INERTIAL EFFECTS ON PARTICLE DYNAMICS

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To my mother and father:

For their loving support and encouragement

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Abstract

While the theory of suspension flows and particle dynamics is well understood under Stokes flow conditions when viscous forces dominate, little is known at finite Reynolds number when the inertial forces of the suspending fluid are important. In the present study, expressions are derived that allow for dynamic calculations of particle, drop, and bubble motion at finite Reynolds number. The results show a significant change in the temporal behavior of the force/velocity relationship from that derived from the unsteady Stokes equations, particularly as a body approaches its steady state. At finite Reynolds number, when the convective inertial effects are included, the hydrodynamic force on a body has much weaker history dependence on the past motion of the body and it reaches its steady state faster than what would be predicted if only the unsteady inertial effects are accounted for. When compared with numerical solutions of the Navier–Stokes equations, the analytical force expressions perform well up to a Reynolds number of 0.5.

A common theme to the derivations is the use of the reciprocal theorem which provides for an efficient and elegant means for computing inertial effects in suspension mechanics. Connections with past approaches are made in light of these new applications of the reciprocal theorem.

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Chapter 1

Introduction

In modeling suspensions, the particular approach one uses is largely determined by the magnitude of the Reynolds number, a measure of the inertial forces relative to the viscous forces in the suspension. The Reynolds number is given by $Re = \rho l_c U_c/\mu$, where l_c and U_c are the characteristic length dimension and velocity of the particles and μ and ρ are the viscosity and density of the suspending fluid. Thus the modeling approach is usually dictated by the particle sizes one is dealing with.

For small particles, say in the micron range, suspended in fluids of moderate viscosities, i.e., liquids, the Reynolds number is generally small ($\ll 1$). Under these conditions, the Stokes equations, which describe the motion of an inertialess fluid, are used to evaluate the particle dynamics in suspensions dominated by viscous forces. That is, the Reynolds number is taken to be identically equal to zero. The Stokes equations have the fortuitous property of being linear in the fluid velocity and lead to a linear relationship between the particle velocities and the hydrodynamic stresses acting on them. A suspension modeling method that fully exploits this linearity property is Stokesian dynamics [6] which is a molecular dynamics-like technique for simulating suspension flows. Another important consequence of the Stokes equations is that the particle dynamics governed by Stokes flow depends only on the instantaneous state of the system; the current dynamics are independent of the path by which it has come to that state, which means there is no history dependence.

On the other end of the spectrum, for particles in the millimeter range or larger, an approach for modelling suspensions is granular dynamics. This approach to describing suspension flows completely neglects the presence of the suspending fluid, usually a gas, under the assumption that particle inertia and solid-body collisions dominate the dynamics of the particle motion. Here, the Reynolds number is generally large so that one is justified in neglecting the viscous forces that act on the particles. Analysis in this regime is often by way of kinetic theory [23, 22].

When the particles are on the order of tens or hundreds of microns and the suspending fluid is a gas, one is often in an intermediate condition where both solid-body collisions and viscous forces are important. The result is due to the fact that the Reynolds number is small owing to the small particle sizes, while the Reynolds number based on the particle density instead of the fluid density, usually referred to as the Stokes number ($St = \rho_p l_c U_c / \mu$), is order one or larger. The Stokes number is a measure of the relative importance of particle inertia to the viscous forces in the fluid. This regime has been recently investigated by Koch [27] also through the use of a kinetic theory approach.

At present, little theory exists which takes into account the effects due to the inertia of the suspending fluid. So in an effort to fill this gap, we began to consider cases when both the viscous and inertial forces of the suspending fluid were important, that is, when the Reynolds number was not infinitesimally small. The conditions to have in mind here are for particle sizes of hundreds of microns or millimeters suspended in liquids of moderate viscosities on the order of a Poise. Our ultimate goal is to extend simulation methods such as Stokesian dynamics to cases where the Reynolds number is small but finite. One could anticipate that this is a difficult task since the governing equations for the fluid are now the full unsteady Navier–Stokes equations, and the linearity properties and lack of history dependence attributed to the Stokes equations no longer exist.

Many interesting phenomena exist which are finite Reynolds number effects, particularly those associate with the lift force on particles perpendicular to the flow direction. Most notably was that observed by Segré and Silberberg [48]. They found that the particles in dilute suspensions of neutrally buoyant spheres flowing through tubes tended to migrate to a preferred position at 0.6 tube radii from the axis independent of their initial position. As can be demonstrated by a reversibility argument, this is a phenomena that the Stokes equations could never predict. The possibility of discovering such peculiar behavior provides additional motivation for investigating suspension flows at non-zero Reynolds number.

With its great utility in the study of Stokes flow problems, the reciprocal theorem, in its more general form, was targeted as the method of choice for studying suspensions at small-but-finite Reynolds number. Its applications spans all areas of transport phenomena. A subset of these is in the computation of surface integrated quantities such as the total heat flow from or the net force on a body suspended in a fluid. The more global uses of the reciprocal theorem in the area of fluid mechanics are well demonstrated in a book by Kim and Karrila [24]. In the next section we shall show the relative ease with which one can derive a well-known result for weak convection effects in heat transfer using the reciprocal theorem approach. We choose the heat flow problem because the governing equations are scalar, as opposed to the vector equations associated with fluid flow, making the analysis easier to follow. This will provide the motivation for employing the approach to study other more difficult, and previously unsolved, problems in suspension mechanics.

1.1 Weak convection effects in heat transfer

Here we will outline the use of the reciprocal theorem to compute the small-but-finite Péclet number correction to the heat flow from a spherical particle in a uniform flow accurate to $O(Pe)$. Although for fluid flow, the following chapter will clarify many of the details of the procedure.

Neglecting any viscous dissipation effects, the steady thermal energy equation for an incompressible fluid satisfying Fourier conduction is in dimensionless form

$$Pe(\mathbf{u} \cdot \nabla T) = \nabla^2 T, \tag{1.1}$$

where $\mathbf{u}$ is the velocity and T is the temperature of the fluid. The dimensionless parameter Pe is the Péclet number and provides a measure of the relative magnitude of convection (due to the uniform flow) compared with conduction as a mechanism of transporting heat. It is defined by $Pe \equiv \rho C_p U a/k$ where ρ, C_p and k are the heat capacity and thermal conductivity of the fluid while U is the magnitude of the free-stream velocity and a is the particle radius. We shall assume all the physical properties of the fluid remain constant.

Consider two temperature fields which satisfy (1.1): one for small Péclet number and denoted by T, the other for Péclet number identically equal to zero (the pure conduction limit) which we denote by $\hat{T}$ and satisfying $\nabla^2 \hat{T} = 0$. Let the boundary conditions for both of these fields be given by

$$\begin{aligned} T, \hat{T} &= 1 \quad \text{on} \quad S, \\ T, \hat{T} &\to 0 \quad \text{at} \quad |\mathbf{x}| \to \infty, \end{aligned} \tag{1.2}$$

where S represents the surface of the sphere and $\mathbf{x}$ is the position vector in a coordinate system with origin at the center of the sphere. The solution for the temperature field

$\hat{T}$ is given simply by $\hat{T} = 1/r$ where $r = |\mathbf{x}|$. The solution for T, on the other hand, is unknown for arbitrary Pe.

Next let us assume the Reynolds number for the flow past the particle is sufficiently small that the velocity field $\mathbf{u}$ can be approximated by Stokes flow. The most important consequence of this is that the velocity field possesses fore-aft symmetry and thus $\mathbf{u}(\mathbf{x}) = \mathbf{u}(-\mathbf{x})$. Because the particle is fixed in a uniform flow, $\mathbf{u} = \mathbf{0}$ on the particle surface, while far from the particle it approaches the uniform stream ($\mathbf{u} \to \mathbf{i}_z$) where $\mathbf{i}_z$ is a unit vector in the z-direction.

In order to evaluate the heat flow from the sphere that takes into account the small-but-finite Péclet number effect, we require the reciprocal theorem expression. It is derived by first noting

$$\begin{aligned}
&\int_V \boldsymbol{\nabla}\cdot \left(\hat{T}\boldsymbol{\nabla} T - T\boldsymbol{\nabla}\hat{T}\right) dV \\
&\quad = \int_V \left(\hat{T}\nabla^2 T + \boldsymbol{\nabla}\hat{T}\cdot\boldsymbol{\nabla} T - \boldsymbol{\nabla} T\cdot\boldsymbol{\nabla}\hat{T} - T\nabla^2\hat{T}\right) dV, \\
&\quad = Pe \int_V \left(\hat{T}(\mathbf{u}\cdot\boldsymbol{\nabla} T)\right) dV,
\end{aligned} \tag{1.3}$$

where V represents the volume of the fluid surrounding the particle. Here we have used the fact that T satisfies (1.1) and $\hat{T}$ satisfies Laplace's equation. Starting from the same point above we can apply the divergence theorem to obtain

$$\begin{aligned}
&\int_V \boldsymbol{\nabla}\cdot \left(\hat{T}\boldsymbol{\nabla} T - T\boldsymbol{\nabla}\hat{T}\right) dV \\
&\quad = \int_{S_p} \mathbf{n}\cdot\boldsymbol{\nabla}\hat{T}\, dS - \int_{S_p} \mathbf{n}\cdot\boldsymbol{\nabla} T\, dS,
\end{aligned} \tag{1.4}$$

where S_p is the surface of the particle and $\mathbf{n}$ is the outer normal to the particle surface pointing into the fluid. Here we have applied the boundary conditions (1.2). If we define the dimensionless heat flux from the particle to the fluid, which is identified as the Nusselt number Nu, as

$$Nu = \frac{-1}{2\pi} \int_{S_p} \mathbf{n} \cdot \nabla T \, dS, \tag{1.5}$$

and use the the fact that $\hat{T} = 1/r$, (1.4) becomes

$$\int_V \nabla \cdot \left(\hat{T} \nabla T - T \nabla \hat{T} \right) dV = -4\pi + 2\pi Nu. \tag{1.6}$$

Now equating the two results (1.3) and (1.6) we have

$$Nu = 2 + \frac{1}{2\pi} Pe \int_V \left((\mathbf{u} \cdot \nabla T) \hat{T} \right) dV. \tag{1.7}$$

The first term is the pure conduction result and the second is the finite Pe contribution due to convection. Note that although we shall apply this reciprocal theorem expression to the case of small Pe, it is actually valid for arbitrary Pe.

For $Pe \ll 1$, one would expect that the convective correction could be approximated simply by replacing T by the pure conduction solution $\hat{T}$ as a regular perturbation approach. However, what one finds is that this results in a conditionally convergent integral. This is because the integrand would be $O(r^{-3})$ and would yield a log singularity at infinity; due to the symmetry properties of both $\hat{T}$ and $\mathbf{u}$, on the other hand, the integrand would be antisymmetric and the angular integration if done first would result in a zero contribution. The reason for this apparent anomaly is that while the pure conduction solution is a valid approximation to the actual temperature field near the particle, its not valid far from the particle where convection becomes important in transporting heat.

To resolve the problem we need to consider the far-field form of the temperature field. Its governing equation is

$$Pe\frac{\partial T}{\partial z} = \nabla^2 T + 4\pi\delta(\mathbf{x}), \tag{1.8}$$

where we have made two approximations. First, we have used $\mathbf{u} \sim \mathbf{i}_z$ in the far-field. Second, we have replaced the surface boundary condition by a point-source description of the particle. In the far-field the particle appears as a point-source of heat to a first approximation of magnitude given by the pure conduction solution ($2\pi Nu \sim 4\pi$). This expression is now amenable to Fourier transforms and thus we may write

$$2\pi \mathrm{i} Pe k_3 \tilde{T} = -4\pi^2 k^2 \tilde{T} + 4\pi, \tag{1.9}$$

so that the Fourier transformed temperature field is

$$\tilde{T} = \frac{2}{2\pi k^2 + \mathrm{i} Pe k_3}. \tag{1.10}$$

Since contributions to the convective correction from the near-field are negligible due to the symmetry argument discussed above, we can extend the volume of integration in (1.7) to the entire domain of space, particle plus fluid, without affecting the result to $O(Pe)$. This allows us to apply the convolution theorem and write (1.7) as

$$Nu = 2 + \frac{1}{2\pi} Pe \int \left((2\pi \mathrm{i} k_3 \tilde{T}) \tilde{\hat{T}} \right) d\mathbf{k}. \tag{1.11}$$

Then substituting (1.10) where $\tilde{\hat{T}}$ is given by (1.10) with $Pe = 0$ we have

$$Nu = 2 + \frac{1}{2\pi} Pe \int \left(\frac{4\mathrm{i} k_3}{k^2(2\pi k^2 + \mathrm{i} Pe k_3)} \right) d\mathbf{k}. \tag{1.12}$$

If we ignore the antisymmetric imaginary part of the integrand and its associated singularity, which is simply due to artificially extending the domain of integration to the center of the particle, we can evaluate the real part to find

$$Nu = 2 + Pe. \tag{1.13}$$

This represents the well-known small-but-finite Péclet number corrected Nusselt number for heat flow.

For the fluid flow problems we consider in the following chapters, the analogous arguments to those presented above can be obtained simply by replacing the idea of heat transport by conduction and convection with vorticity transport by diffusion and convection.

1.2 Thesis overview

The motivation for the next three chapters was to more fully understand particle dynamics in fluids at small-but-finite Reynolds number. Surprisingly, little theory has been developed in this area even for single isolated particles. Hence, these chapters have been devoted towards the study of isolated particles in unbounded domains. New results are derived for the temporal behavior of the force/velocity relationship for particles in unsteady motion. These are most often attributed to the history dependence of this relationship associated with finite Reynolds number flows.

In Chapter 2 we develop the theory for the hydrodynamic force acting on a solid particle in arbitrary curvilinear motion in a time-dependent uniform flow at finite Reynolds number. Several inertesting results from the analysis of this chapter were obtained. As a particle approaches a steady velocity, say in changing from one steady state to another, the unsteady Stokes equations, which neglect the convective inertia of the suspending fluid, would predict a temporal decay of the hydrodynamic force of $t^{-\frac{1}{2}}$. At finite Reynolds number, when the convective inertial effects as well as the unsteady inertial effects are accounted for, the temporal decay of the force is faster

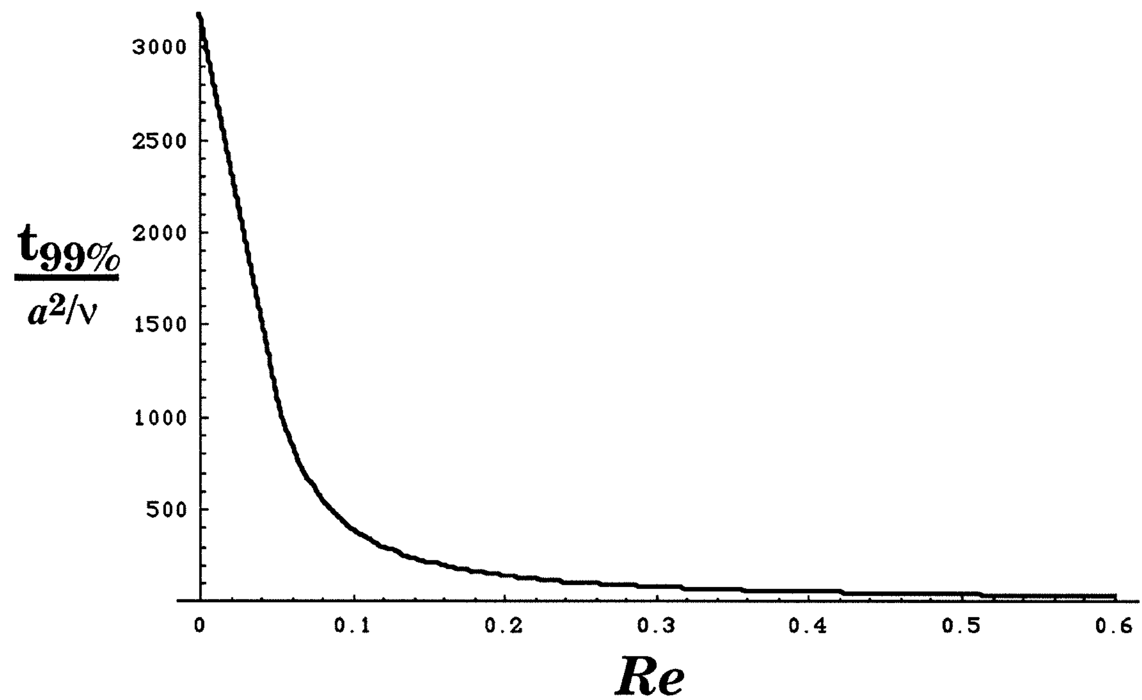

Figure 1.1: The time it takes a sphere of radius a to reach 99% of its terminal velocity after being released from rest in a fluid of kinematic viscosity ν as a function of the Reynolds number Re.

and dependent on the initial and final conditions before and after the particle changed its velocity. One of the more striking consequences of this behavior is that particles, in general, reach their steady state velocity more quickly at finite Reynolds numbers than under zero Reynolds number (unsteady Stokes flow) conditions. This finding is dramatically illustrated in Figure 1.1, where the time it takes a sphere released from rest to reach 99% of its terminal velocity is shown as a function of the Reynolds number Re. For example, the sphere will reach its steady state velocity 40 times faster at $Re = 0.3$ than at zero Reynolds number. An even more intriguing result is that when a particle changes its velocity from one constant, finite value to another, the hydrodynamic force reaches its new steady state exponentially fast at finite Reynolds, as opposed to the "universal" $t^{-\frac{1}{2}}$ approach to steady state predicted by the unsteady Stokes flow solution. The faster temporal decay found here translates into the fact that the velocity history dependence of the force on a particle is much weaker at finite Reynolds number. This observation, in turn, can have important implications for numerical computations as computer time and data storage can be reduced in doing dynamic analyses of particle trajectories in viscous fluids.

In Chapter 3 the analytical force expression from Chapter 2 is applied to the study of oscillatory particle motion. By comparing the results with previously published numerical solutions of the Navier–Stokes equations at various Reynolds numbers, it is found that the analytical force expression works well up to a Reynolds number of about 0.5. This finding is quite suprising considering the fact that the expression is derived on the basis that the Reynolds number is very small.

Chapter 4 represents an extension of the above theory to the case of bubbles and drops. Using the reciprocal theorem, a general hydrodynamic force expression is derived that takes into account drop shape and an arbitrarily imposed flow. This general expression is then applied to the case of a spherical drop moving in a uniform time-dependent flow at small-but-finite Reynolds number. It is found that the low-frequency (long-time) dynamics of the force on a drop is very similar to the case of a solid particle.

In the Chapter 5 we make an effort to extend the ideas of the previous chapters for single particles to multiparticle systems with an eye towards the use of the results to simulate suspension flows. Here, in using the reciprocal theorem approach, one is able to collect in a self-consistent manner all the hydrodynamic terms that affect the dynamics. Included in the expressions are contributions from both the many-body hydrodynamic interactions and the inertia of the suspending fluid (the finite Reynolds number effects).

The last chapter is meant to provide some ideas for the future direction that this work can take. Some of the issues that deserve further investigation have already begun to be addressed, and the preliminary results from these pursuits are presented in this chapter. Specifically, we consider particle dynamics in a linearly varying imposed flows, which can lead to inertial lift forces on particles, and sedimenting cubic latices of particles to find how particle interactions can influence the inertial effects of the suspending fluid.

Chapter 2

The hydrodynamic force on a rigid particle undergoing arbitrary time-dependent motion at small Reynolds number

Summary

The hydrodynamic force acting on a rigid spherical particle translating with arbitrary time-dependent motion in a time-dependent flowing fluid is calculated to $O(Re)$ for small but finite values of the Reynolds number, Re, based on the particle's slip velocity relative to the uniform flow. The corresponding expression for an arbitrarily shaped rigid particle is evaluated for the case when the time scale of variation of the particle's slip velocity is much greater than the diffusive scale, a^2/ν, where a is the characteristic particle dimension and ν is the kinematic viscosity of the fluid. It is found that the expression for the hydrodynamic force is not simply an additive combination of the results from unsteady Stokes flow and steady Oseen flow and that the temporal decay to steady state for small but finite Re is always faster than the $t^{-\frac{1}{2}}$ behavior

of unsteady Stokes flow. For example, when the particle accelerates from rest the temporal approach to steady-state scales as t^{-2}.

2.1 Introduction

This study is concerned with the unsteady motion of a rigid particle in an unbounded incompressible Newtonian fluid. The flow far from the particle may also be unsteady, but is taken to be uniform. In addition, we assume that the particle Reynolds number, $Re = U_c a/\nu$, based on a characteristic particle slip velocity, U_c, remains small throughout the particle motion. Here a denotes the characteristic particle dimension and ν is the kinematic viscosity of the fluid. In particular, we solve for the unsteady hydrodynamic force acting on the particle for small but finite values of the Reynolds number.

The motivation for studying such a problem at small Reynolds number is three-fold. First, there is the basic question of what are the forces acting on a particle undergoing an arbitrary time-dependent motion. How do these forces depend on time and on the Reynolds number? Second, the problem has applications in particle dynamics. For example, in suspension mechanics Stokes flow often dominates the fluid motion, owing to small particle sizes and relatively large fluid viscosities. The inertia of the fluid and particle are then small corrections, and the present analysis represents a first step to the inclusion of both convective and unsteady inertial effects in suspension mechanics. Third, there is considerable interest in the transport of small particles in turbulent flows [44, 36, 53], and a large part of the unsteadiness arises from the turbulent velocity fluctuations. Determining unsteady hydrodynamic forces in these flows is necessary to determine the particle motion, which has obvious implications for particle dispersion and deposition, particle-image velocimetry measurements [2], etc. In addition, through this problem we wish to demonstrate the relative simplicity of using the reciprocal theorem as an approach to computing inertial effects.

To determine the force correct to $O(Re)$, we do not solve for the detailed flow field; instead, we make use of the reciprocal theorem and the known results of steady (and, in some cases, unsteady) Stokes flow to avoid the detailed problem and proceed directly to computing the force. From the reciprocal theorem we obtain results for the hydrodynamic force in terms of inertial corrections to the steady Stokes drag. These corrections come from two sources. One is due to the unsteady nature of the flow and yields such contributions to the hydrodynamic force as the Basset force and the added mass. The other inertial corrections are due to weak convective effects and are the origin of the well-known $O(Re)$ Oseen correction to the steady Stokes drag. When both contributions are considered together, we shall see that the nature of the hydrodynamic force is determined by the characteristic time scale of the motion, τ_c. When τ_c is small (e.g., $O(a^2/\nu)$), the unsteady inertial corrections dominate the hydrodynamic force. Only when τ_c becomes very large (i.e., $\tau_c > O(\nu/{U_c}^2)$) does the convective inertial correction take the form of the Oseen result. Further, and not surprisingly, we shall see that the two inertial contributions, unsteadiness and convection, are not simply additive. Indeed, the convective inertia changes fundamentally the temporal decay of the unsteady inertia from $t^{-\frac{1}{2}}$, characteristic of an unsteady Stokes flow, to t^{-2} when particles accelerate from rest, which has profound implications for the approach to steady-state.

In the area of unsteady inertial effects, several results have been obtained by previous researchers. For example, Basset [3] determined the hydrodynamic force acting on a spherical particle undergoing arbitrary time-dependent motion in an otherwise quiescent fluid. The governing equations were the unsteady Stokes equations for a fluid of density ρ and viscosity μ:

$$-\nabla p + \mu \nabla^2 \mathbf{u} = \rho \frac{\partial \mathbf{u}}{\partial t}, \quad \nabla \cdot \mathbf{u} = 0, \tag{2.1a}$$

with boundary conditions

$$\mathbf{u} = \mathbf{U}_p(t) \quad \text{when} \quad |\mathbf{x}| = a; \quad \mathbf{u},\ p \to 0 \quad \text{as} \quad |\mathbf{x}| \to \infty. \tag{2.1b}$$

Here, $\mathbf{u}$ is the velocity of the fluid, p the pressure, $\mathbf{U}_p(t)$ the velocity of the particle, and a its radius. Basset's result is

$$\mathbf{F}^H_{us}(t) = -6\pi\mu a \mathbf{U}_p(t) - 6\pi\mu a \left(\frac{a^2}{\pi\nu}\right)^{\frac{1}{2}} \int_{-\infty}^{t} \frac{\dot{\mathbf{U}}_p(s)\, d\, s}{(t-s)^{\frac{1}{2}}} - \tfrac{2}{3}\pi\rho a^3 \dot{\mathbf{U}}_p(t), \tag{2.2}$$

where $\dot{\mathbf{U}}_p(t)$ is the acceleration of the particle. The first term is the pseudo-steady Stokes drag. The second is the Basset memory integral, which depends on the past history of the particle motion. It is a combination of both viscous and inertial contributions to the force in that it depends on both the viscosity of the fluid and the acceleration of the particle. The last term, a purely inertial contribution, is the so-called added mass. It represents the additional mass the particle appears to have due to the resistance to acceleration of the surrounding fluid.

Basset's result was extended to conditions where the flow far from the particle was other than uniform [33] (see also Appendix 2.10.) and to particles of non-spherical shape An important result of the analysis by Lawrence and Weinbaum [29] is that the form of the hydrodynamic force becomes significantly more complicated than (2.2) for non-spherical particles. For example, they demonstrated the existence of distinct high- and low-frequency forms for the "Basset" history force associated with the axisymmetric oscillation of a spheroid. Only for the special case of a sphere is the history force the same in the high- and low-frequency limits. Here we refer to the history force as that term proportional to the half power of the frequency in the Fourier transform representation of the solution. In this way it retains the form of the second term of (2.2) in the time domain for arbitrary particle motion. We note also that Gavze [17] recently presented a general theory for the force and torque acting on a particle of arbitrary shape undergoing rigid body motion in unsteady Stokes flow. Gavze derives expressions for the second-rank tensors that relate the force and torque to the particle's velocity and acceleration, and demonstrates their symmetry properties. The author identifies a steady resistance tensor associated with steady Stokes flow, a potential (or "added mass") resistance tensor associated with potential flow, and a Basset resistance tensor that is a function of time.

One of the first treatments of convective inertia was the classic problem solved by Oseen [40, 41] . He computed the first correction to Stokes drag for small but finite values of the Reynolds number for a sphere held fixed in a steady uniform flow, $\mathbf{U}$. Oseen recognized that the governing equations near the particle were adequately described by the Stokes equations, but that far from the particle they were more appropriately given by the so-called Oseen equations:

$$-\boldsymbol{\nabla} p + \mu\nabla^2\mathbf{u} = \rho\mathbf{U}\cdot\boldsymbol{\nabla}\mathbf{u}, \quad \boldsymbol{\nabla}\cdot\mathbf{u} = 0. \tag{2.3}$$

Oseen constructed a uniformly valid leading order approximation to the flow field that satisfies (2.3) everywhere and obtained the hydrodynamic force expression

$$\mathbf{F}^H_{Os} = 6\pi\mu a\mathbf{U}(1 + \tfrac{3}{8}Re), \tag{2.4}$$

where $Re = a|\mathbf{U}|/\nu$. Although this result for the force is correct, Oseen did not compute the velocity field accurate to $O(Re)$. Proudman and Pearson [43] made use of singular perturbation theory to correctly compute the velocity field to this order, and extended Oseen's result to $O(Re^2\log(Re))$. Brenner and Cox [8] generalized these results to particles of arbitrary shape.

Sano [47] extended the Proudman and Pearson result to the unsteady startup problem where $\mathbf{U}(t) = \mathbf{U}H(t)$ and the fluid is stationary everywhere for $t < 0$. ($H(t)$ is the Heaviside function.) His solution for the hydrodynamic force is

$$\begin{aligned}\mathbf{F}^H(t) =& 6\pi\mu a\mathbf{U}\Bigg[H(t) + \tfrac{1}{3}\delta(t) + (\pi t)^{-\frac{1}{2}} + \tfrac{3}{8}Re\Bigg\{\left(1 + \frac{4}{Re^4 t^2}\right)\mathrm{erf}(\tfrac{1}{2}Re\, t^{\frac{1}{2}}) \\ &+ \frac{2}{(\pi t)^{\frac{1}{2}} Re}\left(1 - \frac{2}{Re^2 t}\right)\exp(-\tfrac{1}{4}Re^2 t) - \frac{8}{3(\pi t)^{\frac{1}{2}} Re}\Bigg\} + \tfrac{9}{40}Re^2 \ln Re + O(Re^2)\Bigg].\end{aligned} \tag{2.5}$$

Here, t has been nondimensionalized by a^2/ν, $\delta(t)$ is the dirac delta function, and $\mathrm{erf}(x)$ is the error function. The first three terms are the unsteady Stokes force, while the portion in curly brackets represents the unsteady Oseen contribution. This result was based on Bentwich and Miloh's [5] Laplace transform solution for the unsteady Oseen velocity field. It is Sano's result that most closely corresponds to the current work. We extend his result to conditions where the particle and the far-field flow can have general time dependence and to particles of arbitrary shape. Of particular interest in (2.5) is the fact that with the inclusion of the Oseen convective inertia the force approaches its steady value as t^{-2} in contrast to the $t^{-\frac{1}{2}}$ predicted by the Basset term of (2.2) from just including the unsteady inertia. This much more rapid temporal decay has important implications for unsteady particle motion. Ockendon [39] also found that on long time scales there is a change in the temporal decay due to the contribution from convective inertia. Recent numerical studies by Mei, Lawrence and Adrian [37] and Mei and Adrian [35] have also confirmed this change in temporal decay.

In order to appreciate the different physical processes occurring in unsteady particle motion, a consideration of the various time scales present is warranted. One can define a Strouhal number as $Sl = (a/U_c)/\tau_c$, which is a measure of the time scale of variation, or unsteadiness, relative to the convective time a/U_c. For example, when a particle is accelerating from rest in stationary fluid, the appropriate time scale is $O(a^2/\nu)$, the time it takes vorticity to diffuse over the length a. In this case, $Sl = O(Re^{-1}) \gg 1$, reflecting the fact that unsteady inertia is large compared to convective inertia. If one considers the problem of a particle released from rest and settling under gravity, initially the time scale is $O(a^2/\nu)$, but as the particle approaches its terminal velocity it progresses through an entire range of longer time scales. Thus, the Strouhal number initially would be very large but would become very small as steady state was being attained. In oscillatory motion $\tau_c = O(\omega^{-1})$, where ω is the frequency of oscillation. Here again, the magnitude of Sl may take on very large or small values depending on the magnitude of ω. Steady particle motion, on the other hand, corresponds to an infinite time scale of variation. Thus, for example, oscillatory particle motion coupled with steady motion provides a condition for the existence of duel time scales, one being ω^{-1} and the other infinite. Due to the non-linearity of the Navier-Stokes equations, these two effects cannot in general be separated into isolated contributions. Also, distinct magnitudes of Sl may not be identifiable at a given instant in time. Since we wish to consider *arbitrary* time-dependent motion, we must allow for the fact the relative importance of unsteady and convective inertia, i.e., Sl, may be of arbitrary magnitude and may change as a function of time. Thus, we shall explicitly leave the Strouhal number in the problem.

If we take as a prototypical example the problem of a spherical particle settling under gravity, our goal then is to compute the hydrodynamic force correct to $O(Re)$ as the particle accelerates from rest and approaches its terminal velocity. As we shall see, initially the hydrodynamic force is determined by unsteady Stokes flow, where the Stokes drag, Basset force, and the added mass are of equal importance. At this stage, vorticity has not diffused out to the Oseen distance of $O(aRe^{-1})$, and, since there is no far-field Oseen region, one does not expect an $O(Re)$ Oseen-like correction to the drag. We find, for the case of a spherical particle, that on this time scale ($\tau_c \sim a^2/\nu$) there is no convective inertial correction to the hydrodynamic force at $O(Re)$. (Under these conditions the convective inertial correction can be anticipated to enter at $O(Re^2)$.) In fact, not until $Sl = O(Re)$, or $\tau_c = \nu/{U_c}^2$, which corresponds to the time it takes vorticity to diffuse out to the Oseen distance, does the convective inertia of the fluid make a contribution to the hydrodynamic force at $O(Re)$. Finally, when the particle reaches its terminal velocity, the hydrodynamic force to $O(Re)$ is given by the Stokes drag plus the steady Oseen correction.

As remarked earlier, the Oseen convective inertia changes the approach to steady state from $t^{-\frac{1}{2}}$ to t^{-2}. To understand how this comes about we must first appreciate that the origin of the Basset, $t^{-\frac{1}{2}}$, scaling is due to the uniform (spherically symmetric) diffusion of vorticity generated at the particle surface into regions of irrotational flow. At finite Re, however, this process does not continue indefinitely. Rather, when the vorticity has diffused out to the Oseen distance, it is swept up in the wake region behind the particle where it is transported by convection. As we shall show, this change in the mechanism of vorticity transport accounts for the change in the temporal behavior of the hydrodynamic force. Because of this change in the temporal decay, a particle released from rest at finite Reynolds number will reach its terminal velocity much more rapidly than one would have predicted based on the Basset correction alone. This fact may explain why experimentalists have measured a steady terminal velocity when the length of their apparatus would not have permitted this if the Basset force was correct.

We can at this point estimate the temporal decay of the force from energy dissipation arguments and the known behavior of unsteady Stokes flow and the wake region in Oseen flow. The rate of doing work by the particle on the fluid is proportional to the volume integral over the fluid of the dissipation $\boldsymbol{\sigma}:\nabla\mathbf{u}$. As steady-state is approached the temporal perturbation to the steady force is proportional to the "dissipation deficit" – the dissipation due to the disturbance velocity – in the region of the fluid not yet reached by the transported vorticity. Vorticity is either confined to the vorticity diffusion volume for short time scales or to the truncated Oseen wake at finite Reynold numbers for long time scales. In both cases, in the region yet to be reached by the vorticity, at steady-state the disturbance velocity $\mathbf{u}$ will behave as $1/r$ and the associated stress field $\boldsymbol{\sigma}$ will be $1/r^2$ from Stokes flow, where r is the distance from the center of the particle. When vorticity has not diffused out to the Oseen distance, $\tau_c < \nu/U_c^2$, the velocity disturbance is confined to a (spherical) region of size $a(\nu t)^{\frac{1}{2}}$. Hence, the temporal correction to the force in dimensional form goes as $F \sim \mu a^2 U \int_{(\nu t)^{\frac{1}{2}}}^{\infty} r^{-2} \cdot r^{-2} dV \sim \mu a^2 U (\nu t)^{-\frac{1}{2}} \sim \mu a U (a^2/\nu t)^{\frac{1}{2}}$. This is just the $t^{-\frac{1}{2}}$ approach to steady-state of the Basset force.

However, when vorticity has diffused out to the Oseen distance, $(\nu t)^{\frac{1}{2}} \approx a Re^{-1}$, the velocity disturbance is gathered up by the convective flow and confined to the (unsteady) Oseen wake. Denoting by z the distance behind the particle, the length of the wake grows convectively as $z \sim Ut$, while its width $(\sqrt{A})$ grows diffusively as $(z\nu/U)^{\frac{1}{2}} = (\nu t)^{\frac{1}{2}}$, where A is the cross-sectional area of the wake. Thus, letting $\boldsymbol{\sigma}$ scale as $\nabla \mathbf{u}$ and $|\mathbf{u}| \sim 1/r \approx 1/z$ from Stokes flow, the temporal correction to the dimensional force goes as $F \sim \mu a^2 U \int z^{-2} \cdot z^{-2} dz dA \sim \mu a^2 U z^{-3} A \sim \mu a^2 U (Ut)^{-3} (\nu t) \sim \mu a U (aU/\nu)(\nu/U^2 t)^2$; the advertised t^{-2} temporal decay. We shall see explicitly below by detailed analysis that this is indeed the proper temporal decay.

In what follows, we obtain the form for the hydrodynamic force to $O(Re)$ that spans the transition from the unsteady Stokes force to the Oseen drag for arbitrary time-dependent motion. As expected from the above discussion and from the result of Sano [47] , it is found that the results are not simply additive. We shall also see that the form of the result will greatly simplify for the case of a spherical particle. In addition, we obtain a simplified expression for arbitrarily shaped particles for the case when the time scale is long ($\gg a^2/\nu$). Under this condition, both unsteady and convective inertial corrections can be treated through singular perturbations to steady Stokes flow and will require only the steady Stokes velocity field associated with the translating particle.

