel state information. As mentioned in Chapter 7, the capacity of broadcast channels highly depends on the amount of channel state information in the transmitter (CSI). If perfect CSI is available at the transmitter, the throughput scales linearly with the number of transmit antennas (as the transmit power or the number of users increases). Comparing this result to a MIMO point-to-point system, we see that in the presence of perfect CSI, there is nothing to be lost by lack of cooperation among the receivers. On the other hand, if there is no CSI available at the transmitter, employing multiple antennas does not increase the throughput significantly. This is unlike the capacity result for MIMO point-to-point channels, where it is shown that even if the transmitter (or the receiver) does not know the channel, the capacity still scales with the number of antennas in the system [150, 109, 148, 140]. From a practical point of view, simple and effective scheduling schemes that are robust against noisy channel state information (and/or require partial knowledge of the channel) and also have a good performance are desirable [146]. There has been some progress on devising simple scheduling schemes that operate close to boundary points of the capacity region with limited feedback [121, 142, 145, 143]. However, the requirement of having accurate channel estimation is a strict constraint.

In this Chapter we consider the effect of channel estimation error on the capacity of MIMO Gaussian broadcast channels. We assume that the receivers have access to only an estimate (or noisy version) of their channels, and these estimates are fed back to the transmitter. We propose an achievable region based on the dirty paper coding scheme. This scheme is essentially similar to the one proposed for MIMO point-to-point and multi-access channels with uncertainty in channel measurements [111, 140]. We further show an interesting *duality* between the achievable rate region and the capacity of a multi-access channel where the noise covariance is dependent on the transmit power at different users. This duality is explored to show the effect of the estimation error on the sum-rate for large number of users and in the large power regime. It is shown that, for large number of users, as long as the estimation error is fixed with respect to the number of users, we achieve the same scaling law as if there was no estimation error. Of course, there is a loss due to the estimation error in the sum-rate which is obtained as a function of the covariance of the estimation error. However, in the large power regime, if the quality of our estimate does not increase with the transmit power, our achievable rate does not scale with the number of antennas. This is because in the large power, the system will be in the interference-dominated regime. As a matter of fact, in [126], the authors have shown that, unlike point-to-point systems, the sum-capacity of a broadcast channel with two users and two transmit antennas and with estimation error does not scale linearly with the number of antennas.

Based on the achievable rate region derived earlier, we analyze the performance of a training-based scheme for block fading models. We show that if the transmitter is willing to invest a fixed fraction of power in observing and training the channel, the sum-rate scales with the number of antennas at the transmitter. We also find the optimal training scheme that gives this linear scaling of the sum-rate in terms of the number of antennas.

The remainder of this chapter is organized as follows. After introducing the system model in the next section, we provide an achievable rate region for MIMO Gaussian broadcast channels with estimation error in Section 8.3. Section 8.4 looks at the optimal power allocation. In Section 8.5 we analyze the effect of the estimation error on the asymptotic behavior of the sum-rate for large number of users. Section 8.6 considers the achievable sum-rate based on training schemes and the conclusion comes in Section 8.7.

8.2 System Model

We consider a block fading MIMO Gaussian broadcast channel with channel estimation error. The transmitter employs M transmit antennas. We assume that there are n users in the system, each equipped with $r_i, i = 1, \ldots, n$ antennas. The channel matrix between the transmitter and user i is an $M \times r_i$ matrix and is denoted by H_i. A block fading model with coherence interval of length T is considered. We assume that the channel coefficients for each user are zero-mean jointly Gaussian random variables with covariance matrix $\operatorname{cov}(H_i) = \mathrm{E}\,(\operatorname{vec} H_i)(\operatorname{vec} H_i)^* = R_H$. The received signal at user i is given by

$$Y_i = H_i \underline{x} + \underline{n}_i,$$

where $\underline{n}_i$ is additive white Gaussian noise with zero mean and identity covariance matrix. $\underline{x}$ is the input vector with power constraint $\mathrm{E}\,[\underline{x}^* \underline{x}] \leq P$.

In this chapter, the users and the transmitter do not have exact knowledge of the channel matrices. We assume that user i estimates its channel to $\hat{H}_i$. This estimate is fed back to the transmitter through a perfect channel (see Figure 8.1). The channel estimation error $\tilde{H}_i$ which is equal to $H_i - \hat{H}_i$, is assumed to be uncorrelated from the estimate $\hat{H}_i$ (i.e, MMSE estimation). The coordinates of $\tilde{H}_i$ are assumed to be jointly Gaussian random variables with covariance matrices of form

$$\operatorname{cov}(\tilde{H}_i) = A_i^T \otimes K_i, \tag{8.1}$$

where A_i and K_i are positive semi-definite $M \times M$ and $r_i \times r_i$ matrices. The above

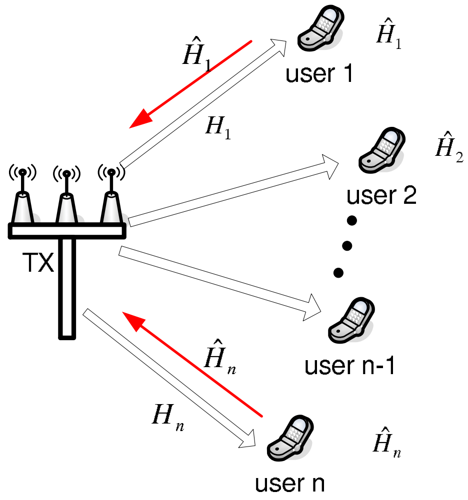


Figure 8.1: Multi-antenna Gaussian broadcast channel with channel estimation error.

