EIGENVALUE PROBLEMS FOR POSITIVE MONOTONIC NONLINEAR OPERATORS

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ABSTRACT

The determination of the set Λ of values of λ for which a family of operators $\{A_\lambda\}$ on a real, partially-ordered Banach space has positive fixed points and the description of the behavior of the fixed points as functions of λ are considered. The operators A_λ are usually assumed to be compact monotonic operators which satisfy $A_\lambda 0 > 0$, and the elements $A_\lambda u$ are assumed to be continuous increasing functions of λ for every positive u. It is shown that Λ is an interval, that for each λ the operator A_λ has a smallest positive fixed point $u^o(\lambda)$, and that $u^o(\lambda)$ is an increasing function of λ which is continuous from the left in λ. Conditions are given which guarantee the uniqueness of the fixed point of A_λ for each λ and permit the precise determination of the set Λ.

When $\sup \Lambda \in \Lambda$ and $A_\lambda u$ satisfies certain differentiability conditions, the behavior of $u^o(\lambda)$ for near $\sup \Lambda$ is described and the existence of a second positive fixed point for λ near $\sup \Lambda$ is proved. The asymptotic behavior of $A_\lambda u$ for large positive u is used to determine the behavior of the fixed points of large norm and the existence and value of a number μ_1 such that the norms of a sequence of fixed points approach infinity as the corresponding values of λ approach μ_1. The existence of a second positive fixed point is proved under various conditions, including the case when the operators A_λ are Fréchet differentiable and $0 < \mu_1 < \sup \Lambda \in \Lambda$. More precise results are obtained when the operators A_λ are concave or convex.

These results are used to study the eigenvalue problem for Hammerstein integral equations and nonlinear ordinary differential

equations. For certain ordinary differential equations with convex nonlinearities, the existence of precisely two positive fixed points is proved. Finally, an independent treatment is given of the eigenvalue problem for the equation $u''+\lambda f(u) = 0$ with the boundary conditions $u(0) = u(1) = 0$; use is made of the first integral of the differential equation and a study of the equation in the phase plane.

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INTRODUCTION

The following nonlinear partial differential equation arises in the study of the steady-state temperature distribution of a physical medium in which heat is being generated nonlinearly:

$$\Delta u(x) + \lambda f(u, u(x)) = 0 , \quad x \in \Omega ,$$

$$\alpha(x)u(x) + \beta(x) \frac{\partial u}{\partial n}(x) = 0 , \quad x \in \partial\Omega ,$$

where Δ denotes the three-dimensional Laplacian operator, Ω is a bounded domain in three-dimensional space with a boundary $\partial\Omega$, and $\partial/\partial n$ denotes the outer normal derivative on the boundary. The unknown $u(x)$ represents the non-negative difference $T(x) - T_o$ between the temperature $T(x)$ at a point x of the medium and the ambient temperature T_o , λ is a positive parameter which can be varied by varying the conditions under which the material is being studied, and $f(x, u)$ represents a physical property of the material which is causing heat to be generated at a point x of the material in a way which depends on the temperature $T = u + T_o$ at that point (Chambré 1952*, Kaganov 1963, Jakob 1959, Joseph 1965).

In each of the references cited, the functions α and β are non-negative, and $f(x, u)$ is a positive monotonically increasing function of u ; in particular, $f(x, 0)$ is in general not zero. The latter fact distinguishes these examples from physical problems which give rise to similar kinds of equations in which $f(x, 0) = 0$

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* A name followed by a date is used to refer to an entry in the list of references.

and which have been extensively studied (see the references in Section I. 2). A study of this type of equation (with the Laplacian Δ replaced by a more general self-adjoint elliptic operator) assuming specifically that $f(x,0) > 0$ has been made recently by Keller and Cohen (1967) under the assumption that the problem can be reformulated as an equivalent integral equation

$$u(x) = \lambda \int_{\Omega} G(x,y) f(y, u(y)) dy ,$$

with an appropriate Green's function $G(x,y)$. Thus, the problem becomes one of finding the fixed points of the operators A_λ defined by

$$A_\lambda u(x) = \lambda \int_{\Omega} G(x,y) f(y, u(y)) dy$$

on the set of non-negative functions in some function space. For the most part, the methods of Keller and Cohen depend on only four properties of the operators A_λ : (1) the operators are compact on the set $C(\overline{\Omega})$ of continuous functions on $\overline{\Omega} = \Omega \cup \partial\Omega$ (see the definition in Section I. 1); (2) for any positive numbers $\lambda_1 < \lambda_2$ and non-negative function u , $A_{\lambda_1} u < A_{\lambda_2} u$; (3) for any positive number λ , $A_\lambda 0 > 0$, where 0 denotes the zero function on $\overline{\Omega}$; and (4) for any positive number λ and non-negative functions $u_1 \le u_2$, $A_\lambda u_1 \le A_\lambda u_2$.

In this thesis, we consider operators A_λ satisfying these conditions on a real, partially-ordered Banach space and investigate the number of fixed points for a given λ and the behavior of the fixed points as functions of λ . In addition to the results of Keller and Cohen, we obtain information on the behavior of the fixed points near bifurcation points and the existence and behavior of fixed points of large norm. We are also able to prove in certain cases the existence of exactly two positive fixed points. A method of proving the existence of at least n fixed points for a given λ , where $n \geq 2$, is given by Krasnosel'skii and Stecenko (1966). For a different method of treating eigenvalue problems for nonlinear elliptic partial differential equations, see Berger (1965a, 1965b) and the references given there.

We have divided the material of this thesis into two parts. In Part I, our principal results are stated and proved for the abstract operators A_λ ; in Part II, we apply these results to a discussion of differential and integral equations of the type described earlier. We assume in Part I that the reader is familiar with the elementary theory of Banach spaces, in particular the notions of continuity, compactness, and the dual space, and the rudiments of the spectral theory of continuous linear operators. It is hoped that, except for the proofs of most of the theorems, Part II can be read independently of Sections I. 4 through I. 10.

Sections I. 1 through I. 3 are introductory, and their contents are adequately described by their titles in the Table of Contents. In Section I. 4 we use the methods of Keller and Cohen to generalize their results to the minimal positive fixed points of operators A_λ satisfying conditions (1) through (4) above; various simple iteration procedures are obtained for constructing solutions. In Section I. 5 we apply the methods of Krasnosel'skii (1964a) to prove the uniqueness and continuous dependence on λ of the fixed points of a large class of operators A_λ which satisfy the condition of equation (I. 5. 1) of Section I. 5. Section I. 6 discusses the Fréchet derivative and the abstract form of the implicit function theorem; this material is well-known and is included for the sake of completeness. Assuming the operators A_λ are differentiable, we are able to describe the behavior of the solutions as functions of λ in more detail in Section I. 7. Most of this section is devoted to a study of this behavior near a value λ_o for which there are two fixed points for λ near λ_o which converge to a single fixed point of A_{λ_o} as λ approaches λ_o; this includes, in particular, the behavior of the solutions near the maximum value λ^* of λ for which there are solutions (if there is such a maximum) (Theorems I. 7-3, I. 7-5, and I. 7-6). The behavior of the solutions of large norm is described in Section I. 8 ; denoting by $u(\lambda)$ a fixed point of A_λ, we are able to give conditions under which there exists a number μ_1 such that $\lim_{\lambda \to \mu_1} \|u(\lambda)\| = \infty$ (Theorems I. 8-1 and I. 8-6) and describe how the behavior of $A_\lambda u$ for large u and small $|\mu_1 - \lambda|$ can be used to determine whether the fixed points of large norm exist for λ greater than μ_1 or for λ less than μ_1 (Theorems I. 8-3 and I. 8-6). Conditions under which one can prove the existence of at least two fixed points for a given λ are described in Section I. 9. In Section I. 10, we consider operators which are convex or concave. Using the results of Section I. 5, we prove the uniqueness of the fixed point for concave operators, and we give a condition for the uniqueness of the fixed point for convex operators. A characterization is given of the maximum value λ^* (if any) of of those λ for which there are fixed points of convex operators A_λ, and it is shown that if there is more than one fixed point corresponding to λ^*, there are infinitely many (Theorems I. 10-15 and I. 10-16). Finally, two examples of convex operators are given for which the fixed points can be found explicitly.

In Section II. 1, we apply the results of Part I to study the eigenvalue problem for the nonlinear Hammerstein integral equation. The development of Section II. 1 largely parallels that of Sections I. 4 - I. 5 and I. 7 through I. 10, and consists of deriving the behavior of the eigenfunctions by stating properties of the function f which imply corresponding properties of the Hammerstein integral operator. Occasionally we are able to extend the results of Part I or give alternative methods of proof by using the specific form of the integral operator.

The results of Section II. 1 are able to be improved when the kernel of the Hammerstein integral operator is the Green's function for an ordinary differential equation, and we discuss some of these improvements in Section II. 2. Here, we are able for the first time to give conditions under which there are exactly two eigenfunctions for certain values of λ, and most of Section II. 2 is devoted to the statement and proof of these results. The simplest such condition is that the function $f(x, u)$ is independent of x, convex, and twice differentiable. Section II. 3 is independent of the preceding work, and derives by different methods some of the results of Section II. 2 for the ordinary differential equation $u'' + \lambda f(u) = 0$ on the unit interval

(0, 1), with the boundary conditions $u(0) = u(1) = 0$. We study the equation in the phase plane, making use of the first integral of the differential equation and the symmetry of the eigenfunctions about the point 1/2. A more detailed discussion of the contents of Sections II. 1 through II. 3 can be found at the beginning of each of these sections.

We will now say a few words about the notation employed. Equations are numbered in the form (x. y), where x indicates the section in which the equation occurs, and y is the number of the equation in that section; when it is necessary in Part II to refer to an equation in Part I, the notation (I. x. y) is used, indicating equation (x. y) of Part I. A similar system is used for numbering theorems, propositions, etc., except that the symbol x-y is used. These numbers do not distinguish between theorems, propositions, lemmas, and corollaries; for example, Theorem 4-4 follows Lemma 4-3, which in turn follows Proposition 4-2 (in Section I. 4). We label as a "theorem" a statement which is concerned with the fixed points of operators or the values of λ for which there are fixed points; "propositions" are auxiliary results which describe conditions under which operators or functions have certain properties. The end of a proof is indicated by the symbol //. The words "function," "mapping," and "operator" are used in the following different contexts: a function has as its domain a subset of $\mathbb{R}^n$ (n-dimensional Euclidean space) and as its range a subset of $\mathbb{R}$ (the real numbers); an operator has as its domain and range subsets of a Banach space; a mapping is an operator to which we do not give a specific name, e.g., the mapping $\lambda \to A_\lambda u$ of a subset of $\mathcal{R}$ into a Banach space $\mathcal{B}$, where the A_λ are operators on $\mathcal{B}$ depending on the real parameter λ. The notation $\equiv$ is usually used to indicate that the symbol on one side of $\equiv$ is being defined by the expression on the other side; occasionally, it is also used in the sense of "identically equal to."

PART I. FIXED POINTS OF FAMILIES OF MONOTONIC FORCED OPERATORS

I. 1. Definitions and Notation

Let $\mathcal{B}$ be a real Banach space; the norm of an element $u \in \mathcal{B}$ will be denoted by $\|u\|$. We say that $\mathcal{B}$ is partially ordered if there is given in $\mathcal{B}$ a subset $\mathcal{C}$, called the positive cone of $\mathcal{B}$, with the following properties:

(a) $\mathcal{C}$ is a convex cone; i. e., for any u, v in $\mathcal{B}$ and any non-negative real numbers α, β, $\alpha u+\beta v \in \mathcal{C}$.

(b) If both u and $-u$ are in $\mathcal{C}$, then $u = 0$.

(c) $\mathcal{C}$ is closed.

The cone $\mathcal{C}$ defines a partial order $\leq$ in $\mathcal{B}$ in the following way: $u \leq v$ if $v-u \in \mathcal{C}$. If $u \leq v$ and $u \neq v$, we write $u < v$. If $0 < u$, then u is positive. Conditions (a) and (b) imply that $\leq$ is indeed a partial order on $\mathcal{B}$ (a reflexive, anti-symmetric, transitive relation on $\mathcal{B}$) which is compatible with the linear structure of $\mathcal{B}$ (inequalities are preserved under addition of an arbitrary element of $\mathcal{B}$ to both sides of the inequality and under multiplication of both sides of the inequality by a positive number). Condition (c) implies that one may pass to the limit in a sequence of inequalities.

For convenience, we shall always assume, unless the contrary is specifically indicated, that the norm is a monotone function on $\mathcal{C}$ and that $\mathcal{C}$ spans $\mathcal{B}$; that is,

(d) If $0 \leq u \leq v$, then $\|u\| \leq \|v\|$. (This condition implies that the cone $\mathcal{C}$ is "normal"; see Krasnosel'skii 1964a.)

(e) $\mathcal{B} = \mathcal{C} - \mathcal{C}$; i.e., for any u in $\mathcal{B}$, there exist u^+, u^- in $\mathcal{C}$ such that $u = u^+ - u^-$.

We denote by $\mathcal{C}^+$ the cone $\mathcal{C}$ with 0 deleted. For any positive number $r \leq \infty$, $\mathcal{C}^r = \{u \in \mathcal{C} : \|u\| < r\}$; $\mathcal{C}^{r+} = \mathcal{C}^r \cap \mathcal{C}^+$. Thus, $\mathcal{C}^\infty = \mathcal{C}$ and $\mathcal{C}^{\infty+} = \mathcal{C}^+$. Similarly, $\mathcal{B}^r = \{u \in \mathcal{B} : \|u\| < r\}$.

For any u, v in $\mathcal{B}$ with $u \leq v$, we define

$$[u, v] = \{w \in \mathcal{B} : u \leq w \leq v\} .$$

Suppose that $g_o > 0$; we say that $u > 0$ is g_o - measurable if there are positive numbers $\alpha(u)$ and $\beta(u)$ such that $u \in [\alpha(u) g_o, \beta(u) g_o]$. The set of g_o - measurable elements in $\mathcal{C}^r$ will be denoted by $\mathcal{C}^r_{g_o}$.

An example of a partially ordered Banach space is the space $C(\overline{\Omega})$ of continuous functions on a bounded closed set $\overline{\Omega}$ of Euclidean n-space, $\mathbb{R}^n$. The norm in $C(\overline{\Omega})$ is defined as

$$\|u\| = \max\{|u(x)| : x \in \overline{\Omega}\} .$$

Unless otherwise mentioned, the positive cone $\mathcal{C}$ in $C(\overline{\Omega})$ is always understood to be the set of non-negative functions in $C(\overline{\Omega})$; the conditions (a) through (e) are readily verified. Note that in our notation, $u > 0$ for $u \in C(\overline{\Omega})$ means only that u is not the zero function and $u(x) \geq 0$ for all x in $\overline{\Omega}$; $u(x)$ may be zero for some, but not all, points x of $\overline{\Omega}$. The set of all functions in $\mathcal{C}$ which are never zero on $\overline{\Omega}$ form the interior Int $\mathcal{C}$ of $\mathcal{C}$.

Other possible choices for the positive cone in $C(\overline{\Omega})$ are the set of all non-negative convex functions on $\overline{\Omega}$ or the set of all non-negative concave functions on $\overline{\Omega}$.

Let S be a subset of a real, partially ordered Banach space $\mathcal{B}_1$, A an operator mapping S into a real, partially ordered Banach space $\mathcal{B}_2$, and g_o a positive element of $\mathcal{B}_2$. We say that A is

<u>bounded on S</u> if A maps every closed bounded subset of S into a bounded subset of $\mathcal{B}_2$;

<u>compact</u> (or <u>completely continuous</u>) <u>on S</u> if A is continuous on S and maps every closed bounded subset of S into a set with compact closure;

<u>monotonic on S</u> if $u \geq v$ $(u, v \in S)$ implies $Au \geq Av$;

<u>strictly monotonic on S</u> if $u > v$ $(u, v \in S)$ implies $Au > Av$;

<u>positive on S</u> if $u \geq 0$ $(u \in S)$ implies $Au \geq 0$;

<u>strictly positive on S</u> if $u > 0$ $(u \in S)$ implies $Au > 0$;

<u>g_o-bounded below (above) on S</u> if for any $u \in S$ there exists a positive number $\alpha(u)$ such that $Au \geq \alpha(u)g_o$ $(Au \leq \alpha(u)g_o)$. If A is g_o-bounded above and below on S, we say that A is <u>g_o-bounded on S</u>.

If the set S is the positive cone $\mathbb{C}$, we will often omit specific reference to S.

If $0 \in S$ and A is positive on S, then we say that A is <u>forced</u> if $A0 > 0$; if $A0 = 0$, then A is <u>unforced</u>.

If J is an interval on the real line and $\{u_\lambda\}$, $\lambda \in J$, is a family of elements of $\mathcal{B}$, then $\{u_\lambda\}$ is said to be an <u>increasing</u> (<u>strictly increasing</u>) <u>family</u> if, whenever λ_1, λ_2 are in J and $\lambda_1 < \lambda_2$, then $u_{\lambda_1} \leq u_{\lambda_2}$ $(u_{\lambda_1} < u_{\lambda_2})$. If $\{A_\lambda\}$, $\lambda \in J$, is a family of operators defined on a subset S of $\mathcal{B}$, then $\{A_\lambda\}$ is said to be an <u>increasing</u> (<u>strictly increasing</u>) <u>family on S</u> if, for each u in S, the family $\{A_\lambda u\}$ is increasing (strictly increasing).

A <u>decreasing family</u> is defined similarly.

If A maps a subset S of the Banach space $\mathcal{B}$ into $\mathcal{B}$, and if there exist $u \neq 0$ in S and a real number λ such that $Au = \lambda u$, then we say that u is an eigenvector and λ is an eigenvalue of A; if $\lambda \neq 0$, then we call λ^{-1} a characteristic value of A. If A is a linear operator defined on $\mathfrak{B}$, then we may extend A to a linear operator $\tilde{A}$ defined on the complex extension $\tilde{\mathfrak{B}} = \mathfrak{B} + i\mathfrak{B}$ of $\mathfrak{B}$. The spectrum of A consists of all complex numbers λ such that $\tilde{A}-\lambda I$ (where I is the identity operator on $\tilde{\mathfrak{B}}$) does not have a continuous inverse defined on $\tilde{\mathfrak{B}}$. By the spectral radius $r_o(A)$ of A we shall mean the spectral radius of $\tilde{A}$, i.e., the supremum of the absolute values of the elements of the spectrum of A. We set $\mu_o(A) = r_o(A)^{-1}$. (Kantorovich and Akilov 1964, Schaefer 1966)

Positive linear continuous operators which are g_o-bounded have the following important properties:

1-1. Theorem. (Krasnosel'skii 1964a, Chapter 2) Let T be a positive linear continuous operator with a positive eigenvector ϕ_o: $T\phi_o = \lambda_o\phi_o$. For an integer $n \geq 1$ and $g_o \in \mathfrak{C}^+$, let T^n be g_o-bounded on $\mathfrak{C}_{g_o}$. Then T^n is ϕ_o-bounded and T has no other linearly independent positive eigenvectors; the eigenvalue λ_o is simple and greater than the absolute value of any other eigenvalue of T.

If there is a non-zero $u \in \mathfrak{C}$ and a number λ such that $Tu \leq \lambda u$, then $\lambda \geq \lambda_o$; if $\lambda = \lambda_o$, then $Tu = \lambda u$ and there is a number α such that $u = \alpha\phi_o$. If there is a non-zero $u \in \mathfrak{C}$ and a number λ such that $Tu \geq \lambda u$, then $\lambda \leq \lambda_o$; if $\lambda = \lambda_o$, then $Tu = \lambda_o u$ and there is a number α such that $u = \alpha\phi_o$.

For any positive continuous linear operator T on $\mathfrak{B}$, if $0 \leq \lambda <$

$\mu_o(T)$ then $(I-\lambda T)^{-1}$ exists and is a positive operator; i.e., $u-\lambda Tu \geq 0$ implies $u \geq 0$ (obviously, then, $u-\lambda Tu > 0$ implies $u>0$) for $0 \leq \lambda < \mu_o(T)$ (Schaefer 1966, Appendix 2.3 ; the proof uses conditions (d) and (e) which we have imposed on the positive cone $\mathcal{C}$).

Certain positive linear operators T satisfy the following positivity assumption (PA), which is a partial converse of this fact:

(PA) If there exist $u \in \mathcal{C}^+$, $\lambda \in \mathbb{R}$, such that $u-\lambda Tu > 0$, then $\lambda < \mu_o(T)$.

If T is a compact positive operator and $\mu_o(T) < \infty$, then by the Krein-Rutman Theorem (see Theorem 1-2 below), the operator T^* adjoint to T has an eigenvector ξ_o, which is a positive linear functional on $\mathcal{B}$, corresponding to the eigenvalue $r_o(T) = \mu_o(T)^{-1}$. Assuming $u \in \mathcal{C}^+$ and λ are such that $u-\lambda Tu > 0$, we have $0 \leq \xi_o(u) - \lambda\xi_o(Tu) = [1-\lambda/\mu_o(T)]\xi_o(u)$. Thus, a compact operator T satisfies (PA) if ξ_o is strictly positive, i.e., if $u \in \mathcal{C}^+$ implies $\xi_o(u)>0$. Without using the linear functional ξ_o, Theorem 1-1 and the Krein - Rutman Theorem show that a compact linear operator, some power of which is g_o-bounded, satisfies (PA). The identity operator I is an example of a non-compact positive linear operator which satisfies (PA).

Related assumptions which we shall at times wish to impose on a positive linear operator T are the following:

(PA_1) If there exist $u \in \mathcal{C}^+$, $\lambda \in \mathbb{R}$, such that $u-\lambda Tu > 0$, then $\lambda \neq \mu_o(T)$.

(PA_2) If there exist $u \in \mathcal{B}$, $\lambda \in \mathbb{R}$, such that $u-\lambda Tu > 0$, then $\lambda \neq \mu_o(T)$.

It is easy to see that if the adjoint of T has a strictly positive eigenvector corresponding to the characteristic value $\mu_o(T) < \infty$, then T satisfies (PA_2). Proposition 1-1 implies that any compact linear operator, some power of which is g_o-bounded, satisfies (PA_1).

We conclude this section by stating the Krein-Rutman Theorem (Krein-Rutman 1950, Theorem 6. 1; Schaefer 1966, Appendix 2. 4), since we shall refer to it several times in the sequel:

<u>1-2 Theorem.</u> (Krein-Rutman Theorem.) Let T be a compact positive linear operator on $\mathfrak{B}$. If the spectral radius $r_o(T)$ is positive (i. e., if T has a non-zero eigenvalue), then $r_o(T)$ is an eigenvalue of T corresponding to a positive eigenvector and an eigenvalue of the adjoint T^* of T corresponding to a positive eigenvector (i. e., a positive linear functional on $\mathfrak{B}$).

Unless otherwise mentioned, we assume in Sections I. 3 through I. 10 that we are considering a real, partially-ordered Banach space $\mathfrak{B}$ with positive cone $\mathfrak{C}$ having the properties (a) through (e) stated at the beginning of this section.

I. 2. Examples

In this section we shall give some examples of operators on the space $C(\overline{\Omega})$ introduced in Section I. 1; the properties of these operators will guide us in the choice of hypotheses made on the abstract operators considered in Sections I. 3 through I. 10.

2-1. Example. We consider the Hammerstein integral equation with a weakly singular kernel

$$u(x) = \lambda \int_{\Omega} K(x, y) f(y, u(y)) dy , \tag{2.1}$$

where Ω is a bounded open connected subset of $\mathbb{R}^n$, $K(x, y)$ is continuous in (x, y) on $\overline{\Omega} \times \overline{\Omega}$ except possibly when $x = y$,

$$|K(x, y)| \leq \frac{\varkappa}{|x-y|^{\alpha}} , \tag{2.2}$$

with $0 \leq \alpha < n$, and the function f is continuous on $\overline{\Omega} \times [0, r)$ for some number $r > 0$. This problem has been considered by many investigators, among them Hammerstein (1930), Tricomi (1957, §4. 6), Dolph and Minty (1963), Schaefer (1963), Kolodner (1964), Krasnosel'skii (1964a, §7. 1, and 1964b), Vainberg (1964, Chapter VII), and Pogorzelski (1966, Chapters IX and X).

In the remainder of this section, all integrations are carried out over Ω unless otherwise noted.

We recall some of the properties of integral equations with weak singularities (Mikhlin 1964, Miranda 1955, Pogorzelski 1966, Smirnov 1964a). The operator $\Gamma : C(\overline{\Omega}) \to C(\overline{\Omega})$ defined by

$$\Gamma u(x) = \int K(x, y) u(y) dy \tag{2.3}$$

is a compact linear operator on $C(\overline{\Omega})$ by Arzéla's Theorem, and is positive if $K(x,y) \geq 0$. We inductively define the m^{th} iterated kernel K_m as

$$K_1(x,y) = K(x,y)$$

and

$$K_m(x,y) = \int K(x,z)K_{m-1}(z,y)dz$$

for $m > 1$; K_m is the kernel of the compact operator $\Gamma^m = \Gamma\Gamma^{m-1}$. Each of the kernels K_m is continuous except when $x = y$, and all kernels K_m are continuous on $\overline{\Omega}\times\overline{\Omega}$ for $m > \frac{n}{n-\alpha}$. It is possible to choose $m > \frac{n}{n-\alpha}$ so that the eigenfunctions of K corresponding to a given eigenvalue of K (characteristic value of Γ), i.e., the non-zero solutions of

$$h(x) = \mu\Gamma h(x) = \mu\int K(x,y)h(y)dy \tag{2.4}$$

for a given μ, are precisely the same as the eigenfunctions of K_m; then λ is an eigenvalue of K_m if and only if one of the m^{th} roots of λ is an eigenvalue of K. In what follows, we denote by m_o such a value of m.

In general, the operator Γ may have no characteristic values. However, if the kernel K is positive on $\Omega\times\Omega$, then it has an eigenvalue. This is seen as follows (cf. Jentzsch 1912): all iterates of K are positive, in particular K_{3m_o}, and therefore, $\int_\Omega K_{3m_o}(x,x)dx > 0$. Since K_{m_o} is bounded, it follows from the Fredholm theory for integral equations with bounded kernels that K_{m_o} has an eigenvalue (Pogorzelski 1966, Ch. VII, §5) and hence K does also. The Krein-Rutman Theorem implies that K has a non-negative eigenfunction ϕ corresponding to the eigenvalue $\mu_o(\Gamma)$ (the largest eigenvalue of K and the reciprocal of the spectral radius of the compact operator Γ). Since $K(x, y) > 0$ for $(x, y) \in \Omega \times \Omega$, it is easily seen that $\phi(x) > 0$ for $x \in \Omega$. The arguments of Jentzsch (1912) applied to the kernel K_{m_o} show that $\mu_o(\Gamma)$ is a simple eigenvalue of K (for the simplicity of $[\mu_o(\Gamma)]^{m_o}$ as an eigenvalue of K_{m_o} implies the simplicity of $\mu_o(\Gamma)$ as an eigenvalue of K), $\mu_o(\Gamma)$ is larger than the absolute value of any other eigenvalue of K, and the positive multiples of ϕ are the only positive eigenfunctions of K. The same reasoning shows that the adjoint kernel $K^*(x, y) = K(y, x)$ has a continuous eigenfunction ψ, positive on Ω, corresponding to the eigenvalue $\mu_o(\Gamma)$. We suppose that ϕ and ψ are chosen so that $\|\phi\| = 1$ and $\int_\Omega \phi(x)\psi(x)dx = 1$. We refer to $\mu_o(\Gamma)$ as the principal eigenvalue of K or of the linear equation (2.4).

Any function $u \in C(\overline{\Omega})$ can obviously be represented in the form

$$u = \xi(u)\phi + Pu ,$$

where

$$\xi(u) = \int \psi(x)u(x)dx$$

is a continuous linear function on $C(\overline{\Omega})$, and

$$Pu = u - \xi(u)\phi$$

is a projection operator on $C(\overline{\Omega})$ $(P^2u = Pu)$ which commutes with Γ $(P\Gamma u = \Gamma Pu)$. It follows from Fredholm's Theorems that for any function $v \in PC(\overline{\Omega}) = \{h \in C(\overline{\Omega}) : h = Ph\}$, the equation

$$h - \mu_o(\Gamma)\Gamma h = v \tag{2.5}$$

has a unique solution $h \in PC(\overline{\Omega})$; conversely, for any $h \in PC(\overline{\Omega})$, the function v of equation (2.5) is in $PC(\overline{\Omega})$. Thus, $I - \mu_o(\Gamma)\Gamma$ restricted to the subspace $PC(\overline{\Omega})$ has an inverse on $PC(\overline{\Omega})$ (cf. equation (7.8) below).

Using the fact that the eigenfunction ψ of the adjoint kernel is non-zero on Ω, it is easy to verify that Γ satisfies the assumptions (PA) and (PA_2) of Section I. 1, i. e., if there exists a function $h \in \mathcal{C}$ such that

$$h(x) - \lambda \int K(x, y)h(y)dy \geq 0 , \tag{2.6}$$

and this expression is actually positive for some $x \in \Omega$, then $\lambda < \mu_o(\Gamma)$; if h is any function in $C(\overline{\Omega})$ and inequality (2. 6) holds, with strict inequality holding for some $x \in \Omega$, then $\lambda \neq \mu_o(\Gamma)$.

The positivity requirements imposed on the kernel can be relaxed somewhat. For example, let p be a function in $\mathcal{C}$ which is not the zero function. Then there is a non-empty open subset $\Omega_1 \subseteq \Omega$ such that $p(x) > 0$ for $x \in \Omega_1$; $p(x) = 0$ for $x \in \Omega_o = \Omega - \Omega_1$. The kernel $N(x, y) \equiv K(x, y)p(y)$ is positive when $x \in \Omega$, $y \in \Omega_1$, if $K(x, y)$ is positive for $x \in \Omega$, $y \in \Omega$, and it is easily seen that all iterates $N_m(x, y)$ are also positive for $x \in \Omega$, $y \in \Omega_1$. Thus, $\int N_{3m_o}(x, y)dy > 0$, and the kernel $N(x, y)$ has an eigenfunction which is ≥ 0 on Ω corresponding to the eigenvalue $\mu_o(T) < \infty$, where T is the compact positive linear operator

$$Th(x) = \int K(x, y)p(y)h(y)dy .$$

Any positive eigenfunction satisfies

$$\phi(x) = \mu \int K(x, y)p(y)\phi(y)dy ,$$

for some number $\mu \neq 0$; if $\phi(x) = 0$ for some $x \in \Omega$, then for this x, $K(x, y) > 0$ for all $y \in \Omega$, and therefore $p(y)\phi(y) = 0$ for all $y \in \Omega$; but then $\phi(x) = 0$ for all $x \in \Omega$, which contradicts the fact that ϕ is an eigenfunction. Thus, any positive eigenfunction of N is strictly positive on Ω. Similarly, we see that any positive eigenfunction of the adjoint kernel $N^*(x, y) = p(x)K(y, x)$ is zero at a point $x \in \Omega$ if and only if $x \in \Omega_o$. Again, the arguments of Jentzsch (1912) show that the kernels N and N^* do not have more than one linearly independent positive eigenfunction, and that $\mu_o(T)$ is a simple eigenvalue of N which is smaller than the absolute value of all other eigenvalues.

The assumptions (PA) and (PA_2) are not satisfied by the operator T unless Ω_o is empty. However, the following weaker forms of the assumptions (PA) and (PA_2) hold:

Suppose that

$$(2.7) \qquad h(x) - \lambda \int K(x, y)p(y)h(y)dy \geq 0 ,$$

with strict inequality holding for some $x \in \Omega_1$; if $h \in \mathcal{C}$, then $\lambda < \mu_o(T)$; if $h \in C(\overline{\Omega})$, then $\lambda \neq \mu_o(T)$. Notice that if strict inequality holds in equation (2.6) for $h \in \mathcal{C}$ and $x = x_o \in \Omega$, then $h(x_o) > 0$.

We return now to the nonlinear equation (2.1), assuming henceforth that $K(x, y) > 0$ for $(x, y) \in \Omega \times \Omega$ and $f(x, \rho) > 0$ for $(x, \rho) \in \overline{\Omega} \times [0, r)$, with $0 < r \leq \infty$. The continuous function f induces an operator $\underline{f}$ defined by $\underline{f}u(x) = f(x, u(x))$ on

$$\mathcal{C}^r = \{u \in C(\overline{\Omega}) : 0 \leq u(x) < r , \ x \in \overline{\Omega}\} ,$$

and $\underline{f}$ is a bounded operator on $\mathcal{C}^r$ as defined in Section I. 1. The operator $A = \Gamma\underline{f}$ defined by

$$(2.8) \qquad Au(x) = \int K(x, y)f(y, u(y))dy$$

is therefore compact on $\mathcal{C}^r$, since Γ is compact on $C(\overline{\Omega})$. The operator A is obviously positive on $\mathcal{C}^r$, and it is monotonic if $f(x, \rho)$ is a non-decreasing function of ρ. Furthermore, A is g_o-bounded, where

$$g_o(x) = \int K(x, y)dy \quad ,$$

since for any function $u \in C^r$,

$$\min\{f(x, \rho): x \in \overline{\Omega}, \ \rho \in [0, \|u\|]\} g_o(x) \le \int K(x, y)f(y, u(y))dy$$
$$\le \max\{f(x, \rho): x \in \overline{\Omega}, \ \rho \in [0, \|u\|]\} g_o(x) \ .$$

We will return to a discussion of equation (2. 1) in Section II. 1.

If the boundary $\partial\Omega$ of Ω is sufficiently regular, the methods used to study equation (2. 1) may also be used to study equations of the form

(2. 9) $$u(x) = \lambda \int_\Omega K_1(x, y)f_1(y, u(y))dy + \int_\Omega K_2(x, y)f_2(y, u(y))dy$$
$$+ \int_{\partial\Omega} K_3(x, y)g(y, u(y))dy \quad ,$$

where K_1, K_2, f_1, and f_2 have the properties discussed previously, g is a positive continuous function on $\partial\Omega \times [0, r)$, and the kernel $K_3(x, y)$ is continuous on $\overline{\Omega} \times \overline{\Omega}$ except possibly when $x = y$ and satisfies

$$|K_3(x, y)| \le \frac{\varkappa}{|x-y|^\beta} \quad ,$$

where $\varkappa > 0$ and $0 \le \beta < n-1$.

Integral equations of the form of equation (2. 1) can be used to treat eigenvalue problems for nonlinear elliptic partial differential equations of the form

(2. 10a) $$Lu(x) = \lambda f(x, u(x)) \quad , \qquad x \in \Omega \ ,$$

with the boundary conditions

(2. 10b) $Bu = g$:

$$\alpha(x)u(x) + \frac{\partial u}{\partial \nu}(x) = g(x, u(x)) \quad , \qquad x \in \partial\Omega_1 \quad ,$$
$$u(x) = 0 \quad , \qquad x \in \partial\Omega_2 \quad ,$$

where L is the uniformly elliptic operator

$$Lu(x) = -\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x)D_iD_ju(x) + \sum_{i=1}^{n} b_i(x)D_iu(x) + c(x) ,$$

with $D_i = \frac{\partial}{\partial x_i}$; Ω is a bounded open connected subset of $\mathbb{R}^n$ with a boundary $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$ which has an outer normal vector $\{n_i(x)\}$ and the inside sphere property (Friedman 1964, p. 55) at each point $x \in \partial\Omega_1$ (either $\partial\Omega_1$ or $\partial\Omega_2$ may be empty); $\partial u/\partial \nu$ is the conormal derivative

$$\frac{\partial u}{\partial \nu}(x) = \sum_{i=1}^{n}\sum_{j=1}^{n} n_i(x)a_{ij}(x)D_ju(x) \quad , \quad x \in \partial\Omega_1 \quad ;$$

a_{ij} , b_i , and c are continuous functions on Ω , with $\{a_{ij}(x)\}$ uniformly positive definite and $c(x) \geq 0$ for $x \in \overline{\Omega}$; α is a non-negative continuous function on $\partial\Omega_1$, with $\alpha(x) > 0$ for some $x \in \partial\Omega_1$ if $\partial\Omega_2$ is empty and $c(x) = 0$ for all $x \in \Omega$; f is a strictly positive continuous function on $\overline{\Omega} \times [0, r)$; and g is a non-negative function which is continuous on $\partial\Omega_1 \times [0, r)$ and zero on $\partial\Omega_2 \times [0, r)$ (cf. Keller and Cohen 1967). If the boundary $\partial\Omega$ and the functions appearing in the operators L and B are sufficiently smooth, then the operator L with the boundary conditions $Bu = 0$ has a Green's function $G(x, y)$ which is continuous for $x \neq y$ and satisfies condition (2.2) with $\alpha = n-2$; any solution $u(x)$ of equations (2.9) satisfies the integral equation

$$(2.11) \quad u(x) = \lambda\int_{\Omega} G(x, y)f(y, u(y))dy + \int_{\partial\Omega_1} G(x, y)g(y, u(y))dy$$

$$\equiv A_\lambda u(x) \quad ;$$

which is of the form of equation (2.9); if f and g satisfy additional smoothness conditions, then any solution of the integral equation (2.11) is also a solution of equations (2.10) (Gevrey 1930, Giraud 1932, Mikhlin 1964, Miranda 1955, Pogorzelski 1966, Sternberg 1924; see also Leray and Schauder 1934, and Example 2-3 below). Moreover $G(x, y) > 0$ for all $x \in \Omega$, $y \in \Omega$ (Protter and Weinberger 1967, p. 88). When $g = 0$, equation (2. 11) reduces to an equation of the form (2. 1).

As a simple example, suppose L is the negative of the n-dimensional Laplacian, $L = -\Delta$; the boundary conditions are Dirichlet conditions, $u(x) = 0$ for $x \in \partial\Omega$; and f is uniformly Hölder continuous on $\overline{\Omega} \times [0, r)$. If $\partial\Omega$ is sufficiently smooth, then the Green's function $G(x, y)$ exists (Hellwig 1964, p. 223) and for any continuous function u, $\int_{\Omega} G(x, y) f(y, u(y)) dy$ is a continuously differentiable function of x on $\overline{\Omega}$ (Hellwig 1964, p. 173; Miranda 1955, §13). If u satisfies

$$u(x) = \lambda \int G(x, y) f(y, u(y)) dy ,$$

then u is continuously differentiable on $\overline{\Omega}$, $f(y, u(y))$ is a uniformly Hölder continuous function of y on $\overline{\Omega}$, and thus u satisfies equations (2. 10) (ibid.).

<u>2-2. Example</u>. Let the elliptic operator L, boundary operator B, domain Ω, and functions f and g be as in the discussion of equation (2. 10). We consider the boundary value problem

$$\text{(2. 12)} \qquad \begin{aligned} Lu(x) - \lambda p(x) u(x) &= f(x, u(x)) , && x \in \Omega , \\ Bu(x) &= g(x, u(x)) , && x \in \partial\Omega , \end{aligned}$$

where p is a positive function on Ω. The positive eigenfunctions and eigenvalues of this problem can be studied by seeking those λ for which the operators A_{λ},

$$(2.13) \qquad A_\lambda u(x) = \lambda\int_\Omega G(x,y)p(y)u(y)dy + \int_{\partial\Omega_1} G(x,y)g(y)u(y))dy$$
$$+ \int_\Omega G(x,y)f(y,u(y))dy \quad ,$$

have positive fixed points.

For $\lambda \geq 0$, $\{A_\lambda\}$ is an increasing family of compact positive monotonic forced operators on $C(\overline{\Omega})$; moreover, for $\lambda > 0$, $\{\lambda^{-1}A_\lambda\}$ is a decreasing family (cf. Lemma 5-5 and Theorem 5-6). It follows from the Positivity Lemma of Keller and Cohen (1967) or Schaefer (1966, App. 2.3) that positive eigenfunctions can exist only for values of λ less than the principal (smallest) eigenvalue μ_1 of

$$(2.14) \qquad \begin{aligned} L\phi(x) - \mu p(x)\phi(x) &= 0 \ , \qquad & x \in \Omega \ , \\ B\phi(x) &= 0 \ , \qquad & x \in \partial\Omega \ . \end{aligned}$$

In contrast with Example 2-1, we may expect positive eigenfunctions for $\lambda \leq 0$; however, our results are not in general applicable to the investigation of the existence and behavior of these fixed points, since A_λ may not be a positive or monotonic operator for $\lambda < 0$. If we choose numbers $\lambda^- < 0$, $\rho_o > 0$ such that

$$\int G(x,y)[f(y,\rho) + \lambda p(y)\rho]dy$$

is a positive increasing function of ρ for all $x \in \Omega$, $0 \leq \rho < \rho_o$, $\lambda > \lambda^-$, then $\{A_\lambda\}$, $\lambda > \lambda^-$, is an increasing family of compact positive monotonic forced operators on C^{ρ_o} in $C(\overline{\Omega})$.

The problem (2.12) can also be treated by using the Green's function $G(x,y;\lambda)$ of the operator $L-\lambda p$ subject to the boundary conditions $Bu = 0$. Since we are interested only in λ less than the principal eigenvalue μ_1 of (2.14), $G(x,y;\lambda)$ is positive on $\Omega \times \Omega$, $\lambda < \mu_1$;

for $\lambda \leq 0$, this is proved just as for the Green's function $G(x,y) = G(x,y;0)$ above, while for $\lambda > 0$, it follows from Schaefer (1966, Appendix 2.3). Thus, the operator A_λ defined by

(2.15) $$A_\lambda u(x) = \int_\Omega G(x,y;\lambda)f(y,u(y))dy + \int_{\partial\Omega_1} G(x,y;\lambda)g(y,u(y))dy$$

is a compact, strictly positive monotonic operator on $C(\bar{\Omega})$ for $\lambda < \mu_1$.

The operators $\{A_\lambda\}$, $\lambda < \mu_1$, form a strictly increasing family. To see this, suppose for simplicity that $g = 0$. Since

$$L(A_\lambda u)(x) = \lambda p(x)A_\lambda u(x) + f(x,u(x)),$$

we have

$$A_\lambda u = \lambda T(A_\lambda u) + f_1(u),$$

where

$$Tv(x) = \int_\Omega G(x,y)p(y)v(y)dy = \Gamma pv(x),$$

$$f_1(u)(x) = \int_\Omega G(x,y)f(y,u(y))dy = \Gamma \underline{f}u(x),$$

and

$$G(x,y) = G(x,y;0);$$

thus,

(2.16) $$A_\lambda u = [I-\lambda T]^{-1} f_1(u) = R(\lambda)\Gamma\underline{f}u$$

if $\lambda < \mu_1$; i.e.,

$$\int_\Omega G(x,y;\lambda)f(y,u(y))dy = A_\lambda u(x) = R(\lambda)\Gamma\underline{f}u(x),$$

where $R(\lambda) = [I-\lambda T]^{-1}$. Using the easily verified functional equation for the resolvent,

$$R(\lambda_1) - R(\lambda_2) = (\lambda_1 - \lambda_2)R(\lambda_1)TR(\lambda_2),$$

we have

$$A_{\lambda_1}u - A_{\lambda_2}u = (\lambda_1-\lambda_2)R(\lambda_1)TR(\lambda_2)f_1(u) = (\lambda_1-\lambda_2)R(\lambda_1)TA_{\lambda_2}u$$

$$= (\lambda_1-\lambda_2)R(\lambda_1)\Gamma[pA_{\lambda_2}u]$$

where we have used equation (2.16). Since

$$R(\lambda_1)\Gamma[pA_{\lambda_2}u](x) = \int_\Omega G(x,\xi;\lambda_1)p(\xi)\int_\Omega G(\xi,y;\lambda_2)f(y,u(y))dyd\xi > 0 ,$$

for $x \in \Omega$, whenever $\lambda_2 < \lambda_1 < \mu_1$ we have

$$A_{\lambda_1}u > A_{\lambda_2}u .$$

This result may also be derived by using the fact that under appropriate smoothness conditions, the partial derivative of $G(x,y;\lambda)$ with respect to λ is given by

$$\text{(2.17)} \qquad \frac{\partial G}{\partial \lambda}(x,y;\lambda) = \int G(x,z;\lambda)G(z,y;\lambda)p(z)dz$$

(cf. Mikhlin 1964, p. 48). Equation (2.17) may be obtained formally by differentiating the bilinear expansion of $G(x,y;\lambda)$ in terms of the eigenfunctions of L with the boundary conditions $Bu = 0$ and comparing with the bilinear expansion of the right hand side of equation (2.17).

In this case, the family $\{\lambda^{-1}A_\lambda\}$, $0 < \lambda < \mu_1$, is not decreasing, since $\lambda^{-1}G(x,y;\lambda) \to +\infty$ as $\lambda \to \mu_1$ from below.

If we extend the domains of definition of $f(x,u)$ and $g(x,u)$ so that they are defined for all u negative and sufficiently close to zero, and if the extension is carried out so that f remains strictly positive and g remains non-negative, then the operators A_λ of equations (2.11) (if $g \not\equiv 0$), (2.13), and (2.15) have the following property: For any $\lambda_o \geq 0$ and any function $u_o \in C^r$, there are positive numbers δ, ε such that if $u \in C(\bar{\Omega})$, $\|u-u_o\| \leq \varepsilon$, and $|\lambda-\lambda_o| \leq \delta$, then $A_\lambda u \in C$.

This result also holds for the operators $A_\lambda = \lambda A$, where A is defined by equation (2.8), if $\lambda_o > 0$. Thus, the families $\{A_\lambda\}$ satisfy the condition (SP) of Section I.6.

<u>2-3. Example</u>. Again, let L, B, and Ω be as in the discussion of equations (2.10). Let h be a continuous real-valued function of $2n+2$ variables defined on $\mathbb{R}^n \times \mathbb{R} \times \overline{\Omega} \times J$, where J is a subinterval of $\mathbb{R}$, and let g be a continuous real-valued function of $n+1$ variables defined on $\overline{\Omega} \times J$. A very general problem which includes Example 2-2 and the partial differential equation of Example 2-1 when the boundary conditions are linear is the following non-linear boundary value problem for the unknown function $\tilde{u}$:

$$(2.18) \qquad \begin{aligned} L\tilde{u}(x) &= h(\nabla\tilde{u}(x), \tilde{u}, x;\lambda), \quad x \in \Omega, \quad \lambda \in J, \\ B\tilde{u}(x) &= g(x;\lambda) \qquad\qquad\quad, \quad x \in \partial\Omega. \end{aligned}$$

If we let $u = L\tilde{u}$, so that

$$(2.19) \quad \tilde{u}(x) = B_\lambda u(x) \equiv \int_\Omega G(x,y)u(y)dy + \int_{\partial\Omega_1} G(x,y)g(y;\lambda)dy + \int_{\partial\Omega_2} \frac{\partial}{\partial\nu_y} G(x,y)g(y;\lambda)dy,$$

then solutions of equations (2.18) are fixed points of the operator A_λ defined by

$$(2.20) \qquad A_\lambda u(x) = h(\nabla B_\lambda u(x), B_\lambda u(x), x;\lambda).$$

Here, $\nabla B_\lambda u(x)$ denotes the n-component vector whose i^{th} component is $D_i B_\lambda u(x)$.

Under rather general conditions, the components of the operator ∇B_λ are compact, as well as the operator B_λ (see the discussion and references in Example 2-1). Under these conditions, if $h(v_1, \ldots, v_n, u, x_1, \ldots, x_n;\lambda)$ is strictly positive, then A_λ is a compact positive forced operator on $C(\overline{\Omega})$ (the compactness may be proved by a simple application of Arzéla's Theorem). It will not in general be monotonic because of the dependence of A_λ on the derivatives of $G(x, y)$. If, however, all the derivatives $\frac{\partial}{\partial x_i} G(x, y)$ and, if $g(x;\lambda) \neq 0$ on $\partial\Omega_2$, the derivatives $\frac{\partial}{\partial \nu_y} G(x, y)$ and $\frac{\partial}{\partial x_i} \frac{\partial}{\partial \nu_y} G(x, y)$ are positive, and if $h(v_1, \ldots, v_n; u; x_1, \ldots, x_n;\lambda)$ is an increasing function of each of the variables $v_1, \ldots, v_n, u$ for all non-negative values of these variables and all $(x_1, \ldots, x_n) \in \overline{\Omega}$, $\lambda \in J$, then A_λ is a positive monotonic operator on C, and the family $\{A_\lambda\}$, $\lambda \in J$, is increasing if $g(x;\lambda)$ and $h(v, u, x;\lambda)$ are increasing functions of λ.

As an example in which the operators A_λ are monotonic, consider the boundary value problem for the ordinary differential equation

$$\begin{aligned} &[p(x)\tilde{u}'(x)]' + h(\tilde{u}'(x), \tilde{u}(x), x;\lambda) = 0 \quad , \quad 0 \le x \le 1 \quad , \\ &\tilde{u}(0) - (\cos\theta)p(0)\tilde{u}'(0) = \gamma_1 \, , \quad \tilde{u}'(1) = \gamma_2 \quad , \end{aligned} \tag{2.21}$$

where $0 \le \theta \le \pi/2$, γ_1 and γ_2 are non-negative constants, p is a strictly positive function in $C[0, 1]$, h is continuous on $[0, \infty) \times [0, \infty) \times [0, 1] \times J$, and $h(v, u, x;\lambda)$ is increasing in v, u, and λ for $0 \le v < +\infty$, $0 \le u < +\infty$, $0 \le x \le 1$, and $\lambda \in J$. The appropriate Green's function is

$$G(x, y) = \begin{cases} \cos\theta + \int_0^x \frac{1}{p(z)} dz \quad , & 0 \le x \le y \le 1 \\ \cos\theta + \int_0^y \frac{1}{p(z)} dz \quad , & 0 \le y \le x \le 1 \, , \end{cases}$$

so

$$\frac{\partial G}{\partial x}(x, y) = \begin{cases} \frac{1}{p(x)} & 0 \le x \le y \le 1 \\ 0 & 0 \le y \le x \le 1 \ . \end{cases}$$

Letting $u = L\widetilde{u} = -(p\widetilde{u}')'$, we obtain for the operator $B_\lambda = B$ defined above:

$$Bu(x) = \int_0^1 G(x, y)u(y)dy + \gamma_2 x + \gamma_1 + \gamma_2 p(0) \cos\theta \ ,$$

and

$$\nabla Bu(x) = \int_x^1 \frac{1}{p(x)} u(y)\,dy + \gamma_2 \ .$$

Each of these operators is positive and monotonic on $C[0, 1]$, and thus the operators A_λ defined by equation (2.20) are compact positive monotonic operators on the positive cone $\mathbb{C}$ in $C[0, 1]$.

Since any solution $\widetilde{u}(x)$ of (2.21) satisfies $\widetilde{u}'(x) > 0$ for $0 \le x < 1$, the problem (2.21) may also be treated as follows. Let $h_1(u_2, u_1, x;\lambda) = h(u_2/p(x), u_1, x;\lambda)$; h_1 is well-defined since $p(x) > 0$ on $[0, 1]$. Let $\{C[0, 1]\}^2$ denote the space of ordered pairs of continuous functions on $[0, 1]$, i.e., $\{C[0, 1]\}^2 = C[0, 1] \times C[0, 1]$, let

$$\mathbb{C} = \{(u_1, u_2) \in \{C[0, 1]\}^2 : u_1(x) \ge 0,\ u_2(x) \ge 0,\ 0 \le x \le 1\} ,$$

and let $\|(u_1, u_2)\| = \|u_1\| + \|u_2\|$ for $(u_1, u_2) \in \{C[0, 1]\}^2$. If we define the operator A_λ on $\mathbb{C}$ by

$$A_\lambda(u_1, u_2) = (v_1, v_2) ,$$

where

$$v_1(x) = \int_0^1 G(x, y)h_1(u_2(y), u_1(y), y;\lambda)dy + \gamma_2 x + \gamma_1 + \gamma_2 p(0) \cos\theta$$

and

$$v_2(x) = p(x)\int_0^1 \frac{\partial G}{\partial x}(x, y)h_1(u_2(y), u_1(y), y;\lambda)dy + p(x)\gamma_2$$

for any pair of functions $(u_1, u_2) \in \mathcal{C}$, then the equations (2.21) are equivalent to the equation

$$(u_1, u_2) = A_\lambda(u_1, u_2) ,$$

with $u_1 = \tilde{u}$ and $u_2 = p\tilde{u}'$. The family $\{A_\lambda\}$, $\lambda \in J$, is an increasing family of compact positive monotonic operators on $\mathcal{C} \subseteq \{C[0,1]\}^2$.

An alternative method of treating equations (2.18) is given by Pogorzelski (1966, §§7, 13); see also Leray and Schauder (1934). This method does not assume the existence of a Green's function for the general elliptic operator L with boundary conditions $Bu = 0$.

2-4. Example. The problem of finding the positive fixed points of the operators

$$A_\lambda u(x) = 1 + \lambda \int_x^1 u(y)u(y-x)dy \tag{2.22}$$

in $C[0,1]$ has been considered by Pimbley (1967). The family $\{A_\lambda\}$, $\lambda \geq 0$, is an increasing family of strictly positive monotonic operators on the positive cone $\mathcal{C}$ of $C[0,1]$. Some properties of these operators are described in Proposition 4-2 and at the beginning of Section I. 10.

I. 3. General Theorems

Before investigating specifically families of positive monotone operators, we shall cite some general principles concerning the set of fixed points of more general families of operators $\{A_\lambda\}$, where λ is an element of an interval $J \subseteq R$. The proofs of the theorems are omitted; the proofs are simple and may be obtained, e. g., by minor modifications of proofs given by Krasnosel'skii (1964a, Chapter V) for theorems on the set of eigenvectors of an operator A.

3-1. Theorem. Let $\{A_\lambda\}$, $\lambda \in J$, be a family of operators defined on a closed set S, and let J be closed and bounded. Let the mapping $(\lambda, u) \to A_\lambda u$ be continuous on $J \times S$. Then the set of fixed points of the operators A_λ is closed. If the operators A_λ have the form $A_\lambda = \lambda A$, then the assumption that J is bounded may be removed.

3-2. Theorem. Let $\{A_\lambda\}$, $\lambda \in J$, be a family of operators on a closed bounded set S such that the mapping $(\lambda, u) \to A_\lambda u$ is compact on $J \times S$. Then the set $\{\lambda \in J: \exists\ u \in S \ni u = A_\lambda u\}$ is closed relative to J.

3-3. Corollary. Let $\{A_\lambda\}$, $\lambda \in J$, be a family of operators on a subset S of the ball $\mathcal{B}^r$ which is closed relative to $\mathcal{B}^r$, and let the mapping $(\lambda, u) \to A_\lambda u$ be compact on $J \times S$. If a point $\lambda_o \in J$ is an accumulation point of $\{\lambda \in J: \exists\ u \in S \ni u = A_\lambda u\}$, then either A_{λ_o} has a fixed point in S or

$$\lim_{\delta \to 0+} \inf\{\|u\| : u = A_\lambda u \in S \text{ for some } \lambda \in J \cap (\lambda_o - \delta, \lambda_o + \delta)\} = r.$$

Proof. Suppose the indicated limit (which certainly exists and is $\leq r$) is less than r. Then there is a positive number $\rho < r$ such that for any integer $n \geq 1$, there is a $\lambda_n \in (\lambda_o - \frac{1}{n}, \lambda_o + \frac{1}{n})$ and a fixed point u_n of A_{λ_n} such that $\|u_n\| \leq \rho$. Applying the Theorem 3-2 to the set $S \cap \bar{\mathcal{B}}^\rho$ (which is closed and bounded), we see that

$$\lim_{n\to\infty} \lambda_n = \lambda_o \in \{\lambda \in J: \exists\, u \in S \ni u = A_\lambda u\}. \qquad //$$

A set $\mathfrak{F}$ of fixed points of a family $\{A_\lambda\}$, $\lambda \in J$, of operators defined on a set S is said to form a continuous branch of length r in S if the boundary of every open set containing 0 and contained in the ball $\mathcal{B}^r$ contains a point of $\mathfrak{F}$. Note that according to this definition a continuous branch of fixed points need not be connected; e. g., the union of two (possibly disjoint) continuous branches of length r is a continuous branch of length r.

The next theorem states conditions under which, from the existence of a continuous branch of fixed points of a family $\{A_\lambda\}$, $\lambda \in J$, one can infer that $\{\lambda \in J: \exists\, u \ni u = A_\lambda u\}$ contains an interval (see also Krasnosel'skii 1964a, Section 5. 1).

3-4. Theorem. Let $\{A_\lambda\}$, $\lambda \in J$, be a family of operators defined on a closed set S containing elements of arbitrary norm; let J be bounded and contain either $\lambda^- = \inf J$ or $\lambda^+ = \sup J$. Let the mapping $(\lambda, u) \to A_\lambda u$ be compact on $J \times S$. Suppose that the set of fixed points of the operators A_λ contains a branch $\mathfrak{F}$ of infinite length in S, and that there are numbers λ_o, λ_∞ in $\overline{J}$ such that the norms of the fixed points of the operators A_λ in $\mathfrak{F}$ approach 0 as λ approaches λ_o and approach ∞ as λ approaches λ_∞. For any λ_1 strictly between λ_o and λ_∞, let the norms of the fixed points of the operators A_λ in $\mathfrak{F}$ be bounded above for λ between λ_o and λ_1 and be bounded away from 0 for λ between λ_1 and λ_∞. Then for every $\lambda \in (\lambda_o, \lambda_\infty)$ the operator A_λ has a fixed point in $\mathfrak{F}$. If $A_\lambda = \lambda A$, J may be taken to be $(0, \infty)$.

Conditions for the existence of a continuous branch of fixed points for operators $A_\lambda = \lambda A$ are given by the following theorems (Krasnosel'skii 1964b, Chapter V, Theorems 1.2 and 2.4; Schaefer 1963).

<u>3-5. Theorem.</u> Let A be a positive compact operator on C^r. If for some positive number $r_1 < r$,

$$\inf\{\|Au\|: u > 0, \|u\| = r_1\} > 0 ,$$

then A has eigenvector $u > 0$ such that $\|u\| = r_1$.

If

$$\inf\{\|Au\|: u \in C^r \cap \partial\mathcal{G}\} > 0$$

for any connected open set $\mathcal{G}$ which contains 0 and is contained together with its boundary $\partial\mathcal{G}$ in the ball $\mathcal{B}^r = \{u: \|u\| < r\}$, then the eigenvectors of A form a branch of length r in C^r.

<u>3-6. Theorem.</u> Let A be a positive compact operator on C^r. Suppose there exists a compact positive linear T with a non-zero eigenvalue such that for every u in C^r we have

$$Au \geq Tu .$$

Then the eigenvectors of A form a continuous branch of length r in C^r. Any eigenvalue λ of A satisfies $\lambda \geq r_o(T)$ if T is g_o-bounded.

I. 4. Minimal Positive Fixed Points

In this section we begin our study of the set of positive fixed points of a family $\{A_\lambda\}$, $\lambda \in J$, of positive monotonic operators. Our primary interest is in the case that the operators are compact on C^r for some $r > 0$; however, for the discussion of minimal positive fixed points in this and the following three sections, it usually suffices that the operators satisfy the following weaker hypothesis (H). We say that an operator defined on a subset S of the cone C satisfies (H) on S or has the property (H) on S if

(H) The operator is a continuous positive monotonic operator on S such that the image of every monotonic sequence contained in a closed bounded subset of S is a convergent sequence.

The requirement that the image of every monotonic sequence be convergent can be replaced by the requirement that the image be weakly convergent, since any monotonic weakly convergent sequence is strongly convergent, by the generalized Dini's Theorem (Schaefer 1966, p. 251). Since any monotonic sequence with compact closure is convergent (Krasnosel'skii 1964a, p. 40), the convergence requirement may also be replaced by the requirement that the image of a monotonic sequence have compact closure. This observation implies:

4-1. Proposition. If A is a compact positive monotonic operator on a subset S of C, then A satisfies (H) on S.