In the next section, the full governing equations for the problem will be posed and the relevant dimensionless parameters identified. In Section 2.3, the reciprocal theorem is presented for computing the inertial correction to the steady Stokes drag. A general derivation of the reciprocal theorem with inertial effects is given in Appendix 2.10. This derivation also generalizes the work of Maxey and Riley [33] . Since the inertial correction in the reciprocal theorem is expressed as a volume integral of functions of the velocity field over the entire fluid domain, we apply scaling arguments in Section 2.4 to obtain a uniformly valid velocity field that is valid for all time scales. In Section 2.5, we combine this velocity field with the reciprocal theorem to obtain an expression for the hydrodynamic force correct to $O(Re)$. In the following three sections, calculations are performed to simplify the expression under various conditions of time scales and particle shape. Included in Section 2.7 is the application of the expression to the numerical calculation of the settling velocity of a spherical particle released from rest. In the last section, we conclude with a discussion of the results.

2.2 Governing equations and boundary conditions

The governing equations for the problem are given by the full Navier-Stokes equations

$$-\boldsymbol{\nabla} p' + \mu \nabla^2 \mathbf{u}' = \rho\left(\frac{\partial \mathbf{u}'}{\partial t} + \mathbf{u}' \cdot \boldsymbol{\nabla} \mathbf{u}'\right), \tag{2.6a}$$

$$\boldsymbol{\nabla} \cdot \mathbf{u}' = 0, \tag{2.6b}$$

with the boundary conditions

$$\mathbf{u}' = \mathbf{U}_p(t) \quad \text{on the surface of the particle,} \tag{2.6c}$$

$$\mathbf{u}' \to \mathbf{U}^\infty(t),\ p' \to p^\infty \quad \text{as} \quad |\mathbf{x} - \mathbf{Y}_p(t)| \to \infty. \tag{2.6d}$$

The uniform undisturbed flow far from the particle is $\mathbf{U}^\infty(t)$, and $\mathbf{Y}_p(t)$ is the position vector for the center of mass of the particle. The pressure p^∞ satisfies

$$-\boldsymbol{\nabla} p^\infty = Re Sl \dot{\mathbf{U}}^\infty(t). \tag{2.7}$$

For our purposes, it is more convenient to pose the problem in a translating coordinate system with the origin at the instantaneous center of the particle. Also, since we wish to deal with a velocity field that decays to zero far from the particle, we consider the disturbance velocity and pressure fields. Thus, letting $\mathbf{r} = \mathbf{x} - \mathbf{Y}_p(t)$, $\mathbf{u} = \mathbf{u}' - \mathbf{U}^\infty(t)$ and $p = p' - p^\infty$, the problem becomes in dimensionless form:

$$-\boldsymbol{\nabla} p + \nabla^2 \mathbf{u} = Re Sl \frac{\partial \mathbf{u}}{\partial t} + Re \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u} - Re \mathbf{U}_s(t) \cdot \boldsymbol{\nabla} \mathbf{u}, \tag{2.8a}$$

$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0, \tag{2.8b}$$

with

$$\mathbf{u} = \mathbf{U}_s(t) \quad \text{on the surface of the particle,} \tag{2.8c}$$

$$\mathbf{u},\ p \to 0 \quad \text{as} \quad |\mathbf{r}| \to \infty. \tag{2.8d}$$

The slip velocity of the particle $\mathbf{U}_s(t)$ is defined by $\mathbf{U}_s(t) = \mathbf{U}_p(t) - \mathbf{U}^\infty(t)$. The Reynolds number is $Re = U_c a/\nu$, and the Strouhal number is $Sl = (a/U_c)/\tau_c$. Here, a, U_c, and τ_c are the characteristic particle dimension, particle slip velocity, and time scale, respectively. $\mathbf{U}_s(t)$ and $\mathbf{u}$ have been nondimensionalized by U_c, $\mathbf{r}$ by a, and p

by $\mu U_c/a$. We do not prescribe an initial condition, but only require that $\mathbf{u}$ remains bounded for all time, $-\infty < t < \infty$.

2.3 The reciprocal theorem

As stated in the introduction, we make use of the reciprocal theorem to compute the hydrodynamic force for the problem stated above. In Appendix 2.10, we derive the reciprocal theorem for the total hydrodynamic force acting on a spherical particle undergoing the time-dependent motion $\mathbf{U}_p(t)$ in a time- and space-dependent flow, $\mathbf{v}^\infty(\mathbf{x}, t)$, and for the total hydrodynamic force on an arbitrarily shaped particle in a uniform time-dependent flow, $\mathbf{U}^\infty(t)$. For a uniform flow $\mathbf{U}^\infty(t)$ about a particle of arbitrary shape with slip velocity $\mathbf{U}_s(t)$, the reciprocal theorem takes the following form:

$$\mathbf{F}^H(t) = \mathbf{F}^H_s(t) + Re\,Sl\,\tilde{V}_p\dot{\mathbf{U}}^\infty(t) - \int_{V_f} \mathbf{f}(\mathbf{u}) \cdot \hat{\mathbf{M}}(\mathbf{r})\, dV, \tag{2.9}$$

where the hydrodynamic force, $\mathbf{F}^H(t)$, has been nondimensionalized by $\mu a U_c$ and the volume of the particle, $\tilde{V}_p$, by a^3. Here, $\mathbf{F}^H_s(t)$ is the pseudo-steady Stokes drag on the particle under the given flow conditions associated with the problem; for example, for a sphere $\mathbf{F}^H_s(t) = -6\pi\mathbf{U}_s(t)$. The function $\mathbf{f}(\mathbf{u})$ is given by the RHS of (2.8a) and satisfies the following series of equalities:

$$\mathbf{f}(\mathbf{u}) = Re\,Sl\frac{\partial \mathbf{u}}{\partial t} + Re\left\{\mathbf{u} \cdot \nabla\mathbf{u} - \mathbf{U}_s(t) \cdot \nabla\mathbf{u}\right\}, \tag{2.10a}$$

$$= \nabla \cdot \boldsymbol{\sigma}, \tag{2.10b}$$

$$= -\nabla p + \nabla^2\mathbf{u}, \tag{2.10c}$$

where $\boldsymbol{\sigma}$ and $\mathbf{u}$ are, respectively, the stress and velocity fields associated with the full disturbance Navier–Stokes problem given by (2.8). The inertia of the fluid is represented by $\mathbf{f}$ in the sense that $\mathbf{f} = 0$ under steady Stokes flow conditions when

$Re \equiv 0$.

The second-order tensor $\hat{\mathbf{M}}(\mathbf{r})$ in (2.9) is defined by the condition that $\hat{\mathbf{M}}(\mathbf{r}) \cdot \mathbf{U}_p$ is the Stokes velocity field for the particle translating with constant velocity $\mathbf{U}_p$ in a stationary fluid. For example, for a spherical particle $\hat{\mathbf{M}}(\mathbf{r})$ is given by

$$\hat{\mathbf{M}}(\mathbf{r}) = \frac{3}{4}\left(\frac{\mathbf{I}}{r} + \frac{\mathbf{rr}}{r^3}\right) + \frac{3}{4}\left(\frac{\mathbf{I}}{3r^3} - \frac{\mathbf{rr}}{r^5}\right), \tag{2.11}$$

where $r = |\mathbf{r}|$ and $\mathbf{I}$ is the idem tensor. We note that the center of the coordinate system is at the particle center and that the volume integral in the reciprocal theorem is over the entire volume of fluid.

As an added note, the reciprocal theorem is an exact result; if $\mathbf{u}$ is known, (2.9) yields the exact hydrodynamic force acting on the particle for *arbitrary* values of the Reynolds number. This is in contrast to a similar analysis presented by Leal [30] for bounded domains which were limited to regular perturbations in small Re. By formulating the disturbance problem, we are able to remove this limitation and treat the general problem.

Our goal now is to approximate the volume integral of (2.9) to give the inertial contribution to the hydrodynamic force correct to $O(Re)$ valid for all times. However, since the volume integral in (2.9) is over the entire fluid domain, the approximation needed must be uniformly valid over this region. It is well known that the solution to Stokes equations do not provide a valid description far from the particle, and singular perturbations will be necessary to construct a uniformly valid approximation.

In order to further motivate the necessity of singular perturbation theory, consider the result of substituting the steady Stokes velocity field for a spherical particle (the solution to (2.8) with $Re \equiv 0$) into the reciprocal theorem through the expression for $\mathbf{f}(\mathbf{u})$ given by (2.10a). This velocity field decays as $1/r$ far from the particle, which is also true of the $\hat{\mathbf{M}}$-field, as can be seen from (2.11). Thus, the integrand associated with the first term of $\mathbf{f}$, $ReSl\partial\mathbf{u}/\partial t \cdot \hat{\mathbf{M}}$, decays as $1/r^2$. (Note that the integrand in this case can be expressed as $ReSl\hat{\mathbf{M}} \cdot \hat{\mathbf{M}} \cdot \dot{\mathbf{U}}_s(t)$.) This term yields a divergent contribution to the volume integral of the reciprocal theorem. The contribution from the third term in $\mathbf{f}$ is conditionally convergent being $1/r^3$.

The next obvious choice to substitute into the reciprocal theorem would be to consider the unsteady Stokes velocity field since it decays as $O(1/r^3)$ and would eliminate the convergence difficulties. As it turns out, however, for the case of a sphere this only yields the unsteady inertial corrections associated with the last two terms of (2.2) and fails to produce convective inertial corrections due to the symmetry of the unsteady Stokes field. Since the unsteady Stokes flow must be linear in $\mathbf{U}_s(t)$, it must be an even function of $\mathbf{r}$, as $\mathbf{r}$ is the only other vector in the problem. Since $\mathbf{f}$ is then odd in $\mathbf{r}$ and $\hat{\mathbf{M}}(\mathbf{r})$ is even, the integrand is odd in $\mathbf{r}$ and the volume integral is zero upon angular integration. Clearly, more detailed considerations of the appropriate "approximate" velocity field are necessary in order to correctly include the convective inertial contributions to $O(Re)$. (Note that although for nonspherical particles convective inertial contributions will be produced by these terms with the unsteady Stokes velocity field, it will not be the complete correction.)

2.4 The uniformly valid velocity field

We see from the governing equation (2.8) that the parameters that determine the form of the flow field are the Reynolds number and the Strouhal number (or, more appropriately, Re and the product $ReSl$). The Reynolds number indicates the magnitude of the convective inertia relative to viscous forces, while $ReSl$ is a measure of the relative magnitude of the unsteady inertia of the fluid. Throughout the analysis that follows we assume that $Re \ll 1$. As we have seen, however, the Strouhal number may take on a range of magnitudes depending on the time scale of variation, and, since we do not restrict the time scale, $ReSl$ may also range in magnitude. In Appendix 2.11, a formal perturbation analysis is used to develop scaling arguments for constructing a velocity field which is uniformly valid in space and time. Here we simply outline the results and present the reasoning used to obtain the uniformly valid field.

The most important point one finds from the scaling analysis is that singular perturbations are not required for particle motions with time scales smaller than ν/U_c^2, i.e., such that $ReSl \gg Re^2$. With these conditions the unsteady Stokes velocity field, which we denote as $\mathbf{v}_0$, is uniformly valid in the entire fluid domain to leading order. There is no Oseen region where the convective inertia is of the same order as unsteady inertia or viscous forces. This feature of the flow is simply a result of the fact that the vorticity generated by such motion cannot diffuse out to the Oseen distance, aRe^{-1}.

For particle motions with longer time scales, such that $ReSl \leq Re^2$, singular perturbations are required in order to take into account the Oseen region. The basic idea in constructing a uniformly valid field with singular perturbation techniques is to add the inner field, valid to the appropriate level of accuracy in the inner region, to the outer field and subtract the parts they have in common in their region of overlap. In the present case, we identify the inner region as that within the Oseen distance, and the outer region as that outside the Oseen distance. The leading order field in the inner region, valid for all time scales, satisfies the unsteady Stokes equations. In the outer region the leading order velocity field is given by the point-forced unsteady Oseen equations (see (2.114)). Only the point-forced equations are required to leading order because the dominate disturbance produced by the particle at large distances is that due to the force monopole. The velocity field that is common to both is that resulting from the point-forced unsteady Stokes equations (see (2.110)), and this field must be subtracted to prevent a double-counting of the contributions from the inner and outer regions. We denote the outer field less its overlap with the inner region as

$Re\mathbf{v}_1^{p+}$.

Although, strictly speaking, the leading order inner field on these long time scales is the *steady* Stokes velocity field, by replacing it with the *unsteady* Stokes field we have actually constructed a field uniformly valid in space *and* time. This is a result of the fact that the unsteady Stokes field on long time scales, $\tau_c \geq \nu/U_c^2$, will reduce to leading order to the steady Stokes field in the inner region and to the point-forced unsteady Stokes field in the outer region, the behavior of which is removed by the common part field discussed above. On the other hand, on short time scales, $\tau_c < \nu/U_c^2$, the "outer" unsteady Oseen field will reduce to the point-forced unsteady Stokes field to leading order, which similarly will be removed by the common part field. In both cases we obtain the desired velocity field to leading order: a steady Stokes inner field plus unsteady Oseen outer field for long time scales, and a uniformly valid unsteady Stokes field for short time scales.

An additional level of accuracy to the unsteady Stokes field is required for particle motions with time scales of a^2/ν or smaller, or $ReSl \geq O(1)$. Under these conditions, the regular perturbation to the unsteady Stokes field for convective inertia (see (2.108)) is also required in general to obtain the proper $O(Re)$ correction to the unsteady term, $ReSl\partial\mathbf{u}/\partial t$, in (2.10a). This field is denoted $Re\mathbf{v}_1$. However, one must be careful in adding this field to prevent double-counting contributions already included by $Re\mathbf{v}_1^{p+}$. Their common part, the point-forced portion of the regular perturbation field denoted by $Re\mathbf{v}_1^p$, must be subtracted. Having taken these requirements into account, the final expression for the uniformly valid velocity field for all time scales is

$$\mathbf{u}^{uv} = \mathbf{v}_0 + Re\mathbf{v}_1^{p+} + Re\mathbf{v}_1 - Re\mathbf{v}_1^p. \tag{2.12}$$

(As a reminder from Appendix 2.11, a subscript "0" indicates a leading order field and a "1" indicates a velocity field contribution due solely to convective inertia, a superscript "p" signifies a point-forced field, and the superscript "+" indicates that convective inertial terms are retained to leading order in the governing equation.) The $\mathbf{v}_0$-field is the unsteady Stokes velocity field and $Re\mathbf{v}_1$ is its regular perturbation for convective inertia. The $Re\mathbf{v}_1^{p+}$-field is the point-forced unsteady Oseen field with the point-forced unsteady Stokes field subtracted off, and this represents a singularly perturbed correction for convective inertia. The governing equation for $Re\mathbf{v}_1$ is given by (2.124). The last term, $Re\mathbf{v}_1^p$, with governing equation given by (2.121), is necessary to prevent a double counting of contributions from $Re\mathbf{v}_1$ and $Re\mathbf{v}_1^{p+}$.

At this point one can derive an additional result from the scaling analysis of Appendix 2.11. Although it is true that when $\tau_c \ll \nu/U_c^2$ there is no region in the fluid domain where convective inertia is of the same magnitude as unsteady inertia, the converse of this condition is not true. That is, when $\tau_c \gg \nu/U_c^2$ there will always be a far-field region where convective and unsteady inertia (as well as viscous forces) are of equal importance. This region is a result of the finite length of the Oseen wake for finite times. One can predict the temporal decay given at the end of Section 2.1 from the scalings for this wake region given by (2.117) when $ReSl \ll Re^2$. That is, if we use the (2.117) scalings in the full integral portion of the reciprocal theorem (2.9) we find

$$\mathbf{f}(\mathbf{u}) \sim O(Sl^3), \tag{2.13a}$$

$$\hat{\mathbf{M}} \sim O(Sl), \tag{2.13b}$$

$$dV = dx\,dy\,dz \sim O((ReSl^2)^{-1}), \tag{2.13c}$$

which when these are combined yield a contribution to the hydrodynamic force that is $O(Sl^2/Re)$.

2.5 Expression for the hydrodynamic force to $O(Re)$

With the above description of an approximation for the velocity field which is uniformly valid in space and time, we now use the reciprocal theorem (2.9) to obtain an expression for the hydrodynamic force, correct to $O(Re)$, which is uniformly valid in time. To evaluate the inertial contributions to the force, we need to approximate the volume integral of (2.9). This calculation is performed by combining the uniformly valid velocity field (2.12) with the first expression for $\mathbf{f}(\mathbf{u})$ given in (2.10a):

$$\mathbf{f}(\mathbf{u}) \sim \mathbf{f}(\mathbf{u}^{uv}) = ReSl\frac{\partial \mathbf{v}_0}{\partial t} + Re\left(ReSl\frac{\partial \mathbf{v}_1}{\partial t} + \mathbf{v}_0 \cdot \boldsymbol{\nabla}\mathbf{v}_0 - \mathbf{U}_s(t)\cdot\boldsymbol{\nabla}\mathbf{v}_0\right)$$
$$+ReSl\left(Re\frac{\partial \mathbf{v}_1^{p+}}{\partial t} - Re\frac{\partial \mathbf{v}_1^{p}}{\partial t}\right) - Re^2\mathbf{U}_s(t)\cdot\boldsymbol{\nabla}\mathbf{v}_1^{p+} + \ldots \quad (2.14)$$

The first four terms on the right-hand side of (5.1) represent contributions from the unsteady Stokes equations and its regular $O(Re)$ perturbation for convection. The additional terms shown only make important contributions in the outer "Oseen" region and only when $ReSl \leq O(Re^2)$. As we shall see in what follows, they are necessary to get the correct force to $O(Re)$. Scaling arguments can be used to show that the terms not shown combine (as in $Re^2\mathbf{U}_s(t)\cdot\boldsymbol{\nabla}(\mathbf{v}_1 - \mathbf{v}_1^p)$) to make contributions to the hydrodynamic force smaller than $O(Re)$ for all time and can therefore be neglected.

When this approximation to $\mathbf{f}(\mathbf{u})$ is combined with the reciprocal theorem (2.9), we obtain

$$\begin{aligned}\mathbf{F}^H(t) =&\mathbf{F}^H_s(t) + Re Sl \tilde{V}_p \dot{\mathbf{U}}^\infty(t) - \int\limits_{V_f} Re Sl \frac{\partial \mathbf{v}_0}{\partial t} \cdot \hat{\mathbf{M}}\, dV \\ &- Re \int\limits_{V_f} \left(Re Sl \frac{\partial \mathbf{v}_1}{\partial t} + \mathbf{v}_0 \cdot \nabla \mathbf{v}_0 - \mathbf{U}_s(t) \cdot \nabla \mathbf{v}_0 \right) \cdot \hat{\mathbf{M}}\, dV \\ &- Re \int\limits_{V_f} \left[Re Sl \left(\frac{\partial \mathbf{v}_1^{p+}}{\partial t} - \frac{\partial \mathbf{v}_1^{p}}{\partial t} \right) - Re \mathbf{U}_s(t) \cdot \nabla \mathbf{v}_1^{p+} \right] \cdot \hat{\mathbf{M}}\, dV + o(Re) \end{aligned} \quad (2.15)$$

The first integral yields precisely the hydrodynamic force due to the unsteady Stokes velocity field, $\mathbf{v}_0$, less the pseudo-steady Stokes drag, $\mathbf{F}^H_s(t)$. This is due to the fact that the reciprocal theorem is an exact result for any inertial velocity field, including the unsteady Stokes field. Indeed, the result that this first integral represents the unsteady inertial contributions to the unsteady Stokes drag was verified by direct calculation using the known solution to the unsteady Stokes equations for a spherical particle. The second integral represents the contribution from the regular perturbation to the unsteady Stokes problem.

The last integral of (2.15) is the contribution from the outer, singularly perturbed, velocity field. The entire point of the scaling analysis in Section 2.4 was to obtain this last term. At large length scales and long times this volume integral is necessary to obtain the correct $O(Re)$ force, which can be seen by noting that the terms in square brackets in (2.15) are $O(Re^2)$ in the outer region when $Sl = Re$ and that $\hat{\mathbf{M}}\, dV$ is $O(Re^{-2})$ in this outer region (distances of $O(Re^{-1})$), so that the volume integral itself is $O(1)$.

Finally, we note that all integrals in (2.15) are absolutely convergent in space for all time. It is important to note that had we used the steady Stokes velocity field, $\mathbf{u}_0$, in place of the unsteady Stokes field, $\mathbf{v}_0$, in the integrals above, which would be a natural thing to do with the reciprocal theorem, the integrals would be conditionally convergent because the term $\mathbf{U}_s(t) \cdot \nabla \mathbf{u}_0 \cdot \hat{\mathbf{M}}$, for example, is $O(1/r^3)$. Using the unsteady Stokes solution removes this difficulty.

In order to greatly ease later calculations, we make a few simplifications of the last integral in (2.15). First, we note that because the contribution to the integral from the inner region (i.e., for distances smaller than $O(Re)$) is of lower order than $O(Re)$, the precise form of $\hat{\mathbf{M}}$ in the inner region is not required. Only the far field form of $\hat{\mathbf{M}}$, which is to leading order given by the Stokeslet field, $\hat{\mathbf{M}}_p$, is required. This field is the solution to

$$-\boldsymbol{\nabla}\mathbf{P} + \nabla^2\hat{\mathbf{M}}_p = -6\pi\boldsymbol{\Phi}\delta(\mathbf{r}), \quad \boldsymbol{\nabla}\cdot\hat{\mathbf{M}}_p = 0. \tag{2.16}$$

Here, $\mathbf{P}$ is a vector and $6\pi\boldsymbol{\Phi}$ is the Stokes resistance tensor associated with the particle: $-6\pi\boldsymbol{\Phi}\cdot\mathbf{U}_p$ is the hydrodynamic force acting on the particle translating with velocity $\mathbf{U}_p$. For a spherical particle, $\hat{\mathbf{M}}_p$ is given by the first term of (2.11).

Next, we note that if we replace $\hat{\mathbf{M}}$ with $\hat{\mathbf{M}}_p$ we may extend the integration volume to include the volume of the particle. This extension may be performed because the $\mathbf{v}_1^{p+}$- and the $\mathbf{v}_1^p$-fields are nonsingular at the origin (i.e., the point force at the origin has been subtracted). Thus, the entire integrand behaves at worst as $O(r^{-2})$ as $r \to 0$, which is integrable. Also, upon integrating over the particle volume we make an error smaller than $O(Re)$, since the integrand is smaller than $O(Re)$ and the volume of integration is $O(1)$. Note that the difference field $\mathbf{v}_1^{p+} - \mathbf{v}_1^p$ is $O(Re)$ in this region.

The final simplification is in neglecting the term involving $Re\mathbf{v}_1^p$. This can be done because the $\mathbf{v}_1^p$ field is antisymmetric (or odd); that is,

$$\mathbf{v}_1^p(-\mathbf{r}) = -\mathbf{v}_1^p(\mathbf{r}), \tag{2.17}$$

as can be seen from the governing equations, (2.110) and (2.121). Thus, since the $\hat{\mathbf{M}}_p$-field is symmetric, that part of integrand involving the $\mathbf{v}_1^p$-field will be strictly antisymmetric and will angularly average to zero when integrated over any spherical surface.

With the above considerations taken into account, the expression to $O(Re)$ for the hydrodynamic force acting on a particle of arbitrary shape translating with a time dependent velocity $\mathbf{U}_s(t)$ relative to a uniform stream, $\mathbf{U}^\infty(t)$, is

$$\begin{aligned}\mathbf{F}^H(t) =& \mathbf{F}^H_{us}(t) + ReSl\widetilde{V}_p\dot{\mathbf{U}}^\infty(t)\\ & -Re\int\limits_{V_f}\left(ReSl\frac{\partial\,\mathbf{v}_1}{\partial\,t} + \mathbf{v}_0\cdot\boldsymbol{\nabla}\mathbf{v}_0 - \mathbf{U}_s(t)\cdot\boldsymbol{\nabla}\mathbf{v}_0\right)\cdot\hat{\mathbf{M}}\,dV\\ & -Re\int\limits_{V_\infty}\left(ReSl\frac{\partial\,\mathbf{v}_1^{p+}}{\partial\,t} - Re\mathbf{U}_s(t)\cdot\boldsymbol{\nabla}\mathbf{v}_1^{p+}\right)\cdot\hat{\mathbf{M}}_p\,dV + o(Re),\end{aligned} \qquad (2.18)$$

where V_∞ represents the entire volume of space. $\mathbf{F}^H_{us}(t)$ is that portion of the hydrodynamic force due solely to the unsteady Stokes problem (*cf* (2.107)); and, with $\mathbf{U}_s(t)$ replacing $\mathbf{U}_p(t)$, it becomes (2.2) for a spherical particle. The second integral is only required if $ReSl \leq O(Re^2)$; that is, this $O(Re)$ contribution comes from integration over the outer region when vorticity has diffused out to an $O(aRe^{-1})$ distance. In the next section we evaluate this outer contribution. The contribution from unsteady Stokes flow and the regular perturbation (the first integral) will be commented on, and simplified for certain cases, in Section 2.7 and Section 2.8.

2.6 Calculation of the unsteady Oseen correction

We write the contribution to the hydrodynamic force from the outer velocity field (the second integral in (2.18)) as

$$\mathbf{F}^H_{out}(t) = -\int\limits_{V_\infty}\mathbf{f}_{out}(Re\mathbf{v}_1^{p+})\cdot\hat{\mathbf{M}}_p\,dV, \qquad (2.19a)$$

where

$$\mathbf{f}_{out}(Re\mathbf{v}_1^{p+}) = Re\left(ReSl\frac{\partial\,\mathbf{v}_1^{p+}}{\partial\,t} - Re\mathbf{U}_s(t)\cdot\boldsymbol{\nabla}\mathbf{v}_1^{p+}\right). \qquad (2.19b)$$

To evaluate this integral we make use of the convolution theorem and Fourier transforms to obtain:

$$\mathbf{F}^H_{out}(t) = -\int_{V_\infty} \widehat{\mathbf{f}_{out}}(Re\mathbf{v}_1^{p+})(\mathbf{k}) \cdot \widehat{\hat{\mathbf{M}}_p}(-\mathbf{k})\, d\,\mathbf{k}, \tag{2.20}$$

where $\widehat{\ }$ indicates the Fourier transform defined by

$$\widehat{g}(\mathbf{k}) = \int_{V_\infty} g(\mathbf{r}) e^{-2\pi \mathrm{i}\mathbf{k}\cdot\mathbf{r}}\, d\,\mathbf{r}. \tag{2.21}$$

The Stokeslet field $\widehat{\hat{\mathbf{M}}_p}$ can be found from the Fourier transform of the governing equations (2.16):

$$\widehat{\hat{\mathbf{M}}_p} = \frac{3}{2\pi k^2}(\mathbf{I} - \mathbf{n}_k\mathbf{n}_k) \cdot \mathbf{\Phi}. \tag{2.22}$$

Here, $\mathbf{n}_k$ is the spherical unit normal in $\mathbf{k}$-space (i.e., $\mathbf{n}_k = \mathbf{k}/k$) and $k = |\mathbf{k}|$. The Fourier transform of $\mathbf{f}_{out}$ is found from the transform of (2.124) with (2.110) (see Appendix 2.12). The result is

$$\begin{aligned}\widehat{\mathbf{f}_{out}}(Re\mathbf{v}_1^{p+}) = &\int_{-\infty}^{t} \frac{2\pi \mathrm{i} Re}{ReSl}\mathbf{k} \cdot \left(\mathbf{U}_s(s) \mathrm{e}^{2\pi \mathrm{i} Re(\mathbf{Y}_s(t)-\mathbf{Y}_s(s))\cdot\mathbf{k}/ReSl} - \mathbf{U}_s(t)\right)\\ &\qquad \times \mathrm{e}^{-4\pi^2 k^2 (t-s)/ReSl} \mathbf{F}^H_s(s) \cdot (\mathbf{I} - \mathbf{n}_k\mathbf{n}_k)\, d\,s\\ &+ \int_{-\infty}^{t} \left[1 - \mathrm{e}^{2\pi i Re(\mathbf{Y}_s(t)-\mathbf{Y}_s(s))\cdot\mathbf{k}/ReSl}\right]\\ &\qquad \times \mathrm{e}^{-4\pi^2 k^2 (t-s)/ReSl} \dot{\mathbf{F}}^H_s(s) \cdot (\mathbf{I} - \mathbf{n}_k\mathbf{n}_k)\, d\,s,\end{aligned} \tag{2.23}$$

where $\mathbf{Y}_s(t) - \mathbf{Y}_s(s)$ is the integrated displacement of the particle relative to the fluid from time s to the current time t, and $\mathbf{F}^H_s(s)$ is the pseudo-steady Stokes drag at time s. Also, it should be noted that in this equation, and in the equations that follow, $\mathbf{Y}_s(t)$ has been nondimensionalized by aSl^{-1} ($= \tau_c U_c$), not a. Since only the symmetric part (which is also the real part) of (6.5) can contribute to the volume integral of (2.19a), we write

$$\widehat{\mathbf{f}_{out}}(Re\mathbf{v}_1^{p+})_{sym} = \int_{-\infty}^{t} \frac{-2\pi Re}{ReSl}\mathbf{k}\cdot\mathbf{U}_s(s)\sin\Big(2\pi Re(\mathbf{Y}_s(t)-\mathbf{Y}_s(s))\cdot\mathbf{k}/ReSl\Big)$$

$$\times e^{-4\pi^2k^2(t-s)/ReSl}\mathbf{F}_s^H(s)\cdot(\mathbf{I}-\mathbf{n}_k\mathbf{n}_k)\,d\,s$$

$$+\int_{-\infty}^{t}\Big[1-\cos\Big(2\pi Re(\mathbf{Y}_s(t)-\mathbf{Y}_s(s))\cdot\mathbf{k}/ReSl\Big)\Big]$$

$$\times e^{-4\pi^2k^2(t-s)/ReSl}\dot{\mathbf{F}}_s^H(s)\cdot(\mathbf{I}-\mathbf{n}_k\mathbf{n}_k)\,d\,s. \tag{2.24}$$

If we combine (2.22) and (2.24) in (2.20) and perform the k-integration, the result is

$$\mathbf{F}_{out}^H(t) = \frac{3Re^2}{16\pi^{\frac{3}{2}}(ReSl)^{\frac{1}{2}}}\int_{-\infty}^{t}\frac{\mathbf{F}_s^H(s)}{(t-s)^{\frac{1}{2}}}\mathbf{U}_s(s)\left(\frac{\mathbf{Y}_s(t)-\mathbf{Y}_s(s)}{t-s}\right)\vdots\mathbf{G}_1(t,s)\,d\,s\cdot\mathbf{\Phi}$$
$$-\frac{3(ReSl)^{\frac{1}{2}}}{8\pi^{\frac{3}{2}}}\int_{-\infty}^{t}\frac{\dot{\mathbf{F}}_s^H(s)}{(t-s)^{\frac{1}{2}}}\cdot\mathbf{G}_2(t,s)\,d\,s\cdot\mathbf{\Phi}, \tag{2.25a}$$

where

$$\mathbf{G}_1(t,s) = \int_{\Omega}\mathbf{n}_k\mathbf{n}_k(\mathbf{I}-\mathbf{n}_k\mathbf{n}_k)e^{-(\mathbf{A}(t,s)\cdot\mathbf{n}_k)^2}\,d\,\Omega, \tag{2.25b}$$

$$\mathbf{G}_2(t,s) = \int_{\Omega}(\mathbf{I}-\mathbf{n}_k\mathbf{n}_k)(1-e^{-(\mathbf{A}(t,s)\cdot\mathbf{n}_k)^2})\,d\,\Omega, \tag{2.25c}$$

and

$$\mathbf{A}(t,s) = \frac{Re}{2}\left(\frac{t-s}{ReSl}\right)^{\frac{1}{2}}\left(\frac{\mathbf{Y}_s(t)-\mathbf{Y}_s(s)}{t-s}\right); \tag{2.25d}$$

here Ω represents the angular integration over a spherical surface.

Now if we consider the situation with $\dot{\mathbf{U}}_s(t) = 0$ we can obtain the steady-state result for $\mathbf{F}_{out}^H$:

$$\mathbf{F}_{out}^H = -6\pi\left(\tfrac{3}{8}Re[\tfrac{3}{2}(\mathbf{\Phi}\cdot\mathbf{p})\cdot\mathbf{\Phi}-\tfrac{1}{2}(\mathbf{\Phi}\cdot\mathbf{p})\cdot\mathbf{pp}\cdot\mathbf{\Phi}]\right)|\mathbf{U}_s|^2, \tag{2.26a}$$

where

$$\mathbf{p} = \frac{\mathbf{U}_s}{|\mathbf{U}_s|}. \tag{2.26b}$$

Here we have used the fact that

$$\mathbf{F}_s^H = -6\pi\mathbf{\Phi}\cdot\mathbf{U}_s. \tag{2.27}$$

This result agrees with the Oseen correction to Stokes drag obtained by Brenner [7] for an arbitrary particle shape.

In what follows we present some simplified expressions for the *unsteady* Oseen correction to the Stokes drag. The unsteady Oseen correction will be defined by

$$\mathbf{F}_{Osc}^H(t) = \mathbf{F}_{out}^H(t) + \left(\frac{Re\,Sl}{\pi}\right)^{\frac{1}{2}}\int_{-\infty}^{t}\frac{\dot{\mathbf{F}}_s^H(s)}{(t-s)^{\frac{1}{2}}}\,d\,s\cdot\mathbf{\Phi}. \tag{2.28}$$

The first term on the right-hand side is given by (2.25). The second term represents the long-time expression of the "Basset" history force for an arbitrarily shaped particle and, as will be shown in Section 2.8, is the dominant long-time temporal contribution from unsteady Stokes flow. This definition is chosen because we wish to show the long-time asymptotic behavior for various particle slip velocities. In all cases, we find that the long-time tail decays faster than the $t^{-\frac{1}{2}}$ predicted by unsteady Stokes flow.