covariance matrix models the possible correlation at the transmitter and the receiver side (see [144]). The capacity region of the aforementioned broadcast channel is known when H_i is available to the transmitter and to the i-th receiver for $i = 1, \ldots, n$ [118]. Assuming the noise covariance matrix N_i for the i-th user, and under the transmit covariance matrix constraint, i.e., $\mathrm{E}\left[\underline{x}\,\underline{x}^*\right] \preceq S$, the capacity region is given by dirty paper coding and can be written as

$$\mathcal{C}(S, \{N_i\}, \{H_i\}) = \mathrm{conv}\left\{ \bigcup_{\pi, \{B_i\}} \mathcal{R}(\pi, \{B_i\}, \{N_i\}, \{H_i\}) \right\},$$

where the union is over all permutations on set $\{1, \ldots, n\}$ and all positive semi-definite covariance matrices $B_1, \ldots, B_n$ such that $\sum_{i=1}^n B_i \preceq S$ and

$$\mathcal{R}(\pi, \{B_i\}, \{N_i\}, \{H_i\}) = \left\{ (R_1, \ldots, R_n) \right| \\ 0 \leq R_{\pi(i)} \leq \log \frac{|N_{\pi(i)} + H_{\pi(i)}(\sum_{k=1}^{i} B_{\pi(k)}) H_{\pi(i)}^*|}{|N_{\pi(i)} + H_{\pi(i)}(\sum_{k=1}^{i-1} B_{\pi(k)}) H_{\pi(i)}^*|} \left. \right\}.$$

Finally, the capacity region of the broadcast channel with average total transmit power constraint P, i.e., $\mathrm{tr}\,(S) \leq P$, is given by the

$$\mathcal{C}(P, \{N_i\}, \{H_i\}) = \bigcup_{S:\mathrm{tr}\,(S)\leq P} \mathcal{C}(S, \{N_i\}, \{H_i\}).$$

In order to compute any point on the boundary of the capacity region, [132, 133] establish a duality between the capacity region of broadcast and multiple access channels under sum power constraints. This duality is considered in a more general scenario and based on the mini-max (and the Lagrangian) duality in [136, 137]. These results are very useful since the multi-access channel capacity region has a nice polymatroid structure [34] that makes it much easier to work with [103].

8.3 Inner Bound on the Capacity Region

In this section we give an inner bound on the capacity region of the MIMO Gaussian broadcast channel with estimation error. Here we assume that the transmitter and the receiver have access to an estimate of the channel rather than the actual channel. The uncertainty in the estimate is modeled as a Gaussian random variable. The results are based on the fact that the worst uncorrelated noise with given covariance matrix has Gaussian distribution. This was in fact used previously to obtain lower bounds on the capacity of MIMO point-to-point channels and multi-access channels in [111, 140].

Theorem 8.1. *Consider a MIMO Gaussian broadcast channel described in Section 8.2 where the estimated channel for the i-th user is denoted by* $\hat{H}_i$*. We assume that* $\hat{H}_i$ *is known to the transmitter and the corresponding user. Then, the capacity region of this broadcast channel includes the capacity region of a MIMO Gaussian broadcast channel with channel matrices* $\hat{H}_i$ *and effective noise covariances* $I + \mathrm{tr}\,(A_i(\sum_{l=1}^{n} B_l))K_i$ *(as shown in Figure 8.2.) In other words, the capacity region*

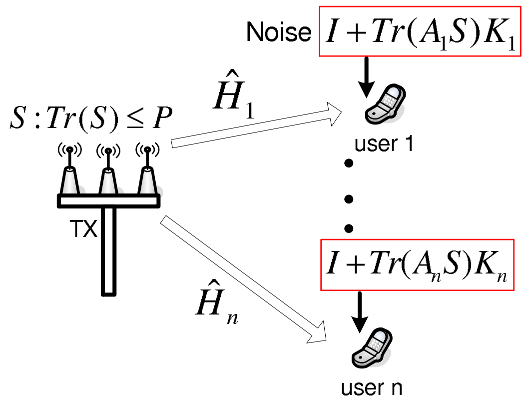


Figure 8.2: Inner bound on the capacity region with estimation error.

includes the following region

$$\mathrm{conv}\Bigg\{ \bigcup_{\substack{\pi,\{B_i\} \\ \mathrm{tr}\,(\sum_i B_i)\le P}} \mathcal{R}\Big(\pi, \{B_i\}, \{I + \mathrm{tr}\,(A_i(\sum_{l=1}^{n} B_l))K_i\}, \{\hat{H}_i\}\Big)\Bigg\}.$$

Proof. The proof follows a similar approach to the dirty paper coding scheme. Note that $\hat{H}_i$ is the measurement (estimate) of the channel H_i, and $\tilde{H}_i$ is the error in measurement. We can write the received signal at user i as

$$\underline{y}_i = \hat{H}_i\underline{x} + \underbrace{\tilde{H}_i\underline{x} + \underline{n}_i}_{\underline{v}_i}. \tag{8.2}$$

Without loss of generality, suppose $\pi(\cdot)$ is the identity permutation, i.e., $\pi(i) = i$ for all $i \in \{1, \dots, n\}$. Let us consider a broadcast channel with no channel estimation error and with channel coefficients $\{\hat{H}_i\}$ and noise covariance matrices $\{N_{i,eq}\}$, where

$$N_{i,eq} = \sum_{l=1}^{n} \mathrm{tr}\,(B_l A_i)K_i + I, \qquad i = 1, \dots, n, \tag{8.3}$$

and $B_1, \dots, B_n$ are positive definite matrices with $\sum_{l=1}^{n} \mathrm{tr}\,(B_l) \le P$. K_i and A_i are defined in (8.1). We will refer to this broadcast channel as BC_{eq}. The received signal at user i in BC_{eq} is

$$\underline{y}_{i,\,eq} = \hat{H}_i\underline{x} + \underline{n}_{i,\,eq}, \tag{8.4}$$

where $\underline{n}_{i,\,eq}$ is zero-mean a Gaussian noise with covariance $N_{i,\,eq}$.