This proposition can be used to deduce that operators of the type considered in Examples 2-1 through 2-3 of Section I. 2 satisfy (H) on any subset of C. We shall give a direct proof that the operator of

Example 2-4 satisfies (H) on $\mathcal{C}$.

4-2. Proposition. Let A be the operator on the space $C[0,1]$ of real valued continuous functions on the interval $[0,1]$ defined by

$$(4.1) \qquad Au(x) = \int_x^1 u(y)u(y-x)dy$$

(see Example 2-4). Then A satisfies (H) on the cone $\mathcal{C}$ of non-negative functions in $C[0,1]$.

Proof. Let $\{u_n\}$ be a bounded monotonically increasing sequence in $\mathcal{C}$ (the proof for decreasing sequences is similar). Then we can define a function u on $[0,1]$ by $u(x) = \lim_{n\to\infty} u_n(x)$, and by the Lebesgue monotone convergence theorem,

$$v(x) \equiv \int_x^1 u(y)u(y-x)dy = \lim_{n\to\infty} \int_x^1 u_n(y)u_n(y-x)dy$$

for each x in $[0,1]$. Since the function u is bounded, it follows from the continuity in the mean of the Lebesgue integral (Goldberg 1962, p. 4) that v is a continuous function on $[0,1]$ (cf. Doetsch 1937, p. 159). Thus, the monotonically increasing sequence of continuous functions $\{Au_n\}$ converges pointwise to a continuous function v; Dini's Theorem implies that the convergence is uniform, and therefore the sequence $\{Au_n\}$ converges in $C[0,1]$. //

Pimbley (1967) asserts that the operator A of equation (4.1) is not compact on $C[0,1]$.

The following lemma, which follows immediately from property (H), is fundamental for the subsequent material in this section.

<u>4-3. Lemma.</u> Let the operator A have the property (H) on a set $S \subseteq \mathcal{C}$. Suppose there are elements u_o, v_o in S such that

$$0 \leq u_o \leq Au_o \leq Av_o \leq v_o .$$

Then A has a fixed point in $[u_o, v_o]$. Each of the sequences defined by

$$u_n = Au_{n-1} , \; n \geq 1 ,$$

and

$$v_n = Av_{n-1} , \; n \geq 1 ,$$

converges to a fixed point of A .

If $u^o > 0$ is a fixed point of an operator A such that for any fixed point $v > 0$ of A, $u^o \leq v$, then u^o is called the <u>minimal positive fixed point of A</u>, or the <u>minimal positive solution of $u = Au$</u>.

We shall investigate the minimal positive solutions of the equations $u = A_\lambda u$ using the methods of Keller and Cohen (1967). The family $\{A_\lambda\}$ will be considered for λ in some interval J of real numbers; we define $\lambda^- = \inf J$, $\lambda^+ = \sup J$. If $A_\lambda = \lambda A$, the interval J is understood to be $J = (0, \infty)$ unless otherwise noted.

<u>4-4. Theorem.</u> Let the forced operator A have the property (H) on $\mathcal{C}^r$ $(0 < r \leq \infty)$. Define the sequence $\{u_n\}$ by $u_o = 0$; for $n \geq 0$, $u_{n+1} = Au_n$ if $\|u_n\| < r$, $u_{n+1} = u_n$ if $\|u_n\| \geq r$.

Then the following statements are equivalent:

(i) The sequence $\{u_n\}$ is bounded in norm by a number $r_1 < r$.

(ii) A has a fixed point in $\mathcal{C}^r$.

(iii) $\lim_{n \to \infty} u_n$ exists in $\mathcal{C}^r$ and equals the minimal positive fixed point of A.

Proof. (i) implies (ii). Since A has the property (H), the bounded monotonic sequence $\{u_n\}$ is mapped into a convergent sequence $\{Au_n\}$; then $\lim u_n = \lim Au_n = A \lim u_n$, so A has the fixed point $\lim u_n > 0$, $\|\lim u_n\| \le r_1 < r$.

(ii) implies (iii). Let v be any fixed point of A in C^r. Then $0 < A0 \le Av = v$, so we may apply Lemma 4-3 to conclude that A has a fixed point $u^o = \lim u_n$ with $0 < u^o \le v$; thus, u^o is the minimal positive fixed point of A.

(iii) implies (i). $\|u_n\| \le \|\lim_{n\to\infty} u_n\| < r$. //

The following Theorem 4-5 and Corollary 4-6 are the fundamental comparison theorems for establishing the existence of minimal positive fixed points of a family $\{A_\lambda\}$ and for obtaining bounds on the values of λ for which A_λ has positive fixed points.

4-5. Theorem. Let A be a forced operator satisfying (H) on C^r. Let the operator B defined on C^r have a fixed point $v > 0$ in C^r such that $v = Bv \ge Av$. Then A has a positive fixed point $u = \lim v_n$ in C^r, where $v_o = v$ and $v_n = Av_{n-1}$ for $n \ge 1$, and the minimal positive fixed point u^o of A satisfies $u^o \le u \le v$.

If, in addition, B is a forced operator satisfying (H) on C^r and $Bu^o \ge Au^o = u^o$, then the minimal positive fixed point v^o of B satisfies $v^o = \lim u_n \ge u^o$, where $u_o = u^o$ and $u_n = Bu_{n-1}$.

Proof. Since $0 < A0 \le Av \le Bv = v$, A has a fixed point $u = \lim v_n$ in $[0, v]$ by Lemma 4-3. By the preceding theorem, A has a minimal positive fixed point $u^o \le u \le v$.

If B satisfies the additional hypotheses, then it is easy to see that $\lim u_n$ converges to a fixed point of B which is less than or equal to any other positive fixed point. Thus, $\lim u_n = v^o$, the minimal positive fixed point. //

If $\{A_\lambda\}$, $\lambda \in J$, is a family of operators defined on $\mathcal{C}^r$, $0 < r \leq \infty$, we set

$$\Lambda_A^r = \{\lambda \in J : \exists\, u \in \mathcal{C}^{r+} \ni A_\lambda u = u\}$$

and let $\Lambda_A = \Lambda_A^\infty$.

<u>4-6. Corollary.</u> Let $\{A_\lambda\}$ and $\{B_\lambda\}$, $\lambda \in J$, be two families of operators on $\mathcal{C}^r$ such that for each $\lambda \in J$, A_λ is forced and has property (H) on $\mathcal{C}^r$, and for each $u > 0$ in $\mathcal{C}^r$, $B_\lambda u \geq A_\lambda u$. Then

$$\Lambda_B^r \subseteq \Lambda_A^r ,$$

and for each $\lambda \in \Lambda_B^r$, the minimal positive fixed point of A_λ is less than or equal to any positive fixed point of B_λ.

<u>4-7. Corollary.</u> Let $\{A_\lambda\}$, $\lambda \in J$, be a family of forced operators satisfying (H) on $\mathcal{C}^r$. Suppose that for each $\lambda \in J$ there is a $b_\lambda > 0$ and a linear continuous positive operator T_λ on $\mathcal{B}$ such that

$$A_\lambda u \leq b_\lambda + T_\lambda u_\lambda$$

for all u in $\mathcal{C}^r$. If $\mu_o(T_\lambda) > 1$ and $r > \|(I-T_\lambda)^{-1} b_\lambda\|$, then $\lambda \in \Lambda_A^r$. For each such λ, one of the fixed points of A_λ is $\lim v_n$, where $v_o = (I-T_\lambda)^{-1} b_\lambda$ and $v_n = A_\lambda v_{n-1}$ for $n \geq 1$.

If there is an operator T such that $T_\lambda = \lambda T$, then $\lambda \in J \cap (0, \mu_o(T))$ and $\|(I-\lambda T)^{-1} b_\lambda\| < r$ imply $\lambda \in \Lambda_A^r$.

<u>Proof.</u> If $\mu_o(T_\lambda) > 1$ for some λ, then $(I-T_\lambda)^{-1}$ exists and the equation

$$v = b_\lambda + T_\lambda v$$

has the unique solution $(I-T_\lambda)^{-1}b_\lambda > 0$ (Krasnosel'skii 1964a, p. 64; Schaefer 1966, App. 2.3). By Theorem 4-5 or Corollary 4-6, $\lambda \in \Lambda_A^r$ if $r > \|(I-T_\lambda)^{-1}b_\lambda\|$, and $\lim v_n$ is a fixed point of A_λ.

If $T_\lambda = \lambda T$, then $\mu_o(T) = \lambda\mu_o(T_\lambda)$ for $\lambda > 0$, so the second assertion of the Corollary follows from the first. //

<u>4-8. Corollary.</u> Let $\{A_\lambda\}$, $\lambda \in J$, be a family of operators on $\mathcal{C}^r$. If for each $\lambda \in J$ there is a $b_\lambda \in \mathcal{C}^+$ and a linear operator T_λ on $\mathcal{B}$ satisfying (PA) and (H) on $\mathcal{C}$ such that $u \in \mathcal{C}^{r+}$ implies

$$A_\lambda u \geq b_\lambda + T_\lambda u,$$

then $\lambda \in \Lambda_A^r$ implies $\mu_o(T_\lambda) > 1$ and any positive fixed point v_o of A in $\mathcal{C}^r$ satisfies

$$v_o \geq (I-T_\lambda)^{-1}b_\lambda .$$

If, in addition, A_λ satisfies (H) on $\mathcal{C}^r$ for each $\lambda \in J$, then for any $\lambda \in \Lambda_A^r$ the minimal positive fixed point of A_λ is given by $u^o(\lambda) = \lim u_n$, where $u_o = (I-T_\lambda)^{-1}b_\lambda$ and $u_n = A_\lambda u_{n-1}$ for $n \geq 1$.

If there is a linear operator T such that $T_\lambda = \lambda T$ for $\lambda > 0$, then

$$\Lambda_A^r \cap (0,\infty) \subseteq J \cap (0, \mu_o(T)) .$$

<u>Proof.</u> According to Theorem 4-5, whenever A_λ has a positive fixed point in $\mathcal{C}^r$ (i.e., whenever $\lambda \in \Lambda_A^r$), the equation $v = b_\lambda + T_\lambda v$ has a positive solution; since T_λ satisfies (PA), this implies that $1 < \mu_o(T_\lambda)$. The rest of the Corollary follows immediately from Theorem 4-5. //

The next two theorems describe the behavior of the set of minimal positive fixed points $\{u^o(\lambda)\}$ of an increasing family $\{A_\lambda\}$ of operators satisfying (H) on C^r. The theorems show that $\{u^o(\lambda)\}$ is an increasing family, and if $A_\lambda u$ is continuous in λ uniformly for u in any subset of C^r of the form $[0, v]$, then $u^o(\lambda)$ is continuous from the left in λ at all points of Λ_A^r. Note that these hypotheses are satisfied by operators $A_\lambda = \lambda A$, $\lambda \in (0, \infty)$, if A satisfies (H) on C^r. Theorem 4-9 also shows that the set Λ_A^r is an interval.

4-9. Theorem. Let $\{A_\lambda\}$, $\lambda \in J$, be a (strictly) increasing family of forced operators which satisfy (H) on C^r. If $\lambda_o \in \Lambda_A^r$, then $J \cap [\lambda^-, \lambda_o] \subseteq \Lambda_A^r$, and the minimal fixed points $\{u^o(\lambda)\}$, $\lambda \in \Lambda_A^r$, form a (strictly) increasing family. For any $\lambda \in J \cap [\lambda^-, \lambda_o]$, the sequence $\{v_n(\lambda)\}$, $n \geq 0$, defined by

$$v_o(\lambda) = u^o(\lambda_o) \quad , \quad v_n(\lambda) = A_\lambda v_{n-1}(\lambda) \quad , \quad n \geq 1 ,$$

is a monotonically decreasing sequence converging to $u^o(\lambda)$, and the sequence $\{u_n(\lambda)\}$, $n \geq 0$, defined by

$$u_o(\lambda) = u^o(\lambda) \quad , \quad u_n(\lambda) = A_{\lambda_o} u_{n-1}(\lambda) , \quad n \geq 1 ,$$

is a monotonically increasing sequence converging to $u^o(\lambda_o)$.

Proof. The conclusion of the Theorem for an increasing family $\{A_\lambda\}$ follows immediately from Theorem 4-5 by identifying A of Theorem 4-5 with A_λ for any $\lambda \in J \cap [\lambda^-, \lambda_o]$, B with A_{λ_o}, and v with the minimal fixed point $u^o(\lambda_o)$ of A_{λ_o}.

If $\{A_\lambda\}$ is strictly increasing, then $\{u^o(\lambda)\}$, $\lambda \in \Lambda_A^r$, is strictly increasing, since for any $\lambda_1, \lambda_2 \in \Lambda_A^r$ such that $\lambda_1 < \lambda_2$, we have $u^o(\lambda_1) \leq u^o(\lambda_2)$ by what has already been shown, and therefore

$$u^o(\lambda_1) = A_{\lambda_1} u^o(\lambda_1) < A_{\lambda_2} u^o(\lambda_2) = u^o(\lambda_2) . \qquad //$$

4-10. Theorem. Let $\{A_\lambda\}$, $\lambda \in J$, be an increasing family of forced operators which satisfy (H) on C^r. For $\lambda_o \in \Lambda_A^r$, $\lambda_o > \lambda^-$, let the mapping $\lambda \to A_\lambda u$ be continuous from the left at λ_o uniformly for u in $[0, u^o(\lambda_o)]$. Then the mapping $\lambda \to u^o(\lambda)$ is continuous from the left at λ_o.

Proof. By Theorem 4-9, Λ_A^r contains the interval $(\lambda^-, \lambda_o]$ if it contains λ_o, so there are minimal positive fixed points $u^o(\lambda)$ with $\lambda < \lambda_o$. To prove the theorem, we show that for any sequence $\{\lambda_n\}$, $n \geq 1$, in (λ^-, λ_o), monotonically increasing to λ_o, the corresponding minimal positive fixed points $\{u^o(\lambda_n)\}$ converge to $u^o(\lambda_o)$. By Theorem 4-9 the sequence $\{u^o(\lambda_n)\}$ is increasing and bounded above by $u^o(\lambda_o)$; since A_{λ_o} satisfies (H), $\{A_{\lambda_o} u_o(\lambda_n)\}$ converges to a limit w_o. Given $\varepsilon > 0$, we can choose an integer n_o such that for all $n \geq n_o$, and for all $u \leq u^o(\lambda_o)$ in C^r, we have

$$\|w_o - A_{\lambda_o} u^o(\lambda_n)\| < \varepsilon/2 \quad \text{and} \quad \|A_{\lambda_o} u - A_{\lambda_n} u\| < \varepsilon/2 .$$

Then $n \geq n_o$ implies

$$\|u^o(\lambda_n) - w_o\| \leq \|A_{\lambda_n} u^o(\lambda_n) - A_{\lambda_o} u^o(\lambda_n)\| + \|A_{\lambda_o} u^o(\lambda_n) - w_o\| < \varepsilon$$

and therefore

$$\lim_{n \to \infty} u^o(\lambda_n) = w_o .$$

Since $0 < u^o(\lambda_n) \leq u^o(\lambda_o)$ for all n, $0 < w_o \leq u^o(\lambda_o)$. But $A_{\lambda_o} w_o = A_{\lambda_o} \lim u^o(\lambda_n) = \lim A_{\lambda_o} u^o(\lambda_n) = w_o$, so $\lim u^o(\lambda_n) = w_o = u^o(\lambda_o)$, since $u^o(\lambda_o)$ is the minimal positive solution. //

The mapping $\lambda \to u^o(\lambda)$ is not necessarily continuous from the right at λ_o, even if the mapping $\lambda \to A_\lambda u$ is. In Part II, we shall give an example of a compact operator on $\mathcal{C}$ for which the minimal eigenvectors $u^o(\lambda)$ are discontinuous from the right at a point $\lambda_o \in \Lambda_A^r$, even though the mapping $\lambda \to A_\lambda u = \lambda Au$ is continuous uniformly for u in any bounded subset of $\mathcal{C}$.

The method of proof of Theorem 4-10 shows that if $\{A_\lambda\}$, $\lambda \in J$, is an increasing family of operators satisfying (H) on $\mathcal{C}^r$, and if for any $\lambda_o \in J$ and $w \in \mathcal{C}^r$ the mapping $\lambda \to A_\lambda u$ is continuous at λ_o uniformly for $u \in [0, w]$, then the mapping $(\lambda, u) \to A_\lambda u$ (which obviously may be considered monotonic) has the following property analogous to (H): if $\{\lambda_k\}$ and $\{u_k\}$ are bounded monotonic sequences whose closures are contained in J and $\mathcal{C}^r$, respectively, then $\{A_{\lambda_k} u_k\}$ converges in $\mathcal{C}^r$. Similarly, we have:

__4-11. Theorem__. Let $\{A_\lambda\}$, $\lambda \in J$, be an increasing family of forced operators satisfying (H) on $\mathcal{C}^r$ $(0 < r \le \infty)$. Let Λ_A^r contain the open interval $J_1 = (\lambda_*, \lambda_o)$, with $\lambda_* < \lambda_o \in J$, and let the minimal positive fixed points $\{u^o(\lambda)\}$, $\lambda \in J_1$, be bounded in norm by a positive number $r_1 < r$. If, for any $w \in \mathcal{C}^r$, the mapping $\lambda \to A_\lambda u$ is continuous from the left at λ_o uniformly for $u \in [0, w]$, then $\lambda_o \in \Lambda_A^r$ and $\lim_{\lambda \uparrow \lambda_o} u^o(\lambda)$ exists and is the minimal positive fixed point of A_{λ_o}.

To study the minimal positive eigenvectors of a forced operator A (i.e., the minimal positive solutions of $u = \lambda Au$), we set $A_\lambda = \lambda A$ in the preceding theorems; if A satisfies (H) on $\mathcal{C}^r$, then $\{A_\lambda\}$, $\lambda > 0$, is an increased family of forced operators which satisfy (H) on $\mathcal{C}^r$. If

$0 < \lambda_o \in \Lambda_A^r$, then by Theorem 4-9, the interval $(0, \lambda_o]$ is in Λ_A^r and the minimal positive eigenvectors $u^o(\lambda)$ satisfy $u^o(\lambda) < u^o(\lambda_o)$ for any $\lambda \in (0, \lambda_o)$. Since $0 < u^o(\lambda) = \lambda A u^o(\lambda) < \lambda A u^o(\lambda_o)$ for $0 < \lambda < \lambda_o$, $\lim_{\lambda \downarrow 0} u^o(\lambda) = 0$. If A is bounded on C^r, then for any sequence $\{u_n\}$ of eigenvectors of A in C^r which are bounded in norm by a number $r_1 < r$ and which correspond to a sequence of characteristic values converging to zero, we have $\lim_{n \to \infty} u_n = 0$. Thus, if $\{u_n\}$ is a sequence of eigenvectors of such an operator corresponding to characteristic values converging to zero, then either $\lim_{n \to \infty} \|u_n\| = r$ or $\{u_n\}$ contains a subsequence converging to zero. This implies:

4-12. Theorem. Let the forced operator A satisfy (H) on C^r $(0 < r \leq \infty)$. Then the minimal positive eigenvectors $\{u^o(\lambda)\}$ satisfy $\lim_{\lambda \downarrow 0} u^o(\lambda) = 0$. If, in addition, A is bounded on C^r, and there are positive numbers ρ and δ such that for $0 < \lambda < \delta$, λA has at most one fixed point in C^ρ, then for any sequence $\{u_n\}$ of non-minimal eigenvectors of A in C^r which correspond to a sequence of characteristic values converging to zero, we have $\lim_{n \to \infty} \|u_n\| = r$.

4-13. Corollary. Let the forced operator A be bounded, satisfy (H) on C^r , and satisfy a uniform Lipschitz condition on C^ρ for some positive number $\rho < r$: i.e., there is a positive number γ such that for $u, v \in C^\rho$,

$$\|Au - Av\| \leq \gamma \|u - v\| \quad . \tag{4.2}$$

Then there is a positive number δ such that $(0, \delta) \subseteq \Lambda_A^r$, and any sequence of non-minimal eigenvectors $\{u_n\}$ in C^r corresponding to characteristic values converging to zero satisfies $\lim_{n\to\infty} \|u_n\| = r$.

<u>Proof.</u> If A satisfies a uniform Lipschitz condition on C^ρ, then there is a positive number $\delta = \frac{\rho}{\gamma\rho + \|A0\|}$ such that $0 < \lambda < \delta$ and $u, v \in C^\rho$ imply $\lambda Au \in C^\rho$ and

$$\|\lambda Au - \lambda Av\| \leq \lambda\gamma\|u-v\|$$

with $\lambda\gamma < 1$. Thus, λA satisfies the hypotheses of the contraction mapping principle (Krasnosel'skii 1964b) on C^ρ and therefore has a unique fixed point in C^ρ for $0 < \lambda < \delta$. //

If we replace the Lipschitz condition (4.2) by the condition $A_\lambda v - A_\lambda u \leq T_\lambda(v-u)$, where T_λ is a continuous positive linear operator and $0 \leq u \leq v \in C^r$, we are able to obtain the much sharper estimate of Theorem 4-14 and Corollary 4-15 for the values of λ for which A_λ has only one positive fixed point. This result cannot be improved without further restrictions on the A_λ. We shall see in Part II that there are operators $A_\lambda = \lambda A$ which satisfy (4.3) for all $u, v \in C$ with $u \leq v$ for which A has more than one positive eigenvector in C corresponding to characteristic values $\lambda \in (\mu_o(T), \sup\Lambda_A)$. A simpler example is obtained by taking A to be the continuously differentiable real valued function $Au = e^u$ for $0 \leq u \leq r$, $Au = e^r(u+1-r)$ for $r \leq u$, and taking $Tu = e^r u$, where r is a positive number. Then equation (4.4) holds, and $\mu_o(T) = e^{-r}$. For $r = 1$, A has infinitely many eigenvectors (all $u \geq 1$) corresponding to $\lambda = \mu_o(T) = e^{-1}$, and for $r > 1$, A has two eigenvectors corresponding to each $\lambda \in (\mu_o(T), e^{-1}) = (e^{-r}, e^{-1})$.

<u>4-14. Theorem.</u> Let $\{A_\lambda\}$, $\lambda \in J$, be a family of forced operators satisfying (H) on C^r $(0 < r \leq \infty)$. Suppose that for each $\lambda \in J$

there is a continuous positive linear operator T_λ such that whenever $u \in \mathcal{C}^r$, $v \in \mathcal{C}^r$, and $u \le v$, then

$$A_\lambda v - A_\lambda u \le T_\lambda(v-u). \tag{4.3}$$

Then A_λ has only one fixed point in $\mathcal{C}^r$ for each $\lambda \in \Lambda_A^r$ such that $1 < \mu_o(T_\lambda)$.

Proof. If $\lambda \in \Lambda_A^r$, then A_λ has a minimal positive fixed point $u^o(\lambda)$ and any other fixed point $v \in \mathcal{C}^r$ satisfies $v \ge u^o(\lambda)$. Since

$$v - u^o(\lambda) = A_\lambda v - A_\lambda u^o(\lambda) \le T_\lambda[v - u^o(\lambda)],$$

or

$$[u^o(\lambda)-v] - T_\lambda[u^o(\lambda)-v] \ge 0,$$

for any fixed point $v \in \mathcal{C}^r$ of A_λ, we have $u^o(\lambda) \ge v$ if $1 < \mu_o(T_\lambda)$ (Schaefer 1966, Appendix 2.3). Thus, A_λ has only one fixed point $u^o(\lambda)$ in $\mathcal{C}^r$. //

4-15. Corollary. Let the forced operator A satisfy (H) on $\mathcal{C}^r$, and let T be a continuous positive linear operator such that

$$Av - Au \le T(v-u) \tag{4.4}$$

whenever $u, v \in \mathcal{C}^r$ and $u \le v$. Then for each $\lambda \in \Lambda_A^r$ for which $\lambda < \mu_o(T)$, A has only one eigenvector in $\mathcal{C}^r$.

Another condition for the uniqueness in $\mathcal{C}^\rho$ of the eigenvectors of A will be given in the next section.

If the operators A_λ have the form $A_\lambda = c + \lambda A$ (i.e., $A_\lambda u = c + \lambda Au$), where $c \in \mathcal{C}$ and A satisfies (H) on $\mathcal{C}^r$, then we take $J = [0, \infty)$ if $c > 0$ or $J = (0, \infty)$ if $c = 0$. It is easily seen that the minimal positive fixed points satisfy $u^o(\lambda) \ge c$ for $\lambda > 0$, and $\lim_{\lambda \downarrow 0} u^o(\lambda) = c$.

Consequently, if there are any positive fixed points of $A_\lambda = c + \lambda A$ in C^r for $\lambda > 0$, we must have $c \in C^r$. For $c \in C^r$ and $\lambda > 0$, the problem of finding the positive fixed points of $c + \lambda A$ is equivalent to the problem of finding the positive eigenvectors of the operator B defined on $\{u \in C: u+c \in C^r\}$ by $Bu = A(c+u)$; u_o is a positive fixed point of $c+\lambda A$ in C^r if and only if $u_o - c$ is a positive eigenvector of B in $\{u \in C: u+c \in C^r\}$ corresponding to the characteristic value λ. Thus, any assertion concerning the positive eigenvectors of an operator (such as Theorem 4-11 and Corollary 4-12) implies a corresponding statement for the positive fixed points of a family of operators $\{A_\lambda\}$ of the form $A_\lambda = c+\lambda A$ for $\lambda > 0$.

I. 5. A Special Class of Operators

We can obtain a rather complete description of the set of positive fixed points of a large class of families $\{A_\lambda\}$ of operators satisfying a condition of the form

$$A_\lambda(\alpha u) \geq \alpha(1+\eta)A_\lambda u \tag{5.1}$$

for every real number α between 0 and 1; the positive number η may depend on α, u, and λ. Our discussion of such operators follows Krasnosel'skii (1964a, Chapter 6; 1964b, p. 277), who calls such operators "concave." We prefer not to use this terminology, since included in this class are certain operators which we call "convex" (see Proposition 10-6), as well as operators we call "concave." Krasnosel'skii considers only operators A_λ of the form $A_\lambda = \lambda A$; we generalize his results to increasing families $\{A_\lambda\}$ such that $\{\lambda^{-1}A_\lambda\}$ is a decreasing family, where $\lambda > 0$ (Theorem 5-6).

If A is a monotonic operator such that

$$A(\alpha u) \geq \alpha A u \tag{5.2}$$

for all $\alpha \in (0, 1)$ and some $u \in C^+$, then $\alpha Au \leq A(\alpha u) \leq Au$; thus, A has the following continuity property:

$$\lim_{\alpha \uparrow 1} A(\alpha u) = Au .$$

Similarly, if f is a monotonic function on $(r_1, r_2) \subseteq \mathbb{R}$ and $f(\alpha u) \geq \alpha f(u)$ for all $u \in (r_1, r_2)$ and all $\alpha \in (0, 1)$ such that $\alpha u \in (r_1, r_2)$, then f is continuous on (r_1, r_2). If $\{A_\lambda\}$, $\lambda \in J$, $0 \leq \lambda^- \notin J$, is an increasing family of operators on a subset S of $\mathcal{B}$ such that $\{\lambda^{-1}A_\lambda\}$, $\lambda \in J$, is a decreasing family, then for any $u \in S$, $A_{\alpha\lambda}u \geq \alpha A_\lambda u$

whenever $\lambda^- < \alpha\lambda \le \lambda \in J$, and the mapping $\lambda \to A_\lambda u$ is continuous on J for each $u \in S$.

The following propositions will give an indication of the importance of this class of operators.

5-1. Proposition. Let Ω be an open connected subset of $\mathbb{R}^n$ and r a positive real number. Let the non-negative function f be continuous on $\overline{\Omega} \times [0, r)$, and for each $x \in \overline{\Omega}$, let $f(x, u)/u$ be a strictly decreasing function of the real variable u, $0 < u < r$. Let K be a non-negative function on $\overline{\Omega} \times \overline{\Omega}$ such that $\int_\Omega K(x, y)dy$ exists for each $x \in \overline{\Omega}$ and such that the operator A, defined by

$$Au(x) = \int_\Omega K(x, y) f(y, u(y))\, dy$$

(for any function $u \in \mathcal{C}^r \subseteq C(\overline{\Omega})$), maps $\mathcal{C}^r$ into $C(\overline{\Omega})$. Then for any number $\alpha \in (0, 1)$ and any function $u \in \mathcal{C}^r$ there is a positive number η such that $A(\alpha u) \ge \alpha(1+\eta)Au$.

Proof. The fact that $f(x, u)/u$ is a strictly decreasing function of u is easily seen to be equivalent to the following: for any $x \in \overline{\Omega}$ and any numbers $\alpha \in (0, 1)$, $\rho \in [0, r)$,

$$f(x, \alpha\rho) - \alpha f(x, \rho) > 0.$$

Let u be a function in $\mathcal{C}^r$; by continuity, for fixed $\alpha \in (0, 1)$, there are numbers $\epsilon > 0$, $M > 0$, such that for all $x \in \overline{\Omega}$, $\rho \in [0, \|u\|]$, we have

$$f(x, \alpha\rho) - \alpha f(x, \rho) \ge \epsilon$$

and

$$0 \le f(x, \rho) < M.$$

Choosing $\eta < \epsilon/M$, we have for any $x \in \overline{\Omega}$,

$A(\alpha u)(x) - \alpha(1+\eta)Au(x)$

$$= \int_\Omega K(x,y)[f(y,\alpha u(y)) - \alpha(1+\eta)f(y,u(y))]dy$$

$$\geq (\epsilon - \eta M)\int_\Omega K(x,y)dy \geq 0 .$$

Thus,

$$A(\alpha u) \geq \alpha(1+\eta)Au . \qquad //$$

5-2. Proposition. Let Ω and r be as in Proposition 5-1. Let f be a strictly positive continuous function on $\overline{\Omega} \times [0, r)$ which satisfies the one-sided Lipschitz condition

$$f(x,\rho) - f(x,\sigma) \leq \gamma(\rho - \sigma) \qquad (\gamma > 0)$$

uniformly for $x \in \overline{\Omega}$, $0 \leq \sigma \leq \rho < r$. Then there is a positive number $r_1 < r$ such that for each $x \in \overline{\Omega}$, $f(x,u)/u$ is a strictly decreasing function of u for $0 < u \leq r_1$.

Proof. For any $\alpha \in (0,1)$ and any $\rho \in (0,r)$, we have

$$f(x,\alpha\rho) - \alpha f(x,\rho) \geq (1-\alpha)[f(x,\rho) - \gamma\rho].$$

Since f is positive and continuous, we can choose $r_1 \in (0,r)$ such that $\rho \in [0,r_1]$ implies $f(x,\rho) - \gamma\rho > 0$ for all $x \in \overline{\Omega}$. Then for all $x \in \overline{\Omega}$, $\alpha \in (0,1)$, and $\rho \in [0,r_1]$,

$$f(x,\alpha\rho) - \alpha f(x,\rho) > 0.$$

Thus, $f(x,u)/u$ is a strictly decreasing function of u for each $x \in \overline{\Omega}$, $0 < u \leq r_1$. //

If, in addition to the hypotheses of Proposition 5-2, $f(x,u)$ has a partial derivative $f_u(x,u)$, then $f(x,u)/u$ is a strictly decreasing function of u for all u such that

$$f(x,u) - uf_u(x,u) = -u^2 \frac{\partial}{\partial u}\frac{f(x,u)}{u} > 0.$$

The next three lemmas are preliminary to the principal theorem of this section, Theorem 5-6. Lemma 5-3 is a comparison theorem similar to Theorem 4-5; we will use it in the proof of Theorem 5-6 by taking A_1 and A_2 to be operators from an increasing family $\{A_\lambda\}$. Observe also that the hypotheses of Lemma 5-3 are satisfied if we take $A_1 = \lambda_1 A$ and $A_2 = \lambda_2 A$, where $0 < \lambda_1 < \lambda_2$ and A is a monotone g_o-bounded operator on C^r satisfying equation (5. 2). Thus, if u_1 and u_2 are any two eigenvectors in C^r of such an operator A corresponding to characteristic values $\lambda_1 < \lambda_2$, then $u_1 < u_2$ (since $\lambda_1 Au_1 < \lambda_2 Au_1$).

<u>5-3. Lemma.</u> Let A_1 and A_2 be positive operators on C^r $(0 < r \le \infty)$ having positive fixed points u_1 and u_2, respectively, in C^r, such that $u_1 = A_1u_1 \le A_2u_1$. Let A_2 be monotone and g_o-bounded for some $g_o \in C^+$. For any number $\alpha \in (0, 1)$, let there be a positive number η such that $A_2(\alpha u_1) \ge (1+\eta)\alpha A_1u_1$. (This condition is satisfied if for any $\alpha \in (0, 1)$ there is an $\eta > 0$ such that either

$$A_2(\alpha u_1) \ge (1+\eta)A_1(\alpha u_1) \quad \text{and} \quad A_1(\alpha u_1) \ge \alpha A_1 u_1$$

or

$$A_2(\alpha u_1) \ge A_1(\alpha u_1) \qquad \text{and} \quad A_1(\alpha u_1) \ge \alpha(1+\eta)A_1u_1 \text{ .)}$$

Then either $u_1 = u_2$ and $A_1u_1 = A_2u_1$, or $u_1 < u_2$.

<u>Proof.</u> Suppose $u_1 \nleq u_2$. Since A_2 is g_o-bounded, there are positive numbers α and β such that $\alpha g_o \le A_2u_2$ and $A_2u_1 \le \beta g_o$. Thus,

$$u_2 = A_2u_2 \ge \alpha g_o \ge (\alpha/\beta)A_2u_1 \ge (\alpha/\beta)A_1u_1 = (\alpha/\beta)u_1 .$$

Since we are assuming $u_1 \nleq u_2$, there is a largest positive number

$\alpha_o < 1$ such that $u_2 \geq \alpha_o u_1$. But, since A_2 is monotone,

$$u_2 = A_2 u_2 \geq A_2(\alpha_o u_1) \geq \alpha_o(1+\eta)A_1 u_1 = \alpha_o(1+\eta)u_1 ;$$

this contradicts the maximality of α_o. Thus, $u_1 \leq u_2$. //

If we set $A_1 = A_2$ in Lemma 5-3, we obtain the following uniqueness result (Krasnosel'skii 1964a, p. 188):

5-4. Lemma. Let A be a positive monotonic operator on C^r which is g_o-bounded on C^r for some $g_o \in C^+$. Let A have a positive fixed point u_1 in C^r, and for any $\alpha \in (0, 1)$, let there be an $\eta > 0$ such that $A(\alpha u_1) \geq \alpha(1+\eta)Au_1$. Then u_1 is the only positive fixed point of A in C^r.

The following theorem gives conditions for the existence of a family of fixed points $\{u(\lambda)\}$ of a family of operators $\{A_\lambda\}$ in the neighborhood of a given fixed point $u(\lambda_o)$; this result is of particular interest because it does not require the Fréchet differentiability of the operators A_λ (cf. the discussion following Theorem 6-5). Equations (5.3) and (5.4) are satisfied for all $\lambda_o \in \Lambda_A^r$ if A satisfies equation (5.1) for all $u \in C^r$.

5-5. Theorem. Let $\{A_\lambda\}$, $\lambda \in J$, $\lambda^- \geq 0$, be an increasing family of operators satisfying (H) on C^r; let the family $\{\lambda^{-1}A_\lambda\}$ be decreasing(for $\lambda > 0$) on C^r. For some λ_o in J, let A_{λ_o} have a positive fixed point $u(\lambda_o)$ in C^r, and suppose that for any $\alpha \in (0, 1)$ there is an $\eta > 0$ for which

$$A_{\lambda_o}(\alpha u(\lambda_o)) \geq \alpha(1+\eta)A_{\lambda_o} u(\lambda_o) \tag{5.3}$$

and

(5.4) $$A_{\lambda_o} u(\lambda_o) \geq \alpha(1+\eta) A_{\lambda_o}(\alpha^{-1} u(\lambda_o))$$

if $\alpha^{-1} u(\lambda_o) \in C^r$.

Then, given $\varepsilon > 0$, there exists a neighborhood N in J of λ_o such that for each $\lambda \in N$ there exists a fixed point $u(\lambda)$ of A_λ with $\|u(\lambda) - u(\lambda_o)\| < \varepsilon$.

Proof. Suppose $\lambda_o > \lambda^-$. For any fixed $\alpha \in (0, 1)$, choose $\eta > 0$ sufficiently small that $\lambda^- \leq \lambda_o/(1+\eta)$ and (5.3) holds. Then, for any $\lambda \in (\frac{\lambda_o}{1+\eta}, \lambda_o)$ we have

$$\begin{aligned} A_\lambda(\alpha u(\lambda_o)) &\geq \frac{\lambda}{\lambda_o} A_{\lambda_o}(\alpha u(\lambda_o)) \\ &\geq \frac{\lambda(1+\eta)}{\lambda_o} \alpha A_{\lambda_o} u(\lambda_o) \\ &\geq \alpha u(\lambda_o), \end{aligned}$$

since $\{\lambda^{-1} A_\lambda\}$ is a decreasing family. Since $\{A_\lambda\}$ is an increasing family, $A_\lambda(u(\lambda_o)) \leq A_{\lambda_o} u(\lambda_o) = u(\lambda_o)$. Thus, we may apply Lemma 4-1 to the operator A_λ on $[\alpha u(\lambda_o), u(\lambda_o)]$ and conclude that A_λ has a fixed point $u(\lambda)$ in $[\alpha u(\lambda_o), u(\lambda_o)]$.

Similarly, if $\lambda_o < \lambda^+$, then for each λ such that $\lambda_o < \lambda < \lambda_o(1+\eta) \leq \lambda^+$, the operator A_λ has a fixed point in $[u(\lambda_o), \alpha^{-1} u(\lambda_o)]$, provided α has been chosen large enough that $\|\alpha^{-1} u(\lambda_o)\| < r$. The theorem follows from the fact that α may be chosen arbitrarily close to 1. //

The preceding lemmas provide the following description of the set of fixed points of the family $\{A_\lambda\}$:

5-6. Theorem. Let $\{A_\lambda\}$, $\lambda \in J$, $\lambda^- \geq 0$, be an increasing family of operators satisfying (H) on C^r $(0 < r \leq \infty)$, and let $\{\lambda^{-1}A_\lambda\}$, $0 < \lambda \in J$, be a decreasing family on C^r. For each $\lambda \in J$, let A_λ be g_λ-bounded for some $g_\lambda \in C^+$. Suppose that Λ_A^r is not empty and that for each $\lambda_o \in \Lambda_A^r$ (with a corresponding positive fixed point $u(\lambda_o)$ of A_{λ_o}) and for each number $\alpha \in (0, 1)$, there exists a positive number η such that equations (5.3) and (5.4) are satisfied.

Then Λ_A^r is an interval which is open relative to J; to each $\lambda \in \Lambda_A^r$ there corresponds a unique fixed point $u(\lambda)$ of A_λ. The mapping $\lambda \to u(\lambda)$ is continuous on Λ_A^r and the family $\{u(\lambda)\}$, $\lambda \in \Lambda_A^r$, is an increasing family. If we let $\lambda^* = \sup \Lambda_A^r$ and $\lambda_* = \inf \Lambda_A^r$, then either $\lambda^* = \lambda^+$ or $\lim_{\lambda \uparrow \lambda^*} \|u(\lambda)\| = r$, and either $\lambda_* = \lambda^-$ or $\lim_{\lambda \downarrow \lambda_*} \|u(\lambda)\| = 0$. (Recall that $\lambda^+ \equiv \sup J$ and $\lambda^- \equiv \inf J$.)

Proof. (Cf. Krasnosel'skii 1964a, pp. 200-201.) The uniqueness of the fixed point of A_λ in C^r follows from Lemma 5-4, the continuous dependence of $u(\lambda)$ on λ and the openness of Λ_A^r in J follow from the uniqueness and Lemma 5-5, and the fact that the family $\{u(\lambda)\}$ is an increasing family is a consequence of Lemma 5-3. Thus, only the last sentence of the theorem remains to be proved.

If $\lambda^* \notin J$, then clearly $\lambda^* = \lambda^+$; thus, we assume $\lambda^* \in J$.

Since $\|u(\lambda)\|$ increases as λ increases, $\lim_{\lambda \uparrow \lambda^*} \|u(\lambda)\|$ exists or is infinite and is $\leq r$. Suppose $\lim_{\lambda \uparrow \lambda^*} \|u(\lambda)\| < r$. Choose a monotonically increasing sequence $\{\lambda_n\}$ converging to $\lambda^* \in J$. Then $u(\lambda_n)$ is a monotonically increasing sequence and is bounded in norm. Thus, the sequence $\{A_{\lambda^*}u(\lambda_n)\}$ has a limit w (since A_{λ^*} satisfies (H)).

Since the family $\{\lambda^{-1}A_\lambda\}$ is decreasing, $\lambda_n < \lambda^*$ implies

$$\frac{\lambda_n}{\lambda^*} A_{\lambda^*} u^o(\lambda_n) \le A_{\lambda_n} u^o(\lambda_n) \le A_{\lambda^*} u^o(\lambda_n) .$$

Passing to the limit, we obtain

$$\lim_{n\to\infty} u^o(\lambda_n) = \lim_{n\to\infty} A_{\lambda_n} u^o(\lambda_n) = w$$

and

$$\|w\| \le \lim_{\lambda\uparrow\lambda^*} \|u^o(\lambda)\| < r .$$

Thus,

$$w = \lim_{n\to\infty} A_{\lambda^*} u^o(\lambda_n)$$

$$= A_{\lambda^*} \lim_{n\to\infty} u^o(\lambda_n) = A_{\lambda^*} w ,$$

so that $\lambda^* \in \Lambda_A^r$ and hence $\lambda^* = \lambda^+$, since Λ_A^r is open relative to J.

A similar argument with a decreasing sequence $\{\lambda_n\}$ converging to λ_* shows that if $\lim_{\lambda\downarrow\lambda_*} \|u(\lambda)\| > 0$ and $\lambda_* \in J$, then $\lambda_* \in \Lambda_A^r$, which is possible only if $\lambda_* = \lambda^-{}^*$. //

The hypotheses of Theorem 5-6 are satisfied by operators A_λ of the form

$$A_\lambda u = A_o u + \lambda A u , \quad \lambda \in J = (0, \infty) ,$$

if A_o and A satisfy (H) on C^r, and one of the operators A_o, A is g_o-bounded and satisfies equation (5.1), while the other satisfies equation (5.2) (e.g., a positive linear operator). If either A_o or A is forced, then A_λ is forced for each $\lambda \in J$, and Λ_A^r, if non-empty, is an open interval $(0, \lambda^*)$, $0 < \lambda^* \le \infty$.

We mention one more important property of positive monotonic g_o-bounded operators satisfying equation (5.1) (Krasnosel'skii 1964a, p. 192). If the operator A is compact, then the positive fixed point of A (if any, there is only one by Lemma 5-4) can be obtained by successive approximations, and one may use any $u_o \in C^{r+}$ as the initial approximation if the successive approximations $u_n = Au_{n-1}$ remain in the domain of definition C^r of A. This result is true even if A is unforced and thus has 0 as a fixed point.

I. 6. Fréchet Derivatives and the Implicit Function Theorem

Let $\{A_\lambda\}$, $\lambda \in J$, be an increasing family of forced operators satisfying (H) on C^r. According to Theorem 4-9, if $\lambda_o \in \Lambda_A^r$, then all λ in J less than λ_o are also in Λ_A^r; thus, $\inf \Lambda_A^r = \inf J = \lambda^-$. For values of λ in J larger than λ_o, it may happen (as in Theorem 5-6) that there are minimal solutions as far as there possibly can be; i.e., either Λ_A^r exhausts J or the minimal fixed points have norms which approach r as λ approaches $\lambda^* = \sup \Lambda_A^r \leq \lambda^+$. It is also possible that the minimal fixed points are bounded in norm by a number less than r and that $\lambda^* < \lambda^+$. In the latter case, one is interested in the behavior of the minimal fixed points as λ approaches λ^*, whether $\lambda^* \in \Lambda_A^r$, and whether there may be non-minimal fixed points for $\lambda < \lambda^*$ which approach the minimal fixed points as λ approaches λ^*. We are able to answer these questions by assuming that the operators A_λ have certain differentiability properties; this is the subject of the next section. At the same time, we discuss the behavior of the solutions near a value of λ at which the minimal solutions $u^o(\lambda)$ are discontinuous (from the right) in λ. In general, one would like to determine the value of λ^* and determine for what values of λ, if any, there are non-minimal fixed points. Some progress in this direction can be made if $r = \infty$ and we again assume certain differentiability properties for the operators A_λ. This matter and the behavior of the fixed points of large norm are discussed in Sections 8 and 9.

In order to have at hand the facts concerning differentiable operators which shall be useful later, we give in this section the definition and some of the properties of the Fréchet derivative and then state and discuss the implicit function theorem for equations in a Banach space. The Fréchet derivative is discussed in all the books referred to in this section.

Let $\mathcal{B}_1, \mathcal{B}_2$ be Banach spaces, A an operator mapping a subset of $\mathcal{B}_1$ into $\mathcal{B}_2$, and $u_o \in \mathcal{B}_1$. The operator A is said to be Fréchet differentiable at u_o if A is defined on a neighborhood of u_o and if there exists a continuous linear operator $A'(u_o)$: $\mathcal{B}_1 \to \mathcal{B}_2$ such that

$$(6.1) \qquad \lim_{\|h\| \to 0} \frac{\|A(u_o+h) - Au_o - A'(u_o)h\|}{\|h\|} = 0 .$$

The operator $A'(u_o)$ is called the Fréchet derivative of A at u_o. If A is defined in a neighborhood of a set S and Fréchet differentiable at each point of S, then A is said to be Fréchet differentiable on S. A is continuously differentiable at u_o if A is Fréchet differentiable on a neighborhood $\mathcal{N}$ of u_o and if the mapping A': $u \to A'(u)$ is continuous at u_o as a mapping from $\mathcal{N}$ into the Banach space $\mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$ of continuous linear operators from $\mathcal{B}_1$ into $\mathcal{B}_2$. If A is Fréchet differentiable on a neighborhood $\mathcal{N}$ of u_o and if the mapping A' is itself differentiable at u_o, then A is said to be twice Fréchet differentiable at u_o. The Fréchet derivative of A' at u_o, denoted by $A''(u_o)$, is a continuous linear operator from $\mathcal{B}_1$ into $\mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$; it may also be considered as a bilinear operator from $\mathcal{B}_1$ into $\mathcal{B}_2$.

If A is compact on a neighborhood of u_o and Fréchet differentiable at u_o, then $A'(u_o)$ is compact. If A is monotonic on a neighborhood of u_o and Fréchet differentiable at u_o, then $A'(u_o)$ is a positive operator (Krasnosel'skii 1964a).

Let S be a subset of a Banach space $\mathfrak{B}$ containing elements of arbitrarily large norm, and let A be an operator mapping a subset of S into $\mathfrak{B}$. We say that A is S-asymptotically linear if there is a positive number r and a continuous linear operator $A'(\infty): \mathfrak{B} \to \mathfrak{B}$, called the S - asymptotic derivative of A, such that A is defined on $\{u \in S: \|u\| \geq r\}$ and

$$\lim_{\|u\| \to \infty} \frac{\|Au - A'(\infty)u\|}{\|u\|} = 0, \tag{6.2}$$

where the limit is taken over all $u \in S$ such that $\|u\| \geq r$. We shall always take S to be either $\mathfrak{B}$ (in which case we shall omit the reference to the set $S = \mathfrak{B}$) or the positive cone $\mathfrak{C}$ of a partially ordered space $\mathfrak{B}$. Under the assumptions we have made on the cone $\mathfrak{C}$ in Section 1, we have:

If A is defined and compact on a neighborhood of $\{u \in S: \|u\| \geq r\}$ and if A is S - asymptotically linear, then $A'(\infty)$ is compact on $\mathfrak{B}$; if A is a positive operator, then $A'(\infty)$ is a positive operator (Krasnosel'skii 1964a).

Let $\{A_\lambda\}$, $\lambda \in J$, be a family of operators defined on a subset S of $\mathfrak{B}$. We define the norm on $\mathfrak{R} \times \mathfrak{B}$ as $\|(\lambda, u)\| = |\lambda| + \|u\|$ and consider the differentiability of the mapping $(\lambda, u) \to A_\lambda u$ at a point $(\lambda_o, u_o) \in J \times S$. Suppose that this mapping is Fréchet differentiable at (λ_o, u_o). Then the above definition implies that the operator A_{λ_o}: $S \to \mathfrak{B}$ is Fréchet differentiable at u_o, and that the mapping $\lambda \to A_\lambda u_o$ of J into $\mathfrak{B}$ is Fréchet differentiable at λ_o; the derivative of the latter mapping may be represented in terms of an element $B_{\lambda_o} u_o \in \mathcal{B}$ as the linear mapping $\delta \to \delta B_{\lambda_o} u_o$, $\delta \in \mathbb{R}$. According to equation (6.1), we have

$$\lim_{|\delta|+\|h\|\to 0} \frac{\|A_{\lambda_o+\delta}(u_o+h)-A_{\lambda_o}u_o-A'_{\lambda_o}(u_o)h-\delta B_{\lambda_o}u_o\|}{|\delta| + \|h\|} = 0 .$$

It is easily seen that if the family $\{A_\lambda\}$ has the form

$$A_\lambda = \sum_{k=0}^{n} \lambda^k A^{(k)}$$

for $\lambda \in J$ and some integer $n \geq 0$, then the mapping $(\lambda, u) \to A_\lambda u$ is Fréchet differentiable at (λ_o, u_o) for any $\lambda_o \in \operatorname{Int} J$ if each of the operators $A^{(k)}$, $k = 0, 1, \ldots, n$, is Fréchet differentiable at u_o.

In the definition of the Fréchet derivative of an operator A at a point u, it is assumed that A is defined on a neighborhood of u. Whenever we speak of the Fréchet derivative in the following, this is assumed to be the case, even if not explicitly mentioned. Thus, a statement such as: "A is Fréchet differentiable on $\mathcal{C}^r$," is to be interpreted as: "A is defined on a neighborhood of $\mathcal{C}^r$ and Fréchet differentiable at each point of $\mathcal{C}^r$."

It is possible to define a Riemann integral for continuous Banach space valued functions of a real variable. The Fréchet derivative and this integral are related by

$$(6.3) \qquad A(u+h) - Au = \int_{\alpha=0}^{1} A'(u+\alpha h)h\, d\alpha ,$$

where it is assumed that A is Fréchet differentiable at each point $u+\alpha h \in \mathcal{B}$, $0 \leq \alpha \leq 1$, and that the mapping $\alpha \to A'(u+\alpha h)h$ of $[0, 1]$ into

$\mathcal{B}$ is continuous on $[0, 1]$ (Liusternik and Sobolev 1961).

As an example of a Fréchet differentiable operator, consider the family of operators $\{A_\lambda\}$ on $C(\bar{\Omega})$ defined by

$$A_\lambda u(x) = \int_\Omega K(x, y;\lambda)f(y, u(y))dy \quad .$$

We assume that f is a continuous real-valued function on $\bar{\Omega} \times (-\infty, +\infty)$, and that the kernel $K(x, y;\lambda)$ is such that the linear operator Γ_λ defined by

$$\Gamma_\lambda u(x) = \int_\Omega K(x, y;\lambda)u(y)dy$$

is a continuous operator on $C(\bar{\Omega})$ for each λ in an interval J. Then $\{A_\lambda\}$, $\lambda \in J$, is a family of continuous operators on $C(\bar{\Omega})$. If, in addition, $f(x, u)$ is continuously differentiable with respect to u with derivative $f_u(x, u)$, for $x \in \Omega$ and $\alpha < u < \beta$, then each operator A_λ is continuously Fréchet differentiable on $\{u \in C(\bar{\Omega}): \alpha < u(x) < \beta , \ x \in \bar{\Omega}\}$; the Fréchet derivative is given by

$$A_\lambda'(u)h(x) = \int_\Omega K(x, y;\lambda)f_u(y, u(y))h(y)dy \quad .$$

These assertions are simple to verify; see, e.g., Krasnosel'skii (1964a, 1964b) and Kantorovich and Akilov (1964).

If, in addition, the kernel $K(x, y;\lambda)$ has a partial derivative $K_\lambda(x, y;\lambda)$ with respect to λ which is continuous in $(x, y;\lambda)$ on $\bar{\Omega}\times\bar{\Omega}\times J$ except for a weak singularity when $x = y$ (Example 2-1),then the mapping $(\lambda, u) \to A_\lambda u$ is Fréchet differentiable at (λ_o, u_o) for all $\lambda_o \in J$ and $u_o \in C(\bar{\Omega})$ such that $\alpha < u_o(x) < \beta$ for $x \in \bar{\Omega}$. The Fréchet derivative is given by the linear mapping $(\delta, h) \to \delta B_{\lambda_o} u_o + A'_{\lambda_o}(u_o)h$ for $\delta \in \mathbb{R}$ and $h \in C(\overline{\Omega})$, where

$$B_{\lambda_o} u_o(x) = \int_{\Omega} K_{\lambda}(x, y; \lambda_o) f(y, u_o(y)) dy \ .$$

If $f(x, u)$ has a second partial derivative $f_{uu}(x, u)$ which is continuous on $\overline{\Omega} \times (\alpha, \beta)$, then each operator A_{λ} is twice Fréchet differentiable, and $A''_{\lambda}(u)$ is the bilinear operator which sends any pair of continuous functions $h, k \in C(\overline{\Omega})$ into the function $A''_{\lambda}(u)hk = A''_{\lambda}(u)kh \in C(\overline{\Omega})$ defined by

$$A''_{\lambda}(u)(hk)(x) = \int_{\Omega} K(x, y; \lambda) f_{uu}(y, u(y)) h(y) k(y) dy$$

for any function u such that $\alpha < u(x) < \beta$ for $x \in \overline{\Omega}$.

We remarked earlier that a differentiable positive monotonic operator A has a positive (hence monotonic) Fréchet derivative which is compact if A is compact, and a $\mathfrak{C}$ - asymptotically linear positive operator A has a positive $\mathfrak{C}$ - asymptotic derivative which is compact if A is compact. The next two propositions show that the property (H) also is inherited by the derivatives of operators satisfying (H). In view of the preceding remarks, it suffices to show that the derivatives map any bounded monotonic sequence in $\mathfrak{C}$ into a convergent sequence.

6-1. Proposition. Let the positive monotonic operator A be defined in a neighborhood $\mathcal{N}$ of a point $u \in \mathfrak{C}$, satisfy (H) on $\mathcal{N} \cap \mathfrak{C}$, and be Fréchet differentiable at u. Then $A'(u)$ satisfies (H) on $\mathfrak{C}$.

Proof. Assume the given neighborhood $\mathcal{N}$ is closed (otherwise, take a smaller closed neighborhood). Let $\{h_n\}$ be a monotonic sequence in $\mathfrak{C}$ bounded in norm by the number $\gamma > 0$. Given $\varepsilon > 0$, there is a positive number ρ such that $\|k\| \leq \rho$ implies $u+k \in \mathcal{N}$ and

$$\|A(u+k) - Au - A'(u)k\| \leq \varepsilon \|k\| / 3\gamma .$$

Let α be a fixed positive number less than ρ/γ. Since $\{u+\alpha h_n\}$ is a bounded monotonic sequence in $\mathcal{N} \cap \mathbf{C}$ and A satisfies (H) on $\mathcal{N} \cap \mathbf{C}$, $\{A(u+\alpha h_n)\}$ converges, and therefore it is possible to find a positive integer n_o such that $n, m \geq n_o$ implies

$$\|A(u+\alpha h_n) - A(u+\alpha h_m)\| \leq \varepsilon\alpha/3 .$$

Thus, $n, m \geq n_o$ implies

$$\begin{aligned}\|A'(u)h_n - A'(u)h_m\| &\leq \frac{1}{\alpha}\|A(u+\alpha h_n) - Au - A'(u)\alpha h_n\| + \frac{1}{\alpha}\|A(u+\alpha h_m) - Au - A'(u)\alpha h_m\| \\ &\qquad + \frac{1}{\alpha}\|A(u+\alpha h_n) - A(u+\alpha h_m)\| \\ &\leq \varepsilon .\end{aligned}$$

Therefore, the sequence $\{A'(u)h_n\}$ converges. //

The proof of the corresponding result for $A'(\infty)$ is similar:

<u>6-2. Proposition.</u> Let the positive operator A satisfy (H) on $\{u \in \mathbf{C} : \|u\| \geq r\}$ $(0 \leq r < \infty)$ and be $\mathbf{C}$-asymptotically linear. Then $A'(\infty)$ satisfies (H) on $\mathbf{C}$.

A well-known application of the Fréchet derivative of an operator A such that $A0 = 0$ is to the determination of the possible values of λ from which eigenvectors can bifurcate from the trivial zero solution of $u = \lambda Au$; if $A'(0)$ exists and is compact, these bifurcation values must be characteristic values of $A'(0)$ (Krasnosel'skii 1964a or 1964b). Theorem 6-3 is an analogous result for the positive eigenvectors of a family $\{A_\lambda\}$ of positive operators, and the simple proof is omitted (cf. Krasnosel'skii 1964a, Section 5.3.5).

6-3. Theorem. Let $\{A_\lambda\}$, $\lambda \in J$, be a family of positive unforced continuous operators on $\mathcal{C}^r$ $(0 < r \leq \infty)$. Suppose there is a sequence $\{\lambda_n\}$ in J converging to $\lambda_o \in J$ such that the operators $\{A_\lambda\}$ have fixed points $u(\lambda_n)$ in $\mathcal{C}^r$ with $\lim_{n\to\infty} \|u(\lambda_n)\| = 0$, and let there be a compact linear operator B such that

$$\lim_{\substack{\|u\|\to 0,\, u\in\mathcal{C}^r \\ \lambda\to\lambda_o}} \frac{\|A_\lambda u - Bu\|}{\|u\|} = 0 . \tag{6.4}$$

Then 1 is an eigenvalue of B corresponding to a positive eigenvector.

Since the limit in equation (6.4) of Theorem 6-3 is taken only over elements of $\mathcal{C}$, the operator B may be interpreted as the derivative $A'_{\lambda_o}(0+)$ of A_{λ_o} from above at 0:

$$\lim_{\substack{\|u\|\to 0 \\ u\in\mathcal{C}^r}} \frac{\|A_{\lambda_o} u - A'_{\lambda_o}(0+)u\|}{\|u\|} = 0 . \tag{6.5}$$

It is easily seen that the hypotheses of Theorem 6-4 are satisfied by a family $\{A_\lambda\}$ of the form

$$A_\lambda = \sum_{i=1}^{N} \Gamma_\lambda^{(i)} A^{(i)} , \tag{6.6}$$

where for each $i = 1, 2, \ldots, N$, $A^{(i)}$ is a continuous positive operator on $\mathcal{C}^r$ for which $A^{(i)'}(0+)$ exists, $\Gamma_\lambda^{(i)}$ is a continuous linear operator on $\mathcal{B}$, either $A^{(i)'}(0+)$ or $\Gamma_{\lambda_o}^{(i)}$ is compact, and the mapping $\lambda \to \Gamma_\lambda^{(i)}$ is continuous at λ_o (i.e.,

$$\lim_{\lambda\to\lambda_o} \|\Gamma_\lambda^{(i)} - \Gamma_{\lambda_o}^{(i)}\| = 0).$$

If A is continuously differentiable on a set S and the norms of the Fréchet derivatives are uniformly bounded on S, then it follows from the integral formula (6.3) that A satisfies a Lipschitz condition on S. If the derivatives satisfy inequality (6.7) below, then condition (e) of Section I.1 and the uniform boundedness principle (Dunford and Schwarz 1958) imply that $\{A'(u) : u \in \mathcal{C}\}$ is bounded; moreover, A satisfies inequality (4.4) on $\mathcal{C}^r$. We therefore obtain as a corollary of Theorem 4-14:

<u>6-4. Theorem.</u> Let A be a continuously Fréchet differentiable forced operator which satisfies (H) on $\mathcal{C}^r$. Let T be a continuous positive linear operator such that for each $u \in \mathcal{C}^r$, $h \in \mathcal{C}$,

$$A'(u)h \leq Th \quad (6.7)$$

Then A has at most one eigenvector in $\mathcal{C}^r$ corresponding to a characteristic value $\lambda \in (0, \mu_o(T))$.