First note that the second integral in the expression for $\mathbf{F}_{out}^H(t)$, (2.25a), may be written as

$$\left[\frac{3(Re\,Sl)^{\frac{1}{2}}}{8\pi^{\frac{3}{2}}}\int_{-\infty}^{t}\frac{\dot{\mathbf{F}}_s^H(s)}{(t-s)^{\frac{1}{2}}}\cdot\int_{\Omega}(\mathbf{I}-\mathbf{n}_k\mathbf{n}_k)\mathrm{e}^{-(\mathbf{A}\cdot\mathbf{n}_k)^2}\,d\,\Omega\,d\,s - \left(\frac{Re\,Sl}{\pi}\right)^{\frac{1}{2}}\int_{-\infty}^{t}\frac{\dot{\mathbf{F}}_s^H(s)}{(t-s)^{\frac{1}{2}}}\,d\,s\right]\cdot\mathbf{\Phi}, \tag{2.29}$$

where it is seen that the second term cancels identically with the long-time "Basset" history force term in (2.28). Next, performing integration by parts in s on the first integral in (2.29) and combining with the first integral of (2.25a), we obtain the following result for $\mathbf{F}^H_{Osc}(t)$:

$$\mathbf{F}^H_{Osc}(t) = \frac{3}{8}\frac{(ReSl)}{\pi^{\frac{3}{2}}}\left\{\int_{-\infty}^{t}\int_{\Omega}\left[\left\{\mathbf{F}^H_s(s)(\mathbf{A}\cdot\mathbf{n}_k)^2 e^{-(\mathbf{A}\cdot\mathbf{n}_k)^2} + \tfrac{1}{2}\left(\mathbf{F}^H_s(t) - \mathbf{F}^H_s(s)e^{-(\mathbf{A}\cdot\mathbf{n}_k)^2}\right)\right\}\cdot\frac{(\mathbf{I}-\mathbf{n}_k\mathbf{n}_k)}{(t-s)^{\frac{3}{2}}}\right] ds\, d\Omega\right\}\cdot\mathbf{\Phi}. \quad (2.30)$$

Further simplification of the angular integration can be achieved by dividing $\mathbf{F}^H_s(s)$ into portions parallel and perpendicular to $\mathbf{A}$, which itself is parallel to the displacement vector $\mathbf{Y}_s(t) - \mathbf{Y}_s(s)$. If we define the direction of the displacement vector by the unit vector $\mathbf{p}(s)$, we can define the following:

$$\mathbf{F}^{H\|}_s(s) = \mathbf{F}^H_s(s)\cdot\mathbf{p}(s)\mathbf{p}(s), \quad (2.31a)$$

$$\mathbf{F}^{H\perp}_s(s) = \mathbf{F}^H_s(s)\cdot\big(\mathbf{I} - \mathbf{p}(s)\mathbf{p}(s)\big), \quad (2.31b)$$

where

$$\mathbf{p}(s) = \frac{\mathbf{Y}_s(t) - \mathbf{Y}_s(s)}{|\mathbf{Y}_s(t) - \mathbf{Y}_s(s)|}. \quad (2.31c)$$

Note that $\mathbf{F}^H_s(s) = \mathbf{F}^{H\|}_s(s) + \mathbf{F}^{H\perp}_s(s)$. By always letting $\mathbf{p}(s)$ point in the z-direction and $\mathbf{F}^{H\perp}_s(s)$ in the x-direction in Ω-space, the angular integration can be reduced to obtain

$$\mathbf{F}^H_{Osc}(t) = \frac{3}{8}\left(\frac{Re\,Sl}{\pi}\right)^{\frac{1}{2}} \Bigg\{ \int_{-\infty}^{t}\int_{0}^{1}\Big[2(1-x^2)\Big\{2\mathbf{F}^{H\|}_s(s)(|\mathbf{A}|x)^2 \mathrm{e}^{-(|\mathbf{A}|x)^2} + \Big(\mathbf{F}^{H\|}_s(t) - \mathbf{F}^{H\|}_s(s)e^{-(|\mathbf{A}|x)^2}\Big)\Big\}$$
$$+(1+x^2)\Big\{2\mathbf{F}^{H\perp}_s(s)(|\mathbf{A}|x)^2\mathrm{e}^{-(|\mathbf{A}|x)^2} + \Big(\mathbf{F}^{H\perp}_s(t) - \mathbf{F}^{H\perp}_s(s)\mathrm{e}^{-(|\mathbf{A}|x)^2}\Big)\Big\}\Big]$$
$$\times\frac{d\,x\,d\,s}{(t-s)^{\frac{3}{2}}}\Bigg\}\cdot\mathbf{\Phi}. \tag{2.32}$$

Here, x represents the Cosine of the angle between $\mathbf{p}(s)$ and $\mathbf{n}_k$. Finally, performing integration by parts in x, we find the following expression for $\mathbf{F}^H_{Osc}(t)$:

$$\mathbf{F}^H_{Osc}(t) = \frac{3}{8}\left(\frac{Re\,Sl}{\pi}\right)^{\frac{1}{2}} \Bigg\{ \int_{-\infty}^{t}\Big[\ \tfrac{2}{3}\mathbf{F}^{H\|}_s(t) - \Big\{\frac{1}{|\mathbf{A}|^2}\Big(\frac{\pi^{\frac{1}{2}}}{2|\mathbf{A}|}\mathrm{erf}(|\mathbf{A}|) - \exp(-|\mathbf{A}|^2)\Big)\Big\}\mathbf{F}^{H\|}_s(s)$$
$$+\tfrac{2}{3}\mathbf{F}^{H\perp}_s(t) - \Big\{\exp(-|\mathbf{A}|^2) - \frac{1}{2|\mathbf{A}|^2}\Big(\frac{\pi^{\frac{1}{2}}}{2|\mathbf{A}|}\mathrm{erf}(|\mathbf{A}|) - \exp(-|\mathbf{A}|^2)\Big)\Big\}\mathbf{F}^{H\perp}_s(s)\Big]$$
$$\frac{2\,d\,s}{(t-s)^{\frac{3}{2}}}\Bigg\}\cdot\mathbf{\Phi}. \tag{2.33}$$

Now in order to investigate the temporal response of this correction, we evaluate (2.33) for various rectilinear slip velocities: $\mathbf{U}_s(t) = U_s(t)\mathbf{p}$. In all cases the time scale, τ_c, is chosen as $4\nu/U_c^2$ so that $Re\,Sl = Re^2/4$. This choice is simply for convenience since we are interested in long-time asymptotic behavior. Also we let $\mathbf{F}^{H\|}_s(t) = \mathbf{F}^{H\|}_s U_s(t)$ and $\mathbf{F}^{H\perp}_s(t) = \mathbf{F}^{H\perp}_s U_s(t)$ where $\mathbf{F}^{H\|}_s = -6\pi(\mathbf{\Phi}\cdot\mathbf{p})\cdot(\mathbf{pp})$ and $\mathbf{F}^{H\perp}_s = -6\pi(\mathbf{\Phi}\cdot\mathbf{p})\cdot(\mathbf{I}-\mathbf{pp})$.

First we consider the slip velocity given by a step change at $t=0$, $U_s(t) = H(t)$. After some manipulation and with the aid of *Mathematica*, we arrive at the following expression:

$$\mathbf{F}^H_{Osc}(t) = \frac{3}{8}Re\boldsymbol{\Phi}\cdot\left[\mathbf{F}^{H\|}_s\left\{\left(1+\frac{1}{4t^2}\right)\mathrm{erf}(t^{\frac{1}{2}}) + \frac{1}{(\pi t)^{\frac{1}{2}}}\left(1-\frac{1}{2t}\right)\exp(-t)\right\}\right.$$
$$\left. +\mathbf{F}^{H\perp}_s\left\{\left(\frac{3}{2}-\frac{1}{8t^2}\right)\mathrm{erf}(t^{\frac{1}{2}}) + \frac{3}{2(\pi t)^{\frac{1}{2}}}\left(1+\frac{1}{6t}\right)\exp(-t)\right\}\right]. \tag{2.34}$$

This result is consistent with that obtained by Sano, (2.5), for spherical particles. Recall that $\boldsymbol{\Phi} = \mathbf{I}$ for spheres. In addition, (2.34) demonstrates the existence of the t^{-2} decay for arbitrarily shaped particles which confirms the scaling analysis in Section 2.3. Note that the above result does not include an "added mass" contribution which would appear simply as a delta function at $t = 0$.

The expression given by (2.34) leads to a long-time asymptotic form for $\mathbf{F}^H_{Osc}(t)$ which is useful when a particle approaches its steady-state velocity monotonically from rest. Consider such a slip velocity, $U_s(t)$, which is zero for $t < 0$ and goes to 1 as $t \to \infty$. Provided that for $O(1)$ values of t on the time scale $4\nu/U_c^2$ we have $|1 - (Y_s(t) - Y_s(0))/U_s(t)t| \ll 1$, the velocity profile will appear as a step change on this time scale. Under these conditions, the long-time asymptotic form can be predicted from (2.34) by replacing U_c with $U_cU_s(t)$ in all terms implicitly scaled by U_c:

$$\mathbf{F}^H_{Osc}(t) = \frac{3}{8}cRe\boldsymbol{\Phi}\cdot\left[\mathbf{F}^{H\|}_s\left\{\left(1+\frac{1}{4c^2t^2}\right)\mathrm{erf}((ct)^{\frac{1}{2}}) + \frac{1}{(\pi ct)^{\frac{1}{2}}}\left(1-\frac{1}{2ct}\right)\exp(-ct)\right\}\right.$$
$$\left. +\mathbf{F}^{H\perp}_s\left\{\left(\frac{3}{2}-\frac{1}{8c^2t^2}\right)\mathrm{erf}((ct)^{\frac{1}{2}}) + \frac{3}{2(\pi ct)^{\frac{1}{2}}}\left(1+\frac{1}{6ct}\right)\exp(-ct)\right\}\right], \tag{2.35}$$

where $c(t) = U_s^2(t)$. Strictly speaking, this form corresponds to the result for a step change in the velocity from zero to $U_s(t)$.

In order to test the validity of (2.35), for various $U_s(t)$ profiles we investigate the coefficient of the $Re\mathbf{\Phi}\cdot\mathbf{F}_s^{H\|}$-term extracted from the complete expression (2.33), defined here as $f(t)$:

$$f(t) \equiv \frac{3}{8}\frac{1}{\pi^{\frac{1}{2}}}\left\{\int_{-\infty}^{t}\left[\ \tfrac{2}{3}U_s(t) - \left\{\frac{1}{|\mathbf{A}|^2}\left(\frac{\pi^{\frac{1}{2}}}{2|\mathbf{A}|}\mathrm{erf}(|\mathbf{A}|) - \exp(-|\mathbf{A}|^2)\right)\right\}U_s(s)\right] \times\frac{d\,s}{(t-s)^{\frac{3}{2}}}\right\}. \qquad (2.36)$$

Equation (2.35) indicates that by plotting $f(t)/c(t)$ versus $c(t)t$, the results should fall on the same curve. Since we wish to justify applying the asymptotic form to particles settling under gravity, we test it with trajectories that possess the long-time asymptotic form predicted by unsteady Stokes flow for a particle released from rest and settling under gravity:

$$\begin{aligned} U_s(t) &= 0, && t<0,\\ &= 1-\frac{1}{\left(1+\frac{\pi t}{Re\,Sl}\right)^{\frac{1}{2}}}, && t>0,\\ &= 1-\frac{1}{\left(1+\frac{4\pi t}{Re^2}\right)^{\frac{1}{2}}}, && t>0. \end{aligned} \qquad (2.37)$$

If we were to choose a trajectory with a t^{-2} decay, it would not be surprising to see a force response as in (2.35) with a t^{-2} decay. However, if we use a trajectory with a much slower decay, $t^{-\frac{1}{2}}$, as in (2.37), and the form (2.35) still holds, this will be a much more severe test since it is a greater deviation from a step change velocity profile. The results for various values of Re are given in figures 2.1 and 2.2, which have unscaled and scaled axes, respectively, by performing the integration in (2.36) numerically. The curve for $Re = 0$ corresponds to the step response given by (2.34). Figure 2.2 shows that the scaled variables collapse the results onto the $Re = 0$ curve quite well. The fact that the curves are uniformly higher for increasing Re reflects the increased

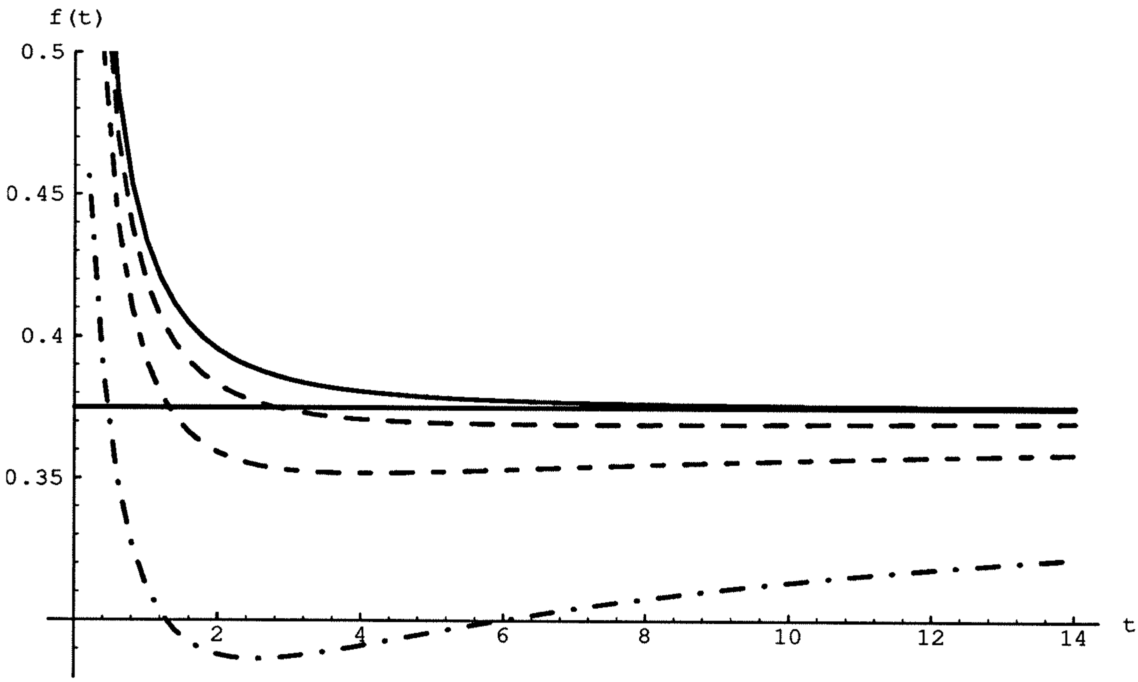

Figure 2.1: Temporal response of the unsteady Oseen correction to the hydrodynamic force, $f(t)$, due to a particle accelerating from rest as $O(t^{-\frac{1}{2}})$. The particle trajectory is given by (2.37) and $f(t)$ is defined by (2.36). Time is scaled by $4\nu/U_c^2$. The curve for Re=0 is the step increase response from (2.34). The Reynolds number, which is a parameter in (2.37), is: —— Re=0; - - Re=0.1; - - - Re=0.3; - · - Re=1.

deviation of the velocity profiles from that of a step change. Additional refinement can be achieved if, instead of $U_s^2(t)$, time was scaled with $(Y_s(t) - Y_s(0))^2/t^2$, which is indicated by the presence of $\mathbf{A}$ in the exact result, (2.33).

Next, in order to test the generality of the t^{-2} decay, we consider the temporal response of $f(t)$ to the following velocity profile:

$$\begin{aligned} U_s(t) &= b, \qquad t < 0, \\ &= 1, \qquad t > 0, \end{aligned} \tag{2.38}$$

where $0 < b < 1$. The results for various values of b are shown in figure 2.3. The exact solution for the case when $b = 0$ is given by (2.34). The long-time asymptotic

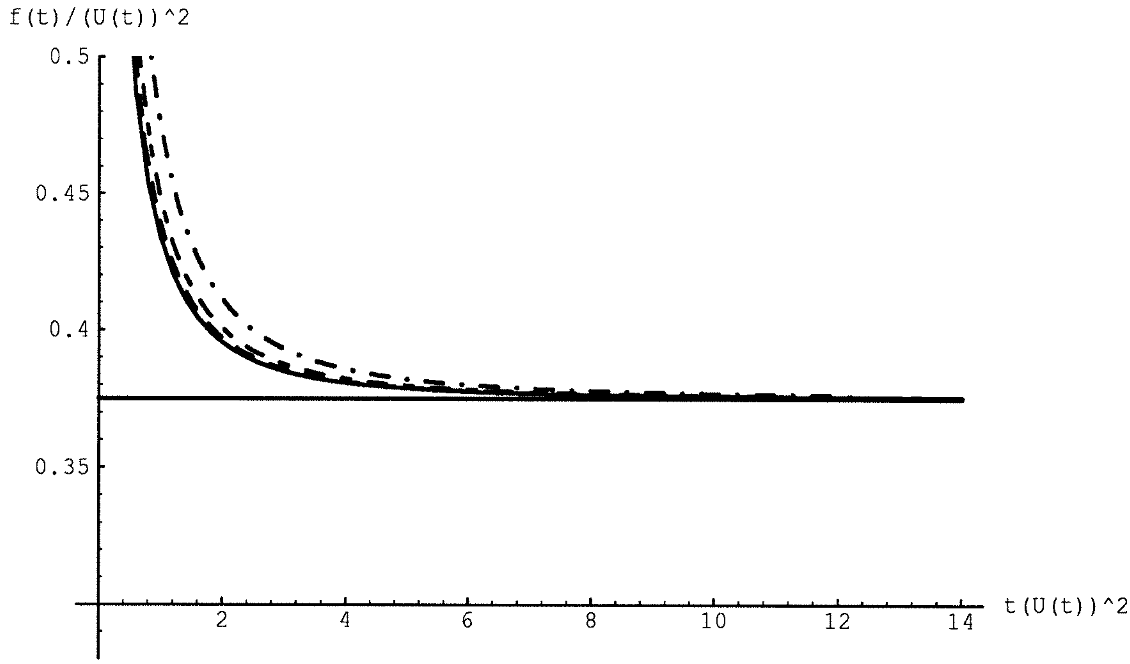

Figure 2.2: Temporal response of the scaled unsteady Oseen correction to the hydrodynamic force, $f(t)/U_s^2(t)$, due to a particle accelerating from rest as $O(t^{-\frac{1}{2}})$. The particle trajectory is given by (2.37) and $f(t)$ is defined by (2.36). Time is scaled by $4\nu/U_c^2$. The curve for Re=0 is the step increase response from (2.34). The Reynolds number, which is a parameter in (2.37), is: —— Re=0; – – Re=0.1; – – – Re=0.3; – · – Re=1.

forms from (2.36) were found analytically to be

$$f(t) = \frac{3}{8}\left[1 + \left(\frac{1}{4} + O(t^{-\frac{1}{2}}e^{-t})\right)\frac{1}{t^2}\right], \qquad b = 0, \quad t \gg 1, \qquad (2.39a)$$

$$= \frac{3}{8}\left[1 + \left(\frac{1}{4(1-b)^2} + O(t^{-1})\right)\frac{\mathrm{e}^{-4b(1-b)t}}{t^2}\right], \qquad 0 < b < \tfrac{1}{2}, \quad t \gg \frac{(1-b)}{b(1-2b)^2}, \qquad (2.39b)$$

$$= \frac{3}{8}\left[1 + \left(\frac{1}{2} + O(t^{-\frac{1}{2}})\right)\frac{\mathrm{e}^{-t}}{t^2}\right], \qquad b = \tfrac{1}{2}, \quad t \gg 1, \qquad (2.39c)$$

$$= \frac{3}{8}\left[1 + \left(\frac{1-b}{2b-1} + O(t^{-1})\right)\frac{\mathrm{e}^{-t}}{\pi^{\frac{1}{2}}t^{\frac{5}{2}}}\right], \qquad \tfrac{1}{2} < b < 1, \quad t \gg \frac{1}{(1-2b)^2}. \qquad (2.39d)$$

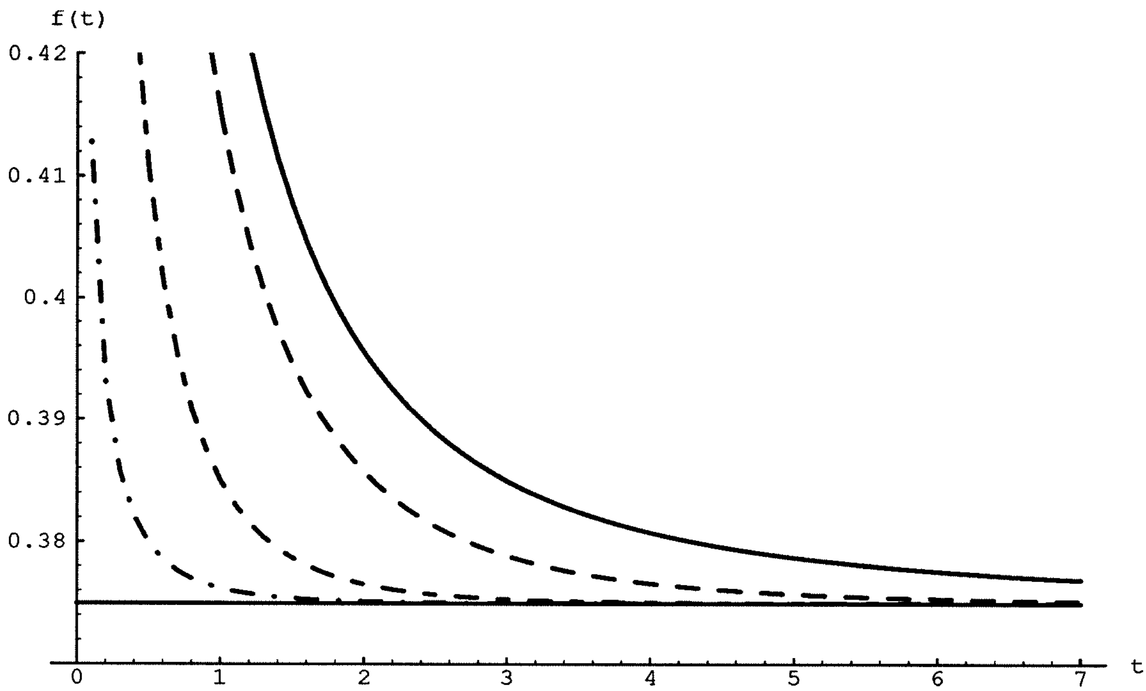

Figure 2.3: Temporal response of the unsteady Oseen correction to the hydrodynamic force, $f(t)$, due to a step change increase in the particle velocity. The particle trajectory is given by (2.38) and $f(t)$ is defined by (2.36). Time is scaled by $4\nu/U_c^2$. The ratio, b, of the particle velocity before the step change to that after is: —— b=0; – – b=0.1; - - - b=0.5; – · – b=0.9.

We see that for a step change from a non-zero constant velocity, the decay to steady-state is ultimately exponential. Note that for $b \ll 1$, $f(t)$ behaves as (2.39a) provided that $1 \ll t \ll 1/(4b)$. Also, when $(2b-1) \ll 1$, $f(t)$ goes as (2.39c) if $1 \ll t \ll (2b-1)^{-2}$.

Finally, we consider the velocity profile for a step change down:

$$
\begin{aligned}
U_s(t) &= 1, \qquad t < 0, \\
&= b, \qquad t > 0. \qquad (2.40)
\end{aligned}
$$

The results for different values of b are shown in figure 2.4. The long-time asymptotic

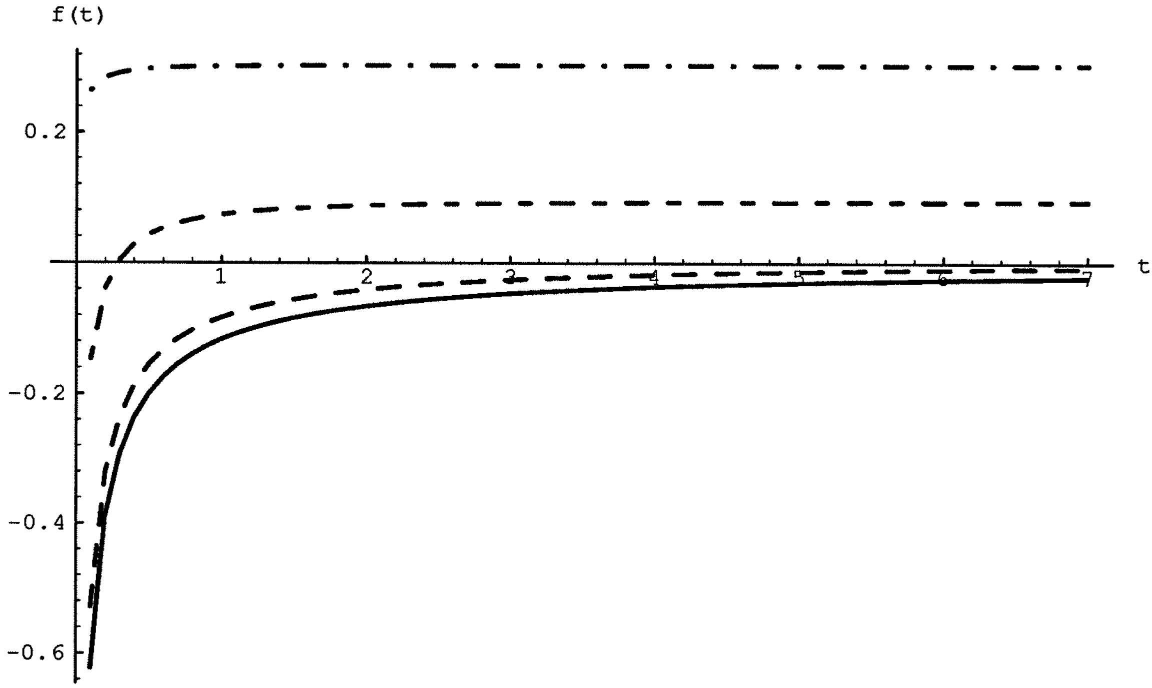

Figure 2.4: Temporal response of the unsteady Oseen correction to the hydrodynamic force, $f(t)$, due to a step change decrease in the particle velocity. The particle trajectory is given by (2.40) and $f(t)$ is defined by (2.36). Time is scaled by $4\nu/U_c^2$. The ratio, b, of the particle velocity after the step change to that before is: —— b=0; – – b=0.1; – - – b=0.5; – · – b=0.9.

forms from (2.36) were found analytically as

$$f(t) = -\frac{3}{8}\left[\frac{1}{2t} - \frac{1}{2\pi^{\frac{1}{2}}t^{\frac{3}{2}}} + \frac{1}{8t^2} - \frac{1}{16\pi^{\frac{1}{2}}t^{\frac{5}{2}}} + O(t^{-3})\right], \qquad b = 0, \quad t \gg 1, \quad (2.41a)$$

$$= \frac{3}{8}b^2\left[1 - \left(\frac{1-b}{2-b} + O((b^2t)^{-1})\right)\frac{e^{-b^2t}}{\pi^{\frac{1}{2}}(b^2t)^{\frac{5}{2}}}\right], \qquad 0 < b < 1, \quad t \gg \frac{1}{b^2} \quad (2.41b)$$

Equation (2.41a) also holds for $b \ll 1$ when $1 \ll t \ll b^{-2}$.

These differences in the temporal decay to steady-state reflects the contrast between the creation or destruction of the wake structure, associated with the algebraic decay, and simply modifying the wake structure already established, which is associated with exponential decay. Step changes from or to zero velocity require the creation or complete destruction of the wake, and this evidently requires an algebraically long-time. Step changes from and to finite velocities maintain the wake structure throughout the process, and the efficient convective transport of vorticity yields an exponential decay or response. These fundamentally different temporal responses – algebraic versus exponential – are quite intriguing and may have important implications for oscillatory flows or the response of particles to time-dependent fluctuating velocities caused, for example, by turbulent flows.

2.7 Expression of the hydrodynamic force acting on a spherical particle

The remaining contribution to the hydrodynamic force is given by

$$\begin{aligned}\mathbf{F}^H_{in}(t) =& \mathbf{F}^H_{us}(t) + Re Sl \tilde{V}_p \dot{\mathbf{U}}^\infty(t) \\ & - Re \int\limits_{V_f} \left(Re Sl \frac{\partial \mathbf{v}_1}{\partial t} + \mathbf{v}_0 \cdot \nabla \mathbf{v}_0 - \mathbf{U}_s(t) \cdot \nabla \mathbf{v}_0 \right) \cdot \hat{\mathbf{M}} \, dV. \end{aligned} \qquad (2.42)$$

For a spherical particle, the evaluation of (2.42) is simple. The entire contribution from the regular perturbation is identically zero. This simplification follows as the $\mathbf{v}_0$-field is strictly symmetric, and the $\mathbf{v}_1$-field is strictly anti-symmetric, as one can observe from their governing equations, (2.107) and (2.108). Hence, the entire integrand of the integral in (2.42) antisymmetric. (One can note that the $\hat{\mathbf{M}}$-field is symmetric from (2.11).) Then, since the volume of integration is spherically symmetric, angular integration will yield zero. For a sphere, the only contribution from (2.42) is the unsteady Stokes force, which is given by (2.2) with $\mathbf{U}_s(t)$ replacing $\mathbf{U}_p(t)$, and the buoyancy force $Re Sl \tilde{V}_p \dot{\mathbf{U}}^\infty(t)$ with $\tilde{V}_p = \frac{4\pi}{3}$. Thus, we have

$$\mathbf{F}^H_{in}(t) = -6\pi \left\{ \mathbf{U}_s(t) + \left(\frac{Re Sl}{\pi} \right)^{\frac{1}{2}} \int_{-\infty}^{t} \frac{\dot{\mathbf{U}}_s(s)}{(t-s)^{\frac{1}{2}}} \, ds \right\} - \frac{2\pi}{3} Re Sl \dot{\mathbf{U}}_s(t) + \frac{4\pi}{3} Re Sl \dot{\mathbf{U}}^\infty(t). \qquad (2.43)$$

Combining $\mathbf{F}^H_{out}(t)$ from (2.28) and (2.33) with (2.43), the expression for the total hydrodynamic force acting on a sphere is

$$\begin{aligned}\mathbf{F}^H(t) = {} & \frac{4\pi}{3} Re\,Sl\,\dot{\mathbf{U}}^\infty(t) - 6\pi\mathbf{U}_s(t) - \frac{2\pi}{3} Re\,Sl\,\dot{\mathbf{U}}_s(t) \\ + {} & \frac{3}{8}\left(\frac{Re\,Sl}{\pi}\right)^{\frac{1}{2}} \left\{ \int_{-\infty}^{t} \left[\tfrac{2}{3}\mathbf{F}^{H\parallel}_s(t) - \left\{ \frac{1}{|\mathbf{A}|^2}\left(\frac{\pi^{\frac{1}{2}}}{2|\mathbf{A}|}\mathrm{erf}(|\mathbf{A}|) - \exp(-|\mathbf{A}|^2)\right)\right\}\mathbf{F}^{H\parallel}_s(s) \right.\right. \\ & \left. + \tfrac{2}{3}\mathbf{F}^{H\perp}_s(t) - \left\{ \exp(-|\mathbf{A}|^2) - \frac{1}{2|\mathbf{A}|^2}\left(\frac{\pi^{\frac{1}{2}}}{2|\mathbf{A}|}\mathrm{erf}(|\mathbf{A}|) - \exp(-|\mathbf{A}|^2)\right)\right\}\mathbf{F}^{H\perp}_s(s) \right] \\ & \left. \frac{2\,ds}{(t-s)^{\frac{3}{2}}} \right\} + o(Re), \end{aligned} \tag{2.44}$$

where we have used $\mathbf{F}^H_s(t) = -6\pi\mathbf{U}_s(t)$. Thus, since the last term of (2.44) is $\mathbf{F}^H_{Osc}(t)$, the asymptotic temporal behaviors described for the step changes in the previous section were actually for the entire hydrodynamic force.

Now we apply (2.44) to the problem of a sphere released from rest and settling under gravity. The equation of motion for a particle immersed in a fluid is given in dimensional form by

$$m_p\dot{\mathbf{U}}_p(t) = \mathbf{F}^{ext}(t) + \mathbf{F}^H(t), \tag{2.45}$$

where $\mathbf{F}^{ext}(t)$ is the external force acting on the particle. For the problem of a particle settling under gravity, the external force is the buoyancy force

$$\mathbf{F}^{ext} = (m_p - m_f)g(-\mathbf{i}_z), \tag{2.46}$$

where m_p is the mass of the particle and m_f is the mass of the fluid displaced by the particle. The hydrodynamic force acting on the sphere to $O(Re)$ is given by (2.44) with $\mathbf{U}^\infty(t) = 0$, $\mathbf{U}_s(t) = \mathbf{U}_p(t)$, $\mathbf{Y}_s(t) = \mathbf{Y}_p(t)$, and $\mathbf{F}^H_s(t) = -6\pi\mathbf{U}_p(t)$.[a] Thus,

[a]In actuality we could equally well replace $\mathbf{F}^H_s(t)$ with the full force monopole, $\mathbf{F}_1(t)$, given by (2.113). In this case, $\mathbf{F}_1(t) = -\mathbf{F}^{ext} + (m_p/m_f - 1)\widetilde{V}_p Re\,Sl\dot{U}_p(t) \approx -\mathbf{F}^{ext}$ for $\tau_c = \nu/U_c^2$. However, we then would need to include the Basset term and the quantity $(Re\,Sl/\pi t)^{\frac{1}{2}}\mathbf{F}^{ext}$ as well in (2.44). Regardless, the difference in choice will alter the result by an amount smaller than $O(Re)$.

letting $\mathbf{U}_p(t) = U_p(t)(-\mathbf{i}_z)$, $\mathbf{Y}_p(t) = Y_p(t)(-\mathbf{i}_z)$, and $\mathbf{F}^H(t) = F^H(t)(-\mathbf{i}_z)$, we have for the sphere released from rest at $t = 0$

$$\frac{F^H(t)}{\mu a U_c} = -6\pi U_p(t) - \tfrac{2}{3}\pi ReSl\dot{U}_p(t)$$
$$-\frac{9}{4}(\pi ReSl)^{\frac{1}{2}}\left\{\int_0^t\left[\ \tfrac{2}{3}U_p(t) - \left\{\frac{1}{A^2}\left(\frac{\pi^{\frac{1}{2}}}{2A}\mathrm{erf}(A) - \exp(-A^2)\right)\right\}U_p(s)\right]\right.$$
$$\left.\frac{2\,ds}{(t-s)^{\frac{3}{2}}} + \frac{8U_p(t)}{3t^{\frac{1}{2}}}\right\}, \tag{2.47a}$$

where

$$A^2 = \frac{Re^2}{4}\frac{t-s}{ReSl}\left(\frac{Y_p(t) - Y_p(s)}{t-s}\right)^2. \tag{2.47b}$$

Combining (2.46) and (2.47) in (2.45) and rearranging, we obtain for the equation of motion of the sphere

$$1 = U_p(t) + \frac{1}{9}\left(1 + 2\frac{\rho_p}{\rho}\right)\dot{U}_p(t)$$
$$+\frac{3}{8\pi^{\frac{1}{2}}}\left\{\int_0^t\left[\ \tfrac{2}{3}U_p(t) - \left\{\frac{1}{A^2}\left(\frac{\pi^{\frac{1}{2}}}{2A}\mathrm{erf}(A) - \exp(-A^2)\right)\right\}U_p(s)\right]\right.$$
$$\left.\frac{2\,ds}{(t-s)^{\frac{3}{2}}} + \frac{8U_p(t)}{3t^{\frac{1}{2}}}\right\}, \tag{2.48}$$

where ρ_p is the density of the particle. Here, we have taken the diffusive time scale, a^2/ν, so that $ReSl = 1$, and the Stokes terminal velocity as the velocity scale:

$$U_c = U_{Stokes} = \frac{(m_p - m_f)g}{6\pi\mu a}. \tag{2.49}$$

We note that $Re = aU_c/\nu = 2(\rho_p - \rho)\rho g a^3/9\mu^2$. Also, with $U_p(0) = 0$, we have

$$\dot{U}_p(0) = \frac{9}{\left(1 + 2\rho_p/\rho\right)}. \tag{2.50}$$

With a simple time stepping, finite difference routine $U_p(t)$ was solved numerically to produce Figure 2.5, with $Re = 0.3$ and $\rho_p/\rho = 1.1$. This value of Re was chosen based on experimental results which showed the steady Oseen formula for the drag to be a good approximation for Reynolds numbers below 0.4 [34]. The curve for pure unsteady Stokes flow (i.e., for $Re = 0$) is included for comparison. By definition, the terminal velocity associated with this curve is 1.0. Numerical difficulties occurred for the $Re = 0.3$ curve after $t \sim 50$ which are believed to be due to the nonlinearity of the governing equation (2.48). The remainder of the curve was obtained using the long-time asymptotic formula (2.35) for the last term of (2.48), the unsteady Oseen correction. Using (2.35) for $t \gg Re^{-2}$ (note that here t must be replaced by $Re^2 t/4$ in (2.35)), we have the following solution for $U_p(t)$:

$$U_p(t) \sim U_{po} - \frac{\frac{3}{2}Re}{(Re^2 t)^2}\left[\frac{1}{U_{po}(2 - U_{po})}\right] + o\Big((Re^2 t)^{-2}\Big), \tag{2.51a}$$

where

$$U_{po} = \frac{4}{3Re}\left(\left(1 + \tfrac{3}{2}Re\right)^{\frac{1}{2}} - 1\right). \tag{2.51b}$$

We find that for $Re = 0.3$ the decay to steady-state (defined as 99% of the terminal velocity) is about forty times faster with the unsteady Oseen correction than with just the unsteady Stokes force. One must be cautious with such conclusions, however, since this sort of temporal decay is bounded by the existence of higher order terms which may have slower than $O(t^{-2})$ decay. We note that the next correction, which is $O(Re^2 \log Re)$, was indicated by Sano [47] to have no temporal nature. This is believed to be related to the fact that the $O(Re^2 \log Re)$ term arises from volume integration in the reciprocal theorem of the inner fields over the inner region where the fluid motion is nearly steady.[b]

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[b]One can compute the $O(Re^2 log Re)$ term from:

$$-Re^2 \int_{r=O(1)}^{r=O(Re^{-1})} \left[-\mathbf{U}_s(t)\cdot\nabla\mathbf{u}_1^{-1} + \mathbf{u}_0^{-1}\cdot\nabla\mathbf{u}_1^{0} + \mathbf{u}_1^{0}\cdot\nabla\mathbf{u}_0^{-1}\right]\cdot\hat{\mathbf{M}}_p \, dV,$$

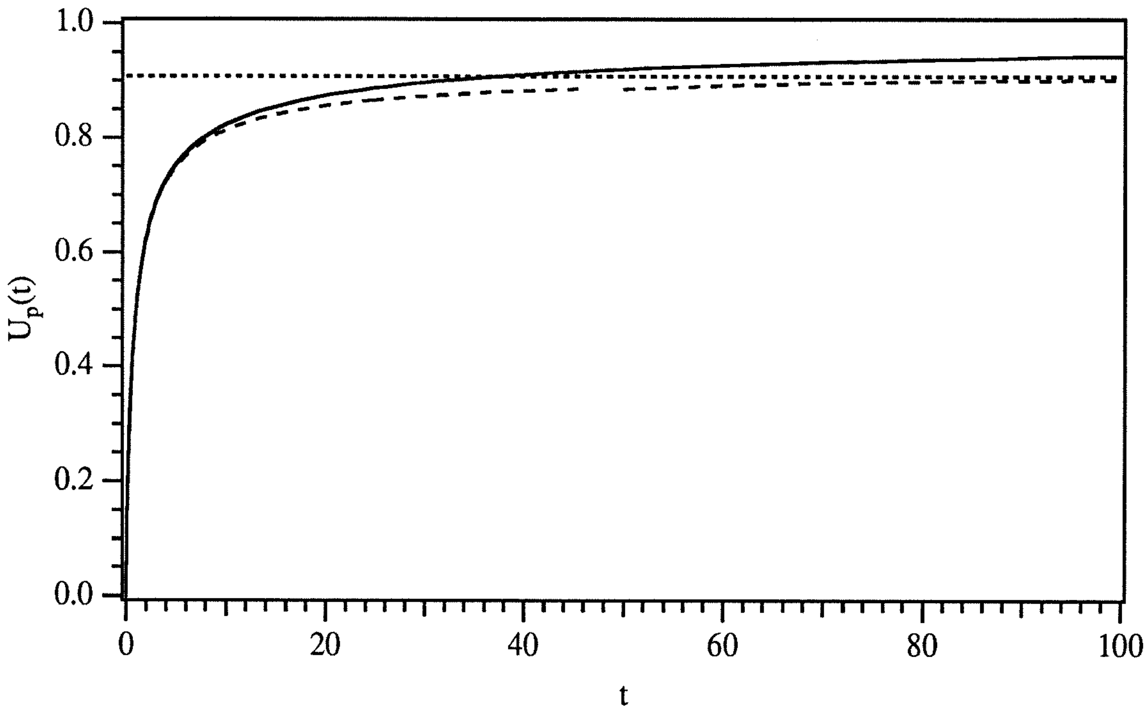


Figure 2.5: Trajectory of a sphere released from rest and settling under gravity. The ratio of particle density to fluid density, ρ_p/ρ_f, is 1.1. Time is scaled by a^2/ν and the velocity by the Stokes terminal velocity. The right side of the curve for *Re*=0.3 is from the long-time asymptotic result given by (2.51). The Reynolds number is: —— *Re*=0; - - *Re*=0.3; . . . . *Re*=0.3 (steady-state solution).