Using the dirty paper coding approach and the result of [138] concerning the capacity of memoryless channels with a random state known non-causally at the transmitter, we know that for BC_{eq} there exist Gaussian random vectors $\{\underline{x}_i,\ i = 1,\dots,n\}$ and $\{\underline{u}_i,\ i = 1,\dots,n\}$ with the following properties

- $\underline{x}_i$s for different is are independent zero-mean Gaussian random vectors with covariance matrix B_i.
- $\underline{u}_i$ is a Gaussian random vector that is a function of $\underline{x}_j$ for $i \le j \le n$ and $\hat{H}_j,\ i \le j \le n$.
- The rate for user i in BC_{eq} can be written as

$$R_{i,\,eq} = I(\underline{u}_i;\underline{y}_{i,\,eq}) - I(\underline{u}_i;\underline{s}_i) = \log\frac{|N_{i,\,eq} + \hat{H}_i(\sum_{k=1}^{i} B_i)\hat{H}_i^*|}{|N_{i,\,eq} + \hat{H}_i(\sum_{k=1}^{i-1} B_i)\hat{H}_i^*|}, \tag{8.5}$$

  where $\underline{s}_i = \sum_{j=i+1}^{n}\underline{x}_j$ is the state known non-causally at the transmitter.

Now, looking back at the original broadcast channel with estimation error, we will use the same set of $\{x_i\}$ and $\{u_i\}$ for coding and show that the rates achieved by the coding scheme of [138] are lower-bounded by the rates achieved for channel BC_{eq}, i.e., $\underline{R}_{eq}$ shown in (8.5). The transmitter will send $\underline{x} = \sum_{i=1}^{n}\underline{x}_i$, where x_i is the signal intended for user i. It can be easily checked that $\underline{v}_i$ in (8.2) and $\underline{x}_j$s are uncorrelated from each other, i.e.,

$$\Psi_{vx} = \mathrm{E}\,\underline{v}_i\underline{x}_j^* = 0, \qquad j = 1,\dots,n.$$

This suggests that $\underline{v}_i$ and $\underline{u}_i$ are also uncorrelated

$$\Psi_{vu} = \mathrm{E}\, \underline{v}_i \underline{u}_i^* = 0. \tag{8.6}$$

Furthermore, the covariance of $\underline{v}_i$ in (8.2) can be written as

$$\begin{aligned} \mathrm{cov}\,(\underline{v}_i) &= \mathrm{E}\, \underline{v}_i + \underline{v}_i^* \\ &= I + \mathrm{E}\, \tilde{H}_i \underline{x}\underline{x}^* \tilde{H}_i^* \\ &= I + \mathrm{E}\, \tilde{H}_i (\sum_{l=1}^{n} B_l) \tilde{H}_i^* \\ &= I + \sum_{l=1}^{n} \mathrm{tr}\,(B_l A_i) K_i \\ &= N_{i,\,eq}. \end{aligned} \tag{8.7}$$

In other words, the covariance matrix of $\underline{v}_i$ is equal to the covariance matrix of the Gaussian noise $\underline{n}_{i,\,eq}$ present in BC_{eq}.

For the covariance of $\underline{y}_i$ in (8.2)we have

$$\mathrm{cov}\,(\underline{y}_i) = \mathrm{cov}\,(\underline{v}_i) + \hat{H}_i (\sum_{l=1}^{n} B_l) \hat{H}_i^* = \mathrm{cov}\,(\underline{y}_{i\,eq}).$$

Therefore, the covariance matrices of the received signals in the original broadcast channel and BC_{eq} are equal.

Now, the rate achieved with coding of [138] is equal to

$$R_i = I(\underline{u}_i; \underline{y}_i) - I(\underline{u}_i; \underline{s}_i) \tag{8.8}$$

for each user i. Comparing (8.5) with (8.8), we see that only the first mutual information term is different for the two channels. The first term in (8.8) can be written as

$$I(\underline{u}_i, \underline{y}_i) = h(\underline{u}_i) - h(\underline{u}_i | \underline{y}_i).$$

We have the following upper bound on $h(\underline{u}_i|\underline{y}_i)$ in terms of the covariance matrix $\mathrm{cov}\,(\underline{u}_i|\underline{y}_i) = \mathrm{E}_{\,|\underline{y}_i}(\underline{u}_i - \mathrm{E}_{\,|\underline{y}_i}\underline{u}_i)(\underline{u}_i - \mathrm{E}_{\,|\underline{y}_i}\underline{u}_i)^*$:

$$h(\underline{u}_i|\underline{y}_i) \leq \log\det \pi e \mathrm{cov}\,(\underline{u}_i|\underline{y}_i), \tag{8.9}$$

since among all random vectors with the same covariance matrix, the one with a Gaussian distribution has the largest entropy.

The following lemma gives an important property of $\mathrm{cov}\,(\underline{u}_i|\underline{y}_i)$, the proof of which can be found in [147].