One of the most important uses we shall make of the Fréchet derivative is in connection with the implicit function theorem; various formulations of the theorem are given by Dieudonne (1960), Graves (1965), Hildebrandt and Graves (1927), Krasnosel'skii (1964b), Liusternik and Sobolev (1961), and Nirenberg (1961). If one seeks to solve the equation $\Phi(x,y)=0$ for y in terms of x, and if it is known that $\Phi(x_o, y_o)=0$, this theorem gives conditions under which one may solve for y as a continuous function of x in a neighborhood of x_o.

We cite the form of the theorem as given by Liusternik and Sobolev. In the statement of the theorem, we will use the Cartesian product $\mathcal{B}_1 \times \mathcal{B}_2$ of two Banach spaces $\mathcal{B}_1, \mathcal{B}_2$, which is to be assumed equipped with the product topology; this is the same as the direct sum $\mathcal{B}_1 \oplus \mathcal{B}_2$ of the two spaces and may be given the norm $\|(x, y)\| = \|x\| + \|y\|$ for $x \in \mathcal{B}_1$ and $y \in \mathcal{B}_2$. For any open subset $\mathcal{G} \subseteq \mathcal{B}_1 \times \mathcal{B}_2$ containing a point (x, y), there are open subsets $\mathcal{G}_1 \subseteq \mathcal{B}_1$, $\mathcal{G}_2 \subseteq \mathcal{B}_2$ such that $(x, y) \in \mathcal{G}_1 \times \mathcal{G}_2 \subseteq \mathcal{G}$.

6-5. Theorem. (Implicit Function Theorem.) Let $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$ be Banach spaces (not necessarily partially ordered), let $\mathcal{G}_1, \mathcal{G}_2$ be open subsets of $\mathcal{B}_1, \mathcal{B}_2$, respectively, and let Φ be a continuous mapping of $\mathcal{G}_1 \times \mathcal{G}_2$ into $\mathcal{B}_3$. Let $(x_o, y_o) \in \mathcal{G}_1 \times \mathcal{G}_2$. Suppose that

(a) $\Phi(x_o, y_o) = 0$;

(b) for each $x \in \mathcal{G}_1$, the mapping $y \to \Phi(x, y)$ of $\mathcal{G}_2$ into $\mathcal{B}_3$ is differentiable; we will denote the derivative mapping by $D_2\Phi(x, y)$;

(c) the mapping $(x, y) \to D_2\Phi(x, y)$ of $\mathcal{G}_1 \times \mathcal{G}_2$ into $\mathcal{L}(\mathcal{B}_2, \mathcal{B}_3)$ is continuous;

(d) the inverse mapping $[D_2\Phi(x_o, y_o)]^{-1}$ exists; that is, $D_2\Phi(x_o, y_o)$ is a one-to-one mapping of $\mathcal{B}_2$ onto $\mathcal{B}_3$.

Then, for any sufficiently small neighborhood $\mathcal{N}_2 \subseteq \mathcal{G}_2$ of y_o, there is a neighborhood $\mathcal{N}_1 \subseteq \mathcal{G}_1$ of x_o and a continuous operator $F : \mathcal{N}_1 \to \mathcal{B}_2$ such that

(i) $F(x_o) = y_o$;

(ii) if $x \in \mathcal{N}_1$, then $\Phi(x, F(x)) = 0$;

(iii) if $(x, y) \in \mathcal{N}_1 \times \mathcal{N}_2$ and $\Phi(x, y) = 0$, then $y = F(x)$.

If, in addition, Φ is continuously differentiable on $\mathcal{G}_1 \times \mathcal{G}_2$, then $\mathcal{N}_1$ may be chosen so that F is continuously differentiable on $\mathcal{N}_1$, and

$$F'(x) = -[D_2\Phi(x, F(x))]^{-1} D_1\Phi(x, F(x))$$

for $x \in \mathcal{N}_1$.

We shall use this theorem with $\mathcal{B}_1 = \mathcal{R}$, $\mathcal{B}_2 = \mathcal{B}_3 = \mathcal{B}$, and $\Phi(\lambda, u) = u - A_\lambda u$. At any u for which A_λ has a Fréchet derivative $A'_\lambda(u)$, $D_2\Phi(\lambda, u)$ exists and $D_2\Phi(\lambda, u) = I - A'_\lambda(u)$, where I is the identity operator of $\mathcal{B}$. Thus, $D_2\Phi(\lambda, u)$ has an inverse on $\mathcal{B}$ if and only if 1 is not in the spectrum of $A'_\lambda(u)$.

Let $\{A_\lambda\}$, $\lambda \in J$, be an increasing family of forced operators satisfying (H) on $\mathcal{C}^r$. Suppose that for $\lambda_1 \in J$, the operator A_{λ_1} has a positive fixed point $u(\lambda_1)$ and a Fréchet derivative $A'_{\lambda_1}(u(\lambda_1))$ at $u(\lambda_1)$. If the implicit function theorem is applicable to the equation $u - A_\lambda u = 0$ for (λ, u) near $(\lambda_1, u(\lambda_1))$ [so that 1 is not in the spectrum of $A'_{\lambda_1}(u(\lambda_1))$], then there is a neighborhood N_1 of λ_1 in J for which we can construct a continuous family $\{u(\lambda)\}$, with $\lambda \in N_1$, of fixed points of A_λ. If $u(\lambda_1)$ happens to be on the boundary of $\mathcal{C}$, then without imposing further conditions we have no assurance that the fixed points obtained in this way belong to $\mathcal{C}$. We shall see below, however, that if we impose the following stronger positivity condition (SP) on the operators A_λ, then the fixed points $u(\lambda)$ constructed by using the implicit function theorem lie in $\mathcal{C}$ if the neighborhood N_1 of λ_1 is taken sufficiently small; this condition says roughly that A_λ maps a neighborhood of the cone $\mathcal{C}$ into $\mathcal{C}$.

(SP) Let $\{A_\lambda\}$, $\lambda \in J$, be a family of operators defined on an open set containing the point $u_o \in \mathcal{C}$ and let $\lambda_o \in J$. Then $\{A_\lambda\}$ satisfies (SP) at (λ_o, u_o) if there is a J-neighborhood N of λ_o and a neighborhood $\mathcal{N}$ of u_o such that for all $(\lambda, u) \in N \times \mathcal{N}$, $A_\lambda u$ is defined and $A_\lambda u \in \mathcal{C}$.

If the operators A_λ are defined in a neighborhood of a point $u_o \in \mathcal{C}$ and if $A_{\lambda_o} u_o \in \text{Int}\,\mathcal{C}$, then this condition is satisfied at (λ_o, u_o) if the mapping $(\lambda, u) \to A_\lambda u$ is continuous at (λ_o, u_o), since there is a neighborhood $\mathcal{N}_o$ of $A_{\lambda_o} u_o$ such that $\mathcal{N}_o \subseteq \mathcal{C}$; by continuity, there is therefore a neighborhood $N \times \mathcal{N}$ of (λ_o, u_o) such that $(\lambda, u) \in N \times \mathcal{N}$ implies $A_\lambda u \in \mathcal{N} \subseteq \mathcal{C}$.

If A_{λ_1} has a positive fixed point $u(\lambda_1)$ and the operators A_λ have fixed points $u(\lambda)$ for λ in some neighborhood N_o of λ_1 such that

$$\lim_{\substack{\lambda \to \lambda_1 \\ \lambda \in N_o}} u(\lambda) = u(\lambda_1),$$

and if the condition (SP) is satisfied by $\{A_\lambda\}$ at $(\lambda_1, u(\lambda_1))$, then we can choose a neighborhood $N \times \mathcal{N}$ of $(\lambda_1, u(\lambda_1))$ such that $(\lambda, u) \in N \times \mathcal{N}$ implies $A_\lambda u \in \mathcal{C}$. We can assume N so small that $\lambda \in N \cap N_o$ implies $u(\lambda) \in \mathcal{N}$, and therefore, $\lambda \in N \cap N_o$ implies $u(\lambda) = A_\lambda u(\lambda) \in \mathcal{C}$. Thus, application of the implicit function theorem at $(\lambda_1, u(\lambda_1))$ will yield positive fixed points $u(\lambda)$ for λ sufficiently close to λ_1.

I. 7. Behavior of Fixed Points Near the Maximum of Λ

We are now able to describe more completely the behavior of the minimal positive fixed points $\{u^o(\lambda)\}$ of an increasing family $\{A_\lambda\}$ of Fréchet differentiable operators satisfying (H): the fact that $1 \leq \mu_o[A'_\lambda(u^o(\lambda))]$ (Theorem 7-1), their continuity and differentiability with respect to λ if $1 < \mu_o[A'_\lambda(u^o(\lambda))]$ (Theorems 7-1 and 7-7), and the precise dependence on λ as λ approaches a value λ_o at which $1 = \mu_o[A'_{\lambda_o}(u^o(\lambda_o))]$ (Theorem 7-3). Roughly speaking, we find that the minimal fixed points are continuous and differentiable with respect to λ unless $1 = \mu_o[A'_\lambda(u^o(\lambda))]$, and that $1 = \mu_o[A'_\lambda(u^o(\lambda))]$ can occur only if a certain expression involving the second Fréchet derivative $A''_\lambda(u^o(\lambda))$ is non-negative. If this quantity is positive at a point λ_o at which $1 = \mu_o[A'_{\lambda_o}(u^o(\lambda_o))]$ and $1 < \mu_o[A'_\lambda(u^o(\lambda))]$ for $\lambda_o - \delta < \lambda < \lambda_o$, then we have what is known as a "bifurcation point," "branch point," or "limiting point" (see, e. g., Krasnosel'skii 1964a and 1964b, Cronin 1950, Bolotin 1963); there is a second positive fixed point for all λ less than and sufficiently close to λ_o, these fixed points approach $u^o(\lambda_o)$ as λ approaches λ_o, and there is a neighborhood of $u^o(\lambda_o)$ in which there are no fixed points of A_λ for $\lambda > \lambda_o$.

Suppose that the operator A_{λ_o} on C^r has a positive fixed point $u_o \in C^r$ and that the mapping $(\lambda, u) \to A_\lambda u$ is Fréchet differentiable at (λ_o, u_o). Then from Section I. 6,

$$(7.1) \qquad A_\lambda u = A_{\lambda_o} u_o + A'_{\lambda_o}(u_o)(u-u_o) + (\lambda-\lambda_o)B_{\lambda_o} u_o + C(\lambda-\lambda_o; u-u_o),$$

where

$$(7.2) \qquad \lim_{(\lambda, u)\to(\lambda_o, u_o)} \frac{C(\lambda-\lambda_o, u-u_o)}{|\lambda-\lambda_o| + \|u-u_o\|} = 0 .$$

Consider a sequence $\{\lambda_n\}$, $n \geq 1$, in Λ_A^r converging to λ_o such that there are corresponding fixed points $u_n \neq u_o$ of A_{λ_n} with $\lim_{n\to\infty} u_n = u_o$. If $A'_{\lambda_o}(u_o)$ is assumed to be a compact linear operator, then there is a subsequence $\{\lambda_{n_k}\}$ such that $A'_{\lambda_o}(u_o)(u_o-u_{n_k})/\|u_o-u_{n_k}\|$ has a limit w. Since $u_o-u_{n_k} = -(A_{\lambda_{n_k}} u_{n_k} - A_{\lambda_o} u_o)$,

$$\frac{u_o-u_{n_k}}{\|u_o-u_{n_k}\|} + B_{\lambda_o} u_o \frac{\lambda_{n_k}-\lambda_o}{\|u_o-u_{n_k}\|} + \frac{C_{\lambda_o}(\lambda_{n_k}-\lambda_o, u-u_o)}{|\lambda_{n_k}-\lambda_o|+\|u_{n_k}-u_o\|} \bullet \frac{|\lambda_o-\lambda_{n_k}|}{\|u_o-u_{n_k}\|}$$

$$= A'_{\lambda_o}(u_o)\frac{u_o-u_{n_k}}{\|u_o-u_{n_k}\|} - \frac{C_{\lambda_o}(\lambda_{n_k}-\lambda_o, u_{n_k}-u_o)}{|\lambda_{n_k}-\lambda_o|+\|u_{n_k}-u_o\|} ,$$

and the right hand side of this equality converges to w as $n_k \to \infty$. Thus, the numerical sequence

$$(7.3) \qquad \{\beta_{n_k}\} = \left\{ \frac{\lambda_o-\lambda_{n_k}}{\|u_o-u_{n_k}\|} \right\}$$

is bounded, and we may assume that $\{\lambda_{n_k}\}$ has been chosen so that $\{\beta_{n_k}\}$ converges to a number β_o.

Then

$$(7.4) \qquad h \equiv \lim_{k\to\infty} \frac{u_o-u_{n_k}}{\|u_o-u_{n_k}\|} = \beta_o B_{\lambda_o} u_o + w$$

$$= \beta_o B_{\lambda_o} u_o + A'_{\lambda_o}(u_o)h .$$

This equation for the case where the $u(\lambda)$ are minimal positive solutions leads to the following theorem.

7-1. Theorem. Let $\{A_\lambda\}$, $\lambda \in J$, be a strictly increasing family of forced operators which satisfy (H) on C^r $(0 < r \leq \infty)$ for each $\lambda \in J$. Let $\lambda_o \in \Lambda_A^r$, $\lambda_o > \inf \Lambda_A^r$, and let $u_o = u^o(\lambda_o)$ be the corresponding minimal fixed point of A_{λ_o}. Let the mapping $(\lambda, u) \to A_\lambda u$ of $J \times C^r$ into C be differentiable at (λ_o, u_o), the mapping $\lambda \to A_\lambda u$ be continuous at λ_o uniformly for u in $[0, u_o]$, and $A'_{\lambda_o}(u_o)$ be a compact linear operator. If $\mu_o[A'_{\lambda_o}(u_0)] < \infty$, let $A'_{\lambda_o}(u_o)$ satisfy (PA) and have a unique positive eigenvector of unit norm. (The Krein-Rutman theorem shows that the unique positive eigenvector of $A'_{\lambda_o}(u_o)$ must correspond to the characteristic value $\mu_o[A'_{\lambda_o}(u_o)]$.)

Then $1 \leq \mu_o[A'_{\lambda_o}(u_o)]$:

If $1 < \mu_o[A'_{\lambda_o}(u_o)]$, then the mapping $\lambda \to u^o(\lambda)$ is differentiable from the left at λ_o; the derivative is the unique solution $\psi = \psi(\lambda_o)$ of

$$\psi - A'_{\lambda_o}(u_o)\psi = B_{\lambda_o} u_o \quad . \tag{7.5}$$

If $1 = \mu_o[A'_{\lambda_o}(u_o)]$, then

$$\lim_{\lambda \uparrow \lambda_o} \frac{u_o - u^o(\lambda)}{\|u_o - u^o(\lambda)\|}$$

exists and is the unique positive eigenvector of unit norm of $A'_{\lambda_o}(u_o)$ corresponding to the eigenvalue 1.

Proof. In the considerations leading to equation (7.3), choose the sequence $\{\lambda_n\}$ to be a monotonic sequence in Λ_A^r increasing to λ_o. Then $\lim_{n\to\infty} u^o(\lambda_n) = u_o$ by Theorem 4-10. Since $\{A_\lambda\}$ is a strictly increasing family, $u^o(\lambda_n) < u_o$, so h in equation (7.3) is well-defined and positive. Since $\beta_o \geq 0$ and $B_{\lambda_o} u_o \geq 0$, (7.6) $$h \geq A'_{\lambda_o}(u_o)h \ .$$

Suppose $\mu_o[A'_{\lambda_o}(u_o)] < \infty$. If equality holds in (7.6), then $1 = \mu_o[A'_{\lambda_o}(u_o)]$ by the Krein-Rutman theorem. If inequality holds, then by (PA) $1 < \mu_o[A'_{\lambda_o}(u_o)]$. Hence, we always have $1 \leq \mu_o[A'_{\lambda_o}(u_o)]$.

If $1 < \mu_o[A'_{\lambda_o}(u_o)]$, then 1 is not a characteristic value of $A'_{\lambda_o}(u_o)$ and $\beta_o B_{\lambda_o} u_o > 0$. Hence, we may write equation (7.4) as

(7.7) $$\frac{h}{\beta_o} = A'_{\lambda_o}(u_o)\frac{h}{\beta_o} + B_{\lambda_o} u_o \ .$$

That is, any sequence $\{\lambda_n\}$ increasing to λ_o contains a subsequence $\{\lambda_{n_k}\}$ such that the limit in equation (7.4) and the limit of the sequence $\{\beta_{n_k}\}$, equation (7.3), exist, and therefore

$$\lim_{k\to\infty} \frac{u_o - u_{n_k}}{\lambda_o - \lambda_{n_k}}$$

also exists and equals the unique solution $\psi(\lambda_o) = h/\beta_o$ of (7.5) or (7.7). Thus,

$$\lim_{\lambda \uparrow \lambda_o} \frac{u_o - u^o(\lambda)}{\lambda_o - \lambda} = \psi(\lambda_o)$$

as asserted.

If $1 = \mu_o[A'_{\lambda_o}(u_o)]$, then it follows from equation (7.4) and the fact that $A'_{\lambda_o}(u_o)$ satisfies (PA) that $\beta_o B_{\lambda_o} u_o = 0$. Thus, any sequence converging to λ_o contains a subsequence $\{\lambda_{n_k}\}$ such that the limit in (7.4) exists and is a positive eigenvector of $A'_{\lambda_o}(u_o)$ of unit norm. If it is assumed that such an eigenvector is unique, then $\lim_{\lambda\uparrow\lambda_o} [u_o - u^o(\lambda)]/\|u_o - u^o(\lambda)\|$ exists and is this eigenvector, with characteristic value 1 . //

If $1 = \mu_o[A'_{\lambda_o}(u_o)]$ and 1 is a simple eigenvalue of $A'_{\lambda_o}(u_o)$, then, since $A'_{\lambda_o}(u_o)$ is a compact operator and the simple eigenvalue 1 corresponds to a positive eigenvector ϕ, every $u \in \mathcal{B}$ may be represented in the form

$$u = Pu + \xi(u)\phi \ , \tag{7.8}$$

where P is a continuous projection operator onto the invariant subspace $P\mathcal{B}$ such that $I-A'_{\lambda_o}(u_o)$ restricted to $P\mathcal{B}$ has an inverse on $P\mathcal{B}$, and ξ is a positive continuous linear functional with $\xi(\phi) = 1$ and $\xi(Pu) = 0$ for all $u \in \mathcal{B}$ (Dieudonné 1960, Krasnosel'skii 1964b, Nirenberg 1961).

7-2. Corollary. Let the hypotheses of Theorem 7-1 hold with $1 = \mu_o[A'_{\lambda_o}(u_o)]$, and let 1 be a simple eigenvalue of $A'_{\lambda_o}(u_o)$ corresponding to the positive eigenvector ϕ of unit norm. Let ξ and P be the linear operators defined above. Then

$$\lim_{\lambda \uparrow \lambda_o} \frac{\xi[u_o - u^o(\lambda)]}{\|u_o - u^o(\lambda)\|} = \xi(\phi) = 1 \tag{7.9}$$

and

$$\lim_{\lambda \uparrow \lambda_o} \frac{P[u_o - u^o(\lambda)]}{\|u_o - u^o(\lambda)\|} = 0 \ . \tag{7.10}$$

Proof. Theorem 7-1 shows that

$$\lim_{\lambda \uparrow \lambda_o} \frac{u_o - u^o(\lambda)}{\|u_o - u^o(\lambda)\|} = \phi \ . \tag{7.11}$$

The Corollary follows from the continuity of ξ and P . //

We consider in more detail the case $1 = \mu_o[A'_{\lambda_o}(u^o(\lambda_o))]$, assuming this is a simple eigenvalue and that A_{λ_o} is twice Fréchet differentiable at u_o ; i.e., we have in place of equation (7.1),

(7.12) $$A_\lambda u = A_{\lambda_o} u_o + (\lambda - \lambda_o) B_{\lambda_o} u_o + A'_{\lambda_o}(u_o)(u-u_o) + \tfrac{1}{2} A''_{\lambda_o}(u_o)(u-u_o)^2 + \omega(\lambda-\lambda_o; u-u_o) ,$$

where ω satisfies

(7.13) $$\lim_{|\delta|+\|h\|\to 0} \frac{\omega(\delta, h)}{|\delta|+\|h\|^2} = 0 .$$

By replacing u by $u^o(\lambda)$ in (7.12) and using

(7.14) $$u^o(\lambda)-u_o = \xi(u^o(\lambda)-u_o)\phi + P(u^o(\lambda)-u_o)$$

and

$$\xi[A'_{\lambda_o}(u_o)(u^o(\lambda)-u_o)] = \xi[u^o(\lambda)-u_o] ,$$

we obtain

(7.15) $$0 = (\lambda-\lambda_o)\xi[B_{\lambda_o} u_o] + \tfrac{1}{2}(\xi[u^o(\lambda)-u_o])^2 \xi[A''_{\lambda_o}(u_o)\phi^2]$$
$$+ \xi[A''_{\lambda_o}(u_o)\{\xi(u^o(\lambda)-u_o)\phi P(u^o(\lambda)-u_o) + \tfrac{1}{2}(P(u^o(\lambda)-u_o))^2\}]$$
$$+ \xi[\omega(\lambda-\lambda_o, u^o(\lambda)-u_o)] .$$

If we divide this equation by $\|u_o - u^o(\lambda)\|^2$, let λ approach λ_o from the left, and use Corollary 7-2 and equation (7.13), we obtain

$$0 \le \lim_{\lambda\uparrow\lambda_o} \frac{\lambda_o - \lambda}{\|u_o-u^o(\lambda)\|^2} = \frac{1}{2} \frac{\xi[A''_{\lambda_o}(u_o)\phi^2]}{\xi[B_{\lambda_o} u_o]}$$

if $\xi[B_{\lambda_o} u_o] \neq 0$. If $\xi[A''_{\lambda_o}(u_o)\phi^2] \neq 0$, it follows from equations (7.9) and (7.11) that

(7.16) $$\lim_{\lambda\uparrow\lambda_o} \frac{\xi[u_o - u^o(\lambda)]}{\sqrt{\lambda_o - \lambda}} = \left\{ 2 \frac{\xi[B_{\lambda_o} u_o]}{\xi[A''_{\lambda_o}(u_o)\phi^2]} \right\}^{\frac{1}{2}}$$

and

$$(7.17) \qquad \lim_{\lambda \uparrow \lambda_o} \frac{u_o - u^o(\lambda)}{\sqrt{\lambda_o - \lambda}} = \left\{ 2 \frac{\xi[B_{\lambda_o} u_o]}{\xi[A''_{\lambda_o}(u_o)\phi^2]} \right\}^{\frac{1}{2}} \phi .$$

Since $\{A_\lambda\}$ is an increasing family, if $\xi[B_{\lambda_o} u_o] \neq 0$ then $\xi[B_{\lambda_o} u_o] > 0$, and therefore $\xi[A''_{\lambda_o}(u_o)\phi^2] \geq 0$.

From the preceding arguments we can deduce that the minimal fixed points $u^o(\lambda)$ are discontinuous in λ from the right at λ_o if $\xi[B_{\lambda_o} u_o]\, \xi[A''_{\lambda_o}(u_o)\phi^2] \neq 0$ (and therefore > 0). For if we suppose there are minimal fixed points for $\lambda > \lambda_o$ such that $\lim_{\lambda \downarrow \lambda_o} u^o(\lambda) = u_o$, then the proof of Theorem 7-1 for $1 = \mu_o[A'_{\lambda_o}(u_o)]$ can be carried through with appropriate changes of sign to conclude that

$$\lim_{\lambda \downarrow \lambda_o} \frac{u^o(\lambda) - u_o}{\|u^o(\lambda) - u_o\|} = \phi .$$

From equation (7. 15) we can obtain the relation analogous to (7. 16),

$$0 \geq \lim_{\lambda \downarrow \lambda_o} \frac{(\xi[u^o(\lambda) - u_o])^2}{\lambda_o - \lambda} = 2 \frac{\xi[B_{\lambda_o} u_o]}{\xi[A''_{\lambda_o}(u_o)\phi^2]} .$$

But we have assumed that the right hand side in this equality is positive. This contradiction shows that, if there are any minimal positive fixed points for $\lambda > \lambda_o$, then there is a positive number ε such that for any $\lambda > \lambda_o$ in Λ_A^r, there is a number $\delta \in (0, \lambda - \lambda_o)$ such that $\|u^o(\lambda_o + \delta) - u_o\| \geq \varepsilon$. Since

$$0 \leq u^o(\lambda_o + \delta) - u_o \leq u^o(\lambda) - u_o$$

by Theorem 4-9, we have

$$\|u^o(\lambda) - u_o\| \geq \varepsilon$$

for all $\lambda > \lambda_o$ in Λ_A^r. This result shows also that if $1 = \mu_o[A'_{\lambda_o}(u_o)]$

and the minimal positive fixed points $u^o(\lambda)$ are continuous (from the right as well as from the left) in λ at λ_o, then we must have $\xi[A''_{\lambda_o}(u_o)\phi^2] = 0$.

Equation (7.17) tells us the behavior of the minimal fixed points $u^o(\lambda)$ as λ approaches from the left a point λ_o such that $1 = \mu_o[A'_{\lambda_o}(u^o(\lambda_o))]$ and $\xi[A''_{\lambda_o}(u^o(\lambda_o))\phi^2] > 0$. Since 1 is an eigenvalue of the operator $A'_{\lambda_o}(u^o(\lambda_o))$ and since $\|u^o(\lambda_o)-u^o(\lambda)\| = O(\sqrt{\lambda_o-\lambda})$ as λ increases to λ_o, with $\xi[u^o(\lambda)-u^o(\lambda_o)] \le 0$, we expect a branch point at $(\lambda_o, u^o(\lambda_o))$ and seek a second fixed point $u(\lambda)$ of A_λ for $\lambda_o-\lambda$ small and positive such that $\|u^o(\lambda_o)-u(\lambda)\| = O(\sqrt{\lambda_o-\lambda})$ and $\xi[u(\lambda)-u^o(\lambda_o)] \ge 0$. Our method of obtaining this solution and proving its uniqueness is modeled after the work of Krasnosel'skii (1964a) on equations of the form $\lambda Au = u$, where $A0 = 0$ (cf. Nirenberg 1961, Ch. VII; Pimbley 1967; Cronin 1950). We prove the existence of positive fixed points of the operators A^o_λ, where

(7.18) $$A^o_\lambda h = A_\lambda[u^o(\lambda)+h] - A_\lambda u^o(\lambda).$$

The equation $A^o_\lambda h = h$ is equivalent to the two equations

(7.19) $$P[A^o_\lambda(Ph+\xi(h)\phi)]-Ph = 0$$

and

(7.20) $$\xi[A^o_\lambda(Ph+\xi(h)\phi)]-\xi h = 0.$$

Making use of the representation (7.12), equation (7.19) may be written as

(7.21) $$y = P\{A'_{\lambda_o}(u_o)y + A''_{\lambda_o}(u_o)[u^o(\lambda)-u_o]h + \tfrac{1}{2}A''_{\lambda_o}(u_o)h^2$$
$$+ \omega(\lambda-\lambda_o, u^o(\lambda)+h-u_o) - \omega(\lambda-\lambda_o, u^o(\lambda)-u_o\}$$

where $y = Ph$. The operator $I - A'_{\lambda_0}(u_0)$ restricted to the invariant subspace $P\mathcal{B}$ has an inverse, R_p, on $P\mathcal{B}$; we set $R = R_p P$. Then (7.21) is equivalent to

(7.22) $$y = T(\beta, \lambda)y \ ,$$

where $\beta = \xi(h)$ and

(7.23) $$T(\beta,\lambda)y = R\{A''_{\lambda_0}(u_0)[u^0(\lambda)-u_0][y+\beta\phi] + \tfrac{1}{2}A''_{\lambda_0}(u_0)(y+\beta\phi)^2 + \omega(\lambda-\lambda_0, u^0(\lambda)+h-u_0) - \omega(\lambda-\lambda_0, u^0(\lambda)-u_0\} \ .$$

We assume that the remainder ω satisfies the following Lipschitz condition: for any sufficiently small positive numbers δ, ρ, there is a positive number $q_\omega(\delta,\rho)$ such that

(7.24) $$\|\omega(\lambda-\lambda_0, h_1)-\omega(\lambda-\lambda_0, h_2)\| \le q_\omega(\delta,\rho)\|h_1-h_2\|$$

whenever $\|h_1\| \le \rho$, $\|h_2\| \le \rho$, and $|\lambda-\lambda_0| \le \delta$, where

(7.25) $$\lim_{\delta+\rho\to 0} \frac{q_\omega(\delta,\rho)}{\sqrt{\delta}+\rho} = 0 \ .$$

(If $A_\lambda = \lambda A$, then equations (7.12), (7.13), (7.24), and (7.25) are satisfied if

(7.26) $$Au = Au_0 + A'(u_0)(u-u_0) + \tfrac{1}{2}A''(u_0)(u-u_0)^2 + \widetilde{\omega}(u-u_0) \ ,$$

where

(7.27) $$\lim_{\|h\|\to 0} \frac{\widetilde{\omega}(h)}{\|h\|^2} = 0 \ ,$$

(7.28) $$\|\widetilde{\omega}(h_1) - \widetilde{\omega}(h_2)\| \le \widetilde{q}_\omega(r)\|h_1-h_2\|$$

for $\|h_1\| \le r$, $\|h_2\| \le r$, and

(7.29) $$\lim_{r\to 0} \frac{\widetilde{q}_\omega(r)}{r} = 0 \ .)$$

The expression $\omega(\lambda-\lambda_0, h)$ is indeed well defined by equation (7.12) for all h of sufficiently small norm, since the differentiability of the mapping $(\lambda, u) \to A_\lambda u$ at (λ_o, u_o) means that $A_\lambda u$ is defined for all λ sufficiently close to λ_o and all u in a neighborhood of u_o.

The implicit function theorem as given by Nirenberg (1961) then implies that (7.21) has a solution $y_{\beta,\lambda}$ which is unique in some sufficiently small ball about the origin, whenever $|\beta|$ and $|\lambda-\lambda_o|$ are sufficiently small. We shall give a constructive proof of this fact by showing that $T(\beta,\lambda)$ is a contraction mapping on a sufficiently small ball, if $|\beta|$ and $|\lambda-\lambda_o|$ are small, since we will later wish to use some of the estimates obtained in the course of the proof to prove the uniqueness of the solution $h(\lambda)$ of $A_\lambda^o h = h$.

From equations (7.13), (7.24), and (7.25), it is clear that it is possible to choose positive numbers δ and ρ such that $0 \le \lambda_o - \lambda \le \delta$, $\|y\| \le \rho$, and $|\beta| \le \rho$ imply $\|T(\beta,\lambda)y\| \le \rho$. Similarly, since

$$T(\beta,\lambda)y_1 - T(\beta,\lambda)y_2$$

$$= R\{A''_{\lambda_o}(u_o)[u^o(\lambda)-u_o][y_1-y_2] + \tfrac{1}{2}A''_{\lambda_o}(u_o)(y_1^2-y_2^2)$$

$$+ \beta A''_{\lambda_o}(u_o)\phi(y_1-y_2)$$

$$+ \omega(\lambda-\lambda_o, u^o(\lambda)+y_1+\beta\phi-u_o) - \omega(\lambda-\lambda_o, u^o(\lambda)+y_2+\beta\phi-u_o)\},$$

equations (7.24) and (7.25) show that we may assume δ and ρ have been chosen so that

$$\|T(\beta,\lambda)y_1 - T(\beta,\lambda)y_2\| \le q(\delta,\rho)\|y_1-y_2\|, \tag{7.30}$$

if $|\beta| \le \rho$, $\|y_1\| \le \rho$, $\|y_2\| \le \rho$, $0 \le \lambda_o - \lambda \le \delta$, where $0 < q(\delta,\rho) < 1$ and $\lim_{\delta+\rho\to 0} q(\delta,\rho) = 0$. Thus, for each such β, λ, $T(\beta,\lambda)$ is a contraction mapping of $\overline{\mathbb{B}}^\rho = \{y : \|y\| \le \rho\}$ into itself and therefore has a fixed point $y_{\beta,\lambda}$; this fixed point depends continuously on β, λ for $|\beta| \le \rho$, $0 \le \lambda_o - \lambda \le \delta$, since the constant $q(\delta, \rho)$ in (7.30) is independent of β and λ and $T(\beta,\lambda)y$ is a continuous function of β and λ for any $y \in \overline{\mathbb{B}}^\rho$.

For later use, we note the following inequalities. From the triangle inequality,

$$\| T(\beta,\lambda)y\| \le \| T(\beta,\lambda)0\| + \| T(\beta,\lambda)y - T(\beta,\lambda)0\| ,$$

and therefore equations (7.23) and (7.30) imply that for any fixed point $y_{\beta,\lambda} = T(\beta,\lambda)y_{\beta,\lambda}$,

(7.31) $$\| y_{\beta,\lambda}\| \le |\beta|(\lambda_o - \lambda)^{\frac{1}{2}}\theta_1(\delta,\rho) + \beta^2\theta_2(\delta,\rho) ,$$

if $|\beta| < \rho$ and $0 \le \lambda_o - \lambda \le \delta$, where $\theta_1(\delta,\rho)$ and $\theta_2(\delta,\rho)$ are positive bounded functions of (δ,ρ) for δ and ρ sufficiently small.

If ρ and δ have been chosen small enough, then $y_{\beta,\lambda}$ is Lipschitz continuous in β, since

$$\begin{aligned} y_{\beta_1,\lambda} - y_{\beta_2,\lambda} &= T(\beta_1,\lambda)y_{\beta_1,\lambda} - T(\beta_1,\lambda)y_{\beta_2,\lambda} \\ &\quad + T(\beta_2,\lambda)[y_{\beta_2,\lambda} + (\beta_1 - \beta_2)\phi] - T(\beta_2,\lambda)y_{\beta_2,\lambda} , \end{aligned}$$

so

$$\| y_{\beta_1,\lambda} - y_{\beta_2,\lambda}\| \le q(\rho,\delta)\| y_{\beta_1,\lambda} - y_{\beta_2,\lambda}\| + q(3\rho,\delta)|\beta_1 - \beta_2| .$$

Thus,

(7.32) $$\| y_{\beta_1,\lambda} - y_{\beta_2,\lambda}\| \le \frac{q(3\rho,\delta)}{1 - q(\rho,\delta)} |\beta_1 - \beta_2| .$$

Having established the existence of a solution $y_{\beta,\lambda}$ of equation (7.22) for $|\beta| \le \rho$, $0 \le \lambda_o - \lambda \le \delta$, we seek $\beta \in (0,\rho]$ so that

(7.33) $$\xi[A_\lambda^o(y_{\beta,\lambda} + \beta\phi)] - \beta = 0$$

for $\lambda \in (0, \delta]$ (see equation 7.20). According to equation (7.12), equation (7.33) is equivalent to

(7.34) $$\xi\{A''_{\lambda_o}(u_o)[u^o(\lambda)-u_o][y_{\beta,\lambda}+\beta\phi] + \tfrac{1}{2}A''_{\lambda_o}(u_o)[y_{\beta,\lambda}+\beta\phi]^2\}$$
$$+\xi\{\omega(\lambda-\lambda_o, u^o(\lambda)+y_{\beta,\lambda}+\beta\phi-u_o)-\omega(\lambda-\lambda_o, u^o(\lambda)-u_o)\}$$
$$= 0 .$$

Using arguments similar to those used above, and assuming that $\xi[B_{\lambda_o}u_o]\xi[A''_{\lambda_o}(u_o)\phi^2] > 0$, we write equation (7.34) in the form

(7.35) $$\tfrac{1}{2}\alpha[1+f_1(\alpha,\lambda_o-\lambda)] = \xi_o[1+f_2(\alpha,\lambda_o-\lambda)] , \quad \beta = \alpha\sqrt{\lambda_o-\lambda} ,$$

where the continuous functions f_i , $i = 1, 2$, satisfy

$$|f_i(\alpha,\lambda_o-\lambda)| \le g_i(\alpha,\delta)$$

if $0 \le \lambda_o-\lambda \le \delta$, and for any $\alpha_o > 0$,

$$\lim_{\delta\downarrow 0} g_i(\alpha,\delta) = 0$$

uniformly for $\alpha \in [0, \alpha_o]$, and

$$\xi_o = \lim_{\lambda\uparrow\lambda_o} \frac{\xi[u_o-u^o(\lambda)]}{\sqrt{\lambda_o-\lambda}} = \left\{2\,\frac{\xi[B_{\lambda_o}u_o]}{\xi[A''_{\lambda_o}(u_o)\phi^2]}\right\}^{\frac{1}{2}} .$$

Choose $\alpha_o = 6\xi_o$ and δ so small that $0 < \lambda_o-\lambda \le \delta \le \rho^2/\alpha_o^2$ and $0 \le \alpha \le \alpha_o$ imply $g_i(\alpha,\delta) \le \frac{1}{2}$, $i = 1, 2$. For any such λ and δ , $\xi_o \le 2\xi_o[1+f_2(\alpha_1,\lambda_o-\lambda)] \le 3\xi_o$; since $\alpha_o[1+f_1(\alpha_o,\lambda_o-\lambda)] \ge \frac{1}{2}\alpha_o = 3\xi_o$, the function $\alpha[1+f_1(\alpha,\lambda_o-\lambda)]$ assumes all values between 0 and $3\xi_o$ as α varies between 0 and α_o , for each $\lambda \in [\lambda_o-\delta, \lambda)$. Thus, for each such λ , equation (7.35) has a solution $\alpha(\lambda)$, $0 \le \alpha(\lambda) \le \alpha_o$.

It follows that $\beta(\lambda) = \alpha(\lambda)\sqrt{\lambda_o - \lambda} = \alpha_o\sqrt{\delta} \le \rho$, and therefore $h(\lambda) = y_{\beta(\lambda),\lambda} + \beta(\lambda)\phi$ provides a non-zero solution of $A^o_\lambda h = h$ for each λ, $0 < \lambda_o - \lambda \le \delta$; thus, $u(\lambda) = u^o(\lambda) + h(\lambda)$ provides a second solution of $A_\lambda u = u$. Moreover, since it is possible to make $f_1(\alpha, \lambda_o - \lambda)$ and $f_2(\alpha, \lambda_o - \lambda)$ as small as desired for $0 \le \alpha \le \alpha_o$, $0 \le \lambda_o - \lambda \le \delta$, by taking δ sufficiently small, it is clear from (7.35) that it is possible to choose the solutions $\alpha(\lambda)$ of (7.35) so that $\lim_{\lambda \uparrow \lambda_o} \alpha(\lambda) = 2\xi_o$.

From equation (7.31),

$$\lim_{\lambda \uparrow \lambda_o} y_{\beta(\lambda),\lambda} / \sqrt{\lambda_o - \lambda} = 0 ,$$

and therefore

$$\begin{aligned} u(\lambda) - u_o &= h(\lambda) + u^o(\lambda) - u_o \\ &= \xi_o \sqrt{\lambda_o - \lambda}\, \phi + z(\lambda) \end{aligned}$$

where $\|z(\lambda)\| = o(\sqrt{\lambda_o - \lambda})$.

In order to be assured that the solutions $h(\lambda)$ are positive (which they must be if $u^o(\lambda) + h(\lambda)$ is to be positive) for $\lambda_o - \lambda$ sufficiently small, we assume that the condition (SP) of Section I. 6 is satisfied by the operators $\{A_\lambda\}$ at $(\lambda_o, u^o(\lambda_o))$. Then the fixed points $u(\lambda) = A_\lambda u(\lambda)$ are in C^r for $0 < \lambda_o - \lambda \le \delta$ if δ is sufficiently small.

We summarize the results of the construction of the fixed points $u(\lambda) = u^o(\lambda) + h(\lambda)$ in the following theorem.

7-3. Theorem. Let $\{A_\lambda\}$, $\lambda \in J$, be a strictly increasing family of forced operators which satisfy (H) on C^r for each $\lambda \in J$. For $\lambda_o \in \Lambda^r_A$, $\lambda_o > \inf \Lambda^r_A$, let the mapping $(\lambda, u) \to A_\lambda u$ be differentiable at $(\lambda_o, u^o(\lambda_o))$, the mapping $\lambda \to A_\lambda u$ be continuous at λ_o uniformly for u in $[0, u^o(\lambda_o)]$, and A_{λ_o} be twice Fréchet differentiable at $u^o(\lambda_o)$, so that equation (7.12) holds, with the remainder ω satisfying equation (7.13). Let $A'_{\lambda_o}(u^o(\lambda_o))$ be compact, satisfy (PA), and have a unique positive eigenvector ϕ of unit norm corresponding to the simple characteristic value $\mu_o[A'_{\lambda_o}(u^o(\lambda_o))]$, so that equation (7.8) holds. Assume that $\xi[B_{\lambda_o} u^o(\lambda_o)] > 0$. If $\mu_o[A'_{\lambda_o}(u^o(\lambda_o))] = 1$, then $\xi[A''_{\lambda_o}(u^o(\lambda_o))\phi^2] \geq 0$; if $\xi[A''_{\lambda_o}(u^o(\lambda_o))\phi^2] > 0$, then the minimal positive fixed points $u^o(\lambda_o)$ satisfy

$$\lim_{\lambda \uparrow \lambda_o} \frac{u^o(\lambda_o)-u^o(\lambda)}{\sqrt{\lambda_o-\lambda}} = \left\{ 2\, \frac{\xi[B_{\lambda_o} u^o(\lambda_o]}{\xi[A''_{\lambda_o}(u^o(\lambda_o))\phi^2]} \right\}^{\frac{1}{2}} \phi , \tag{7.17}$$

and there is a positive number $\varepsilon > 0$ such that for $\lambda_o < \lambda \in J$, any positive fixed point $u(\lambda)$ of A_λ satisfies

$$\|u(\lambda)-u_o\| \geq \varepsilon .$$

If, in addition, the remainder ω in equation (7.12) satisfies the Lipschitz condition of equations (7.24) and (7.25), then for each $\lambda < \lambda_o$ with $\lambda_o-\lambda$ sufficiently small, the operator A_λ has a second fixed point $u(\lambda)$, which is positive if $\{A_\lambda\}$ satisfies the positivity condition (SP) at $(\lambda_o, u^o(\lambda_o))$, and

$$\lim_{\lambda \uparrow \lambda_o} \frac{u(\lambda)-u^o(\lambda_o)}{\sqrt{\lambda_o-\lambda}} = \left\{ 2\, \frac{\xi[B_{\lambda_o} u^o(\lambda_o)]}{\xi_o[A''_{\lambda_o}(u^o(\lambda_o))\phi^2]} \right\}^{\frac{1}{2}} \phi . \tag{7.36}$$

We now investigate the uniqueness of the fixed point $h(\lambda)$ of A_{λ_o} which we have found above. Let ρ and δ be small positive numbers chosen as above, and let

(7.37) $$\sigma = \min\left\{ \frac{1}{\|P\|}\rho , \frac{1}{\|\xi\|}\rho \right\} .$$

If A^o_λ has a fixed point h with $\|h\| \le \sigma$ for some $\lambda \in [\lambda_o - \delta, \lambda_o]$, then $\|Ph\| \le \rho$ and $|\xi(h)| \le \rho$, so $Ph = y_{\xi(h),\lambda}$, the unique fixed point of $T(\xi(h),\lambda)$ in the ball $\overline{\mathbb{B}}^\rho$ (equation (7.22).

We first show that $u^o(\lambda_o)$ is an isolated fixed point of A_{λ_o}. Any fixed point h_o of $A^o_{\lambda_o}$ in the ball $\overline{\mathbb{B}}^\sigma$ satisfies equation (7.31) with $\beta = \xi(h_o)$, $\lambda = \lambda_o$, and $y_{\beta,\lambda_o} = y_{\xi(h_o),\lambda_o} = Ph_o$. Thus, $h_o = 0$ if and only if $\beta = \xi(h_o) = 0$. If equation (7.34) had a solution $\beta \ne 0$ for $\lambda = \lambda_o$ with $|\beta| \le \rho$, we may divide through by $\frac{1}{2}\beta^2$ to obtain

(7.38) $$\xi[A''_{\lambda_o}(u_o\phi^2]$$

$$= -\xi[A''_{\lambda_o}(u_o)\{2\phi(y_{\beta,\lambda_o}/\beta) + (y_{\beta,\lambda_o}/\beta)^2\}]$$

$$-2\beta\xi[\frac{1}{\beta^2}\omega(0, y_{\beta,\lambda_o} + \beta\phi)] .$$

But according to equation (7.31), $\|y_{\beta,\lambda_o}\| \le \beta^2\theta_2(\delta,\rho)$, so by choosing ρ sufficiently small, the right hand side of equation (7.38) could be made less than the left hand side $\xi[A''_{\lambda_o}(u_o)\phi^2]$ (which is assumed positive). Thus, for all sufficiently small $\rho > 0$, the only solution β of (7.33) or (7.34) for $\lambda = \lambda_o$ with $|\beta| \le \rho$ is $\beta = 0$, and therefore 0 is the only fixed point of $A^o_{\lambda_o}$ in $\overline{\mathbb{B}}^\sigma$, with σ defined by equation (7.37) for sufficiently small ρ. Thus, $u^o(\lambda_o) = u_o$ is an isolated fixed point A_{λ_o}.

Consider next the operator A^o_λ for fixed $\lambda \in [\lambda_o - \delta, \lambda_o)$. If A^o_λ has two fixed points h_1, h_2 in $\overline{\mathbb{B}}^\sigma$, then from equation (7.32) we have

$$\|Ph_1 - Ph_2\| \le \frac{q(3\rho, \delta)}{1-q(\rho, \delta)} \, |\xi(h_1) - \xi(h_2)| \quad . \tag{7.39}$$

We prove that A_λ^o has at most one non-zero fixed point in $\overline{\mathfrak{B}}^\sigma$ for sufficiently small ρ by using this inequality and the following lemma (Krasnosel'skii 1964a, p. 207).

7-4. Lemma. Let A be a positive monotone operator defined on a subset S of a Banach space $\mathfrak{B}$ with a positive cone $\mathfrak{C}$. Let either of the following two conditions be satisfied:

(a) for every number $t \in [0, 1]$, $tS \subseteq S$, and for any $u \in S$ and any number $t \in (0, 1)$, there is a positive number η (which may depend on u and t) such that

$$A(tu) \le (1-\eta)tAu \quad ;$$

(b) for every number $t \ge 1$, $tS \subseteq S$, and for any $u \in S$ and any number $t > 1$, there is a positive number η (which may depend on u and t) such that

$$A(tu) \ge (1+\eta)tAu \quad .$$

If A has two non-zero fixed points u_1, u_2 in S, then neither $u_1 - u_2$ nor $u_2 - u_1$ are interior points of $\mathfrak{C}$.

(It is possible to strengthen the hypotheses of the lemma slightly and conclude that neither $u_1 - u_2$ nor $u_2 - u_1$ are positive; see Krasnosel'skii (1964a) for details.)

Thus, to prove the uniqueness of the non-zero fixed point of A_λ^o, we assume the existence of two fixed points h_1, h_2 and attempt to show that either $h_1 - h_2$ or $h_2 - h_1$ must be an interior element of the cone $\mathfrak{C}$; if A_λ^o satisfies the condition (a), this would contradict the lemma. The principal difficulty here is proving that the difference of two fixed points of A_λ^o is either positive or negative. It is possible, however, to introduce new cones relative to which this is trivial; one then must show that A_λ^o satisfies the hypotheses of Lemma 7-4 relative to at least one of these cones. The Lipschitz condition (7.39) suggests the introduction of the cones $\mathcal{C}(\phi, \gamma)$ defined by

$$(7.40) \qquad \mathcal{C}(\phi, \gamma) = \{u: \|Pu\| \le \gamma\xi(u)\} \qquad (\gamma > 0).$$

(If $\mathfrak{B}$ is a Hilbert space with inner product $\langle\ ,\ \rangle$ and if $\xi(u) = \langle\phi, u\rangle$, then $\mathcal{C}(\phi, \gamma)$ may be interpreted as a cone having its axis in the direction of ϕ and consisting of all $u \in \mathfrak{B}$ such that $0 \le \tan(u, \phi) \le \gamma$.)

We note the following properties of the cones $\mathcal{C}(\phi, \gamma)$. Clearly, $\mathcal{C}(\phi, \gamma_1) \subseteq \mathcal{C}(\phi, \gamma_2)$ if $0 < \gamma_1 < \gamma_2$. Also, if $\|Pu\| < \gamma\xi(u)$, then u is an interior element of $\mathcal{C}(\phi, \gamma)$, since $\|Pu\| < \gamma\xi(u)$ and

$$\|h\| < \frac{1}{2}\,\frac{\gamma\xi(u) - \|Pu\|}{(1+\gamma)(\|P\| + \|\xi\|)}$$

imply $u+h \in \mathcal{C}(\phi, \gamma)$.

However, we cannot expect A_λ^o to be monotonic relative to any of these new cones without further modifications. To see this, we observe that for small $(\lambda_o - \lambda)$ and for elements h of small norm, we have $A_\lambda^o h \simeq A'_{\lambda_o}(u_o)h$. But $u_2 - u_1 \in \mathcal{C}(\phi, \gamma)$ implies $\|A'_{\lambda_o}(u_o)P(u_2-u_1)\| \le \gamma\|A'_{\lambda_o}(u_o)\|_P\,\xi(u_2-u_1)$, where $\|A'_{\lambda_o}(u_o)\|_P = \sup\{\|A'_{\lambda_o}(u_o)Pu\| : \|Pu\| = 1\}$;

hence, $A'_{\lambda_o}(u_o)(u_2-u_1) \in \mathcal{C}(\phi, \gamma)$ (and therefore $A'_{\lambda_o}(u_o)$ is monotone) if $\|A'_{\lambda_o}(u_o)\|_P \le 1$. This will not be true in general; however, it is possible to introduce a new norm, equivalent to the old one, for which this inequality holds, and then we can expect $A'_{\lambda_o}(u_o)$ and A^o_λ to be monotone on elements of sufficiently small norm relative to the cones $\mathcal{C}(\phi, \gamma)$ defined by using the new norm.

To define the new norm, we assume that 1 is larger in absolute value than all other (real or complex) eigenvalues of the compact operator $A'_{\lambda_o}(u_o)$ (since the corresponding eigenvector ϕ is positive, this will be true if $A'_{\lambda_o}(u_o)$ is g_o-bounded for some $g_o \in \mathcal{C}^+$ by Theorem 1-1). Since this eigenvalue has been assumed to be simple, the largest eigenvalue of $A'_{\lambda_o}(u_o)$ restricted to the invariant subspace $P\mathcal{B}$ is less than 1. Then it is possible to introduce a new norm $\| \ \|_o$, equivalent to the old one, such that

$$\|A'_{\lambda_o}(u_o)Pu\|_o \le \varkappa_o \|Pu\|_o , \quad \varkappa_o < 1 ,$$

$$\|u\|_o = \|Pu\|_o + |\xi(u)| ,$$

and

$$\|Pu\| \le \|Pu\|_o \le m_o \|Pu\|$$

for some positive number m_o (Krasnosel'skii 1964a, pp. 88 and 217). We denote the cones defined by equation (7.40) using this new norm $\| \ \|_o$ by $\mathcal{C}_o(\phi, \gamma)$. Note that $\mathcal{C}_o(\phi, \gamma) \subseteq \mathcal{C}(\phi, \gamma) \subseteq \mathcal{C}_o(\phi, m_o\gamma)$.

Now Lemma 7-4 may be applied to the operators A^o_λ for an appropriate choice of the cone $\mathcal{C}_o(\phi, \gamma)$. To show this, we use the following two assertions.

(I) For any positive numbers γ_1, γ_3, there are positive numbers r_o, ε such that for all $\gamma \in [\gamma_1, \gamma_3]$ and all $\lambda \in [\lambda_o - \varepsilon, \lambda_o]$, $A^o_\lambda h_1 - A^o_\lambda h_2 \in \mathcal{C}_o(\phi, \gamma)$ whenever $h_1 - h_2 \in \mathcal{C}_o(\phi, \gamma)$ and $\|h_1\| \le r_o$, $\|h_2\| \le r_o$.

(II) (We are assuming that $\xi(A''(u_o)\phi^2) > 0$.) There are positive numbers γ_1, $\gamma_2 \geq \gamma_1$, r_o, and ε such that for $\gamma \geq \gamma_2$, $h \in \mathcal{C}(\phi, \gamma_1)$, $\|h\| \leq r_o$, $\lambda \in [\lambda_o - \varepsilon, \lambda_o]$, and $t \in (0, 1)$, there exists a positive number η such that $(1-\eta)tA_\lambda^o u - A_\lambda^o(tu) \in \mathcal{C}_o(\phi, \gamma)$.

The proofs of these assertions are slight modifications of the proofs of Krasnosel'skii's Lemmas 6.6 and 6.8 (1964a); we omit the simple verification.

Choose γ_1, γ_2, r_o, and ε as in (II) with $r_o \leq \sigma$ (equation 7.37) and $\varepsilon \leq \delta$. Assume r_o and ε small enough so that for any fixed point h of A_λ^o with $\|h\| \leq r_o$, $0 \leq \lambda_o - \lambda \leq \varepsilon$, we have (see equation 7.31)

$$\|Ph\| \leq \frac{\gamma_1}{m_o}\xi(h),$$

so that

$$\|Ph\|_o \leq \gamma_1 \xi(h),$$

i.e., $h \in \mathcal{C}_o^{r_o}(\phi, \gamma_1) = \{u \in \mathcal{C}_o(\phi, \gamma_1) : \|u\| \leq r_o\}$. Choose

$$\gamma_3 > \max\left\{\gamma_2, \frac{q(3\rho, \delta)}{1-q(\rho, \delta)}\right\}$$

(see equation 7.39), and assume that r_o and ε have been taken small enough that the conclusion of (I) holds. Then for $0 \leq \lambda_o - \lambda \leq \varepsilon$, A_λ^o has at most one non-zero fixed point in the ball $\overline{\mathfrak{B}}^{r_o}$. For if A_λ^o had two non-zero fixed points h_1, h_2 in $\overline{\mathfrak{B}}^{r_o}$, then, as we have just seen, both h_1 and h_2 would be in $\mathcal{C}_o^{r_o}(\phi, \gamma_1)$. According to equation (7.39), if $h_1 - h_2 \neq 0$, either $h_1 - h_2$ or $h_2 - h_1$ would be an interior point of $\mathcal{C}_o(\phi, \gamma_3)$, since $[q(3\rho, \delta)]/[1-q(\rho, \delta)] < \gamma_3$. But we can apply Lemma 7-4 to A_λ^o, taking $S = \mathcal{C}_o^{r_o}(\phi, \gamma_1)$ and using $\mathcal{C}_o(\phi, \gamma_3)$ as the positive cone, for assertions (I) and (II) show that A_λ^o satisfies condition (a) of

Lemma 7-4 on $\mathcal{C}_o^{r_o}(\phi, \gamma_1)$ relative to the ordering generated by $\mathcal{C}_o(\phi, \gamma_3)$. Thus, neither $h_1 - h_2$ nor $h_2 - h_1$ can be interior to $\mathcal{C}_o(\phi, \gamma_3)$, so $h_1 = h_2$ and A_λ^o has only one non-zero fixed point in $\overline{\mathfrak{B}}^{r_o}$. We have, therefore, the following uniqueness result:

7-5. Theorem. Let the hypotheses of Theorem 7-3 be satisfied, and let 1 be greater in absolute value than all other eigenvalues of $A'_{\lambda_o}(u_o)$. Then there is a ball $\overline{\mathfrak{B}}^\rho$ centered at $u^o(\lambda_o)$ and a number $\varepsilon > 0$ such that $u^o(\lambda_o)$ is the only fixed point of A_{λ_o} in $\overline{\mathfrak{B}}^\rho$, and for each $\lambda \in (\lambda_o - \varepsilon, \lambda_o)$, A_λ has exactly one fixed point $u(\lambda) > u^o(\lambda)$ in $\overline{\mathfrak{B}}^\rho$.

The preceding existence and uniqueness proofs do not depend in any essential way on the positivity or monotonicity of the operators A_λ. We shall briefly describe the modifications necessary to prove results corresponding to Theorems 7-1, 7-3, and 7-5 for more general operators, no longer assuming that the given set of fixed points are minimal positive fixed points. We return to equation (7. 4), assuming that $\{A_\lambda\}$, $\lambda \in J$, is an increasing family of continuous operators defined on an open subset S of $\mathfrak{B}$ and having a set of fixed points $\{u(\lambda)\}$ depending continuously on λ for λ in a closed interval J_1, with $\lambda_o \in J_1$, either $\lambda_o = \max J_1$ or $\lambda_o = \min J_1$, and $u(\lambda) \neq u(\lambda_o)$ for $\lambda_o \neq \lambda \in J_1$. We denote by $\lim_{J_1}$ a limit taken as λ approaches λ_o in J_1. We assume that $A'_{\lambda_o}(u_o)$ is compact and has the simple eigenvalue 1 corresponding to an eigenvector ϕ of unit norm, so that equation (7. 8) holds for any $u \in \mathfrak{B}$, and we assume that from equation (7. 4) it follows that h is an eigenvector of $A'_{\lambda_o}(u_o)$ corresponding to the eigenvalue 1 (i.e., $\beta_o B_{\lambda_o} u_o = 0$).

Then any sequence in J_1 converging to λ_o contains a subsequence $\{\lambda_n\}$ such that $h_n \equiv [u_o - u(\lambda_n)]/\|u_o - u(\lambda_n)\|$ converges either to $+\phi$ or to $-\phi$; thus, $\xi(h_n)$ converges either to $+1$ or to -1 , and $|\xi(h_n)|$ converges to $+1$. Consequently, $|\xi[u_o - u(\lambda)]/\|u_o - u(\lambda)\||$ converges to $+1$ as λ approaches λ_o in J_1 , so we may assume J_1 has been chosen small enough that $|\xi[u_o - u(\lambda)]/\|u_o - u(\lambda)\||$ is bounded below by a positive number for all $\lambda \in J_1$. Any sequence converging to λ_o therefore contains a subsequence $\{\lambda_n\}$ such that $\xi[u_o - u(\lambda_n)]$ either is positive or is negative, but is never zero (at least for sufficiently large n); since $\xi[u_o - u(\lambda)]$ is a continuous real-valued function of λ on J_1 which is never zero, we clearly must have $\xi[u_o - u(\lambda_n)]$ positive for all sequences $\{\lambda_n\}$ in J_1 converging to λ_o, or $\xi[u_o - u(\lambda_n)]$ negative for all sequences $\{\lambda_n\}$ in J_1 converging to λ_o. Thus, either

$$\lim_{J_1} \frac{\xi[u_o - u(\lambda)]}{\|u_o - u(\lambda)\|} = +1$$

or

$$\lim_{J_1} \frac{\xi[u_o - u(\lambda)]}{\|u_o - u(\lambda)\|} = -1 ,$$

and either

$$\lim_{J_1} \frac{u_o - u(\lambda)}{\|u_o - u(\lambda)\|} = +\phi ,$$

or

$$\lim_{J_1} \frac{u_o - u(\lambda)}{\|u_o - u(\lambda)\|} = -\phi .$$

All the considerations of Theorems 7-1, 7-3, and 7-5, for the case $1 = \mu_o[A'_{\lambda_o}(u^o(\lambda_o))]$, can now be carried through in the present case. Thus, we have:

7-6. Theorem. Let $\{A_\lambda\}$, $\lambda \in J$, be a strictly increasing family of continuous operators defined on an open subset S of $\mathcal{B}$ and having a set of fixed points $\{u(\lambda)\}$ depending continuously on λ for $\lambda \in J_1$, with $\lambda_o = \max J_1$ or $\lambda_o = \min J_1$, and $u(\lambda) \neq u(\lambda_o) \equiv u_o$ for $\lambda_o \neq \lambda \in J_1$. Let the mapping $(\lambda, u) \to A_\lambda u$ be differentiable at (λ_o, u_o), and let $A'_{\lambda_o}(u_o)$ be a compact linear operator with the simple eigenvalue 1 corresponding to an eigenvector ϕ of unit norm; let there be no $u \in \mathcal{B}$ such that $u - A'_{\lambda_o}(u_o)u > 0$. Then

$$\lim_{J_1} \frac{u_o - u(\lambda)}{\|u_o - u(\lambda)\|} = \eta\phi ,$$

where η is either +1 or -1.