2.8 Expression of the hydrodynamic force acting on an arbitrarily shaped particle: long-time limit

As can be seen from (2.42), to evaluate the hydrodynamic force to $O(Re)$ in general would require both the unsteady Stokes velocity field, $\mathbf{v}_0$, and its regular perturbation, $\mathbf{v}_1$. Computing such fields for non-spherical particles is a difficult task. Few solutions

where $\mathbf{u}_1$ is the solution to the regular perturbation to steady Stokes flow and the above superscripts indicate that the velocity fields are homogeneous in that power of $|\mathbf{r}|$, i.e., $\mathbf{u}_1^{-1}$ represents those terms of $\mathbf{u}_1$ that are $O(r^{-1})$. By analogy, we anticipate that this term will be $O(Re^2 \log((ReSl)^{-\frac{1}{2}}))$ for time scales such that $a^2/\nu \ll \tau_c \ll \nu/U_c^2$, since the velocity fields in the integral are only valid out to the distance to which vorticity has diffused, $O((ReSl)^{-\frac{1}{2}})$.

exist for the unsteady Stokes field created by the motion of a non-spherical particle. Most notable is the rather complex formulation presented by Lawrence and Weinbaum [28] for the axisymmetric motion of a spheroid. We are unaware, however, of any solutions for the regular perturbation field.

For a non-spherical particle, however, one can use the general approach of the reciprocal theorem and the idea of a uniformly valid velocity field to find the form of the unsteady Stokes corrections to the pseudo-steady Stokes drag in the limit of $ReSl \ll 1$. This is the limit when the time scale of variation is much longer than the diffusive scale a^2/ν. The Basset history force in the long-time limit, which, in general, may be different in the short-time limit, is identified as the term that is $O((ReSl)^{\frac{1}{2}})$. The acceleration reaction (for lack of a better name) will represent the $O(ReSl)$ term since, as we shall see below, it is proportional to the particle acceleration. These terms are contained in $\mathbf{F}^H_{us}$, so for now we only consider this term and not the integral in (2.42).

We begin with the reciprocal theorem expression for the unsteady Stokes force from (2.15):

$$\mathbf{F}^H_{us}(t) = \mathbf{F}^H_s(t) - ReSl \int_{V_f} \frac{\partial \mathbf{v}_0}{\partial t} \cdot \hat{\mathbf{M}}\, dV, \tag{2.52}$$

where $\mathbf{v}_0$ is the unsteady Stokes velocity field (i.e., the solution of (2.107)). When $ReSl \ll 1$, near to the particle the velocity field will be dominated by steady Stokes flow, while far from the particle at distances of $O(a(ReSl)^{-\frac{1}{2}})$, the flow to leading order is given by the point-forced unsteady Stokes solution. Thus, we approximate $\mathbf{v}_0$ with a uniformly valid velocity field as

$$\mathbf{v}_0 \sim \mathbf{v}_0^{uv} = (\mathbf{u}_0 - \mathbf{u}_0^p) + \mathbf{v}^p, \tag{2.53}$$

where $\mathbf{u}_0$ is the steady Stokes velocity field, and $\mathbf{u}_0^p$ is the solution to the point-forced Stokes field given by (2.119). The solution to the full point-forced unsteady Stokes velocity field, $\mathbf{v}^p$, is given by the solution to

$$-\boldsymbol{\nabla}\Pi^p + \nabla^2\mathbf{v}^p = ReSl\frac{\partial\,\mathbf{v}^p}{\partial\,t} + \mathbf{F}_1^{us}(t)\delta(\mathbf{r}), \quad \boldsymbol{\nabla}\cdot\mathbf{v}^p = 0, \tag{2.54}$$

which is identified as the outer velocity field. Here we retain the entire force monopole to ensure the desired accuracy in the hydrodynamic force; that is,

$$\mathbf{F}_1^{us}(t) = \mathbf{F}_{us}^H(t) - ReSl\widetilde{V}_p\dot{\mathbf{U}}_s(t). \tag{2.55}$$

In the presence of an $O(1)$ dipole contribution to the velocity field, which would arise from an external torque or straining motion, this uniformly valid field is not accurate to $O(ReSl)$, but only to $O((ReSl)^{\frac{1}{2}})$. We assume for now there is no dipole contribution, but will return to this issue later.

If we define $\hat{\mathbf{M}}_s$ as

$$\hat{\mathbf{M}}_s = \hat{\mathbf{M}} - \hat{\mathbf{M}}_p, \tag{2.56}$$

and note that

$$\mathbf{u}_0 - \mathbf{u}_0^p = (\hat{\mathbf{M}} - \hat{\mathbf{M}}_p)\cdot\mathbf{U}_s(t), \tag{2.57}$$

then (2.53) becomes

$$\mathbf{v}_0^{uv} = \hat{\mathbf{M}}_s\cdot\mathbf{U}_s(t) + \mathbf{v}^p. \tag{2.58}$$

Using this approximation for the unsteady Stokes velocity field we obtain the following expression for the unsteady Stokes drag to $O(ReSl)$:

$$\mathbf{F}_{us}^H(t) = \mathbf{F}_s^H(t) - ReSl\dot{\mathbf{U}}_s(t)\cdot\int_{V_f}\hat{\mathbf{M}}_s^T\cdot\hat{\mathbf{M}}dV - ReSl\int_{V_f}\frac{\partial\,\mathbf{v}^p}{\partial\,t}\cdot\hat{\mathbf{M}}dV + o(ReSl), \tag{2.59}$$

where $\hat{\mathbf{M}}_s^T$ is the transpose of $\hat{\mathbf{M}}_s$.

One can see that if there were a dipole contribution to $\hat{\mathbf{M}}_s$, the first integral of (2.59) would be conditionally convergent, as the dipole is antisymmetric and $O(r^{-2})$, while the monopole contribution from $\hat{\mathbf{M}}$ is $O(r^{-1})$ and symmetric. That is, if we perform the radial integration first we obtain a log singularity at infinity; while angular integration, done first over any spherical surface, removes the log singularity. Note that without the dipole $\hat{\mathbf{M}}_s$ is $O(r^{-3})$.

This apparent problem is simply a shortcoming of approximating the actual velocity by a field with too low an accuracy in $ReSl$, only to $O((ReSl)^{\frac{1}{2}})$, in the outer region. To be mathematically rigorous, one could include the dipole field in the outer velocity field in the same manner as was done for the monopole. This creates unnecessary labor, however. Instead, if there is a dipole contribution, we may force a convergent order of integration by writing (2.59) as

$$\mathbf{F}^H_{us}(t) = \mathbf{F}^H_s(t) - ReSl\dot{\mathbf{U}}_s(t) \cdot \lim_{R\to\infty} \int\limits_{V_f(R)} \hat{\mathbf{M}}^T_s \cdot \hat{\mathbf{M}}\, dV - ReSl \int\limits_{V_f} \frac{\partial \mathbf{v}^p}{\partial t} \cdot \hat{\mathbf{M}}\, dV + o(ReSl), \tag{2.60}$$

where now $V_f(R)$ is the volume of fluid surrounding the particle and bounded by a spherical surface of radius R. This is justified by the fact that the actual dipole contribution to the unsteady Stokes field is anti-symmetric but only $O(r^{-4})$ for large r. As an added note, a dipole contribution will only exist if the particle can exert a torque (or stresslet) on the fluid by its translational motion. This would be the case, for example, for a screw-shaped particle if there was an external torque to prevent it from rotating.

Now we shall consider how to simplify the calculation of the second integral of (2.60), the contribution of $\mathbf{v}^p$ to $\mathbf{F}^H_{us}$. First we decompose it into the following two parts:

$$ReSl \int\limits_{V_f} \frac{\partial \mathbf{v}^p}{\partial t} \cdot \hat{\mathbf{M}}\, dV = ReSl \int\limits_{V_f} \frac{\partial \mathbf{v}^p}{\partial t} \cdot \hat{\mathbf{M}}_p\, dV + ReSl \int\limits_{V_f} \frac{\partial \mathbf{v}^p}{\partial t} \cdot \hat{\mathbf{M}}_s\, dV. \tag{2.61}$$

In the last integral of (2.61), we may replace $\mathbf{v}^p$ with $\mathbf{u}^p_0$ because in the outer region, where this is an invalid approximation, the error that it makes in the hydrodynamic force is $o(ReSl)$. Here again the dipole contribution may be dealt with as above to obtain

$$ReSl \int\limits_{V_f} \frac{\partial \mathbf{v}^p}{\partial t} \cdot \hat{\mathbf{M}}_s \, dV = ReSl \dot{\mathbf{U}}_s(t) \cdot \lim_{R \to \infty} \int\limits_{V_f(R)} \hat{\mathbf{M}}_p^T \cdot \hat{\mathbf{M}}_s \, dV + o(ReSl). \tag{2.62}$$

The first integral in (2.61) may be rewritten as an integral over the entire region of space minus that over the volume of the particle and approximated as

$$ReSl \int\limits_{V_f} \frac{\partial \mathbf{v}^p}{\partial t} \cdot \hat{\mathbf{M}}_p \, dV = ReSl \int\limits_{V_\infty} \widehat{\frac{\partial \mathbf{v}^p}{\partial t}}(\mathbf{k}) \cdot \widehat{\hat{\mathbf{M}}_p}(-\mathbf{k}) \, d\mathbf{k} - ReSl \dot{\mathbf{U}}_s(t) \cdot \int\limits_{V_p} \hat{\mathbf{M}}_p^T \cdot \hat{\mathbf{M}}_p \, dV + o(ReSl). \tag{2.63}$$

In the integral over the volume of the particle we have replaced $\mathbf{v}^p$ with $\mathbf{u}_0^p$, since this makes an error that is $o(ReSl)$. The integral over all space was rewritten in Fourier space with the use of the convolution theorem. Combining (2.62) and (2.63) in (2.61), this contribution to the unsteady Stokes force becomes

$$- ReSl \int\limits_{V_f} \frac{\partial \mathbf{v}^p}{\partial t} \cdot \hat{\mathbf{M}} \, dV = -ReSl \int\limits_{V_\infty} \widehat{\frac{\partial \mathbf{v}^p}{\partial t}}(\mathbf{k}) \cdot \widehat{\hat{\mathbf{M}}_p}(-\mathbf{k}) \, d\mathbf{k} + ReSl \dot{\mathbf{U}}_s(t) \cdot \int\limits_{V_p} \hat{\mathbf{M}}_p^T \cdot \hat{\mathbf{M}}_p \, dV - ReSl \dot{\mathbf{U}}_s(t) \cdot \lim_{R \to \infty} \int\limits_{V_f(R)} \hat{\mathbf{M}}_p^T \cdot \hat{\mathbf{M}}_s \, dV + o(ReSl). \tag{2.64}$$

The integral in (2.64) involving the Fourier transforms can be evaluated in the same manner as was done in Section 2.6. Following the procedure in Appendix 2.12, we obtain

$$ReSl \widehat{\frac{\partial \mathbf{v}^p}{\partial t}} = \int_{-\infty}^{t} \dot{\mathbf{F}}_1^{us}(s) \cdot (\mathbf{n}_k \mathbf{n}_k - \mathbf{I}) \mathrm{e}^{-4\pi^2 k^2 (t-s)/ReSl} \, ds. \tag{2.65}$$

Combining (2.22) for $\widehat{\hat{\mathbf{M}}_p}$ and (2.65) and performing the integration over $\mathbf{k}$-space gives

$$Re\,Sl \int\limits_{V_\infty} \widehat{\frac{\partial \mathbf{v}^p}{\partial t}}(\mathbf{k}) \cdot \widehat{\hat{\mathbf{M}}_p}(-\mathbf{k})\, d\mathbf{k} = -\left(\frac{Re\,Sl}{\pi}\right)^{\frac{1}{2}} \int_{-\infty}^{t} \frac{\dot{\mathbf{F}}_1^{us}(s)}{(t-s)^{\frac{1}{2}}}\, ds \cdot \mathbf{\Phi}. \tag{2.66}$$

Now we must consider carefully what form $\mathbf{F}_1^{us}(t)$ must take in order to obtain the hydrodynamic force to $O(Re\,Sl)$. Note that $\mathbf{F}_1^{us}(t)$ represents the exact hydrodynamic force acting on the particle due to the unsteady Stokes field plus a term that is $O(Re\,Sl)$. To obtain the contribution to the hydrodynamic force from (2.66) to $O(Re\,Sl)$, $\mathbf{F}_1^{us}(t)$ must be replaced with the particle drag to $O((Re\,Sl)^{\frac{1}{2}})$. The O(1) contribution is the pseudo-steady Stokes drag, $\mathbf{F}_s^H(t)$. The $O((Re\,Sl)^{\frac{1}{2}})$ contribution to $\mathbf{F}_1^{us}(t)$ can be found by evaluating (2.66) with $\mathbf{F}_1^{us}(t)$ replaced by $\mathbf{F}_s^H(t)$. Thus the proper form of $\mathbf{F}_1^{us}(t)$ to be used in (2.66) is

$$\mathbf{F}_1^{us}(t) = \mathbf{F}_s^H(t) + \left(\frac{Re\,Sl}{\pi}\right)^{\frac{1}{2}} \int_{-\infty}^{t} \frac{\dot{\mathbf{F}}_s^H(s)}{(t-s)^{\frac{1}{2}}}\, ds \cdot \mathbf{\Phi} + o\Big((Re\,Sl)^{\frac{1}{2}}\Big). \tag{2.67}$$

Evaluating (2.66), using the above expression for $\mathbf{F}_1^{us}(t)$ we obtain

$$\begin{aligned} -\,Re\,Sl \int\limits_{V_\infty} \widehat{\frac{\partial \mathbf{v}^p}{\partial t}} \cdot \widehat{\hat{\mathbf{M}}_p} d\mathbf{k} &= \left(\frac{Re\,Sl}{\pi}\right)^{\frac{1}{2}} \int_{-\infty}^{t} \frac{\dot{\mathbf{F}}_s^H(s)}{(t-s)^{\frac{1}{2}}}\, ds \cdot \mathbf{\Phi} \\ &\quad + Re\,Sl\dot{\mathbf{F}}_s^H(t) \cdot \mathbf{\Phi} \cdot \mathbf{\Phi} + o(Re\,Sl). \end{aligned} \tag{2.68}$$

Finally, combining the above result with (2.64) in the expression of the total unsteady Stokes force (2.60), we obtain, correct to $O(Re\,Sl)$,

$$\begin{aligned} \mathbf{F}_{us}^H(t) = &-6\pi\mathbf{\Phi} \cdot \left\{ \mathbf{U}_s(t) + \left(\frac{Re\,Sl}{\pi}\right)^{\frac{1}{2}} \mathbf{\Phi} \cdot \int_{-\infty}^{t} \frac{\dot{\mathbf{U}}_s(s)}{(t-s)^{\frac{1}{2}}}\, ds + Re\,Sl\mathbf{\Phi} \cdot \mathbf{\Phi} \cdot \dot{\mathbf{U}}_s(t) \right\} \\ &- Re\,Sl \left\{ \lim_{R\to\infty} \left(\int\limits_{V_f(R)} \hat{\mathbf{M}}^T \cdot \hat{\mathbf{M}}\, dV - \frac{9\pi}{2} \mathbf{\Phi} \cdot \mathbf{\Phi} R \right) \right\} \cdot \dot{\mathbf{U}}_s(t) \\ &+ o(Re\,Sl), \end{aligned} \tag{2.69}$$

where we have used the fact that $\int_{V_f(R)+V_p} \hat{\mathbf{M}}_p^T \cdot \hat{\mathbf{M}}_p \, dV = \frac{9\pi}{2} \mathbf{\Phi} \cdot \mathbf{\Phi} R$. The first term is the pseudo-steady Stokes drag. The second is identified as the Basset history force in the long-time limit. This form of the history force was first noted by Williams [51] . The remaining terms combine to contribute to what we have referred to as the acceleration reaction, being proportional to the particle acceleration relative to the imposed flow. It is the counterpart to the added mass in the short-time limit associated with potential flow. The entire result agrees in form with the expression obtained by Pozrikidis [42] for the low-frequency oscillation of a particle. However, Pozrikidis' expression requires the solution of an integral equation for the given particle. It is also interesting to point out that the above resultant "acceleration reaction" resistance tensor is symmetric, which agrees with the finding of Gavze [17] .

To find the acceleration reaction correction in this limit of small $ReSl$, we only require the steady Stokes velocity field created by the translating particle at time t and the corresponding Stokes drag. Indeed, if we use the steady Stokes velocity field for a translating sphere we obtain $-\frac{2}{3}\pi ReSl\dot{\mathbf{U}}_p(t)$, the added mass of a sphere, which agrees with the fact that the acceleration reaction and the added mass are the same for the special case of a spherical particle. If we use the Stokes velocity field and drag for an oblate spheroid translating along its axis of symmetry, given in the text by Happel and Brenner [18] , we obtain the following expression for the acceleration reaction in dimensional variables when $ReSl \ll 1$:

$$\mathbf{F}^H_{acc}(t) = -\tfrac{2}{3}\pi\rho b^3 \dot{\mathbf{U}}_s(t)\left\{ 9\left(\frac{\frac{4}{3}\lambda^{-1}}{\lambda - (\lambda^2-1)\cot^{-1}\lambda}\right)^3 \right.$$
$$+ \frac{2\lambda^{-3}}{(\lambda - (\lambda^2-1)\cot^{-1}\lambda)^2}\left[2\lambda + 4\lambda^3 + (2 + 5\lambda^2 - 3\lambda^6)\cot^{-1}\lambda\right.$$
$$\left.\left. - (1 - 2\lambda^2 + \lambda^4)\left(\int_\lambda^\infty [(\cot^{-1}x)^2 + 3(x\cot^{-1}x)^2 - 3]\,dx - 3\lambda\right)\right]\right\}, \tag{2.70a}$$

where

$$\lambda = b/(a^2 - b^2)^{\frac{1}{2}}. \tag{2.70b}$$

Here a and b the are major and minor semi-axes of the spheroid, respectively. For a slightly oblate spheroid we define its nonsphericity, ϵ, by

$$a = b(1+\epsilon), \quad \epsilon \ll 1. \tag{2.71}$$

Then the acceleration reaction becomes

$$\mathbf{F}^H_{am}(t) = -\tfrac{2}{3}\pi\rho b^3\left\{1 + \frac{16}{5}\epsilon + \frac{628}{175}\epsilon^2 + o(\epsilon^2)\right\}\dot{\mathbf{U}}_s(t), \tag{2.72}$$

which agrees with the result of Lawrence and Weinbaum [28] in the low-frequency limit of an oscillating slightly oblate spheroid for the part of the force proportional to the first power of the frequency. We note that the added mass found by Lawrence and Weinbaum [28] in the high-frequency limit differs from (2.72) at $O(\epsilon^2)$, which again demonstrates the uniqueness of the result (2.2) for a perfect solid sphere.

Finally, for $ReSl \ll 1$, the contribution to the hydrodynamic force from the regular perturbation to unsteady Stokes flow (the integral portion of (2.42)) may be simplified. Clearly, the part from the term $Re(ReSl)\partial \mathbf{v}_1/\partial t$ is only a small correction to the already small $O(Re)$ correction to the drag evaluated in Section 2.6 and can therefore be neglected. In addition, in the terms involving $Re\mathbf{v}_0 \cdot \nabla \mathbf{v}_0$ and $Re\mathbf{U}_s(t) \cdot \nabla \mathbf{v}_0$, $\mathbf{v}_0$ could be replaced with the approximate field given by (2.53). However, even this level of accuracy is unnecessary. If we simply use the steady Stokes field, $\mathbf{u}_0$, we can obtain the correction to $O(Re)$, but to only $O(1)$ in $ReSl$. In doing so, however, one must recognize that the integrand, $Re\mathbf{U}_s(t) \cdot \nabla \mathbf{u}_0 \cdot \hat{\mathbf{M}}$, is conditionally convergent, since it is $O(r^{-3})$ and antisymmetric at large distances from the particle. Thus, we simply approximate this entire contribution by writing

$$Re \int_{V_f} \left(Re\,Sl \frac{\partial \mathbf{v}_1}{\partial t} + \mathbf{v}_0 \cdot \nabla \mathbf{v}_0 - \mathbf{U}_s(t) \cdot \nabla \mathbf{v}_0 \right) \cdot \hat{\mathbf{M}}\, dV =$$

$$Re \lim_{R \to \infty} \int_{V_f(R)} \left(\mathbf{u}_0 \cdot \nabla \mathbf{u}_0 - \mathbf{U}_s(t) \cdot \nabla \mathbf{u}_0 \right) \cdot \hat{\mathbf{M}}\, dV + o(Re). \quad (2.73)$$

The integral on the right-hand side of (2.73) may be shown to only contribute a force orthogonal to the direction of motion of the particle – a lift force. The portion parallel to the direction of motion is given by the volume integration of

$$\left(\mathbf{u}_0 \cdot \nabla \mathbf{u}_0 - \mathbf{U}_s(t) \cdot \nabla \mathbf{u}_0 \right) \cdot \hat{\mathbf{M}} \cdot \mathbf{U}_s(t) = \left[\nabla \cdot \left(\mathbf{u}_0 \mathbf{u}_0 - \mathbf{U}_s(t) \mathbf{u}_0 \right) \right] \cdot \mathbf{u}_0$$
$$= \frac{1}{2} \nabla \cdot \left(\mathbf{u}_0 \mathbf{u}_0 \cdot \mathbf{u}_0 - \mathbf{U}_s(t) \mathbf{u}_0 \cdot \mathbf{u}_0 \right), \quad (2.74)$$

where we have used the continuity equation and that $\hat{\mathbf{M}} \cdot \mathbf{U}_s(t) = \mathbf{u}_0$. Application of the divergence theorem in the volume integral of this quantity and using the boundary condition on the surface of the particle yields surface integrals which are identically zero. Thus, the contribution given by (2.73) can only yield a transverse or side force. This result agrees with the conclusion reached by Brenner and Cox [8] . This form of the term represented by (2.73) was first noted in a paper by Cox [11] .

If we combine the results of Section 2.6 and Section 2.8, we obtain the expression of the hydrodynamic force acting on an arbitrarily shaped rigid particle in the limit when the time scale associated with the particle's slip velocity is much greater than the diffusive scale. Combining (2.28) and (2.33) with (2.42), (2.69), and (2.73) the hydrodynamic force is

$$\mathbf{F}^H(t) = ReSl\widetilde{V}_p\dot{\mathbf{U}}^\infty(t) + \mathbf{F}_s^H(t)$$
$$- ReSl\Bigg\{6\pi\mathbf{\Phi}\cdot\mathbf{\Phi}\cdot\mathbf{\Phi} + \lim_{R\to\infty}\Bigg(\int_{V_f(R)} \hat{\mathbf{M}}^T\cdot\hat{\mathbf{M}}\,dV - \frac{9\pi}{2}\mathbf{\Phi}\cdot\mathbf{\Phi}R\Bigg)\Bigg\}\cdot\dot{\mathbf{U}}_s(t)$$
$$+\frac{3}{8}\left(\frac{ReSl}{\pi}\right)^{\frac{1}{2}}\Bigg\{\int_{-\infty}^{t}\Bigg[\ \tfrac{2}{3}\mathbf{F}_s^{H\|}(t) - \Bigg\{\frac{1}{|\mathbf{A}|^2}\Big(\frac{\pi^{\frac{1}{2}}}{2|\mathbf{A}|}\mathrm{erf}(|\mathbf{A}|) - \exp(-|\mathbf{A}|^2)\Big)\Bigg\}\mathbf{F}_s^{H\|}(s)$$
$$+\tfrac{2}{3}\mathbf{F}_s^{H\perp}(t) - \Bigg\{\exp(-|\mathbf{A}|^2) - \frac{1}{2|\mathbf{A}|^2}\Big(\frac{\pi^{\frac{1}{2}}}{2|\mathbf{A}|}\mathrm{erf}(|\mathbf{A}|) - \exp(-|\mathbf{A}|^2)\Big)\Bigg\}\mathbf{F}_s^{H\perp}(s)\Bigg]$$
$$\times\frac{2\,ds}{(t-s)^{\frac{3}{2}}}\Bigg\}\cdot\mathbf{\Phi}$$
$$- Re\lim_{R\to\infty}\int_{V_f(R)}\Big(\mathbf{u}_0\cdot\nabla\mathbf{u}_0 - \mathbf{U}_s(t)\cdot\nabla\mathbf{u}_0\Big)\cdot\hat{\mathbf{M}}\,dV + o(ReSl) + o(Re). \qquad (2.75)$$

The first term of this expression is due to an accelerating reference frame, the second is the pseudo-steady Stokes drag, and the third has been labeled as the acceleration reaction. The fourth term represents the unsteady Oseen correction to the hydrodynamic force. It is a new history integral that replaces the Basset history force in the long-time limit at finite Reynolds number. The last term of this expression can only contribute a force perpendicular to the slip velocity of the particle. In order to make use of this expression for a given particle we only require the steady Stokes drag and the corresponding steady Stokes velocity field created by the translating particle.

2.9 Results and discussion

When the time scale is $O(\nu/U_c^2)$, or longer, the history force represented by the unsteady Oseen correction is $O(Re)$ and has a temporal behavior very different from the Basset term; for shorter time scales it behaves simply as the Basset history force in the long-time limit. This is due to the fact that the time scale must be of this large magnitude for vorticity to have diffused out to the Oseen distance of aRe^{-1}.

When this has occurred vorticity is swept up in a wake region behind the particle and transported by convection. Also, on this long time scale, the acceleration reaction (the third term of (2.75)) is negligible, being $O(Re^2)$. Due to the change in the mechanism of vorticity transport – convection in the wake versus radial diffusion – we have shown that for time scales greater than $O(\nu/U_c^2)$ the temporal behavior of the force will decay faster than the $t^{-\frac{1}{2}}$ predicted by unsteady Stokes flow.

It is important to note that (2.75) is valid for *all* time scales for the case of a rigid spherical particle. This condition exists because the asymptotic form of the unsteady Stokes force to $O(ReSl)$ for large time scales ($\tau_c \gg a^2/\nu$) is the same for arbitrary time scales. The corresponding expression for a spherical particle is given by (2.44). This expression would be appropriate for the calculation of the hydrodynamic force to $O(Re)$ acting on a sphere undergoing any time dependent motion in a prescribed uniform flow when the particle Reynolds number is small but finite. As demonstrated in the previous section, it can also be used for the dynamic calculation of the trajectory of the particle under the action of an external force.

To reiterate the approach we have used, the analysis begins by constructing a uniformly valid velocity field which is then used in the appropriate reciprocal theorem. This uniformly valid velocity field is constructed by summing the leading order field valid close to the particle with that valid far from the particle and subtracting the parts common to both fields. The near field is given by the steady (or unsteady) Stokes equations. It is this field which takes into account the finite size and shape of the particle. The far field, when one exists, is given by a point-forced equation with only the dominant inertial terms retained; in the far field the particle appears as a force monopole to leading order. Then the common part is simply the point-forced Stokes (or unsteady Stokes) velocity field.

In performing calculations, the way these fields are distributed is dictated by the necessity to maintain convergent volume integrations. In Appendix 2.11, the outer field, $\mathbf{v}_1^{p+}$, was formed by subtracting the point-forced unsteady Stokes field so that the Stokeslet nature of the field at the origin was removed. This rearrangement allowed integration to be extended to the center of the particle and the subsequent application of Fourier transforms and the convolution theorem in Section 2.6. Had this rearrangement not been performed, the resulting term, $\mathbf{U}_s(t) \cdot \nabla \mathbf{v}_0^{p+} \cdot \hat{\mathbf{M}}_p$, would have been a conditionally convergent anti-symmetric integrand at the origin, being $O(|\mathbf{r}|^{-3})$. Also, in grouping the velocity fields the way we did, the unsteady Stokes force was retained as a separate contribution, which could be treated on its own.

On the other hand, when the unsteady Stokes force was evaluated in Section 2.8, the outer field was taken as the point-forced unsteady Stokes field. Since it formed a convergent integrand both at the origin and at infinity, this again allowed the application of Fourier transforms and the convolution theorem. In this way the common part, the Stokeslet, was subtracted from the inner steady Stokes field to form the new field given by $\hat{\mathbf{M}}_s \cdot \mathbf{U}_s(t)$. Then once proper account was taken for possible dipole contributions, this field yielded a convergent contribution to the hydrodynamic force when used in the reciprocal theorem. Had the Stokeslet not been subtracted from the inner field, the resulting integrand, $\partial \mathbf{u}_0 / \partial t \cdot \hat{\mathbf{M}}$, would not have been convergent when integrated to infinity, being $O(r^{-2})$ at large distances from the particle.

In using this approach to obtain inertial corrections to the steady Stokes force acting on the particle, one is able to attribute the various contributions directly to regions in the fluid domain and the corresponding velocity field. This method contrasts the usual approach of applying matching conditions and the subsequent evaluation of contributions to the hydrodynamic force from higher order inner fields. In a sense, the matching procedure is done implicitly by the formation of a uniformly valid velocity field. An advantage to this approach is that, in general, the evaluation of higher order fields is not required to obtain the leading order corrections to the steady Stokes force. In addition, only the Fourier space solution to the outer field

(2.124) is needed to perform the calculations. As an added comment, we note that the Laplace transform (in time) method of Bentwich and Miloh [5] could not be used here because of the problem of dealing with the time-dependent coefficient, $\mathbf{U}_s(t)$, in the convective terms of the outer velocity field.

The equation for the outer contribution to the hydrodynamic force, (2.20), can be shown to be equivalent to previous researchers' findings [9, 46] in that it represents a term proportional to a uniform flow created by the outer velocity field at the center of the particle. The governing equations for the $Re\mathbf{v}_1^{p+}$-field dictate that

$$\widehat{\mathbf{f}_{out}}(Re\mathbf{v}_1^{p+})(\mathbf{k}) = Re\left(ReSl\frac{\partial \widehat{\mathbf{v}_1^{p+}}}{\partial t} - 2\pi \mathrm{i} Re\mathbf{U}_s(t)\cdot\mathbf{k}\widehat{\mathbf{v}_1^{p+}}\right) \tag{2.76a}$$

$$= -4\pi^2 Rek^2\widehat{\mathbf{v}_1^{p+}} - 2\pi\mathrm{i}Re\mathbf{U}_s(t)\cdot\mathbf{k}\widehat{\mathbf{v}^p}. \tag{2.76b}$$

Thus, if (2.76b) is combined with (2.22), the Fourier transform of the $\hat{\mathbf{M}}_p$-field, in (2.20), we find

$$\mathbf{F}^H_{out}(t) = 6\pi\mathbf{\Phi}\cdot\int\limits_{V_\infty}\left(Re\widehat{\mathbf{v}_1^{p+}} - \frac{\mathrm{i}\mathbf{U}_s(t)\cdot\mathbf{k}}{2\pi k^2}\widehat{\mathbf{v}^p}\right)d\mathbf{k}. \tag{2.77}$$

The second term of this expression has no effect except to remove the conditionally convergent antisymmetric contribution that was discussed above. Therefore, (2.77) demonstrates that this contribution is simply the dot product of $6\pi\mathbf{\Phi}$ with the uniform flow created by the disturbance field $Re\mathbf{v}_1^{p+}$ at the center of the particle.

The extension of this analysis to include particle rotation does not significantly alter the derivation provided we have

$$\frac{|\mathbf{\Omega}_p(t)|a}{|\mathbf{U}_s(t)|} \leq O(1), \tag{2.78}$$

where $\mathbf{\Omega}_p(t)$ is the angular velocity of the particle. This condition implies that the Reynolds number based on the characteristic slip velocity is of the same magnitude as that based on the characteristic angular velocity of the particle. The only adjustment for the derivation of the outer contribution from Section 2.6 is to recognize the fact that $\mathbf{F}_s^H(t)$ is given by

$$\mathbf{F}_s^H(t) = -6\pi\boldsymbol{\Phi}(t)\cdot\mathbf{U}_s(t) - \mathbf{R}_{F\Omega}(t)\cdot\boldsymbol{\Omega}_p(t), \tag{2.79}$$

where $\mathbf{R}_{F\Omega}(t)$ is the resistance matrix relating particle rotation to hydrodynamic force. For nonspherical particles both resistance matrices may be time dependent due to the rotation of the particle.

In the presence of particle rotation, the boundary conditions on the surface of the particle must be appropriately modified in the governing equations for the inner velocity fields. With particle rotation the symmetry arguments described in Section 2.7 for a spherical particle no longer hold for the contributions from the regular perturbation to unsteady Stokes flow. These will make an $O(Re)$ contribution in the presence of particle rotation. Thus, the conclusion that the expression given by (2.44) holds for all time scales would no longer be valid. However, for it to be valid to $O(Re)$ and to $O(ReSl)$ when $ReSl \ll O(1)$, the only modification required is to include the contribution from the expression given by (2.73). That is,

$$-Re\lim_{R\to\infty}\int_{V_f(R)}\left(\mathbf{u}_0\cdot\nabla\mathbf{u}_0 - \mathbf{U}_s(t)\cdot\nabla\mathbf{u}_0\right)\cdot\hat{\mathbf{M}}\,dV = \pi Re\boldsymbol{\Omega}_p(t)\times\mathbf{U}_s(t), \tag{2.80}$$

where $\boldsymbol{\Omega}_p(t)$ has been nondimensionalized by U_c/a and $\times$ indicates a vector cross product. Here, $\mathbf{u}_0$ is the steady Stokes velocity field created by the sphere translating and rotating in a stationary fluid. This represents the lift force first obtained by Rubinow and Keller [45] . We note that in this derivation the $\hat{\mathbf{M}}$-field remains the same as previously described, i.e., that for particle translation only.