Lemma 8.2. *Let $\hat{\underline{u}}_i = f(\underline{y}_i)$ be any estimate of $\underline{u}_i$ given $\underline{y}_i$. Then we have*

$$\mathrm{cov}\,(\underline{u}_i|\underline{y}_i) \leq \mathrm{E}\,(\underline{u}_i - \hat{\underline{u}}_i)(\underline{u}_i - \hat{\underline{u}}_i)^*.$$

Substituting the Linear MMSE estimate $\hat{\underline{u}}_i = \Psi_{uy}\mathrm{cov}\,(\underline{y}_i)\underline{y}_i$ (Ψ_{uy} denotes the cross-covariance of $\underline{u}_i$ and $\underline{y}_i$) in the above lemma yields

$$\mathrm{cov}\,(\underline{u}_i|\underline{y}_i) \leq \mathrm{cov}\,(\underline{u}_i) - \Psi_{uy}\mathrm{cov}\,(\underline{y}_i)\Psi_{yu}. \tag{8.10}$$

Ψ_{uy} can be calculated as follows:

$$\begin{aligned}
\Psi_{uy} &= \mathrm{E}\,\underline{u}_i\underline{y}_i^* \\
&= \mathrm{E}\,\underline{u}_i(\hat{H}_i\underline{x} + \underline{v}_i)^* \\
&= \Psi_{ux}\hat{H}_i^* + \Psi_{uv} \\
&= \Psi_{ux}\hat{H}_i^* \\
&= \mathrm{E}\,\underline{u}_i\underline{y}_{i,\,eq} = \Psi_{uy_{eq}}.
\end{aligned}$$

Therefore, the cross-covariance of $\underline{u}_i$ and $\underline{y}_i$ is equal to the cross-covariance of $\underline{u}_i$ and $\underline{y}_{i,eq}$ in BC$_{eq}$. Combining (8.10) and (8.9) we have

$$\begin{aligned} h(\underline{u}_i|\underline{y}_i) &\leq \log\det \pi e \mathrm{cov}\,(\underline{u}_i|\underline{y}_i) \\ &\leq \log\det \pi e(\mathrm{cov}\,(\underline{u}_i) - \Psi_{uy}\mathrm{cov}\,(\underline{y}_i)\Psi_{yu}) \\ &\overset{(a)}{=} \log\det \pi e(\mathrm{cov}\,(\underline{u}_i) - \Psi_{uy_{eq}}\mathrm{cov}\,(\underline{y}_{i,\,eq})\Psi_{yu_{eq}}) \\ &\overset{(b)}{=} h(\underline{u}_i|\underline{y}_{i,\,eq}), \end{aligned}$$

where in (a) we have used the facts that covariance matrices of y_i and $y_{i,\,eq}$ are the same and $\Psi_{uy} = \Psi_{uy_{eq}}$. Also, (b) is a consequence of the random vector $\underline{u}_i$ given $\underline{y}_{i,\,eq}$ being a Gaussian and, therefore, MMSE estimate being the optimal estimate.

The above inequality suggests that

$$I(\underline{u}_i;\underline{y}_i) \geq I(\underline{u}_i;\underline{y}_{i,\,eq}),$$

and hence, from (8.8) and (8.5) we have

$$R_i \geq R_{i,\,eq}$$

for all users. This completes the proof. ∎

Remark 8.3. *Note that although we proved Theorem 8.1 for estimation error covariances of form $A_i^T \otimes K_i$, the theorem can be easily generalized for error covariance matrices of general form. In the general case, the equivalent noise covariance is given by (8.7), and the achievable set of rates is given in (8.5).*

8.4 Optimal Power Allocation

In the previous section an achievable rate region for MIMO broadcast channels with estimation error was given. This region is based on dirty paper coding and is equal to the capacity region of a broadcast channel with noise covariances that depend on the covariance of the transmitted signal. It is well known that the dirty paper coding region is not convex in input covariance matrices, and finding the boundary points of the capacity region directly from the dirty paper coding regime is not computationally tractable. However, using the duality of the broadcast and multiple access channels [132, 133] and the mini-max duality introduced in [136, 137], it is possible to find the boundary points of the capacity region under some class of power constraints using convex optimization. In this section we consider finding the power allocation for any boundary point on the achievable rate region described in Theorem 8.1.

It is worth mentioning that since in our case the effective noise covariance matrices also depend on the input covariance matrices, the transformations used in [133] do not go through. As a matter of fact, the transformation used in [133] is valid only for a sum power constraint. For the presentation of this chapter we provide duality results in the following two cases.

For all users $A_i = I$:

It can be easily shown that any boundary point on the region described in Theorem 8.1 is achieved when $\sum_{i=1}^{n} \operatorname{tr}(B_i) = P$. Therefore, if $A_i = I$ for all the channels, the effective noise of (8.3) does not depend on B_is anymore and is given by $I + PK_i$. In this case, one can use the duality of multiple access and broadcast channels with sum power constraints. Hence, the region of Theorem 8.1 is equal to the capacity region of a Gaussian Multiple access channel with sum power constraint P and channel coefficients $\hat{H}_i^*(I + PK_i)^{-\frac{1}{2}}$. Therefore, any point on the boundary can be computed

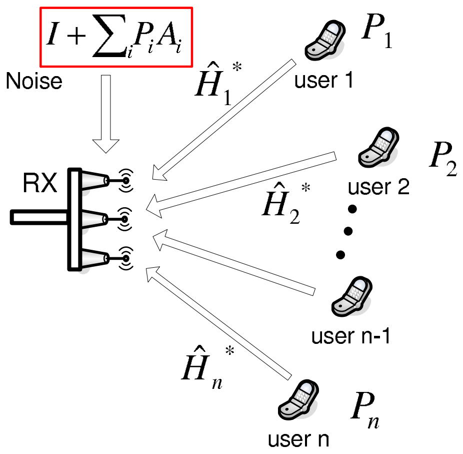


Figure 8.3: A dual representation of the inner bound of the capacity region with estimation error.

using convex optimization. We have summarized this result as follows.