Let ξ be the linear functional of equation (7.8). If, in addition, equations (7.12), (7.13), (7.24), and (7.25) hold, and if $\xi[A''_{\lambda_o}(u_o)\phi^2] \neq 0$, then

$$(7.41) \qquad (\lambda_o - \lambda) \frac{\xi[B_{\lambda_o} u_o]}{\xi[A''_{\lambda_o}(u_o)\phi^2]} \geq 0 \qquad (\lambda \in J_1)$$

and

$$(7.42) \qquad \lim_{J_1} \frac{u_o - u(\lambda)}{\sqrt{|\lambda_o - \lambda|}} = \left\{ 2 \frac{\lambda_o - \lambda}{|\lambda_o - \lambda|} \frac{\xi[B_{\lambda_o} u_o]}{\xi[A''_{\lambda_o}(u_o)\phi^2]} \right\}^{\frac{1}{2}} \eta\phi , \quad \lambda_o \neq \lambda \in J_1 .$$

If $\xi[B_{\lambda_o} u_o] \neq 0$, then for each λ in J_1 sufficiently close to λ_o, $\lambda \neq \lambda_o$, the operator A_λ has a second fixed point $u^{(2)}(\lambda) \neq u(\lambda)$ such that

$$(7.43) \qquad \lim_{J_1} \frac{u_o - u^{(2)}(\lambda)}{\sqrt{|\lambda_o - \lambda|}} = -\lim_{J_1} \frac{u_o - u(\lambda)}{\sqrt{|\lambda_o - \lambda|}} .$$

The fixed point $u_o = u(\lambda_o)$ is an isolated fixed point of A_{λ_o}, and if 1 is greater in absolute value than all other eigenvalues of $A'_{\lambda_o}(u_o)$, then there is a neighborhood $\mathcal{N}$ of $u(\lambda_o)$ and a number $\varepsilon > 0$ such that for each λ in J_1 with $0 < |\lambda_o - \lambda| < \varepsilon$, A_λ has exactly one fixed point different from $u(\lambda)$ in $\mathcal{N}$.

If the fixed points $u(\lambda)$ are positive and condition (SP) is satisfied at (λ_o, u_o), then the fixed points $u^{(2)}(\lambda)$ are positive for $|\lambda - \lambda_o|$ sufficiently small.

The preceding theorem can be applied in particular to the case of most interest to us, when the forced operators A_λ satisfy (H) on C^r and the fixed points $u(\lambda)$ are not necessarily minimal fixed points. If A_λ is Fréchet differentiable on C^r for $\lambda \in J$, and if the mapping $(\lambda, u) \to A'_\lambda(u)$ is continuous on $J \times C^r$, then the implicit function theorem and the fixed points $u^o(\lambda) + h(\lambda)$ of Theorem 7-3 can be used to construct a new branch of fixed points $u^{(1)}(\lambda)$ for decreasing λ, provided 1 is not in the spectrum of $A'_\lambda(u^{(1)}(\lambda))$ for any of the fixed points $u^{(1)}(\lambda)$ used in the construction; the fixed points will be positive if the family $\{A_\lambda\}$ satisfies the condition (SP) at all points of $J \times C^r$. If the mapping $(\lambda, u) \to A_\lambda u$ is compact, the construction can be continued until either $\lim_{\lambda \downarrow \lambda_1} \|u^{(1)}(\lambda)\| = r$ or 1 is in the spectrum of $A'_{\lambda_1}(u^{(1)}(\lambda_1))$ for some $\lambda_1 \in J$ (see Theorem 3-2). If at such a λ_1, $1 = \mu_o[A'_{\lambda_1}(u^{(1)}(\lambda_1))]$ and this is a simple eigenvalue of $A'_{\lambda_1}(u^{(1)}(\lambda_1))$, and if $A'_{\lambda_1}(u^{(1)}(\lambda_1))$ satisfies (PA_2) (Section I. 1), then Theorem 7-6 applies and the eigenvector ϕ and the linear functional ξ (equation

7.8) may be taken to be positive; if $\xi(B_{\lambda_1}(u^{(1)}(\lambda_1)) > 0$, then equation (7.41) of Theorem 7-6 shows that we must have $\xi[A''_{\lambda_1}(u^{(1)}(\lambda_1))\phi^2] \leq 0$. If $\xi[A''_{\lambda_1}(u^{(1)}(\lambda_1))\phi^2] < 0$, then there is a new branch $\{u^{(2)}(\lambda)\}$ of fixed points satisfying equation (7.43) for λ increasing from λ_1. (It follows from Theorem 10-15 below that for convex operators A_λ for which all derivatives $A'_\lambda(u)$ satisfy (PA_2), any value of λ such that $1 = \mu_o[A'_\lambda(u(\lambda))]$ is the maximum of Λ^r_A, and thus the situation just described cannot occur.)

We return to the consideration of the minimal positive fixed points $u^o(\lambda)$. So far, we have shown continuity from the left in λ (Theorem 4-10) and differentiability from the left in λ if $1 < \mu_o[A'_{\lambda_o}(u^o(\lambda))]$ (Theorem 7-1). Stronger results on the continuity and the differentiability may be obtained by using the implicit function theorem. We first prove the following:

<u>7-7. Lemma</u>. Let $\{A_\lambda\}$, $\lambda \in J$, be an increasing family of forced operators satisfying (H) on C^r. Let $\lambda_o \in \Lambda^r_A$ and suppose there exists a number $\delta > 0$ such that for each $\lambda \in (\lambda_o, \lambda_o+\delta)$ the operator A_λ has a fixed point $v(\lambda) \in C^r$ with $\lim_{\lambda \downarrow \lambda_o} v(\lambda) = u^o(\lambda_o)$. Let the mapping $\lambda \to A_\lambda u$ be continuous uniformly for u in $[0, u^o(\lambda_o)]$. Then $(\lambda_o, \lambda_o+\delta) \subseteq \Lambda^r_A$, and the mapping $\lambda \to u^o(\lambda)$ is continuous at λ_o.

<u>Proof.</u> It is clear that $(\lambda_o, \lambda_o+\delta) \subseteq \Lambda^r_A$, and it follows from Theorem 4-10 that the mapping $\lambda \to u^o(\lambda)$ is continuous from the left at λ_o (if $\lambda_o > \inf \Lambda^r_A$). The continuity from the right follows immediately by letting λ decrease to λ_o in the inequalities

$$u^o(\lambda_o) \leq u^o(\lambda) \leq v(\lambda) . \qquad //$$

<u>7-8. Theorem.</u> Let $\{A_\lambda\}$, $\lambda \in J$, be an increasing family of forced operators on a neighborhood of C^r $(0 < r \le \infty)$ which satisfy (H) on C^r. For $\lambda_o \in \Lambda_A^r \cap (\lambda^-, \lambda^+)$ (where $\lambda^- = \inf J$, $\lambda^+ = \sup J$), let the mapping $\lambda \to A_\lambda u$ be continuous uniformly for $u \in [0, u^o(\lambda_o)]$; for some neighborhood $N_1 \times \mathcal{N}_1$ of $(\lambda_o, u^o(\lambda_o))$, let A_λ be Fréchet differentiable on $\mathcal{N}_1$ for each $\lambda \in N_1$ and the mapping $(\lambda, u) \to A'_\lambda(u)$ be continuous on $N_1 \times \mathcal{N}_1$. Let the family $\{A_\lambda\}$ satisfy condition (SP) at $(\lambda_o, u^o(\lambda_o))$. If $1 \ne \mu_o[A'_{\lambda_o}(u^o(\lambda_o))]$, then for any sufficiently small neighborhood $\mathcal{N}$ of $u^o(\lambda_o)$ there is a neighborhood N of λ_o such that $N \subseteq \Lambda_A^r$, and for each $\lambda \in N$, $u^o(\lambda) \in \mathcal{N}$. The mapping $\lambda \to u^o(\lambda)$ is continuous on N. If, in addition, the mapping $(\lambda, u) \to A_\lambda u$ is continuously Fréchet differentiable on $N_1 \times \mathcal{N}_1$, then the neighborhood N of λ_o may be chosen so that the mapping $\lambda \to u^o(\lambda)$ is continuously differentiable on N, and the derivative at λ_o is

$$\psi(\lambda_o) = \lim_{\lambda \to \lambda_o} \frac{u^o(\lambda) - u^o(\lambda_o)}{\lambda - \lambda_o} = [I - A'_{\lambda_o}(u^o(\lambda_o))]^{-1} B_{\lambda_o} u^o(\lambda_o) ,$$

i.e., the solution of the equation

$$\psi - A'_{\lambda_o}(u^o(\lambda_o))\psi = B_{\psi_o} u^o(\lambda_o) ,$$

where

$$B_{\lambda_o} u^o(\lambda_o) = \lim_{\lambda \to \lambda_o} \frac{A_\lambda u^o(\lambda_o) - A_{\lambda_o} u^o(\lambda_o)}{\lambda - \lambda_o} .$$

<u>Proof.</u> Since $1 \ne \mu_o[A'_{\lambda_o}(u^o(\lambda_o))]$, it follows from Theorem 7-1 that $1 < \mu_o[A'_{\lambda_o}(u^o(\lambda_o))]$; thus, the operator $I - A'_{\lambda_o}(u^o(\lambda_o))$ has an inverse. The implicit function theorem 6-5 implies that, given any sufficiently small neighborhood $\mathcal{N}$ of $u^o(\lambda_o)$, there is a neighborhood

N of λ_o such that for each $\lambda \in N$ the equation $u - A_\lambda u = 0$ has a solution $u = v(\lambda)$ which is unique in $\mathcal{N}$ and continuous (or continuously differentiable under the additional hypothesis) with respect to λ. Condition (SP) guarantees that $v(\lambda) \in C$ for sufficiently small N. From Lemma 7-7, the minimal fixed points $u^o(\lambda)$ are continuous in λ at λ_o. Therefore, we can choose the neighborhood N of λ_o so small that $u^o(\lambda) = v(\lambda)$ for all $\lambda \in N$. The last equation of the Theorem follows from the implicit function theorem or by differentiating $u^o(\lambda) - A_\lambda u^o(\lambda) = 0$. //

If the operators A_λ have the form $A_\lambda = A_o + \lambda A$, then the differentiability requirements of the preceding theorem are satisfied if A_o and A are continuously differentiable in a neighborhood of $u^o(\lambda_o)$.

If it is known that the continuity and differentiability hypotheses (including the condition (SP)) of Theorem 7-8 are satisfied for all $\lambda \in \Lambda_A^r$ and $u^o(\lambda) \in C^r$, and if $\lambda_o \in \Lambda_A^r \cap (\lambda^-, \lambda^+)$ is such that $1 \neq \mu_o[A'_{\lambda_o}(u^o(\lambda_o))]$, then the preceding theorem shows that there is a neighborhood $N \subseteq J$ of λ_o such that $N \subseteq \Lambda_A^r$ and $u^o(\lambda)$ depends continuously on λ for $\lambda \in N$. From Theorem 4-11 we see that either $\sup N = \sup \Lambda_A^r = \lambda^+ \notin J$, or $\lim_{\lambda \uparrow \sup N} \|u^o(\lambda)\| = r$, or $\sup N \in \Lambda_A^r$. If $\lambda^+ > \sup N \in \Lambda_A^r$, then we can reapply Theorem 7-8 at $\sup N$ if $1 \neq \mu_o[A'_{\lambda_o}(u^o(\lambda_o))]$ to conclude that there are minimal positive fixed points $u^o(\lambda)$ for λ in a neighborhood of $\sup N$. Proceeding in this way, we see that there is a maximal open interval J_1 containing λ_o such that $J_1 \subseteq \Lambda_A^r$ and the minimal positive fixed points $u^o(\lambda)$ depend continuously on λ for $\lambda \in J_1$. At the right endpoint $\lambda_1 = \sup J_1$, either $\lambda_1 = \lambda^+$, or $\lim_{\lambda \uparrow \lambda_1} \|u^o(\lambda)\| = r$, or $\lambda_1 \in \Lambda_A^r$, $1 = \mu_o[A'_{\lambda_1}(u^o(\lambda_1))]$, and $\lim_{\lambda \uparrow \lambda_1} u^o(\lambda) = u^o(\lambda_1)$. If the last alternative holds and A_{λ_1} is twice Fréchet differentiable, we may use Theorems 7-3 and 7-5 to investigate the behavior of the set of fixed points of A_λ for λ near λ_1.

I. 8. Behavior of Fixed Points of Large Norm

In this section, we develop necessary and sufficient conditions for the existence of a number $\mu_1 \in J$ such that the family $\{A_\lambda\}$, $\lambda \in J$, has a set of fixed points $\{u(\lambda)\}$ whose norms approach infinity as λ approaches μ_1. We shall see that if the operators A_λ have a behavior on vectors of large norm which can be expressed roughly as

$$\lim_{\|u\| \to \infty} [A_\lambda u - A_\lambda'(\infty)u - b_\lambda] = 0 ,$$

where the mapping $\lambda \to A_\lambda'(\infty)$ is differentiable at μ_1, then the operators A_λ have positive fixed points $u(\lambda)$ such that $\|u(\lambda)\| \to \infty$ as $\lambda \to \mu_1$ if and only if $A_{\mu_1}'(\infty)$ has the eigenvalue 1 corresponding to a positive eigenvector. Moreover, the fixed points exist for $\lambda > \mu_1$ if $b_{\mu_1} < 0$ and for $\lambda < \mu_1$ if $b_{\mu_1} > 0$.

Our first theorem gives a necessary condition for the existence of fixed points $u(\lambda)$ such that $\|u(\lambda)\| \to \infty$ as λ approaches a number μ_1.

<u>8-1. Theorem.</u> Let $\{A_\lambda\}$, $\lambda \in J$, be a family of continuous operators on C. Suppose that there is a sequence $\{\lambda_n\}$ in J converging to $\mu_1 \in J$ such that the operators A_{λ_n} have fixed points $u(\lambda_n)$ in C with $\lim_{n \to \infty} \|u(\lambda_n)\| = \infty$, and A_{μ_1} has a compact C-asymptotic derivative $A_{\mu_1}'(\infty)$ such that

$$\lim_{\substack{\|u\| \to \infty \\ u \in C \\ \lambda \to \mu_1}} \frac{A_\lambda u - A_{\mu_1}'(\infty)u}{\|u\|} = 0 . \tag{8.1}$$

Then 1 is an eigenvalue of $A_{\mu_1}'(\infty)$ corresponding to a positive eigenvector.

Proof. Since $A'_{\mu_1}(\infty)$ is compact, we can suppose that the sequence $\{\lambda_n\}$ has been chosen so that $A'_{\mu_1}(\infty)u_n/\|u_n\|$ (where $u_n = u(\lambda_n)$) converges to w, say. Then $u_n/\|u_n\|$ also converges to w, since

$$\lim_{n\to\infty} \left\| \frac{u_n}{\|u_n\|} - w \right\| \le \lim_{n\to\infty} \left\{ \left\| \frac{A_{\lambda_n} u_n}{\|u_n\|} - \frac{A'_{\mu_1}(\infty)u_n}{\|u_n\|} \right\| + \left\| \frac{A'_{\mu_1}(\infty)u_n}{\|u_n\|} - w \right\| \right\}.$$

Hence, $A'_{\mu_1}(\infty)w = w$, and 1 is an eigenvalue of $A'_{\mu_1}(\infty)$ corresponding to the positive eigenvector w. //

Note that if w is the unique positive eigenvector of $A'_{\mu_1}(\infty)$ of unit norm, then $\lim_{n\to\infty} u(\lambda_n)/\|u(\lambda_n)\| = w$ for any sequence $u(\lambda_n)$ such that $\lim_{n\to\infty} \|u(\lambda_n)\| = \infty$ and $\lim_{n\to\infty} \lambda_n = \mu_1$.

Theorem 8-1 has the following well-known corollary (see, e.g., Krasnosel'skii 1964b, IV. 3).

8-2. Corollary. Let A be a continuous operator on $\mathbb{C}$ with a compact $\mathbb{C}$-asymptotic derivative $A'(\infty)$. Suppose there is a convergent sequence $\{\lambda_n\}$ of positive characteristic values of A corresponding to positive eigenvectors $u(\lambda_n)$ such that $\lim_{n\to\infty} \|u(\lambda_n)\| = \infty$. Then $\lim_{n\to\infty} \lambda_n$ is a characteristic value of $A'(\infty)$ corresponding to a positive eigenvector.

Proof. The Corollary follows from Theorem 8-1 if we show that $\lim_{n\to\infty} \lambda_n > 0$. This is shown by passing to the limit in the following inequalities:

$$\lambda_n^{-1} = \frac{\|Au_n\|}{\|u_n\|} \le \frac{\|Au_n - A'(\infty)u_n\|}{\|u_n\|} + \frac{\|A'(\infty)u_n\|}{\|u_n\|}$$

$$\le \frac{\|Au_n - A'(\infty)u_n\|}{\|u_n\|} + \|A'(\infty)\| ,$$

where $u_n = u(\lambda_n)$. //

The next theorem describes in more detail the behavior of a sequence $\{u_n\}$ of fixed points with $\|u_n\| \to \infty$. It is easily seen that the hypotheses of this theorem imply that the conditions of Theorem 8-1 are satisfied. We omit the proof of the theorem, since it is quite similar to the proofs of Theorems 7-1 and 7-3.

<u>8-3. Theorem.</u> Let $\{A_\lambda\}$, $\lambda \in J$, be a family of positive continuous operators on $\mathcal{C}$. Suppose there is a $\mu_1 \in J$ and a sequence $\{\lambda_n\}$ in J converging to μ_1 such that the operators A_{λ_n} have fixed points u_n in $\mathcal{C}$ with $\lim_{n\to\infty} \|u_n\| = \infty$. If the operators A_λ have the form

$$(8.2) \qquad A_\lambda u = \bar{A}_{\mu_1} u + (\lambda - \mu_1) B_{\mu_1} u + C_\lambda u + D_\lambda u + \omega_\lambda u \ ,$$

where the $\mathcal{C}$-asymptotic derivative $\bar{A}_{\mu_1} = A'_{\mu_1}(\infty)$ of A_{μ_1} is compact and has a unique positive eigenvector ϕ of unit norm, the characteristic value $\mu_0[A'_{\mu_1}(\infty)]$ is simple, B_{μ_1} is a continuous linear operator, $\{C_\lambda\}$ is a family of operators on $\mathcal{C}$ homogeneous of degree s (i.e., for $h \in \mathcal{C}$, $\lambda \in J$, $\alpha > 0$, $C_\lambda(\alpha h) = \alpha^s C_\lambda h$) for some number $s \in [0, 1)$, the mapping $(\lambda, u) \to C_\lambda u$ of $J \times \mathcal{C}$ into $\mathcal{C}$ is continuous, the family $\{D_\lambda\}$ of continuous linear operators satisfies

$$(8.3) \qquad \lim_{\lambda \to \mu_1} \frac{\|D_\lambda\|}{\lambda - \mu_1} = 0 ,$$

and the family $\{\omega_\lambda\}$ satisfies

$$(8.4) \qquad \lim_{\|u\| \to \infty} \frac{\|\omega_\lambda u\|}{\|u\|^s} = 0$$

uniformly for λ in an open subset (relative to J) of J containing μ_1, then

(8.5) $$\lim_{n\to\infty} (\lambda_n-\mu_1)\|u_n\|^{1-s} = -\frac{\xi(C_{\mu_1}\phi)}{\xi(B_{\mu_1}\phi)},$$

where ξ is the positive linear functional of equation (7.8).

If $\xi(B_{\mu_1}\phi) \neq 0$, and $s' = (1-s)^{-1}$, then

(8.6) $$\lim_{n\to\infty} |\lambda_n-\mu_1|^{s'} u_n = \lim_{n\to\infty} [|\lambda_n-\mu_1|^{s'}\xi(u_n)]\phi = \left|\frac{\xi(C_{\mu_1}\phi)}{\xi(B_{\mu_1}\phi)}\right|^{s'} \phi$$

and, with $R = [I-\bar{A}_{\mu_1}]^{-1}_{P\mathcal{B}}P$,

(8.7) $$\lim_{n\to\infty} \frac{Pu_n}{\|u_n\|^s} = R\left\{-\frac{\xi(C_{\mu_1}\phi)}{\xi(B_{\mu_1}\phi)} B_{\mu_1}\phi + C_{\mu_1}\phi\right\}.$$

If $\xi(B_{\mu_1}\phi)\xi(C_{\mu_1}\phi) \neq 0$, then there are positive numbers r and δ such that for all λ with

(8.8) $$0 \leq (\lambda-\mu_1) \operatorname{sgn} [\xi(B_{\mu_1}\phi)\xi(C_{\mu_1}\phi)] \leq \delta,$$

the operator A_λ has no fixed points in $\mathcal{C}$ with norm greater than r.

If we take $A_\lambda = \lambda A$ for $\lambda > 0$, where A is $\mathcal{C}$-asymptotically linear, then $A'_{\mu_1}(\infty) = \mu_1 A'(\infty)$, $B_{\mu_1} = A'(\infty)$, and the number μ_1 must be $\mu_o[A'(\infty)]$ by Corollary 8-2 (if $A'(\infty)$ has a unique positive eigenvector of unit norm). Thus, we obtain:

<u>8-4. Corollary</u>. Let A be a positive continuous $\mathcal{C}$-asymptotic linear operator on $\mathcal{C}$ which has the form

(8.9) $$Au = A'(\infty)u + Cu + \omega u$$

where $A'(\infty)$ is a compact linear operator which has a unique positive eigenvector ϕ and simple characteristic value $\mu_o[A'(\infty)]$, C is a continuous operator which is homogeneous of degree $s \in [0, 1)$, and the operator ω satisfies

(8.10) $$\lim_{\|u\| \to \infty} \frac{\|\omega(u)\|}{\|u\|^s} = 0 .$$

If $\{\lambda_n\}$ is a bounded sequence of characteristic values of $\mathcal{C}$ to which there correspond eigenvectors $\{u_n\}$ with $\lim_{n\to\infty} \|u_n\| = \infty$, then

$$\lim_{n\to\infty} \lambda_n = \mu_o[A'(\infty)] ,$$

and

$$\lim_{n\to\infty} (\lambda_n - \mu_o[A'(\infty)])\|u_n\|^{1-s} = -(\mu_o[A'(\infty)])^2 \xi(C\phi) .$$

If $\xi(C\phi) > 0$, then for any positive number $\rho > \mu_o[A'(\infty)]$ there is a positive number r such that A has no positive eigenvectors with norms greater than r corresponding to $\lambda \in [\mu_o[A'(\infty)], \rho]$; if $\xi(C\phi) < 0$, then there is a positive number r such that A has no positive eigenvectors with norms greater than r corresponding to $\lambda \in [0, \mu_o[A'(\infty)]]$.

When $s = 0$, the conditions (8.4) or (8.10) on the remainder ω_λ or ω are not appropriate for the applications we shall wish to make of the results of Theorem 8-3 and Corollary 8-4. For example, in Proposition 8-8 below, the operator A is the integral operator of Example 2-1, and

$$f(x, u) = m(x)u + b(x) + o(1)$$

as $u \to +\infty$, uniformly for $x \in \overline{\Omega}$; in this case, the operator ω defined by

$$\omega(u)(x) = \int_{\Omega} K(x,y)[f(y,u(y))-m(y)u(y)-b(y)]dy$$

does not satisfy equation (8.10) in general. To see this, take $\overline{\Omega} = [0,1] \subseteq \mathbb{R}$,

$$f(x,u) = mu + c + \frac{1}{1+u}$$

(independent of x) for $x \in [0,1]$, $u \geq 0$, and

$$K(x,y) = \min(x,y)$$

for $x, y \in [0,1]$, so that

$$\omega(u)(x) = \int_0^1 [\min(x,y)][1+u(y)]^{-1}dy \ .$$

If we consider the sequence $\{u_\eta\}$ of positive continuous functions

$$u_\eta(x) = \begin{cases} \eta-\eta^2 x \ , & 0 \leq x \leq \eta^{-1} \\ 0 & \eta^{-1} \leq x \leq 1 \end{cases} ,$$

then a simple calculation shows that $\lim_{\eta\to+\infty} \|\omega(u_\eta)\| \neq 0$, although $\lim_{\eta\to+\infty} \|u_\eta\| = \infty$. In other words,

$$\lim_{\rho\to+\infty} [f(x,\rho) - m(x)\rho - b(x)] = 0 ,$$

uniformly for $x \in \overline{\Omega}$, does not necessarily imply

$$\lim_{\substack{\|u\|\to\infty \\ u \in C}} \max_{x\in\overline{\Omega}} \left| \int_{\Omega} K(x,y)[f(y,u(y))-m(y)u(y)-b(y)]dy \right| = 0 \ .$$

The difficulty here is that, although $\|u_\eta\| \to \infty$ as $\eta \to +\infty$, the functions u_η are non-zero only on an arbitrarily small subset of $\overline{\Omega} = [0,1]$ as $\eta \to +\infty$; hence, the operator ω acting on the functions u_η is not able to take advantage of the asymptotic behavior of $f(x,\rho)$ as $\rho \to +\infty$. We need to restrict the limit in equation (8.4) or (8.10) to functions u which become large as $\|u\| \to \infty$ on a subset of Ω which is sufficiently large that the asymptotic behavior of $f(x, \rho)$ as $\rho \to +\infty$ determines the asymptotic behavior of Au and $\omega(u)$ as $\|u\| \to \infty$.

In order to obtain a more appropriate condition for ω_λ, we observe that if we replace equation (8.4) by the requirement that $\|\omega_\lambda u\| / \|u\|^s$ be uniformly bounded for $u \in C^+$ and λ in some J-neighborhood of μ_1, then equation (8.1) of Theorem 8-1 still holds, and the order of magnitude estimates of Theorem 8-3 remain valid if $\xi(B_{\mu_1}\phi) \neq 0$ (also when $s > 0$); i.e., as $\lambda_n \to \mu_1$, the quantities $|\lambda_n - \mu_1| \; \|u_n\|^{1-s}$, $|\lambda_n-\mu_1| [\xi(u_n)]^{1-s}$, $\|Pu_n\| [\xi(u_n)]^{-s}$, and $\|Pu_n\| \; |\lambda_n - \mu_1|^{ss'}$ remain bounded. One easily sees that consequently the results of Theorem 8-3 hold if, in addition to the boundedness of $\|\omega_\lambda u\| / \|u\|^s$, we replace equation (8.4) by the following condition: for any positive number r,

$$\lim_{\beta \to +\infty} \beta^{-s} \|\omega_\lambda(\beta\phi + \beta^s h)\| = 0 \tag{8.11}$$

uniformly for $\|h\| \leq r$, provided $\beta\phi + \beta^s h \in C$ for all sufficiently large β.

The condition (8.11) can be verified for operators of the form of equation (2.8) or (8.19) for suitable functions ϕ; e.g., if ϕ is positive almost everywhere on $\overline{\Omega}$, then for all h in any bounded subset of $C(\overline{\Omega})$, $\beta\phi + \beta^s h$ can be made arbitrarily large except on an arbitrarily small subset of $\overline{\Omega}$, if β is sufficiently large (see Proposition 8-8 below for the verification of equation (8.11) when $s = 0$).

<u>8-5. Theorem.</u> Let the conditions of Theorem 8-3 hold, except that the condition (8.4) is replaced by the following: for λ in a

neighborhood N (relative to J) of μ_1 in J,

(a) the set $\{\|\omega_\lambda u\|/\|u\|^s : u \in \mathcal{C}^+,\ \lambda \in N\}$ is bounded, and

(b) for any positive number r, the limit in equation (8.11) exists uniformly for $\lambda \in N$ and $h \in \mathcal{B}^r$ such that $h+\beta\phi \in \mathcal{C}$ for all sufficiently large $\beta > 0$.

Then all the conclusions of Theorem 8-3 are valid.

Proof. Since equation (8.1) of Theorem 8-1 is satisfied by the operators A_λ and $\bar{A}_{\mu_1} = A'_{\mu_1}(\infty)$, 1 is an eigenvalue of $\bar{A}_{\mu_1}$ corresponding to the positive eigenvector $\phi = \lim_{n\to\infty} u_n/\|u_n\|$. From

$$u_n = A_{\lambda_n} u_n = \bar{A}_{\mu_1} u_n + (\lambda_n - \mu_1) B_{\mu_1} u_n + C_{\lambda_n} u_n + D_{\lambda_n} u_n + \omega_{\lambda_n} u_n ,$$

we obtain

$$0 = (\lambda_n - \mu_1)\xi(B_{\mu_1} u_n) + \xi(C_{\lambda_n} u_n) + \xi(D_{\lambda_n} u_n) + \xi(\omega_{\lambda_n} u_n) .$$

Dividing by $(\lambda_n - \mu_1)\|u_n\|$ and letting $n \to \infty$, we have

$$0 = \xi(B_{\mu_1}\phi) + \lim_{n\to\infty} \frac{1}{(\lambda_n - \mu_1)\|u_n\|^{1-s}} \xi\left(C_{\lambda_n} \frac{u_n}{\|u_n\|}\right) + \frac{1}{\|u_n\|^s} \xi(\omega_{\lambda_n} u_n) \quad .$$

If $\xi(B_{\mu_1}\phi) \neq 0$, it follows from hypothesis (a) that $\limsup_{n\to\infty} |\lambda_n - \mu_1| \, \|u_n\|^{1-s} < \infty$. Similarly, if we divide

$$Pu_n = R\{(\lambda_n - \mu_1) B_{\mu_1} u_n + C_{\lambda_n} u_n + D_{\lambda_n} u_n + \omega_{\lambda_n} u_n\}$$

by $\|u_n\|^s$, we obtain

$$\frac{Pu_n}{\|u_n\|^s} = R\Big\{(\lambda_n-\mu_1)\|u_n\|^{1-s}\frac{B_{\mu_1}u_n}{\|u_n\|} + C_{\lambda_n}\Big(\frac{u_n}{\|u_n\|}\Big)$$

$$+\frac{1}{\lambda_n-\mu_1}D_{\lambda_n}\Big(\frac{u_n}{\|u_n\|}\Big)(\lambda_n-\mu_1)\|u_n\|^{1-s}+\frac{\omega_{\lambda_n}u_n}{\|u_n\|^s}\Big\} \ ,$$

so

$$\limsup_{n\to\infty}\frac{\|Pu_n\|}{\|u_n\|^s} < \infty \ .$$

Since $\lim_{n\to\infty}\xi(u_n)/\|u_n\| = 1$, we also have

$$\limsup_{n\to\infty}\frac{\|Pu_n\|}{\xi(u_n)^s} < \infty \ .$$

Let $\beta_n = \xi(u_n)$, $h_n = Pu_n/\beta_n^s$. Then there is a positive number r such that $\|h_n\| \le r$ for all n. In view of the uniformity of the limit in equation (8.11), we have

$$\lim_{n\to\infty}\|u_n\|^{-s}\omega_{\lambda_n}u_n = \lim_{n\to\infty}\beta_n^{-s}\omega_{\lambda_n}(\beta_n\phi+\beta_n^s h_n) = 0 \ .$$

Equations (8.5), (8.6), and (8.7), and the condition associated with inequality (8.8) then follow from the equations above. //

We next show that if $C_\lambda u$ is a constant b_λ for $u \in C$, and if the operators ω_λ satisfy the asymptotic condition (8.11) with $s = 0$, then we can establish a converse of the preceding results for operators of the form of equation (8.2): if $\overline{A}_{\mu_1}$ has the simple characteristic value 1 to which there corresponds a positive eigenvector, then the operators A_λ have fixed points of arbitrarily large norm for λ near μ_1.

<u>8-6. Theorem.</u> Let $\{A_\lambda\}$, $\lambda \in J$, be a family of continuous operators on a cone C_1 containing C^+ in its interior, with $A_\lambda C_1 \subseteq C$

for $\lambda \in J$, and let the operators A_λ have the form of equation (8.2) for some $\mu_1 \in J$, where $\bar{A}_{\mu_1}$ and B_{μ_1} are continuous linear operators, $C_\lambda u = b_\lambda \in \mathcal{B}$ for all u and all $\lambda \in J$, the mapping $\lambda \to b_\lambda$ of J into $\mathcal{B}$ is continuous, D_λ is a continuous linear operator which satisfies equation (8.3), and the operators ω_λ satisfy conditions to be specified later. Let $\bar{A}_{\mu_1}$ have the simple eigenvalue 1 corresponding to a positive eigenvector ϕ such that every $u \in \mathcal{B}$ is representable in the form

$$u = \xi(u)\phi + Pu ,$$

where P is a projection of $\mathcal{B}$ onto the subspace $P\mathcal{B}$, invariant under $\bar{A}_{\mu_1}$, on which the operator $I-\bar{A}_{\mu_1}$ has an inverse R_p, and ξ is a positive continuous linear functional with $\xi(Pu) = 0$ for all $u \in \mathcal{B}$ and $\xi(\phi) = 1$. Let ω_λ satisfy the following conditions for any sufficiently small number $r > 0$ and for λ in an open subset N (relative to J) of J containing μ_1:

$$(8.12) \qquad \lim_{\substack{\beta \to +\infty \\ \beta \in \mathbb{R}}} \|\omega_\lambda(\beta\phi+h)\| = 0$$

uniformly for $h \in \mathcal{B}^r = \{u \in \mathcal{B} : \|u\| < r\}$ and $\lambda \in N$; and

$$(8.13) \qquad \|\omega_\lambda(\beta_1\phi+h_1) - \omega_\lambda(\beta_2\phi+h_2)\|$$

$$\leq q_\omega(\beta_1, \beta_2; h_1, h_2; \lambda)\|(\beta_1\phi+h_1)-(\beta_2\phi+h_2)\|$$

where $q_\omega(\beta_1, \beta_2; h_1, h_2; \lambda)$ is a real-valued positive function of the numbers β_1, β_2, the vectors h_1, h_2, and the number λ, such that

$$(8.14) \qquad \lim_{\substack{\beta_2 \to +\infty \\ \beta_1 > \beta_2}} q_\omega(\beta_1, \beta_2; h_1, h_2; \lambda) = 0$$

uniformly for $h_1, h_2 \in \mathcal{B}^r$ and $\lambda \in N$.

Then there exists a number $\delta > 0$ such that for each $\lambda \in J$ with

(8.15) $$0 < (\mu_1 - \lambda)\, \mathrm{sgn}[\xi(b_{\mu_1})\ \xi(B_{\mu_1}\phi)] < \delta$$

A_λ has a fixed point $u(\lambda_n) \in C^+$, and for any sequence $\{\lambda_n\}$, the elements of which satisfy relation (8.15), the corresponding fixed points $u(\lambda_n) \equiv u_n$ of A_{λ_n} satisfy equations (8.5), (8.6), and (8.7), with $s = 0$.

Proof. The proof is very similar to that of Theorem 7-3, and we shall give only a brief outline. We seek a solution u of the equation $A_\lambda u - u = 0$ in the form $u = y + \beta\phi$ by first attempting to solve

(8.16) $$P(A_\lambda u - u) = P[A_\lambda(y+\beta\phi) - (y+\beta\phi)] = 0$$

for y in terms of β, and then choosing β so that the equation

(8.17) $$\xi(A_\lambda u - u) = \xi[A_\lambda(y+\beta\phi) - (y+\beta\phi) = 0$$

is satisfied. Then $u = y + \beta\phi$ will satisfy $A_\lambda u - u = 0$.

Equation (8.16) is equivalent to

(8.18) $$z = T(\alpha, \lambda) z ,$$

where

$$\alpha = (\lambda - \mu_1)\beta , \qquad z = Pu - k_{\alpha,\lambda} ,$$

$$k_{\alpha,\lambda} = \alpha(\lambda - \mu_1)^{-1} R[(\lambda - \mu_1) B_{\mu_1} + D_\lambda]\phi + R b_\lambda ,$$

and

(8.19) $$T(\alpha,\lambda) z = R[(\lambda - \mu_1) B_{\mu_1}(z + k_{\alpha,\lambda}) + D_\lambda(z + k_{\alpha,\lambda}) + \omega_\lambda(z + k_{\alpha,\lambda} + \alpha(\lambda - \mu_1)^{-1}\phi)] .$$

For any positive number θ and sufficiently small positive η, it is possible to find a neighborhood N_1 of μ_1 in J such that for $|\alpha| \le \theta$, $\lambda \in N$, and $\alpha(\lambda - \mu_1) > 0$, $T(\alpha, \lambda)$ is a contraction mapping of $\overline{\mathbb{B}}^\eta$ into itself. Thus, equation (8. 18) has a solution $z_{\alpha,\lambda}$, which depends continuously on α for each $\lambda \in N_1$.

Setting $y_{\alpha,\lambda} = z_{\alpha,\lambda} + k_{\alpha,\lambda}$ and substituting for y in equation (8. 17), we find that for N_1 sufficiently small, $|\alpha| \leq \theta$, and $\beta = \alpha(\lambda - \mu_1)^{-1} > 0$, equation (8. 17) has a solution $\alpha(\lambda)$ for $\lambda \in N$ with $|\alpha(\lambda)| \leq \theta$ and

$$\operatorname{sgn} \alpha(\lambda) = \operatorname{sgn} (\lambda - \mu_1) = -\operatorname{sgn}[\xi(b_{\mu_1})\xi(B_{\mu_1}\phi)] ,$$

for appropriate choice of θ. Thus, there is a positive number δ such that for any λ satisfying inequality (8. 15), the operator A_λ has a fixed point

$$u(\lambda) = \tilde{y}_{\alpha(\lambda),\lambda} + \frac{\alpha(\lambda)}{\lambda - \mu_1}\phi .$$

From equations (8. 12), (8. 13), (8. 14), and (8. 3), the solution $\alpha(\lambda)$ of (8. 17) may be taken arbitrarily close to $-\xi(b_{\mu_1})/\xi(B_{\mu_1}\phi)$ for λ sufficiently close to μ_1. Then

$$\lim_{\lambda \to \mu_1} \xi[u(\lambda)](\lambda - \mu_1) = \lim_{\lambda \to \mu_1} \alpha(\lambda) = -\xi(b_{\mu_1})/\xi(B_{\mu_1}\phi) .$$

Similarly, equation (8. 7), and thence equations (8. 5) and (8. 6), may be obtained from equation (8. 17). //

We now give some conditions under which Hammerstein integral operators have the properties assumed in Theorems 8-3, 8-5, and 8-6.

<u>8-7. Proposition.</u> Let the operators A_λ, B_λ defined by

(8. 20a) $$A_\lambda u(x) = \int_\Omega K(x, y; \lambda) f(y, u(y)) dy$$

(8.20b) $$B_\lambda u(x) = \int_\Omega K(x,y;\lambda)g(y,u(y))dy$$

be continuous operators on $C(\overline{\Omega})$ for each λ in some interval J. Here, Ω is a bounded domain in $\mathbb{R}^n$, f and g are continuous functions on $\overline{\Omega}\times(-\infty,\infty)$, and there is a positive constant $\varkappa_o$ such that

$$|K(x,y;\lambda)| \le \frac{\varkappa_o}{|x-y|^\alpha}$$

for some number $\alpha \in [0,n)$, all $x,y \in \overline{\Omega}$, and λ in a subset N of J. Let

$$\lim_{\rho\to+\infty} [f(x,\rho) - g(x,\rho)] = 0$$

uniformly for $x\in\overline{\Omega}$. Then for any function $\phi\in C(\overline{\Omega})$ which is positive almost everywhere on $\overline{\Omega}$, and for every positive number r, we have

$$\lim_{\beta\to+\infty} \|A_\lambda(\beta\phi+h) - B_\lambda(\beta\phi+h)\| = 0$$

uniformly for $\lambda\in N$ and $h\in C(\overline{\Omega})$ such that $\|h\| \le r$.

Proof. Let

$$f_m(r) = \max\{|f(x,v)| : x\in\overline{\Omega},\ 0\le v\le r\} ,$$

$$g_m(r) = \max\{|g(x,v)| : x\in\overline{\Omega},\ 0\le v\le r\} ,$$

and

$$\gamma = \sup\{\int_\Omega |K(x,y;\lambda)|dy : x\in\overline{\Omega},\ \lambda\in J\} .$$

Let ε and r be given positive numbers and choose $r' > 0$ such that $|f(x,\rho)-g(x,\rho)| \le \varepsilon/\gamma$ for $\rho \ge r'$ and $x\in\Omega$. Choose $\delta > 0$ such that

$$\int_{|x-y|\le\delta} |K(x,y;\lambda)|dy \le \varepsilon[f_m(r')+g_m(r')]^{-1}$$

for $x\in\Omega$, and choose $\beta_o > 0$ such that

$$\text{meas}\,\{x\in\overline{\Omega}:\phi(x)\le \frac{r'+r}{\beta_o}\} \le \frac{\varepsilon}{\kappa_o}\,\frac{\delta^{\alpha}}{f_m(r')+g_m(r')} \quad .$$

If $\beta \ge \beta_o$ and $\|h\| \le r$, then $u(y) = \beta\phi(y)+h(y) \le r$ implies

$$\phi(y) \le \beta^{-1}[r'-h(y)] \le \beta_o^{-1}[r'+r] ;$$

setting

$$\Omega_1 = \{y\in\Omega : u(y) \le r' ,\ |x-y| > \delta\} ,$$

$$\Omega_2 = \{y\in\Omega : u(y) \le r' ,\ |x-y| \le \delta\} ,$$

and

$$\Omega_3 = \{y\in\Omega : u(y) > r'\} ,$$

we have

$$\begin{aligned}
&|A_\lambda(\beta\phi+h)(x) - B_\lambda(\beta\phi+h)(x)| \\
&\quad \le \int_{\Omega_1} + \int_{\Omega_2} + \int_{\Omega_3} |K(t,s)[f(s,u(s))-g(s,u(s))]|\,ds \\
&\quad \le \frac{\kappa_o}{\delta^{\alpha}}[f_m(r')+g_m(r')]\,\frac{\varepsilon}{\kappa_o}\,\frac{\delta^{\alpha}}{f_m(r')+g_m(r')} \\
&\qquad + \frac{\varepsilon}{f_m(r')+g_m(r')}[f_m(r')+g_m(r')] + \frac{\varepsilon}{\gamma}\cdot\gamma \\
&\quad = 3\varepsilon \ .
\end{aligned}$$

Thus, $\lim_{\beta\to+\infty} \|A_\lambda(\beta\phi+h)-B_\lambda(\beta\phi+h)\| = 0$ uniformly for $\lambda\in N$ and $\|h\| \le r$. //

8-8. Proposition. Let the kernel $K(x,y;\lambda)$, the function f, and the operator A_λ have the properties described in the first two sentences of the previous proposition. Let $f(x,u)$ have a continuous partial derivative f_u such that

$$\lim_{\rho\to+\infty} [f_u(x,\rho)] = m(x)$$

uniformly for $x \in \overline{\Omega}$. Define the operator ω_λ on $C(\overline{\Omega})$ by

$$\omega_\lambda u(x) = \int_\Omega K(x, y;\lambda)[f(y, u(y)) - m(y)u(y) - b(y)]dy \quad .$$

Then ω_λ satisfies conditions (8.12), (8.13), and (8.14) of Theorem 8-6 for any function $\phi \in C(\overline{\Omega})$ which is positive almost everywhere on $\overline{\Omega}$.

Proof. The preceding proposition, with $g(x, u) = m(x)u + b(x)$, shows that ω_λ satisfies condition (8.12). Condition (8.13) is proved in a similar way: using the mean value theorem, we have for any functions $u_1, u_2 \in C(\overline{\Omega})$,

$$|f(y, u_1(y)) - m(y)u_1(y) - f(y, u_2(y)) + m(y)u_2(y)|$$
$$= |f_u(y, \tilde{u}(y)) - m(y)|\,|u_1(y) - u_2(y)| ,$$

where

$$\min\{u_1(y), u_2(y)\} \le \tilde{u}(y) \le \max\{u_1(y), u_2(y)\} \quad .$$

As in the proof of the preceding proposition, we can then show that, given positive numbers ε, r, it is possible to find a number β_o such that whenever $\beta_1 \ge \beta_2 \ge \beta_o$, $\|h_1\| \le r$, $\|h_2\| \le r$, we have

$$\|D_\lambda(\beta_1\phi + h_1) - D_\lambda(\beta_1\phi + h_2)\|$$
$$\le \sup_x \int_\Omega |K(x, y;\lambda)|\,|f_u(y, \tilde{u}(y)) - m(y)|\,|u_1(y) - u_2(y)|\,dy$$
$$\le \varepsilon \|(\beta_1\phi + h_1) - (\beta_2\phi + h_2)\|$$

(where $u_i = \beta_i\phi + h_i$, $i = 1, 2$). This shows that conditions (8.13) and (8.14) are satisfied. //

8-9. Proposition. Let the kernel $K(x, y;\lambda)$, the function f, and the operator A_λ have the properties described in the first two sentences of Proposition 8-7. For functions $m \in C(\overline{\Omega})$, $c \in C(\overline{\Omega})$, and a number s , $0 < s < 1$, let

$$\lim_{\rho \to +\infty} [f(x,\rho) - m(x)\rho - c(x)\rho^s] = 0$$

uniformly for $x \in \overline{\Omega}$. Define

$$C_\lambda u(x) = \int_\Omega K(x,y;\lambda)c(y)[u(y)]^s dy$$

and

$$\omega_\lambda u(x) = \int_\Omega K(x,y;\lambda)[f(y,u(y)) - m(y)u(y) - c(y)[u(y)]^s]dy$$

for $u \in C(\overline{\Omega})$. Then C_λ is homogeneous of degree s and ω_λ satisfies both condition (8.4) of Theorem 8-3 and condition (8.11) of Theorem 8-5.

Proof. See Proposition 8-7 and Krasnosel'skii (1964a, p. 242). //

The final theorem of this section and its corollary treat operators $\{A_\lambda\}$, $\lambda \in J$, which may be described as asymptotically superlinear; equation (8.21) of Theorem 8-10 holds for arbitrarily large β and appropriate $b_{\lambda,\beta}$. In this case, if we have a convergent sequence $\{\lambda_n\}$ such that the operators $\{A_{\lambda_n}\}$ have fixed points $\{u(\lambda_n)\}$ with $\lim_{n\to\infty} \|u(\lambda_n)\| = \infty$, then $\lim_{n\to\infty} \lambda_n \notin J$. The conditions of Theorem 8-10 are quite similar to the conditions of Corollary 9-2, in which we prove the existence of non-minimal positive fixed points.

8-10. Theorem. Let $\{A_\lambda\}$, $\lambda \in J$, be a family of positive operators on $\mathfrak{C}$. Suppose there is a positive number r and a family $\{T_\lambda\}$, $\lambda \in J$, of positive linear compact operators for which the mapping $\lambda \to T_\lambda$ is continuous on J , such that for any positive number β there exists $b_{\lambda,\beta} \in \mathfrak{B}$ so that

(8.21) $$A_\lambda u \geq \beta T_\lambda u + b_{\lambda,\beta}$$

for all $u > 0$ with $\|u\| \geq r$. For each β and any closed bounded subset J_1 of J, let the set $\{b_{\lambda,\beta}: \lambda \in J_1\}$ be bounded in norm. If there is a convergent sequence $\{\lambda_n\}$ in J to which there corresponds a sequence $\{u_n\}$ of positive fixed points of the operators A_{λ_n} such that $\lim_{n\to\infty} \|u_n\| = \infty$, and if there is a $g_\lambda \in C^+$ and an integer $m \geq 1$ such that $T_\lambda^m u_n \geq \|u_n\| g_\lambda$ for each $\lambda \in J$, then $\lim_{n\to\infty} \lambda_n$ is not in J.

Proof. Suppose $\lim_{n\to\infty} \lambda_n = \lambda_\infty$ is in J. Then the numbers λ_n are contained in a closed bounded subset J_1 of J. Since T_{λ_∞} is compact, we can suppose that the sequences $\{\lambda_n\}$ and $\{u_n\}$ have been chosen so that $\{T_{\lambda_\infty} u_n / \|u_n\|\}$ converges, say to h, and $\|u_n\| \geq r$. Since for each n, $T_{\lambda_\infty}^m u_n \geq \|u_n\| g_{\lambda_\infty}$, it follows that $h > 0$. Choose $\beta > 1/\|h\|$. Then

$$\frac{u_n}{\|u_n\|} = \frac{A_{\lambda_n} u_n}{\|u_n\|} \geq \beta (T_{\lambda_n} - T_{\lambda_\infty}) \frac{u_n}{\|u_n\|} + \beta T_{\lambda_o} \frac{u_n}{\|u_n\|} + \frac{b_{\lambda_n,\beta}}{\|u_n\|} ,$$

so

$$1 + \beta \| T_{\lambda_n} - T_{\lambda_\infty} \| \geq \beta \| T_{\lambda_\infty} \frac{u_n}{\|u_n\|} \| - \frac{\|b_{\lambda_n,\beta}\|}{\|u_n\|} .$$

Letting $n \to \infty$, using the continuity of the T_λ in λ and the boundedness of $\{b_{\lambda,\beta}: \lambda \in J_1\}$, we obtain the contradiction:

$$1 \geq \beta \|h\| .$$

Thus, $\lim_{n\to\infty} \lambda_n$ is not in J. //

Corollary 8-11. Let A be a positive operator on $\mathcal{C}$. Suppose there is a positive number r and a positive linear compact operator T such that for any positive number α there exists $b_\alpha \in \mathcal{B}$ so that

$$\text{(8. 22)} \qquad Au \geq \alpha Tu + b_\alpha$$

for all $u > 0$ with $\|u\| \geq r$. If there is a sequence $\{u_n\}$ of positive eigenvectors of A such that $\lim_{n\to\infty} \|u_n\| = \infty$, and if there is a $g_o \in C^+$ and an integer $m \geq 1$ such that $T^m u_n \geq \|u_n\| g_o$ for every n, then the corresponding sequence of characteristic values $\{\lambda_n\}$ $(u_n = \lambda_n A u_n)$ converges to zero.

Proof. Taking $J = (0, \infty)$ in Theorem 8-10, we see that any limit point of the sequence $\{\lambda_n\}$ must be either 0 or ∞. If the sequence $\{\lambda_n\}$ is not bounded, then there is a subsequence $\{\lambda_{n_k}\}$ converging to ∞ such that $\{Tu_{n_k} / \|u_{n_k}\|\}$ converges to a positive vector h (since T is compact and $T^m u_{n_k} / \|u_{n_k}\| \geq g_o$). But

$$\lambda_{n_k}^{-1} = \frac{\|Au_{n_k}\|}{\|u_{n_k}\|} \geq \alpha \frac{\|Tu_{n_k}\|}{\|u_{n_k}\|} - \frac{\|b_\alpha\|}{\|u_{n_k}\|}$$

for any positive number α; taking the limit in this inequality, we obtain

$$\liminf_{k\to\infty} \lambda_{n_k}^{-1} \geq \alpha \|h\| > 0,$$

which contradicts $\lim_{k\to\infty} \lambda_{n_k} = \infty$. Thus, the sequence $\{\lambda_n\}$ is bounded and has 0 as its unique limit point. //

I. 9. Existence of a Second Fixed Point

In the following theorem and its corollaries, we develop further criteria for the existence for forced operators of positive fixed points larger than the minimal positive fixed points. The general method may be described as follows: given a positive monotonic forced operator A with a minimal positive fixed point u^o, we seek a positive fixed point of the unforced operator A^o defined by $A^o h = A(u^o+h)-Au^o$ for $h \geq 0$. A positive fixed point h_o of A^o provides us with a positive fixed point $u^o+h_o > u^o$ of A.

The significance of the hypotheses of the following theorem may be understood by considering the unforced operator A to be a positive real-valued function. A positive linear operator T may be interpreted as a straight line through the origin of slope $\mu_o(T)^{-1}$. Then condition (a_o) states that at $u = r_o$, the graph of A lies below that of a straight line through the origin of slope $\mu_o(T_o)^{-1} \leq 1$, and condition (a_∞) states that the graph of A eventually lies above that of a straight line of slope $\mu_o(T)^{-1} > 1$. It is then clear geometrically that the graph of A intersects the straight line through the origin of slope 1 at some positive value of u. This argument also shows that the condition $A0 = 0$ cannot be dispensed with. Conditions (b_o) and (b_∞) can be given a similar graphical interpretation.

The requirement that $T^m h \geq \|h\| g_o$ given under condition (a_∞) is needed to assure that the sequence $Tv_i / \|v_i\|$, constructed in the next to the last part of the proof, does not converge to a limit 0.

<u>9-1. Theorem.</u> Let A be a compact positive unforced operator on $\mathcal{C}$. Suppose that either of the following conditions is satisfied:

(a_o) There exist a continuous positive linear operator T_o with $\mu_o(T_o) \geq 1$ and a positive number r_o such that

(9.1) $$Au \leq T_o u$$

for all $u \in \mathcal{C}$ with $\|u\| = r_o$;

(b_o) A has a compact Fréchet derivative $A'(0)$ at the origin, and $\mu_o[A'(0)] > 1$.

Suppose further that either of the following conditions is satisfied:

(a_∞) there exist a compact positive linear operator T, with $\mu_o(T) < 1$, satisfying (PA) and having no positive eigenvector with eigenvalue 1 ; a positive vector g_o and an integer m such that for all $h \in \mathcal{C}$ for which there is a positive number $\alpha(h)$ such that $h \geq \alpha(h) g_o$, we have

(9.2) $$T^m h \geq \|h\| g_o ;$$

and a positive number r_1 and a vector $b \in \mathcal{B}$ such that

(9.3) $$Au \geq Tu + b$$

for all $u \in \mathcal{C}$ with $\|u\| \geq r_1$;

(b_∞) A has a compact $\mathcal{C}$-asymptotic derivative $A'(\infty)$, with $\mu_o(A'(\infty)) < 1$, which has no positive eigenvector with eigenvalue 1 .

Then A has at least one fixed point in $\mathcal{C}^+$; if condition (a_o) is satisfied, then A has a positive fixed point with norm greater than r_o ; if equation (9.1) is satisfied for all u in $\mathcal{C}^{r_o}$ and if $\mu_o(T_o) > 1$, then A has no non-zero fixed point in $\mathcal{C}$ with norm less than r_o .

Proof. (Cf. Krasnosel'skii 1964a, Theorem 4.16.) The idea of the proof is the following: for sufficiently large integers n, we construct new operators A_n which differ from A on elements of large and small norm in C, but agree with A on elements of "intermediate" norm. The operators A_n are constructed so that we know they have fixed points in C^+. Using condition (a_o) or (b_o), we show that for sufficiently large n, these fixed points do not occur among the elements of small norm on which A_n differs from A; using condition (a_∞) or (b_∞), we show that these fixed points also cannot occur among the elements of large norm on which A_n differs from A, for sufficiently large n. Thus, for large n, the fixed points of A_n occur among those elements u such that $A_n u = Au$, and therefore A has at least one positive fixed point.

The proof under conditions (b_o) and (b_∞) is given by Krasnosel'skii (1964a, Theorem 4.16, or 1964b, Theorem V.3.4). We carry through the proof under conditions (a_o) and (a_∞). Since the proof concerning elements of small norm and that concerning elements of large norm are independent, the other cases may be handled by using a combination of Krasnosel'skii's proof and ours.

Without loss of generality, we assume that $r_1 > r_o$. For $n \geq \max\{3, r_1\}$, we define

$$A_n u = \begin{cases} \dfrac{2\|u\|-r_o}{r_o} A\left(\dfrac{r_o u}{\|u\|}\right), & \dfrac{r_o}{2} \le \|u\| \le r_o \\ Au, & r_o \le \|u\| \le n \\ \dfrac{\|u\|}{n} A\left(\dfrac{nu}{\|u\|}\right) + (\|u\|-n)^2 \dfrac{g_o}{\|g_o\|}, & n \le \|u\| \le n^2 \\ \dfrac{\|u\|}{n} \dfrac{n^3-\|u\|}{n^3-n^2} A\left(\dfrac{nu}{\|u\|}\right) + (n^2-n)^2 \dfrac{g_o}{\|g_o\|}, & n^2 \le \|u\| \le n^3 . \end{cases}$$

The operators A_n are compact positive operators defined on the set of all C such that $r_o/2 \le \|u\| \le n^3$. When $\|u\| = r_o/2$, $\|A_n u\| = 0 < r_o/2$; when $\|u\| = n^3$, then $\|A_n u\| = (n^2-n)^2 > n^3$. It follows from the theorem of Krasnosel'skii (1964a, p. 147) on operators expanding the cone that each A_n has a positive fixed point u_n in its domain of definition.

We now apply condition (a_o) to show that we cannot have $r_o/2 \le \|u_n\| \le r_o$. For if this equality were satisfied by some u_n, then we would have

$$(9.4) \qquad 0 < v_n = \frac{r_o u_n}{\|u_n\|} = \alpha_n A v_n \le \alpha_n T_o v_n ,$$

where

$$\alpha_n = \frac{2\|u_n\|-r_o}{\|u_n\|} < 1 .$$

As in the proof of Theorem 4-13, it follows that $\mu_o(T_o) \le \alpha_n < 1$, which contradicts the assumption $\mu_o(T_o) \ge 1$. Thus, all fixed points of the operators A_n satisfy $\|u_n\| > r_o$.

Suppose next that there is a sequence $\{u_{n_i}\}$ of fixed points of the operators A_{n_i} such that $n_i \le \|u_{n_i}\| \le n_i^2$. Then

$$\frac{v_i}{\|v_i\|} = \frac{1}{i} A\left(\frac{iv_i}{\|v_i\|}\right) + \beta_i \frac{g_o}{\|g_o\|} \geq T\left(\frac{v_i}{\|v_i\|}\right) + \frac{b}{i} + \beta_i \frac{g_o}{\|g_o\|} ,$$

where $v_i = u_{n_i}$ and

$$\beta_i = \frac{(\|v_i\| - n_i)^2}{\|v_i\|} \geq 0 ;$$

thus

(9.5) $$v_i \geq \|v_i\| \beta_i \frac{g_o}{\|g_o\|} .$$

Since T is a positive linear operator,

(9.6) $$T \frac{v_i}{\|v_i\|} \geq T^2 \frac{v_i}{\|v_i\|} + \frac{1}{i} Tb .$$

Using the fact that T is compact, we may assume that the sequence $\{u_{n_i}\}$ has been chosen so that $\{Tv_i / \|v_i\|\}$ converges to w ; from equations (9.5) and (9.2), $w > 0$. Passing to the limit in equation (9.6), we obtain

$$w \geq Tw .$$

Since T satisfies (PA) and does not have a positive eigenvector corresponding to the eigenvalue 1 , $w \geq Tw$ implies $\mu_o(T) > 1$, which contradicts the assumption of the theorem that $\mu_o(T) < 1$. Thus, there are no fixed points of A_n with $n \leq \|u_n\| \leq n^2$ for sufficiently large n .