To modify the expression given by (2.75) for nonspherical particle rotation one must first apply the $\mathbf{F}_s^H(t)$ as given by (2.79) where the resistance matrices may also be time dependent. The acceleration reaction contribution, given as the third term of (2.75), requires detailed modifications of the analysis given in Section 2.8. That is, in (2.75) we have assumed $\mathbf{\Phi}$ is not a function of time. If, however, the particle was nonspherical and as it translated it was tumbling about an axis that was not an axis of symmetry, $\mathbf{\Phi}$ would be time dependent. Under these conditions the acceleration reaction would be replaced by

$$ReSl\left\{\lim_{x\to-\infty}\frac{1}{\pi}\left[\int_x^t \ddot{\mathbf{F}}_s^H(q)\cdot\mathbf{J}(q,t)\,dq+\dot{\mathbf{F}}_s^H(x)\cdot\mathbf{J}(x,t)\right]\right.$$
$$\left.-\lim_{R\to\infty}\int_{V_f(R)}\left(\frac{\partial\mathbf{u}_0}{\partial t}\cdot\hat{\mathbf{M}}-\frac{\partial\mathbf{u}_0^p}{\partial t}\cdot\hat{\mathbf{M}}_p\right)dV+\int_{V_p}\frac{\partial\mathbf{u}_0^p}{\partial t}\cdot\hat{\mathbf{M}}_p\,dV\right\},\quad(2.81a)$$

where

$$\mathbf{J}(x,t)=\int_x^t\frac{\mathbf{\Phi}(s)}{((t-s)(s-x))^{\frac{1}{2}}}\,ds\cdot\mathbf{\Phi}(t).\qquad(2.81b)$$

When $\mathbf{\Phi}$ is independent of time, $\mathbf{J}(x,t)$ is equal to $\pi\mathbf{\Phi}\cdot\mathbf{\Phi}$. In the above expression, as well as in the last term of (2.75), the steady Stokes velocity field for the particle both translating and rotating in a stationary fluid is required for the $\mathbf{u}_0$-field. It should also be noted that the Stokes velocity fields $\mathbf{u}_0$ and $\mathbf{u}_0^p$ are in general functions of time not only because of the time dependent nature of $\mathbf{U}_s(t)$ and $\mathbf{\Omega}_p(t)$ but also because of the change in orientation of the particle with time. For example, if $\mathbf{u}_0 = \hat{\mathbf{M}}\cdot\mathbf{U}_s(t)$, then $\hat{\mathbf{M}}$ may also be a function of time through the particle orientation dependence relative to $\mathbf{U}_s(t)$. As a final note, the second-order tensor $\mathbf{\Phi}$ in these modified expressions remains as defined above since the $\hat{\mathbf{M}}$-field is expressed in terms of the particle translating in the given orientation at time t.

Another possibility for extension of these results would be to consider the case of non-steady particle motion in a linear flow. Maxey and Riley [33] have obtained a solution for the corresponding unsteady Stokes problem associated with a translating sphere in a time-dependent non-uniform flow. Their analysis shows there is no significant modification of the unsteady Stokes force from that of (2.2) for the case of a linear flow, the only difference being that the particle velocity is given as relative to the undisturbed flow evaluated at the particle center. It is believed that the inclusion of the convective inertial corrections would be of more significance in this case than in the uniform flow problem analyzed here, because the critical time scale for which an outer region develops, where the convective inertia is important to leading order, is $\dot{\gamma}^{-1}$, where $\dot{\gamma}$ is the characteristic strain rate of the external flow. This is a more relevant time scale since it does not depend on the fluid viscosity nor on any external forces that act on the particle. The dominant convective correction to the unsteady Stokes force would in general come from the outer contribution since it is $O(Re_{\dot{\gamma}}^{\frac{1}{2}})$ where $Re_{\dot{\gamma}} = a^2\dot{\gamma}/\nu$ [46]. This correction could be solved for, in principle, by the analysis presented here, but it would ultimately require the solution of a partial differential equation for the velocity field in Fourier space.

As a final note, the use of the present approach to study oscillatory motion and the motion of bubbles and drops at small Reynolds number will be carried out in Chapters 3 and 4, respectively.

2.10 Appendix: The reciprocal theorem and expressions for the particle force

It is necessary to briefly consider the governing equations and general features of the fluid motion about a rigid particle translating with velocity $\mathbf{U}_p$ in a general time-dependent flow. Relative to a coordinate system fixed in the laboratory, the undisturbed fluid motion is denoted $\mathbf{v}^\infty(\mathbf{x},t)$. Referred to a coordinate system instantaneously fixed to the particle, the undisturbed fluid motion is denoted $\mathbf{u}^\infty(\mathbf{r},t)$, where $\mathbf{v}^\infty(\mathbf{x},t) = \mathbf{U}^\infty + \mathbf{u}^\infty(\mathbf{r},t)$ and $\mathbf{U}^\infty = \mathbf{v}^\infty(\mathbf{Y}_p(t),t)$ is the fluid velocity evaluated at the current particle location.

The disturbance flow equations relative to a particle-fixed observer enter frequently in the analysis below. Denoting the disturbance velocity, pressure, and stress by $(\mathbf{u},p,\boldsymbol{\sigma})$, the disturbance flow satisfies

$$\nabla^2\mathbf{u} - \boldsymbol{\nabla} p = \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} = Re\left[Sl\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u} + \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u}^\infty + \mathbf{u}^\infty\cdot\boldsymbol{\nabla}\mathbf{u} + \left(\mathbf{U}^\infty - \mathbf{U}_p\right)\cdot\boldsymbol{\nabla}\mathbf{u}\right],$$
$$= \mathbf{f}(\mathbf{u},\mathbf{u}^\infty), \tag{2.82a}$$

and

$$\boldsymbol{\nabla}\cdot\mathbf{u} = 0, \tag{2.82b}$$

where we have assumed $\mathbf{v}^\infty$ satisfies the Navier–Stokes equations. The function $\mathbf{f}$ will be used throughout to represent the inertial terms in the equations of motion. This term clearly depends on the details of the undisturbed fluid motion and the disturbance flow generated by the particle. Also, the disturbance flow problem satisfies the boundary conditions

$$\mathbf{u} = \mathbf{U}_p - \mathbf{v}^\infty(\mathbf{x},t) \qquad \text{for } \mathbf{r} = \mathbf{x} - \mathbf{Y}_p(t) \in S_p \tag{2.83a}$$

and

$$\mathbf{u},\ p \to 0 \qquad \text{as } r = |\mathbf{r}| \to \infty. \tag{2.83b}$$

Notice that the disturbance velocity field $\mathbf{u}(\mathbf{r}, t)$ describes fluid motion in the vicinity of the particle relative to an observer moving along the streamline, which, at time t, passes through the particle center. The final term in equation (2.82a) represents a slip between the actual particle velocity and the undisturbed flow evaluated at the instantaneous location of the particle.

Translational accelerations of the particle arise to balance the difference between the hydrodynamic and external forces, $\mathbf{F}^{ext}$. The hydrodynamic force $\mathbf{F}^{H}$ exerted by the fluid on the particle is given by

$$Re\,Sl\,\widetilde{m}_p\dot{\mathbf{U}}_p - \mathbf{F}^{ext} = \int_{S_p} \mathbf{n}\cdot\boldsymbol{\sigma}_T\,dS = \mathbf{F}^{H}, \tag{2.84}$$

where $\widetilde{m}_p$ is the mass of the particle nondimensionalized by the product of the density of the fluid, ρ, and the cube of the characteristic particle dimension, a^3. Here, $\boldsymbol{\sigma}_T$ denotes the stress tensor for the actual flow about the particle.

We now outline the use of the reciprocal theorem to derive a general expression, including inertial effects, for the hydrodynamic force as a function of the time-dependent undisturbed velocity field. This application of the reciprocal theorem is similar to the procedure used by previous investigators [12, 20, 30]. These earlier studies were interested in calculating the inertial contribution to particle translational and rotational velocities for situations where the Reynolds number was sufficiently small that a rigid boundary was reached prior to an $O(Re^{-\frac{1}{2}})$ distance where fluid inertia becomes significant. Consequently, the boundary was located in the "inner" region and a regular perturbation method accounting for inertial effects was applicable. Contrary to statements made in earlier studies, however, here we show it is possible to use the reciprocal theorem in those instances where fluid inertia enters directly and, in principle, a solution for the detailed flow field would require singular perturbation methods. In this case, we require that the "outer" region where fluid inertia becomes significant is closer to the particle than any boundaries; i.e., $O(Re^{-\beta}) < L/a$, where L is a representative geometric length scale and where, for example, $\beta = 1$ for uniform flow and $\beta = 1/2$ for linear flow.

Consider two solutions of the Navier–Stokes equations $(\mathbf{u}, \boldsymbol{\sigma}, \mathbf{f})$ and $(\hat{\mathbf{u}}, \hat{\boldsymbol{\sigma}}, \hat{\mathbf{f}})$ where $\mathbf{f}$ and $\hat{\mathbf{f}}$ represent the inertial (steady and unsteady) terms. Also, $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$ and $\boldsymbol{\nabla} \cdot \hat{\mathbf{u}} = 0$. The reciprocal theorem states that at any time t,

$$\int_S (\mathbf{n} \cdot \boldsymbol{\sigma}) \cdot \hat{\mathbf{u}} \, dS + \int_{V_f} \mathbf{f} \cdot \hat{\mathbf{u}} \, dV = \int_S (\mathbf{n} \cdot \hat{\boldsymbol{\sigma}}) \cdot \mathbf{u} \, dS + \int_{V_f} \hat{\mathbf{f}} \cdot \mathbf{u} \, dV \tag{2.85}$$

where S represents all bounding surfaces, $S = S_p + S_\infty$ and $\mathbf{n}$ in the unit outward normal directed from S into the fluid volume V_f. We will neglect the presence of any walls since here we are concerned with an isolated particle in an unbounded prescribed flow.

Let the unhatted flow correspond to the disturbance flow problem defined earlier by equation (2.82). Let $(\hat{\mathbf{u}}, \hat{\boldsymbol{\sigma}}, \hat{\mathbf{f}})$ correspond to the *Stokes flow problem* of a rigid particle translating with velocity $\hat{\mathbf{U}}$:

$$\boldsymbol{\nabla} \cdot \hat{\mathbf{u}} = 0, \quad \boldsymbol{\nabla} \cdot \hat{\boldsymbol{\sigma}} = 0, \quad \hat{\mathbf{f}} = 0, \tag{2.86a}$$

with

$$\hat{\mathbf{u}} = \hat{\mathbf{U}} \quad \text{for} \quad \mathbf{r} \in S_p \quad \text{and} \quad \hat{\mathbf{u}} \to 0 \quad \text{as} \quad |\mathbf{r}| \to \infty. \tag{2.86b}$$

Owing to the linearity of the governing equations and boundary conditions, the solution of this fundamental Stokes flow problem will take the form

$$\hat{\mathbf{u}}(\mathbf{r}) = \hat{\mathbf{M}} \cdot \hat{\mathbf{U}} \tag{2.87}$$

where the second-rank tensor $\hat{\mathbf{M}}$ is a function of position; for a spherical particle it has the well-known form

$$\hat{\mathbf{M}} = \frac{3}{4}\Big(\frac{\mathbf{I}}{r} + \frac{\mathbf{rr}}{r^3}\Big) + \frac{1}{4}\Big(\frac{\mathbf{I}}{r^3} - \frac{3\mathbf{rr}}{r^5}\Big). \tag{2.88}$$

The disturbance Stokes flow problem has the familiar properties

$$|\hat{\mathbf{u}}| \sim O(1/r) \quad \text{as} \quad r \to \infty, \qquad |\hat{\boldsymbol{\sigma}}| \sim O(1/r^2) \quad \text{as} \quad r \to \infty. \tag{2.89}$$

Although (2.82) cannot be solved exactly (unless $Re \equiv 0$), nevertheless for any Reynolds number it is clear that for large r

$$|\mathbf{u}| \sim O(r^{-\alpha}), \qquad \text{with} \quad \alpha > 0. \tag{2.90}$$

Hence, the integral over S_∞ in (2.85) involving $(\mathbf{n} \cdot \hat{\boldsymbol{\sigma}}) \cdot \mathbf{u}$ will vanish as the surface S_∞ moves off to infinity. Similarly, the viscous stress associated with $\boldsymbol{\sigma}$ will scale as $|\boldsymbol{\nabla}\mathbf{u}| \sim O(r^{-\alpha-1})$, and thus this part of $(\mathbf{n} \cdot \boldsymbol{\sigma}) \cdot \hat{\mathbf{u}}$ on S_∞ will also give zero. Thus, we are left to consider the integral

$$-\int_{S_\infty} p\mathbf{n} \cdot \hat{\mathbf{u}} \; dS.$$

If in the far field viscous terms are larger than or the same order as inertial terms, then p will also scale as $r^{-\alpha-1}$ and this integral will not contribute. If there are regions in the outer flow that are inviscid, then either these regions will be irrotational or rotational and the pressure must be estimated from Bernoulli's equation. In this case it is not obvious that p decays sufficiently rapidly to neglect this integral, although it seems most likely that this will be the case.

For the calculations performed in this paper, which are perturbations about the zero-Reynolds-number solution, p scales viscously and we may safely neglect all terms which involve S_∞. Thus, the reciprocal theorem (2.85) may now be written in the simpler 'disturbance' form

$$\int_{S_p} (\mathbf{n} \cdot \boldsymbol{\sigma}) \cdot \hat{\mathbf{u}}\, dS + \int_{V_f} \mathbf{f} \cdot \hat{\mathbf{u}}\, dV = \int_{S_p} (\mathbf{n} \cdot \hat{\boldsymbol{\sigma}}) \cdot \mathbf{u}\, dS, \tag{2.91}$$

which, with the caveat about inviscid flow, is an exact statement valid at any time t for an isolated rigid particle in a prescribed unbounded flow at arbitrary particle Reynolds number Re.

We will now make use of the definition of the disturbance flow problems $\mathbf{u}$, which is the flow involving inertia, and $\hat{\mathbf{u}}$, which is the fundamental solution to typical Stokes flow problems, in order to derive a general equation relating the hydrodynamic force on the particle, to the undisturbed fluid motion $\mathbf{v}^\infty(\mathbf{x}, t)$.

Using (2.86b) for the boundary condition on $\hat{\mathbf{u}}$, it follows that

$$\int_{S_p} (\mathbf{n} \cdot \boldsymbol{\sigma}) \cdot \hat{\mathbf{u}}\, dS = \int_{S_p} \mathbf{n} \cdot \boldsymbol{\sigma}\, dS \cdot \hat{\mathbf{U}}. \tag{2.92}$$

Now it only remains to identify the disturbance flow stress as $\boldsymbol{\sigma} = \boldsymbol{\sigma}_T - \boldsymbol{\sigma}^\infty$, where $\boldsymbol{\sigma}_T$ is the stress tensor for the actual flow about the particle, $\boldsymbol{\sigma}^\infty$ is the stress tensor for the undisturbed motion, and so involves $\mathbf{v}^\infty$, and $\boldsymbol{\sigma}$ is defined by the disturbance flow problem (2.82). It follows that the term on the right-hand side of (2.92) is related to the particle force (see (2.84).). Thus, (2.92) becomes

$$\int_{S_p} (\mathbf{n} \cdot \boldsymbol{\sigma}) \cdot \hat{\mathbf{u}}\, dS = \mathbf{F}^H \cdot \hat{\mathbf{U}} - \int_{S_p} (\mathbf{n} \cdot \boldsymbol{\sigma}^\infty)\, dS \cdot \hat{\mathbf{U}}. \tag{2.93}$$

Applying the divergence theorem to the integral involving the undisturbed flow, we obtain

$$\int_{S_p} (\mathbf{n} \cdot \boldsymbol{\sigma}) \cdot \hat{\mathbf{u}} \, dS = \mathbf{F}^H \cdot \hat{\mathbf{U}} - \int_{V_p} (\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}^\infty) \, dV \cdot \hat{\mathbf{U}}. \tag{2.94}$$

The evaluation of this integral involving the undisturbed motion is most easily performed by referring all velocities to the laboratory frame. In this case $\mathbf{v}^\infty(\mathbf{x}, t)$ satisfies

$$\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}^\infty = Re \frac{D^\infty \mathbf{v}^\infty}{Dt} \tag{2.95a}$$

where

$$\frac{D^\infty}{Dt} \equiv Sl \frac{\partial}{\partial t} + \mathbf{v}^\infty(\mathbf{x}, t) \cdot \boldsymbol{\nabla}. \tag{2.95b}$$

The right-hand side of (2.95a) is a function of position $\mathbf{x}$ and when substituted in (2.94) yields

$$\int_{S_p} (\mathbf{n} \cdot \boldsymbol{\sigma}) \cdot \hat{\mathbf{u}} \, dS = \mathbf{F}^H \cdot \hat{\mathbf{U}} - Re \int_{V_p} \frac{D^\infty \mathbf{v}^\infty}{Dt} \, dV \cdot \hat{\mathbf{U}}. \tag{2.96}$$

Next consider the integral on the right-hand side of (2.91). Using the boundary condition on the surface of the particle associated with the disturbance flow problem (2.83a), we have

$$\int_{S_p} (\mathbf{n} \cdot \hat{\boldsymbol{\sigma}}) \cdot \mathbf{u} \, dS = \int_{S_p} \mathbf{n} \cdot \hat{\boldsymbol{\sigma}} \, dS \cdot \mathbf{U}_p - \int_{S_p} (\mathbf{n} \cdot \hat{\boldsymbol{\sigma}}) \cdot \mathbf{v}^\infty \, dS, \tag{2.97}$$

or

$$\int_{S_p} (\mathbf{n} \cdot \hat{\boldsymbol{\sigma}}) \cdot \mathbf{u} \, dS = \hat{\mathbf{F}} \cdot \mathbf{U}_p - \int_{S_p} (\mathbf{n} \cdot \hat{\boldsymbol{\sigma}}) \cdot \mathbf{v}^\infty \, dS \tag{2.98}$$

where $\hat{\mathbf{F}}$ is the force acting on the particle associated with the disturbance Stokes flow problem.

To proceed further to simplify (2.98) one can either use a known solution for the Stokes flow about the particle or one must have knowledge of the nature of the undisturbed motion and appeal to the symmetry properties of the resistance tensors associated with the particle force. We pursue the first approach now by considering the flow about a spherical particle. We will return to the second approach later for the case when the undisturbed flow is uniform.

On the surface of a sphere, the force per unit area in Stokes flow is given simply by

$$\mathbf{n}\cdot\hat{\boldsymbol{\sigma}}|_{r=1,\mathbf{r}=\mathbf{n}} = -\tfrac{3}{2}\hat{\mathbf{U}}. \tag{2.99}$$

Substituting (2.99) into the integral on the right-hand side of (2.98) yields

$$\int_{S_p}(\mathbf{n}\cdot\hat{\boldsymbol{\sigma}})\cdot\mathbf{u}\,dS = \left[-6\pi\mathbf{U}_p + \frac{3}{2}\int_{S_p}\mathbf{v}^\infty\,dS\right]\cdot\hat{\mathbf{U}}. \tag{2.100}$$

Here we have used the known result for a sphere that $\hat{\mathbf{F}} = -6\pi\hat{\mathbf{U}}$. Eliminating $\hat{\mathbf{U}}$ from the problem, we find that when (2.96) and (2.100) are combined in the reciprocal theorem (2.91) it yields the following relationship for a sphere:

$$\mathbf{F}^H - Re\int_{V_p}\frac{D^\infty\mathbf{v}^\infty}{Dt}\,dV = -6\pi\mathbf{U}_p + \frac{3}{2}\int_{S_p}\mathbf{v}^\infty\,dS - \int_{V_f}\mathbf{f}\cdot\hat{\mathbf{M}}\,dV. \tag{2.101}$$

Each of the integrals involving the undisturbed motion in equation (2.101) can be evaluated by using a Taylor series expansion about the instantaneous center of the particle $\mathbf{Y}_p(t)$ (assuming the velocity gradient variations are small over the length scale of the particle) and performing the integrations; here, quadratic variations are retained. Using (2.84), which balances the particle translational acceleration against the net forces, leads to the general expression for a sphere:

$$\begin{aligned}\frac{4\pi}{3}Re\left[\frac{\rho_p}{\rho_f}\dot{\mathbf{U}}_p - \left(\frac{D^\infty\mathbf{v}^\infty}{Dt}\right)_{\mathbf{Y}_p(t)} - \frac{1}{10}\nabla^2\left(\frac{D^\infty\mathbf{v}^\infty}{Dt}\right)_{\mathbf{Y}_p(t)}\right] &= \mathbf{F}^{ext} \\ -6\pi\left[\mathbf{U}_p - \mathbf{U}^\infty - \frac{1}{6}\nabla^2\mathbf{v}^\infty(\mathbf{Y}_p(t),t)\right] - \int_{V_f}\mathbf{f}\cdot\hat{\mathbf{M}}\,dV, &\end{aligned} \tag{2.102}$$

where ρ_p is the density of the particle. In principle, equation (2.102) is valid for any Reynolds number *Re* at any time *t*. Specifically, we have found this formulation useful for an examination of inertial corrections at low Reynolds numbers. It is a generalization incorporating all inertial influences of a result of Maxey and Riley [33], which was restricted to the unsteady Stokes equations.

Now for a *nonspherical* particle in a *uniform* undisturbed flow ($\mathbf{v}^\infty \equiv \mathbf{U}^\infty(t)$, $\mathbf{u}^\infty = 0$) (2.98) becomes

$$\int_{S_p} (\mathbf{n} \cdot \hat{\boldsymbol{\sigma}}) \cdot \mathbf{u} \, dS = \hat{\mathbf{F}} \cdot (\mathbf{U}_p - \mathbf{U}^\infty). \tag{2.103}$$

Using the resistance tensor form for Stokes flow, $\hat{\mathbf{F}}$ may be expressed as

$$\hat{\mathbf{F}} = -\mathbf{R}_{FU} \cdot \hat{\mathbf{U}}, \tag{2.104}$$

where $\mathbf{R}_{FU}$ is the second-rank tensor relating particle velocity to the drag. Since $\mathbf{R}_{FU}$ is a symmetric tensor, (2.103) is now given by

$$\int_{S_p} (\mathbf{n} \cdot \hat{\boldsymbol{\sigma}}) \cdot \mathbf{u} \, dS = \mathbf{F}_s^H \cdot \hat{\mathbf{U}}, \tag{2.105}$$

where $\mathbf{F}_s^H (= -\mathbf{R}_{FU} \cdot (\mathbf{U}_p - \mathbf{U}^\infty))$ is the actual Stokes drag acting on the particle. If we combine (2.96) with (2.105) in the reciprocal theorem (2.91) and eliminate $\hat{\mathbf{U}}$, we find the following expression for the hydrodynamic force acting on an arbitrarily shaped particle translating in a uniform flow:

$$\mathbf{F}^H(t) = \mathbf{F}_s^H(t) + Re\,Sl\, \widetilde{V}_p \dot{\mathbf{U}}^\infty(t) - \int_{V_f} \mathbf{f} \cdot \hat{\mathbf{M}} \, dV. \tag{2.106}$$

Here, $\widetilde{V}_p$ is the volume of particle nondimensionalized by a^3, and $\hat{\mathbf{M}} \cdot \hat{\mathbf{U}}$ is the Stokes velocity field caused by the nonspherical particle translating with velocity $\hat{\mathbf{U}}$.

As a final note, analogous expressions for the hydrodynamic torque and particle stresslet have also been derived with this approach by introducing a rotational velocity and rate-of-strain tensor, respectively, into the surface boundary condition (2.86b) for the disturbance Stokes flow problem [49].

2.11 Appendix: Scaling arguments for the uniformly valid velocity field

We identify three cases to be considered based on the relative magnitude of Re and $ReSl$: $ReSl \gg O(Re^2)$, $ReSl = O(Re^2)$, and $ReSl \ll O(Re^2)$. In each case, we attempt to rescale length to find if a dominant balance exists in the governing equation such that convective inertia is retained to leading order. That is, we wish to find "outer" velocity fields where convective inertial terms are of equal magnitude as viscous or unsteady inertial terms. In all cases the pressure will be rescaled so that it remains to satisfy continuity. Also, in order to obtain further scaling information for the outer velocity fields, we shall employ the method used by Saffman [46] in replacing the boundary conditions on the surface of the particle with force monopoles, dipoles, etc. at the particle center.

We shall be considering various velocity fields associated with different governing equations, and each field will have different dependent variables. The general convention for the pressure and velocity symbols are as follows: (*a*) (Π, $\mathbf{v}$) are for unsteady fields, while (p, $\mathbf{u}$) for steady fields; (*b*) a subscript "0" denotes a leading order field, while a subscript "1" denotes a corresponding velocity field contribution due solely to the convective inertia of the fluid, which is identically zero for zero Reynolds number; (*c*) a superscript "p" denotes a monopole, or point-force field; (*d*) a superscript "+" signifies that a convective inertial term is retained in the governing equation for the leading order field. Note that the unsub-(or unsuper-)scripted (p, $\mathbf{u}$) will continue to represent the solution to the full Navier–Stokes problem presented in Section 2.2.

2.11.1 $Sl \gg Re$:

When $Sl \gg Re$, or the characteristic time $\tau_c \ll \nu/U_c^2$, there is no rescaling of the governing equations for which the convective terms can be the same order as the viscous or unsteady terms. This reflects the fact that vorticity has not diffused out to the Oseen distance aRe^{-1}. Thus, a uniformly valid first approximation to the velocity field under these conditions will always be given by the unsteady Stokes equations

$$-\boldsymbol{\nabla}\Pi_0 + \nabla^2 \mathbf{v}_0 = ReSl\frac{\partial \mathbf{v}_0}{\partial t}, \tag{2.107a}$$

$$\boldsymbol{\nabla} \cdot \mathbf{v}_0 = 0, \tag{2.107b}$$

with

$$\mathbf{v}_0 = \mathbf{U}_s(t) \quad \text{on the surface of the particle,} \tag{2.107c}$$

$$\mathbf{v}_0,\ \Pi_0 \to 0 \quad \text{as} \quad |\mathbf{r}| \to \infty. \tag{2.107d}$$

The contribution from the convective inertia will then be simply an $O(Re)$ regular perturbation to unsteady Stokes flow. That is, if we take $(\Pi_0, \mathbf{v}_0)$ as the solution to the above unsteady Stokes equations, then the $O(Re)$ contribution to the velocity field due to convection, $(Re\Pi_1, Re\mathbf{v}_1)$, must satisfy

$$-\boldsymbol{\nabla}\Pi_1 + \nabla^2 \mathbf{v}_1 - ReSl\frac{\partial \mathbf{v}_1}{\partial t} = \mathbf{v}_0 \cdot \boldsymbol{\nabla}\mathbf{v}_0 - \mathbf{U}_s(t) \cdot \boldsymbol{\nabla}\mathbf{v}_0, \tag{2.108a}$$

$$\boldsymbol{\nabla} \cdot \mathbf{v}_1 = 0, \tag{2.108b}$$

$$\mathbf{v}_1 = 0 \quad \text{on the surface of the particle,} \tag{2.108c}$$

$$\mathbf{v}_1,\ \Pi_1 \to 0 \quad \text{as } |\mathbf{r}| \to \infty. \tag{2.108d}$$

Although this correction is $O(Re)$ and therefore one would expect that when using the reciprocal theorem (2.9) this field could be neglected, it is needed in order to get the proper $O(Re)$ correction to the unsteady, $ReSl\partial\mathbf{u}/\partial t$, term in (2.10).

Other than the requirement that $ReSl \gg O(Re^2)$, the above arguments are for arbitrary magnitudes of $ReSl$. If, however, we have the additional condition that $ReSl \ll O(1)$, which is equivalent to having a time scale much longer than the diffusive scale a^2/ν, one can apply a singular perturbation analysis to the solution of the unsteady Stokes equations. This is done in Section 2.8 in the main text for the case of non-spherical particles. We only note here that if $ReSl \ll O(1)$, the governing equations for $\mathbf{v}_0$ near the particle will be to leading order the steady Stokes equations:

$$-\boldsymbol{\nabla} p_0 + \nabla^2 \mathbf{u}_0 = 0, \tag{2.109a}$$

$$\boldsymbol{\nabla} \cdot \mathbf{u}_0 = 0, \tag{2.109b}$$

$$\mathbf{u}_0 = \mathbf{U}_s(t) \quad \text{on the surface of the particle}, \tag{2.109c}$$

$$\mathbf{u}_0,\ p_0 \to 0 \quad \text{as} \quad |\mathbf{r}| \to \infty. \tag{2.109d}$$

While far from the particle, at distances of $O(a(ReSl)^{-\frac{1}{2}})$, they will be given to leading order by point-forced unsteady Stokes equations:

$$-\boldsymbol{\nabla}\Pi_0^p + \nabla^2 \mathbf{v}_0^p = ReSl\frac{\partial \mathbf{v}_0^p}{\partial t} + \mathbf{F}_s^H(t)\delta(\mathbf{r}), \tag{2.110a}$$

$$\boldsymbol{\nabla} \cdot \mathbf{v}_0^p = 0, \tag{2.110b}$$

where $\mathbf{F}_s^H(t)$ is the dimensionless pseudo-steady Stokes drag acting on the particle and $\delta(\mathbf{r})$ is the three dimensional dirac delta function. Convection will enter as a regular perturbation to these fields in their respective regions of validity.

2.11.2 $Sl = O(Re)$:

When $Sl = O(Re)$, there is a dominant balance between the viscous and both the unsteady and convective inertial terms. This signals the existence of a singular perturbation expansion in small Re. The time scale corresponding to this condition is $\tau_c \approx \nu/{U_c}^2$, and was also identified by Bentwich and Miloh [5] in their analysis. If we define outer variables as

$$\overline{\mathbf{r}} = Re\mathbf{r}, \quad \overline{p} = Re^{-2}p, \quad \overline{\mathbf{u}} = Re^{-1}\mathbf{u}, \tag{2.111}$$

the full governing equations for this "outer" region become

$$\begin{aligned} -\overline{\boldsymbol{\nabla}}\overline{p} + \overline{\nabla}^2\overline{\mathbf{u}} =& \frac{\partial \overline{\mathbf{u}}}{\partial t} - \mathbf{U}_s(t)\cdot\overline{\boldsymbol{\nabla}}\overline{\mathbf{u}} + \mathbf{F}_1(t)\delta(\overline{\mathbf{r}}) \\ &+ Re\overline{\mathbf{u}}\cdot\overline{\boldsymbol{\nabla}}\overline{\mathbf{u}} + Re\mathbf{F}_2(t)\cdot\overline{\boldsymbol{\nabla}}\delta(\overline{\mathbf{r}}) + ..., \end{aligned} \tag{2.112a}$$

$$\overline{\boldsymbol{\nabla}}\cdot\overline{\mathbf{u}} = 0. \tag{2.112b}$$

Here, we have replaced the boundary condition on the surface of the particle by a series of multipoles. The first multipole $\mathbf{F}_1(t)$ is given by integrating (2.112a) over the volume of the particle to obtain

$$\mathbf{F}_1(t) = \mathbf{F}^H(t) - ReSl\widetilde{V}_p\dot{\mathbf{U}}_p(t), \tag{2.113}$$

where $\mathbf{F}^H(t)$ is the full hydrodynamic force acting on the particle, nondimensionalized by $\mu a U_c$. The volume of the particle $\widetilde{V}_p$ has been nondimensionalized by a^3. In this case, $\mathbf{F}_1(t)$ can be approximated by the pseudo-steady Stokes drag, $\mathbf{F}^H_s(t)$, to leading order. The second multipole $\mathbf{F}_2(t)$ is a second-order tensor which is given by the stresslet and rotlet acting on the particle. The remaining terms (not shown) represent the higher order multipoles which are of lower order in Re. Thus, our leading order outer equations when $ReSl = O(Re^2)$ are given, in the original scaled variables, by

$$-\nabla\Pi_0^{p+} + \nabla^2\mathbf{v}_0^{p+} = ReSl\frac{\partial\,\mathbf{v}_0^{p+}}{\partial\,t} - Re\mathbf{U}_s(t)\cdot\nabla\mathbf{v}_0^{p+} + \mathbf{F}_s^H(t)\delta(\mathbf{r}), \tag{2.114a}$$

$$\nabla\cdot\mathbf{v}_0^{p+} = 0. \tag{2.114b}$$

The governing equations for the inner region (i.e., for $|\mathbf{r}| < O(Re^{-1})$) are the steady Stokes equations given by (2.109).

2.11.3 $Sl \ll Re$:

When $Sl \ll Re$, or $\tau_c \gg \nu/U_c^2$, again, to leading order, the velocity field near to the particle is given by the steady Stokes equations. There are two possibilities, however, for producing a balance of terms through rescaling that includes convective and viscous terms. The first results from the rescaling given by (2.111), which yields the steady Oseen equations to leading order:

$$-\nabla p_0^+ + \nabla^2\mathbf{u}_0^+ = -Re\mathbf{U}_s(t)\cdot\nabla\mathbf{u}_0^+ + \mathbf{F}_s^H(t)\delta(\mathbf{r}), \quad \nabla\cdot\mathbf{u}_0^+ = 0. \tag{2.115}$$

The other possibility comes from a balance of the unsteady and convective inertial terms with viscous terms in a region that is $O(aSl^{-1})$ from the particle.

To discover this second outer region, the length in the direction of $\mathbf{U}_s(t)$, considered constant on this long time scale, is rescaled to balance the unsteady and convective inertial terms. Then, the other two mutually orthogonal directions are rescaled to retain diffusive (or viscous) terms, i.e., the classic scalings for an unsteady laminar wake. The pressure gradient scale is determined by using these length scales in the appropriate expression for the pressure created by a Stokeslet (i.e., $\mathbf{F}_s^H\cdot\mathbf{x}/4\pi r^3$), as the pressure variation takes this form to leading order in the outer region. Finally, the velocity scales are determined by balancing dominant terms with the pressure gradient in the individual component momentum equations. For example, if we take the z-direction as the direction of $\mathbf{U}_s(t)$, then the z-momentum equation becomes to leading order

$$-\frac{\partial \overline{p}}{\partial \overline{z}}+\left(\frac{\partial^2}{\partial \overline{x}^2}+\frac{\partial^2}{\partial \overline{y}^2}\right)\overline{u}_z=\left(\frac{\partial}{\partial t}-U_s\frac{\partial}{\partial \overline{z}}\right)\overline{u}_z. \tag{2.116}$$

The rescaling in this equation is given by

$$(x,\ y)=(ReSl)^{-\frac{1}{2}}(\overline{x},\ \overline{y}),\quad z=Sl^{-1}\overline{z}, \tag{2.117a}$$

and

$$\frac{\partial p}{\partial z}=Sl^3\frac{\partial \overline{p}}{\partial \overline{z}},\quad u_z=\frac{Sl^2}{Re}\overline{u}_z. \tag{2.117b}$$

The length scales associated with these equations are larger than those in the steady Oseen equations above, $O(aSl^{-1})$ vs. $O(aRe^{-1})$. Thus they are associated with a farther field region than the Oseen region which applies to the wake immediately behind the particle. As seen in Section 2.4, the inclusion of the transient term in the governing equation for this new region of the flow has consequence for the temporal decay of the drag.

The above scaling analysis has identified the form of the outer region depending on the relative magnitudes of $ReSl$ and Re^2. With this information we now proceed to construct a velocity field that is uniformly valid in space for all times. This velocity field will be used in the volume integral over the entire fluid region when computing the inertial correction to the steady Stokes drag through the reciprocal theorem (2.9).

The field given by the unsteady Stokes equations, $\mathbf{v}_0$, is uniformly valid for short times, $ReSl \gg O(Re^2)$, and we can add to this the $O(Re)$ regular perturbation, $\mathbf{v}_1$, to include convective effects; specifically, to get the $O(Re)$ correction to $\partial \mathbf{u}/\partial t$. Our uniformly valid velocity field for short time scales is then

$$\mathbf{u}_s^{uv}=\mathbf{v}_0+Re\mathbf{v}_1, \tag{2.118}$$

where $\mathbf{v}_0$ and $\mathbf{v}_1$ are given by solutions to (2.107) and (2.108), respectively.

For long times when $ReSl \leq O(Re^2)$, we must recognize that we have a singular asymptotic expansion. Near the particle we have the field governed by the steady Stokes equations (2.109). Far from the particle we have, at most general, the unsteady Oseen equations (2.114). (Note that the unsteady Oseen flow will asymptote to (2.115) or (2.116) at long times, $ReSl \ll Re^2$.) Under these conditions, to construct a uniformly valid velocity field, one must add these two velocity fields and subtract their common parts. Since the unsteady Oseen equation is point-forced, the associated velocity field will reduce to the point-forced Stokes field, or Stokeslet, as the particle center is approached. Similarly, the inner Stokes field will reduce to the same Stokeslet at large distances from the particle. It is this Stokeslet that is common to both the inner and outer fields in their region of overlap, when $|\mathbf{r}| = O(aRe^{-1})$. This Stokeslet field is given by the solution to

$$-\nabla p_0^p + \nabla^2 \mathbf{u}_0^p = \mathbf{F}_s^H(t)\delta(\mathbf{r}), \quad \nabla \cdot \mathbf{u}_0^p = 0. \tag{2.119}$$

Thus, the uniformly valid velocity field for long time scales is given by

$$\mathbf{u}_l^{uv} = \mathbf{u}_0 + \mathbf{v}_0^{p+} - \mathbf{u}_0^p, \tag{2.120}$$

where $\mathbf{u}_0$ and $\mathbf{v}_0^{p+}$ are given by solutions to (2.109) and (2.114), respectively.