Theorem 8.4. *Consider the setting of Theorem 8.1. Further assume that the covariance matrix of the estimation error for i-th channel is of the form* $\text{cov}(\tilde{H}_i) = I \otimes K_i$*. Then, the capacity region of the channel includes the capacity region of a multiple access channel with sum power constraint* P *and channel coefficients* $\hat{H}_i^*(I+PK_i)^{-\frac{1}{2}}$*.*

MISO Broadcast with Estimation Error, $r_i = 1$:

In the rest of this chapter we consider the achievable rates for MISO broadcast channels with estimation error. For this case, we can state the achievable region based on the capacity region of a dual multiple access channel.

Theorem 8.5. *Consider a MISO Gaussian broadcast channel with estimation error covariance* $A_i \succ 0$ *for user* i *and total transmit power constraint of* P*. Then, the capacity region includes the capacity region of a multiple access channel shown in Figure 8.3 with one antenna at each transmitter and* M *antennas at the receiver. The channel coefficient vector for transmitter* i *is* $\hat{H}_i^*$*. The noise covariance is* $I + \sum_{i=1}^n P_i A_i$*, where* P_i *is the transmit power for user* i*, with* $\sum_i P_i \leq P$*.*

Proof. We will use the approach taken in [136]. Instead of looking at the achievable rates, we will look at the feasibility of a set of SINR constraints in the broadcast and the dual multi-access scenario. Similar to [137], we consider beam-forming with dirty paper precoding. The transmitted signal in this case can be written as

$$x = \sum_i \underline{\phi}_i s_i,$$

where $\underline{\phi}_i$ is the i-th beam that carries s_i, the information for user i. Without loss of generality we assume that $\mathrm{E}\,|s_i|^2 = 1$. Looking back at the definition of B_i in Theorem 1, here we have assumed that $B_i = \underline{\phi}_i \underline{\phi}_i^*$. Next we consider the problem of minimizing total transmit power subject to a set of SINR constraints for broadcast channel

$$\begin{aligned} L^{BC} = \min_{P,\, \underline{\phi}_i} \quad & P \\ \text{subject to} \quad & \frac{|h_i \underline{\phi}_i|^2}{\sum_{j>i} |h_i \underline{\phi}_j|^2 + 1 + \sum_{l=1}^n w_l^* A_i \underline{\phi}_l} \geq \gamma_i \\ & \sum_{i=1}^n \underline{\phi}_i^* \underline{\phi}_i \leq P. \end{aligned}$$

Similarly, we can write the following problem for the dual multiple access channel. The SINRs shown below are achieved by using $\underline{\varphi}_i$ as a filter for i-th user information and using interference cancelation.

$$\begin{aligned} L^{MA} = \min_{P_i \geq 0,\, \underline{\varphi}_i} \quad & \sum_{i=1}^n P_i \\ \text{subject to} \quad & \frac{P_i |h_i \underline{\varphi}_i|^2}{\sum_{j<i} |h_j \underline{\varphi}_i|^2 + \underline{\varphi}_i^* (\sum_{l=1}^n P_l A_l) \underline{\varphi}_i} \geq \gamma_i \end{aligned}$$

Following the steps of [137], we can show that both of the above problems have the following dual

$$\begin{aligned} \max_{P_i \geq 0} & \quad \sum_{i=1}^{n} P_i \\ \text{subject to} & \quad \sum_{j<i} P_j h_j^* h_j + \sum_{l=1}^{n} P_l A_l + I \succeq \frac{P_i}{\gamma_i} h_i^* h_i. \end{aligned}$$

Furthermore, the strong duality holds, and the two problems have the same minimum power and are equivalent. Therefore, the achievable rate region of MISO broadcast channel is equivalent to the capacity region of a multiple access channel with M antennas at the receiver, total power constraint P, and a noise covariance Q_{eq} that depends on the different users' transmit powers in the following form

$$Q_{\text{eq}} = I + \sum_{l=1}^{n} P_l A_l,$$

and this proves Theorem 8.5. ■

Clearly, Theorem 8.5 implies that for a homogeneous system, where $A_i = A$ for all users, the capacity region of this channel includes the capacity region of a multiple access channel with total transmit power P and noise covariance matrix $I + PA$.

8.5 Scaling Laws of the Achievable Sum-rate

Using Theorem 8.5, we know that the following sum-rate is achievable for homogeneous MISO broadcast channels:

$$R_{sum} = \max_{\substack{P_i \geq 0 \\ \sum_{i=1}^{n} P_i \leq P}} \log \frac{|I + PA + \sum_{i=1}^{n} P_i \hat{H}_i^* \hat{H}_i|}{|I + PA|} \tag{8.11}$$

This optimization problem is convex in the P_is and can be therefore solved when n is not too large. The achievable ergodic sum-rate for fading channels is just the expectation of R_{sum} over all channel realizations. Clearly, when n is large, computing the average sum-rate becomes computationally intensive. In what follows, we obtain the scaling law of the ergodic sum-rate for large number of users.

Defining $G_i = \hat{H}_i(I + PA)^{-\frac{1}{2}}$, the ergodic sum-rate is given by

$$R^\star = \mathrm{E}(R_{sum}) = \mathrm{E} \max_{P_i \geq 0, \sum_{i=1}^n P_i \leq P} \log |I + \sum_{i=1}^n P_i G_i^* G_i|, \tag{8.12}$$

where the expectation is over G_is for $i = 1, \ldots, n$. Here G_is are independent Gaussian vectors with covariance matrix

$$\mathrm{E}\left(G_i^* G_i\right) = (I + PA)^{-\frac{1}{2}}(R_H - A)(I + PA)^{-\frac{1}{2}}. \tag{8.13}$$

Note that (8.12) is in fact the ergodic sum-capacity of a MISO broadcast channel where channels have a Gaussian distribution with covariance matrix given in (8.13). The ergodic sum-capacity of MISO broadcast channels with spatial correlation in channel coefficients is analyzed for a large number of users in [139]. Assuming that R_H and A are fixed (in terms of n), one can use the result of [139] to state the following Theorem.