Finally, suppose there is a sequence $\{u_{n_i}\}$ of fixed points of A_{n_i} such that $n_i^2 \leq \|u_{n_i}\| \leq n_i^3$. With $v_i = u_{n_i}$, we have

$$\frac{v_i}{\|v_i\|} = \frac{\gamma_i}{\|v_i\|} A_{v_i} + \delta_i g_o \geq \gamma_i\left(T \frac{v_i}{\|v_i\|} + b\right) + \delta_i g_o ,$$

where

$$0 \leq \gamma_i = \frac{n_i^3 - \|v_i\|}{n_i^3 - n_i^2} \leq 1 ,$$

$$0 \leq \delta_i = \frac{(n_i^2 - n_i)^2}{\|v_i\|} \geq \frac{(n_i - 1)^2}{n_i} \to \infty \quad \text{as } i \to \infty .$$

But

$$0 \leq \delta_i g_o \leq \frac{v_i}{\|v_i\|} - \gamma_i (T \frac{v_i}{\|v_i\|} + b)$$

implies

$$\delta_i = |\delta_i| \leq 1 + \|T\| + \|b\| ,$$

which contradicts $\delta_i \to \infty$ as $i \to \infty$. Thus, there are no fixed points of A_n with $n^2 \leq \|u_n\| \leq n^3$ for sufficiently large n.

Hence, for sufficiently large n, all fixed points u_n of A_n satisfy $r_o \leq \|u_n\| \leq n$, and they are therefore fixed points of A.

If $Au \leq T_o u$ for all u in C^{r_o}, then any fixed point u_o of A in C^{r_o} satisfies $u_o \leq T_o u_o$, and it follows as in the discussion of equation (9.4) that $\mu_o(T_o) \leq 1$. Thus, if $\mu_o(T_o) > 1$, A has no positive fixed points with norm less than r_o. //

If A is a compact positive monotonic forced operator on C with a positive fixed point u^o, then the preceding theorem may be used to deduce the existence of a second positive fixed point $> u^o$ by applying it to the operator A^o as described just before Theorem 9-1. The following corollaries are applications of this type of argument.

<u>9-2. Corollary.</u> Let $\{A_\lambda\}$, $\lambda \in J$, be a family of compact positive monotonic forced operators on C satisfying the conditions of the first paragraph of Theorem 7-1 for each $\lambda_o \in \Lambda_A$. For each

$\lambda \in \Lambda_A$, let T_λ be a compact positive linear operator satisfying (PA), such that there is a $g_\lambda \in C^+$ and an integer $m_\lambda \geq 1$ with $T_\lambda^{m_\lambda} h \geq \|h\| g_\lambda$ for every h for which there is a positive number $\alpha(h)$ with $h \geq \alpha(h) g_\lambda$. Suppose that for any number $\beta > 0$ and any $\lambda \in \Lambda_A$, there is a $b_{\lambda,\beta} \in \mathcal{B}$ and a number r_λ such that

(9.7) $$A_\lambda u \geq \beta T_\lambda u + b_{\lambda,\beta}$$

for all $u \in C$ with $\|u\| \geq r$.

Then, for any $\lambda \in \Lambda_A$, $\lambda > \inf \Lambda_A$, with $1 \neq \mu_o[A'(u^o(\lambda))]$, the operator A_λ has a second positive fixed point.

Proof. If $1 \neq \mu_o[A_\lambda'(u^o(\lambda))]$, it follows from Theorem 7-1 that $1 < \mu_o[A_\lambda'(u^o(\lambda))]$; therefore, condition (b_o) of Theorem 9-1 holds for the operator A_λ^o defined by $A_\lambda^o u = A_\lambda(u^o(\lambda)+u) - A_\lambda u^o(\lambda)$, since $A_\lambda^{o'}(0) = A_\lambda'(u^o(\lambda))$. Since $T_\lambda^{m_\lambda} g_\lambda \geq \|g_\lambda\| g_\lambda$, T_λ has a positive eigenvalue (Krein-Rutman 1950, Theorem 6.2), and therefore $\mu_o(\beta T_\lambda) = \beta^{-1}\mu_o(T_\lambda)$ can be made less than 1 for sufficiently large β. Equation (9.7) implies that $A_\lambda^o u = A_\lambda(u^o(\lambda)+u) - A_\lambda u^o(\lambda) \geq \beta T_\lambda h + [\beta T_\lambda u^o(\lambda) + b_{\lambda,\beta} - u^o(\lambda)]$; thus, A_λ^o also satisfies condition (a_∞) of Theorem 9-1 if we take $T = \beta T_\lambda$ for sufficiently large β. Therefore, A_λ^o has a positive fixed point and A_λ has a second positive fixed point. //

If we use condition (b_∞) instead of (a_∞), we obtain

9-3. Corollary. Let the family $\{A_\lambda\}$ satisfy the conditions of the preceding corollary. Then for any $\lambda \in \Lambda_A$, $\lambda > \inf \Lambda_A$, for which A_λ has a C-asymptotic derivative $A_\lambda'(\infty)$ having no positive eigenvector with eigenvalue 1, A_λ has at least two positive fixed points if

$$\mu_o[A_\lambda'(\infty)] < 1 < \mu_o[A_\lambda'(u^o(\lambda))] .$$

9-4. Corollary. Let A be a compact positive monotonic operator on $\mathcal{C}$, let $c \in \mathcal{C}$, and let $c+\lambda A0 > 0$ for $\lambda > 0$; for each $\lambda \in \Lambda_A = \{\lambda > 0 : \exists\, u \in \mathcal{C}^+ \ni u = c+\lambda Au\}$, let A have a Fréchet derivative $A'(u^o(\lambda))$ which satisfies (PA) and has a unique positive eigenvector of unit norm. Let T be a compact positive linear operator satisfying (PA) and the condition associated with equation (9.2) of Theorem 9-1. If there is a number $r_1 > 0$ such that for any number α there is a vector $b_\alpha \in \mathcal{B}$ so that

$$(9.8) \qquad Au \geq \alpha Tu + b_\alpha$$

for all $u \in \mathcal{C}$ with $\|u\| > r_1$, then for any $\lambda \in \Lambda_A$ with $\lambda \neq \mu_o[A'(u^o(\lambda))]$, the equation $u = c+\lambda Au$ has at least two positive solutions.

9-5. Corollary. Let A be as described in the first sentence of the preceding corollary. Let A have a compact $\mathcal{C}$-asymptotic derivative $A'(\infty)$ having no positive eigenvectors except those corresponding to the characteristic value $\mu_o[A'(\infty)]$ (assumed finite). Then for any $\lambda \in \Lambda_A$ such that $\lambda > \mu_o[A'(\infty)]$ and $\lambda \neq \mu_o[A'(u^o(\lambda))]$, the equation $u = c+\lambda Au$ has at least two positive solutions.

I. 10. Concave and Convex Operators

Let the operator A be defined on a convex set S. Consider the following inequality for elements $u \neq v$ of S:

(10.1) $$A(\alpha u + (1-\alpha)v) \leq \alpha Au + (1-\alpha)Av,$$

where $\alpha \in (0, 1)$. If equation (10.1) holds for all $\alpha \in (0, 1)$ and all u, v in S such that either $u < v$ or $v < u$, we say that A is convex in the direction of $\mathcal{C}$ on S; if (10.1) holds for all u, v in S, then A is convex on S. If (10.1) holds with $\leq$ replaced by $<$ for all $\alpha \in (0, 1)$, then the convexity of A is described as strict. If an operator A on S is such that $-A$ is convex (in the direction of $\mathcal{C}$) on S, then A is called concave (in the direction of $\mathcal{C}$) on S.

An (inhomogeneous) linear operator $A = c+T$, where T is linear and c is a constant vector ($\neq 0$), is both concave and convex. Thus, all results of this section on (forced) concave and convex positive monotonic operators are valid for positive operators $c+T$ (with $c > 0$).

The operator of Example 2-4 provides an example of an operator which is convex in the direction of $\mathcal{C}$, but not convex, on $\mathcal{C}$. Let

$$A\,u(x) = \int_x^1 u(y)u(y-x)dy$$

on $C[0, 1]$. Then

(10.2) $$A(\alpha u+(1-\alpha)v)(x)-\alpha Au(x)-(1-\alpha)Av(x)$$

$$= -\alpha(1-\alpha)\int_x^1 [u(y-x)-v(y-x)][u(y)-v(y)]dy \;.$$

Thus, if either $u > v$ or $u < v$, we have

$$A(\alpha u+(1-\alpha)v) \leq \alpha Au+(1-\alpha)Av$$

for $0 < \alpha < 1$, i.e., A is convex in the direction of $\mathcal{C}$. Suppose, however, we choose u, v such that

$$u(x) > v(x)\,,\quad 0 \le x < \tfrac{1}{2}\quad ;\quad u(x) < v(x)\,,\quad \tfrac{1}{2} < x \le 1$$

Then $u(y-\frac{1}{2})-v(y-\frac{1}{2}) > 0$ for $\frac{1}{2} < y \le 1$, and $u(y)-v(y) < 0$ for $\frac{1}{2} < y \le 1$, so

$$-\alpha(1-\alpha)\int_{\frac{1}{2}}^{1}[u(y-\tfrac{1}{2})-v(y-\tfrac{1}{2})][u(y)-v(y)]dy \;>\; 0$$

and therefore (see equation (10.2))

$$A(\alpha u+(1-\alpha)v) \nleq \alpha Au+(1-\alpha)Av \;.$$

The following propositions give some of the properties of differentiable convex or concave operators which will be useful in our later discussion of the set of fixed points of families of such operators.

<u>10-1. Proposition.</u> Let the operator A be Fréchet differentiable on an open convex set S. Then A is convex (in the direction of $\mathcal{C}$) on S if and only if

$$A'(u)(v-u) \le Av - Au \tag{10.3}$$

for every u, v in S (such that $u \le v$ or $v \le u$).

The convexity of A is strict on S if and only if

$$A'(u)(v-u) < Av - Au \tag{10.4}$$

for every u, v in S (such that $u < v$ or $v < u$).

Proof. If A is convex (in the direction of $\mathcal{C}$), then

$$\begin{aligned} A'(u)(v-u) &= \lim_{\alpha\to 0}\frac{A(u+\alpha h)-Au}{\alpha} \qquad\qquad (h = v-u)\\ &\le \lim_{\alpha\to 0}\frac{\alpha A(u+h)+(1-\alpha)Au-Au}{\alpha}\\ &\le Av - Au \end{aligned}$$

for any u, v in S (such that $u \le v$ or $v \le u$). (So far, we have used only the differentiability of A at u .)

Conversely, if (10.2) holds, then

$$(10.5) \quad \alpha[Au-A(\alpha u+(1-\alpha)v)] + (1-\alpha)[Av-A(\alpha u+(1-\alpha)v)]$$
$$\geq \alpha[A'(\alpha u+(1-\alpha)v)(u-\alpha u-(1-\alpha)v)]$$
$$+ (1-\alpha)[A'(\alpha u+(1-\alpha)v)(v-\alpha u-(1-\alpha)v)]$$
$$= 0 ,$$

whenever u, v are in S (and $u \leq v$ or $v \leq u$); this proves the convexity of A (in the direction of $\mathcal{C}$) on S.

If $A'(u)(v-u) = Av-Au$, then by equations (10.1) and (10.3), $(1-\alpha)(Av-Au) \geq A(\alpha u+(1-\alpha)v)-Au \geq A'(u)(1-\alpha)(v-u) = (1-\alpha)(Av-Au)$; thus, the existence of any $\alpha \in (0, 1)$ such that strict inequality holds in (10.1) implies equation (10.4); a fortiori, strict convexity implies (10.4). The converse follows by replacing $\geq$ in equation (10.5) by $>$. //

<u>10-2. Proposition.</u> Let the operator A be defined and Fréchet differentiable on an open convex set $S \subseteq \mathcal{B}$. If A is convex in the direction of $\mathcal{C}$ on S, then $A'(u+\alpha(v-u))(v-u)$ (which is defined at least for $\alpha \in [0, 1]$) is an increasing function of α for any u, v in S with $u < v$.

Conversely, if $A'(u+\alpha(v-u))(v-u)$ is a continuous increasing function of α whenever u, v are in S and $u \leq v$, then A is convex in the direction of $\mathcal{C}$ on S.

<u>Proof.</u> Define $\psi(\alpha) = A(u+\alpha(v-u))$ for u, v in S with $v > u$. Then $\psi'(\alpha) = A'(u+\alpha(v-u))(v-u)$. If A is convex, then ψ is convex, and by an easy generalization of the well-known result for real-valued functions (Choquet 1966), ψ' is an increasing function.

Conversely, if ψ' is increasing and continuous, then for

$0 \le \alpha \le 1$,

$$A(\alpha v+(1-\alpha)u)-Au = \alpha\int_0^1 \psi'(\beta\alpha)d\beta \le \alpha\int_0^1 \psi'(\beta)d\beta = \alpha[Av-Au] ,$$

which proves the convexity of A in the direction of $\mathcal{C}$. //

10-3. Proposition. Let the operator A be convex in the direction of $\mathcal{C}$ on $\mathcal{C}$ and $\mathcal{C}$-asymptotically linear. Then whenever $0 \le u \le v$,

$$Av-Au \le A'(\infty)(v-u) .$$

Proof. Let $h = v-u$, $\alpha > 1$. Then from equation (10. 1),

$$A(u+\frac{1}{\alpha}(\alpha h))-Au \le \frac{1}{\alpha}A(u+\alpha h)+(1-\frac{1}{\alpha})Au-Au = \frac{1}{\alpha}[A(u+\alpha h)-Au] .$$

If $h > 0$, we let $\alpha \to +\infty$ and obtain

$$A(u+h)-Au \le A'(\infty)h . \qquad //$$

10-4. Proposition. Let the operator A be Fréchet differentiable on $\mathcal{C}$, convex in the direction of $\mathcal{C}$, and $\mathcal{C}$-asymptotically linear. Then for any u and h in $\mathcal{C}$, we have $A'(u)h \le A'(\infty)h$.

Proof. From Proposition 10-3, for any $\alpha > 0$ we have

$$\frac{1}{\alpha}[A(u+\alpha h)-Au] \le A'(\infty)h .$$

Letting $\alpha \to 0$, we obtain

$$A'(u)h \le A'(\infty)h . \qquad //$$

Results analogous to Propositions 10-1 through 10-4 hold for concave operators.

In Theorem 7-3, the sign of $\xi(A''(u)\phi^2)$, where ξ is a positive linear functional and $\phi \in \mathcal{C}^+$, is important. For convex and concave operators, we can determine the sign of this quantity (see also Theorem 10-14).

10-5. Proposition. Let A be Fréchet differentiable on a convex neighborhood $\mathcal{N}$ of a point $u \in \mathcal{C}$ and twice Fréchet differentiable at u, and let $h \in \mathcal{C}$. If A is convex in the direction of $\mathcal{C}$ on $\mathcal{N}$, then $A''(u)h^2 \geq 0$; if A is concave on $\mathcal{N}$, then $A''(u)h^2 \leq 0$.

Proof. If, e.g., A is convex, then $A'(u+\alpha h)h$ is defined for sufficiently small $\alpha \geq 0$, $h > 0$, and is an increasing function of α by Proposition 10-2. Thus,

$$A''(u)h^2 = \frac{d}{d\alpha} A'(u+\alpha h)h \Big|_{\alpha=0} \geq 0$$

for sufficiently small positive h and consequently for all positive h. //

Our next result shows that the theory of Section I. 5 can be applied to monotonic g_o-bounded forced operators which are concave in the direction of $\mathcal{C}$.

10-6. Proposition. Let A be a monotonic forced operator on $\mathcal{C}^r$ which is concave in the direction of $\mathcal{C}$. Then for any number $\alpha \in (0, 1)$ and any $u \in \mathcal{C}^r$, $A(\alpha u) > \alpha Au$. If, in addition, A is g_o-bounded on $\mathcal{C}^r$ for some $g_o \in \mathcal{C}^+$, then for any $\alpha \in (0, 1)$ and $u \in \mathcal{C}^r$, there is a number $\eta > 0$ such that $A(\alpha u) \geq \alpha(1+\eta)Au$.

Proof. From the concavity in the direction of $\mathcal{C}$, we have

$$A(\alpha u) \geq \alpha Au+(1-\alpha)A0 > \alpha Au$$

since A is forced. If A is g_o-bounded on $\mathcal{C}^r$, then there are positive numbers β and β' such that $A0 \geq \beta g_o \geq \beta' Au$, and therefore

$$A(\alpha u) \geq \alpha Au+(1-\alpha)\beta' Au = \alpha[1+\alpha^{-1}(1-\alpha)\beta']Au . \qquad //$$

The theory of Section I. 5 also applies in certain cases to convex operators. For any operator A which is Fréchet differentiable at u, we define $\hat{A}u = Au - A'(u)u$. (If A is a real-valued function of a real variable, then $\hat{A}u$ is the intersection with the vertical axis of the tangent to the graph of A.) If A is convex in the direction of $\mathcal{C}$ and Fréchet differentiable, then, using Proposition 10-1, we have

$$A(\alpha u) - \alpha Au \geq A'(u)(\alpha u - u) + (1-\alpha)Au = (1-\alpha)\hat{A}u ,$$

if $0 \leq \alpha \leq 1$. This gives us the following:

<u>10-7. Proposition.</u> Let the operator A be convex in the direction of $\mathcal{C}$ and Fréchet differentiable on $\mathcal{C}^r$. If $\hat{A}$ is a positive operator on $\mathcal{C}^r$, then $A(\alpha u) \geq \alpha Au$ for any u in $\mathcal{C}^r$ and any $\alpha \in [0,1]$. If, in addition, both A and $\hat{A}$ are g_o-bounded on $\mathcal{C}^+$, then for any u in $\mathcal{C}^r$ and any $\alpha \in (0,1)$, there exists a positive number η such that $A(\alpha u) \geq \alpha(1+\eta)Au$.

Theorem 5-4 implies that an operator A satisfying all the hypotheses of Propositions 10-6 or 10-7 has at most one positive fixed point in $\mathcal{C}^r$. We can also prove uniqueness for differentiable concave operators or convex operators for which $\hat{A}$ is strictly positive if we replace the requirement that A (and $\hat{A}$) be g_o-bounded by the requirement that the derivatives $A'(u)$ satisfy (PA). The following simple lemma is the key to the uniqueness proofs.

<u>10-8. Lemma.</u> Let A be a positive monotonic operator on $\mathcal{C}^r$ which has a positive fixed point $u \in \mathcal{C}^r$. Let the Fréchet derivative $A'(u)$ exist and satisfy (PA). If $\hat{A}u > 0$, then $1 < \mu_o[A'(u)]$. If $\hat{A}u \geq 0$ and $A'(u)$ has positive eigenvectors only for the characteristic

value $\mu_o[A'(u)]$, then $1 \leq \mu_o[A'(u)]$.

Proof. If $u = Au \in \mathcal{C}^{r+}$ and $\hat{A}u = u - A'(u)u > 0$; then it follows from (PA) that $1 < \mu_o[A'(u)]$. If $\hat{A}u \geq 0$, then either $u - A'(u)u > 0$ (whence $1 < \mu_o[A'(u)]$ by (PA)), or $u - A'(u)u = 0$, so $1 = \mu_o[A'(u)]$ if $A'(u)$ has positive eigenvectors only for the characteristic value $\mu_o[A'(u)]$. //

10-9. Theorem. Let A be a monotonic forced operator on $\mathcal{C}^r$ which is concave in the direction of $\mathcal{C}$ and has a fixed point $u \in \mathcal{C}^r$. Let $A'(u)$ exist and satisfy (PA). Then A has no fixed point greater than u in $\mathcal{C}^r$.

Proof. Since A is concave in the direction of $\mathcal{C}$ and Fréchet differentiable at u, $\hat{A}u = Au - A'(u)u \geq A0 > 0$ (cf. Proposition 10-1). By Lemma 10-8, $1 < \mu_o[A'(u)]$. If A had a fixed point $v \geq u$, then

$$u - v = Au - Av \geq A'(u)(u-v)$$

(Proposition 10-1). The argument used in the proof of our Theorem 4-13 (Schaefer 1966, Appendix 2.3) shows that $1 < \mu_o[A'(u)]$ implies $u - v \geq 0$, so $u = v$. //

10-10. Theorem. Let A be a monotonic forced operator on $\mathcal{C}^r$ which is convex in the direction of $\mathcal{C}$ and has a fixed point $u_o \in \mathcal{C}^r$. For any fixed point u of A, let $A'(u)$ exist and satisfy (PA), and let $\hat{A}u > 0$. Then A has no fixed point in $\mathcal{C}^r$ greater than u_o.

If, instead of $\hat{A}u > 0$, we have $\hat{A}u \geq 0$ for any fixed point $u \in \mathcal{C}^r$, if $A'(u)$ has positive eigenvectors only for the characteristic value $\mu_o[A'(u)]$, and if A is strictly convex in the direction of $\mathcal{C}$, then A has no fixed point in $\mathcal{C}^r$ greater than u_o.

Proof. The proof is similar to that of Theorem 10-9, using Proposition 10-1 and Lemma 10-8. //

If the operator A of Theorem 10-9 or 10-10 has a minimal positive fixed point (Theorem 4-4), then it is the only fixed point of A in C^r.

Propositions 10-1 through 10-4 can be used with Theorem 4-5 and its corollaries to obtain better estimates of the size of Λ_A^r for differentiable concave or convex operators $\{A_\lambda\}$. Assuming that the indicated derivatives exist, we have

(10.6) $$A'(v)u+\hat{A}v \le Au \le A'(\infty)(u-v)+Av$$

for convex A, and

(10.7) $$A'(\infty)(u-v)+Av \le Au \le A'(v)u+\hat{A}v$$

for concave A.

10-11. Theorem. Let A be a forced operator satisfying (H) on C^r.

If A is convex (in the direction of C) and has a fixed point u in C^r, and if there is a v ($\ge u$ or $\le u$) in C^r with $\hat{A}v > 0$ such that $A'(v)$ exists and satisfies (PA), then $1 < \mu_o[A'(v)]$.

Let $r = \infty$.

If A is convex in the direction of C and C-asymptotically linear, then $\mu_o[A'(\infty)] > 1$ implies that A has exactly one fixed point in C.

If A is concave in the direction of C and has a fixed point in C, and if $A'(\infty)$ exists and satisfies (PA), then $1 < \mu_o[A'(\infty)]$.

If A is concave and there is a $v \in C$ such that $A'(v)$ exists and

$\mu_o[A'(v)] > 1$, then A has a fixed point in $\mathcal{C}$.

Proof. Since $A'(v)$ and $A'(\infty)$ satisfy (H) on $\mathcal{C}$ by Propositions 6-1 and 6-2, the theorem follows from the inequalities (10.6) and (10.7) and Corollaries 4-7, 4-8, and 4-15. //

Theorem 4-5 and its corollaries also provide iteration procedures for obtaining fixed points of A , starting with the solution h of one of the linear equations

$$A'(v)h + Av = h$$

or

$$A'(\infty)h + A0 = h ,$$

and using the inequalities (10.6) and (10.7).

10-12. Corollary. Let A be a positive monotonic operator satisfying (H) on $\mathcal{C}^r$, let $A_\lambda = c+\lambda A$, $\lambda > 0$, where c is a constant vector in $\mathcal{C}^r$, and suppose that $A_\lambda 0 > 0$.

If A is convex (in the direction of $\mathcal{C}$) and $\lambda \in \Lambda_A^r$, and if there is a v ($\geq u^o(\lambda)$) in $\mathcal{C}^r$ such that $A'(v)$ exists and satisfies (PA) and $\hat{A}v > 0$, then $\lambda < \mu_o[A'(v)]$; thus, $\sup \Lambda_A^r \leq \mu_o[A'(v)]$.

Let $r = \infty$.

If A is convex in the direction of $\mathcal{C}$ and $\mathcal{C}$-asymptotically linear, then $\mu_o[A'(\infty)] > \lambda > 0$ implies $\lambda \in \Lambda_A$; thus, $\sup \Lambda_A \geq \mu_o[A'(\infty)]$. A_λ has exactly one positive fixed point for each $\lambda \in (0, \mu_o[A'(\infty)])$.

If A is concave in the direction of $\mathcal{C}$, $\mathcal{C}$-asymptotically linear, and $A'(\infty)$ satisfies (PA), then $\lambda \in \Lambda_A$ implies $\lambda < \mu_o[A'(\infty)]$; thus, $\sup \Lambda_A = \mu_o[A'(\infty)]$.

If A is concave and there is a v in $\mathcal{C}$ such that $A'(v)$ exists, then $\mu_o[A'(v)] > \lambda > 0$ implies $\lambda \in \Lambda_A$; thus, $\sup \Lambda_A \geq \mu_o[A'(v)]$.

Proof. For concave operators, sup $\Lambda_A \leq \mu_o[A'(\infty)]$ follows from Theorem 10-11, and then sup $\Lambda_A = \mu_o[A'(\infty)]$ follows from the discussion at the end of Section I. 4, Theorem 3-5, and Theorem 8-1. The remaining assertions are immediate consequences of Theorem 10-11. //

In Theorem 7-1 we have shown that for a minimal positive fixed point $u^o(\lambda)$ of an operator A_λ we have $1 \leq \mu_o[A_\lambda'(u^o(\lambda))]$. If A_λ is convex, we have the following partial converse of this result.

10-13. Theorem. Let A be convex and monotonic on C^r and have a fixed point u in C^r such that the Fréchet derivative $A'(u)$ exists and $1 < \mu_o[A'(u)]$. Then u is less than any other fixed point of A in C^r.

Proof. Let u be any fixed point of A in C^r. Using Proposition 10-1, we have

$$u_1 - u = Au_1 - Au \geq A'(u)(u_1 - u) .$$

Since $1 < \mu_o[A'(u)]$, $u_1 - u \geq 0$ (cf. the proofs of Theorems 4-13 and 10-10). //

From Proposition 10-1, Theorem 10-13, and assumption (PA), we obtain:

10-14. Theorem. Let the monotonic operator A on C^r have two fixed points $u_1 < u_2$ in C^r and have a Fréchet derivative $A'(u_1)$ at u_1 which satisfies (PA). If A is strictly convex in the direction of C on C^r, then $1 < \mu_o[A'(u_1)]$, and thus u_1 is less than any other fixed point of A. If A is convex in the direction of C on C^r and $A'(u_1)$ has positive eigenvectors only for the characteristic value

$\mu_o[A'(u_1)]$, then $1 \le \mu_o[A'(u_1)]$.

Under the conditions of Theorem 7-3, if $1 = \mu_o[A'_{\lambda_o}(u^o(\lambda_o))]$ for for some $\lambda_o \in \Lambda_A^r$, then there are no positive fixed points of A_λ close to $u^o(\lambda_o)$ for $\lambda > \lambda_o$. The next theorem shows that if the operators A_λ are convex, then such a λ_o must in fact be the maximum of Λ_A^r.

<u>10-15. Theorem</u>. Let $\{A_\lambda\}$, $\lambda \in J$, be a strictly increasing family of forced operators which satisfy (H) on $\mathcal{C}^r$ for each $\lambda \in J$. Suppose that there is $\lambda^* \in \Lambda_A^r$ with a corresponding minimal positive fixed point $u^* = u^o(\lambda^*)$ such that A_{λ^*} is convex in the direction of $\mathcal{C}$ and has a Fréchet derivative $A'_{\lambda^*}(u^*)$ satisfying (PA_1), with $1 = \mu_o[A'_{\lambda^*}(u^*)]$. Then λ^* is the maximum element of Λ_A^r. If, in addition, A_{λ^*} is strictly convex in the direction of $\mathcal{C}$, then u^* is the only positive fixed point of A_{λ^*}. If, for each $\lambda \in \Lambda_A^r$, A_λ is convex and has a Fréchet derivative $A'_\lambda(u(\lambda))$ satisfying (PA_2) at any positive fixed point $u(\lambda)$, then there is at most one number λ such that $1 = \mu_o[A'_\lambda(u(\lambda))]$, and such a λ is the maximum of Λ_A^r.

<u>Proof</u>. Suppose there were a $\lambda > \lambda^*$ in Λ_A^r ; let $u_o = u^o(\lambda)$ be the corresponding minimal positive fixed point. Then $u_o > u^*$ and from Proposition 10-1 ,

$$u_o - u^* - A'_{\lambda^*}(u^*)(u_o - u^*) \ge A_\lambda u_o - A_{\lambda^*} u_o \quad . \tag{10.8}$$

Since $\{A_\lambda\}$ is a strictly increasing family, the right hand side of (10.8) is positive; this violates condition (PA_1). If A_{λ^*} has a second positive fixed point u_o and is strictly convex in the direction of $\mathcal{C}$, then the left hand side of (10.8) is positive, which again contradicts condition (PA_1).

If the derivatives $A'_\lambda(u(\lambda))$ each satisfy (PA_2), then an argument similar to that used in connection with equation (10.8) shows that any number λ^* such that there is a corresponding fixed point $u(\lambda^*)$ with $1 = \mu_o[A'_{\lambda^*}(u(\lambda^*))]$ (whether or not it is the minimal positive fixed point) is the maximum of Λ^r_A, and hence there is only one such λ^*. //

If the operators A_λ are unforced and we take $u^* = 0$ in the hypotheses of Theorem 10-10, then the proof shows that $\lambda^* \geq \sup \Lambda^r_A$, where $\Lambda^r_A = \{\lambda \in J : \exists\ u \in C^r \ni u = A_\lambda u > 0\}$.

From Theorems 10-13, 10-14, and 10-15, we obtain:

<u>10-16. Theorem.</u> Let $\{A_\lambda\}$, $\lambda \in J$, be a strictly increasing family of forced operators which satisfy (H) on C^r and are convex in the direction of C. Let A_λ have a Fréchet derivative $A'_\lambda(u(\lambda))$ at each positive fixed point $u(\lambda)$ of A_λ, $\lambda \in \Lambda^r_A$, and let $A'_\lambda(u(\lambda))$ satisfy (PA). If, for some $\lambda \in \Lambda^r_A$ with $\lambda < \sup \Lambda^r_A$, A_λ has a fixed point $u^{(1)}(\lambda) > u^o(\lambda)$, then A_λ does not have a third fixed point $u^{(2)}(\lambda)$ such that $u^{(2)}(\lambda) > u^{(1)}(\lambda)$ if either of the following conditions is satisfied:

(a) A_λ is strictly convex in the direction of C;

(b) $A'_\lambda(u(\lambda))$ satisfies (PA_2) and has positive eigenvectors only for the characteristic value $\mu_o[A'_\lambda(u(\lambda))]$.

If $\lambda^+ > \lambda^* = \sup \Lambda^r_A \in \ {}^r_A$, the operators A_λ are Fréchet differentiable on C^r for λ in a neighborhood N of λ^*, the mapping $(\lambda, u) \to A'_\lambda(u)$ is continuous for $\lambda \in N$ and $u \in C^r$, and all derivatives $A'_{\lambda^*}(u)$, $u \in C^r$, satisfy (PA_2), then $\lambda^* = \max \Lambda^r_A$, and the existence of two fixed points $u^{(1)}(\lambda^*) = u_1$ and $u^o(\lambda^*) = u_o$ implies that all vectors $u_\alpha \equiv \alpha u_1 + (1-\alpha)u_o$, $0 \le \alpha \le 1$, are fixed points of A_{λ^*}; moreover, in this case, all vectors $\hat{A}u_\alpha$ are neither positive nor negative.

Proof. If $\lambda < \sup \Lambda_A^r$, then by Theorem 10-15, $1 < \mu_o[A_\lambda'(u^o(\lambda))]$. If A_λ had two non-minimal fixed points $u^{(2)}(\lambda) > u^{(1)}(\lambda)$, then $1 < \mu_o[u^{(1)}(\lambda)]$ by Theorem 10-14 if (a) is satisfied, and by Theorem 10-13, $u^{(1)}(\lambda)$ is minimal, which contradicts the assumption $u^{(1)}(\lambda) > u^o(\lambda)$. If (b) is satisfied, then Theorem 10-14 implies that $1 \le \mu_o[A_\lambda'(u^{(1)}(\lambda))]$; since $A_\lambda'(u^{(1)}(\lambda))$ satisfies (PA_2), Theorem 10-15 implies $1 < \mu_o[A_\lambda'(u^{(1)}(\lambda))]$, and we reach a contradiction to Theorem 10-13 as before.

If $\lambda^+ > \lambda^* = \sup \Lambda_A^r \in \Lambda_A^r$ and the mapping $(\lambda, u) \to A_\lambda'(u)$ is continuous, then the implicit function theorem implies that $1 = \mu_o[A_{\lambda^*}(u_o)]$. For convenience, we will denote A_{λ^*} by A for the rest of the proof. If A has two fixed points u_1, u_o, then

$$u_1 - u_o = Au_1 - Au_o = \int_0^1 A'(u_\alpha)(u_1-u_o)d\alpha \geq A'(u_o)(u_1-u_o)$$

by Proposition 10-1. By (PA_2), we cannot have inequality here, and thus

$$u_1 - u_o = \int_0^1 A'(u_\alpha)(u_1-u_o)d\alpha = A'(u_o)(u_1-u_o) \quad . \tag{10.9}$$

By Proposition 10-2, $A'(u_\alpha)(u_1-u_o)$ is an increasing function of α; if for some $\alpha_o \in (0,1)$, $A'(u_{\alpha_o})(u_1-u_o) > A'(u_o)(u_1-u_o)$, then

$$\int_0^1 A'(u_\alpha)(u_1-u_o)d\alpha \geq \int_0^{\alpha_o} A'(u_o)(u_1-u_o)d\alpha + \int_{\alpha_o}^1 A'(u_{\alpha_o})(u_1-u_o)d\alpha$$
$$> A'(u_o)(u_1-u_o),$$

which contradicts equation (10.9). Thus, for $\alpha \in [0,1)$,

$$A'(u_o+\alpha(u_1-u_o))(u_1-u_o) = A'(u_o)(u_1-u_o),$$

and by continuity,

(10.10) $$u_1-u_o = A'(u_\alpha)(u_1-u_o) = Au_1-Au_o.$$

for $0 \le \alpha \le 1$. By Proposition 10-1,

$$A'(u_\alpha)\alpha(u_1-u_o) \ge Au_\alpha-Au_o \ge A'(u_o)\alpha(u_1-u_o),$$

so from equation (10.10),

$$Au_\alpha = Au_o+\alpha A'(u_o)(u_1-u_o) = u_o+\alpha(u_1-u_o) = u_\alpha .$$

We also have

$$u_o = u_\alpha-\alpha A'(u_o)(u_1-u_o) = u_\alpha-\alpha A'(u_\alpha)(u_1-u_o) = A'(u_\alpha)u_o+\hat{A}u_\alpha ;$$

since $1 = \mu_o[A'(u_\alpha)]$ by the implicit function theorem, assumption (PA_2) for $A'(u_\alpha)$ implies that we cannot have $\hat{A}u_\alpha > 0$ or $\hat{A}u_\alpha < 0$ (we must have, in fact, $\xi_\alpha(\hat{A}u_\alpha) = 0$ for any eigenvector ξ_α of the adjoint of $A'(u_\alpha)$ corresponding to the characteristic value $\mu_o[A'(u_\alpha)]$). //

The proof of the theorem shows that any fixed point of A_{λ^*} must have the form $u^o(\lambda^*)+\alpha\phi$, where $\alpha \ge 0$ and ϕ is a positive eigenvector of $A'_{\lambda^*}(u^o(\lambda^*))$ corresponding to the characteristic value $1 = \mu_o[A'_{\lambda^*}(u^o(\lambda^*))]$.

If A is convex and has a minimal positive fixed point u^o, then the operator A^o defined by $A^oh = A(u^o+h)-Au^o$ satisfies

$$A^o(\alpha h) \le \alpha A^oh$$

for any $\alpha \in [0,1]$, $h \in C^r-u^o \equiv \{u\in C : u^o+u\in C^r\}$. If, for any $\alpha \in (0,1)$, $h\in C^r-u^o$, there is a number $\eta > 0$ such that

$$A^{o}(\alpha h) \le \alpha(1-\eta)A^{o}h \; ,$$

then one may establish results similar to those of Theorem 10-16 on the non-existence of non-minimal positive fixed points $u^{(2)} > u^{(1)}$ by using Lemma 7-4 (Krasnosel'skii 1964a, §§ 6.3.2, 7.1.11). We shall apply this type of argument in Section II. 2 to the eigenfunctions of nonlinear ordinary differential equations.

In Part II we shall see that when A is the Hammerstein integral operator of Example 2-1, and the kernel is the Green's function of certain boundary value problems for ordinary differential equations, then the convexity in u of $f(x, u)$ implies that A has at most two eigenvectors for any characteristic value λ ; if, moreover, $\lim_{u \to +\infty} f(x, u)/u = \infty$, then A has exactly two eigenvectors for each λ in the open interval $(0, \sup \Lambda_A)$. We shall now give two examples to illustrate other types of behavior for convex operators.

Let f be a continuous function on $[0, 1] \times [0, r)$, $0 < r \le \infty$, and let $f(x, u)$ be convex and monotonic in u . Then the operator A defined by

$$Au(x) = \int_0^x f(y, u(y))dy \; ,$$

is a compact positive monotonic convex operator on $\mathbf{C}^r \subseteq C[0, 1]$ (Example 2-1), and $u \in \mathbf{C}^r$ is a fixed point of A if and only if $u'(x) = f(x, u(x))$, $u(0) = 0$. Since $f(x, u)$ is convex in u ,

$$0 \le f(x, u) - f(x, v) \le f_u(x, u+)(u-v) \; ,$$

where $f_u(x, u+)$ is the partial derivative from the right of $f(x, u)$ with respect to u (Choquet 1966). If for all positive numbers $r_1 < r$,

$f_u(x, u+)$ is bounded for $0 \le x \le 1$, $0 \le u \le r_1$, then from the well-known uniqueness theory for the initial value problem for ordinary differential equations (Ince 1956), A has at most one fixed point in C^r. This uniqueness result also follows from Corollary 4-15, since we have

$$Au \le T(r_1)u$$

for all $u \in C^{r_1}$, where $T(r_1)$ is the linear operator

$$T(r_1)u(x) = \int_0^x M(r_1)u(y)dy$$

and $M(r_1) = \sup\{f_u(x, u+) : 0 \le x \le 1,\ 0 \le u \le r_1\}$. The operator $T(r_1)$ has $\mu_o[T(r_1)] = +\infty$, since $\mu T(r_1)u(x) = u(x)$ implies $u(x) = 0$, $0 \le x \le 1$, for any finite number μ. Thus, the operators λA have at most one fixed point in C^{r_1} for any $r_1 < r$, and thus at most one fixed point in C^r, for each $\lambda > 0$.

As a particular example, consider the Riccati differential equation

$$u'(x) = \lambda(1+u^2(x)), \quad 0 \le x \le 1,$$

with the initial condition

$$u(0) = 0 .$$

This can be written as the integral equation

$$u(x) = A_\lambda u(x) = \lambda Au(x) = \lambda\int_0^x [1+u^2(y)]dy, \quad 0 \le x \le 1 .$$

It is easy to verify that this equation has continuous positive solutions only for $0 < \lambda < \pi/2$, and that the solutions are

$$u^o(\lambda;x) = \tan(\lambda x) .$$

Thus, for each $\lambda \in \Lambda_A = (0, \pi/2)$, there is exactly one eigenfunction

$u^o(\lambda)$, and $\lim_{\lambda \uparrow \pi/2} \|u^o(\lambda)\| = \infty$.

For any function $u \in C$, the forced operator A has the compact positive Fréchet derivative $A'(u) : C[0,1] \to C[0,1]$ given by

$$A'(u)h(x) = 2\int_0^x u(y)h(y)dy \quad .$$

Clearly, there are no eigenfunctions h such that $h = \mu A'(u)h$ for finite μ ; thus, $\mu_o[A'(u)] = \infty$, and $A'(u)$ satisfies (PA) and (PA_2).

The operator $\hat{A}$ is given by

$$\hat{A}u(x) = \int_0^x [1-u^2(y)]dy \quad .$$

This is not a positive operator on C, yet the eigenfunctions of A are unique for each $\lambda \in \bar{\Lambda}_A$; thus, the converse of Theorem 10-10 does not hold. For the eigenfunctions $u^o(\lambda)$ we have

$$\hat{A}u^o(\lambda;x) = 2x - \lambda^{-1}\tan\lambda x \quad .$$

The operator A is not C-asymptotically linear, for if it had a C-asymptotic derivative $A'(\infty)$, then $\lim_{\alpha \to +\infty} \alpha^{-1}A(\alpha h) = A'(\infty)h$ for any function $h \in C$. But $\lim_{\alpha \to +\infty} \alpha^{-1}A(\alpha h)(x) = \infty$ for all $x \in (0,1]$. This shows that the existence of a C-asymptotic derivative is not a necessary condition for the existence of a number $\mu_1 > 0$ such that $\lim_{\lambda \to \mu_1} \|u(\lambda)\| = \infty$ (cf. Corollary 8-2).

We shall now give an example of a convex forced operator which has four eigenvectors for small values of λ ; this is a modification of an example given by Krasnosel'skii (1964a, p. 206).

Let $\mathcal{B} = \mathcal{R}^2$, i.e., the vector space of two-component vectors, and take as the positive cone C the first quadrant, i.e.,

$$\mathbb{C} = \{(\xi_1, \xi_2) \in \mathbb{R}^2 : \xi_1 \geq 0 , \ \xi_2 \geq 0\} .$$

Take

$$A(\xi_1, \xi_2) = (\xi_1^2 + \xi_2 + \epsilon^2 a^2, \ \epsilon^{-1}\xi_2^2 + \epsilon^2 \xi_1 + \epsilon^3 a^2) ,$$

where ϵ and a are arbitrary fixed positive numbers. It is easily verified that A is strictly convex on $\mathbb{C}$. The eigenvectors of A satisfying $x = \lambda Ax$ are $x^o(\lambda) = (\xi_1^o(\lambda), \xi_2^o(\lambda))$ defined by

$$\xi_1^o(\lambda) = \tfrac{1}{2}[(\lambda^{-1}-\epsilon) - \sqrt{(\lambda^{-1}-\epsilon)^2 - 4a^2\epsilon^2}\ ] \ ,$$

$$\xi_2^o(\lambda) = \epsilon \xi_1^o(\lambda)$$

for $0 < \lambda \leq \lambda^* \equiv (\epsilon + 2a\epsilon)^{-1}$; $x^{(1)}(\lambda) = (\xi_1^{(1)}(\lambda), \xi_2^{(1)}(\lambda))$ defined by

$$\xi_1^{(1)}(\lambda) = \tfrac{1}{2}[(\lambda^{-1}-\epsilon) + \sqrt{(\lambda^{-1}-\epsilon)^2 - 4a^2\epsilon^2}\ ]$$

$$\xi_2^{(1)}(\lambda) = \epsilon \xi_1^{(1)}(\lambda)$$

for $0 < \lambda \leq \lambda^*$; and $x^{(\pm)}(\lambda) = (\xi_1{}^{(\pm)}(\lambda), \xi_2{}^{(\pm)}(\lambda)$ defined by

$$\xi_1{}^{(\pm)}(\lambda) = \tfrac{1}{2}[(\lambda^{-1}+\epsilon) \pm \sqrt{(\lambda^{-1}+\epsilon)(\lambda^{-1}-3\epsilon) - 4a^2\epsilon^2}\ ]$$

$$\xi_2{}^{(\pm)}(\lambda) = \epsilon \xi_1{}^{(\mp)}(\lambda)$$

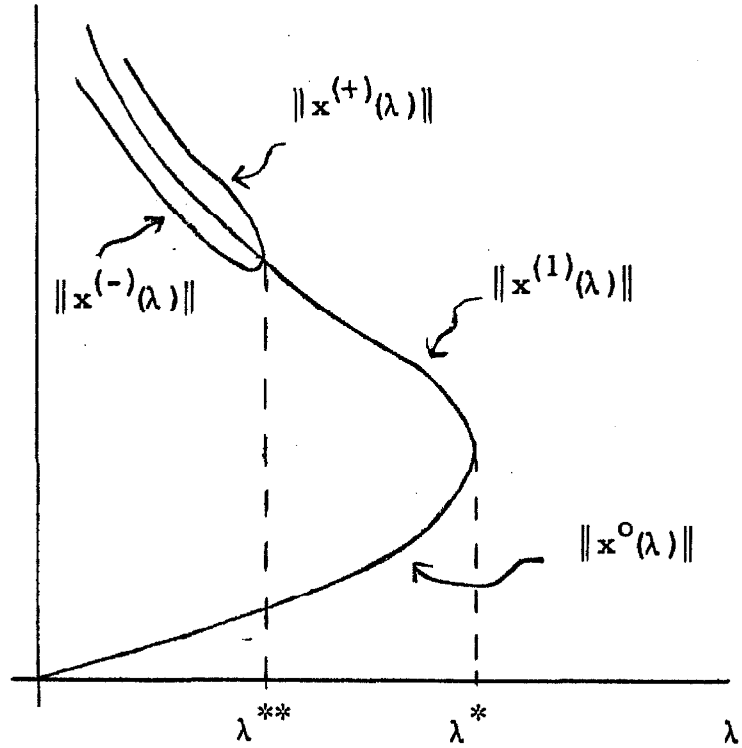
for $0 < \lambda \quad \lambda^{**} \equiv \epsilon^{-1}[1 + 2\sqrt{1+a^2}\,]^{-1} < \lambda^*$. (These eigenvectors may be obtained by making the change of variables $\eta_1 = \epsilon^{-1}\xi_1$, $\eta_2 = \epsilon^{-2}\xi_2$, $\overline{\lambda} = \epsilon\lambda$ in the equations $(\xi_1, \xi_2) = \lambda A(\xi_1, \xi_2)$. The equations in η_1, η_2 so obtained are the equations for determining the eigenvectors of A when $\epsilon = 1$.) Notice that

$$\lim_{\lambda \uparrow \lambda^*} x^o(\lambda) = \lim_{\lambda \uparrow \lambda^*} x^{(1)}(\lambda) = x^o(\lambda^*) = x^* = \epsilon a(1, \epsilon)$$

and

$$\lim_{\lambda \uparrow \lambda^{**}} x^{(+)}(\lambda) = \lim_{\lambda \uparrow \lambda^{**}} x^{(-)}(\lambda) = \lim_{\lambda \to \lambda^{**}} x^{(1)}(\lambda) = x^{**} = \epsilon[1 + \sqrt{1+a^2}\,](1, \epsilon) .$$

If we define $\|(\xi_1, \xi_2)\| = \max(\xi_1, \xi_2)$, and assume $0 < \epsilon \leq 1$, then a graph of the norms of the eigenvectors against λ has roughly the appearance shown:



If we represent the elements of $\mathbb{R}^2$ by column vectors, the Fréchet derivative $A'(x)$ of the operator A at any point $x = (\xi_1, \xi_2) \in \mathbb{R}^2$ is the linear operator which is represented by the matrix

$$A'(x) = \begin{pmatrix} 2\xi_1 & 1 \\ \varepsilon^2 & 2\varepsilon^{-1}\xi_2 \end{pmatrix}.$$

The second Fréchet derivative $A''(x)$ is a bilinear operator on $\mathbb{R}^2$; for any vector $d = (\delta_1, \delta_2) \in \mathbb{R}^2$, $A''(x)$ generates the linear operator $A''(x)d$ represented by

$$A''(x)d = \begin{pmatrix} 2\delta_1 & 0 \\ 0 & 2\delta_2\varepsilon^{-1} \end{pmatrix}.$$

The eigenvalues of $A'(x)$ $(A'(x)h = \mu^{-1}h)$ are

$$\mu^{-1} = \xi_1+\epsilon^{-1}\xi_2 \pm \sqrt{(\xi_1-\epsilon^{-1}\xi_2)^2+\epsilon^2} ;$$

for the solutions $x^o(\lambda)$ and $x^{(1)}(\lambda)$, $\xi_1 = \epsilon^{-1}\xi_2 = \|x\|$, so

$$\mu_1[x^o(\lambda)]^{-1} = 2\|x^o(\lambda)\|+\epsilon = \mu_o[A'(x^o(\lambda))]^{-1}$$

$$\mu_2[x^o(\lambda)]^{-1} = 2\|x^o(\lambda)\|-\epsilon ,$$

and similarly for $x^{(1)}(\lambda)$. Thus,

$$\mu_1[x^o(\lambda)]^{-1} = \lambda^{-1} - \sqrt{(\lambda^{-1}-\epsilon)^2-4a^2\epsilon^2} < \lambda^{-1}$$

except for $\lambda = \lambda^*$, when $\mu_1[x^o(\lambda^*)] = \lambda^*$ (see Theorem 7-1). Also,

$$\mu_1[x^{(1)}(\lambda)]^{-1} = \lambda^{-1} + \sqrt{(\lambda^{-1}-\epsilon)^2-4a^2\epsilon^2} > \lambda^{-1}$$

for $0 < \lambda < \lambda^*$ (see Theorem 10-13).

The eigenvectors h_1 and h_2 of $A'[x^o(\lambda)]$ corresponding to the eigenvalues $\mu_1[x^o(\lambda)]$, $\mu_2[x^o(\lambda)]$ are

$$h_1 = (1, \epsilon) , \quad h_2 = (1, -\epsilon) ,$$

respectively, and the linear functional (which we have denoted by ξ in Theorem 7-3) corresponding to the positive eigenvector h_1 is the mapping

$$z = (\zeta_1, \zeta_2) \to (\epsilon\zeta_1+\zeta_2)/2\epsilon = h_1^* \cdot z ,$$

where $h_1^* = (\epsilon, 1)/2\epsilon$ is an eigenvector of the transposed matrix $A'(x^o(\lambda))^t$ corresponding to the characteristic value $(2\|x^o(\lambda)\|+\epsilon)^{-1}$, and $h_1^* \cdot z$ denotes the dot product of h_1^* and z. Thus, any vector $z=(\zeta_1, \zeta_2) \in \mathbb{R}^2$ has the representation

$$z = (h_1^* \cdot z)h_1 + Pz ,$$

where

$$P = \frac{1}{2}\begin{pmatrix} 1 & -\epsilon^{-1} \\ -\epsilon & 1 \end{pmatrix}$$

(see equation (7.8)). It is straightforward to verify that

$$\lim_{\lambda\uparrow\lambda^*} \frac{x^*-x^o(\lambda)}{\sqrt{\lambda^*-\lambda}} = \lim_{\lambda\uparrow\lambda^*} \frac{x^{(1)}(\lambda)-x^*}{\sqrt{\lambda^*-\lambda}} = \left\{2\,\frac{h_1^*\cdot x^*}{h_1^*\cdot A''(x^*)h_1h_1}\right\}^{\frac{1}{2}} h_1$$

$$(10.10) \qquad\qquad = \sqrt{a\epsilon}\; h_1$$

(Theorem 7-3).

For the non-minimal eigenvectors $x^{(1)}(\lambda)$, the second characteristic value $\mu_2[x^{(1)}(\lambda)]$ is related to λ by

$$\mu_2[x^{(1)}(\lambda)]^{-1} = \lambda^{-1}-2\epsilon+\sqrt{(\lambda^{-1}-\epsilon)^2-4a^2\epsilon^2}$$

so

$$\mu_2[x^{(1)}(\lambda)] \begin{cases} >\lambda & \text{for } \lambda^{**}<\lambda\le\lambda^* \\ =\lambda & \text{for } \lambda=\lambda^{**} \\ <\lambda & \text{for } 0<\lambda<\lambda^{**}\ . \end{cases}$$

As predicted by the implicit function theorem 6-5 and Theorem 10-15, the bifurcation point at $\lambda = \lambda^{**} < \lambda^*$ occurs when λ is a characteristic value of $A'[x^{(1)}(\lambda)]$ different from $\mu_o[A'(x^o(\lambda))]^{-1}$, i.e., $\lambda = \mu_2[x^{(1)}(\lambda)]$; this case is not covered by the theory of Section I. 7. If h_2^* is the eigenvector of $A'(x^{**})^t$ corresponding to $\mu_2[x^{**}]$, then $h_2^*\cdot A''(x^{**})h_2h_2 = 0$, and $\frac{\partial}{\partial\lambda}x^{(1)}(\lambda)$ exists at λ^{**}. On the other hand, the eigenvectors $x^{(\pm)}(\lambda)$ are not differentiable at λ^{**}, but

$$\lim_{\lambda\uparrow\lambda^{**}} \frac{x^{(\pm)}(\lambda)-x^{**}}{\sqrt{\lambda^{**}-\lambda}} = \pm\frac{\sqrt{\epsilon\sqrt{1+a^2}}}{\lambda^{**}}\, h_2\ ,$$

which is similar to equation (10.10) above for the behavior of $x^o(\lambda)$ and $x^{(1)}(\lambda)$ when λ is near λ^*.

PART II. EIGENVALUE PROBLEMS FOR NONLINEAR INTEGRAL AND DIFFERENTIAL EQUATIONS

II. 1. Hammerstein Integral Equations

We shall consider in detail how the results of Part I can be applied to the investigation of the behavior of the eigenfunctions of the nonlinear integral equation of Example I. 2. 1,

$$u(x) = \lambda \int_{\Omega} K(x,y) f(y, u(y)) dy \,, \tag{1.1}$$

where Ω is an open bounded domain in $\mathbb{R}^n$; $K(x, y)$ is positive for $(x, y) \in \Omega \times \Omega$, continuous on $\bar{\Omega} \times \bar{\Omega}$ except possibly when $x = y$, and for a positve number $\varkappa$ and a non-negative number $\alpha < n$,

$$0 \le \bar{K}(x,y) < \frac{\varkappa}{|x-y|^{\alpha}} \tag{1.2}$$

for $(x, y) \in \bar{\Omega} \times \bar{\Omega}$; and for some positive number r , $0 < r \le \infty$, f is a positive continuous function of $n+1$ variables on $\bar{\Omega} \times [0, r)$ such that

$$0 < f(x, \rho_1) \le f(x, \rho_2)$$

whenever $x \in \bar{\Omega}$ and $0 \le \rho_1 \le \rho_2 < r$. We set

$$f_{min}(\rho) = \min \{f(x, \rho) : x \in \bar{\Omega}\}$$

and

$$f_{max}(\rho) = \max \{f(x, \rho) : x \in \bar{\Omega}\}$$

for $0 \le \rho < r$. Since $f_{min}(0) > 0$, we may assume that f is extended to a continuous positive function $\tilde{f}$ defined on $\bar{\Omega} \times (-r, r)$, and that the extended function $\tilde{f}$ retains all the continuity and differentiability properties of f at or near the origin which we may later impose on f . For example, suppose that $f(x, u)$ has a continuous partial derivative $f_u(x, u)$ for $(x, u) \in \bar{\Omega} \times [0, r)$. If we define

$$\tilde{f}(x,u) = \begin{cases} f(x,u) & , \ 0 \le u < r \\ \frac{1}{2} f(x,0)\left[1 + \exp\left(\frac{2f_u(x,0)}{f(x,0)} u\right)\right] & , \ u < 0 \end{cases}$$

then $\tilde{f}$ and $\tilde{f}_u$ are continuous and positive on $\bar{\Omega} \times (-\infty, r)$, with $\tilde{f}(x,u) \ge \frac{1}{2} f(x,0) \ge f_{min}(0) > 0$.

Henceforth we denote the extended function $\tilde{f}$ by f. These assumptions on the set Ω, kernel K, and function f in equation (1.1) are retained throughout Section II.1 unless the contrary is specifically indicated, and all integrations are to be carried out over the set Ω unless otherwise noted.

As noted in the Introduction, the development of this section closely parallels that of Part I. We first prove that (1.1) has eigenfunctions of arbitrary norm (Theorem 1-1). Theorem 1-2 shows the uniqueness of the eigenfunctions for each λ when $f(x,u)/u$ is a decreasing function of u. When f(x, u) is twice continuously differentiable with respect to u, Theorem 1-5 describes the behavior of the minimal positive eigenfunctions for λ near the maximum λ^* of the eigenvalues λ of (1.1) (if there is such a maximum), and it proves the existence of a second fixed point for λ near λ^*.

Theorems 1-7 and 1-8 describe the behavior of eigenfunctions of large norm; if $f(x,u) \sim m(x)u + c(x)u^s$ for $0 \le s < 1$, we are able to determine the value $\mu_1 > 0$ such that $\lim_{\lambda \to \mu_1} \|u(\lambda)\| = \infty$ (where $u(\lambda)$ denotes an eigenfunction of (1.1) corresponding to the eigenvalue λ), and using the sign of c(x) we can describe whether the eigenfunctions of large norm exist for $\lambda > \mu_1$ or $\lambda < \mu_1$. If $\lambda^* > \mu_1 > 0$

and $f(x, u)$ is continuously differentiable with respect to u, Theorem 1-10 shows that there are at least two eigenfunctions for all λ such that $\mu_1 < \lambda < \lambda^*$. By putting further restrictions on the kernel $K(x, y)$, we are able to state in Theorems 1-12 and 1-14 conditions under which there are at least two eigenfunctions for each λ with $0 < \lambda < \lambda^*$, and $\mu_1 = 0$.

The rest of the section is devoted to a discussion of equation (1.1) when $f(x, u)$ is convex in u, usually under the assumption that $f(x, u)$ is continuously differentiable with respect to u. Theorem 1-15 describes the general properties of the eigenfunctions $u(\lambda)$ as functions of λ and characterizes the maximum value λ^* (if any). Theorems 1-18 and 1-19 show that for a given λ there cannot be three distinct eigenfunctions satisfying $u_2 \geq u_1 \geq u^o$. This result can be obtained by using the methods and results of Section I. 10; we give here an alternative derivation for the case when $K(x, y)$ is symmetric. We will use this result in Section II. 2 to prove that in certain cases (1.1) has at most two eigenfunctions for each λ. These theorems also show that there is either only one or infinitely many eigenfunctions corresponding to λ^*. Theorems 1-23 and 1-24 give conditions under which there is only one eigenfunction for λ^*; these conditions have the very simple form given in Theorem 1-24 when $f(x, u)$ is independent of u. In Section II. 2 we shall show that the non-existence of three eigenfunctions $u_2 \geq u_1 \geq u^o$ can be proven for $f(x, u)$ convex in u without the differentiability assumption if the kernel satisfies condition (2.7) of that section.

The results of Keller and Cohen (1967) on nonlinear elliptic partial differential equations can be easily generalized to apply to the problem (1.1). In the following discussion we shall therefore emphasize results which are not included in the work of Keller and Cohen.

We denote by $C(\overline{\Omega})$ the partially ordered Banach space of continuous functions on $\overline{\Omega}$, with the norm

$$\|u\| = \max\{|u(x)| : x \in \overline{\Omega}\}$$

for any function $u \in C(\overline{\Omega})$, and set

$$\mathcal{B}^r = \{u \in C(\overline{\Omega}) : \|u\| < r\},$$

$$\mathcal{C} = \{u \in C(\overline{\Omega}) : u(x) \geq 0, \ x \in \overline{\Omega}\},$$

and

$$\mathcal{C}^r = \mathcal{B}^r \cap \mathcal{C}.$$

Then the operator A defined on $\mathcal{B}^r$ by

$$(1.3) \qquad Au(x) = \int K(x,y) f(y, u(y)) dy$$

is a positive monotonic operator on $\mathcal{C}^r$ which, as we have seen in Section I.2, is compact on $\mathcal{B}^r$, and the problem of finding the positive eigenfunctions of equation (1.1) is equivalent to the problem of finding the positive fixed points of the strictly increasing family $\{A_\lambda\}$, $\lambda > 0$, of positive monotonic operators, where $A_\lambda = \lambda A$. Since f is positive, $A_\lambda \mathcal{B}^r \subseteq -\mathcal{C}$ for $\lambda \leq 0$ and $A_\lambda \mathcal{B}^r \subseteq \mathcal{C}$ for $\lambda > 0$; thus, A_λ has no positive fixed points for $\lambda \leq 0$, and any fixed point of A_λ in $\mathcal{B}^r$ for $\lambda > 0$ is positive. The operator A is g_o-bounded on $\mathcal{C}^r$; i.e., for any function $u \in C(\overline{\Omega})$, there are positive numbers $\alpha(u)$, $\beta(u)$ such that

$$\alpha(u) g_o \leq Au \leq \beta(u) g_o,$$

where g_o is the continuous non-negative function defined by

$$g_o(x) = \int K(x, y)dy \quad .$$

Since $g_o(x) > 0$ for $x \in \Omega$, all eigenfunctions of equation (1.1) are strictly positive on Ω.