Now, to construct a velocity field valid for all time scales, we must consider the two velocity fields, (2.118) and (2.120), which are uniformly valid in space albeit for different time scales, and determine those features which are common to both. For long times (i.e., $ReSl = Re^2$), $\mathbf{v}_0$ will reduce to $(\mathbf{u}_0 + \mathbf{v}_0^p - \mathbf{u}_0^p)$, where $(\mathbf{v}_0^p - \mathbf{u}_0^p)$ represents the singular perturbation to Stokes flow from weak unsteadiness. Recall $\mathbf{v}_0^p$ is given by the solution to point-forced unsteady Stokes equations (2.110). For short times (i.e., $ReSl \gg Re^2$), $\mathbf{u}_l^{uv}$ will reduce to $(\mathbf{u}_0 + \mathbf{v}_0^p - \mathbf{u}_0^p)$ to leading order as well. This condition exists because the convective term in the governing equation for $\mathbf{v}_0^{p+}$, (2.114), is only a small correction that would not modify the $\mathbf{v}_0^p$-field to leading order anywhere in the fluid domain, as discussed in Section 2.11.1 above. Thus, $\mathbf{v}_0^{p+}$ reduces to $\mathbf{v}_0^p$ under the short time scales corresponding to this condition.

In addition, there is another portion of $\mathbf{v}_0^{p+}$ that is also common to $Re\mathbf{v}_1$. That is, the point-forced regular perturbation field, $Re\mathbf{v}_1^p$, given by

$$-\boldsymbol{\nabla}\Pi_1^p + \nabla^2\mathbf{v}_1^p - ReSl\frac{\partial \mathbf{v}_1^p}{\partial t} = -\mathbf{U}_s(t)\cdot\boldsymbol{\nabla}\mathbf{v}_0^p, \quad \boldsymbol{\nabla}\cdot\mathbf{v}_1^p = 0. \tag{2.121}$$

Recall that the superscript "+" means that convective terms are retained at leading order. Thus, the uniformly valid velocity field valid for all time scales is given by

$$\begin{aligned}\mathbf{u}^{uv} &= \mathbf{u}_s^{uv} + \mathbf{u}_l^{uv} - (\mathbf{u}_0 + \mathbf{v}_0^p - \mathbf{u}_0^p) - Re\mathbf{v}_1^p \\ &= \mathbf{v}_0 + Re\mathbf{v}_1 + Re\mathbf{v}_1^{p+} - Re\mathbf{v}_1^p,\end{aligned} \tag{2.122}$$

where

$$Re\mathbf{v}_1^{p+} = \mathbf{v}_0^{p+} - \mathbf{v}_0^p, \tag{2.123}$$

which is given by the solution to

$$-\boldsymbol{\nabla}\Pi_1^{p+} + \nabla^2\mathbf{v}_1^{p+} - ReSl\frac{\partial \mathbf{v}_1^{p+}}{\partial t} + Re\mathbf{U}_s(t)\cdot\boldsymbol{\nabla}\mathbf{v}_1^{p+} = -\mathbf{U}_s(t)\cdot\boldsymbol{\nabla}\mathbf{v}_0^p, \tag{2.124a}$$

$$\boldsymbol{\nabla}\cdot\mathbf{v}_1^{p+} = 0. \tag{2.124b}$$

2.12 Appendix: The Fourier transformed outer velocity field

In this section we solve for the Fourier transformed outer velocity field, $\widehat{\mathbf{v}_1^{p+}}$, and the resulting quantity, $\mathbf{f}_{out}(\widehat{Re\mathbf{v}_1^{p+}})$. We begin by taking the Fourier transform of (2.110a):

$$-2\pi\mathrm{i}\mathbf{k}\widehat{\Pi_0^p} - 4\pi^2k^2\widehat{\mathbf{v}_0^p} = ReSl\frac{\partial\widehat{\mathbf{v}_0^p}}{\partial t} + \mathbf{F}_s^H(t). \tag{2.125}$$

If we then make use of the continuity equation, which takes the form in Fourier space

$$\mathbf{k}\cdot\widehat{\mathbf{v}_0^p} = 0, \tag{2.126}$$

one finds the Fourier transformed pressure by taking the dot product of (2.125) with $\mathbf{k}$:

$$\widehat{\Pi_0^p} = \frac{\mathbf{k}\cdot\mathbf{F}_s^H(t)}{-2\pi\mathrm{i}k^2}. \tag{2.127}$$

Combining this with (2.125) and rearranging we have

$$\frac{\partial\widehat{\mathbf{v}_0^p}}{\partial t} + \frac{4\pi^2k^2}{ReSl}\widehat{\mathbf{v}_0^p} = \frac{\mathbf{F}_s^H(t)}{ReSl}\cdot(\mathbf{n}_k\mathbf{n}_k - \mathbf{I}), \tag{2.128}$$

where $\mathbf{n}_k = \mathbf{k}/k$. The solution to this equation is then our solution for $\widehat{\mathbf{v}_0^p}$:

$$\widehat{\mathbf{v}_0^p} = \int_{-\infty}^{t}\frac{\mathbf{F}_s^H(s)}{ReSl}\mathrm{e}^{-4\pi^2k^2(t-s)/ReSl}\,ds\cdot(\mathbf{n}_k\mathbf{n}_k - \mathbf{I}). \tag{2.129}$$

Next we take the Fourier transform of (2.124a) and obtain

$$-2\pi\mathrm{i}\mathbf{k}\widehat{\Pi_1^{p+}} - 4\pi^2k^2\widehat{\mathbf{v}_1^{p+}} = ReSl\frac{\partial\widehat{\mathbf{v}_1^{p+}}}{\partial t} - 2\pi\mathrm{i}Re\mathbf{U}_s(t)\cdot\mathbf{k}\widehat{\mathbf{v}_1^{p+}} - 2\pi\mathrm{i}\mathbf{U}_s(t)\cdot\mathbf{k}\widehat{\mathbf{v}_0^p}. \tag{2.130}$$

From the application of the continuity equation we find

$$\widehat{\Pi_1^{p+}} = 0. \tag{2.131}$$

Now (2.130) takes the following form

$$\frac{\partial \widehat{\mathbf{v}_1^{p+}}}{\partial t} + (\frac{4\pi^2 k^2 - 2\pi \mathrm{i} Re \mathbf{U}_s(t)\cdot\mathbf{k}}{Re Sl})\widehat{\mathbf{v}_1^{p+}} = \frac{2\pi \mathrm{i} \mathbf{U}_s(t)\cdot\mathbf{k}}{Re Sl}\widehat{\mathbf{v}_0^{p}}, \tag{2.132}$$

and performing the integration and substituting for $\widehat{\mathbf{v}_0^p}$ from (2.129), we obtain

$$\begin{aligned}\widehat{\mathbf{v}_1^{p+}} =& \Bigg[\int_{-\infty}^{t} \frac{2\pi \mathrm{i} \mathbf{U}_s(s)\cdot\mathbf{k}}{Re Sl}\left(\int_{-\infty}^{s}\frac{\mathbf{F}_s^H(q)}{Re Sl}\mathrm{e}^{-4\pi^2k^2(t-q)/Re Sl}\,d\,q\right) \\ &\times \mathrm{e}^{2\pi \mathrm{i} Re(\mathbf{Y}_s(t)-\mathbf{Y}_s(s))\cdot\mathbf{k}/Re Sl}\,d\,s\Bigg]\cdot(\mathbf{n}_k\mathbf{n}_k - \mathbf{I}).\end{aligned} \tag{2.133}$$

Now we can compute

$$\mathbf{f}_{out}(\widehat{Re\mathbf{v}_1^{p+}}) = Re\left(Re Sl\frac{\partial \widehat{\mathbf{v}_1^{p+}}}{\partial t} - 2\pi \mathrm{i} Re \mathbf{U}_s(t)\cdot\mathbf{k}\widehat{\mathbf{v}_1^{p+}}\right), \tag{2.134}$$

to find

$$\begin{aligned}\mathbf{f}_{out}(\widehat{Re\mathbf{v}_1^{p+}}) =& 2\pi \mathrm{i} Re \mathbf{U}_s(t)\cdot\mathbf{k}\int_{-\infty}^{t}\frac{\mathbf{F}_s^H(s)}{Re Sl}\mathrm{e}^{-4\pi^2k^2(t-s)/Re Sl}\,d\,s\cdot(\mathbf{n}_k\mathbf{n}_k - \mathbf{I}) \\ &-4\pi^2k^2\Bigg[\int_{-\infty}^{t}\frac{2\pi \mathrm{i} Re \mathbf{U}_s(s)\cdot\mathbf{k}}{Re Sl}\left(\int_{-\infty}^{s}\frac{\mathbf{F}_s^H(q)}{Re Sl}\mathrm{e}^{-4\pi^2k^2(t-q)/Re Sl}\,d\,q\right) \\ &\times \mathrm{e}^{2\pi \mathrm{i} Re(\mathbf{Y}_s(t)-\mathbf{Y}_s(s))\cdot\mathbf{k}/Re Sl}\,d\,s\Bigg]\cdot(\mathbf{n}_k\mathbf{n}_k - \mathbf{I}).\end{aligned} \tag{2.135}$$

Next we note that by performing an integration by parts we have

$$\begin{aligned}\int_{-\infty}^{s}\frac{\mathbf{F}_s^H(q)}{Re Sl}\mathrm{e}^{-4\pi^2k^2(t-q)/Re Sl}\,d\,q =& \frac{\mathbf{F}_s^H(s)}{4\pi^2k^2}\mathrm{e}^{-4\pi^2k^2(t-s)/Re Sl} \\ &- \int_{-\infty}^{s}\frac{\dot{\mathbf{F}}_s^H(q)}{4\pi^2k^2}\mathrm{e}^{-4\pi^2k^2(t-q)/Re Sl}\,d\,q.\end{aligned} \tag{2.136}$$

Thus combining (2.136) with (2.135) we have finally

$$
\begin{aligned}
\mathbf{f}_{out}(\widehat{Re\mathbf{v}_1^{p+}}) =& 2\pi\mathrm{i}Re\mathbf{U}_s(t)\cdot\mathbf{k}\int_{-\infty}^{t}\frac{\mathbf{F}_s^H(s)}{ReSl}\mathrm{e}^{-4\pi^2k^2(t-s)/ReSl}\,d\,s\cdot(\mathbf{n}_k\mathbf{n}_k-\mathbf{I})\\
&-\int_{-\infty}^{t}\frac{2\pi\mathrm{i}Re\mathbf{U}_s(s)\cdot\mathbf{k}}{ReSl}\mathbf{F}_s^H(s)\mathrm{e}^{-4\pi^2k^2(t-s)/ReSl}\\
&\qquad\times\mathrm{e}^{2\pi\mathrm{i}Re(\mathbf{Y}_s(t)-\mathbf{Y}_s(s))\cdot\mathbf{k}/ReSl}\,d\,s\cdot(\mathbf{n}_k\mathbf{n}_k-\mathbf{I})\\
&+\int_{-\infty}^{t}\int_{-\infty}^{s}\frac{2\pi\mathrm{i}Re\mathbf{U}_s(s)\cdot\mathbf{k}}{ReSl}\mathrm{e}^{2\pi\mathrm{i}Re(\mathbf{Y}_s(t)-\mathbf{Y}_s(s))\cdot\mathbf{k}/ReSl}\dot{\mathbf{F}}_s^H(q)\\
&\qquad\times\mathrm{e}^{-4\pi^2k^2(t-q)/ReSl}\,d\,q\,d\,s\cdot(\mathbf{n}_k\mathbf{n}_k-\mathbf{I}).
\end{aligned}
\tag{2.137}
$$

If we change the order of integration in the last integral of (2.137) and integrate with respect to s, we obtain the result in (2.23) in the main text.

Chapter 3

The force on a sphere in a uniform flow with small amplitude oscillations at finite Reynolds number

Summary

The unsteady force acting on a sphere that is held fixed in a steady uniform flow with small amplitude oscillations is evaluated to $O(Re)$ for small Reynolds number, Re. Good agreement is shown with the numerical results of Mei, Lawrence, and Adrian [37] up to $Re \approx 0.5$. The analytical result is transformed by Fourier inversion to allow for an arbitrary time-dependent motion which is small relative to the steady uniform flow. This yields a history-dependent force which has an integration kernel that decays exponentially for large time.

3.1 Introduction

Recently Mei, Lawrence, and Adrian [37] (hereinafter referred to as MLA) numerically computed the unsteady force acting on a spherical particle fixed in a fluid which has small fluctuations about its steady free-stream velocity. Specifically, the force was obtained numerically for the following imposed flow:

$$U^{\infty\prime}(t') = U(1 + \alpha_1 e^{-i\omega t'}), \tag{3.1}$$

with the condition $\alpha_1 \ll 1$. The primes are used to indicate dimensional quantities when there exists a corresponding nondimensional quantity elsewhere in the paper. The Reynolds number, Re, based on the particle radius, a, and free-stream velocity, U, ranged from zero up to 50 in their numerical study. For the low-frequencies, their results indicated that the force has a much shorter memory than that predicted by the Basset history integral from the unsteady Stokes solution.

Later, Mei and Adrian [35] (henceforth referred to as MA) evaluated the force analytically at small Reynolds number and low-frequency, ω, for the above imposed flow. A matched asymptotic solution was used in the limit $Sl_\omega \ll Re \ll 1$, where Sl_ω is the Strouhal number ($a\omega/U$). The results agreed well with the previous numerical study of MLA in this limit. Based on the results from both the numerical and analytical studies, a modified expression for the history force was proposed in the time domain. It had an integration kernel that decayed as t^{-2} at large time for both small and finite Reynolds numbers, as opposed to the $t^{-\frac{1}{2}}$ decay, associated with the Basset term for zero Reynolds number.

In the present study, we extend the above analytical results to arbitrary frequency (or Sl_ω), maintaining the requirement of $Re \ll 1$. This analysis is accomplished by making use of the previously obtained expression from Chapter 2 for the unsteady force acting on a particle in arbitrary motion (relative to the fluid) accurate to $O(Re)$. The derivation combines the general reciprocal theorem for the Navier–Stokes equations with a uniformly valid asymptotic expansion for the flow field. When the result is applied to the motion given by (3.1), it is found that the force agrees with both the analytical results of MA and the numerical results of MLA up to $Re \approx 0.5$. However, when the expression is transformed to account for arbitrary time-dependent motion, a history-dependent force with an integration kernel that decays exponentially at large time is obtained, in contrast to the proposed expression of MA which decays algebraically.

In what follows, we first derive the force expression in the frequency domain for the flow given by (3.1) and compare it to the results of MLA and MA. Next, in § 3, we generalize the expression to arbitrary time-dependent motion through Fourier inversion and evaluate the behavior at large times. We conclude in § 4 with a discussion of the results.

3.2 Evaluation of the force expression in the frequency domain

For a fixed spherical particle in a rectilinear imposed flow, $U^\infty(t)$, the hydrodynamic force derived in Chapter 2 reduces to

$$F^H(t) = 6\pi U^\infty(t) + 2\pi Re Sl\, \dot{U}^\infty(t) + \frac{9}{2}(Re Sl\,\pi)^{\frac{1}{2}}\left\{\int_0^1 \int_{-\infty}^t \left[\frac{2}{3}U^\infty(t) - \left\{\frac{1}{A^2}\left(e^{-A^2x^2} - e^{-A^2}\right)\right\}U^\infty(s)\right]\frac{d\,s}{(t-s)^{\frac{3}{2}}}\,d\,x\right\}, \tag{3.2}$$

where

$$A(t,s) = \frac{1}{2}\left(\frac{Re}{Sl}\right)^{\frac{1}{2}}\frac{\int_s^t U^\infty(q)\,d\,q}{(t-s)^{\frac{1}{2}}}. \tag{3.3}$$

The Reynolds number and Strouhal number are defined by

$$Re = \frac{aU_c}{\nu}, \quad Sl = \frac{a/U_c}{\tau_c}, \tag{3.4}$$

where U_c and τ_c are the characteristic velocity and time scales of the imposed flow and ν is the kinematic viscosity of the fluid. The force, $F^H(t)$, has been nondimensionalized by $a\mu U_c$, where μ is the viscosity of the fluid. The first term of (3.2) is the steady Stokes drag; the second represents a combination of the added mass and the force due to the accelerating imposed flow (which would have been exerted on the fluid displaced by the sphere); and the last term is a new history integral: it reduces to the steady Oseen correction for steady motion, and to the Basset history integral for short-time unsteady motion.

For the flow given by (3.1), we let $U_c = U$ and $\tau_c = \omega^{-1}$, allowing the dimensionless imposed flow to be expressed as

$$U^\infty(t) = (1 + \alpha_1 \mathrm{e}^{-\mathrm{i}t}). \tag{3.5}$$

If we use this flow in the force expression (3.2) and take the limit of $\alpha_1 \ll 1$, we obtain to $O(\alpha_1)$ after some tedious, but straightforward, manipulation and change of variables

$$\begin{aligned} F^H(t) = {} & 6\pi\left(1 + \frac{3}{8}Re + \alpha_1 \mathrm{e}^{-\mathrm{i}t}\right) - 2\pi \mathrm{i}\, Re Sl_\omega (\alpha_1 \mathrm{e}^{-\mathrm{i}t}) \\ & + \frac{9}{2} Re \pi^{\frac{1}{2}} \alpha_1 \mathrm{e}^{-\mathrm{i}t} \int_0^1 \int_0^\infty \left(\frac{1 - \mathrm{e}^{\mathrm{i}s\gamma_\omega}}{s\gamma_\omega}\right)\left(\frac{3}{2}\frac{\mathrm{e}^{-sx^2} - \mathrm{e}^{-s}}{s} - \mathrm{e}^{-s}\right)\frac{d\,s}{s^{\frac{3}{2}}}\,d\,x \\ & + \frac{9}{4} Re \pi \frac{1}{2} \alpha_1 \mathrm{e}^{-\mathrm{i}t} \int_0^1 \int_0^\infty \left[\frac{2}{3} - \frac{\mathrm{e}^{\mathrm{i}s\gamma_\omega}}{s}\left(\mathrm{e}^{-sx^2} - \mathrm{e}^{-s}\right)\right]\frac{d\,s}{s^{\frac{3}{2}}}\,d\,x, \end{aligned} \tag{3.6}$$

where $\gamma_\omega = 4Sl_\omega/Re$, and $Sl_\omega = a\omega/U$. We note that by taking the limit of small α_1 we have linearized the relationship between the time-dependent part of the velocity and the force, which will allow for Fourier inversion to the time domain in the next section. The above integrations were carried out using *Mathematica* to obtain

$$F^H(t) = 6\pi\left(1 + \frac{3}{8}Re + \alpha_1 \mathrm{e}^{-\mathrm{i}t}\right) - 2\pi \mathrm{i}\, Re Sl_\omega (\alpha_1 \mathrm{e}^{-\mathrm{i}t})$$

$$+6\pi Re\alpha_1 e^{-it}\frac{2^{\frac{1}{2}}(1-i)(\gamma_\omega+i)^{\frac{3}{2}}-i2}{4\gamma_\omega}. \tag{3.7}$$

If we expand this expression for small γ_ω (i.e., for small frequency such that $Sl_\omega \ll Re$), the force to $O(Sl_\omega)$ is

$$F^H(t) = 6\pi\left(1+\frac{3}{8}Re+\alpha_1 e^{-it}\right) - 2\pi i\, ReSl_\omega(\alpha_1 e^{-it})$$
$$+6\pi Re\alpha_1 e^{-it}\left(\frac{3}{4}-i\frac{3}{4}\frac{Sl_\omega}{Re}\right). \tag{3.8}$$

This expression agrees with the analytical result of MA. In addition, further terms in the low-frequency expansion are in integer powers of the frequency; the even powers are associated with the real coefficients of the imposed flow ($\alpha_1 e^{-it}$) and the odd powers with the imaginary coefficients.

To compare (3.7) with the numerical work of MLA, we define the following quantities based on their equivalence to those in MLA:

$$D_{1RAC} = \mathrm{Re}\left\{Re\frac{2^{\frac{1}{2}}(1-i)(\gamma_\omega+i)^{\frac{3}{2}}-i2}{4\gamma_\omega}\right\} - \frac{3}{4}Re, \tag{3.9}$$

$$\Delta_{1I} = -\mathrm{Im}\left\{Re\frac{2^{\frac{1}{2}}(1-i)(\gamma_\omega+i)^{\frac{3}{2}}-i2}{4\gamma_\omega}\right\}. \tag{3.10}$$

Here, D_{1RAC} represents the real part of the frequency-dependent drag coefficient and Δ_{1I} is the imaginary part of the frequency-dependent drag coefficient excluding the $-2\pi i\, ReSl_\omega$-term, both of which are nondimensionalized by $6\pi\mu a$. In figures 3.1(*a, b*) and 3.2(*a, b*) these quantities are plotted as a function of γ_ω for various Reynolds numbers, with the numerical data from MLA included for comparison. The same quantities scaled by the Reynolds number are presented as well to show the results may be collapsed on a single curve for small *Re*. The figures show good agreement of the analytical and numerical results up to $Re \approx 0.5$. This might appear somewhat surprising given that the force expression is valid strictly for the limit of infinitesimally small Reynolds number, its accuracy being only to $O(Re)$. We note, however, that a similar finding was made by Maxworthy [34] who determined that the experimentally observed terminal settling velocity of spheres were adequately predicted by the $O(Re)$-accurate Oseen approximation up to $Re \approx 0.4$.

3.3 Generalization of the force expression to arbitrary time-dependent motion

In order to evaluate the force for a small general time-dependent flow, we must consider α_1 as the Fourier transform of a small unsteady velocity, $U_1(t)$, which is superimposed on the steady uniform flow U, under the condition that $U_1(t) \ll U$ for all t. Then α_1 is a function of ω and is related to $U_1(t)$ by

$$U_1(t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} \alpha_1 \mathrm{e}^{-\mathrm{i}\omega t'}\, d\,\omega, \quad \alpha_1(\omega) = \int_{-\infty}^{\infty} U_1(s')\mathrm{e}^{-\mathrm{i}\omega s'}\, d\,s'. \qquad (3.11)$$

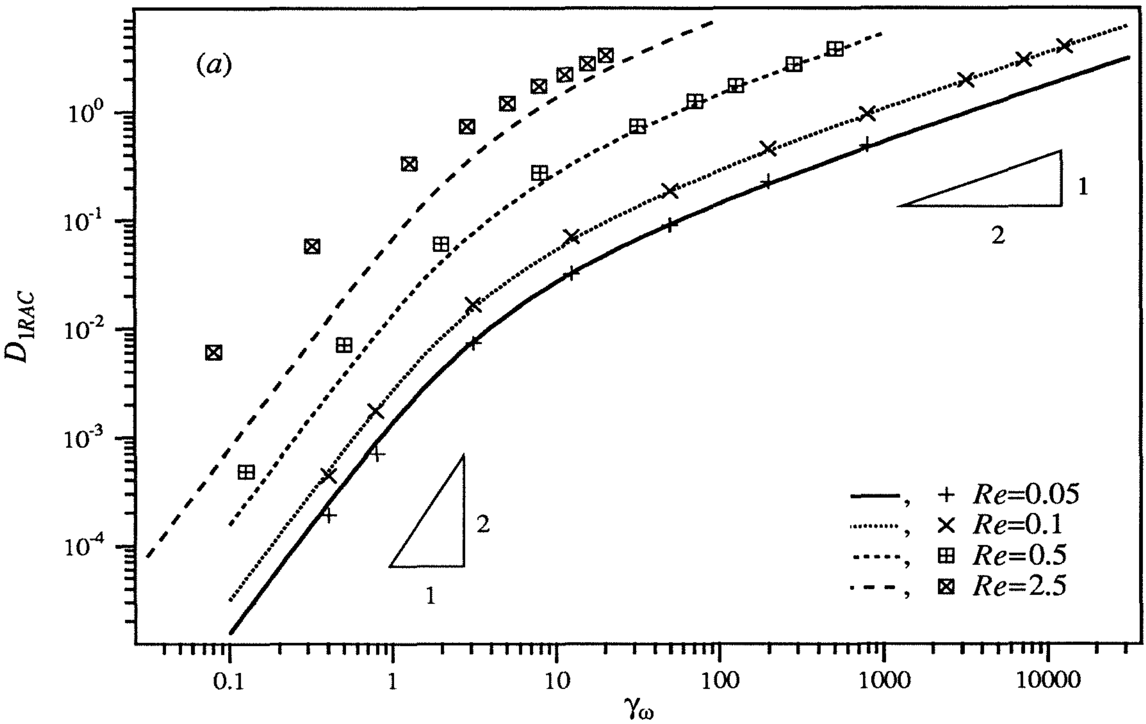
Thus the α_1-dependent part of the force expression (3.7) may be readily transformed to the time domain by integration with respect to ω to obtain

$$F^H(t) = 6\pi\left(1 + U_1(t) + \frac{3}{8}Re\,(1 + 2U_1(t)) + F'(t)\right) + 2\pi Re\,Sl\,\dot{U}_1(t), \qquad (3.12)$$

where

$$F'(t) = \frac{Re}{2\pi}\int_{-\infty}^{\infty} \dot{U}_1(s)\int_{-\infty}^{\infty} \frac{2 - 3\gamma_\omega \mathrm{i} + 2^{\frac{1}{2}}(1+\mathrm{i})(\gamma_\omega + \mathrm{i})^{\frac{3}{2}}}{4\gamma_\omega^2} \times \mathrm{e}^{-\mathrm{i}\gamma_\omega (t-s)/\gamma}\, d\,\gamma_\omega\, d\,s. \qquad (3.13)$$

Here Sl is as defined in (3.4) and $\gamma = 4Sl/Re$.



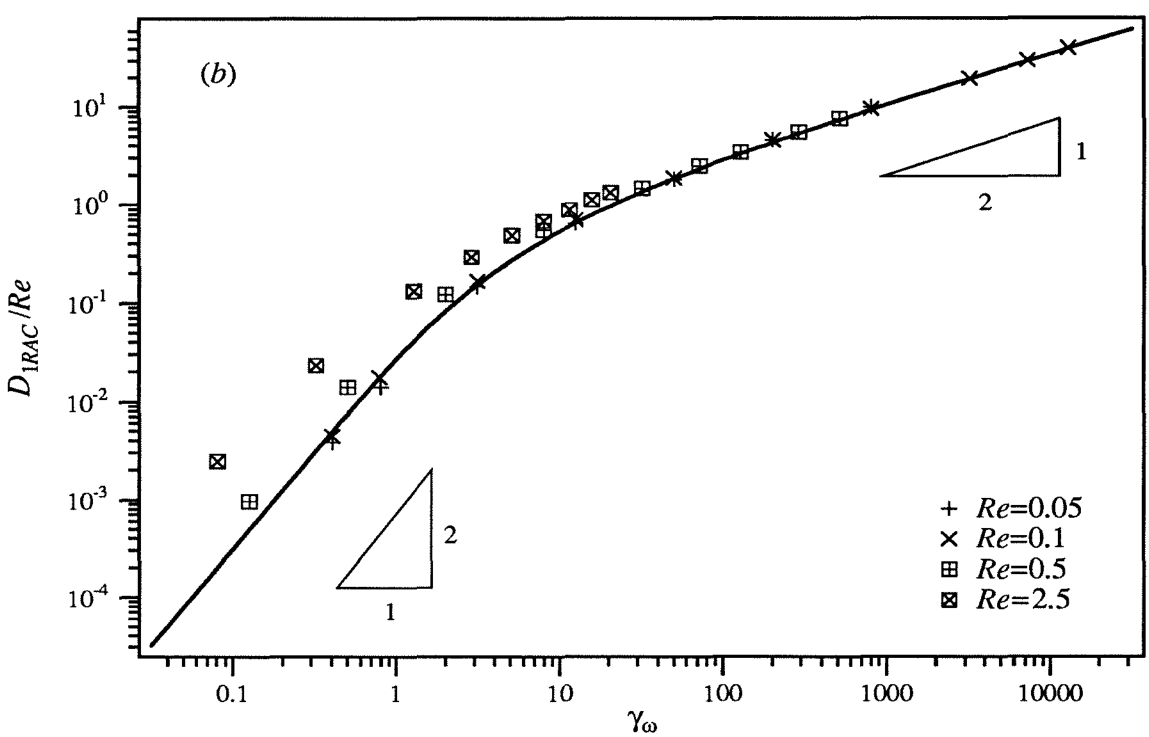


Figure 3.1: The real part of the acceleration-dependent drag coefficient for small amplitude oscillations about a uniform flow past a sphere as a function of the dimensionless frequency at various Reynolds numbers, (a) unscaled; (b) scaled. The lines are the analytical result (3.9) and the symbols are the numerical results of MLA.

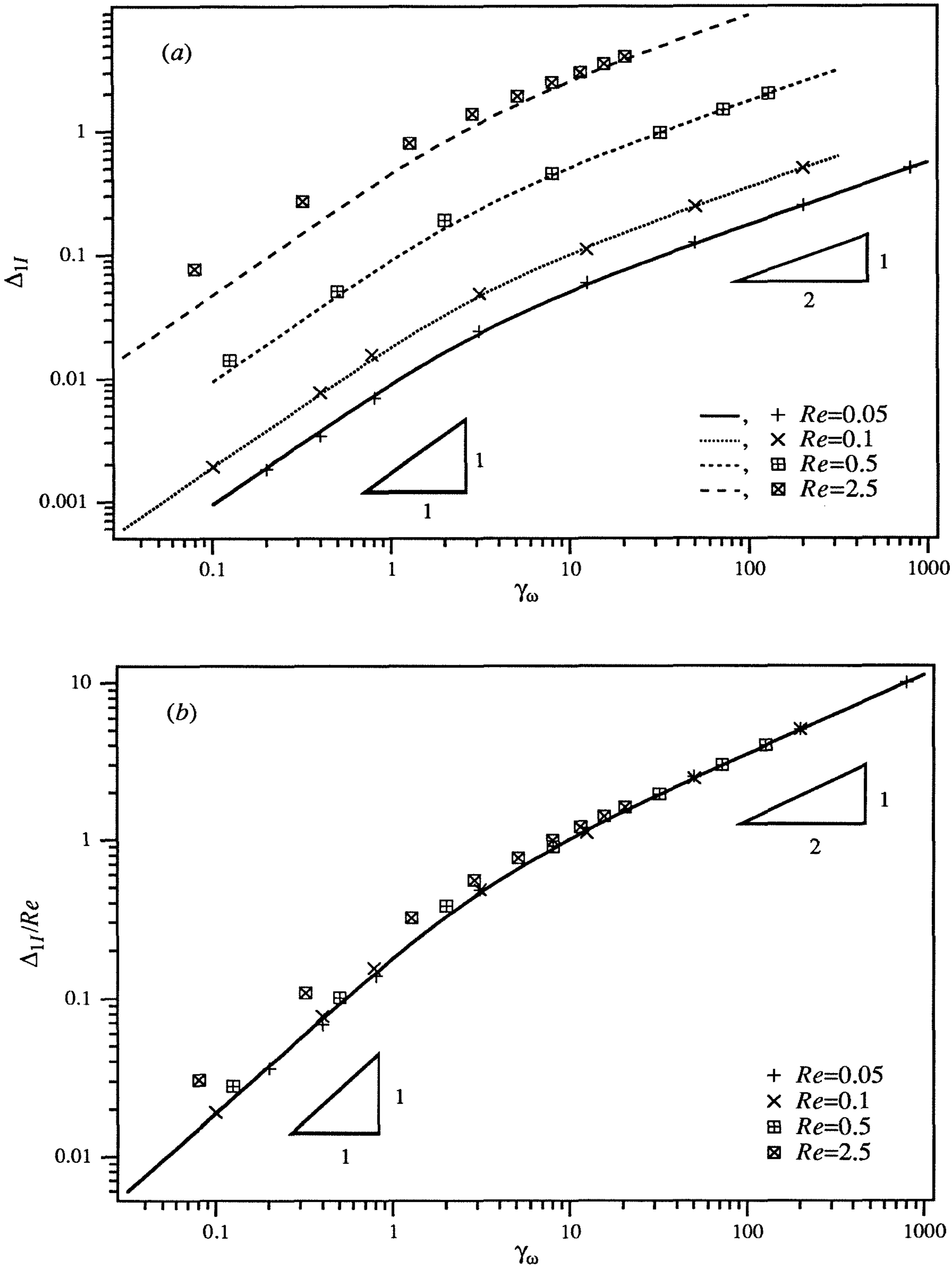


Figure 3.2: The difference between the imaginary part of the acceleration-dependent drag coefficient and $-2\pi \mathrm{i} Re Sl_\omega$ for small amplitude oscillations about a uniform flow past a sphere as a function of the dimensionless frequency at various Reynolds numbers, (a) unscaled; (b) scaled. The lines are the analytical result (3.10) and the symbols are the numerical results of MLA.

The γ_ω-integration in the expression for $F'(t)$ may be simplified by contour integration. The branch cut for the square root in the complex γ_ω-plane originates at $\gamma_\omega = -\mathrm{i}$ and extends along the negative imaginary axis to $-\mathrm{i}\infty$. The boundedness of the integrand in (3.13), particularly at the origin, means that its integration along any closed contour not crossing the branch cut must be zero. Therefore, the appropriate contours for $s > t$ and $s < t$ are in the upper and lower half-planes, respectively. The radius of the semicircular portions of the contours are taken to the limit of infinity, and it can be seen that there is no contribution from the integration along these parts of the contours. As expected, this implies there is no contribution to the integral when $s > t$. When $s < t$, the γ_ω-integration reduces to two integrals along each side of the branch cut:

$$\begin{aligned} F'(t) = \frac{Re}{2\pi} \int_{-\infty}^{t} & \dot{U}_1(s) \\ & \left[- \int_{-\mathrm{i}-\epsilon}^{-\mathrm{i}\infty-\epsilon} \frac{2 - 3\gamma_\omega \mathrm{i} + 2^{\frac{1}{2}}(1+\mathrm{i})(\gamma_\omega + \mathrm{i})^{\frac{3}{2}}}{4\gamma_\omega^2} \mathrm{e}^{-\mathrm{i}\gamma_\omega (t-s)/\gamma}\, d\gamma_\omega \right. \\ & \left. - \int_{-\mathrm{i}\infty+\epsilon}^{-\mathrm{i}+\epsilon} \frac{2 - 3\gamma_\omega \mathrm{i} + 2^{\frac{1}{2}}(1+\mathrm{i})(\gamma_\omega + \mathrm{i})^{\frac{3}{2}}}{4\gamma_\omega^2} \mathrm{e}^{-\mathrm{i}\gamma_\omega (t-s)/\gamma}\, d\gamma_\omega \right] d s, \end{aligned} \tag{3.14}$$

where ϵ is an infinitesimally small, real, positive number. If we set $x = \mathrm{i}\gamma_\omega - 1$ this expression simplifies to

$$F'(t) = \frac{Re}{2\pi} \int_{-\infty}^{t} G(t-s) \dot{U}_1(s)\, d s. \tag{3.15}$$

The integration kernel for this history force is given by

$$\begin{aligned} G(t) &= \mathrm{e}^{-t/\gamma} \int_0^\infty \frac{x^{\frac{3}{2}}}{(1+x)^2} \mathrm{e}^{-xt/\gamma}\, d x \\ &= \mathrm{e}^{-t/\gamma} \Gamma(\frac{5}{2}) \Psi(\frac{5}{2}, \frac{3}{2}, t/\gamma), \end{aligned} \tag{3.16}$$

where Ψ is a confluent hypergeometric function, sometimes known as the Kummer function [1].

The asymptotic properties of $G(t)$ for small and large time are

$$G(t) = (\frac{\gamma\pi}{t})^{\frac{1}{2}} - \frac{3}{2}\pi + O((\frac{t}{\gamma})^{\frac{1}{2}}), \quad \frac{t}{\gamma} \ll 1, \tag{3.17}$$

$$G(t) = \left[\frac{3\pi^{\frac{1}{2}}}{4}(\frac{\gamma}{t})^{\frac{5}{2}} + O((\frac{\gamma}{t})^{\frac{7}{2}})\right] \mathrm{e}^{-t/\gamma}, \quad \frac{t}{\gamma} \gg 1. \tag{3.18}$$

In dimensional time these limits are $t' \ll \nu/U^2$ and $t' \gg \nu/U^2$, where ν/U^2 represents the time it takes vorticity to diffuse out to, or be convected through, the Oseen distance ν/U. The integration kernel behaves as that in the Basset history integral for small time, but shows exponential decay for large time. Note that the second term of (3.17) will result in the canceling of the $\frac{3}{4}Re\, U_1(t)$-term in the other part of the force expression (3.12) when the time scale of the motion is small. We note also that the behavior for large time is in exact agreement with the result obtained in Chapter 2 wherein the temporal response was observed for the force when the velocity made a step change from one non-zero velocity to another.