Theorem 8.6. *Consider the setting of Theorem 8.5. Assume the channel covariance matrix is R_H and estimation error covariance is $A \preceq R_H$. Then, as the number of users n goes to infinity, the achievable sum-rate scales like*

$$\begin{aligned} R^\star &= M \log\log n + M \log \frac{P}{M} \\ &\quad + \log\det(R_H - A) - \log\det(I + PA) + o(1). \end{aligned} \tag{8.14}$$

Theorem 8.6 suggests that as long as the estimation error covariance matrix is fixed in terms of n, one gets the same scaling as the case where the channel is known perfectly at the receivers and the transmitter. In fact, the effect of estimation error shows up as a constant hit in the achievable rate.

At the end, we should remark that since for a homogeneous network, the equivalent noise in Theorem 8.5 is linear in the transmit power, in the high SNR regime (and for a fixed number of users), the achievable sum-rate will be of constant order (see also [126]).

8.6 Training

The results obtained so far are based on a given estimation error covariance. To estimate the channel, a training phase is often required. During this phase, some portion of the transmission interval and transmit power is used to send known training signals. In this section we consider training for block fading MISO broadcast channels with M transmit antennas, coherence interval of $T \geq M$, and total transmit power of P. We further assume that the channel coefficients are independent zero mean unit variance Gaussian random variables. We find the optimum amount of time and power that should be allocated for training to maximize our achievable sum-rate.

During the training phase, the transmitter sends T_τ training vectors with total transmit energy of $P_\tau T_\tau$. Let X_τ be the $M \times T_\tau$ matrix consisting of the training vectors. We have

$$\operatorname{tr}\left(X_\tau^* X_\tau\right) = P_\tau T_\tau. \tag{8.15}$$

The received signal at user i can be written as

$$y_{i,\tau} = h_i X_\tau + v_{i,\tau}.$$

At the end of the training phase, each user finds the LMMSE estimate of its channel and feeds it back to the transmitter. In order to obtain a meaningful estimate of h_i,[1] we need at least as many measurements as unknowns, which implies that $T_\tau \geq M$. The estimate can be written as

$$\hat{h}_i = y_{i,\tau}(I + X_\tau^* X_\tau)^{-1} X_\tau^*.$$

Note that $y_{1,\tau}, \ldots, y_{n,\tau}$ are independent and identically distributed. The estimation error covariance for each user is

$$\begin{aligned} A_\tau = \text{cov}(\tilde{h}_i) &= I - X_\tau^*(I + X_\tau^* X_\tau X_\tau)^{-1} X_\tau \\ &= (I + X_\tau X_\tau^*)^{-1}. \end{aligned}$$

Let $T_d = T - T_\tau$ and $P_d T_d = PT - P_\tau T_\tau$. After the training phase, the transmitter starts sending data over the T_d time samples left and with total transmitter energy $P_d T_d$. Therefore, for a fixed P_τ, T_τ, using the result of Theorem 8.5, the following sum-rate is achievable:

$$R_\tau = \frac{T_d}{T} \text{E} \max_{P_i \geq 0, \sum_i P_i \leq P_d} \log \frac{|I + P_d A_\tau + \sum_{i=1}^n P_i \hat{h}_i^* \hat{h}_i|}{|I + P_d A_\tau|}, \tag{8.16}$$

where h_is are independent vectors whose elements are jointly Gaussian random variables with covariance matrix $I - A_\tau$ (which follows from the orthogonality principle). Now consider the eigenvalue decomposition of $X_\tau X_\tau^* = U \Omega U^*$, where U is unitary and Ω is diagonal. From (8.15) we have $\text{tr}\,(\Omega) \leq P_\tau T_\tau$. After some manipulation of (8.16) we can rewrite the achievable rate as

$$R_\tau = \frac{T_d}{T} \text{E} \max_{\substack{P_i \geq 0 \\ \sum_i P_i \leq P_d}} \log \frac{|I + (1 + P_d)\Omega^{-1} + \sum_{i=1}^n P_i g_i^* g_i|}{|I + (1 + P_d)\Omega^{-1}|}. \tag{8.17}$$

[1] Throughout this section we use h_i rather than H_i to represent the channel vector for i-th user.

The g_is are independent vectors whose elements are independent zero mean unit variance Gaussian random variables, and the expectation is over g_i. Now let us consider the case where Ω is a scaled version of identity. Using the trace constraint we have

$$\Omega = \frac{P_\tau T_\tau}{M} I.$$

This Ω corresponds to the case where the training matrix X_τ is a multiple of a matrix with orthonormal columns. Using this Ω and simplifying (8.17), the following rate is achievable:

$$R_\tau = \frac{T_d}{T} \mathrm{E} \max_{P_i \geq 0, \sum_i P_i \leq P_{\text{eff}}} \log |I + \sum_{i=1}^{n} P_i g_i^* g_i|, \tag{8.18}$$

where for each i, g_i is a vector of i.i.d zero-mean unit-variance Gaussian random variables. P_{eff} is the effective power and is given as

$$P_{\text{eff}} = \frac{P_d P_\tau T_\tau}{P_\tau T_\tau + (1 + P_d)M}.$$

We can maximize the achievable lower bound of (8.18) over power and time allocated for training. Note that for a fixed T_τ (and T_d), the optimal power allocation is one that maximizes the effective transmit power P_{eff}. By maximizing P_{eff} over P_τ and P_d we get

$$P_{\text{eff}}^\star(T_d) = \frac{(PT)^2}{\left(\sqrt{(PT + T_d)M} + \sqrt{(M + PT)T_d}\right)^2}. \tag{8.19}$$