For any function $u \in C^r$, we have

$$\|Au\| \geq \|A0\| \geq f_{min}(0)\|g_o\| > 0 \ ,$$

and therefore, by Theorem I. 3-5:

<u>1-1. Theorem.</u> The problem (1.1) possesses a continuous branch of eigenfunctions of length r (see Section I. 3); in particular, for any positive number $\rho < r$, there is an eigenfunction whose norm is ρ.

The assumption that f is monotonically increasing in u can be dispensed with if $f(x, u)$ is uniformly bounded below by a positive number for $x \in \Omega$, $0 \leq u < r$.

We denote by Λ_f^r the set of positive numbers λ for which (1.1) possesses a positive solution with norm $< r$, and refer to the elements of Λ_f^r as either eigenvalues (of the problem (1.1)) or characteristic values (of A). If r and f are fixed throughout a discussion, we will replace the notation Λ_f^r by Λ. Since for any eigenvalue λ and corresponding eigenfunction u we have $\lambda = \|u\| / \|Au\| \leq \|u\| / \|A0\|$, it is clear that as the norms of the eigenfunctions go to zero, the corresponding eigenvalues approach 0. It follows from Theorem 1-1 and Theorem I. 4-9 that Λ_f^r is a non-empty interval with $\inf \Lambda_f^r = 0 \notin \Lambda_f^r$, and that for each $\lambda \in \Lambda_f^r$ the problem (1.1) has a minimal positive eigenfunction $u^o(\lambda)$ such that for any other eigenfunction $u(\lambda)$ corresponding to the eigenvalue λ, $u^o(\lambda) \le u(\lambda)$; if $f(x, u)$ is a strictly increasing function of u for all $x \in \Omega$ and $0 < u < r$, then $u^o(\lambda;x) < u(\lambda;x)$ for all $x \in \Omega$. The minimal eigenfunctions $u^o(\lambda)$ are continuous from the left in λ and increase as λ increases by Theorems I. 4-9 and I. 4-10; from the assumptions that we have made on the kernel $K(x, y)$, we actually have $u(\lambda_1;x) < u(\lambda_2;x)$ for all $x \in \Omega$ whenever $0 < \lambda_1 < \lambda_2 \in \Lambda_f^r$. The minimal eigenfunctions $u^o(\lambda)$ may be constructed by a simple iteration procedure (Keller and Cohen 1967, or our Theorem I. 4-4). Various restrictions on the size of Λ_f^r can be obtained from inequalities satisfied by f with the help of Theorem I. 4-5 and its corollaries; the most general one is that if $f(x, u) \le g(x, u)$ for $x \in \Omega$, $0 \le u < r$, then $\Lambda_g^r \subseteq \Lambda_f^r$ (Corollary I. 4-6; Keller and Cohen 1967, Theorem 3. 3).

We consider first the case when the function f satisfies the condition

(1. 4) $\qquad f(x, \alpha u) > \alpha f(x, u)\,,\quad 0 < \alpha < 1\,,\quad x \in \Omega\,,\ 0 < u < r$

(see Section I. 5). We have shown in Proposition I. 5-2 that if f satisfies a one-sided Lipschitz condition

(1. 5) $\qquad f(x, u) - f(x, v) \le g(x)(u-v)\,,$

for $x \in \Omega$ and $0 < v \le u < r$, where $g \in \mathbb{C}$, then there is a number $r_1 \in (0, r)$ such that equation (1. 4) is satisfied for $0 < u < r_1$. Geometrically, condition (1. 4) says that a line drawn to the origin from any point on the graph of $f(x, u)$ against u (for fixed x) lies strictly below the graph of $f(x, u)$ against u. This condition is easily seen to be equivalent to the condition that, for each $x \in \Omega$, $f(x, u)/u$ is a strictly decreasing function of u for $0 < u < r$; if f has a partial derivative with respect to u, this is true if $f(x, u) - uf_u(x, u) > 0$ for $x \in \Omega$ and $0 < u < r$. Since $f(x, u)/u$ is a decreasing function of u, it has a limit $m(x)$ as $u \to +\infty$ (if $r = \infty$). If this limit is approached uniformly for $x \in \Omega$, then A has a $\mathbf{C}$-asymptotic derivative $A'(\infty)$ (see Section I. 6) given by

$$(1.6) \qquad A'(\infty)h(x) = \int K(x, y)m(y)h(y)dy$$

for any $h \in C(\overline{\Omega})$ (Krasnosel'skii 1964a, §7. 1. 5; cf. our Propositions I. 8-7 and I. 8-9); the operator $A'(\infty)$ satisfies

$$\lim_{\substack{\|h\| \to \infty \\ h \in \mathbf{C}}} \frac{\|Ah - A'(\infty)h\|}{\|h\|} = 0 .$$

As shown in Example I. 2-1, if $m(x) > 0$ for some $x \in \Omega$, then $A'(\infty)$ has a simple characteristic value $\mu_o[A'(\infty)] \equiv \mu_1(\infty)$; $\mu_1(\infty)$ is a positive eigenvalue of the linear integral equation

$$(1.7) \qquad h(x) = \mu \int K(x, y)m(y)h(y)dy ,$$

to which there corresponds a unique positive eigenvector φ_∞ of unit norm, and $\mu_1(\infty)$ is smaller than the absolute value of all other eigenvalues of equation (1.7). Using Theorem I. 5-6, we obtain:

<u>1-2. Theorem.</u> Let $f(x, \rho)/\rho$ be a strictly decreasing function of ρ for each $x \in \overline{\Omega}$ and $0 < \rho < r$. Then for each $\lambda \in \Lambda$, the problem (1. 1) has a unique eigenfunction $u(\lambda) = u^o(\lambda)$ which increases as λ increases and depends continuously on λ; moreover,

$$\lim_{\lambda \to 0} u(\lambda) = 0 \quad \text{and} \quad \lim_{\lambda \uparrow \lambda^*} \|u(\lambda)\| = r ,$$

where $\lambda^* = \sup \Lambda$.

If $r = \infty$ and $\lim_{u \to +\infty} f(x,u)/u = m(x)$ uniformly for $x \in \Omega$, and $m \neq 0$, then $\lambda^* = \mu_1(\infty)$, the principal eigenvalue of equation (1.7), and

$$\lim_{\lambda \uparrow \lambda^*} \frac{u(\lambda)}{\|u(\lambda\|} = \varphi_\infty .$$

If $\lim_{u \to +\infty} f(x,u)/u = 0$ for $x \in \bar{\Omega}$, then $\lambda^* = +\infty$.

Proof. According to Proposition I. 5-1, for any $u \in C^r$, $\alpha \in (0,1)$, there is an $\eta > 0$ such that $A(\alpha u) \geq \alpha(1+\eta)Au$ (equation (I. 5. 1)). The remaining hypotheses of Theorem I. 5-6 are easily verified, and the first part of the theorem follows. The assertions made for $r = \infty$ and $m \neq 0$ follow from equation (1. 6), Corollary I. 8-2, and the remarks of Example I. 2-1 . When $m(x) = 0$ for all $x \in \bar{\Omega}$, then $A'(\infty) = 0$ and $\lim_{\|u\| \to \infty} \|Au\| / \|u\| = 0$; since $\lambda^{-1} = \|Au^o(\lambda)\| / \|u^o(\lambda)\|$ and $\lim_{\lambda \uparrow \lambda^*} \|u^o(\lambda)\| = \infty$, $\lambda^* = +\infty$. //

When $f(x,u)/u$ is not a decreasing function of u , we can establish the uniqueness of the eigenfunctions for at least a subset of Λ using the Lipschitz condition (1. 5) and Corollary I. 4-15 (cf. Tricomi 1957, p. 212):

1-3. Theorem. Let f satisfy the Lipschitz condition (1. 5) for $0 \leq v \leq u < r$, for some function $g \in C(\bar{\Omega})$. Then equation (1. 1) has at most one positive eigenfunction for each λ such that $0 < \lambda < \mu_1$, where μ_1 is the principal eigenvalue of the linear integral equation

(1. 8) $$h(x) = \mu \int K(x,y) g(y) h(y) dy .$$

(If $g = 0$, $\mu_1 = \infty$.)

Proof. When $g(x) > 0$ for some $x \in \Omega$, the theorem follows immediately from Corollary I. 4-15, with the linear operator T given by

$$Th(x) = \int K(x,y)g(y)h(y)dy .$$

If $g(x) = 0$ for all $x \in \Omega$, then $f(x,u)$ is independent of u and the result is trivial. //

With further conditions on f, we are able to extend the uniqueness interval to include the endpoint μ_1.

1-4. Theorem. Let f satisfy the Lipschitz condition (1. 5). Let strict inequality hold in (1. 5) whenever $x \in \Omega$ and $0 < v < u < r$; or let $f(x,u)$ be a strictly increasing function of u for all $x \in \Omega$, $0 < u < r$, and strict inequality hold in (1. 5) for some $x \in \Omega$ when $0 < v < u < r$. Then equation (1. 1) has at most one eigenfunction for each $\lambda \in (0, \mu_1]$, where μ_1 is defined as in Theorem 1-3.

Proof. The uniqueness for $\lambda \in (0, \mu_1)$ is established in Theorem 1-3. If there are two distinct eigenfunctions u and $u_o = u^o(\mu_1)$ corresponding to $\lambda = \mu_1$, then there is some $y \in \Omega$ such that $u_o(y) < u(y)$, and this holds for all $y \in \Omega$ if $f(y,u)$ is strictly increasing in u for all $y \in \Omega$. Thus, under either of the hypotheses of the theorem, there is some $y \in \Omega$ such that

$$f(y,u(y))-f(y,u_o(y)) < g(y)[u(y)-u_o(y)] .$$

By continuity,

$$u(x)-u_o(x) < \mu_1 \int K(x,y)g(y)[u(y)-u_o(y)]dy$$

for all $x \in \Omega$. But this is impossible (see the discussion of (PA_2) in Example I. 2-1). There is, therefore, at most one positive eigenfunction for $\lambda = \mu_1$. //

It is possible to find functions f and g and a kernel K such that equation (1. 5) is satisfied (for all $x \in \Omega$, $0 \le u \le v < \infty$) for which there are infinitely many eigenfunctions of (1. 1) corresponding to $\lambda = \mu_1$. For example, take $\Omega = (0, 1)$, $K(x, y) \equiv 1$, $f(x, u) = f(u) = e^{u-1}$ when $x \in [0, 1]$, $0 \le u \le 1$, and $f(x, u) = f(u) = u$ when $x \in [0, 1]$ and $u \ge 1$. Then $0 < v < u$ implies $f(x, u)-f(x, v) \le u-v$, so (1. 5) is satisfied with $g(x) \equiv 1$. The principal (and only) eigenvalue of the linear integral equation (1. 8),

$$h(x) = \mu \int_0^1 h(y)dy ,$$

is $\mu_1 = 1$. The nonlinear equation (1. 1) ,

$$u(x) = \lambda \int_0^1 f(y, u(y))dy = \lambda \int_0^1 f(u(y))dy ,$$

has as eigenfunctions the constant functions $u(\lambda;x) \equiv \lambda\alpha(\lambda)$, where $\alpha(\lambda)$ is a solution of $\alpha = f(\lambda\alpha)$. When $\lambda = 1$, the eigenfunctions are all constant functions $u(x) \equiv \alpha$, $\alpha \ge 1$. (Cf. the discussion preceding Theorem I. 4-14.)

Under the assumptions of Theorem 1-4, the interval of uniqueness $(0, \mu_1]$ is also, in general, as large as possible; there are functions f and g satisfying the hypotheses of Theorem 1-4 for which $\sup \Lambda > \mu_1$, and for which there are at least two eigenfunctions for all $\lambda \in (\mu_1, \sup \Lambda]$. For example, take f(u) as above for $u \le 2$, and $f(u) = \frac{9}{8} \frac{u^2}{1+u} + \frac{1}{2}$ for $u \ge 2$; (1. 5) holds with $g(x) \equiv 9/8$. Then $\mu_1 = 8/9$, $\sup \Lambda = 1$, and there are at least two eigenfunctions for each $\lambda \in (8/9, 1]$. See also Theorems 1-10 and 1-13.

We will now assume that $f(x,u)$ has a continuous partial derivative $f_u(x,u)$ for $x \in \bar{\Omega}$, $0 \le u < r$, and that f_u is continuous on $\bar{\Omega} \times [0,r)$. Since $f(x,u)$ is an increasing function of u, $f_u(x,u) \ge 0$. Theorem 1-3 can then be used to investigate the uniqueness of the eigenfunctions of equation (1.1). We define

$$g(x;s) = \max\{f_u(x,\rho) : 0 \le \rho \le s\}$$

for $0 \le s < r$ and $x \in \bar{\Omega}$, and

$$g(x;r) = \sup\{f_u(x,\rho) : 0 \le \rho < r\} ;$$

then from Theorem 1-3 or Theorem I. 6-4, it follows that equation (1.1) has at most one eigenfunction in C^s $(0 < s < r$, and also $s = r$ if $g(x;r)$ is finite) corresponding to any positive λ less than the principal eigenvalue of

$$h(x) = \mu \int K(x,y)g(y;s)h(y)dy .$$

Together with any eigenfunction $u(x)$ of equation (1.1) in C^r, we consider the variational equation

$$h(x) = \mu \int K(x,y)f_u(y,u(y))h(y)dy ; \tag{1.9}$$

this problem is the characteristic value problem $h = \mu A'(u)h$ for the Fréchet derivative of A at u (Section I. 6):

$$A'(u)h(x) = \int K(x,y)f_u(y,u(y))h(y)dy . \tag{1.10}$$

As discussed in Example I. 2-1, $A'(u)$ is a compact linear operator; if $f_u(y,u(y)) > 0$ for some $y \in \Omega$, then the principal eigenvalue of equation (1.9), which we denote by $\mu_1[u] = \mu_o[A'(u)]$, is simple and smaller than the absolute value of all other eigenvalues of

(1.9), and the unique positive eigenfunction φ with $\|\varphi\| = 1$ of (1.9) corresponds to the eigenvalue $\mu_1[u]$. If $f_u(y, u(y)) = 0$ for all $y \in \overline{\Omega}$, we set $\mu_1[u] = \mu_o[A'(u)] = \infty$.

If $f_u(x, \rho) > 0$ for $x \in \Omega$, $0 < \rho < r$, then $A'(u)$ satisfies (PA) for any function $u \in C^r$, and it follows from Theorem I. 7-1 that $\lambda \leq \mu_1[u^o(\lambda)]$ for all $\lambda \in \Lambda$. On the other hand, if for some $\lambda \in \Lambda$, $f_u(x, u^o(\lambda;x)) = 0$ for all $x \in \Omega$, then clearly $\lambda < \mu_1[u^o(\lambda)] = \infty$. If $f_u(x, u^o(\lambda;x)) = 0$ for some (but not all) $x \in \Omega$, then Theorem I. 7-1 is not immediately applicable, since $A'[u^o(\lambda)]$ does not necessarily satisfy (PA). The result $\lambda \leq \mu_1[u^o(\lambda)]$ is still true, however, and can be established using the weakened form of (PA) given in Example I. 2-1. As in the considerations leading to equation (I. 7. 4), for any $\lambda_o \in \Lambda$ we find a monotonically increasing sequence $\{\lambda_k\}$, $k \geq 1$, converging to λ_o, such that

$$h = \beta_o A u_o + \lambda_o A'(u_o) h ,$$

where $u_o = u^o(\lambda_o)$,

$$h = \lim_{k \to \infty} \frac{u_o - u^o(\lambda_k)}{\|u_o - u^o(\lambda_k)\|} ,$$

and

$$\beta_o = \lim_{k \to \infty} \frac{\lambda_o - \lambda_k}{\|u_o - u^o(\lambda_k)\|} .$$

If $\beta_o > 0$, then $h(x) - \lambda_o A'(u_o) h(x) = \beta_o A u_o(x) > 0$ for all $x \in \Omega$; since $A'(u_o)$ satisfies the weakened form of (PA), $\lambda_o < \mu_1[u^o(\lambda_o)] = \mu_o[A'(u_o)]$. If $\beta_o = 0$, then h is a positive eigenvector of $A'(u_o)$, and therefore $\lambda_o = \mu_1[u^o(\lambda_o)]$. Conversely, if $\lambda_o = \mu_1[u^o(\lambda_o)]$, then we must have $\beta_o = 0$, again by the weakened form of (PA).

Thus, the conclusions of Theorem I. 7-1 and all other theorems of Section I. 7 are valid.

We have, therefore, $\lambda \leq \mu_1[u^o(\lambda)]$ for all minimal solutions $u^o(\lambda)$; if $\lambda < \mu_1[u^o(\lambda)]$, the minimal solutions are differentiable with respect to λ , the derivative being the unique solution $\psi(\lambda)$ of the equation

$$(1.11) \qquad \psi(\lambda;x) - \lambda\int K(x,y)f_u(y,u^o(\lambda;y))\psi(\lambda;y)dy = \int K(x,y)f(y,u^o(\lambda;y))dy .$$

For $\lambda < \mu_1[u^o(\lambda)]$, the minimal solutions are isolated; i. e., for each such λ , there is a number δ_λ such that (1. 1) has no eigenfunction u corresponding to the eigenvalue λ with $\|u-u^o(\lambda)\| \leq \delta_\lambda$ (Theorem I. 7-8).

The set of eigenvalues Λ_f^r either exhausts the positive real numbers (in which case we must have $r = \infty$, since if $r < \infty$ we obviously have

$$f(x,u) \geq \frac{1}{3} f_{min}(0)[\frac{u}{r} + 1] ,$$

which implies that Λ_f^r is bounded by Corollary I. 4-8) , or $\lambda^* \equiv \sup \Lambda_f^r$ is a finite number at which one or both of the following alternatives occur: either $\lim_{\lambda\uparrow\lambda^*}\|u^o(\lambda)\| = r \leq \infty$, or $\lambda^* = \mu_1(u^o(\lambda^*))$ (see the discussion of the implicit function theorem following Theorem I. 7-8). Since the minimal solutions are continuous from the left in λ and increasing in λ , we cannot have $\lambda^* \in \Lambda_f^r$ if the former of these alternatives occurs; similarly, there is no $\lambda_o < \lambda^*$ such that $\lim_{\lambda\uparrow\lambda_o} \|u^o(\lambda)\| = r$.

The only points in Λ_f^r at which $u^o(\lambda)$ can fail to be continuous or differentiable are points λ_o such that $\lambda_o = \mu_1[u^o(\lambda_o)]$. We now discuss the behavior of the eigenfunctions and eigenvalues near such a point; the discussion will include, in particular, the case that $\lambda_o = \lambda^* = \sup \Lambda_f^r \in \Lambda_f^r$. Let $u_o = u^o(\lambda_o)$ be the minimal solution of (1.1) corresponding to the eigenvalue λ_o, and let φ be the eigenfunction of the variational problem (1.9) (with u replaced by u_o) corresponding to the eigenvalue λ_o. In order to apply Theorems I.7-3 and I.7-5, we assume that $f(x,u)$ has a second derivative $f_{uu}(x,u)$ which is continuous on $\overline{\Omega}\times[0,r)$; then, according to Section I.6, A has a second Fréchet derivative $A''(u)$, which is the bilinear operator given by

(1.12) $$(A''(u)hk)(x) = \int K(x,y)f_{uu}(y,u(y))h(y)k(y)dy$$

for any $u \in C^r$ and any h, k in $C(\overline{\Omega})$.

For simplicity in the formulation of the theorems which follow, we will assume that the kernel K in equation (1.1) is symmetric: $K(x,y) = K(y,x)$. Then the equation adjoint to the linear equation (1.9) has the positive eigenfunction $\varphi(x)f_u(x,u(x))$.

1-5. Theorem. Let $f(x,u)$ be twice continuously differentiable with respect to u on $\overline{\Omega}\times(-r,r)$. Suppose that there is a point $\lambda_o \in \Lambda$ with a corresponding minimal solution $u_o = u^o(\lambda_o)$ such that $\lambda_o = \mu_1[u_o]$, and let φ be the eigenfunction of the problem (1.9) corresponding to the principal eigenvalue $\lambda_o = \mu_1[u_o]$. Then

$$\int \varphi^3(y)f_{uu}(y,u_o(y))dy \geq 0 .$$

If

(1.13) $$\int \varphi^3(y)f_{uu}(y,u_o(y))dy > 0 ,$$

then

$$(1.14)\quad \lim_{\lambda\uparrow\lambda_o} \frac{u_o-u^o(\lambda)}{\sqrt{\lambda_o-\lambda}} = \left\{2\frac{\int\varphi(x)f_u(x,u_o(x))u_o(x)dx}{\lambda_o\int\varphi^3(x)f_{uu}(x,u_o(x))dx}\right\}^{\frac{1}{2}}\varphi .$$

Moreover, for each λ sufficiently close to λ_o, with $\lambda < \lambda_o$, the problem (1. 1) has a second positive eigenfunction $u(\lambda)$ which for each λ is the only eigenfunction other than the minimal eigenfunction in a sufficiently small ball about u_o in $C(\overline{\Omega})$, and

$$\lim_{\lambda\uparrow\lambda_o} \frac{u(\lambda)-u_o}{\sqrt{\lambda_o-\lambda}} = \lim_{\lambda\uparrow\lambda_o} \frac{u_o-u^o(\lambda)}{\sqrt{\lambda_o-\lambda}} .$$

The eigenfunction u_o can be enclosed in a ball in $C(\overline{\Omega})$ in which it is the only eigenfunction of (1. 1) corresponding to the eigenvalue λ_o, and there are no eigenfunctions in this ball corresponding to $\lambda > \lambda_o$.

Proof. Since $f(x, u)$ is twice continuously differentiable with respect to u, we have

$$f(x,u+h) = f(x,u) + f_u(x,u)h + \tfrac{1}{2}f_{uu}(x,u)h^2 + g(x,u;h)h^2$$

for any $x \in \overline{\Omega}$ and any u, h such that $-r < u-|h| \leq u+|h| < r$, where

$$g(x,u;h) = \int_0^1\int_0^1 [f_{uu}(x,u+\alpha\beta h) - f_{uu}(x,u)]\beta d\alpha d\beta .$$

Since for any positive number $r_1 < r$, f_{uu} is uniformly continuous on $\overline{\Omega}\times[-r_1, r_1]$, we have

$$\lim_{h\to 0} g(x,u;h) = 0$$

uniformly for $(x,u) \in \overline{\Omega}\times[-r_1, r_1]$.

Moreover,

$$g(x,u;h_1)h_1^2 - g(x,u;h_2)h_2^2$$
$$= \{[\tfrac{1}{2}f_{uu}(x,u+h_2)-\tfrac{1}{2}f_{uu}(x,u)+g(x,u+h_2;h_1-h_2)](h_1-h_2)^2$$

$$+ \nu(x,u;h_2)h_2\}(h_1-h_2) \;.$$

where

$$\nu(x,u;h_2) = \int_0^1 [f_{uu}(x,u+\alpha h)-f_{uu}(x,u)]d\alpha \;.$$

For any positive numbers ρ_1, ρ, with $\rho_1 < \rho_1+\rho < r$, we define $\gamma(\rho_1,\rho)$ to be the maximum of the absolute value of the expression in $\{\quad\}$ above for $x \in \overline{\Omega}$, $|u| \le \rho_1$, $|h_1| \le \rho$, $|h_2| \le \rho$. Then

$$|g(x,u;h_1)h_1^2 - g(x,u;h_2)h_2^2| \le \gamma(\rho_1,\rho)|h_1-h_2| \;,$$

where

$$\lim_{\rho\to 0} \frac{\gamma(\rho_1,\rho)}{\rho} = 0 \;.$$

If we set

$$\omega_1(h)(x) = \int K(x,y)g(y,u_o(y);h(y))h^2(y)dy$$

for functions $u_o \in C^r$, $h \in C^{r-\|u_o\|}$, then

$$\lim_{\|h\|\to 0} \frac{\|\omega_1(h)\|}{\|h\|^2} \le \lim_{\|h\|\to 0} \max_{x\in\overline{\Omega}} \left|\int K(x,y)g(y,u_o(y);h(y))dy\right| = 0$$

and

$$\|\omega_1(h_1)-\omega_1(h_2)\| \le \gamma(\|u_o\|,\rho) \max_{x\in\overline{\Omega}} \left|\int K(x,y)dy\right| \|h_1-h_2\|$$

$$= \gamma_1(\rho)\|h_1-h_2\|$$

where $\gamma_1(\rho) = \gamma(\|u_o\|,\rho) \max \left|\int K(x,y)dy\right|$ and

$$\lim_{\rho\to 0} \frac{\gamma_1(\rho)}{\rho} = 0 \;.$$

We define

$$\omega(\delta,h) = \delta[A'(u_o)h + \tfrac{1}{2} A''(u_o)h^2] + (\lambda_o+\delta)\omega_1(h) \;;$$

then for $\lambda > 0$, $u \in C^r$,

$$\lambda Au = \lambda_o Au_o + (\lambda-\lambda_o)Au_o + \lambda_o A'(u_o)(u-u_o) + \tfrac{1}{2}\lambda_o A''(u_o)(u-u_o)^2 + \omega(\lambda-\lambda_o, u-u_o),$$

and the operators $A_{\lambda_o} = \lambda_o A$, $B_{\lambda_o} = A$, $A'_{\lambda_o}(u_o) = \lambda_o A'(u_o)$, $A''_{\lambda_o}(u_o) = \lambda_o A''(u_o)$, and ω have all the properties assumed in Theorems I. 7-1, I. 7-3, and I. 7-5. In particular, the required properties of $A'_{\lambda_o}(u_o)$ follow from the discussion above of the properties of $A'(u_o)$ and the fact that $\lambda_o = \mu_1[u_o] < \infty$ implies $f_u(y, u_o(y)) > 0$ for some $y \in \Omega$.

The linear functional ξ of equation (I. 7. 8) and Theorem I. 7-3 is given by

$$\xi(h) = \eta(\varphi)^{-1}\int\varphi(x)f_u(x, u_o(x))h(x)dx$$

for any function $h \in C(\overline{\Omega})$, where

$$\eta(\varphi) = \int\varphi^2(x)f_u(x, u_o(x))dx ;$$

since for any functions $h_1, h_2 \in C(\overline{\Omega})$,

$$\int h_1(x)\int K(x, y)h_2(y)dydx = \int h_2(y)\int K(x, y)h_1(x)dxdy$$

(Smirnov 1964a), we have

$$\xi[A''(u_o)\varphi^2]\eta(\varphi) = \int\varphi(x)f_u(x, u_o(x))\int K(x, y)f_{uu}(y, u_o(y))\varphi^2(y)dydx$$
$$= \lambda_o^{-1}\int\varphi^3(y)f_{uu}(y, u_o(y))dy$$

and

$$\xi[B_{\lambda_o}u_o] = \xi[Au_o] = \lambda_o^{-1}\xi(u_o) = \eta(\varphi)^{-1}\lambda_o^{-1}\int\varphi(x)f_u(x, u_o(x))u_o(x)dx > 0,$$

the conclusions of the theorem now follow immediately from Theorems I. 7-1, I. 7-3, and I. 7-5. //

If condition (1. 13) is not satisfied, then information about the behavior of the solutions near $(\lambda_o, u^o(\lambda_o))$ can be obtained by investigating higher derivatives of f. This does not seem to be a very worthwhile endeavor unless $f_{uu}(x,u)$ is never zero for $x \in \Omega$, $0<u<r$, since otherwise in order to know whether (1.13) is satisfied, one must know the eigenfunction $u^o(\lambda_o)$.

When $\lambda_o = \mu_1[u^o(\lambda_o)]$, the minimal eigenfunctions cannot have a finite derivative with respect to λ at λ_o, for if they did we would have

$$\frac{\partial u^o}{\partial \lambda}(\lambda_o) = Au^o(\lambda_o) + \lambda_o A'[u^o(\lambda_o)]\,\frac{\partial u^o}{\partial \lambda}(\lambda_o)$$

(equation (1.11)), which is impossible when $\lambda=\lambda_o$, since $Au^o(\lambda_o;x) = \lambda_o^{-1}u^o(\lambda_o;x)>0$ for all $x\in\Omega$ (see assumption (PA_2)).

If condition (1.13) is not satisfied when $\lambda_o=\mu_1[u^o(\lambda_o)]$, then there may or may not be a second solution for $\lambda<\lambda_o$, there may or may not be solutions for $\lambda>\lambda_o$, and the solution at λ_o may or may not be isolated. For example, if in the example following Theorem 1-4 we take $f(u) = \frac{1}{2}(u-\cos u) + \frac{\pi}{4}$ for $0\le u\le\pi$, and $f(u) = 1+\frac{3\pi}{4} - \frac{1}{2}e^{-u+\pi}$ for $\pi\le u$, then f is twice continuously differentiable. For $\lambda=1$, equation (1.1) has the minimal eigenfunction $u^o(\lambda;x)\equiv\frac{\pi}{2}$, and $\lambda=1=\mu_1[u^o(\lambda)]$. We have $\int_0^1 \varphi^3(x)f''(u^o(\lambda;x))\,dx=0$ when $\lambda=1$, since $f''(u^o(1;x))=f''(\frac{\pi}{2})=0$. There is no bifurcation at $\lambda=1$; for each $\lambda>0$, there is a unique eigenfunction $u^o(\lambda)$ depending continuously on λ, and $\lim_{\lambda\to+\infty}\|u^o(\lambda)\| = \infty$ (this follows from Theorem 1-2, since $f(u)/u$ is a strictly decreasing function of u which converges to 0 as $u\to+\infty$). The eigenfunctions are differentiable with respect to λ for all $\lambda>0$, $\lambda\neq1$, but have an infinite derivative with respect to λ at $\lambda=1$.

On the other hand, if we take $f(u) = \frac{1}{2}(u - \cos u) + \frac{\pi}{4}$ for $0 \le u \le \frac{\pi}{2}$ and $f(u) = u^3 - \frac{3\pi}{2}u^2 + (1 + \frac{3\pi^2}{4})u - \frac{\pi^3}{8}$ for $u \ge \frac{\pi}{2}$, then f is twice continuously differentiable, and when $\lambda = 1$, we have $u^o(\lambda;x) \equiv \frac{\pi}{2}$, $\lambda = \mu_1[u^o(\lambda)] = 1$, and $\int_o^1 \varphi^3(x)\, f''(u^o(\lambda;x))\, dx = 0$. In this case $\lambda = 1 = \max \Lambda$ and there are two eigenfunctions for all $\lambda \in (0, 1)$.

If we take $f(u)$ as above for $0 \le u \le \frac{\pi}{2}$, and $f(u) = u$ for $u \ge \frac{\pi}{2}$, then f'' is continuous, $\lambda = \mu_1[u^o(\lambda)]$ when $\lambda = 1$, the eigenfunctions are unique for $0 < \lambda < 1 = \max \Lambda$, and there are infinitely many eigenfunctions $u(x) \equiv \alpha$, $\alpha \ge \frac{\pi}{2}$, corresponding to $\lambda = 1$.

According to Theorem I. 7-6, analogous results hold for the behavior of the eigenfunctions near any point at which $\lambda = \mu_1[u(\lambda)]$, whether or not $u(\lambda)$ is a minimal positive eigenfunction. For example, if equation (1.1) has eigenfunctions $u^{(1)}(\lambda)$ for λ in an interval $[\lambda_1, \lambda_1 + \delta]$, and if $\lim_{\lambda \downarrow \lambda_1} u^{(1)}(\lambda) = u^{(1)}(\lambda_1)$ with $\lambda_1 = \mu_1[u^{(1)}(\lambda_1)]$, then $\int \varphi_1^3(x)\, f_{uu}(x, u^{(1)}(\lambda_1;x)) \le 0$, where φ_1 is the positive eigenfunction of equation (1.9) with $u(\lambda)$ replaced by $u^{(1)}(\lambda)$. If $\int \varphi_1^3(x) f_{uu}(x, u^{(1)}(\lambda_1;x))\, dx < 0$, then for each $\lambda > \lambda_1$ sufficiently close to λ_1 there is another eigenfunction $u^{(2)}(\lambda) \ne u^{(1)}(\lambda)$, and

$$\lim_{\lambda \downarrow \lambda_1} \frac{u^{(1)}(\lambda) - u^{(1)}(\lambda_1)}{\sqrt{\lambda - \lambda_1}} = -\lim_{\lambda \downarrow \lambda_1} \frac{u^{(2)}(\lambda) - u^{(1)}(\lambda_1)}{\sqrt{\lambda - \lambda_1}}$$

$$= \pm \left\{ 2 \frac{\int \varphi_1(x) f_u(x, u^{(1)}(\lambda_1;x))\, u^{(1)}(\lambda_1;x)\, dx}{\lambda_1 \int \varphi_1^3(x)\, f_{uu}(x, u^{(1)}(\lambda;x))\, dx} \right\}^{\frac{1}{2}} \varphi_1.$$

We now take $r = \infty$, i.e., f is defined on $\overline{\Omega} \times (-\infty, \infty)$, and discuss the behavior of the eigenfunctions of large norm. The following elementary proposition, whose proof is omitted, lists some properties of f which will be useful in this connection.

1-6. Proposition. Let $f(x,u)$ be continuously differentiable with respect to u for each $x \in \Omega$, $0 < u < \infty$. If $\lim_{u \to +\infty} f_u(x,u) \equiv m(x) \le \infty$ exists (uniformly for $x \in \overline{\Omega}$), then $\lim_{u\to+\infty} f(x,u)/u$ exists (uniformly for $x \in \overline{\Omega}$) and equals $m(x)$.

If $\lim_{u\to+\infty} u^{-1}[f(x,u) - u f_u(x,u)] = 0$ (uniformly for $x \in \overline{\Omega}$), then $\lim_{u\to+\infty} f(x,u)/u$ and $\lim_{u\to+\infty} f_u(x,u)$ exist (uniformly for $x \in \overline{\Omega}$) and are finite and equal. (The converse is obvious.)

If for some number $s \in [0,1)$, $\lim_{u\to+\infty} u^{-s}[f(x,u) - u f_u(x,u)] \equiv (1-s)c(x)$ exists and is finite (uniformly for $x \in \overline{\Omega}$), then $\lim_{u\to+\infty} u^{-s}[f(x,u) - m(x)u]$ exists and equals $c(x)$ (uniformly for $x \in \overline{\Omega}$).

The functions $f_1(u) = u + \frac{1}{2}\sin u$, $f_2(u) = u^2 + u(1+\sin 2u) + \frac{1}{2}\cos 2u$, and $f_3(u) = u + \frac{1}{2}\int_0^u \frac{\sin v}{v}\,dv$ are continuously differentiable positive increasing functions which show that the converses of the first and last assertions of Proposition 1-6 are not valid. We have $\lim_{u\to+\infty} f_1(u)/u = 1$, but $\lim_{u\to+\infty} f_1'(u)$ does not exist; $\lim_{u\to+\infty} f_2(u)/u = +\infty$, but $\lim_{u\to+\infty} f_2'(u)$ does not exist; and $m = \lim_{u\to+\infty} f_3'(u) = \lim_{u\to+\infty} f_3(u)/u = 1$, $\lim_{u\to+\infty} [f_3(u) - m\,u] = \frac{\pi}{4}$, but $\lim_{u\to+\infty} [f_3(u) - u f_3'(u)]$ does not exist.

If $f(x,u)$ is continuous, monotone in u, and strictly positive for all $x \in \overline{\Omega}$, $0 \le u < \infty$, then it follows from Theorem 1-1 that equation (1.1) has eigenfunctions of arbitrary norm. Suppose that $\lim_{u\to+\infty} f(x,u)/u = m(x)$ exists uniformly for $x\in\overline{\Omega}$, where $0\le m(x)<\infty$ for each $x\in\overline{\Omega}$. If $m(x) = 0$ for all $x\in\overline{\Omega}$, then for any $\epsilon>0$, there is a $\rho>0$ such that $f(x,u)\le\epsilon u$ for $u\ge\rho$. Thus $f(x,u)\le \epsilon u + f_{max}(\rho)$ for all $u\ge 0$. Since the principal eigenvalue of the linear equation

$$h(x) = \mu \int K(x,y)\,\epsilon\, h(y)\, dy$$

can be made arbitrarily large by taking ϵ sufficiently small, it follows from Corollary I. 4-7 (cf. Keller and Cohen 1967, Corollary 3.3.3) that in this case equation (1.1) has solutions for all $\lambda>0$; i.e., $\Lambda=(0,\infty)$ (see also the proof of Theorem 1-2). Since there are eigenfunctions of arbitrary norm and the minimal eigenfunctions are increasing and continuous from the left, it follows that if we associate to each $\lambda\in\Lambda$ an eigenfunction $u(\lambda)$, then $\lim_{\lambda\to\mu_1}\|u(\lambda)\| = \infty$ if and only if $\mu_1=+\infty$.

On the other hand, if $m(\hat{x})>0$ for some $\hat{x}\in\overline{\Omega}$, then Λ is bounded. In fact, we can find a $\rho>0$ such that $f_{min}(0)<\frac{1}{2}m(\hat{x})\rho$ and $u\ge\rho$ implies $f(x,u)\ge\frac{1}{2}m(x)u$; it is easy to see that $f(x,u)\ge p(x)(u+\rho)$ for all (x,u), where $p(x) = \min\{\frac{1}{3\rho}f_{min}(0), \frac{1}{4}m(x)\}$. It follows from Corollary I. 4-8 (cf. Keller and Cohen 1967, Corollary 3.3.4) that Λ is bounded. Since there are eigenfunctions of arbitrarily large norm, and since Λ is bounded, there is a convergent sequence of eigenvalues $\{\lambda_n\}$ of (1.1) with corresponding eigenfunctions u_n such that $\lim_{n\to\infty}\|u_n\| = \infty$. It follows from Corollary I. 8-2 that $\lim_{n\to\infty}\lambda_n$ is positive and is an eigenvalue of the linear equation (1.7) corresponding to a positive eigenfunction;

i.e., $\lim_{n\to\infty} \lambda_n = \mu_1[\infty]$, the principal eigenvalue of equation (1.7). This establishes the first part of the following:

<u>1-7. Theorem.</u> Let $\lim_{u\to+\infty} f(x,u)/u = m(x)$ exist uniformly for $x\in\overline{\Omega}$, with $0\le m(x)<\infty$ for $x\in\overline{\Omega}$. Then there is one and only one number $\mu_1[\infty]$ for which there is a sequence $\{\lambda_n\}$ of eigenvalues of equation (1.1) converging to $\mu_1[\infty]$ such that the corresponding eigenfunctions $\{u_n\}$ satisfy $\lim_{n\to\infty} \|u_n\| = \infty$. If $m(x) = 0$ for all $x\in\overline{\Omega}$, then $\mu_1[\infty] = \infty$ and $\Lambda = (0,\infty)$; if $m(x)>0$ for some $x\in\Omega$, then $\mu_1[\infty]$ is positive and finite and is the principal eigenvalue of equation (1.7); the corresponding eigenfunction φ_∞ of (1.7) can be normalized so that

$$\lim_{n\to\infty} \frac{u_n}{\|u_n\|} = \varphi_\infty \quad ,$$

and $\varphi_\infty(x)>0$ for $x\in\Omega$.

If, furthermore, for some number $s\in[0,1)$, $\lim_{u\to+\infty} u^{-s}[f(x,u) - m(x)u] = c(x)$ exists uniformly for $x\in\Omega$ (where $m(x)>0$ for some $x\in\Omega$), then

$$\lim_{n\to\infty} (\lambda_n - \mu_1[\infty])\|u_n\|^{1-s} = \mu_1[\infty]\,\xi_\infty ,$$

where

$$\xi_\infty = \frac{\int[\varphi_\infty(x)]^{1+s} c(x)\,dx}{\int[\varphi_\infty(x)]^2 m(x)dx} .$$

If $\xi_\infty \neq 0$ (in particular, if $c(x)$ is not identically zero and does not change sign on $\overline{\Omega}$), then

$$u_n(x) = \left(\frac{\mu_1[\infty]\,\xi_\infty}{\mu_1[\infty]-\lambda_n}\right)^{s'} \varphi_\infty(x) + o(|\mu_1[\infty]-\lambda_n|^{-s'})$$

as $n \to \infty$, uniformly for $x \in \overline{\Omega}$, where $s' = (1-s)^{-1}$. If $\xi_\infty > 0$ (or $\xi_\infty < 0$), then there is a number $r > 0$ such that equation (1.1) has no positive eigenfunctions with norm greater than r corresponding to eigenvalues λ in $[\mu_1[\infty], \infty)$ (or $[0, \mu_1[\infty]]$, respectively).

Proof. When $m(x) = 0$ for all $x \in \Omega$, the remarks preceding the theorem show that $\Lambda = (0, \infty)$. When $m(x) > 0$ for some $x \in \Omega$, then we have

$$Au = A'(\infty)u + Cu + \omega(u),$$

where

$$A'(\infty)u(x) = \int K(x,y)\, m(y)\, u(y)\, dy,$$

$$C\,u(x) = \int K(x,y)\, c(y)\, [u(y)]^s\, dy,$$

and

$$\omega(u) = Au - A'(\infty)u - Cu.$$

By Proposition I.8-8, equation (I.8-12) is satisfied by $\omega(u)$. Since

$$\int \varphi_\infty^2(x) m(x)\,dx\, \xi(C\varphi_\infty) = \int \varphi_\infty(x) m(x) \int K(x,y)\, c(y)\, [\varphi_\infty(y)]^s\, dy\, dx$$

$$= \frac{1}{\mu_1} \int [\varphi_\infty(x)]^{1+s}\, c(x)\, dx,$$

the theorem follows from Theorem I.8-3 and Corollaries I.8-2 and I.8-4. //

The requirement that $s \geq 0$ in Theorem 1-7 is probably unnecessary; however, when $s < 0$, the integral in the definition of the operator C becomes improper whenever the function u is zero at any point of $\overline{\Omega}$, so the above proof does not apply directly. Related results are given in Theorems 1-9 and 1-13 below.

As usual, we have assumed without stating so explicitly that $f(x, u)$ is a strictly positive continuous function which is monotonically increasing in u. The fact that f is strictly positive is used only to guarantee the existence of eigenfunctions of arbitrarily large norm (Theorem 1-1). If $m(x) > 0$ for some $x \in \Omega$, the theorem holds as stated, even if $f(x,0) = 0$ for some or all $x \in \Omega$, whenever the equation (1.1) has positive eigenfunctions of arbitrarily large norm and the corresponding eigenvalues of (1.1) are bounded above (see Theorems I. 3-5 and I. 3-6, and also Theorem 2-1 below).

Without assuming <u>a priori</u> the existence of eigenfunctions of arbitrarily large norm or the boundedness of Λ, we have the following result from Theorem I. 8-6 and Proposition I. 8-8 (we do not assume here that f is strictly positive):

<u>1-8.Theorem.</u> Let $f(x, u)$ be non-negative and continuously differentiable with respect to u, with $f_u(x, u) \geq 0$, for $(x, u) \in \overline{\Omega} \times (0, \infty)$, and let the limits

$$\lim_{u \to +\infty} f_u(x, u) = m(x)$$

and

$$\lim_{u \to +\infty} [f(x, u) - m(x)\, u] = b(x)$$

exist uniformly for $x \in \overline{\Omega}$, where $m(x) > 0$ for some $x \in \Omega$. Let φ_∞ be the positive eigenfunction of unit norm of the linear equation (1.7), corresponding to the eigenvalue $\mu_1[\infty]$. If $\int \varphi_\infty(x) b(x) dx \neq 0$, then there exists a number δ having the same sign as $\int \varphi_\infty(x) b(x) dx$ such that for each λ between μ_1 and $\mu_1 + \delta$, there is an eigenfunction $u(\lambda)$ of (1.1) such that $\lim_{\lambda \to \mu_1} \|u(\lambda)\| = \infty$, and there is a positive number r such that for each λ between μ_1 and $\mu_1 - \delta$, there are no eigenfunctions of (1) with norm greater than r. The eigenfunctions $u(\lambda)$ satisfy

$$u(\lambda;x) = \frac{\mu_1[\infty]}{\mu_1[\infty]-\lambda} \quad \xi_\infty \, \varphi_\infty(x) \; + \; o\left(|\mu_1[\infty]-\lambda|^{-1}\right)$$

as $\lambda \to \mu_1[\infty]$, uniformly for $x \in \overline{\Omega}$, where

$$\xi_\infty = \frac{\int \varphi_\infty(y)\, b(y)\, dy}{\int \varphi_\infty^2(y) m(y) dy} \quad .$$

The following theorem gives another non-existence result for eigenfunctions of large norm; the theorem assumes less about f but more about the eigenfunctions than Theorem 1-7. The theorem assumes the existence of a function g, positive a. e. on Ω, such that the eigenfunctions satisfy $u \geq \|u\| g$; a function g such that all eigenfunctions satisfy this inequality can be found if further assumptions are made on the kernel $K(x, y)$; see Theorems 1-12, 1-13, and 1-14 below. When such a function g exists, the theorem implies that for any continuous function m which is non-negative and not identically zero on Ω, if $f(x,u) - m(x)u$ is of one sign and bounded away from zero for all sufficiently large numbers u, then all eigenfunctions of sufficiently large norm of equation (1.1) must correspond to eigenvalues λ such that

$$\operatorname{sgn}(\mu_1 - \lambda) = \operatorname{sgn}[f(x,u) - m(x)u]$$

for large u. The theorem as stated holds if our standard assumption $f(x,u) > 0$ for $x \in \overline{\Omega}$, $u \geq 0$, is weakened to $f(x,u) \geq 0$ for $x \in \overline{\Omega}$, $u \geq 0$.

1-9. Theorem. Let m be a continuous non-negative not identically zero function on $\overline{\Omega}$, and let φ be a positive eigenfunction of equation (1.7) corresponding to the eigenvalue μ_1. Suppose there is a positive number r and an integrable function c on $\overline{\Omega}$ such that for all $u \geq r$, $x \in \overline{\Omega}$, $\int \varphi(x) c(x)\, dx \neq 0$ and

$$\eta\,[f(x,u) - m(x)u - c(x)] \geq 0, \tag{1.15}$$

where $\eta = \operatorname{sgn} \int \varphi(x) c(x)\, dx$. Let g be a continuous function on $\overline{\Omega}$ which is positive almost everywhere on $\overline{\Omega}$. Then there is a number $\rho > 0$ such that equation (1.1) has no eigenfunctions u corresponding to eigenvalues λ such that

$$\|u\| \geq \rho,$$

$$u \geq \|u\|\, g,$$

and

$$\eta\,(\lambda - \mu_1) \geq 0.$$

Proof. Choose $\beta > 0$ such that

(1.16) $$|f(x,u) - m(x)\,u| \le \beta$$

for $x \in \overline{\Omega}$, $0 \le u \le r_1$. Let $\Omega_\epsilon = \{x \in \Omega : 0 \le g(x) \le \epsilon\}$. Since g is positive a.e. on Ω, the measure of Ω_ϵ satisfies $\lim_{\epsilon \to 0} [\text{meas}\, \Omega_\epsilon] = 0$. Thus we can choose a number $\rho > r_1$ such that

(1.17) $$\int_{\Omega_{r_1/\rho}} \varphi(y)\,[\beta + \eta\, c(y)]\,dy < \left|\int \varphi(y)\,c(y)\,dy\right| .$$

Suppose (1.1) has an eigenfunction u such that $u \ge \|u\|\, g \ge \rho g$. Then

$$\eta \int \varphi(y)\,[f(y,u(y)) - m(y)\,u(y)]\,dy$$

$$\ge -|\eta| \int_{u(y)\le r_1} \varphi(y)\,|f(y,u(y)) - m(y)u(y)|\,dy$$

$$+ \eta \int_{u(y)\ge r_1} \varphi(y)\,[f(y,u(y)) - m(y)\,u(y)]\,dy$$

$$\ge -\beta\,|\eta| \int_{u(y)\le r_1} \varphi(y)\,dy + \eta \int_{u(y)\ge r_1} \varphi(y)c(y)\,dy$$

$$\ge -\int_{\Omega_{r_1/\rho}} \varphi(y)[\beta + \eta\, c(y)]\,dy + \eta \int \varphi(y)\,c(y)\,dy$$

$$> 0 ,$$

where we have used equations (1.15), (1.16) and (1.17), and the fact that $u(y) \le r_1$ implies $g(y) \le u(y)/\|u\| \le r_1/\rho$. Since

$$\frac{1}{\lambda} \int u(x)m(x)\varphi(x)\,dx$$

$$= \int \varphi(x)m(x) \int K(x,y)\, f(y,u(y))\, dy\, dx$$

$$= \frac{1}{\mu_1} \int \varphi(y)\, f(y,u(y))\, dy,$$

we have

$$\eta \left(\frac{1}{\lambda} - \frac{1}{\mu_1}\right) \int u(x)\, m(x)\, \varphi(x)\, dx$$

$$= \frac{\eta}{\mu_1} \int \varphi(y)\, [f(y,u(y)) - m(y)\, u(y)]\, dy$$

$$> 0.$$

Thus $\eta(\mu_1 - \lambda) > 0$. //

The hypothesis that m is continuous can be weakened; it suffices to require that m be a non-negative not identically zero function on Ω such that equation (1.7) has a non-negative integrable eigenfunction φ, with the product $\varphi \cdot c$ integrable, corresponding to the eigenvalue μ_1, and that equation (1.15) holds a.e. on $\overline{\Omega}$. It follows from Tonelli's Theorem (Smirnov 1964b) that the product $u \cdot m \cdot \varphi$ is integrable for any eigenfunction u of equation (1.1), since

$$\int u(x)m(x)\, \varphi(x)\, dx$$

$$\leq \lambda \int [K(x,y)\, f_{max}(\|u\|)\, dy]\, m(x)\, \varphi(x)\, dx$$

$$= \lambda f_{max}(\|u\|) \int [\int K(y,x)m(x)\, \varphi(x)\, dx]\, dy,$$

$$= \lambda f_{max}(\|u\|) \int \varphi(y)\, dy/\mu_1 .$$

and φ is integrable by assumption.

Suppose the functions m and φ in Theorem 1-9 are the same as the functions m and φ_∞, respectively, of Theorem 1-7. If $\int [\varphi_\infty(x)]^{1+s} c(x)\,dx > 0$ in Theorem 1-7 or, with s = 0, in Theorem 1-9, then the eigenfunctions of large norm correspond to eigenvalues $\lambda > \mu_1[\infty]$ and therefore cannot all be minimal solutions, since the norms of these eigenfunctions approach ∞ as λ decreases to μ_1, while minimal solutions increase as λ increases. Thus there are values of λ greater than $\mu_1[\infty]$ for which (1.1) has at least two eigenfunctions, and $\mu_1[\infty] < \lambda^* = \sup \Lambda$.

This result can be sharpened by using Corollary I. 9-3:

1-10. Theorem. Let f be as described in the first sentence of Theorem 1-7, with $m(x) > 0$ for some $x \in \Omega$, and let $f(x, u)$ have a partial derivative $f_u(x, u)$ which is continuous on $\overline{\Omega} \times [0, \infty)$. If the principal eigenvalue $\mu_1[\infty]$ of equation (1.7) is less than $\lambda^* = \sup \Lambda$, then the problem (1.1) has at least two solutions for all $\lambda \in (\mu_1[\infty], \lambda^*)$ such that $\lambda \neq \mu_1[u^o(\lambda)]$, and $\lambda^* = \mu_1[u^o(\lambda^*)] \in \Lambda$.

Proof. This last assertion that $\lambda^* = \mu_1[u^o(\lambda^*)] \in \Lambda$ follows from the fact that since $\mu_1[\infty] < \lambda^*$, $\lim_{\lambda \uparrow \lambda^*} \|u^o(\lambda)\| \neq \infty$ (see Theorems I. 3-2 and I. 3-3). The rest of the theorem is an immediate consequence of the Fréchet differentiability of A and Corollary I. 9-3. //

The assumption $\lambda \neq \mu_1[u^o(\lambda)]$ of the proceding theorem is probably not necessary. For example, if Theorem 1-5 is applicable at a point $\lambda_o \in (\mu_1[\infty], \lambda^*)$ such that $\lambda_o = \mu_1[u^o(\lambda_o)]$, then the minimal positive eigenfunctions $u^o(\lambda)$ are discontinuous from the right at λ_o, and $\lim_{\lambda \downarrow \lambda_o} u^o(\lambda)$ exists and is a non-minimal eigenfunction corresponding to the eigenvalue λ_o (see the remarks following Theorem I. 7-6).

Theorem 1-10 may be applicable also when $\int [\varphi_\infty(x)]^{1+s} c(x)dx \le 0$ in Theorem 1-7. For example, by bounding f above by a continuous function g for which $\sup \Lambda_g$ can be determined, and using the fact that $f(x,u) \le g(x,u)$ implies $\sup \Lambda_f \ge \sup \Lambda_g$, it may be possible to show that $\mu_1[\infty] < \sup \Lambda_f$. Then there will be a second positive eigenfunction $u^{(1)}(\lambda) > u^o(\lambda)$ for all $\lambda \in (\mu_1[\infty], \sup \Lambda_f)$ such that $\lambda \ne \mu_1[u^o(\lambda)]$, although we will not have $\lim_{\lambda \downarrow \mu_1[\infty]} \|u^{(1)}(\lambda)\| = \infty$ if $\int [\varphi_\infty(x)]^{1+s} c(x)\, dx < 0$.

As $m(x)$ increases, the principal eigenvalue $\mu_1[\infty]$ of equation (1. 7) decreases. This suggests the following conjecture, which we have been unable to prove in general (but see Corollary I. 8-10 and Theorems 1-12 and 1-14; also, Levinson 1962, Berger 1965a and 1965b).

<u>1-11. Conjecture.</u> If $\lim_{u \to +\infty} f(x,u)/u = \infty$ uniformly for $x \in \overline{\Omega}$, then there is a sequence $\{\lambda_n\}$ of eigenvalues of (1. 1) converging to 0 to which there correspond eigenfunctions $\{u_n\}$ such that $\lim_{n \to \infty} \|u_n\| = \infty$. Furthermore, if $\{\lambda_n\}$ is any sequence of eigenvalues of (1. 1) to which there correspond eigenfunctions $\{u_n\}$ such that $\lim_{n \to \infty} \|u_n\| = \infty$, then $\lim_{n \to \infty} \lambda_n = 0$.

This conjecture can be proved if we make further assumptions on the kernel $K(x,y)$. We first consider the case when $K(x,y)$ is bounded above and below by positive constants.

Let

$$\sigma = \frac{\max\{K(x,y) : (x,y) \in \overline{\Omega} \times \overline{\Omega}\}}{\min\{K(x,y) : (x,y) \in \overline{\Omega} \times \overline{\Omega}\}},$$

$$\Gamma u(x) = \int K(x,y)\, u(y)\, dy,$$

and

$$\tilde{C} = \left\{ u \in C : \|u\| \le \sigma \min\{u(x) : x \in \overline{\Omega}\} \right\}.$$

Then $\tilde{C}$ is a closed convex cone in $C(\overline{\Omega})$ (Krasnosel'skii 1964a), and the operator A (equation (1.3)), the operator A^o_λ defined by

$$A^o_\lambda h = \lambda A[u^o(\lambda) + h] - \lambda A u^o(\lambda) \tag{1.18}$$

for any $\lambda \in \Lambda$, and any linear operator T of the form

$$T \cdot u(x) = \int K(x,y)\, p(y) u(y)\, dy, \tag{1.19}$$

where $p \in C(\overline{\Omega})$ is positive on Ω, each map the cone C of non-negative functions into the cone $\tilde{C}$. Moreover, for any function $u \in \tilde{C}$, we have

$$\begin{aligned} \Gamma u(x) &= \int K(x,y)\ u(y)\ dy \\ &> \sigma^{-1} \|u\| \int K(x,y)\ dy; \end{aligned} \tag{1.20}$$

this inequality permits the application of Corollary I.8-10 and Theorem I.9-1 to prove the following:

<u>1-12. Theorem.</u> Let $f(x,u)$ have a continuous partial derivative $f_u(x,u)$ which is positive on $\Omega \times (0, \infty)$, and let $\lim_{u \to +\infty} f(x,u)/u = \infty$ uniformly for $x \in \overline{\Omega}$. Let the kernel $K(x,y)$ be continuous and uniformly bounded below by a positive number for $(x,y) \in \overline{\Omega} \times \overline{\Omega}$. Then for any $\lambda \in \Lambda$ such that $\lambda < \mu_1 [u^o(\lambda)]$, the problem (1.1) has at least two positive eigenfunctions. For any sequence $\{u(\lambda_n)\}$ of non-minimal eigenfunctions of (1.1) such that $\lim_{n \to \infty} \lambda_n = 0$, we have $\lim_{n \to \infty} \| u(\lambda_n) \| = \infty$; conversely, for any sequence $\{u(\lambda_n)\}$ of eigenfunctions such that $\lim_{n \to \infty} \| u(\lambda_n) \| = \infty$, we have $\lim_{n \to \infty} \lambda_n = 0$.

<u>Proof.</u> (Cf. Corollary I. 9-4.) Since $f(x,u)/u$ approaches ∞ uniformly, given any positive α there is a positive number $r_\alpha \geq f_{min}(0)/\alpha$ such that $f(x,u) \geq \alpha u$ for all $u \geq r_\alpha$. Then for any $x \in \overline{\Omega}$,

$$f(x,u) \geq \alpha u + f_{min}(0) - \alpha r_\alpha .$$

Define the compact positive linear operator $\Gamma_\lambda = \lambda \Gamma$ by

$$\Gamma_\lambda h(x) = \lambda \int_0^1 K(x,y)\, h(y)\, dy$$

for $h \in C(\overline{\Omega})$, and define the function $b_{\lambda,\alpha} \in C(\overline{\Omega})$ by

$$b_{\lambda,\alpha}(x) = \lambda \int_0^1 K(x,y) \, [\alpha u^o(\lambda;y) + f_{min}(0) - \alpha r_\alpha] \, dy.$$

The characteristic value problem $\mu \alpha \Gamma_\lambda h = h$ for $\alpha \Gamma_\lambda$ is equivalent to the eigenvalue problem for the integral equation (for fixed λ, α) $h(x) = \mu \int \alpha \lambda K(x,y) h(y)\, dy$, $x \in \overline{\Omega}$. All eigenvalues for this problem are positive; by choosing α sufficiently large, the principal eigenvalue can be made less than 1. For α so chosen and fixed λ, it is easy to see that all conditions of (a_∞) of Theorem I. 9-1 are satisfied by the operator $T_\lambda = \alpha\Gamma_\lambda$ on the cone $\tilde{C}$, with the function $b = b_{\lambda,\alpha} - u^o(\lambda)$, and the function g_o defined by $g_o(x) = (\alpha\lambda/\sigma)\int_o^1 K(x,y)\,dy$ (see inequality (1.20)).

Condition (b_o) of Theorem I. 9-1 is satisfied by the Fréchet derivative $A_\lambda^{o\prime}(0)$ given by

$$A_\lambda^{o\prime}(0)h(x) = \lambda\int_0^1 K(x,y)f_u(x,u^o(\lambda;y))h(y)\,dy$$

$$= \lambda A'(u^o(\lambda))\,h(x).$$

If $\lambda<\mu_1[u^o(\lambda)]$, then $1<\mu_o[A_\lambda^{o\prime}(0)] = \lambda^{-1}\mu_o[A'(u^o(\lambda))] = \lambda^{-1}\mu_1[u^o(\lambda)]$.