3.4 Discussion of results

The reason MA obtained the algebraic decay t^{-2} instead of exponential decay for their integration kernel can be explained as follows: Their result is based on the inversion of a function that interpolates only the one-term asymptotic forms of the imaginary part of the history force in the low- and high-frequency limits. The problem with this is that the one term in the low-frequency limit, $-\frac{3}{4}Sl_\omega \mathrm{i}$, is insufficient to predict the long-time behavior of the integration kernel. Indeed, when inverted for time-dependent motion this term would yield the acceleration at the current time, which has no history dependence. Thus, their resultant integration kernel depends critically on the choice of interpolating functions; one can obtain a different decay by choosing a different interpolating function. In addition, by their own principle of causality, the imaginary part of the history force must be an odd function of the frequency. However, if their interpolated expression is expanded for low-frequency, an expansion in all powers of the frequency is obtained, not just the odd powers.

It is interesting to note that the force *does* decay as t^{-2} for a step change from a *zero* velocity, as can be observed from the result of Sano [47] . This distinction in decay rates is the result of the difference between the physical processes of the growth of the Oseen wake into essentially irrotational fluid, which is associated with algebraic decay, and the modification of the wake already established to infinite length, which is associated with exponential decay. In the case here, the wake clearly has been established by the uniform bulk flow U. Once the disturbance created by the small unsteady flow has diffused through the viscous Stokes region surrounding the particle, it is balanced exponentially fast by modification of the wake structure through convective transport mechanisms.

Chapter 4

The force on a bubble, drop, or particle in arbitrary time-dependent motion at small Reynolds number

Summary

The hydrodynamic force on a body that undergoes translational acceleration in an unbounded fluid at low Reynolds number is considered. The results extend the prior analysis of Chapter 2 for rigid particles to drops and bubbles. Similar behavior is shown in that, with the inclusion of convective inertia, the long-time temporal decay of the force (or the approach to steady state) at finite Reynolds number is faster than the $t^{-\frac{1}{2}}$ predicted by the unsteady Stokes equations.

4.1 Introduction

In Chapter 2 we analyzed the force on a rigid particle in arbitrary time-dependent motion in a time-dependent uniform flow for small, but finite, Reynolds number, *Re*. The primary conclusion of that study was that the long-time temporal behavior of the hydrodynamic force decays faster than the $t^{-\frac{1}{2}}$ decay associated with the Basset history integral from unsteady Stokes flow. (For short time scale motion, however, the unsteady Stokes results are valid.) This change in the temporal decay for long-time is the result of a transition in the mechanism of vorticity transport: from a symmetric diffusion of vorticity generated at the particle surface to convection of vorticity in the familiar Oseen wake behind the particle.

The motivation for extending the study to drops is to investigate the similarities and differences of the results for solid particles with those for drops and bubbles. Also, it is of value to have an expression for the unsteady force on a drop, which is useful in studies requiring the equation of motion of bubbles, drops, or particles at small-but-finite Reynolds number.

In what follows, we consider the hydrodynamic force for a drop in arbitrary time-dependent motion in an unbounded Newtonian fluid undergoing a time- and spatial-dependent flow. This derivation is accomplished through the use of the reciprocal theorem. We first derive the expression in general terms, and then simplify it for particular cases of drop composition, shape and imposed flow. The results for spatially uniform flow are shown to follow directly from those for a rigid particle.

4.2 Reciprocal theorem expression for the force

Consider a drop of density ρ^* and viscosity μ^* in a fluid of density ρ and viscosity μ. Let $\lambda = \mu^*/\mu$ and $\beta = (\nu^*/\nu)^{\frac{1}{2}}$ where ν^* and ν are the kinematic viscosities of the drop and exterior fluid, respectively. The drop is translating with a time-dependent, center-of-mass velocity $\bar{\mathbf{U}}(t)$ in an imposed flow $\mathbf{u}^\infty(\mathbf{x},t)$. We begin by writing the Navier-Stokes equations for the fluids inside and outside the drop, where an asterisk (*) is used to denote variables and parameters associated with the interior fluid of the drop:

$$\nabla\cdot\boldsymbol{\sigma}^* = \rho^*\frac{D\,\mathbf{u}^*}{D\,t}, \quad \nabla\cdot\mathbf{u}^* = 0 \quad \text{inside the drop;} \tag{4.1}$$

$$\nabla\cdot\boldsymbol{\sigma} = \rho\frac{D\,\mathbf{u}}{D\,t}, \quad \nabla\cdot\mathbf{u} = 0 \quad \text{outside the drop.} \tag{4.2}$$

Here, $\boldsymbol{\sigma} = -p\mathbf{I} + \mu(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$ is the stress tensor for a Newtonian fluid, and the pressure p includes the effect of a uniform body force (e.g., gravity). Although the velocities are those relative to the fixed laboratory frame, the origin of the coordinate system is at the instantaneous center of mass of the drop, so that

$$\frac{D\,\mathbf{u}}{D\,t} = \frac{\partial\,\mathbf{u}}{\partial\,t} + \mathbf{u}\cdot\nabla\mathbf{u} - \bar{\mathbf{U}}(t)\cdot\nabla\mathbf{u}. \tag{4.3}$$

A position vector in this coordinate system will be denoted by $\mathbf{x}$. If we assume immiscible fluids with constant surface tension γ, the appropriate boundary conditions at the interface of the drop and the exterior fluid are continuity of velocity and shear stress:

$$\mathbf{u} = \mathbf{u}^*, \quad \mathbf{n}\cdot(\boldsymbol{\sigma} - \boldsymbol{\sigma}^*)\cdot(\mathbf{I} - \mathbf{n}\mathbf{n}) = \mathbf{0} \quad \text{on } S_d, \tag{4.4}$$

where $\mathbf{n}$ is the normal to the interface pointing into the exterior fluid and S_d represents the surface of the drop. The second equation of (4.4) is the tangential stress balance. In addition, the velocity normal to the interface may be given by

$$\mathbf{n}\cdot\mathbf{u} = \mathbf{n}\cdot\mathbf{u}^* = \mathbf{n}\cdot\mathbf{U}(\mathbf{x}_s,t) \quad \text{on } S_d, \tag{4.5}$$

where the velocity of the interface, $\mathbf{U}(\mathbf{x}_s, t)$, may be a function of the position on the surface, $\mathbf{x}_s$. The conditions to be satisfied far from the drop are

$$\mathbf{u} \rightarrow \mathbf{u}^\infty, \quad p \rightarrow p^\infty \quad \text{as } r \rightarrow \infty, \tag{4.6}$$

where $r = |\mathbf{x}|$, and the imposed flow ($\mathbf{u}^\infty$, p^∞) satisfies the Navier-Stokes equations. Additionally, the velocity and pressure inside the drop are required to remain bounded.

To determine the drop shape the normal stress balance is also required:

$$[\mathbf{n} \cdot (\boldsymbol{\sigma} - \boldsymbol{\sigma}^*)] \cdot \mathbf{n} = \gamma(\boldsymbol{\nabla}\cdot\mathbf{n}) + (\mathbf{f}_b - \mathbf{f}_b^*) \cdot \mathbf{x} \quad \text{on } S_d, \tag{4.7}$$

where $\mathbf{f}_b$ and $\mathbf{f}_b^*$ are the uniform body forces per unit volume acting on the fluid exterior to and inside the drop, respectively, which are necessary here because they have been incorporated in the pressure term of the stress tensors. Although it does not directly influence the derivation that follows, the normal stress balance is included for completeness. For the low-Reynolds-number flows considered here, viscous forces dominate and the critical parameter determining the drop shape is the capillary number, $Ca = \mu U_c/\gamma$, where U_c is the characteristic velocity of the drop relative to the imposed flow. For shear flows, U_c may be replaced by aG where a is the characteristic particle size and G is the local shear rate. Under unsteady Stokes flow conditions,[a] the spherical drop in a time-dependent uniform flow can be shown to be a shape which satisfies the governing equations and boundary conditions independent of Ca [10]. For small Ca, the drop tends to remain spherical in the presence of a non-uniform flow or for finite Re conditions. The effect of a small but finite Reynolds number (i.e., the effect of the convective terms of the Navier–Stokes equations) on the

[a]In unsteady Stokes flow the convective terms of the Navier–Stokes equations (the last two terms of (4.3)) are neglected owing to the smallness of the Reynolds number while the time derivative in (4.3) is retained due to the unsteadiness of the flow.

deformation and drag of a translating drop has been studied by Taylor and Acrivos [50], although they identified the Weber number as the critical parameter that must be small to maintain a near spherical drop shape. The Weber number is equal to the product of *Ca* and *Re*. The effect of a linear flow on the deformation of a drop for small *Ca* has been treated by Leal [31], which also has references to earlier works on drop deformation and breakup.

In order to make use of the reciprocal theorem for an unbounded domain, we require the disturbance quantities which decay at infinity. The disturbance quantities are defined by

$$\mathbf{u}' = \mathbf{u} - \mathbf{u}^{\infty}, \quad p' = p - p^{\infty}, \quad \boldsymbol{\sigma}' = \boldsymbol{\sigma} - \boldsymbol{\sigma}^{\infty}. \tag{4.8}$$

The governing equations for the disturbance fields are:

$$\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}' = \rho\frac{D\,\mathbf{u}'}{D\,t} + \rho\boldsymbol{\nabla}\cdot(\mathbf{u}'\mathbf{u}^{\infty} + \mathbf{u}^{\infty}\mathbf{u}'), \quad \boldsymbol{\nabla}\cdot\mathbf{u}' = 0; \tag{4.9}$$

and the boundary conditions become

$$\mathbf{u}' = \mathbf{u}^* - \mathbf{u}^{\infty}, \quad \mathbf{n}\cdot(\boldsymbol{\sigma}' + \boldsymbol{\sigma}^{\infty} - \boldsymbol{\sigma}^*)\cdot(\mathbf{I} - \mathbf{n}\mathbf{n}) = \mathbf{0} \quad \text{on } S_d, \tag{4.10}$$

and

$$\mathbf{u}' \to \mathbf{0}, \quad p' \to 0 \quad \text{as } r \to \infty. \tag{4.11}$$

We shall also require the disturbance Stokes flow fields for the translating drop for use in the reciprocal theorem below. Denoting these field with a "hat" (ˆ), the governing equations and boundary conditions are

$$\boldsymbol{\nabla}\cdot\hat{\boldsymbol{\sigma}}^* = \mathbf{0}, \quad \boldsymbol{\nabla}\cdot\hat{\mathbf{u}}^* = 0 \quad \text{inside the drop,} \tag{4.12}$$

$$\boldsymbol{\nabla}\cdot\hat{\boldsymbol{\sigma}} = \mathbf{0}, \quad \boldsymbol{\nabla}\cdot\hat{\mathbf{u}} = 0 \quad \text{outside the drop,} \tag{4.13}$$

$$\hat{\mathbf{u}} = \hat{\mathbf{u}}^*, \quad \mathbf{n}\cdot(\hat{\boldsymbol{\sigma}} - \hat{\boldsymbol{\sigma}}^*)\cdot(\mathbf{I} - \mathbf{nn}) = \mathbf{0}, \quad \mathbf{n}\cdot\hat{\mathbf{u}}^* = \mathbf{n}\cdot\hat{\mathbf{U}} \quad \text{on } S_d, \tag{4.14}$$

where $\hat{\mathbf{U}}$ is a constant, and

$$\hat{\mathbf{u}} \to \mathbf{0}, \quad \hat{p} \to 0 \quad \text{as } r \to \infty. \tag{4.15}$$

Using the velocity and stress fields defined above, the reciprocal theorems inside and outside the drop take the following form:

$$\int_{S_d} (\mathbf{n}\cdot\boldsymbol{\sigma}^*)\cdot\hat{\mathbf{u}}^*\, dS - \int_{V_d} (\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}^*)\cdot\hat{\mathbf{u}}^*\, dV = \int_{S_d} (\mathbf{n}\cdot\hat{\boldsymbol{\sigma}}^*)\cdot\mathbf{u}^*\, dS, \tag{4.16}$$

and

$$\int_{S_d} (\mathbf{n}\cdot\boldsymbol{\sigma}')\cdot\hat{\mathbf{u}}\, dS + \int_{V_f} (\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}')\cdot\hat{\mathbf{u}}\, dV = \int_{S_d} (\mathbf{n}\cdot\hat{\boldsymbol{\sigma}})\cdot\mathbf{u}'\, dS, \tag{4.17}$$

where we have assumed that by using disturbance quantities there is no contribution from the surface at infinity.[b] Here, V_d and V_f denote the volume of the drop and exterior fluid, respectively. Following a procedure similar to that used by Leal (1980) for bounded domains, we subtract (4.16) from (4.17), and apply the boundary conditions (4.10) and (4.14) on the surface of the drop, obtain

$$\int_{S_d} \mathbf{n}\cdot(\boldsymbol{\sigma}' - \boldsymbol{\sigma}^*)\cdot\hat{\mathbf{u}}^*\, dS + \int_{V_f} (\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}')\cdot\hat{\mathbf{u}}\, dV + \int_{V_d} (\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}^*)\cdot\hat{\mathbf{u}}^*\, dV =$$
$$\int_{S_d} \mathbf{n}\cdot(\hat{\boldsymbol{\sigma}} - \hat{\boldsymbol{\sigma}}^*)\cdot\mathbf{u}^*\, dS - \int_{S_d} (\mathbf{n}\cdot\hat{\boldsymbol{\sigma}})\cdot\mathbf{u}^\infty\, dS. \tag{4.18}$$

[b]As discussed in Chapter 2, the requirement is that the disturbance pressure p' decays faster than r^{-1}, which is justified for the low-Reynolds-number flows to be considered here.

The first integral on the LHS of (4.18) may be simplified by noting that

$$\begin{aligned}\int_{S_d} \mathbf{n}\cdot(\boldsymbol{\sigma}'-\boldsymbol{\sigma}^*)\cdot\hat{\mathbf{u}}^*\,dS &= \int_{S_d}\mathbf{n}\cdot(\boldsymbol{\sigma}-\boldsymbol{\sigma}^*)\cdot\hat{\mathbf{u}}^*\,dS - \int_{S_d}\mathbf{n}\cdot(\boldsymbol{\sigma}^\infty)\cdot\hat{\mathbf{u}}^*\,dS\\ &= \int_{S_d}\mathbf{n}\cdot(\boldsymbol{\sigma}-\boldsymbol{\sigma}^*)\,dS\cdot\hat{\mathbf{U}}\\ &\quad -\int_{V_d}(\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}^\infty)\cdot\hat{\mathbf{u}}^*\,dV - \int_{V_d}\boldsymbol{\sigma}^\infty : \boldsymbol{\nabla}\hat{\mathbf{u}}^*\,dV\\ &= \mathbf{F}_d^H\cdot\hat{\mathbf{U}} - \int_{V_d}\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}^*\,dV\cdot\hat{\mathbf{U}}\\ &\quad -\int_{V_d}(\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}^\infty)\cdot\hat{\mathbf{u}}^*\,dV - \frac{1}{\lambda}\int_{S_d}\mathbf{n}\cdot\hat{\boldsymbol{\sigma}}^*\cdot\mathbf{u}^\infty\,dS, \end{aligned} \qquad (4.19)$$

where $\mathbf{F}_d^H$ $(=\int_{S_d}\mathbf{n}\cdot\boldsymbol{\sigma}\,dS)$ is the total hydrodynamic force acting on the drop. The first equality is obtained simply by using the definition of $\boldsymbol{\sigma}'$ (4.8). The second equality is obtained by an application of the tangential (or shear) stress balance (4.4), the use of the drop surface boundary condition (4.14), and another application of the tangential stress balance, as follows:

$$\begin{aligned}\int_{S_d}\mathbf{n}\cdot(\boldsymbol{\sigma}-\boldsymbol{\sigma}^*)\cdot\hat{\mathbf{u}}^*\,dS &= \int_{S_d}\mathbf{n}\cdot(\boldsymbol{\sigma}-\boldsymbol{\sigma}^*)\cdot((\mathbf{I}-\mathbf{nn})+\mathbf{nn})\cdot\hat{\mathbf{u}}^*\,dS\\ &= \int_{S_d}\mathbf{n}\cdot(\boldsymbol{\sigma}-\boldsymbol{\sigma}^*)\cdot\mathbf{nn}\cdot\hat{\mathbf{u}}^*\,dS\\ &= \int_{S_d}\mathbf{n}\cdot(\boldsymbol{\sigma}-\boldsymbol{\sigma}^*)\cdot\mathbf{nn}\cdot\hat{\mathbf{U}}\,dS\\ &= \int_{S_d}\mathbf{n}\cdot(\boldsymbol{\sigma}-\boldsymbol{\sigma}^*)\cdot((\mathbf{I}-\mathbf{nn})+\mathbf{nn})\,dS\cdot\hat{\mathbf{U}}\\ &= \int_{S_d}\mathbf{n}\cdot(\boldsymbol{\sigma}-\boldsymbol{\sigma}^*)\,dS\cdot\hat{\mathbf{U}}. \end{aligned} \qquad (4.20)$$

The divergence theorem is also applied to obtain the two volume integrals in the second equality and the first volume integral in the last equality of (4.19). We note that if one makes use of the normal stress balance (4.7) in the third equality of (4.20), it will ultimately lead to an equation of motion for the drop instead of a derivation for the hydrodynamic force. This is the result of the fact that the first two terms in the last equality of (4.19), representing the total hydrodynamic force less the inertia of the drop, would be replaced by the negative of the external body forces acting on the drop. We will return to the derivation of the equation of motion of the drop in Section 4.5. Finally, the last integral of (4.19) was arrived at by the following series of steps:

$$\int_{V_d} \boldsymbol{\sigma}^\infty : \nabla \hat{\mathbf{u}}^* \, dV = \int_{V_d} \mu(\nabla \mathbf{u}^\infty + \nabla \mathbf{u}^{\infty T}) : \nabla \hat{\mathbf{u}}^* \, dV$$

(using the definition of a Newtonian fluid and the fact $p^\infty \mathbf{I} : \nabla \hat{\mathbf{u}}^* = 0$)

$$= \int_{V_d} \mu(\nabla \mathbf{u}^\infty) : (\nabla \hat{\mathbf{u}}^* + \nabla \hat{\mathbf{u}}^{*T}) \, dV$$

(an identity)

$$= \frac{1}{\lambda} \int_{V_d} \nabla \mathbf{u}^\infty : (-\hat{p}^* \mathbf{I} + \mu^* (\nabla \hat{\mathbf{u}}^* + \nabla \hat{\mathbf{u}}^{*T})) \, dV$$

(from $\hat{p}^* \mathbf{I} : \nabla \mathbf{u}^\infty = 0$)

$$= \frac{1}{\lambda} \int_{V_d} \nabla \mathbf{u}^\infty : \hat{\boldsymbol{\sigma}}^* \, dV$$

(using the definition of a Newtonian fluid)

$$= \frac{1}{\lambda} \int_{V_d} \nabla \cdot (\hat{\boldsymbol{\sigma}}^* \cdot \mathbf{u}^\infty) \, dV$$

(from $\nabla \cdot \hat{\boldsymbol{\sigma}}^* = 0$)

$$= \frac{1}{\lambda} \int_{S_d} \mathbf{n} \cdot \hat{\boldsymbol{\sigma}}^* \cdot \mathbf{u}^\infty \, dS, \qquad (4.21)$$

where the last step is obtained by applying the divergence theorem.

The first integral of the RHS of (4.18) may be reexpressed using the same steps as in (4.20) to obtain:

$$
\begin{aligned}
\int_{S_d} \mathbf{n}\cdot(\hat{\boldsymbol{\sigma}}-\hat{\boldsymbol{\sigma}}^*)\cdot\mathbf{u}^*\,dS &= \int_{S_d} \mathbf{n}\cdot(\hat{\boldsymbol{\sigma}}-\hat{\boldsymbol{\sigma}}^*)\cdot\mathbf{U}(\mathbf{x}_s,t)\,dS \\
&= \int_{S_d} \mathbf{n}\cdot(\hat{\boldsymbol{\sigma}}-\hat{\boldsymbol{\sigma}}^*)\cdot\bar{\mathbf{U}}(t)\,dS \\
&\qquad + \int_{S_d} \mathbf{n}\cdot(\hat{\boldsymbol{\sigma}}-\hat{\boldsymbol{\sigma}}^*)\cdot\mathbf{U}'(\mathbf{x}_s,t)\,dS \\
&= \hat{\mathbf{F}}_d^H\cdot\bar{\mathbf{U}}+\int_{S_d} \mathbf{n}\cdot(\hat{\boldsymbol{\sigma}}-\hat{\boldsymbol{\sigma}}^*)\cdot\mathbf{U}'(\mathbf{x}_s,t)\,dS, \qquad (4.22)
\end{aligned}
$$

where $\hat{\mathbf{F}}_d^H$ $(=\int_{S_d}\mathbf{n}\cdot\hat{\boldsymbol{\sigma}}\,dS)$ is the steady Stokes drag for the drop translating with velocity $\hat{\mathbf{U}}$, and $\mathbf{U}'(\mathbf{x}_s,t)$ $(=\mathbf{U}(\mathbf{x}_s,t)-\bar{\mathbf{U}}(t))$ is the velocity of the interface relative to that of the center of mass of the drop. To arrive at the last equality we have also used the fact that $\int_{S_d}\mathbf{n}\cdot\hat{\boldsymbol{\sigma}}^*\,dS=0$ from an application of the divergence theorem.

Combining (4.19) and (4.22) in (4.18) we have

$$
\begin{aligned}
\mathbf{F}_d^H\cdot\hat{\mathbf{U}}+\int_{V_f}(\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}')\cdot\hat{\mathbf{u}}\,dV+\int_{V_d}(\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}^*)\cdot(\hat{\mathbf{u}}^*-\hat{\mathbf{U}})\,dV \\
-\int_{V_d}(\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}^\infty)\cdot\hat{\mathbf{u}}^*\,dV = \\
\hat{\mathbf{F}}_d^H\cdot\bar{\mathbf{U}}-\int_{S_d}(\mathbf{n}\cdot\hat{\boldsymbol{\sigma}})\cdot\mathbf{u}^\infty\,dS+\frac{1}{\lambda}\int_{S_d}\mathbf{n}\cdot\hat{\boldsymbol{\sigma}}^*\cdot\mathbf{u}^\infty\,dS \\
+\int_{S_d}\mathbf{n}\cdot(\hat{\boldsymbol{\sigma}}-\hat{\boldsymbol{\sigma}}^*)\cdot\mathbf{U}'\,dS. \qquad (4.23)
\end{aligned}
$$

Noting that all the disturbance Stokes fields are linear in $\hat{\mathbf{U}}$, we define the following:

$$
\begin{aligned}
\hat{\mathbf{u}}=\hat{\mathbf{M}}\cdot\hat{\mathbf{U}}, &\quad \hat{\mathbf{u}}^*=\hat{\mathbf{M}}^*\cdot\hat{\mathbf{U}}, \\
\hat{\boldsymbol{\sigma}}=\hat{\mathbf{T}}\cdot\hat{\mathbf{U}}, &\quad \hat{\boldsymbol{\sigma}}^*=\hat{\mathbf{T}}^*\cdot\hat{\mathbf{U}}, \qquad (4.24)
\end{aligned}
$$

where $\hat{\mathbf{M}}$ and $\hat{\mathbf{M}}^*$ are second rank tensors and $\hat{\mathbf{T}}$ and $\hat{\mathbf{T}}^*$ are third rank tensors, all of which are functions of position. Also by linearity, the steady Stokes drag may be expressed as

$$
\hat{\mathbf{F}}_d^H=-\hat{\mathbf{R}}_{FU}\cdot\hat{\mathbf{U}}, \qquad (4.25)
$$

where $\hat{\mathbf{R}}_{FU}$ is the symmetric, second rank resistance tensor which is a function of the drop shape as well as the viscosity ratio λ.[c] Thus, since all terms of (4.23) are linear in the arbitrary vector, $\hat{\mathbf{U}}$, the vector may be eliminated from (4.23) to obtain

$$\begin{aligned}\mathbf{F}_d^H + \int_{V_f} (\nabla\cdot\boldsymbol{\sigma}') \cdot \hat{\mathbf{M}}\, dV + \int_{V_d} (\nabla\cdot\boldsymbol{\sigma}^*) \cdot (\hat{\mathbf{M}}^* - \mathbf{I})\, dV \\ - \int_{V_d} (\rho \frac{D\,\mathbf{u}^\infty}{D\,t}) \cdot \hat{\mathbf{M}}^*\, dV = \\ -\hat{\mathbf{R}}_{FU} \cdot \bar{\mathbf{U}} - \int_{S_d} \mathbf{u}^\infty \cdot (\mathbf{n} \cdot \hat{\mathbf{T}})\, dS + \frac{1}{\lambda} \int_{S_d} \mathbf{u}^\infty \cdot (\mathbf{n} \cdot \hat{\mathbf{T}}^*)\, dS \\ + \int_{S_d} \mathbf{U}' \cdot (\mathbf{n} \cdot (\hat{\mathbf{T}} - \hat{\mathbf{T}}^*))\, dS. \end{aligned} \qquad (4.26)$$

Here $\mathbf{u}^\infty$ satisfies the Navier–Stokes equations and thus:

$$\nabla\cdot\boldsymbol{\sigma}^\infty = \rho \frac{D\,\mathbf{u}^\infty}{D\,t} = \rho(\frac{\partial\,\mathbf{u}^\infty}{\partial\,t} + \mathbf{u}^\infty \cdot \nabla\mathbf{u}^\infty - \bar{\mathbf{U}} \cdot \nabla\mathbf{u}^\infty). \qquad (4.27)$$

Equation (4.26) is a general expression of the hydrodynamic force acting on a drop of arbitrary shape in an arbitrarily imposed flow, with, of course, the restriction that the particular drop shape satisfy the normal stress balance for the given imposed flow. Also, as yet, no restriction has been placed on the magnitude of the Reynolds number. The first volume integral on the LHS of (4.26) represents the inertial contributions to the force from the disturbance flow outside the drop. For a solid sphere under unsteady Stokes flow conditions it yields the familiar added mass and Basset force, which has been evaluated, for example, by Maxey and Riley [33]. For small-but-finite Reynolds number, this integral is also the origin of the Oseen correction [40, 41] for steady uniform flow and the Saffman lift force [46] for steady simple shear flow. The second volume integral on the LHS of (4.26) is unique to a drop of finite viscosity, since, as will be shown in Section 4.3, it is identically zero in the limit of a solid

[c] Note that all the Stokes tensor quantities are evaluated at the current time, and thus may be a function of time if the drop is deforming.

particle or a bubble (an inviscid drop). This term is necessary, however, to obtain the correct force expression for a drop of arbitrary viscosity; as shown in Section 4.4, it combines with the first integral to produce the unsteady Stokes force acting on a drop. The last integral on the LHS of (4.26) represents the contribution to the hydrodynamic force from the inertia of the imposed flow. The first two integrals on the RHS are those due to the viscous effects of the imposed flow which, as we shall see in Section 4.3, lead to the Faxen-like corrections to the steady Stokes drag $-\hat{\mathbf{R}}_{FU} \cdot \bar{\mathbf{U}}$. The last integral is the contribution to the hydrodynamic force resulting from the drop changing shape with time.

4.3 Further simplifications of the reciprocal theorem

For a solid particle[d], $\hat{\mathbf{M}}^* = \mathbf{I}$, $\mathbf{U}' = 0$, and $1/\lambda \to 0$ so that (4.26) becomes

$$\mathbf{F}_p^H + \int_{V_f} (\nabla \cdot \boldsymbol{\sigma}') \cdot \hat{\mathbf{M}} \, dV - \int_{V_p} (\rho \frac{D\,\mathbf{u}^\infty}{D\,t}) \, dV = -\hat{\mathbf{R}}_{FU} \cdot \bar{\mathbf{U}} - \int_{S_p} \mathbf{u}^\infty \cdot (\mathbf{n} \cdot \hat{\mathbf{T}}) \, dS. \qquad (4.28)$$

For a zero-viscosity bubble, $\lambda \to 0$ (i.e., $\mu^* \to 0$ for fixed μ) and $\hat{\mathbf{T}}^* \to \mathbf{0}$.[e] Thus, equation (4.26) may be expressed as

$$\mathbf{F}_b^H + \int_{V_f} (\nabla \cdot \boldsymbol{\sigma}') \cdot \hat{\mathbf{M}} \, dV - \int_{V_b} (\rho \frac{D\,\mathbf{u}^\infty}{D\,t}) \cdot \hat{\mathbf{M}}^* \, dV = -\hat{\mathbf{R}}_{FU} \cdot \bar{\mathbf{U}} - \int_{S_b} \mathbf{u}^\infty \cdot (\mathbf{n} \cdot \hat{\mathbf{T}}) \, dS + \int_{S_b} \mathbf{u}^\infty \cdot (\mathbf{n} \cdot \frac{1}{\lambda} \hat{\mathbf{T}}^*) \, dS + \int_{S_b} \mathbf{U}' \cdot (\mathbf{n} \cdot \hat{\mathbf{T}}) \, dS, \qquad (4.29)$$

[d]In this case, $\mathbf{U}'$ could represent solid body rotation, allowing the last integral of (4.26) to yield the contribution to the hydrodynamic force from, for example, a rotating, screw-shaped particle.

[e]The quantity $\hat{\mathbf{T}}^*$ may actually tend to a constant associated with the pressure inside the bubble, but a constant tensor here does not affect the force expression (4.26).

where the second integral of (4.26) was eliminated by noting the following:

$$\int_{V_b} (\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}^*)\cdot(\hat{\mathbf{M}}^* - \mathbf{I})\, dV = \int_{V_b} (-\nabla p^*)\cdot(\hat{\mathbf{M}}^* - \mathbf{I})\, dV$$

(for a bubble $\boldsymbol{\sigma}^* = -p^*\mathbf{I}$)

$$= \int_{V_b} -\boldsymbol{\nabla}\cdot((\hat{\mathbf{M}}^* - \mathbf{I})p^*)\, dV$$

(using $\boldsymbol{\nabla}\cdot(\hat{\mathbf{M}}^* - \mathbf{I}) = \mathbf{0}$)

$$= \int_{S_b} -\mathbf{n}\cdot(\hat{\mathbf{M}}^* - \mathbf{I})p^*\, dS$$

(applying the divergence theorem)

$$= \mathbf{0}, \tag{4.30}$$

where the last step used the condition that $\mathbf{n}\cdot\hat{\mathbf{M}}^* = \mathbf{n}\cdot\mathbf{I}$ on the bubble surface. The second and fourth integrals of (4.29) are evaluated in the limit as $\mu^* \to 0$. Alternatively, one can replace these two integral with their original form, $\int_{S_b}(\mathbf{n}\cdot\boldsymbol{\sigma}^\infty)\cdot\hat{\mathbf{M}}\, dS$, on the RHS of (4.29), although this is not explicit in $\mathbf{u}^\infty$. Note also that the fourth integral of (4.29) is a bounded quantity since $\hat{\mathbf{T}}^*$ scales linearly with μ^* and thus $\frac{1}{\lambda}\hat{\mathbf{T}}^*$ scales with μ, independent of μ^*.

In the case of a spherical drop, the tensors associated with the disturbance Stokes flow problem are known from the Hadamard-Rybczyński solution of (4.12)-(4.15) with (4.24) and (4.25). They are given by

$$\hat{\mathbf{M}}^* = \frac{1}{2(\lambda+1)}\left\{\left(2\lambda + 3 - 2\frac{r^2}{a^2}\right)\mathbf{I} + \frac{\mathbf{xx}}{a^2}\right\}, \tag{4.31}$$

$$\hat{\mathbf{M}} = \frac{3\lambda+2}{4(\lambda+1)}\frac{a}{r}\left(\mathbf{I} + \frac{\mathbf{xx}}{r^2}\right) + \frac{\lambda}{4(\lambda+1)}\frac{a^3}{r^3}\left(\mathbf{I} - 3\frac{\mathbf{xx}}{r^2}\right), \tag{4.32}$$

$$\mathbf{n}\cdot\hat{\mathbf{T}}|_{r=a} = -\frac{3\mu^*}{2a(\lambda+1)}\mathbf{I} - \frac{6\mu}{2a(\lambda+1)}\mathbf{nn}, \tag{4.33}$$

$$\mathbf{n}\cdot\hat{\mathbf{T}}^*|_{r=a} = -\frac{3\mu^*}{2a(\lambda+1)}\mathbf{I} + \frac{9\mu^*}{2a(\lambda+1)}\mathbf{nn}, \tag{4.34}$$

and

$$\hat{\mathbf{R}}_{FU} = 6\pi\mu a\left(\frac{\lambda+2/3}{\lambda+1}\right)\mathbf{I}. \tag{4.35}$$

where a is the radius of the drop. Thus, (4.26) may expressed for a spherical drop as

$$\begin{aligned}\mathbf{F}_d^H + \int_{V_f}(\nabla\cdot\boldsymbol{\sigma}')\cdot\hat{\mathbf{M}}\,dV + \int_{V_d}(\nabla\cdot\boldsymbol{\sigma}^*)\cdot(\hat{\mathbf{M}}^* - \mathbf{I})\,dV \\ - \int_{V_d}(\rho\frac{D\,\mathbf{u}^\infty}{D\,t})\cdot\hat{\mathbf{M}}^*\,dV = \\ -6\pi\mu a\left(\frac{\lambda+2/3}{\lambda+1}\right)\bar{\mathbf{U}} + \frac{3\mu}{2a}\left(\frac{\lambda-1}{\lambda+1}\right)\int_{S_d}\mathbf{u}^\infty\,dS \\ + \frac{15\mu}{2a^2(\lambda+1)}\int_{V_d}\mathbf{u}^\infty\,dV,\end{aligned} \tag{4.36}$$

where we have used the fact that for a sphere $a\int_{S_d}\mathbf{u}^\infty\cdot\mathbf{nn}\,dS = \int_{V_d}\mathbf{u}^\infty\,dV$. The last integral of (4.26) is zero because the drop shape is fixed (or because $\mathbf{U}'$ has zero center-of-mass velocity).

To simplify (4.36) further, we can express $\mathbf{u}^\infty$ and $D\,\mathbf{u}^\infty/D\,t$ as multipole expansions about the center of mass of the drop, assuming the variation of the imposed flow is small over the dimensions of the drop:

$$\mathbf{u}^\infty(\mathbf{x},t) = \mathbf{U}^\infty(t) + \mathbf{x}\cdot\nabla\mathbf{u}^\infty + \frac{\mathbf{xx}}{2!}:\nabla\left(\nabla\mathbf{u}^\infty\right) + \cdots; \tag{4.37}$$

$$\frac{D\,\mathbf{u}^\infty}{D\,t}(\mathbf{x},t) = \frac{D\,\mathbf{u}^\infty}{D\,t}(\mathbf{0},t) + \mathbf{x}\cdot\nabla\frac{D\,\mathbf{u}^\infty}{D\,t} + \frac{\mathbf{xx}}{2!}:\nabla\left(\nabla\frac{D\,\mathbf{u}^\infty}{D\,t}\right) + \cdots, \tag{4.38}$$

where $\mathbf{U}^\infty(t) = \mathbf{u}^\infty(\mathbf{0}, t)$ and the higher order derivatives are evaluated at the instantaneous center of mass of the drop at time t. Using (4.37) and (4.38) in (4.36) and retaining terms up to those including quadratic variations in $\mathbf{u}^\infty$, we have for a spherical drop

$$\mathbf{F}_d^H + \int_{V_f} (\nabla\cdot\boldsymbol{\sigma}') \cdot \hat{\mathbf{M}}\, dV + \int_{V_d} (\nabla\cdot\boldsymbol{\sigma}^*) \cdot (\hat{\mathbf{M}}^* - \mathbf{I})\, dV$$
$$- \frac{4\pi}{3} a^3 \rho \left\{ \frac{D\,\mathbf{u}^\infty}{D\,t} + \left(\frac{\lambda - 1/2}{\lambda+1}\right) \frac{a^2}{10} \nabla^2 \frac{D\,\mathbf{u}^\infty}{D\,t} \right\}\bigg|_{\mathbf{x}=\mathbf{0}} =$$
$$-6\pi\mu a \left(\frac{\lambda + 2/3}{\lambda+1}\right) \left\{ \bar{\mathbf{U}} - \mathbf{U}^\infty - \frac{3\lambda}{3\lambda+2} \frac{a^2}{6} \nabla^2 \mathbf{u}^\infty\Big|_{\mathbf{x}=\mathbf{0}} \right\}, \tag{4.39}$$

where we have used the following equalities to show the two forms of the quadratic variation in $D\,\mathbf{u}^\infty/D\,t$ are equivalent up to quadratic variations in $\mathbf{u}^\infty$:

$$\nabla^2 \frac{D\,\mathbf{u}^\infty}{D\,t} = \frac{1}{\rho} \nabla^2(-\nabla p^\infty + \mu \nabla^2 \mathbf{u}^\infty); \tag{4.40}$$

$$\nabla\nabla\cdot \frac{D\,\mathbf{u}^\infty}{D\,t} = -\frac{1}{\rho} \nabla\nabla^2 p^\infty, \tag{4.41}$$

where in (4.41) we have used the condition that $\nabla\cdot\mathbf{u}^\infty = 0$.