Also, the maximizing P_τ is given by

$$P_\tau^*(T_d) = \frac{PT\sqrt{(T_d + PT)M}}{(T - T_d)(\sqrt{(PT + T_d)M} + \sqrt{(M + PT)T_d})}. \tag{8.20}$$

In order to maximize the achievable rate over T_d we have to solve the following optimization problem:

$$R^{\star} = \max_{T_d, 0 \leq T_d \leq T-M} \frac{T_d}{T} \mathrm{E}_{\{g_i\}} f(P_{\text{eff}}^{\star}(T_d)), \tag{8.21}$$

where $P_{\text{eff}}^{\star}(T_d)$ is given in (8.19), and $f(x)$ is defined as

$$f(x) = \max_{p_i, \sum_i p_i \leq 1} \log |I + x \sum_{i=1}^{n} p_i g_i^* g_i|.$$

It is shown in the following lemma that the cost function in (8.21) is increasing in T_d. This suggests that the optimal T_d is $T - M$.

Lemma 8.7. *The cost function of (8.21) is an increasing function in T_d.*

Proof. The claim is that the following function is increasing in T_d:

$$C(T_d) = \frac{T_d}{T} \mathrm{E}_{\{g_i\}} f(P_{\text{eff}}^{\star}(T_d)),$$

where $f(x)$ is defined in the equation right after (8.21). To show this, we differentiate with respect to T_d:

$$\frac{d}{dT_d} C(T_d) = \frac{1}{T} \mathrm{E}_{\{g_i\}} \left[f(P_{\text{eff}}^{\star}(T_d)) + T_d \frac{d}{dT_d} P_{\text{eff}}^{\star}(T_d) f'(P_{\text{eff}}^{\star}(T_d)) \right].$$

In order to show that $C(T_d)$ is increasing (or equivalently $\frac{d}{dT_d} C(T_d) > 0$), it suffices to show that the term inside the expectation is greater than zero for any value of $P_{\text{eff}}^{\star}(T_d)$. Using (8.19) after some manipulation we have

$$T_d \frac{d}{dT_d} P_{\text{eff}}^{\star}(T_d) = -P_{\text{eff}}^{\star}(T_d) \underbrace{\frac{\sqrt{T_d(M+PT)} + \sqrt{\frac{MT_d^2}{T_d+PT}}}{\sqrt{T_d(M+PT)} + \sqrt{M(PT+T_d)}}}_{z}.$$

It can be readily verified that $z < 1$. Therefore, it is enough to show that

$$f(x) - xf'(x) \geq 0,$$

or equivalently to show that $\frac{f(x)}{x}$ is decreasing. Note that $\frac{\log(1+ax)}{x}$ is decreasing in x. Suppose we write the $f(x+y)$ in terms of the eigenvalues of the optimal matrix. Thus,

$$\begin{aligned} f(x+y) &= \sum_{i=1}^{n} \log(1 + (x+y)\lambda_i) \\ &\leq \sum_{i=1}^{n} \frac{x+y}{x} \log(1 + x\lambda_i) \\ &\leq \frac{x+y}{x} \log\det(I + x\sum_{i=1}^{n} p_i g_i^* g_i) \\ &\leq \frac{x+y}{x} \max_{p_i \sum_{i=1}^{n} p_i \leq 1} \log\det(I + x\sum_{i=1}^{n} p_i g_i^* g_i) \\ &= \frac{x+y}{x} f(x). \end{aligned}$$

This completes the proof. ■

The next theorem summarizes the above arguments.

Theorem 8.8. *Consider a block fading MISO broadcast channel with M transmit antennas, coherence interval of $T \geq M$, and total transmit power of P. Further, assume that the channel coefficients are independent zero-mean unit-variance Gaussian random variables. Then, the following sum-rate is achievable using training:*

$$R^\star = \frac{T-M}{T} \mathrm{E}_{\{g_i\}} \max_{p_i, \sum_i p_i \leq 1} \log|I + P_{\text{eff}}^\star(T-M) \sum_{i=1}^{n} p_i g_i^* g_i|, \tag{8.22}$$

where $P_{\text{eff}}^\star(\cdot)$ is defined in (8.19). Furthermore, this rate is achieved by using orthogonal and fixed power training vectors over the first M time samples and transmitting *data over the remaining portion of the coherence interval. The power of each training vector is* $P^{\star}_{\tau}(T-M)$ *and is given in (8.20).*

The following Corollary gives further insights on the behavior of the sum-rate in different regimes.

Corollary 8.9. *Consider the MISO broadcast channel model described in Theorem 8.8. Then, the achievable sum-rate*

- *for large* P *and fixed* n *scales like*

$$R^{\star} = \min\{M, n\}(1 - \frac{M}{T}) \log P;$$

- *for small* P *and fixed* n *scales like*

$$R^{\star} = \frac{Tc \log e}{4M} P^2,$$

  where c *is the mean of the maximum of* n *i.i.d random variables with* $\chi^2(2M)$ *distribution; and*

- *for large number of users,* n*, and fixed* P *scales like*

$$R^{\star} = M(1 - \frac{M}{T}) \log(1 + P^{\star}_{\text{eff}}(T-M) \log n).$$

Corollary 8.9 shows that using training-based schemes, one can achieve the multiplexing gain of a MIMO point-to-point channel with M transmit and n receive antennas in the high SNR regime. However, the power invested in the training phase increases linearly with P (see (8.20) for large P). Also, the required feedback rate for sending the estimates to the transmitter should increase with P. Therefore, the above scheme suggests that as long as the transmitter is willing to invest a fixed fraction of the available power for learning the channel, good multiplexing gains can be achieved.