Theorem I. 9-1 thus shows that if $\lambda<\mu_1[u^o(\lambda)]$, then the operator A_λ^o has a non-zero fixed point $h(\lambda)$ in $\tilde{C}$, and therefore problem (1.1) has an eigenfunction $u^o(\lambda) + h(\lambda) > u^o(\lambda)$ corresponding to the eigenvalue λ.

The fact that $\lim_{n\to\infty}\|u(\lambda_n)\| = \infty$ implies $\lim_{n\to\infty}\lambda_n = 0$ follows directly from Corollary I. 8-10 since the operator A of equation (1.3) maps C into $\tilde{C}$ and satisfies condition (I. 9.8) with the function b_α given by

$$b_\alpha(x) = \int_o^1 K(x,y)\,[f_{min}(0) - \alpha r]\,dy \quad . //$$

For the possibility of removing the condition $\lambda < \mu_1[u^o(\lambda)]$ when $0<\lambda<\lambda^* = \sup\Lambda$, see the remarks following Theorem 1-10.

The introduction of the cone $\tilde{C}$ is suggested by Krasnosel'skii (1964a, p. 246), who considers the case that $f(x, u) \geq a\, u^{1+\epsilon} + b$ for positive constants a, ϵ, and $f(x, 0) \equiv 0$. See also Krasnosel'skii (1964b, p. 276).

When the kernel $K(x, y)$ satisfies the conditions of the preceding theorem, all eigenfunctions u of equation (1. 1) satisfy the condition $u \geq \|u\|\, g$ of Theorem 1-9, where g is the constant function $g(x) \equiv \sigma^{-1}$. Thus Theorem 1-9 provides additional information about the behavior of eigenfunctions of large norm when $\lim_{u \to \infty} f(x,u)/u \neq \infty$. A similar result is given in the following:

<u>1-13. Theorem.</u> Let the kernel $K(x, y)$ in (1. 1) be as in Theorem 1-12, and let m and φ be functions as in Theorem 1-9. Suppose there is a positive number r_1 and a continuous function c on $\bar{\Omega} \times [r_1, \infty)$ such that for each $u \geq r_1$, $c(x,u)$ does not change sign on $\bar{\Omega}$, $|c(x, u)|$ is a non-increasing function of u for each $x \in \bar{\Omega}$, $\int \varphi(x)\, c(x, u)\, dx \neq 0$, and $\eta\, [f(x, u) - m(x)\, u - c(x, u)] \geq 0$, where $\eta = \operatorname{sgn} \int \varphi(x)\, c(x, u)\, dx$. Then there is a positive number ρ such that equation (1. 1) has no eigenfunctions u corresponding to eigenvalues λ such that $\|u\| \geq \rho$ and $\eta(\lambda - \mu_1) \geq 0$.

(For example, the function c may have the form $c(x, u) = c_1(x)\, u^{-s}$, where c_1 is a non-negative or non-positive function on $\bar{\Omega}$, $\int \varphi(x)\, c_1(x)\, dx \neq 0$, and $s \geq 0$).

<u>Proof</u>. Since $u(x) \geq \|u\|/\sigma$, it is possible to choose ρ such that there is no $x \in \Omega$ with $u(x) \leq r_1$ for $\|u\| \geq \rho$. The proof is then similar to that of Theorem 1-9. //

An analogous theorem holds when the domain Ω is an interval of real numbers and the kernel $K(x, y)$ is concave in x, i. e., for any $x_1, x_2, y \in \Omega$ and any number $\alpha \in [0, 1]$, $K(\alpha x_1 + (1-\alpha) x_2, y) \geq \alpha K(x_1, y) + (1-\alpha) K(x_2, y)$. (The Green's function for the operator $-\frac{d^2}{dx^2}$ subject to homogeneous unmixed boundary conditions is an example of such a kernel (see Section II . 2).) For convenience, we will take $\Omega = (0, 1)$. In this case, any solution of equation (1. 1) is a non-negative concave function on $[0, 1]$, and the difference $h(x)$ between any solution and the minimal positive solution for the same λ is also a non-negative concave function, since it satisfies the equation

$$h(x) = \lambda \int K(x, y) [f(y, u^o(\lambda; y) + h(y)) - f(y, u^o(\lambda; y))] \, dy.$$

Thus any solution of (1. 1) is an element of the cone $\mathcal{C}_c$ of non-negative concave functions in $C[0, 1]$, and the operators A (equation (1. 3)), A^o_λ (equation (1. 18), and T (equation (1. 19) each map $\mathcal{C}$ into $\mathcal{C}_c$.

By considering the graph of a continuous non-negative concave function on $[0, 1]$, it is clear that such functions satisfy the following important inequality:

(1. 21) $$u(x) \geq \| u \| g_1(x),$$

where

$$g_1(x) = \tfrac{1}{2} - |x - \tfrac{1}{2}|.$$

To prove this analytically, we use the fact that by definition a concave function satisfies

$$u(\alpha x_o + (1-\alpha)y) \geq \alpha u(x_o) + (1-\alpha)u(y)$$

for any x_o, y, and α in $[0,1]$. Taking x_o to be a point at which the non-negative continuous function u assumes its maximum, $y = 0$, and $\alpha = x/x_o$ if $x_o \neq 0$, we obtain

$$u(x) \geq \frac{x}{x_o} \| u \|, \quad 0 \leq x \leq x_o ,$$

Similarly, if $x_o \neq 1$,

$$u(x) \geq \frac{1-x}{1-x_o} \| u \|, \; x_o \leq x \leq 1 .$$

It is easy to see that for any $x_o \in (0,1]$, $(1-x)/(1-x_o) \geq \frac{1}{2} - |x - \frac{1}{2}|$. This proves equation (1.21).

Thus for any function u in C_c, any operator T of the form given by equation (1.19) satisfies inequality (I.9.2) of Theorem I.9-1 as follows:

$$Tu(x) = \int_0^1 K(x,y)p(y)u(y) \; dy$$

$$\geq \|u\| \int_0^1 K(x,y)p(y)g_1(y)\,dy.$$

The proof of the following Theorem is therefore essentially the same as the proof of Theorem 1-12:

1-14. Theorem. Let the function f be as in Theorem 1-12, let the kernel K(x, y) be a concave function of x, and let Ω be a bounded interval in $\mathbb{R}$. For any $\lambda \in \Lambda$ such that $\lambda < \mu_1 [u^o(\lambda)]$, the problem (1. 1) has at least two positive solutions. Any sequence $\{u(\lambda_n)\}$ of non-minimal eigenfunctions of (1. 1) satisfies $\lim_{n\to\infty} \|u(\lambda_n)\| = \infty$ if and only if $\lim_{n\to\infty} \lambda_n = 0$.

Note: The cones $\tilde{C}$ and C_c do not satisfy condition (e) of Section I. 1 in $C(\overline{\Omega})$; i. e., not every function in $C(\overline{\Omega})$ can be represented as the difference of two functions in $\tilde{C}$ or two functions in C_c. However, once we have established that the eigenfunctions of A and the positive eigenfunctions of Γ must lie in $\tilde{C}$ or C_c, then we may restrict our considerations to these cones, and the condition (e) is unnecessary.

We now suppose that f(x, u) is a convex function of u on $[0,\infty)$; i. e., for all $u, v \in [0,\infty)$ and all $\alpha \in [0,1]$, $f(x, \alpha u+(1-\alpha)v) \le \alpha f(x,u) + (1-\alpha) f(x,v)$. The following proposition lists some of the properties of convex functions which are useful in the application of Theorem 1-16 below.

1-15. Proposition. Let f be a monotonically increasing convex function on $[0,\infty)$. Then f has left- and right-hand derivatives $f'(u-)$ and $f'(u+)$ for all $u\in(0,\infty)$ and these derivatives are increasing functions of u; whenever $0 \le v \le u$,

(1.22) $$f'(v+)(u-v) \le f(u) - f(v) \le f'(u-)(u-v);$$

$\lim_{u\to+\infty} f(u)/u$ exists as a finite number or $+\infty$; and

(1.23) $$\lim_{u\to+\infty} \frac{f(u)}{u} = \lim_{u\to+\infty} f'(u-) = \lim_{u\to+\infty} f'(u+).$$

If $\lim_{u\to+\infty} f(u)/u = m<\infty$, then for any number $s \ge 0$, $u^{-s}[f(u) - mu]$ is a decreasing function of u. If there is a number $r \ge 0$ such that f is differentiable on (r, ∞), then for any $s \ge 0$, $u^{-s}[f(u) - uf'(u)]$ is a decreasing function of u for $u > r$, and $\lim_{u\to+\infty} u^{-s}[f(u) - mu] = (1-s)^{-1}\lim_{u\to+\infty} u^{-s}[f(u) - uf'(u)]$ for $s \ne 1$.

Proof. The assertions of the theorem through equation (1.22) are proven, e.g., by Choquet (1966). If we divide equation (1.22) by u, let $u\to+\infty$, and then let $v\to+\infty$, we obtain

$$\lim_{v\to+\infty} f'(v+) \le \liminf_{v\to+\infty} \frac{f(u)}{u} \le \limsup_{u\to+\infty} \frac{f(u)}{u} \le \lim_{u\to+\infty} f'(u-);$$

since $f'(u-) \le f'(u+)$ for all u, this establishes equation (1.23). The rest of the theorem follows from equation (1.22) and l'Hospital's rule. //

1-15. Theorem. Let $f(x,u)$ be a convex function of u for $x \in \overline{\Omega}$, $0 \le u < \infty$, and let $\lim_{u\to+\infty} f(x,u)/u = m(x) < \infty$ exist uniformly for $x \in \overline{\Omega}$. If $f(x,u)/u$ is a strictly decreasing function of u for

$x \in \overline{\Omega}$, $0<u<\infty$, or if for some number $s \in [0,1)$, $\lim_{u\to+\infty} u^{-s}[f(x,u)-m(x)u] = c(x)$ exists uniformly for $x\in\overline{\Omega}$ and $\int [\varphi_\infty(x)]^{1+s} c(x)\,dx > 0$ (where φ_∞ is a positive eigenfunction of equation (1.7) corresponding to the eigenvalue $\mu_1[\infty]$), then $\Lambda = (0, \mu_1[\infty])$, and for each $\lambda \in \Lambda$ there is a unique eigenfunction.

On the other hand, if $\lim_{u\to+\infty} u^{-s}[f(x,u) - m(x)u] = c(x)$ exists uniformly for $x\in\overline{\Omega}$ and $\int [\varphi_\infty(x)]^{1+s} c(x)\,dx < 0$, then $\lambda^* = \sup \Lambda > \mu_1[\infty]$. If f has a continuous partial derivative $f_u(x,u)$ on $\overline{\Omega} \times [0, \infty]$, then for each $\lambda \in (\mu_1[\infty], \lambda^*)$ there are at least two eigenfunctions. The only value of λ for which there is an eigenfunction $u(\lambda)$ such that $\lambda = \mu_1[u(\lambda)]$ is λ^*, and $\lambda^* = \mu_1[u^o(\lambda^*)]$. If $f(x,u)$ is strictly convex in u for all x in Ω, then there is only one eigenfunction corresponding to the eigenvalue λ^*. (By strict convexity, we mean that $0<v<u$ and $0<\alpha<1$ imply $f(x, \alpha u + (1-\alpha)v) < \alpha f(x,u) + (1-\alpha) f(x,v)$.)

Proof. If $f(x,u)/u$ is a strictly decreasing function of u, the assertions of the theorem follow from Theorem 1-2. The uniqueness of the eigenfunctions for $\lambda \in (0, \mu_1[\infty])$ is proved in Corollary I. 10-12, since $f(x,u) - f(x,v) \le m(x)(u-v)$ for $0 \le v \le u$.

If $\int [\varphi_\infty(x)]^{1+s} c(x)\,dx > 0$, then Theorems 1-1 and 1-7 show that there are eigenfunctions whose norms approach infinity as λ approaches $\mu_1[\infty]$ from below. Since the eigenfunctions are unique for $\lambda < \mu_1[\infty]$, these must be minimal positive eigenfunctions; i. e., $\lim_{\lambda \uparrow \mu_1} \|u^o(\lambda)\| = \infty$. Therefore there can be no minimal eigenfunctions for $\lambda \ge \mu_1[\infty]$, since $u^o(\lambda)$ increases as λ increases, and thus $\Lambda = (0, \mu_1[\infty])$.

When $\int [\varphi_{\infty}(x)]^{1+s} c(x)\, dx < 0$, the rest of the theorem follows from Theorems 1-7, I.10-12, and I.10-15. //

When the assumptions of Theorem 1-12 or Theorem 1-14 are satisfied by the kernel K(x, y) and the function f, and f(x, u) is also convex in u, then we can obtain a result similar to Theorem 1-16 for the case $\lim_{u \to +\infty} f(x,u)/u = \infty$:

<u>1-17. Theorem</u>. Suppose that in either Theorem 1-12 or Theorem 1-14, f satisfies the following additional hypothesis: f(x, u) is a convex function of u for $x \in \overline{\Omega}$, $u > 0$. Then there is a number $\lambda^* = \max \Lambda$ with the following property: For any $\lambda \in (0, \lambda^*)$, there are at least two positive eigenfunctions; for $\lambda = \lambda^*$, there is at least one positive eigenfunction; and for $\lambda > \lambda^*$, there are no positive eigenfunctions. If to each $\lambda \in (0, \lambda^*)$ we assign a non-minimal eigenfunction $u(\lambda)$, then $\lim_{\lambda \to \lambda_1} \|u(\lambda)\| = \infty$ if and only if $\lambda_1 = 0$.

We will consider the question of the uniqueness of the eigenfunctions corresponding to λ^* more carefully below (see Theorems 1-23 and 1-24).

The assumption that f(x, u) is continuously differentiable with respect to u in Theorems 1-10, 1-16, and 1-17 is used only to assure that the Fréchet derivatives $A'(u^o(\lambda))$ exist, so that condition (b_o) of Theorem I. 9-1 holds. If f is not differentiable, condition (a_o) of Theorem I. 9-1 may be applicable, in place of condition (b_o). Another approach which does not necessarily require differentiability is to use Theorems I. 3-4 and I. 3-6. If for fixed λ, one can show that the operator A_λ^o defined by equation (1.18) has a set of positive eigenfunctions corresponding to characteristic values in an interval which contains the number 1, then A_λ^o has a positive fixed point $h(\lambda)$, and $u^o(\lambda) + h(\lambda)$ is a second positive eigenfunction of A corresponding to the characteristic value λ.

Further information concerning the behavior of the eigenfunctions of (1.1) when $f(x,u)$ is convex in u is contained in the following theorem. The theorem follows directly from Theorem I. 10-16 if $f_u(x,u)$ is strictly positive on Ω for each u, so that the Fréchet derivatives $A'(u)$ satisfy (PA_2). The proof under the weaker condition $f_u(x,u) \geq 0$ is given below. In this theorem, we again assume $f(x,u)$ defined only for $u<r$, where $0<r\leq+\infty$.

<u>1-18. Theorem.</u> Let $f(x,u)$ be convex in u and have a continuous partial derivative $f_u(x,u) \geq 0$ for all $x \in \overline{\Omega}$, $0 \leq u < r$. Then for any $\lambda < \sup \Lambda = \lambda^*$, equation (1.1) does not have three eigenfunctions $u^o(\lambda)$, $u_1(\lambda)$, $u_2(\lambda)$ in $\mathcal{C}^r$ satisfying $u_2(\lambda) > u_1(\lambda) > u^o(\lambda)$. If $\lambda^* \in \Lambda$, and equation (1.1) has more than one eigenfunction in $\mathcal{C}^r$ corresponding to λ^*, it has an infinite number, and all such eigenfunctions have the form $u^o(\lambda^*) + \alpha\varphi^*$, where $\alpha \geq 0$ and φ^* is the positive eigenfunction of unit norm of the linear variational equation (1.9) with u replaced by $u^o(\lambda^*)$; moreover,

there is an $\alpha_o \in (0, \infty]$ such that all functions $u^o(\lambda^*) + \alpha\varphi^*$ with $0 \le \alpha < \alpha_o$ are eigenfunctions of equation (1.1). If we denote by φ^*_α the positive eigenfunction of unit norm of equation (1.9) with u replaced by $u^*_\alpha \equiv u^o(\lambda^*) + \alpha\varphi^*$, then we must have

$$\int \varphi^*_\alpha(x)\, f(x, u^*_\alpha(x))\, dx = 0.$$

for $0 \le \alpha < \alpha_o$.

Since we are assuming that the kernel $K(x, y)$ is symmetric, these results can be derived by using the variational characterization of the smallest eigenvalue of equation (1.9). This method of proving Theorem 1-16 does not require that $f_u(x, u)$ be strictly positive. The proof is based on the following theorem; in this theorem, we assume only the conditions on f explicitly stated in the theorem, but retain the assumption of symmetry of the kernel and the assumptions on the kernel made at the beginning of this section.

1-19. Theorem. Let the function g be continuous on $\overline{\Omega}$. For real numbers $a < b$, let f be continuous on $\overline{\Omega} \times [a, b)$ and have a continuous derivative $f_u(x, u) \ge 0$ with respect to u for x in $\overline{\Omega}$ and $a \le u < b$. Suppose that whenever v_1 and v_2 satisfy $a < v_1 < v_2 < b$, then $f_u(x, v_1) \le f_u(x, v_2)$ for all $x \in \overline{\Omega}$. Then there do not exist three distinct functions u_i, $i = 0, 1, 2$, defined on $\overline{\Omega}$ and satisfying

(1.24) $$u_i(x) = \int K(x,y)\, f(y, u_i(y))\, dy + g(x)$$

and

(1.25) $$a \le u_o(x) \le u_1(x) \le u_2(x) < b, \quad x \in \overline{\Omega},$$

unless the linear equation

(1.26) $$h(x) = \int K(x,y)\, f_u(y, u_o(y))\, h(y)\, dy$$

has a positive solution.

If equation (1.26) does have a positive solution φ for a function u_o satisfying equation (1.24), then either u_o is the only solution of equation (1.24) or this equation has an infinite number of solutions u_α, all of which are of the form $u_\alpha = u_o + \alpha\varphi$, where $0 \le \alpha < \alpha_o$ for some number $\alpha_o \in (0, \infty]$.

The proof of Theorem 1-19 will be established with the help of the following two lemmas. We will use the notation

$$g_1(x) \le\le g_2(x), \quad x \in \Omega,$$

if g_1 and g_2 are two functions on Ω for which $g_1(x) \le g_2(x)$ for all $x \in \Omega$ and strict inequality holds on a subset of Ω of positive measure. Thus, if g_1 and g_2 are continuous on Ω, $g_1(x) \le\le g_2(x)$, $x \in \Omega$, is equivalent to $g_1 < g_2$ in the notation of Section I.1, and in this case $\int [g_2(x) - g_1(x)]\, dx > 0$.

<u>1.20. Lemma.</u> Let r_1 and r_2 be continuous non-negative functions on $\overline{\Omega}$ satisfying

$$0 \leq r_1(x) \leq\leq r_2(x),\ x \in \Omega .$$

Then the principal eigenvalues $\mu^{(1)}$ and $\mu^{(2)}$ of

(1.27) $$h(x) = \mu \int K(x,y)\ r_i(y)h(y)\ dy,\ i = 1,2,$$

satisfy $\mu^{(1)} > \mu^{(2)}$. (If $r_1(x) = 0$ for all $x \in \Omega$, then we take $\mu^{(1)} = \infty$.)

Proof. The lemma is obvious if $r_1(x) = 0$ for all $x \in \Omega$, since then $\mu^{(2)} < \infty = \mu^{(1)}$. If $r_1(x) \geq\geq 0$ for $x \in \Omega$, the smallest eigenvalues of (1.27) are also the smallest eigenvalues of

$$h(x) = \mu \int \sqrt{r_i(x)}\ K(x,y)\sqrt{r_i(y)}\ h(y)\ dy$$

$$= \mu\ \bar{T}(r_i)\ h(x)\ ;$$

since all iterates of the kernel $\sqrt{r_i(x)}\ K(x,y)\sqrt{r_i(y)}$ of the linear integral operator $T(r_i)$ are symmetric, we have

(1.28) $$0 < \left[\frac{1}{\mu^{(1)}}\right]^m = \max \langle h, [T(r_i)]^m h \rangle$$

for a sufficiently large integer m, where $\langle h, k \rangle$ denotes the inner product $\langle h, k \rangle = \int h(x)k(x)\ dx$, and the maximum is taken over a suitable class of functions h satisfying $\langle h, h \rangle = 1$ (Mikhlin 1964). If we let φ_1 be a positive eigenfunction of equation (1.27), then $\varphi_1(x) > 0$ for $x \in \Omega$, and, setting $h = h_1 \equiv \sqrt{r_1}\varphi_1$ in (1.28), we have

$$\left[\frac{1}{\mu^{(1)}}\right]^m = \langle h_1, [T(r_1)]^m h_1\rangle$$

$$< \langle h_1, [T(r_2)]^m h_1\rangle \quad \leq \left[\frac{1}{\mu^{(2)}}\right]^m .$$

Thus $\mu^{(1)} > \mu^{(2)}$. //

1-21. Lemma. Let f and g be as in Theorem 1-19; let u_o and u_1 be distinct functions defined in $\overline{\Omega}$ and satisfying (1-24) and (1-25); let $\mu^{(0)}$ and $^{(1)}$ be the first eigenvalues of the linear boundary value problems (1.9) with u replaced by u_o and u_1, respectively. Then $u_o(x) < u_1(x)$ for all $x \in \Omega$, and either $\mu^{(0)} = 1 = \mu^{(1)}$ or $\mu^{(0)} > 1 > \mu^{(1)}$. If $\mu^{(0)} = 1 = \mu^{(1)}$, then $u_1(x) = u_o(x) + \varphi$, where φ is a positive solution of equation (1.26) for $i = 0$ or 1, and all functions $u_o(x) + \alpha\varphi$, $0 \leq \alpha \leq 1$, are solutions of equation (1.24).

Proof. Let

$$\tilde{f}(x) = \int_0^1 f_u(x, \alpha u_1(x) + (1-\alpha) u_o(x))\, d\alpha .$$

Clearly, for any $x \in \overline{\Omega}$, either $f_u(x, u_o(x)) < \tilde{f}(x) < f_u(x, u_1(x))$ or $f_u(x, u_o(x)) = \tilde{f}(x) = f_u(x, u_1(x))$. From equation (1.24),

$$(1.29) \quad u_1(x) - u_o(x) = \int K(x,y)\tilde{f}(y)[u_1(y) - u_o(y)]\, dy, \; x \in \Omega ;$$

thus $0 \leq \leq \tilde{f}(y) \leq f_u(y, u_1(y))$, $y \in \Omega$, and $u_1 - u_o$ is a positive eigenfunction of the linear equation (1.27) with $r_i(y)$ replaced by $\tilde{f}(y)$, corresponding to the eigenvalue 1. Hence $u_2(x) - u_1(x) > 0$ for all $x \in \Omega$. If $f_u(x, u_1(x)) = \tilde{f}(x) = f_u(x, u_2(x))$ for all $x \in \Omega$, then $\mu^{(0)} = 1 = \mu^{(1)}$. If not, then

$$(1.30) \qquad 0 \le f_u(x, u_o(x)) \le\le \tilde{f}(x) \le\le f_u(x, u_1(x)) \quad , \quad x \in \Omega ,$$

and the preceding lemma shows that

$$\infty \ge \mu^{(0)} > 1 > \mu^{(1)}$$

If $\mu^{(0)} = 1 = \mu^{(1)}$, then $f_u(x, u_o(x)) = \tilde{f}(x) = f_u(x, u_1(x))$; from equation (1.29) it follows that $u_1 - u_o$ is a positive solution of equation (1.26) and just as in the proof of Theorem I.10-16, it follows that $u_o(x) + \alpha\varphi = \alpha u_1(x) + (1-\alpha) u_o(x)$ is a solution of equation (1.24) for $0 \le \alpha \le 1$. //

Proof of Theorem 1-19. If three distinct functions satisfying (1.22) and (1.23) were to exist, then by Lemma 1-19 we would have either $\mu^{(0)} > 1 > \mu^{(1)} > 1 > \mu^{(2)}$, which is clearly impossible (here $\mu^{(i)}$ are the first eigenvalues of the problems (1.27) for $i = 0, 1, 2$), or $\mu^{(0)} = 1 = \mu^{(1)} = \mu^{(2)}$, and then the assertions of the theorem follow from Lemma 1-21. //

1-22. Corollary. Consider the non-linear eigenvalue problem

$$(1.31) \qquad u(x) = g(x) + \lambda \int K(x, y)\, f(y, u(y))\, dy,$$

where f and g satisfy the conditions of Theorem 1-19. Suppose that for each fixed $\lambda > 0$, any two solutions u_o and u_1 of (1.31) satisfy $u_0(x) \le u_1(x)$ (or $u_1(x) \le u_o(x)$), $x \in \Omega$. Then for any $\lambda > 0$ such that the linear equation (1.9) does not have λ as its principal eigenvalue, there exist at most two solutions of (1.31), and if for some such $\lambda > 0$, there are two solutions u_0 and u_1, then $u_0(x) < u_1(x)$ (say) for x in Ω, and $\mu^{(0)} > \lambda > \mu^{(1)}$, where $\mu^{(i)}$ are the first eigenvalues of the problems (1.9) with u replaced by u_i, for $i = 0, 1$.

<u>Proof of Theorem 1-18.</u> Theorem 1-19 and Lemma 1-20, together with the fact that $\lambda = \mu_1[u^o(\lambda)]$ if and only if $\lambda = \max \Lambda$, imply all of Theorem 1-18 except the last equation of this theorem; this equation follows from

$$u^o(\lambda^*;x) = \lambda^* \int K(x,y)\, f_u(y, u_\alpha^*(y))\, u^o(\lambda^*;y)\, dy$$

$$+ \lambda^* \int K(x,y)\, f(y, u_\alpha^*(y))\, dy$$

(see equation (1.29)). //

Since by Lemma 1-21 any two solutions $u_o \le u_1$ of (1.24) satisfy $u_o(x) < u_1(x)$ for all $x \in \Omega$, it is clear that if there is an $x \in \Omega$ such that $f_u(x,u)$ is a strictly increasing function of u for $0 < u < r$, then equation (1.30) of Lemma 1-21 is satisfied, and therefore $u_1 - u_o$ is not a positive solution of the linear equation (1.26), $\mu^{(0)} > 1 > \mu^{(1)}$, and there is no function u_2 satisfying equations (1.24) and (1.25). By imposing further conditions on the kernel $K(x,y)$, we are able to relax the assumption that $f_u(x,u)$ is a strictly increasing function of u for some $x \in \Omega$, and still conclude that if there are two solutions $u_1 \ge u_o$, then $\mu^{(0)} > 1 > \mu^{(1)}$ and there cannot be three functions satisfying (1.24) and (1.25).

For simplicity, we will specialize the conditions of Theorem 1-19 to the case $f(x, u) \geq 0$ for $x \in \overline{\Omega}$, $0 \leq u < r$, $g(x) = 0$ for $x \in \Omega$, and $[a, b) = [0, r)$. Then (1.24) reduces to (1.1) with $\lambda = 1$.

Suppose that the kernel $K(x, y)$ is zero at some point $x_o \in \partial\Omega$ for all $y \in \Omega$. Then all solutions of (1.24) are also zero at x_o. If $0 \leq f_u(x, v) \leq f_u(x, u)$, whenever $x \in \Omega$ and $0 \leq v \leq u < r$, and if there is an open connected subset Ω_1 of Ω with $x_o \in \partial\Omega_1$, and a number $\rho \in (0, r)$ such that $f_u(x, v) < f_u(x, u)$ whenever $x \in \Omega_1$ and $0 < v < u < \rho$, then equation (1.30) of Lemma 1-20 is still valid, and therefore $\mu^{(0)} > 1 > \mu^{(1)}$ for two solutions $u_1 > u_o$ of equation (1.24). The validity of equation (1.30) in this case is shown by the following argument: Let u, v be continuous functions on $\overline{\Omega}$ such that $0 \leq v(x) < u(x)$ for all $x \in \Omega$, and $v(x_o) = u(x_o) = 0$. Then there is a number $\epsilon > 0$ such that $v(x) < u(x) < \rho$ whenever $x \in \Omega_\epsilon = \{x \in \Omega_1 : |x - x_o| < \epsilon\}$, and Ω_ϵ has positive measure. When $x \in \Omega_\epsilon$,

$$f_u(x, v(x)) < f_u(x, u(x)),$$

and therefore

$$f_u(x, v(x)) \leq\leq f_u(x, u(x)), \quad x \in \Omega.$$

The monotonicity requirement on $f_u(x, u)$ can be modified in another way when $K(x_o, y) = 0$ for $x_o \in \partial\Omega$ and all $y \in \Omega$. We assume that some point $x \in \Omega$ can be joined with x_o by a continuous curve γ such that $\{y \in \gamma : y \neq x_o\} \subseteq \Omega$; we will then describe x_o as <u>attainable from x in Ω</u>, or simply as <u>attainable from Ω</u> (since the open connected set Ω is arcwise connected, x_o is attainable from any point in Ω), and say that γ <u>attains x_o from x in Ω</u>. If $f(x,u)$ is a non-homogeneous linear function of u for all $u \in [0, r)$, say $f(x,u) = b(x)u + f(x,0)$, with $f(x,0) \geq\geq 0$ for $x \in \Omega$, then the uniqueness of the positive solution of (1.1) or (1.24) follows from the theory of linear integral equations. If $f(x,u)$ is linear for small u, say

$$f(x, u) = b(x)u + f(x, 0), \quad 0 \leq u \leq \rho < r,$$

where $f(x,0) \geq \geq 0$, and if

$$f_u(x, \rho) < f_u(x, v) \leq f_u(x, u)$$

whenever $x \in \Omega$ and $0 < \rho < v \leq u < r$, then equation (1.30) holds for any two solutions of equation (1.24) with $u_1(x) > u_o(x), x \in \Omega$. To see this, we first note that equation (1.24) has at most one positive solution with norm $\leq \rho$. Thus in equation (1.24), $\|u_1\| > \rho$, and for all $\alpha < 1$ sufficiently close to 1, $\|\alpha u_1 + (1-\alpha) u_o\| > \rho$. Then equation (1.30) is proved by the following argument: Let u, v be continuous functions on $\overline{\Omega}$ with $0 < v(x) < u(x)$ for $x \in \Omega$, $\|u\| > \rho$, and let $u(x_o) = v(x_o) = 0$. We consider first the case $\|v\| \geq \rho$. Using the continuity of u, v and the fact that x_o is attainable from Ω, we can find a point $x_1 \in \Omega$ and a continuous curve γ which attains x_o from x_1 in Ω such that for all $x \in \gamma$, $x \neq x_o, x_1$, we have $0 = v(x_o) < v(x) < v(x_1) = \rho < u(x_1) \equiv \rho + 2\epsilon$. We can find an open neighborhood N_1 of x_1 in Ω such that $u(x) \geq \rho + \epsilon$ for $x \in N_1$; choosing any point $x_2 \in N_1 \cap \gamma$, we have $v(x_2) < \rho$. Thus there is an open neighborhood

$N_2 \subseteq N_1$ of x_2 such that $v(x) < \rho < \rho + \varepsilon \le u(x)$ for $x \in N_2$, and therefore

$$f_u(x, v(x)) < f_u(x, u(x))$$

for $x \in N_2$, so

$$f_u(x, v(x)) \lneq f_u(x, u(x)) \;, \; x \in \Omega.$$

If $\|v\| < \rho$, then we take x_1 to be any point in Ω such that $u(x_1) > \rho$. Choosing N_1 as before, we have $v(x) < \rho < u(x)$ for $x \in N_1$, and the desired result follows.

These remarks imply the following theorem on the uniqueness of the eigenfunction corresponding to λ^*:

<u>1-23. Theorem</u>. Suppose that $f(x,u)$ is convex in u and has a continuous partial derivative $f_u(x, u) \ge 0$ for all $x \in \overline{\Omega}$, $u \in [0, r)$. If $\lambda^* = \sup \Lambda \in \Lambda$, then any of the following assumptions imply the uniqueness of the eigenfunction of equation (1.1) corresponding to λ^*:

(a) There is an $x \in \Omega$ such that $f_u(x,u)$ is a strictly increasing function of u for $0 < u < r$.

(b) There is an $x_o \in \partial\Omega$ such that $K(x_o, y) = 0$ for all $y \in \Omega$, and there is an open connected subset $\Omega_1 \subseteq \Omega$, with $x_o \in \partial\Omega_1$, and a positive number $\rho < r$ such that $f_u(x, u)$ is a strictly increasing function of u for all $x \in \Omega_1$, $0 \le u \le \rho$.

(c) There is an $x_o \in \partial\Omega$ which is attainable from Ω and a positive number $\rho < r$ such that $K(x_o, y) = 0$ for all $y \in \Omega$, $f(x, u) = b(x)\, u + f(x, 0)$ for $0 \le u \le \rho$, and $f_u(x, \rho) = b(x) < f_u(x, u)$ for $\rho < u < r$, all $x \in \Omega$.

According to the proof of Lemma 1-21, the existence of two eigenfunctions u_o and u_1 corresponding to λ^* implies that $f_u(x, u_\alpha(x))$ is independent of α, where $u_\alpha(x) = \alpha u_1(x) + (1-\alpha)u_o(x)$. If $f(x, u)$ is independent of x (we then write $f(x, u) = f(u)$ and $f_u(x, u) = f'(u)$), this clearly implies that $f'(\rho)$ is a constant for ρ in some interval contained in the interval $[\min_{x \subset \overline{\Omega}} u_o(x), \|u_1\|]$. We can, in fact, show that $f'(\rho)$ is constant for all $\rho \in [\min_x u_o(x), \|u_1\|]$. Choose any $x_o \in \Omega$ and consider the set

$$\Delta = \left\{(\alpha, x) : 0 < \alpha < 1,\ x \in \Omega,\ f'(u_\alpha(x)) = f'(u_{\frac{1}{2}}(x_o))\right\}.$$

Since $f'(u_\alpha(x))$ is continuous in (α, x), Δ is closed relative to $(0,1) \times \Omega$. We show that Δ is open. For any $(\alpha_1, x_1) \in \Delta$, choose δ such that $0 < \alpha_1 - \delta < \alpha_1 + \delta < 1$; then

$$0 < u_{\alpha_1 - \delta}(x_1) < u_{\alpha_1}(x_1) < u_{\alpha_1 + \delta}(x_1),$$

so we may choose $\epsilon > 0$ such that

$$\min\left\{u_{\alpha_1 + \delta}(x) : |x - x_1| < \epsilon\right\} > u_{\alpha_1}(x_1)$$

and

$$\max\left\{u_{\alpha_1 - \delta}(x) : |x - x_1| < \epsilon\right\} < u_{\alpha_1}(x_1)$$

For any x such that $|x - x_1| < \epsilon$,

$$u_{\alpha_1 - \delta}(x) < u_{\alpha_1}(x_1) < u_{\alpha_1 + \delta}(x),$$

so there is an α_x, $|\alpha_x - \alpha_1| < \delta$, such that $u_{\alpha_x}(x) = u_{\alpha_1}(x_1)$, and therefore

$$f'(u_\alpha(x)) = f'(u_{\alpha_x}(x)) = f'(u_{\alpha_1}(x_1)) = f'(u_{\frac{1}{2}}(x_o)).$$

Thus all (α, x) with $|\alpha - \alpha_1| < \delta$, $|x - x_1| < \epsilon$, are in Δ, and therefore Δ is open. Since $(0,1)\times\Omega$ is connected and Δ is both open and closed relative to $(0,1)\times\Omega$, $\Delta = (0,1)\times\Omega$. By continuity, $f'(u_\alpha(x)) = f'(u_{\frac{1}{2}}(x_o)) \equiv m$, say, for $0 \le \alpha \le 1$, $x \in \overline{\Omega}$. In particular, $f'(\min_x u_o(x)) = f'(\|u_1\|) = m$, and thus $f'(\rho) = m$ and $f(\rho) = m\rho + b$, say, for $\min_x u_o(x) \le \rho \le \|u_1\|$. From Theorem 1-18,

$$0 = \int \varphi_o(x)\,[\,f(u_o(x)) - u_o(x)\,f'(u_o(x))\,]\,dx$$

$$= b \int \varphi_o(x)\,dx$$

so $b = 0$, and $f(\rho) = m\rho$. This proves:

<u>1-24. Theorem.</u> Let $f(x,u) = f(u)$ be independent of x, convex, and continuously differentiable for $0 \le u < r$. If $\lambda^* \in \Lambda$, then the equation (1.1) has more than one eigenfunction corresponding to λ^* only if there are numbers ρ_1, ρ_2, and m, with $0 \le \rho_1 \le \rho_2 \le r$ and $m > 0$, such that $f(\rho) = m\rho$ for $\rho_1 \le \rho \le \rho_2$; all eigenfunctions $u(\lambda^*)$ of (1.1) corresponding to λ^* satisfy $\rho_1 \le u(\lambda^*; x) \le \rho_2$ for all $x \in \overline{\Omega}$ and are also eigenfunctions of the linear problem (1.9), with $f_u(x, u(x))$ replaced by m, corresponding to the principal eigenvalue λ^*.

<u>1-25. Corollary.</u> Let f be as in Theorem 1-24, and let the kernel $K(x,y)$ satisfy $K(x_o, y) = 0$ for some $x_o \in \partial\Omega$ and all $y \in \Omega$. If

$\lambda^* \in \Lambda$, then equation (1.1) has only one eigenfunction corresponding to λ^*.

Of course, it is implicitly assumed in Theorem 1-24 and Corollary 1-25 that $f(0) > 0$. Corollary 1-25 follows from Theorem 1-24 since $u^o(\lambda^*; x_o) = 0$; if there were more than one eigenfunction for λ^*, we would have $f(\rho) = m\rho$, $\rho_1 \le \rho \le \rho_2$, and $\rho_1 \le u^o(\lambda^*; x_o) = 0$, i.e., $f(\rho) = m\rho$, $0 \le \rho \le \rho_2$, which is impossible.

Without assuming the symmetry of the kernel K(x, y), the results of Theorems 1-19, 1-23, and 1-24 may be established by applying the methods of Section I.10 directly to the integral equation (1.1).

If f(x, u) is not differentiable with respect to u, results similar to those of Theorem 1-18 can be obtained using Lemma I.7-4. We will make use of this fact in Section II.2.

Another result on Hammerstein integral equations with a convex nonlinearity is given by Krasnosel'skii (1964a, §7.1.11).

II.2. Applications to Ordinary Differential Equations.

We consider in this section the boundary value problem for the ordinary differential equation

(2.1a) $$Lu(t) = \lambda f(t,u(t)), \quad 0 \leq t \leq 1,$$

where L is the Sturm-Liouville operator

(2.2) $$Lu(t) \equiv -[p(t)\,u'(t)]' + q(t)\,u(t),$$

with the boundary conditions

$$B(\theta,\psi)u = 0:$$

(2.1b) $$(\sin\theta)u(0) - p(0)(\cos\theta)u'(0) = 0$$

$$(\sin\psi)u(1) + p(1)(\cos\psi)u'(1) = 0.$$

We assume throughout Section II.2 that the function $q \in C[0,1]$ is non-negative on $[0,1]$; the function $p \in C[0,1]$ is strictly positive and continuously differentiable on $[0,1]$; the function f is non-negative and continuous on $[0,1] \times (-\infty,\infty)$ and $f(t,0) > 0$ for some $t \in (0,1)$; for each $t \in [0,1]$, $f(t,u)$ is a non-decreasing function of u for all u; and the numbers θ and ψ satisfy

$$0 \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq \psi \leq \frac{\pi}{2}$$

and

$$\theta + \psi > 0 \quad \text{if} \quad q(t) = 0 \quad \text{for all} \quad t \in [0,1].$$

Under these conditions, the Green's function G for the operator L exists and is continuous on the square $[0,1] \times [0,1]$, and $G(t,s) > 0$ for $0 < t < 1$, $0 < s < 1$ (see Example I.2-1).

This section is primarily concerned with the discussion of equations (2.1) when $Lu = -u''$ and $f(t,u)$ is convex in u, although some of the results are more general. With these assumptions on L and f, we show in the discussion leading to Theorem 2-5 that if f(t,u) is continuously differentiable with respect to u and non-increasing in t, and if $\theta = 0$ and $\psi > 0$ in the boundary conditions (2.1b), then equations (2.1) have at most two eigenfunctions for each $\lambda < \lambda^* = \sup \Lambda$; the proof uses Theorem 1-19. We then show that a result similar to Theorem 1-19 can be established without the differentiability assumption on f (Theorem 2-6). However, since we no longer have the characterization of Theorem 1-16 for $\lambda^* = \max \Lambda$, a result similar to Theorem 2-5 cannot be obtained from Lemmas 2-3 and 2-4; we can assert only that (2.13) has at most two or infinitely many eigenfunctions for each λ. We also state in Theorem 2-6 further assumptions which rule out the possibility of infinitely many eigenfunctions for any λ. Theorem 2-5 does remain valid without the assumption that f is differentiable (for the special cases of the boundary conditions (2.1b) being considered) if f(t,u) is independent of t; in this case, we are able to show that if there are infinitely many eigenfunctions for some λ, then $\lambda = \lambda^* = \max \Lambda$.

In Theorem 2-8 and its proof we construct the solution of the variational equation associated with equations (2.1) when f(t,u) is independent of t. Using this result, we show that if f is strictly convex and twice continuously differentiable, then there are at most two eigenfunctions for each λ (for the general boundary conditions (2.1b)), and for such f we are able to give a rather complete description of the behavior of the eigenfunctions as functions of λ in Theorem 2-11. We conclude the section by using some of the preceding results to construct an example showing that the minimal positive eigenfunctions are not necessarily continuous functions of λ.

Since the Green's function G is positive on $(0,1) \times (0,1)$, the theory of Section II.1 applies to the eigenfunctions of equations (2.1) if $f(t,u) > 0$ for all $t \in [0,1]$, $u \geq 0$; the problem (1.1) is equivalent to the integral equation

$$(2.4) \qquad u(t) = \lambda \int_0^1 G(t,s)\, f(s,u(s))\, ds \equiv \lambda A u(t).$$

Thus all eigenfunctions of the problem (2.1) or (2.4) are positive on $(0,1)$. When $f(t,u)$ has a continuous partial derivative $f_u(t,u)$ for $0 \leq t \leq 1$, $-\infty < u < \infty$, the variational equation associated with (2.4),

$$(2.5) \qquad h(t) = \mu \int_0^1 G(t,s)\, f_u(s,u(s))\, h(s)\, ds \equiv \mu A'(u)\, h(t),$$

is equivalent to the Sturm-Liouville problem

$$(2.6) \qquad Lh(t) = \mu f_u(t,u(t))\, h(t), \qquad B(\theta,\psi)h = 0.$$

No non-zero non-negative solution of (2.1) can have $u(0)=u'(0)=0$ or $u(1) = u'(1) = 0$ (Protter and Weinberger 1967, p. 7). It is not difficult to show that for any subinterval I of $[0,1]$, there is an $\varepsilon > 0$ such that

$$(2.7) \qquad \int_I G(t,s)\, ds \geq \varepsilon \int_0^1 G(t,s)\, ds, \quad 0 \leq t \leq 1$$

(cf. Krasnosel'skii 1964a,p.258). Thus the linear operator T,

$$(2.8) \qquad Th(t) = \int_0^1 G(t,s)\, r(s)\, h(s)\, ds,$$

where r is a continuous strictly positive function on $[0,1]$, is g_0 - bounded on C , where

$$(2.9) \qquad g_0(t) = \int_0^1 G(t,s)\, ds$$

(Krasnosel'skii 1964a, p. 259), and the operator A of equation (2.4) is g_0- bounded on $\mathbf{C}$, since we have assumed $f(t,0) > 0$ for some $t \in (0,1)$.

We first consider the case when $0 \leq \theta < \frac{\pi}{2}$ and $0 \leq \psi < \frac{\pi}{2}$ in the boundary conditions (2.1b). Then for any positive solution of (2.1) we must have $u(0) > 0$ and $u(1) > 0$. Thus the Green's function for the operator L with the boundary conditions $B(\theta, \psi)u = 0$ is a strictly positive continuous function on the closed unit square $[0,1] \times [0,1]$, and the conclusions of Theorems 1-12, 1-13, and 1-17 hold in this case for the problem (2.1) (in addition to all those theorems of Section II.1 which do not make special assumptions on the kernel; we shall see below that the assumption $f(\tilde{t},0) > 0$ for all $t \in [0,1]$ can be weakened to the assumption made at the beginning of this section).

Similarly, if the Green's function $G(t,s)$ is concave in t, then Theorems 1-9, 1-14, and 1-17 apply to the eigenfunctions of (2.1). This is certainly the case when L is the second derivative operator, $Lu = -u''$, since in this case the Green's function is

$$G(t,s) = \frac{[\cos\theta + (\sin\theta)t_<][\cos\psi + (\sin\psi)(1-t_>)]}{\sin(\theta+\psi) + \sin\theta\sin\psi}$$

where $t_< = \min(t,s)$, $t_> = \max(t,s)$, and $\theta + \psi > 0$.

Theorems 1-9, 1-14, and 1-17 also apply when $Lu(t) = -[p(t)u'(t)]'$, i.e., when $q(t) = 0$, $0 \leq t \leq 1$. In this case we introduce the new variable

$$s = p_0 \int_0^t \frac{dt'}{p(t')} , \qquad (2.10)$$

where

$$p_0^{-1} = \int_0^1 \frac{dt}{p(t)} .$$

Then the equation

(2.11) $$\frac{d}{dt}\left[p(t)u'(t)\right] + \lambda f(t,u(t)) = 0, \quad 0 \leq t \leq 1,$$

is transformed into

(2.12a) $$\frac{d^2}{ds^2} v(s) + \lambda p_0^{-2} p(T(s)) f(T(s), v(s)) = 0, \quad 0 \leq s \leq 1,$$

where $T(s)$ is the inverse of the mapping $t \to s$ defined by equation (2.6) and $v(s) = u(T(s))$, and the boundary conditions (2.1b) become

(2.12b)
$$(\sin\theta)\, v(0) - p_0(\cos\theta)\, v'(0) = 0$$
$$(\sin\psi)\, v(1) + \tilde{p}_0(\cos\psi)\, v'(1) = 0.$$

Thus equation (2.11), with $Lu = -(pu')'$, and all properties of the set Λ and the eigenfunctions u for equations (2.11) - (2.1b) can be determined from the corresponding properties for equations (2.12).

In each of the preceding cases ($0 \leq \theta < \frac{\pi}{2}$ and $0 \leq \psi < \frac{\pi}{2}$, or $Lu = -(pu')'$), all functions in the cones $\tilde{C}$ or C_c (see the remarks preceding Theorems 1-12 and 1-14) satisfy a relation of the form

$$u \geq ||u||\, g ,$$

where $g(t) > 0$ for $0 < t < 1$. Thus, if $u \in \tilde{C}$ or $u \in C_c$, and $||u|| = \rho$, we have

$$Au(t) = \int_0^1 G(t,s)\, f(s,u(s))\, ds$$
$$\geq \int_0^1 G(t,s)\, f(s,\rho g(s))\, ds \equiv g_2(t),$$

so $\|Au\| \geq \|g_2\| > 0$ for all such u, provided $f(s,v) > 0$ for all $v > 0$ and some fixed $s \in (0,1)$. From Theorem I.3-5, we obtain:

2-1. Theorem. Suppose $q(t) = 0$ for all $t \in [0,1]$ in (2.2), or $0 \leqslant \theta < \frac{\pi}{2}$, $0 \leqslant \psi < \frac{\pi}{2}$ in (2.1b). For some $s_0 \in (0,1)$, let $f(s_0,v) > 0$ for all $v > 0$. Then for any number $\rho > 0$, there exists a number $\lambda > 0$ and a function $u \in C[0,1]$, with $u(t) > 0$ for $0 < t < 1$, which satisfy equations (2.1) and $\|u\| = \rho$.

This result is much stronger than Theorem 1-1, since it holds also when $f(t,0) = 0$ for all $t \in [0,1]$.

Since the assumption $f(t,0) > 0$ for all $t \in \overline{\Omega}$ which we made in Section II.1 was used only to establish Theorem 1-1 on the existence of eigenfunctions of arbitrary norm and to establish the g_0-boundedness of the operator A (equation (1.3) or (2.4)), and since these results have been established under the weaker hypothesis that $f(t,0) > 0$ for some $t \in \Omega$, the results of Section II.1 can be applied to the special cases just discussed under this weaker hypothesis.

Theorem 1-19 may be used to investigate the number of positive solutions for a given λ of the following special case of (2.1) or (2.7),

$$u''(t) + \lambda f(t,u(t)) = 0, \quad 0 \leqslant t \leqslant 1, \quad \lambda > 0, \tag{2.13}$$
$$u'(0) = u(1) + \beta u'(1) = 0, \quad 0 \leqslant \beta < \infty.$$

The Green's function in this case is

$$G(t,s) = \begin{cases} \beta + 1 - s, & 0 \leqslant t < s \leqslant 1 \\ \beta + 1 - t, & 0 \leqslant s < t \leqslant 1. \end{cases} \tag{2.14}$$

Since $u''(t) = -\lambda f(t,u(t)) \leqslant 0$, every positive eigenfunction has its maximum at $t=0$ and decreases from $u(0) = \|u\|$ to $u(1) = -\beta u'(1) \geqslant 0$. Thus it is possible to define the function T inverse to u by $T(u(t)) = t$ and $u(T(v)) = v$, $0 \leq t \leq 1$, $u(1) \leq v \leq u(0)$. Then (2.13) may be written in the form

(2.15)
$$\tfrac{1}{2}u'(t)^2 - \tfrac{1}{2}u'(t_0)^2 = \lambda \int_{u(t)}^{u(t_0)} f(T(v), v)\, dv$$

for any t and t_0 between 0 and 1.

<u>2-2. Lemma.</u> Let f_1 and f_2 be continuous and non-negative on $[0,1] \times [0,\rho]$ $(\rho > 0)$; suppose that

(2.16)
$$f_1(0,\rho) < f_2(0,\rho)$$

and

(2.17)
$$f_1(t_1,v) \leq f_2(t_2,v)$$

whenever $0 \leq t_2 < t_1 \leq 1$ and $0 \leq v \leq \rho$. Then there do not exist two non-negative functions u_1 and u_2 on $[0,1]$ such that

(2.18)
$$u_i''(t) + f_i(t,u_i(t)) = 0, \quad 0 \leq t \leq 1$$

$(i = 1, 2)$

(2.19)
$$u_i'(0) = u_i(1) + \beta u_i(1) = 0$$

(2.20)
$$u_1(0) = u_2(0) = \rho.$$

<u>Proof.</u> Suppose there were two functions u_1 and u_2 satisfying (2.18) - (2.20). The discussion following (2.14) applies to both u_1 and u_2; thus we may define the functions T_1 and T_2 inverse to u_1 and u_2. From (2.18) and (2.16) we obtain $u_1''(0) > u_2''(0)$; it follows then from (2.20) and (2.19) that there is an interval $(0,t_0)$, $t_0 \leq 1$, in which $u_1(t) > u_2(t)$. As in (2.15), we have

(2.21)
$$\tfrac{1}{2}u_1'(t_0)^2 = \int_{u_1(t_0)}^{\rho} f_1(T_1(v), v)\, dv$$
$$< \int_{u_1(t_0)}^{\rho} f_2(T_2(v), v)\, dv$$

because of (2.16) and (2.17). We may choose t_0 such that either

$u_1(t_0) = u_2(t_0)$ or $t_0 = 1$ and $u_1(1) \geq u_2(1)$. Thus we may change the lower limit in the second integral in (2.21) to $u_2(t_0)$. Hence

$$u_1'(t_0)^2 < u_2'(t_0)^2.$$

But this is impossible, either if $u_1(t_0) = u_2(t_0)$ or, according to (2.19), if $t_0 = 1$ and $u_1(1) > u_2(1)$. //

2-3. Lemma. Suppose that f satisfies a Lipschitz condition $|f(t,u_1) - f(t,u_2)| \leq q(\rho)\,|u_1 - u_2|$ for $0 \leq u_1, u_2 \leq \rho$, and that $f(t,u)$ is non-increasing in t for each fixed u. Then for any number $\rho > 0$, there exists at most one number λ and one function u, $u(0) = \rho$, satisfying (2.13).

Proof. Two solutions corresponding to different λ's would contradict the preceding lemma; two solutions corresponding to the same λ would violate the uniqueness theorem for the initial value problem corresponding to (2.13) and $u(0) = \rho$, $u'(0) = 0$. //

Using methods similar to those used in the proof of Lemma 2-2, one may prove (take $t=1$ in (2.15)):

2-4. Lemma. Suppose that f satisfies the hypotheses of Lemma 2-3. If, for some $\lambda > 0$, the problem (2.13) has two non-negative solutions u_1 and u_2 with $u_1(0) > u_2(0)$, then $u_1(t) > u_2(t)$, $0 \leq t < 1$.

Combining the preceding results with Theorem 1-19, we have:

2-5. Theorem. Let $f(t,u)$ be convex in u, continuously differentiable with respect to u, and non-increasing in t for each u, $0 \leq t \leq 1$, $u \geq 0$. Then for each $\rho > 0$, there is exactly one number $\lambda > 0$ and one (non-negative) function u which satisfy (2.13) and $\|u\| = \rho$. For each $\lambda \in \Lambda$, (2.13) has at most two eigenfunctions. If

$\lambda^* = \sup \Lambda \in \Lambda$, then there is either one or infinitely many eigenfunctions corresponding to λ^*. If there are infinitely many eigenfunctions for $\lambda = \lambda^*$, then they have the form $u^o(\lambda^*) + \alpha\varphi$, where $\alpha \geq 0$ and φ is a positive eigenfunction of the linear problem

$$\varphi'' + \mu f_u(t, u^o(\lambda^*;t))\varphi = 0$$

$$\varphi'(0) = 0 = \varphi(1) + \beta\varphi'(1)$$

corresponding to the eigenvalue λ^*.

Theorems 1-9, 1-16, and 1-17 can be used to determine the values of λ, if any, for which there are exactly two eigenfunctions.

Some of the results of Theorems 1-18, 1-23, and 2-5 can be obtained without the differentiability requirement on f by using arguments similar to those used to prove Lemmas I.5-3 and I.7-4. We return to the general equations (2.1) - (2.2), assuming that f is continuous and non-negative and that $f(t,u)$ is increasing and convex in u for each $t \in [0,1]$ and $u \geq 0$.

Suppose equations (2.1) for $\lambda = 1$ have three distinct solutions u^o, $u^o + h_0$, and $u^o + h_1$, with $0 \leq h_0(t) \leq h_1(t)$. We first show that it is possible to choose a number $\alpha \in (0,1)$ such that $h_0 \leq \alpha h_1$. If $h_0(t) \leq h_1(t)$, then

$$L[h_1(t) - h_0(t)] = f(t, u^0(t) + h_1(t)) - f(t, u^0(t) + h_0(t))$$

$$\geq 0,$$

$$B(h_1 - h_0) = 0,$$

so either $h_1(t) = h_0(t)$ for all $t \in [0,1]$, or $h_1(t) > h_0(t)$ for all $t \in (0,1)$ (Protter and Weinberger 1967, pp.6-7). Suppose $h_1(t) > h_0(t)$ for all $t \in (0,1)$. At either boundary point, either both h_0 and h_1 are zero, or neither is zero. If both are zero, then each has a non-zero derivative at that boundary point, and $h_1 - h_0$ also has a non-zero derivative (Protter and Weinberger, pp. 6-7). If neither h_0 nor h_1 is zero at a boundary point, then they are not equal at this point; for if they were equal, the boundary conditions would imply that the derivatives were equal, and this contradicts $h_1 > h_0$, $L(h_1 - h_0) > 0$, by the same argument as used in the preceding sentence. Thus

$$\lim_{t \downarrow 0} \frac{h_0(t)}{h_1(t)} \quad \text{and} \quad \lim_{t \uparrow 1} \frac{h_0(t)}{h_1(t)}$$

exist and are less than one. There is, therefore, a number $\alpha < 1$ such that $h_0(t) \leq \alpha h_1(t)$ for all $t \in [0,1]$, and we may choose α to be the smallest such number; i.e., there is no number $\beta < \alpha$ such that $h_0(t) \leq \beta h_1(t)$.

If $f(t,u)$ is convex in u for all $t \in (0,1)$ and strictly convex for some $t \in (0,1)$, then

$$f(t, u_0(t) + \alpha h_1(t)) \leq (1-\alpha) f(t, u_0(t)) + \alpha f(t, u_0(t) + h_1(t)), \tag{2.22}$$

and strict inequality holds for some $t \in (0,1)$, since $h_1(t) > 0$ for all $t \in (0,1)$. We set $f^0(t,\sigma) = f(t, u^0(t) + \sigma) - f(t, u^0(t))$. Then

$$\begin{aligned} h_0(t) &= \int_0^1 G(t,s) f^0(s, h_0(s))\, ds \\ &\leq \int_0^1 G(t,s) f^0(s, \alpha h_1(s))\, ds \\ &\leq -\int_I G(t,s) \left[\alpha f^0(s, h_1(s)) - f^0(s, \alpha h_1(s))\right] ds \\ &\qquad + \alpha \int_0^1 G(t,s) f^0(s, h_1(s))\, ds \end{aligned}$$

where I is a closed interval on which strict inequality holds in (2.22). Using inequality (2.7), we can find an $\varepsilon > 0$ such that

$$\begin{aligned} h_0(t) &\leq -\varepsilon \int_0^1 G(t,s)\, ds + \alpha \int_0^1 G(t,s) f^0(s, h_1(s))\, ds \\ &\leq (\alpha - \eta) \int_0^1 G(t,s) f^0(s, h_1(s))\, ds \\ &= (\alpha - \eta)\, h_1(t), \end{aligned}$$

where

$$\eta = \varepsilon \left[\max \{ f^{o}(t,\rho) \colon 0 \leq t \leq 1, \ 0 \leq \rho \leq \|h_1\| \ \} \right]^{-1} .$$

This contradicts the assumption that α is the smallest number such that $h_0(t) \leq \alpha h_1(t)$. Thus there do not exist three solutions $u^{o}(t)$, $u^{o}(t) + h_0(t)$, and $u^{o}(t) + h_1(t)$, with $h_0(t) \leq \ h_1(t) > 0$ for all $t \epsilon (0,1)$ (cf. Theorems 1-18 and 1-19).

This result holds whenever, for all $\alpha \epsilon (0,1)$, there is a $t \epsilon [0,1]$ such that strict inequality holds in (2.22). As in the discussion preceding Theorem 1-23, it is readily shown that this is the case if either $\theta = \frac{\pi}{2}$ or $\psi = \frac{\pi}{2}$ in the boundary conditions (2.1b), so that all solutions of (2.1) vanish at a boundary point $t_0 (=0$ or $1)$, and either of the following conditions is satisfied in addition to the convexity of $f(t,u)$ in u:

(a) There is an interval I in $(0,1)$, with t_0 as one of its boundary points, and a number $\rho > 0$ such that $f(t,u)$ is strictly convex in u for all $t \epsilon I$ and $0 \leq u \leq \rho$.

(b) There is a number $\rho > 0$ and non-negative not identically zero functions m and b on $[0,1]$ such that

$$f(t,u) = m(t)u + b(t) , \quad 0 \leq u \leq \rho, \quad 0 \leq t \leq 1 ,$$

and

$$f(t, \alpha u + (1-\alpha)\rho) < \alpha f(t,u) + (1-\alpha) f(t,\rho), \quad u \geq \rho, \ 0 \leq t \leq 1.$$

If there are three distinct solutions $u^{o}(t)$, $u^{o}(t) + h_0(t)$, and $u^{o}(t) + h_1(t)$ of (2.1), with $0 \leq h_0(t) \leq h_1(t)$, then equality holds in (2.22) for all $t \epsilon [0,1]$. Since $f(t,u)$ is convex in u, equality must hold in (2.22) for all $\alpha \epsilon [0,1]$; therefore

$$f^{o}(t, \alpha h_1(t)) = \alpha f^{o}(t, h_1(t)), \ 0 \leq \alpha \leq 1, \quad 0 \leq t \leq 1,$$

and

$$L(\alpha h_1)(t) = \alpha f^o(t, h_1(t)) = f^o(t, \alpha h_1(t)).$$

Thus for $0 \leq \alpha \leq 1$, $u^o + \alpha h_1$ is a solution of (2.1). We define

$$m(t;h_1) = \frac{f^o(t,h_1(t))}{h_1(t)}$$

for $0 < t < 1$, and also at the boundary points 0 and 1 if $h_1(t)$ is not zero there. If $h_1(t)$ is zero at a boundary point t_0, then $u_0(t_0)=0$ and $f^o(t_0, h_1(t_0)) = 0$, and we define

$$m(t_0;h_1) = f_u(t_0, 0+)$$

(this exists since $f(t,u)$ is convex in u; Proposition 1-15). The function $m(t;h_1)$ so defined is continuous for $t \in [0,1]$, and h_1 is a positive eigenfunction of the linear problem

$$(2.23) \qquad \begin{aligned} Lh(t) &= \mu m(t;h_1)\, h(t) \\ Bh &= 0 \end{aligned}$$

corresponding to the eigenvalue 1.

We have proven the following:

<u>2-6. Theorem.</u> Consider the problem (2.1), where f is continuous on $[0,1] \times [0,\infty)$, $f(t,u)$ is monotonically non-decreasing and convex in u, and $f(t,0) \geq 0$. Then for any $\lambda > 0$, equations (2.1) do not have three solutions $u^o(\lambda)$, $u_1(\lambda)$, $u_2(\lambda)$, satisfying $0 \leq u^o(\lambda) < u_1(\lambda) < u_2(\lambda)$, unless $u_2(\lambda) - u^o(\lambda)$ is an eigenfunction of the linear problem (2.23) corresponding to the principal eigenvalue λ; in the latter case, all functions $u^o(\lambda) + \alpha[u_2(\lambda) - u^o(\lambda)]$, $0 \leq \alpha \leq 1$, are also solutions of the nonlinear problem (2.1). This case does not occur if $f(t,u)$ is strictly convex in u for some $t \in (0,1)$, or if $\theta = \frac{\pi}{2}$ or $\psi = \frac{\pi}{2}$ in the boundary conditions (2.1b) and f satisfies either of conditions (a) (with $t_0 = 0$ or 1 according as $\theta = \frac{\pi}{2}$ or $\psi = \frac{\pi}{2}$) or (b) above.

If $f(t,u) = f(u)$ is independent of t and equality holds in (2.22) for all $t \in [0,1]$ and some $\alpha \in (0,1)$, then

$$f(u^{o}(t) + \alpha h_1(t)) = (1-\alpha) f(u^{o}(t)) + \alpha f(u^{o}(t) + h_1(t))$$

for all $t \in [0,1]$. It follows that f is linear on the interval $[u^{o}(t), u^{o}(t)+h_1(t)]$ for each $t \in [0,1]$ (Choquet 1966); as in the discussion preceding Theorem 1-24, we see that f must have the form

$$f(\rho) = m\rho + b,$$

for all $\rho \in \left[\min \{u^{o}(t) : 0 \leq t \leq 1\}, \|u^{o} + h_1\|\right]$, where $m > 0$ and $b \geq 0$. Thus the functions u^{o}, $u^{o} + h_0$, and $u^{o} + h_1$, are each a solution of

$$\text{(2.24)} \quad \begin{aligned} Lu(t) &= mu(t) + b, \quad 0 \leq t \leq 1, \\ Bu &= 0, \end{aligned}$$

(assuming $\lambda = 1$ in (2.1)); therefore h_0 and h_1 are each positive solutions of

$$\text{(2.25)} \quad \begin{aligned} Lh(t) &= \mu m h(t), \quad 0 \leq t \leq 1 \\ Bh &= 0, \end{aligned}$$

for $\mu = 1$, and 1 is the principal eigenvalue of (2.25). Thus b in (2.24) must be zero, and u^{o}, $u^{o} + h_0$, and $u^{o} + h_1$ must be eigenfunctions of (2.25) corresponding to the eigenvalue 1. If we remove the assumption $\lambda = 1$, then the existence of three such solutions u^{o}, $u^{o} + h_0$, and $u^{o} + h_1$ for some λ implies λ is the principal eigenvalue of (2.25). As in Theorem I.10-15, it follows that $\lambda = \max \Lambda$.

Thus Theorem 1-24 with $\Omega = (0,1)$ and $K = G$, the Green's function for L, B (equations (2.1b) - (2.2)), is valid without the assumption that f is differentiable, Theorem 2-5 holds for equations (2.13) if $f(t,u)$ is independent of t without the assumption that f is differentiable, and we also have:

2-7. Theorem. Let the function $f(t,u)$ in equation (2.1) be independent of t and convex (in u). If $\lambda^* = \sup \Lambda \in \Lambda$, then the eigenfunction $u^o(\lambda^*)$ is the only eigenfunction of (2.1) corresponding to λ^* if either of the following conditions is satisfied:

(a) There does not exist a non-empty interval $[\rho_1, \rho_2]$, with $0 \leq \rho_1 < \rho_2$, in which f has the form $f(u) = mu$, $\rho_1 \leq u \leq \rho_2$, for some positive number m.

(b) The boundary conditions (2.1b) imply that all eigenfunctions of (2.1) are zero at one of the boundary points 0 or 1.

Notice that the boundary conditions $u(0) - \beta u'(0) = 0 = u(1) + \beta u'(1)$ can be included in Theorem 2-5 when $f(t,u)$ is independent of t, since by symmetry we can reduce this problem to one on the interval $[\frac{1}{2}, 1]$ with the boundary conditions $u'(\frac{1}{2}) = u(1) + \beta u'(1) = 0$, which are of the form of equation (2.13) (cf. Section II. 3).

If neither of conditions (a) or (b) is satisfied, then it is possible for there to be infinitely many eigenfunctions corresponding to $\lambda = \lambda^*$. For example, let f be the continuously differentiable function $f(u) = e^{u-1}$, $0 \leq u \leq 1$, $f(u) = u$, $1 \leq u < r$, for some number $r > 1$; let the boundary conditions be $u'(0) = u(1) + \beta u'(1) = 0$, with $0 \leq \beta < \infty$; and let $Lu = -u''$. Let λ^* be the smallest positive root of the equation $\cot\sqrt{\lambda^*} = \beta\sqrt{\lambda^*}$, and choose $r > \sec\sqrt{\lambda^*}$. Then $\Lambda^r{}_f = (0, \lambda^*]$, and all functions $u = \|u\| \cos\sqrt{\lambda^*}\, t$ are solutions of (2.1) for $\sec\sqrt{\lambda^*} \leq \|u\| < r$.

When $f(t,u)$ is independent of t and $Lu = -u''$, it is possible to construct explicitly the eigenfunctions of (2.6) in terms of a given solution $u(\lambda)$ of (2.1) for any value of $\lambda \epsilon \Lambda$ which is also an eigenvalue of the variational problem (2.6). The eigenfunctions so constructed have no zeroes on (0,1) and therefore must correspond to the smallest eigenvalue $\mu_1[u(\lambda)]$ of (2.6). This result will enable us to give a proof that in this case equations (2.1) have at most two solutions for each λ when f is convex; the proof works for the general boundary conditions (2.1b), whereas in Theorem 2-5 we assumed the special boundary conditions of the form in equations (2.13). On the other hand, we must apply Theorem 1-5 in this method of proof and therefore will need to assume that f is twice continuously differentiable and that either condition (a) or condition (b) of Theorem 2-7 is satisfied.

The construction of the eigenfunctions and proof that they are not zero on (0,1) is carried out in the proof of the following theorem, which considers the more general equation $u'' + g(u,u') = 0$. We will use only conclusion (i) of this theorem, but we include also (ii) through (iv) since they follow immediately from the proof and are of interest in themselves. Conclusion (ii), for example, shows that if f is positive and decreasing in equation (2.29) below, then λ is never an eigenvalue of the variational problem (2.30).

<u>2-8. Theorem.</u> Let u satisfy

(2.26) $u''(t) + g(u(t), u'(t)) = 0, \quad u(t) \geqslant 0, \quad 0 \leqslant t \leqslant 1,$

with the boundary conditions (2.1b), where $g(u,v)$ is positive and continuous for $u \geqslant 0$, $-\infty < v < \infty$, and has continuous partial derivatives $D_1 g(u,v) = \frac{\partial}{\partial u} g(u,v)$ and $D_2 g(u,v) = \frac{\partial}{\partial v} g(u,v)$. Let

(2.27a) $\varphi''(t) + D_2 g(u(t), u'(t))\, \varphi'(t) + D_1 g(u(t), u'(t))\, \varphi(t) = 0,$

with the boundary conditions

(2.27b) $B(\theta, \psi)\varphi = 0,$

be the corresponding variational equation. Then

(i) if $D_1 g(u,v) \geqslant 0$ for $u \geqslant 0$, $-\infty < v < \infty$, any non-zero solution of (2.27) has no zeroes on $(0,1)$;

(ii) if $D_1 g(u,v) \leqslant 0$ for $u \geqslant 0$, $-\infty < v < \infty$, then equations (2.27) have no non-zero solution;

(iii) if either $\theta = 0$ or $\psi = 0$ in the boundary conditions (2.1b) and (2.27b), then any non-zero solution of (2.27) has no zeroes on $[0,1)$ or $(0,1]$, respectively;

(iv) if either $\theta = 0$ or $\psi = 0$ and if $D_1 g(u,v) \leqslant 0$ for $u \geqslant 0$ and $v \leqslant 0$ or $v \geqslant 0$, respectively, then equations (2.27) have no non-zero solution.

Proof. If u satisfies equation (2.26), then $v = u'$ satisfies

(2.28) $v'' + D_2 g(u,v) v' + D_1 g(u,v) v = 0.$

To find the general solution of equation (2.27a), we set $\varphi(t) = c(t)v(t)$, substitute in (2.27a), use (2.26), and solve for $c(t)$:

$$c(t) = K \int_0^t \frac{\Gamma(s)}{v^2(s)}\, ds$$

$$= -K \left[\frac{\Gamma(s)}{v(s)\, v'(s)} \Big|_0^t - \int_0^t \frac{1}{v(s)} \frac{d}{ds} \frac{\Gamma(s)}{v'(s)}\, ds \right],$$

where K is the arbitrary constant and

$$\Gamma(s) = \exp\left\{-\int_0^s D_2 g(u(t), v(t))dt\right\}.$$

Thus

$$v(t)c(t) = K\left[\int_0^t \frac{D_1 g(u(s), v(s))}{v'(s)^2}\Gamma(s)ds - \frac{\Gamma(t)}{v'(t)} + \frac{v(t)\Gamma(0)}{v(0)v'(0)}\right],$$

assuming $v(0) = u'(0) \neq 0$ (i.e., $\theta \neq 0$). Therefore when $u'(0) \neq 0$, the general solution of (2.27a) is

$$\varphi(t) = K_1\left\{\frac{v(t)}{v(0)v'(0)} + v(t)\int_0^t \frac{D_1 g(u(s), v(s))}{v'(s)^2}\Gamma(s)ds - \frac{\Gamma(t)}{v'(t)}\right\} + K_2 v(t)$$

$$= K_1\left\{\frac{-v(t)}{u'(0)g(u(0), u'(0))} + v(t)\int_0^t \frac{D_1 g(u(s), v(s))}{g^2(u(s), v(s))}\Gamma(s)ds + \frac{\Gamma(t)}{g(u(t), v(t))}\right\} + K_2 v(t),$$

where K_1 and K_2 are constants.

Using the fact that v satisfies equation (2.28), we obtain

$$\varphi'(t) = v'(t)\left\{\frac{K_1}{v(0)v'(0)} + K_2 + K_1\int_0^t \frac{D_1 g(u(s), v(s))}{v'(s)^2}\Gamma(s)ds\right\}$$

We now impose the boundary condition $\alpha\varphi(0) - \alpha'\varphi'(0) = 0$ on φ, where $\alpha = \sin\theta$, $\alpha' = \cos\theta$; we obtain

$$K_2 = \frac{\alpha'}{v(0)A(v;0)} K_1$$

where

$$A(v;t) = \alpha v(t) - \alpha' v'(t) = \alpha v(t) + \alpha' g(u(t), v(t))$$

(note that $A(v;0) > 0$). Thus

$$\varphi'(t) = K_1 v'(t)\left\{ \frac{1}{v(0)\,v'(0)} + \frac{\alpha'}{v(0)\,A(v;0)} + \int_0^t \frac{D_1 g(u(s),v(s))}{v\epsilon(s)^2}\Gamma(s)\,ds\right\}$$

$$= -K_1 g(u(t),\ v(t))\left\{\frac{-\alpha}{g(u(0),\ v(0))A(v,0)} + \int_0^t \frac{D_1 g(u(s),v(s))}{v'(s)^2}\Gamma(s)ds\right\}$$

Clearly, if $D_1 g(u,v) \leq 0$, φ' had no zeroes on $(0,1]$; if $D_1 g(u,v) \geq 0$, φ' has at most one zero on $[0,1]$.

It is easily seen that if any differentiable function $\varphi \neq 0$ has a zero on $(0,1)$ and satisfies the boundary conditions (2.27b), then φ' has at least two zeroes on $[0,1]$. Thus if $D_1 g(u,v) \geq 0$, any non-zero solution of (2.27) has no zeroes on $(0,1)$. Similarly (since $\alpha = \sin\theta \neq 0$), if $D_1 g(u,v) \leq 0$, equations (2.27) have no non-zero solutions.

If $u'(0) = v(0) = 0$, then we have $\alpha = \sin\theta = 0$, and $u'(t) = v(t) < 0$ for $0 < t \leq 1$; the general solution of (2.27a) is

$$\varphi(t) = \begin{cases} -K_1 v(t)\displaystyle\int_t^1 \frac{\Gamma(s)}{v^2(s)}\,ds + K_2 v(t) & ,\quad t > 0 \\ -\dfrac{K_1}{v'(0)} & ,\quad t = 0. \end{cases}$$

The boundary condition $\beta\varphi(1) + \beta'\varphi'(1) = 0$ (where $\beta = \sin\psi$, $\beta' = \cos\psi$) implies $K_2 = -\beta'\Gamma(1)K_1/v(1)B(v;1)$, where $B(v;t) = \beta v(t) + \beta' v'(t)$ and $B(v;1) = \beta v(1) - \beta' g(u(1),\ v(1)) < 0$. Thus

$$\varphi(t) = \begin{cases} -K_1 v(t)\left\{\displaystyle\int_t^1 \frac{\Gamma(s)\,ds}{v^2(s)} + \frac{\beta'\Gamma(1)}{v(1)B(v;1)}\right\} & ,\quad t > 0 \\ -\dfrac{K_1}{v'(0)} & ,\quad t = 0. \end{cases}$$

Since $v(t) = u'(t) < 0$ for $1 \geq t > 0$ and $v(1)B(v;1) > 0$, φ is never 0 for $1 > t > 0$ if $K_1 \neq 0$.

The boundary condition $\varphi'(0) = 0$ implies $\varphi(0) \neq 0$ (otherwise, $\varphi(t) \equiv 0$) and

$$\varphi'(0) = -\frac{\beta v'(0)\Gamma(1)}{v'(1)B(v;1)} + v'(0)\int_0^1 \frac{\Gamma(s)\, D_1 g(u(s), v(s))}{[v'(s)]^2}\, ds$$

$$= 0.$$

Since $-\frac{\beta v'(0)\,\Gamma(1)}{v'(1)B(v;1)} > 0$ and $v'(0) = -g(u(0), v(0)) < 0$, this equation cannot be satisfied if $D_1 g(u,v) \leqslant 0$.

In this case ($\alpha = \sin\theta = 0$), therefore, any non-zero solution of (2.27) has no zeroes on $[0,1)$, and if $D_1 g(u,v) \leqslant 0$ for $u \geqslant 0$, $v \leqslant 0$, then equations (2.27) have no non-zero solution.

A similar argument shows that when $\beta = \sin\psi = 0$, any non-zero solution of (2.27) has no zeroes on $(0,1]$, and if $D_1 g(u,v) \leqslant 0$ for $u \geqslant 0$, $v \geqslant 0$, then equations (2.27) have no non-zero solution. //

<u>2-9. Corollary.</u> Let $u(\lambda)$ satisfy the differential equation

(2.29) $$u'' + \lambda f(u) = 0, \quad 0 \leqslant t \leqslant 1,$$

with the boundary conditions $B(\theta, \psi)u = 0$. Assume that $f(u)$ is continuously differentiable for $u \geqslant 0$, with $f(0) > 0$ and $f'(u) \geqslant 0$, $0 \leqslant u$.

If λ is an eigenvalue of the variational problem

(2.30) $$\begin{aligned} h'' + \mu f'(u(\lambda))\,h &= 0 \\ B(\theta,\psi)h &= 0, \end{aligned}$$

then λ is the principal eigenvalue, i.e., $\lambda = \mu_1[u(\lambda)]$. (If $f'(u(\lambda;t))=0$ for all $t \in [0,1]$, then equations (2.30) have only the zero solution, and λ is not an eigenvalue of (2.30); we may take $\mu_1[u(\lambda)] = \infty$.)

This result shows that Theorem I.7-6 is applicable whenever λ is an eigenvalue of (2.30) and f is twice differentiable.

If in addition f is convex, then Corollary 2-9 and Theorem 1-16 imply the following:

2-10. Theorem. Let f be a continuously differentiable function on $[0,\infty)$ with $f(0) > 0$, and let f' be non-negative and non-decreasing on $[0,\infty)$ in the problem (2.29) - (2.1b). If $\lambda_1 \in \Lambda$ and u_1 is any corresponding eigenfunction, then one of the following two mutually exclusive alternatives holds: either

(a) $\lambda_1 = \mu_1[u_1] = \max \Lambda$,

or

(b) the variational problem (2.30) with $\lambda = \lambda_1$ has no non-zero solution; if u_1 is different from the minimal solution $u^0(\lambda_1)$, then

$$\mu_1[u_1] < \lambda_1 < \mu_1[u^0(\lambda_1)] \quad ;$$

there is a neighborhood N of λ_1 and $\mathcal{N}$ of u_1 such that for each $\lambda \in N$ equations (2.29) - (2.1b) have a solution $u(\lambda)$, depending continuously on λ, such that $u(\lambda_1) = u_1$, and $u(\lambda)$ is the only eigenfunction of (2.29) - (2.1b) in $\mathcal{N}$.

Proof. Suppose $\lambda_1 \neq \max \Lambda$. Since f is convex, it follows from Theorem 1-16 that $\lambda_1 < \mu_1[u^0(\lambda_1)]$ and $\lambda_1 \neq \mu_1[u_1]$. By Corollary 2-9, λ_1 is not an eigenvalue of the variational problem (2.30), and the assertion concerning the existence of solutions $u(\lambda)$ for λ near λ_1 follows from the implicit function theorem I.6-5. Since $\lambda_1 \neq \mu_1[u_1]$, $\lambda_1 > \mu_1[u_1]$ if u_1 is not the minimal eigenfunction $u^0(\lambda_1)$ by Theorem I.10-13. //

Consider now any non-minimal solution u_1 for $\lambda = \lambda_1 < \lambda^* = \sup \Lambda$, where f satisfies the hypotheses of the preceding theorem. Using this theorem and the compactness of the operator A (equation (2.4)) we can construct a continuous family $\{u^{(1)}(\lambda)\}$ of eigenfunctions of equations (2.29) - (2.1b) in a maximal open interval (λ_2, λ_1) to the left of λ_1, such that $\lim_{\lambda \uparrow \lambda_1} u^{(1)}(\lambda) = u_1$ and either $\lim_{\lambda \downarrow \lambda_2} u^{(1)}(\lambda) = 0$ or $\lim_{\lambda \downarrow \lambda_2} \|u^{(1)}(\lambda)\| = \infty$ (Corollary I.3-3). If $\lim_{\lambda \downarrow \lambda_2} u^{(1)}(\lambda) = 0$, then $\lambda_2 = 0$ (since $u^{(1)}(\lambda) \geq u^{o}(\lambda) > 0$ for $\lambda > 0$) and some of the eigenfunctions $u^{(1)}(\lambda)$ are minimal eigenfunctions, since the minimal eigenfunctions are the only small eigenfunctions for small λ (Corollary I.4-11 and Theorem 1-3). Since both $u^{(1)}(\lambda)$ and $u^{o}(\lambda)$ depend continuously on λ, there is in this case a maximum positive number $\lambda_3 < \lambda_1$ such that $u^{(1)}(\lambda_3) = u^{o}(\lambda_3)$ and $u^{(1)}(\lambda) > u^{o}(\lambda)$ for $\lambda_3 < \lambda < \lambda_2$. This, however, contradicts the last assertion of the preceding theorem applied to $\lambda = \lambda_3$. Thus we must have

$$\lim_{\lambda \downarrow \lambda_2} \|u^{(1)}(\lambda)\| = \infty.$$

According to Theorems 1-17 and 1-16, this implies that either $\lim_{u \to \infty} f'(u) = \infty$ and $\lambda_2 = 0$, or $\lim_{u \to \infty} f'(u) = m < \infty$ and $\lambda_2 = \mu_1[\infty]$, the principal eigenvalue of

$$h''(t) + \mu m h(t) = 0. \qquad (2.31)$$

$$B(\theta, \psi) h = 0.$$

Similarly, there is a maximal open interval (λ_1, λ_4) to the right of λ_1 in which a family $\{u^{(1)}(\lambda)\}$ of eigenfunctions is defined such that $\lim_{\lambda \downarrow \lambda_1} u^{(1)}(\lambda) = u_1$ and either $\lim_{\lambda \uparrow \lambda_4} \|u^{(1)}(\lambda)\| = \infty$ or $\lambda_4 \in \Lambda$ (Corollary I.3-3). The former alternative is not possible, since there is at most one asymptotic bifurcation point (either $\lambda = 0$ or $\lambda = \mu_1[\infty]$), and we have just seen that this is less than λ_1. Thus $\lambda_4 \in \Lambda$; since (λ_1, λ_4) is maximal, Theorem 2-10 shows that $\lambda_4 = \lambda^*$ (otherwise, we could continue $u^{(1)}(\lambda)$ for larger values of λ).

We now wish to apply Theorem 1-5 to conclude that for each λ sufficiently close to λ^*, $u^{(1)}(\lambda)$ is the only non-minimal eigenfunction in a sufficiently small ball about $u^0(\lambda^*)$, and that $u^0(\lambda^*) = u^{(1)}(\lambda_4)$ ($= \lim_{\lambda \downarrow \lambda_4} u^{(1)}(\lambda)$); thus we must assume that f is twice continuously differentiable and that $\int_0^1 \varphi^3(t) f''(u^0(\lambda^*;t))\,dt \neq 0$, where φ is a positive eigenfunction of (2.30). Since $\varphi(t) > 0$ for $0 < t < 1$ and f is convex, this integral is zero only if $f''(u) = 0$ for all numbers $u \in [\min_t u^0(\lambda^*;t), \|u^0(\lambda^*)\|]$. Using the fact that $f(0) > 0$, it is not difficult to see that this situation can occur only if both conditions (a) and (b) of Theorem 2-7 are not satisfied (compare the discussion following equation (3.21) in Section II.3 below). If condition (b) is satisfied, then $\int_0^1 \varphi^3(t) f''(u^0(\lambda^*;t))\,dt = 0$ only if $f''(u) = 0$ for all $u \in [0, \|u^0(\lambda^*)\|]$; but then it is impossible for λ^* to be simultaneously an eigenvalue of (2.29) - (2.1b) and of (2.30). Similarly, it is impossible for λ^* to be simultaneously an eigenvalue of (2.29)-(2.1b) and of (2.30) if condition (a) of Theorem 2-7 is satisfied and $f''(u)=0$ for all $u \in \left[\min_t u^0(\lambda^*;t), \|u^0(\lambda^*)\|\right]$

In other words, if there is a second eigenfunction $u^{(1)}(\lambda)$ for some $\lambda < \lambda^*$, then $\lambda^* \in \Lambda$; in this case, if either condition (a) or (b) of Theorem 2-7 is satisfied, then $\int_0^1 \varphi^3(t) f''(u^0(\lambda^*;t))\,dt > 0$. An analysis similar to that given above shows that we cannot have $u^{(1)}(\lambda) = u^0(\lambda)$ for any $\lambda \in (\lambda_1, \lambda_4) = (\lambda_1, \lambda^*)$, and therefore (assuming the conditions of Theorem 2-7) $\lambda_4 = \lambda^* = \mu_1[u^0(\lambda^*)]$, $\lim_{\lambda \downarrow \lambda^*} u^{(1)}(\lambda) = u^0(\lambda^*)$, and for each λ sufficiently close to λ^*, $u^{(1)}(\lambda)$ is the only non-minimal eigenfunction in a sufficiently small ball about $u^0(\lambda^*)$.

Suppose there were a second non-minimal eigenfunction corresponding to λ_1; we could then construct a second family $\{u^{(2)}(\lambda)\}$ of eigenfunctions for $\lambda_1 < \lambda < \lambda^*$. Under the preceding conditions, we would have $\lim_{\lambda \uparrow \lambda^*} u^{(2)}(\lambda) = u^o(\lambda^*)$ and $u^{(2)}(\lambda) = u^{(1)}(\lambda)$ for λ sufficiently near λ^*, by Corollary I.3-3 and Theorem 1-5. Thus there would be a minimum positive number λ_5, $\lambda_1 < \lambda_5 < \lambda^*$, such that $u^{(1)}(\lambda_5) = u^{(2)}(\lambda_5)$ and $u^{(1)}(\lambda) \neq u^{(2)}(\lambda)$ for $\lambda_1 < \lambda < \lambda_5$; this contradicts the last assertion of Theorem 2-10. There is, therefore, only one non-minimal eigenfunction corresponding to $\lambda = \lambda_1$.

We have proved the following:

2-11. Theorem. Consider the problem (2.29) - (2.1b). Let f be defined, positive, monotonically non-decreasing, convex, and continuously differentiable on $[0,\infty)$, and assume that either of conditions (a) or (b) of Theorem 2-7 is satisfied. Define $\mu_1[\infty]$ to be the principal eigenvalue of (2.30) if $0 < \lim_{u\to\infty} f'(u) \equiv m < \infty$; otherwise, let $\mu_1[\infty] = m^{-1}$.

If $f(u) - uf'(u) \geq 0$ for $0 \leq u < \infty$, then equations (2.29)-(2.1b) have precisely one positive eigenfunction for each $\lambda \in (0, \mu_1[\infty])$ and no positive eigenfunctions for any other values of λ. Moreover, $\lim_{\lambda \uparrow \mu_1[\infty]} \|u^o(\lambda)\| = \infty$.

Suppose now that f is also twice continuously differentiable. If $f(u) - uf'(u) < 0$ for sufficiently large u and if $\lim_{u\to\infty} f'(u) = m < \infty$, then there is a number $\lambda^* > \mu_1[\infty]$ such that equations (2.29)-(2.1b) have precisely one positive eigenfunction for $0 < \lambda \leq \mu_1[\infty]$ and for $\lambda = \lambda^*$; precisely two positive eigenfunctions for $\mu_1[\infty] < \lambda < \lambda^*$; and no positive eigenfunctions for other values of λ.

If $\lim_{u\to\infty} f'(u) = \infty$, then there is a number $\lambda^* > 0$ such that equations (2.29) - (2.1b) have precisely two positive eigenfunctions for $0 < \lambda < \lambda^*$, precisely one positive eigenfunction for $\lambda = \lambda^*$, and no positive eigenfunctions for other values of λ.

If we let $\varphi(\lambda)$ be the positive eigenfunction of unit norm of (2.30) and let $\mu_1[u^0(\lambda)]$ be the corresponding (principal) eigenvalue, then all minimal positive eigenfunctions of (2.29) - (2.1b) satisfy $\lambda < \mu_1[u^0(\lambda)]$ for $0 < \lambda < \lambda^*$, and $\lambda^* = \mu_1[u^0(\lambda^*)]$. If we denote by $\{u^{(1)}(\lambda)\}$, $\mu_1[\infty] < \lambda < \lambda^*$, the family of non-minimal positive eigenfunctions, then the mapping $\lambda \to u^{(1)}(\lambda)$ is continuous on $(\mu_1[\infty], \lambda^*)$,

$$\lim_{\lambda \downarrow \mu_1[\infty]} \| u^{(1)}(\lambda) \| = \infty ,$$

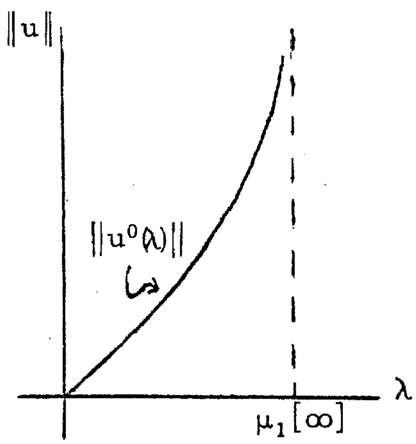
and

$$\lim_{\lambda \uparrow \lambda^*} \frac{u^{(1)}(\lambda) - u^*}{\sqrt{\lambda^* - \lambda}} = \lim_{\lambda \uparrow \lambda^*} \frac{u^* - u^0(\lambda)}{\sqrt{\lambda^* - \lambda}}$$

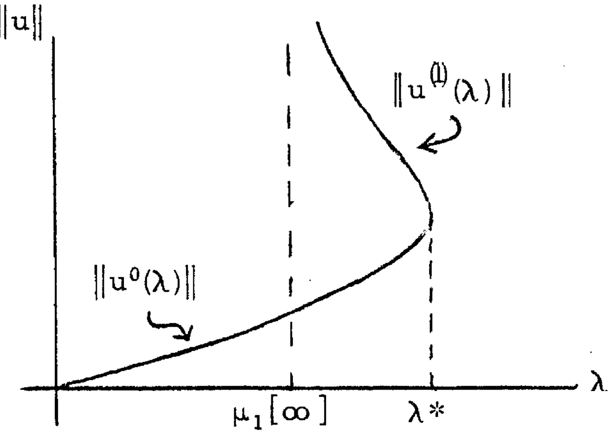
(2.32)

$$= \varphi^* \left[2 \frac{\int_0^1 \varphi^*(t) f'[u^*(t)] \, u^*(t) \, dt}{\lambda^* \int_0^1 [\varphi^*(t)]^3 f''[u^*(t)] \, dt} \right]^{1/2} ,$$

where $\varphi^* = \varphi(\lambda^*)$ and $u^* = u^0(\lambda^*)$.



$f(u) - uf'(u) \geq 0$



$\lim_{u \to \infty} [f(u) - uf'(u)] < 0$, $\lim_{u \to \infty} f'(u) < \infty$.

From Theorems 1-16, 2-5, 2-7, I.6-5, and the remark following Theorem 2-7, all assertions of the preceding theorem except equation (2.32) are valid without the assumption that f is twice differentiable unless θ, ψ, and $\theta-\psi$ are each different from zero in the boundary conditions (2.1b). We therefore conjecture that Theorem 2-11 (except equation (2.32)) holds for the arbitrary boundary condition (2.1b) without the assumption that f is twice differentiable.

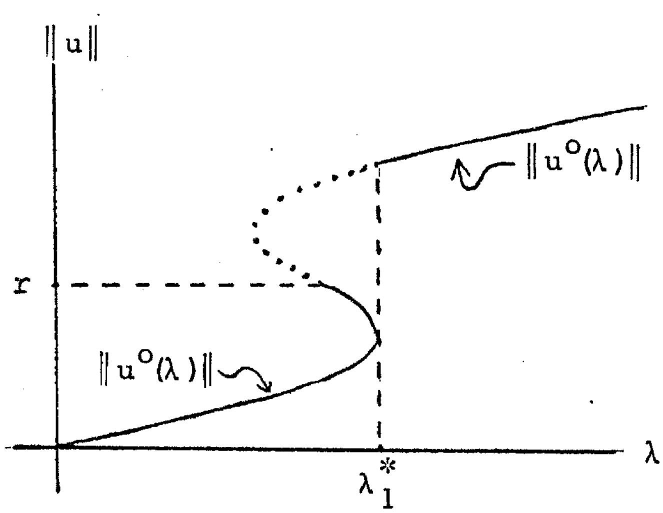
We shall conclude this section by giving the example promised after Theorem I.4-10 illustrating the fact that the minimal positive eigenfunctions $u^o(\lambda)$ may be discontinuous (from the right) in λ at some number $\lambda_0 \epsilon (0, \sup \Lambda)$ at which $\lambda_0 = \mu_1[u^o(\lambda_0)]$. (Thus Theorem 4.1 of Keller and Cohen (1967) is incorrect and should be replaced by a statement similar to our Theorem I.7-1.)

Consider the differential equation (2.29) with the boundary conditions $u'(0) = u(1) = 0$. Let f_1 be any continuously differentiable convex function with $f_1(u) > 0$, $f_1'(u) > 0$ for $u \geq 0$, and $\lim_{u \to +\infty} f_1'(u) = \infty$ (for example, $f_1(u) = e^u$; in this case, the solutions of (2.29) with $u'(0) = u(1) = 0$ can be determined explicitly (Bratu 1914)). According to Theorems 2-5, 2-7, and 1-16, or 2-11 and the remark following it, if we replace f by f_1 in (2.29), there is a number λ_1^* and corresponding eigenfunction $u^o(\lambda^*) = u^*$ such that $\lambda_1^* = \max \Lambda_{f_1} = \mu_1[u^*]$, there are two eigenfunctions $u^o(\lambda)$ and $u^{(1)}(\lambda)$ for each positive λ less than λ^*, and $\|u^{(1)}(\lambda)\| > \|u^*\|$ (since there is only one eigenfunction with given norm; Lemma 2-3 or Theorem 2-5). Let r be any number greater than $\|u^*\|$, and let λ_r be that value of λ for which

$\|u^{(1)}(\lambda_r)\| = r$ (λ_r exists by Theorem 2-5). Let g be any continuously differentiable bounded function with $g(u) > 0$ and $g'(u) > 0$ for $u \geq 0$. Since $g'(u) \to 0$ as $u \to +\infty$, we can assume r chosen so large that $g'(r) < g(r)$. Define

$$f(u) = \begin{cases} f_1(u), & 0 \leq u \leq r \\ f_1(r)\left[\dfrac{g(u) - g(r)}{g'(r)} + 1\right], & u \geq r. \end{cases}$$

Then f is continuously differentiable, bounded, and $f'(u) \to 0$ as $u \to +\infty$. Thus equations (2.29) - (2.1b) have a solution for every $\lambda > 0$ (Keller and Cohen 1967, Corollary 3.3.2, or our Corollary I.4-7), and there are at least two eigenfunctions for each $\lambda \in (\lambda_r, \lambda_1^*)$. Since for such λ the second eigenfunctions $u^{(1)}(\lambda)$ satisfy $\lim_{\lambda \downarrow \lambda_r} \|u^{(1)}(\lambda)\| = r$, and since $\|u^o(\lambda_1^*)\| < r$, all minimal positive eigenfunctions for $\lambda > \lambda_1^*$ satisfy $\|u^o(\lambda)\| \geq r > \|u^*\| = \|u^o(\lambda_1^*)\|$. Thus the minimal eigenfunctions $u^o(\lambda)$ are discontinuous from the right at λ_1^*. (This result also follows from Theorem I.7-3 or Theorem 1-5 if f_1 is twice continuously differentiable.)



II. 3. An Alternative Treatment of $u'' + \lambda f(u) = 0$, $u(0) = u(1) = 0$.

In this section we will give a treatment of equations (3. 1) below which is independent of the preceding investigations. The treatment is based on the fact that we can obtain a solution of (3. 1) and an expression for the eigenvalues λ as a function of the norms $\|u\|$ of the eigenfunctions u of (3. 1) (see equations (3. 7) and (3. 8)), and on a study of (3. 1) in the phase plane (cf. Pimbley 1962). All the results stated in Theorem 3-11 at the end of this section have been obtained previously by different methods (see, e. g., Theorem 2-11).

We consider the following non-linear eigenvalue problem:

$$(3.1)\qquad \begin{aligned} &u'' + \lambda f(u) = 0, \quad 0 \le t \le 1, \quad \lambda \ge 0 \\ &u(0) = u(1) = 0 \end{aligned}$$

where f is a non-negative continuous function defined on $(-\infty, r)$, with $0 < r \le \infty$, and $f(u) > 0$ for $0 < u < r$. We assume that f is Lipschitz continuous on every closed subinterval of $(-\infty, r)$. This problem is equivalent to the integral equation

$$(3.2)\qquad u(s) = \lambda \int_0^1 G_1(s,t)\, f(u(t))\, dt$$

with the Green's function

$$G_1(s,t) = \begin{cases} t(1-s), & 0 \le t < s \le 1 \\ s(1-t), & 0 \le s < t \le 1 . \end{cases}$$

It follows that for non-negative λ, all eigenfunctions are non-negative; moreover, since $u''(t) = -\lambda f(u(t)) < 0$ for $u(t) > 0$ and positive λ, every eigenfunction has only one maximum in $0 \le t \le 1$. The system (3. 1) may be integrated once to give:

(3.3) $$\tfrac{1}{2}(u')^2 + \lambda F(u) = \tfrac{1}{2}(u'(0))^2$$

where $F(u) = \int_0^u f(x)\,dx$. If $u(t)$ is a solution of (3.1) corresponding to some λ, then so is $\bar{u}(t) \equiv u(1-t)$; from (3.3) it follows that

$$u'(0)^2 = u'(1)^2 = \bar{u}'(0)^2 ,$$

so that $u'(0) = \bar{u}'(0)$, since each of these quantities must be positive. From the uniqueness theorem for initial value problems, we conclude that $u(t) = \bar{u}(t) = u(1-t)$, and any solution of (3.1) is symmetric about $t = \frac{1}{2}$. Thus (3.1) and (3.2) may be reformulated as the half-interval problems

(3.4) $$\begin{aligned} &u'' + \lambda f(u) = 0, \quad 0 \le t \le \tfrac{1}{2}, \\ &u(0) = u'(\tfrac{1}{2}) = 0 \end{aligned}$$

or

(3.5) $$u(s) = \lambda \int_0^{\frac{1}{2}} G(s,t)\, f(u(t))\, dt$$

with the Green's function

$$G(s,t) = \begin{cases} t, & 0 \le t < s \le \frac{1}{2} \\ s, & 0 \le s < t \le \frac{1}{2} . \end{cases}$$

Problems (3.4) or (3.5) are equivalent to (3.1) or (3.2) in the sense that if u is a solution of (3.1) or (3.2) for some λ, then u restricted to $0 \le t \le \frac{1}{2}$ is a solution of (3.4) or (3.5) for the same λ, and conversely if $u_{\frac{1}{2}}$ is a solution of (3.4) or (3.5) for some λ, then u defined by $u(t) = u_{\frac{1}{2}}(t)$, $0 \le t \le \frac{1}{2}$, and $u(t) = u_{\frac{1}{2}}(1-t)$, $\frac{1}{2} \le t \le 1$, is a solution of (3.1) or (3.2) for the same λ.

Any eigenfunction of (3.4) is positive and strictly increasing on $0 < t \le \frac{1}{2}$. Thus (3.4) may be integrated twice as follows:

$$\tfrac{1}{2}(u')^2 + \lambda F(u) = \tfrac{1}{2}(u'(0))^2 = \lambda F(\|u\|) \tag{3.6}$$

$$t(2\lambda)^{\frac{1}{2}} = \int_0^u \frac{dv}{\sqrt{F(\|u\|) - F(v)}} \tag{3.7}$$

with

$$\left(\frac{\lambda}{2}\right)^{\frac{1}{2}} = \int_0^{\|u\|} \frac{du}{\sqrt{F(\|u\|) - F(u)}} \tag{3.8}$$

Here $F(u)$ is defined as in (3.3) and $\|u\|$ denotes the maximum value of $u(t)$ for $0 \leq t \leq \frac{1}{2}$. The integral in (3.8) is improper at the upper limit $u = \|u\|$; near this limit, F behaves like $F(u) = F(\|u\|) + f(\|u\|)(u-\|u\|) + o(u-\|u\|)$, so the integral is convergent if $f(\|u\|) > 0$.

We may use (3.7) and (3.8) to construct a solution of (3.4). Given any number $\|u\| \in (0, r)$, define λ by (3.8); then (3.7) defines a one-to-one relation between t and u for $0 \leq t \leq \frac{1}{2}$ and $0 \leq u \leq \|u\|$ such that $t = 0$ iff $u = 0$ and $t = \frac{1}{2}$ iff $u = \|u\|$. The function $u(t)$ so defined is easily seen to be twice differentiable and satisfy (3.4). We obtain the following:

<u>3-1. Theorem</u>. For any number $\rho \in (0, r)$, there exists exactly one number $\lambda\{\rho\} > 0$ and one (non-negative) function $u\{\rho\}$ on $[0, 1]$ which satisfies (3.1) and $\|u\{\rho\}\| = \rho$; $\lambda\{\rho\}$ is a continuous function of ρ.

We have the following comparison theorem for solutions of (3.1) or (3.4) corresponding to a given λ:

<u>3-2. Theorem</u>. Let u_1 and u_2 be two solutions of (1) for a fixed $\lambda > 0$ such that $\|u_1\| < \|u_2\|$. Then

$$u_1(t) < u_2(t) \text{ for } 0 < t < 1.$$

Proof. Since $\|u_1\| < \|u_2\|$, $F(\|u_1\|) < F(\|u_2\|)$. Then the existence of a c, $0 < c < \frac{1}{2}$, such that $u_1(c) = u_2(c)$ contradicts (3.7). Since $u_1(\frac{1}{2}) = \|u_1\| < u_2(\frac{1}{2}) = \|u_2\|$, $u_1(t) < u_2(t)$ for $0 < t \leq \frac{1}{2}$, and by symmetry for $0 < t < 1$. //

Using the integral formulation (6), we may obtain the behavior of λ for large and for small $\|u\|$ (cf. Joseph 1965). Equation (3.5) implies

$$\|u\| = u(\tfrac{1}{2}) = \lambda \int_0^{\frac{1}{2}} t\, f(u(t))\, dt ,$$

or

$$\lambda^{-1} = \int_0^{\frac{1}{2}} t\, f(u(t))\, dt \,/\, \|u\|. \tag{3.9}$$

Since u is concave downwards, $u(t) \geq 2t\|u\|$ for $0 \leq t \leq \frac{1}{2}$. The next result then follows directly from (3.9).

3-3. Theorem. If $f(u) \geq au + b$ for $0 \leq u < r$, where $a \geq 0$, $b \geq 0$, then for any eigenfunction u and corresponding eigenvalue $\lambda\{\|u\|\}$ of (3.1), $\dfrac{1}{\lambda\{\|u\|\}} \geq \dfrac{a}{12} + \dfrac{b}{8\|u\|}$, if $\|u\| < r$.

In particular, if $f(u) > 0$ for $u \geq 0$, then $\lambda\{\|u\|\} \to 0$ as $\|u\| \to 0$.

3-4. Theorem. Suppose that $f(u)/u \to \infty$ as $u \to r$ from below. Then $\lambda\{\|u\|\} \to 0$ as $\|u\| \to r$.

Proof. For any number $m > 0$, there is a positive number $r_1 < r$ such that $f(u) > mu$ for $r_1 \leq u < r$. Choose s such that $r_1 < s < r$ and $r_1 < s < 2r_1$, and set $\alpha = r_1/2s$, so that $\frac{1}{4} < \alpha < \frac{1}{2}$. Then whenever $\alpha \leq t < \frac{1}{2}$ and $\|u\| > s$, we have $u(t) \geq 2t\,\|u\| \geq 2\alpha s = r_1$, so

$$\lambda^{-1} \geq \frac{1}{\|u\|} \int_{\alpha}^{\frac{1}{2}} tf(u(t))\, dt$$

$$\geq 2m \int_{\alpha}^{\frac{1}{2}} t^2\, dt$$

$$\geq \tfrac{1}{4}m.$$

Since m can be chosen arbitrarily large, this shows that $\lambda\{\|u\|\} \to 0$ as $\|u\| \to r$. //

The next theorem obtains another bound on the values of λ for which equation (3. 1) has a positive solution; in this case, the bound is obtained directly from (3. 1).

3-5. Theorem. Suppose that for some positive numbers a, b, f(u) satisfies $f(u) \geq au + b$ for $0 > u > r$. Then the eigenvalues satisfy $\lambda\{\|u\|\} < \frac{\pi^2}{a}$ for $0 < \|u\| < r$.

Proof. Let $v(t) = \sin \sqrt{\lambda a t}$, and let u be a solution of (3. 1) corresponding to the eigenvalue λ. Define $W(t) = u(t)\, v'(t) - v(t)\, u'(t)$. If $\pi^2/\lambda a \leq 1$, then

$$\frac{dW}{dt} = u(t)v''(t) - v(t)u''(t)$$

$$\geq - u(t)\, \lambda a v(t) + \lambda a u(t)v(t) + \lambda b v(t) = \lambda b v(t)$$

for $0 \leq t \leq \pi/\sqrt{\lambda a}$, so

$$u\left(\frac{\pi}{\sqrt{\lambda a}}\right) = -\frac{1}{\sqrt{\lambda a}} \left[W\left(\frac{\pi}{\sqrt{\lambda a}}\right) - W(0)\right]$$

$$= -\frac{1}{\sqrt{\lambda a}} \int_0^{\pi/\sqrt{\lambda a}} \frac{dW}{dt}\, dt$$

$$< 0,$$

which is impossible. Thus $\pi^2 > \lambda a$. //

Solutions of the boundary value problems (3.1) and (3.4) can be obtained from solutions U(t;m) of the initial value problem

$$U'' + f(U) = 0, \quad t \geq 0, \tag{3.10}$$
$$U(0;m) = 0, \quad U'(0;m) = m > 0,$$

where primes on U denote differentiation with respect to t. We assume that f is twice continuously differentiable and that $f(u) > 0$, $f'(u) \geq 0$ for all u.

Let us consider (3.10) in the phase plane by introducing

$$U'(t) = V(t)$$
$$V'(t) = -f(U(t)) \tag{3.11}$$
$$U(0) = 0, \quad V(0) = m > 0.$$

In the phase plane, a solution of (3.11) starts at the point (0, m) and moves into the first quadrant with its tangent pointing down and to the right. The sign of the curvature of this curve is the same as the sign of

$$U'(t)\,V''(t) - V'(t)U''(t)$$
$$= -U'^2 f'(U) - f^2(U) \;;$$

this is always negative if $f'(U)$ is non-negative, and the curve is concave as viewed from the origin. Since f(u) is positive, there can be no horizontal tangents, and therefore only one intersection with the negative V-axis. This gives us:

<u>3-6. Lemma.</u> For each number $m > 0$, the solution of (3.10) has exactly one positive zero $z(m) > 0$:

(3.12) $$U(z(m);m) \equiv 0.$$

Thus each value of $m > 0$ yields exactly one solution of (3.1),

(3.13) $$u(t) \equiv U(tz(m);m),$$

corresponding to the eigenvalue $\lambda = z^2(m)$. For $m = 0$, we take $z(0) = 0$ and $u(t) \equiv 0$.

From (3.10) we obtain (cf. (3.6)):

$$\tfrac{1}{2}(U')^2 + F(U) = \tfrac{1}{2}(U'(0))^2 = F(U_{max}(m)),$$

so

(3.14) $$m = \sqrt{2F(U_{max}(m))} = \sqrt{2F(\|u\|)} ,$$

where u is defined by (3.13) and $U_{max}(m) \equiv \max\{U(t;m): 0 \leq t \leq z(m)\}$. This shows that m is a strictly increasing function of $\|u\|$ (and conversely) for $m > 0$, and $m = 0$ iff $\|u\| = 0$.

We conclude that the behavior of λ as a function of $\|u\|$ may be obtained by studying the behavior of

(3.15) $$z(m) = z\,(\sqrt{2F(\|u\|)}\,) = \sqrt{\lambda}$$

as a function of m. From (3.15) and (3.8) we see that z is twice differentiable with respect to m; from (3.10), U(t;m) is twice differentiable with respect to m. Letting $\rho = \|u\|$, we have

$$\frac{d\lambda}{d\rho} = z'(m)\left[\sqrt{2}\, z(m) \frac{f(\rho)}{\sqrt{F(\rho)}}\right],$$

so

$$\operatorname{sgn}\left[\frac{d\lambda}{d\rho}\right] = \operatorname{sgn}\left[z'(m)\right].$$

If (3.12) is differentiated with respect to m, we obtain

(3.16) $$z'(m) = \frac{H(z(m);m)}{m} ,$$

where

$$H(t;m) = \frac{\partial}{\partial m} U(t;m)$$

and we have used $U'(z(m);m) = -m$ (by symmetry; see the discussion following (3.3)).

If we let $x(m)$ be the value of t at which the unique maximum of $U(t;m)$ between $t = 0$ and $t = z(m)$ occurs, we obtain from $U'(x(m);m) \equiv 0$,

$$x'(m) = \frac{K(x(m);m)}{f(U(x(m);m))}$$

where

$$K(t;m) = \frac{\partial}{\partial m} U'(t;m) = H'(t;m)$$

Moreover, from symmetry we conclude that $x(m) = \frac{1}{2}z(m)$.

By differentiating (3.10) with respect to m, we obtain the differential equation satisfied by $H(t;m)$:

(3.18a) $$H'' + f'(U(t;m))H = 0, \quad t \geq 0$$

with the initial conditions

(3.18b) $$H(0;m) = 0, \; H'(0;m) = 1,$$

which can be written as

$$H'(t;m) = K(t;m)$$
$$K'(t;m) = -f'(U(t;m))\, H(t;m)$$
$$H(0;m) = 0, \quad K(0;m) = 1.$$

Let the successive positive zeroes (if any) of H and K be $a_1(m)$, $a_2(m)$, . . ., and $b_1(m)$, $b_2(m)$, . . .; clearly $0 < b_1(m) < a_1(m)$, and none of the zeroes of H can coincide with the zeroes of $K = H'$, by the uniqueness theorem for the initial value problem for (3.18a). Since the zeroes $a_k(m)$ are solutions of $H(t;m) = 0$ and we never have $H'(t;m) = 0$ for $t = a_k(m)$, the implicit function theorem shows that the zeroes $a_k(m)$ depend continuously on m.

3-7. Theorem. For sufficiently small $m > 0$, z is an increasing function of m. (Thus for sufficiently small $\|u\|$, $\lambda\{\|u\|\}$ is an increasing function of $\|u\|$.)

Proof. From (3.16) we see that it suffices to consider the sign of $H(z(m);m)$. Keep m fixed and let

$$W(U,H;t) = U(t)H'(t) - H(t)U'(t) .$$

Using (3.10) and (3.18a), we obtain

$$\frac{\partial W}{\partial t}(U,H;t) = H(t)\left[f(U(t)) - U(t)\, f'(U(t))\right],$$

and therefore

$$W(U,H;t) = \int_0^t H(s)\left[f(U(s)) - U(s)\, f'(U(s))\right] ds, \tag{3.20}$$

since

$$W(U,H;0) = 0.$$

Because f(u) is positive, $f(u) - uf'(u)$ is positive for sufficiently small $|u|$; thus for $0 < t \leq a_1(m)$, the integral in (3.20) is positive for sufficiently small $U_{max}(m)$, i.e., for sufficiently small $m > 0$. Therefore

$$W(U,H;a_1(m)) = U(a_1(m))\ H'(a_1(m)) > 0;$$

since $H'(a_1(m)) < 0$, we must have $U(a_1(m)) < 0$ and therefore $z(m) < a_1(m)$, i.e., $H(z(m)) > 0$, for sufficiently small m.

Hence $z'(m)$ is positive for sufficiently small m, as was to be proved. //

3-8. Theorem. If $f(u) - uf'(u) \geq 0$ for $u > 0$, then z is an increasing function of m for all $m > 0$ (and therefore $\lambda\{\|u\|\}$ is an increasing function of $\|u\|$ for all $\|u\|$).

Proof. This follows from equation (3.20) of the preceding proof. //

From equations (3.16) and (3.17) we obtain the following important lemma.

3-9. Lemma. $z'(m)$ is zero iff $z(m) = a_i(m)$ for some i; $x'(m) = 0$ iff $x(m) = b_j(m)$ for some j.

The lemma shows that if $z'(m) = 0$ (i.e., if $\frac{d\lambda\{\rho\}}{d\rho} = 0$), then $H(t;m)$ is an eigenfunction of the linear equation

$$H'' + \mu f'(U(t;m)) H = 0$$

with the boundary conditions

$$H(0) = H(z(m)) = 0 ,$$

corresponding to the eigenvalue $\mu = 1$; thus $h(t) \equiv H(tz(m);m)$ is an eigenfunction of the variational problem associated with (3.1),

$$h'' + \mu f'(u(t)) h = 0 ,$$

$$h(0) = h(1) = 0,$$

where $u(t)$ is defined by (3.13) and satisfies (3.1), corresponding to the eigenvalue $\mu = \lambda = z^2(m)$.

Suppose now that $f''(u) \geq 0$ for $u \geq 0$. We wish to investigate the values of m (if any) for which $z'(m) = 0$. For such values of m, we obtain from (3.16),

$$z''(m) = \frac{1}{m} \frac{\partial H}{\partial m}(z(m);m) \qquad (z'(m) = 0). \tag{3.21}$$

Differentiating equations (3.18) with respect to m, we find that $\frac{\partial H}{\partial m}(t;m) \equiv H_m(t;m)$ satisfies

$$\begin{aligned} &H_m''(t;m) + f'(U(t;m))\, H_m(t;m) = -f''(U(t;m))\, H^2(t;m) \\ &H_m(0;m) = H_m'(0;m) = 0 \,. \end{aligned} \tag{3.22}$$

We can have $z'(m) = 0$ only if $z(m) = a_i(m)$ for some i (Lemma 3-9); since z and a_k are continuous, $z(0) = 0$, and $a_k(m) < a_i(m)$ for $k < i$, we can have an extremum of z only if, for some m, $z(m) = a_1(m)$. The smallest value of m for which $z(m) = a_1(m)$ will be the smallest value of m for which $z'(m) = 0$.

Let us consider an m (if one exists) where $z(m) = a_1(m)$; then $H(t;m)$ satisfies the equation (3.18a) with the boundary conditions

$$H(0;m) = H(z(m);m) = 0.$$

If for this m we also had $z''(m) = 0$, then (see (3.21)) $H_m(t;m)$ would satisfy equation (3.22) with the boundary conditions

$$H_m(0;m) = H_m(z(m);m) = 0.$$

Since equation (3.18a) is the homogeneous equation corresponding to the differential equation (3.22), this would mean that H is orthogonal to the right hand side of (3.22) on $0 \leq t \leq z(m)$. Since $H(t;m)$ is positive on $0 \leq t < z(m)$, this orthogonality is impossible unless $f''(U(t;m)) = 0$ for all t, $0 \leq t \leq z(m)$. But then f has the form f(u)

$= au + b$, $0 \leq u \leq U_{max}(m)$, for some constants $a \geq 0$, $b > 0$; in this case, equation (3.18a) is the homogeneous equation corresponding to equation (3.1), and therefore $b = 0$, which contradicts $f(0) > 0$ (this shows that we cannot have $z(m) = a_1(m)$ when $f(u) = au + b$ for $0 \leq u \leq U_{max}(m)$). Thus $z(m) = a_1(m)$ implies $z'(m) = 0$ and $z''(m) \neq 0$. Moreover, an analysis similar to that of the proof of Theorem 3-7, applied to (3.18) and (3.22), shows that in fact $H_m(a_1(m);m) < 0$ for any m if $f''(u) \geq 0$ and $f''(u) \not\equiv 0$ for $0 < u < U_{max}(m)$, when $z(m) = a_1(m)$; therefore $z''(m) < 0$ (i.e., z has a maximum) whenever $z(m) = a_1(m)$, by equation (3.21).

Consider now an m (if any) such that, for some $k > 1$, $z(m) = a_k(m)$. Since $U(z(m)-t;m) = U(t;m)$, it is easy to see that $H(a_k(m)-t;m) \equiv \hat{H}(t;m)$ satisfies (3.18a) along with $H(t;m)$; since $\hat{H}(0;m) = 0 = H(0;m)$, H and $\hat{H}$ must be linearly dependent (their Wronskian vanishes). Thus for some constant c we have $H(a_k(m)-t;m) = H(t;m)c$. In particular, if $z(m) = a_2(m)$, then $0 = cH(a_1(m);m) = H(a_2(m)-a_1(m);m)$, so that $a_2(m)-a_1(m) = a_1(m)$ and $a_2(m) = 2a_1(m)$. But then $x(m) = \frac{1}{2}z(m) = \frac{1}{2}a_2(m) = a_1(m)$, and $x(m) = b_j(m)$ for some j by equation (3.17), Lemma 3-9, and the fact that $x'(m) = \frac{1}{2}z'(m) = 0$. As pointed out above, we cannot have $a_1(m) = b_j(m)$, and therefore we cannot have $z(m) = a_2(m)$; by continuity, there is no m such that $z(m) = a_k(m)$ for $k > 1$.

The preceding two paragraphs imply the following:

<u>3-10. Lemma</u>. Suppose f(u) is positive and f'(u) is non-negative for all u, and f''(u) is non-negative for positive u. Then z has at most

one extremum, which must be a maximum.

Proof. Since for no m do we have $z(m) = a_2(m)$, we must have $0 \leq z(m) < a_2(m)$. Hence z can have extrema only if, for some m, $z(m) = a_1(m)$. The condition on $f''(u)$ implies that any of these extrema must be maxima, and therefore there can be at most one. //

From Theorems 3-1, 3-4, 3-7, and 3-8, and Lemma 3-10, we obtain the following description of the set of eigenfunctions of equations (1.1) (we use the notation of Theorem 3-1); note that if $f(u) > 0$ and $f'(u) \geq 0$ for $u \geq 0$, then f can be extended to a differentiable function on $(-\infty, \infty)$, which we also denote by f, such that $f(u) > 0$ and $f'(u) \geq 0$ for all u.

3-11. Theorem. Let f be positive and f' be continuous and non-negative on $[0, \infty)$. Then $\lambda\{\rho\} \to 0$ as $\rho \to 0$ and is an increasing function of ρ for small ρ. If $f(u) - uf'(u) \geq 0$ for all $u > 0$, then $\lambda\{\rho\}$ is an increasing function of ρ, and for each $\lambda > 0$, equations (3.1) have at most one (non-negative) solution. If $f''(u) \geq 0$ for all $u \geq 0$, then $\lambda\{\rho\}$ has at most one maximum for $\rho \geq 0$. If $f(u)/u \to \infty$ as $u \to \infty$, then $\lambda\{\rho\} \to 0$ as $\rho \to \infty$. If $f''(u) \geq 0$ for $u \geq 0$ and $f(u)/u \to \infty$ as $u \to \infty$, then $\lambda\{\rho\}$ has exactly one maximum, say $\lambda\{\rho^*\} = \lambda^*$, and for each $\lambda \epsilon (0, \lambda^*)$, equations (3.1) have exactly two eigenfunctions, while for $\lambda = \lambda^*$, equations (3.1) have exactly one eigenfunction.

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