8.7 Conclusion

This chapter considers an important design issue in the downlink of cellular systems, that is the effect of channel estimation error on the capacity region of MIMO Gaussian broadcast channels. In practical systems, the mobile users and the base station (transmitter) have only an estimate of the channel available. For this case, an achievable rate region based on dirty paper coding is derived. It is further shown that for MISO case, this region is equivalent to the capacity region of a multi-access channel with noise covariance matrix that depends on the transmit power and the estimation error. A training-based scheme for block fading MISO Gaussian broadcast channels is analyzed, and the optimal length of training interval and the power used for training is derived. Designing practical schemes in the presence of channel estimation error is an important future work. Also, finding outer bounds on the capacity region of broadcast channels with estimation error is an interesting problem (see [126]).

Chapter 9

Future Work

Networks, and especially wireless networks, are changing the very fabric of our lives in different aspects. We are moving towards a *networked world* where everyone at any place and at all times is connected and can access and process information seamlessly. However, in order to reach the promised networked world, there are many issues regarding the capabilities and the design of wireless networks that should be addressed.

In this thesis, we analyze the performance limit of a few applications over special classes of wireless networks and look at a few design issues in cellular wireless networks. The results in the thesis have brought up a few interesting open problems.

- **QoS Provisioning in Cellular Systems:** Because of the highly heterogeneous nature of today's cellular systems, Quality of Service (QoS) provisioning is of great importance. In Chapter 7, we looked at this problem by modeling various QoS demands with differentiated rate requirements among users. However, in many applications rate requirement is not the main issue. Delay and reliability requirements can be strict and more important for delay-sensitive and robust applications. An important open problem is proposing a unified framework that integrates all different requirements and allows scheduling of heterogeneous networks with diverse demands.

In Chapter 8 we gave an inner bound on the capacity region of broadcast channels with channel estimation error. Finding outer bounds on the capacity region of broadcast channels with channel estimation error and with limited feedback from the users to the transmitter is also a challenging and important task.

- **Optimization Theory View of Multi-terminal Information Theory:** As mentioned earlier, multi-terminal Information Theory has fallen short of characterizing the absolute limits of communication in general networks. Part of the problem arises from the fact that the current standard tools for providing outer bounds on the achievable rates in a network setup are based on arguments developed for point-to-point communication systems. Usually, the outer bounds are based on information-theoretic cut-set bounds that assume full cooperation of users in each side of the cut. The multicast min-cut bound of Chapter 3 is an example of these bounds. In these bounds the fact that information is distributed in the network is neglected.

  One major challenge is to provide a methodology that considers the distributed nature of information in networks and gives potentially tighter outer bounds on the set of achievable rates in a multi-terminal setup. One promising approach is applying optimization theory tools (such as duality and convex optimization) for providing outer bounds on the capacity region of multi-user and network setups. A special case of this method was proposed and used in Chapter 3. A first attempt could be finding equivalent forms of information theory techniques (such as cut-set bounding, Data Processing inequality, etc.) in the optimization domain.

- **Analysis of Coding over Networks:** The benefits of coding at intermediate nodes of the network is mostly investigated for the multicast scenario. In Chapter 3 we looked at the capacity region of broadcast problems over wireless erasure networks and provide inner and outer bounds. However, as we saw in Chapter 2, in a general setting destinations can demand different subsets of information messages, and there can be multiple copies of an information message available in the network. It is important to explore the benefits of coding for general network problems in wired and wireless networks.

The first step in analyzing the performance of coding over wireless networks is the modeling of the interference present in these networks. One possible approach is to investigate models that incorporate the interference in the network layer rather than the physical layer, which deserves further investigation. For instance, in the wireless erasure network model introduced in this chapter, a possible way to take into account the interference is through the erasure probabilities.

Another interesting topic in this area is identifying the critical side-information in a network setup. In Chapter 2 it was shown that by providing a certain side-information to the destination node we can achieve the min-cut upperbound. Presence of this side-information is critical for achieving the capacity. It would be interesting to find similar side-information for other types of networks.

- **Communication, Control, and Computation in Wireless Ad-hoc Networks:** Sensor and ad-hoc wireless networks are moving toward systems of interconnected devices (such as sensors, actuators, and controllers) that are capable of communicating among themselves and can perform computational tasks and estimation in a distributed manner.

  In Chapter 5 of this thesis, we looked at estimation of a *single* dynamical process at a remote location that is connected to a sensor through an erasure wireless network. We characterize the optimal (minimum) steady-state error and its dependency on the parameters of the network. As a continuation of this work, the following important problem can be considered:

- What are optimal coding schemes for control and estimation of *multiple* dynamical systems in more general networks?

At a more general level, for a resource-efficient operation in these networks, a unified view of computation, communication, and control is required. Therefore, one fundamental challenge is to provide a theoretical framework that integrates various tasks and makes the analysis of different performance measures (and their trade-offs) possible. The work in Chapters 2, 5, and 6 is a first step towards this goal. The following important extensions of the results of these chapters merits further investigation:

- What is a practical model for analyzing power-efficient schemes that take into account the computation and communication power consumption simultaneously?
- How can one efficiently compute a function of the information available at different nodes in a distributed fashion in wireless networks?

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