4.4 The force acting a drop translating in a uniform flow at small Reynolds number

To evaluate exactly the first two integrals of the generalized expression for the hydrodynamic force, (4.26), we would require the solution to the full Navier–Stokes equations for the translating drop. Although we shall not attempt to solve the Navier–Stokes equations for a general imposed flow, we can make some progress for the condition of a *uniform*, time-dependent imposed flow, where $\mathbf{u}^\infty = \mathbf{U}^\infty(t)$.

For uniform flow and arbitrary, but fixed drop shape (a condition generally satisfied if $Ca \ll 1$), (4.26) becomes

$$\mathbf{F}_d^H + \int_{V_f} (\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}') \cdot \hat{\mathbf{M}}\, dV + \int_{V_d} (\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}^*) \cdot (\hat{\mathbf{M}}^* - \mathbf{I})\, dV - \rho V_d \dot{\mathbf{U}}^\infty(t) = -\hat{\mathbf{R}}_{FU} \cdot \bar{\mathbf{U}}_s(t), \qquad (4.42)$$

where $\bar{\mathbf{U}}_s(t)$ $(= \bar{\mathbf{U}}(t) - \mathbf{U}^\infty(t))$ is the slip velocity of the drop. Here we have used the fact that the first two integrals on the RHS of (4.26) may be simplified by noting $\int_{S_d} (\mathbf{n} \cdot \hat{\mathbf{T}})\, dS = -\hat{\mathbf{R}}_{FU}$ and $\int_{S_d} (\mathbf{n} \cdot \hat{\mathbf{T}}^*)\, dS = \mathbf{0}$. The goal now is to estimate the contributions from the two integrals in (4.42) with the condition that the Reynolds number ($Re = aU_c/\nu$) for the flow inside and outside the drop, based on the drop's slip velocity, is small but finite.[f] In so doing, we will obtain an expression for the hydrodynamic force acting on the drop to $O(Re)$ for arbitrary time-dependent motion.

First note the following equalities for the fluid exterior to the drop:

$$\begin{aligned} \mathbf{f}' &\equiv \rho \left(\frac{\partial \mathbf{u}'}{\partial t} - \bar{\mathbf{U}}_s \cdot \boldsymbol{\nabla}\mathbf{u}' + \mathbf{u}' \cdot \boldsymbol{\nabla}\mathbf{u}' \right) \\ &= -\nabla p' + \mu \nabla^2 \mathbf{u}' \\ &= \boldsymbol{\nabla}\cdot\boldsymbol{\sigma}'. \end{aligned} \qquad (4.43)$$

And if we define

$$p^{*\prime} = p^* - \frac{\rho^*}{\rho} p^\infty, \quad \mathbf{u}^{*\prime} = \mathbf{u}^* - \mathbf{U}^\infty(t), \qquad (4.44)$$

[f] For a nonspherical body, a denotes the characteristic drop dimension; otherwise it is the drop radius.

we can note the following equalities for the fluid inside the drop:

$$\begin{aligned}
\mathbf{f}^{*\prime} &\equiv \rho^* \left(\frac{\partial \mathbf{u}^{*\prime}}{\partial t} - \bar{\mathbf{U}}_s \cdot \nabla \mathbf{u}^{*\prime} + \mathbf{u}^{*\prime} \cdot \nabla \mathbf{u}^{*\prime} \right) \\
&= -\nabla p^{*\prime} + \mu^* \nabla^2 \mathbf{u}^{*\prime} \\
&= \nabla \cdot \boldsymbol{\sigma}^* - \rho^* \dot{\mathbf{U}}^\infty(t),
\end{aligned} \tag{4.45}$$

where we have applied the condition that $-(\rho^*/\rho)\nabla p^\infty = \rho^* \dot{\mathbf{U}}^\infty(t)$. Using $\mathbf{f}'$ and $\mathbf{f}^{*\prime}$ to signify the inertial terms from the first equalities of (4.43) and (4.45), the hydrodynamic force can now be expressed as

$$\mathbf{F}_d^H - \rho V_d \dot{\mathbf{U}}^\infty(t) = -\hat{\mathbf{R}}_{FU} \cdot \bar{\mathbf{U}}_s(t) - \int_{V_f} \mathbf{f}' \cdot \hat{\mathbf{M}}\, dV - \int_{V_d} \mathbf{f}^{*\prime} \cdot (\hat{\mathbf{M}}^* - \mathbf{I})\, dV. \tag{4.46}$$

Here we have used arguments similar to (4.30) to show

$$\begin{aligned}
\int_{V_d} (\nabla \cdot \boldsymbol{\sigma}^*) \cdot (\hat{\mathbf{M}}^* - \mathbf{I})\, dV - \int_{V_d} \mathbf{f}^{*\prime} \cdot (\hat{\mathbf{M}}^* - \mathbf{I})\, dV \\
&= \rho^* \dot{\mathbf{U}}^\infty(t) \cdot \int_{V_d} (\hat{\mathbf{M}}^* - \mathbf{I})\, dV \\
&= \rho^* \dot{\mathbf{U}}^\infty(t) \cdot \int_{V_d} \nabla \cdot \left[(\hat{\mathbf{M}}^* - \mathbf{I})\mathbf{r} \right] dV \\
&= \rho^* \dot{\mathbf{U}}^\infty(t) \cdot \int_{S_d} \mathbf{n} \cdot \left[(\hat{\mathbf{M}}^* - \mathbf{I})\mathbf{r} \right] dS \\
&= \mathbf{0}.
\end{aligned} \tag{4.47}$$

Now since the boundary conditions for the "primed" fields are

$$\begin{aligned}
\mathbf{n} \cdot (\mu(\nabla \mathbf{u}' + \nabla \mathbf{u}'^T) - \mu^*(\nabla \mathbf{u}^{*\prime} + \nabla \mathbf{u}^{*\prime T})) \cdot (\mathbf{I} - \mathbf{nn}) = \mathbf{0}, \\
\mathbf{u}' = \mathbf{u}^{*\prime}, \quad \mathbf{n} \cdot \mathbf{u}' = \mathbf{n} \cdot \mathbf{u}^{*\prime} = \mathbf{n} \cdot \bar{\mathbf{U}}_s \quad \text{on } S_d
\end{aligned} \tag{4.48a}$$

and

$$\mathbf{u}' \to \mathbf{0}, \quad p' \to 0 \quad \text{as } r \to \infty \tag{4.48b}$$

it can be seen that the two volume integrals in (4.46) represent the inertial corrections to the steady Stokes drag for a drop translating with velocity $\bar{\mathbf{U}}_s$ in a quiescent fluid. In addition, other than the presence of the integral over the volume of the drop, (4.46) is identical to the expression for a solid particle. Thus, with appropriate modifications, we can make use of the results for solid particles from Chapter 2. We will summarize the general ideas from that chapter to show the similarities and differences with the current derivation. The interested reader is referred to the original work for further details.

For small Reynolds number and short time scale motion ($\tau_c < \nu/U_c^2$ where τ_c is the time scale for the change in the drop's slip velocity), the flow is governed, to leading order in *Re*, by the unsteady Stokes equations throughout the fluid domain:

$$-\nabla p' + \mu\nabla^2\mathbf{u}' = \rho\frac{\partial\,\mathbf{u}'}{\partial\,t}, \quad \boldsymbol{\nabla}\cdot\mathbf{u}' = 0. \tag{4.49}$$

This approximation is appropriate for the flow inside as well as outside the drop. The convective terms of the Navier-Stokes equations, $\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u}$ and $\bar{\mathbf{U}}_s\cdot\boldsymbol{\nabla}\mathbf{u}$, are everywhere smaller than the viscous or the unsteady inertial terms, because the vorticity produced at the surface of the drop has not diffused out to the Oseen distance, ν/U_c, where convection becomes important as a transport mechanism. Under these conditions, the contributions from the convective terms are obtained solely from a regular perturbation analysis.

On the other hand, for long time scale motion ($\tau_c \geq \nu/U_c^2$) the flow in the near-field region (for length scales shorter than the Oseen distance ν/U_c) is governed by the steady Stokes equations, while that in the far-field (defined by distances from the drop of $O(\nu/U_c)$ or greater) is determined by the unsteady Oseen equations to leading order:

$$-\nabla p' + \mu \nabla^2 \mathbf{u}' = \rho \left(\frac{\partial \mathbf{u}'}{\partial t} - \bar{\mathbf{U}}_s \cdot \boldsymbol{\nabla} \mathbf{u}' \right) + \mathbf{F}^H_{St} \delta(\mathbf{x}), \quad \boldsymbol{\nabla} \cdot \mathbf{u}' = 0. \tag{4.50}$$

Here, the boundary conditions at the drop surface are replaced by the presence of the force monopole in the governing equation; that is, to leading order in the far-field region, the particle appears as a point-force disturbance of magnitude the pseudo-steady Stokes drag $\mathbf{F}^H_{St}$ $(= -\hat{\mathbf{R}}_{FU} \cdot \bar{\mathbf{U}}_s(t))$. Also, in this case, diffusion *and* convection are of equal importance in the transport of vorticity.

For motion of arbitrary time scales, the unsteady Stokes equations describe the flow to leading order everywhere, except in the far field where the unsteady Oseen equations govern the flow when the time scale of the motion is large. Thus, in evaluating the volume integrals of (4.46), one is able to identify three sources of inertial terms that can contribute to the hydrodynamic force to $O(Re)$: those from unsteady Stokes flow, those from applying regular perturbation techniques to the unsteady Stokes equations in order to account for the convective terms, and those from unsteady Oseen flow. After taking the proper precautions to prevent a double-counting of contributions from these sources, one arrives at the following expression for the hydrodynamic force acting on the drop (basically by analogy with the results from Chapter 2):

$$\begin{aligned} \mathbf{F}^H_d - \rho V_d \dot{\mathbf{U}}^\infty(t) = \mathbf{F}^H_{USt} &- \int_{V_f} \rho \left(\frac{\partial \mathbf{u}'_1}{\partial t} - \bar{\mathbf{U}}_s \cdot \boldsymbol{\nabla} \mathbf{u}'_0 + \mathbf{u}'_0 \cdot \boldsymbol{\nabla} \mathbf{u}'_0 \right) \cdot \hat{\mathbf{M}} \, dV \\ &- \int_{V_d} \rho^* \left(\frac{\partial \mathbf{u}^{*\prime}_1}{\partial t} - \bar{\mathbf{U}}_s \cdot \nabla \mathbf{u}^{*\prime}_0 + \mathbf{u}^{*\prime}_0 \cdot \nabla \mathbf{u}^{*\prime}_0 \right) \cdot \hat{\mathbf{M}}^* \, dV \\ &- \sqrt{\frac{a^2}{\pi \nu}} \int_{-\infty}^{t} \frac{\dot{\mathbf{F}}^H_{St}(s)}{\sqrt{t-s}} ds \cdot \boldsymbol{\Phi} + \mathbf{F}^H_{Os}. \end{aligned} \tag{4.51}$$

This expression retains the leading order effects of the convective inertia of the fluid for small Re, accurate to $O(\mu a U_c Re)$. The quantity $\mathbf{F}^H_{USt}$, henceforth referred to as the unsteady Stokes force, represents the hydrodynamic force acting the drop translating with velocity $\bar{\mathbf{U}}_s(t)$ in a quiescent fluid as determined by the unsteady Stokes equations (4.49).

The two volume integrals of (4.51) are from the regular perturbation to unsteady Stokes flow. The velocity fields $\mathbf{u}_0'$ and $\mathbf{u}_0^{*\prime}$ are the solutions to (4.49) with the boundary conditions given by (4.48). The velocity fields $\mathbf{u}_1'$ and $\mathbf{u}_1^{*\prime}$ are the regular perturbation to unsteady Stokes flow for convection. They satisfy

$$-\nabla p_1' + \mu\nabla^2\mathbf{u}_1' = \rho\left(\frac{\partial\,\mathbf{u}_1'}{\partial t} - \bar{\mathbf{U}}_s\cdot\boldsymbol{\nabla}\mathbf{u}_0' + \mathbf{u}_0'\cdot\boldsymbol{\nabla}\mathbf{u}_0'\right), \quad \boldsymbol{\nabla}\cdot\mathbf{u}_1' = 0, \tag{4.52}$$

for $\mathbf{u}_1'$ and the same equations for $\mathbf{u}_1^{*\prime}$ by replacing all quantities in (4.52) with those corresponding to the fluid in the drop, which are denoted by an asterisk. The boundary conditions are:

$$\begin{aligned} \mathbf{n}\cdot(\mu(\boldsymbol{\nabla}\mathbf{u}_1' + \boldsymbol{\nabla}\mathbf{u}_1'^{T}) - \mu^*(\nabla\mathbf{u}_1^{*\prime} + \nabla\mathbf{u}_1^{*\prime T}))\cdot(\mathbf{I}-\mathbf{nn}) = \mathbf{0}, \\ \mathbf{u}_1' = \mathbf{u}_1^{*\prime}, \mathbf{n}\cdot\mathbf{u}_1' = \mathbf{n}\cdot\mathbf{u}_1^{*\prime} = 0 \quad \text{on } S_d. \end{aligned} \tag{4.53}$$

The last two terms of (4.51) are attributed to the unsteady Oseen flow, the first of which is the negative of the long-time asymptotic form of the history force from unsteady Stokes flow, where the second rank tensor $\boldsymbol{\Phi}$ is defined by

$$\boldsymbol{\Phi} = \frac{\hat{\mathbf{R}}_{FU}}{6\pi\mu a}. \tag{4.54}$$

The last term $\mathbf{F}_{Os}^{H}$, referred to as the unsteady Oseen force, is a new history integral which can be expressed by

$$\mathbf{F}^H_{Os}(t) =$$
$$\frac{3}{8}\sqrt{\frac{a^2}{\pi\nu}}\Bigg\{\int_{-\infty}^{t}\Bigg[\ \frac{2}{3}\mathbf{F}^{H\|}_{St}(t) - \Bigg\{\frac{1}{A^2}\Big(\frac{\sqrt{\pi}}{2A}\mathrm{erf}(A) - \exp(-A^2)\Big)\Bigg\}\mathbf{F}^{H\|}_{St}(s)$$
$$+\frac{2}{3}\mathbf{F}^{H\perp}_{St}(t) - \Bigg\{\exp(-A^2) - \frac{1}{2A^2}\Big(\frac{\sqrt{\pi}}{2A}\mathrm{erf}(A) - \exp(-A^2)\Big)\Bigg\}\mathbf{F}^{H\perp}_{St}(s)\Bigg]$$
$$\frac{2ds}{(t-s)^{3/2}}\Bigg\}\cdot\mathbf{\Phi}. \qquad (4.55)$$

Here, A has the definition

$$A = \frac{1}{2}\sqrt{\frac{t-s}{\nu}}\left(\frac{|\mathbf{Y}_s(t) - \mathbf{Y}_s(s)|}{t-s}\right), \qquad (4.56)$$

where the displacement vector, $\mathbf{Y}_s(t) - \mathbf{Y}_s(s)$, is the time integration of $\bar{\mathbf{U}}_s$ from s to t. The quantities $\mathbf{F}^{H\|}_{St}$ and $\mathbf{F}^{H\perp}_{St}$ are the components of the pseudo-steady Stokes force $\mathbf{F}^H_{St}$ parallel and perpendicular to this displacement vector. For short time scale motion ($< \nu/U_c^2$), $\mathbf{F}^H_{Os}$ behaves as the negative of the history integral in (4.51) so that their combined contribution to the hydrodynamic force is smaller than $O(\mu a U_c\, Re)$. For long time scale motion on the other hand, the history-dependent part of $\mathbf{F}^H_{USt}$ will cancel with this history integral in (4.51), and the dominant history dependence of the hydrodynamic force comes from $\mathbf{F}^H_{Os}$.

If one has the unsteady Stokes solution for the translating drop, (4.51) can be used to obtain a closed-form expression for the hydrodynamic force for small but finite Reynolds number. In the case of a spherical drop, for example, the analysis is simplified by the fact that the contributions from the regular perturbation to unsteady Stokes flow (the two volume integrals of (4.51)) are identically zero, as can be seen from a symmetry argument. The unsteady Stokes force for a spherical drop in the frequency domain has been analyzed by Kim and Karrila [25]. Their result implicitly assumes the kinematic viscosities of the fluid inside and outside the drop are equal, but this is easily generalized to arbitrary kinematic viscosity ratios; the corrected result as a function of the frequency ω is

$$\tilde{\mathbf{F}}^{H}_{USt}(\omega) = -6\pi\mu a\tilde{\bar{\mathbf{U}}}_s(\omega)\left(1+\alpha+\frac{\alpha^2}{9}-\frac{(1+\alpha)^2 f(\alpha\beta)}{\lambda g(\alpha\beta)+(3+\alpha)f(\alpha\beta)}\right), \tag{4.57}$$

where

$$f(\alpha) = \alpha^2\tanh\alpha - 3\alpha + 3\tanh\alpha, \tag{4.58}$$

and

$$g(\alpha) = \alpha^3 + 6\alpha - 6\tanh\alpha - 3\alpha^2\tanh\alpha. \tag{4.59}$$

Here $\alpha = \sqrt{-i\omega a^2/\nu}$ is the dimensionless frequency parameter and $\beta = \sqrt{\nu/\nu^*}$. The primary result of (4.57) is that the history integral for unsteady Stokes flow for a drop is not of the same form as that for a solid sphere because of the fourth term of the expression, which vanishes in the case of a solid. Although the resulting memory kernel continues to behave as $t^{-\frac{1}{2}}$ in both the limit as $t \to 0$ and as $t \to \infty$, the coefficient of the $t^{-\frac{1}{2}}$ term is different in the long-time asymptotic expression from that for short time scales ($t \ll a^2/\nu$), except for a bubble, which tends toward a constant as $t \to 0$ [52].

For a spherical bubble (4.57) simplifies considerably, allowing one to obtain a closed form expression for the unsteady Stokes force for motion of arbitrary time scale. Using the analytical result from Yang and Leal [52] for the unsteady Stokes force, (4.51) for a spherical bubble becomes

$$\begin{aligned}\mathbf{F}^{H}_{b} - \frac{4\pi}{3}a^3\rho\dot{\mathbf{U}}^{\infty}(t) &= -4\pi\mu a\bar{\mathbf{U}}_s(t) - \frac{2}{3}\pi\rho a^3\dot{\bar{\mathbf{U}}}_s(t)\\ &\quad - 8\pi\mu a\int_{-\infty}^{t} e^{9\nu(t-s)/a^2}\operatorname{erfc}\left(\sqrt{9\nu(t-s)/a^2}\right)\dot{\bar{\mathbf{U}}}_s(s)\,ds\\ &\quad +\frac{8\pi\mu a}{\sqrt{\pi}}\int_{-\infty}^{t}\frac{\dot{\bar{\mathbf{U}}}_s(s)}{\sqrt{9\nu(t-s)/a^2}}ds + \mathbf{F}^{H}_{Os},\end{aligned} \tag{4.60}$$

where in (4.55), the expression for $\mathbf{F}^H_{Os}$, $\mathbf{F}^H_{St}(t)$ is replaced by $-4\pi\mu a\bar{\mathbf{U}}_s(t)$ and $\mathbf{\Phi}$ by $\frac{2}{3}\mathbf{I}$. By an asymptotic analysis it can be seen that the memory kernel of the first integral of (4.60) tends to a constant for small time and behaves as the negative of the second memory function for large-time.

For a drop of arbitrary shape, one can obtain an expression for the hydrodynamic force in terms of the results from steady Stokes flow in the limit when the time scale τ_c of the variation of the drop's slip velocity satisfies $\tau_c \gg a^2/\nu$. Following the procedure from Chapter 2, we obtain

$$
\begin{aligned}
\mathbf{F}^H_d(t) &= \rho V_d \dot{\mathbf{U}}^\infty(t) + \mathbf{F}^H_{St}(t) + \mathbf{F}^H_{Os}(t) \\
&- \rho \left\{ 6\pi \mathbf{\Phi}\cdot\mathbf{\Phi}\cdot\mathbf{\Phi} + \lim_{R\to\infty} \left[\int\limits_{V_f(R)} \hat{\mathbf{M}}^T \cdot \hat{\mathbf{M}}\, dV - \frac{9\pi}{2}\mathbf{\Phi}\cdot\mathbf{\Phi} R \right] \right\} \cdot \dot{\bar{\mathbf{U}}}_s(t) \\
&- \rho^* \left\{ \int\limits_{V_d} \hat{\mathbf{M}}^{*T} \cdot \hat{\mathbf{M}}^*\, dV - V_d \mathbf{I} \right\} \cdot \dot{\bar{\mathbf{U}}}_s(t) \\
&- \rho \lim_{R\to\infty} \int\limits_{V_f(R)} \left(\mathbf{u}'_{St} \cdot \nabla \mathbf{u}'_{St} - \bar{\mathbf{U}}_s \cdot \nabla \mathbf{u}'_{St} \right) \cdot \hat{\mathbf{M}}\, dV \\
&- \rho^* \int\limits_{V_d} \left(\mathbf{u}^{*\prime}_{St} \cdot \nabla \mathbf{u}^{*\prime}_{St} - \bar{\mathbf{U}}_s \cdot \nabla \mathbf{u}^{*\prime}_{St} \right) \cdot \hat{\mathbf{M}}^*\, dV \\
&\qquad + o(\mu a U_c Re) + o(\mu a U_c \frac{a^2/\nu}{\tau_c}),
\end{aligned}
\tag{4.61}
$$

where $V_f(R)$ represents a large spherical volume of radius R with origin at the center of the drop, and $\mathbf{u}'_{St}$ and $\mathbf{u}^{*\prime}_{St}$ are the steady Stokes solutions to the disturbance flow problem.

For a spherical drop the two terms in large curly braces from (4.61) combine to yield

$$
-6\pi a^3 \left\{ \rho \frac{3\lambda^3 - 3\lambda - 1}{27(\lambda+1)^3} + \rho^* \frac{1}{63(\lambda+1)^2} \right\} \cdot \dot{\bar{\mathbf{U}}}_s(t). \tag{4.62}
$$

This result agrees with the low-frequency (long-time) limit of the $O(\alpha^2)$- term from the result given by (4.57). This term is very different from the added mass in the high-frequency (short-time) limit, $-\frac{2}{3}\pi\rho a^3\dot{\mathbf{U}}_s(t)$, and reflects the uniqueness of the solid sphere which happens to have the same value in the low- and high-frequency limits. Recall the that the contribution from the inertial terms inside the drop (the source of the second term of (4.62)) are identically zero for the case of a solid particle or bubble. Note also that since we have assumed the flow inside the drop is described by the steady Stokes equations to leading order in the long-time limit, we require that $\mu^* \gg \rho^* a U_c$. Thus, the second term of (4.62) does indeed go to zero in the limit of a bubble ($\mu^* \to 0$) since ρ^* must approach zero accordingly.

It can be seen in (4.61) that the long-time temporal response of the hydrodynamic force is dictated by the properties of $\mathbf{F}^H_{Os}$. This term, identified as the unsteady Oseen correction to the hydrodynamic force, was analyzed in Chapter 2. It was shown that its decay to steady state is algebraic for a step change from or to a zero velocity: behaving as t^{-2} when the drop accelerates from rest and as t^{-1} when it comes to rest. However, when the step changes are between finite velocities the ultimate decay of the hydrodynamic force is exponential. This contrast in temporal decay reflects the distinction between the creation (or destruction) of the wake structure, associated with the algebraic decay, and the modification of the wake structure already established, which leads to exponential decay. The fact that in all cases the temporal decay is faster than the $t^{-\frac{1}{2}}$ associated with unsteady Stokes flow reflects the efficient mechanism of convective transport of vorticity relative to that of diffusion. It should be reiterated that this behavior is observed on long-time scales ($> \nu/U_c^2$) and thus for small time the decay will go as $t^{-\frac{1}{2}}$.

4.5 Results and discussion

To provide some confirmation of the validity of the results we have obtained, we compare with the recently published numerical work of Mei and Klausner [38]. They evaluated the force on a spherical bubble held fixed in a uniform flow with small fluctuations, a flow given by

$$U^{\infty}(t) = U(1 + \delta e^{-i\omega t}), \tag{4.63}$$

with the condition $\delta \ll 1$ and results for the drag evaluated to $O(\delta)$. By letting $U_s(t) = -U^{\infty}(t)$ in the force expression for a bubble, (4.60), and with the aid of the frequency domain expression (4.57) and the use of *Mathematica* to carry out the appropriate integrations, we arrive at the following expression for the force accurate to $O(\delta)$ and to $O(Re)$

$$\begin{aligned} \frac{F_b^H(t)}{6\pi\mu a U} &= \frac{2}{3} + \frac{2}{3}\delta\, e^{-i\omega t} + \frac{1}{3}\alpha^2 \delta\, e^{-i\omega t} \\ &+ \frac{4}{9}\delta\, e^{-i\omega t}\left(\frac{\alpha}{1+\alpha/3} - \alpha\right) \\ &+ \frac{1}{6} Re + \frac{4}{9} Re\, \delta\, e^{-i\omega t} \frac{2^{\frac{1}{2}}(1-i)(\gamma_\omega + i)^{\frac{3}{2}} - 2i}{4\gamma_\omega}. \end{aligned} \tag{4.64}$$

The Reynolds number is defined by $Re = aU/\nu$ and $\gamma_\omega = 4\omega\nu/U^2$ is a dimensionless low-frequency parameter. The first two terms originate from the pseudo-steady Stokes drag, the third from the added mass and the acceleration of the imposed flow, and the fourth from the first two history integrals of (4.60): the unsteady Stokes history force less its low-frequency asymptote. The last two terms of (4.64) are from the unsteady Oseen correction, $\mathbf{F}_{Os}^H$.

If we define the history force as the part of (4.64) that results from subtracting off both the finite zero-frequency components and the $O(\alpha^2)$ term ($\sim \omega$), the dimensionless history force may be expressed as

$$\frac{F_{bh}^H(t)}{6\pi\mu aU} = \frac{4}{9}\delta\,\mathrm{e}^{-\mathrm{i}\omega t}\left(\frac{\alpha}{1+\alpha/3} - \alpha\right) + \frac{4}{9}Re\,\delta\,\mathrm{e}^{-\mathrm{i}\omega t}\left(\frac{2^{\frac{1}{2}}(1-\mathrm{i})(\gamma_\omega+\mathrm{i})^{\frac{3}{2}} - 2\mathrm{i}}{4\gamma_\omega} - \frac{3}{4}\right), \tag{4.65}$$

where the 3/4-term in (4.65) is necessary to remove the zero frequency asymptote of the last term of (4.64). The real and imaginary parts of the history force coefficient have been evaluated numerically by Mei and Klausner [38] for Re=0.05, 2.5, and 20. (Note that their definition of the Reynolds number is based on the bubble diameter, not the radius as is done here, so their values are reported as 0.1, 5, and 40.) Using the notation of Mei and Klausner [38], we define the following history force coefficients as a function of the frequency parameter $\varepsilon = (\omega a^2/2\nu)^{\frac{1}{2}}$:

$$D_{1RH}(\varepsilon) = \mathrm{Re}\left[\frac{4}{9}\left(\frac{(-2\mathrm{i})^{\frac{1}{2}}\varepsilon}{1+(-2\mathrm{i})^{\frac{1}{2}}\varepsilon/3} - (-2\mathrm{i})^{\frac{1}{2}}\varepsilon\right) + \frac{4}{9}Re\left(\frac{2^{\frac{1}{2}}(1-\mathrm{i})(8\varepsilon^2/Re^2+\mathrm{i})^{\frac{3}{2}} - 2\mathrm{i}}{32\varepsilon^2/Re^2} - \frac{3}{4}\right)\right], \tag{4.66}$$

and

$$D_{1IH}(\varepsilon) = \mathrm{Im}\left[\frac{4}{9}\left(\frac{(-2\mathrm{i})^{\frac{1}{2}}\varepsilon}{1+(-2\mathrm{i})^{\frac{1}{2}}\varepsilon/3} - (-2\mathrm{i})^{\frac{1}{2}}\varepsilon\right) + \frac{4}{9}Re\left(\frac{2^{\frac{1}{2}}(1-\mathrm{i})(8\varepsilon^2/Re^2+\mathrm{i})^{\frac{3}{2}} - 2\mathrm{i}}{32\varepsilon^2/Re^2} - \frac{3}{4}\right)\right], \tag{4.67}$$

where we have used the fact that $\alpha = (-2\mathrm{i})^{\frac{1}{2}}\varepsilon$ and $\gamma_\omega = 8\varepsilon^2/Re^2$. The case of $Re = 0.05$ is relevant to the current small Reynolds number study and is shown in Figures 4.1(a, b) for the real and imaginary parts, D_{1RH} and D_{1IH}, respectively. Also included is the unsteady Stokes result given by the first term of (4.66) and (4.67), which represents the asymptotic limit of the expressions as $Re \to 0$ for fixed ε. The

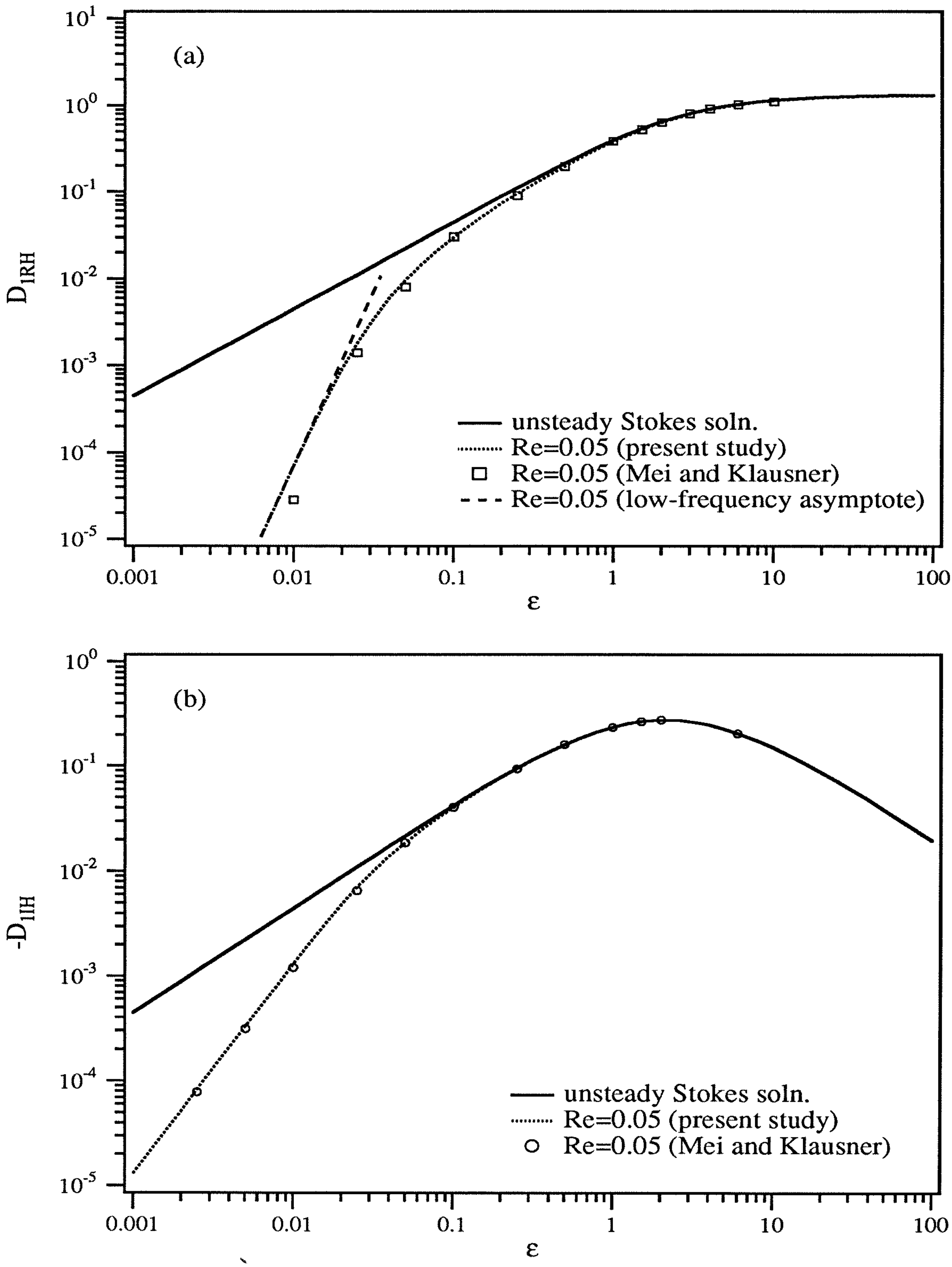


Figure 4.1: Comparison of the (a) real and (b) imaginary parts of the history force coefficient for a bubble, D_{1RH} and D_{1IH} respectively, with the numerical results of Mei and Klausner [38] as a function of the frequency parameter ε.

low-frequency asymptote in Fig. 4.1(a) is given by

$$\lim_{\varepsilon < Re \ll 1} D_{1RH}(\varepsilon) = \frac{8}{9}\frac{\varepsilon^4}{Re^3}. \tag{4.68}$$

The results show very good agreement with the numerical work of Mei and Klausner over the entire range of frequency. This agreement is consistent with the findings of Chapter 3 for solid spheres which showed good agreement with the numerical work of Mei *et al.* [37] up to $Re \sim 0.5$.

Also in the analysis Chapter 3 is the inversion of the frequency-dependent result to the time domain for a general time-dependent motion, when the unsteady portion is small. A similar derivation can be performed here. The interested reader is referred to the previous chapter for the details. The result is

$$\begin{aligned} F_b^H(t) = {} & 6\pi\mu a U\left(\frac{2}{3} + \frac{2}{3}U_1(t) + \frac{1}{6}Re\,(1 + 2U_1(t))\right) + 2\pi\rho a^3 \dot{U}_1(t) \\ & + 8\pi\mu a \int_{-\infty}^{t} e^{9\nu(t-s)/a^2} \mathrm{erfc}\left(\sqrt{9\nu(t-s)/a^2}\right) \dot{U}_1(s)\, d\,s \\ & - \frac{8\pi\mu a}{\sqrt{\pi}} \int_{-\infty}^{t} \frac{\dot{U}_1(s)}{\sqrt{9\nu(t-s)/a^2}} ds \\ & + 6\pi\mu a U \frac{2}{9}\frac{Re}{\pi} \int_{-\infty}^{t} G(t-s)\dot{U}_1(s)\, d\,s, \end{aligned} \tag{4.69}$$

with

$$G(t) = e^{-tU^2/4\nu} \int_0^\infty \frac{x^{\frac{3}{2}}}{(1+x)^2} e^{-xtU^2/4\nu}\, d\,x, \tag{4.70}$$

where the imposed flow $U^\infty(t) = U(1 + U_1(t))$ has the condition $U_1(t) \ll 1$ for all time.

Equations (4.66) and (4.67) also show good agreement at higher Reynolds number for high frequency motion. This condition exists because the unsteady Stokes solution is valid even for moderate Re, provided $\varepsilon > Re$. At low frequency ($\varepsilon < Re$), good qualitative agreement is achieved at higher Reynolds number only when the unsteady Oseen force is left by itself to predict the history force. The corrections from the unsteady Stokes solution (the terms contained in the first set of large parentheses in (4.66) and (4.67)), which are of higher order than $O(Re)$ at low frequency, are not properly matched because (4.60) is strictly valid for very small Re and is only accurate to $O(Re)$. Although they are smaller than $O(Re)$, the unsteady Stokes corrections apparently erroneously alter the behavior of the history force at low frequency. This is not a problem for solid spheres because the unsteady Stokes history force is uniquely simple, being the same in both the high- and the low-frequency limit. The history force for a solid sphere can then be completely accounted for by the unsteady Oseen correction; there are no required unsteady Stokes corrections to the history force. This problem with bubbles (which also exists for drops) can be illustrated by noting that the unsteady Stokes corrections at low frequency contribute the quantity $-8\varepsilon^3/81$ to D_{1RH}. By comparison with (4.68), we see that it will actually dominate and incorrectly change the sign of D_{1RH} when $\varepsilon < Re^3/9$, however, at that point D_{1RH} is less than $O(Re^9)$. Thus, if one is studying low-frequency behavior at finite Re, it is advisable to use only the unsteady Oseen correction to represent the history force for a bubble or drop.

Next in order to show the variation of the force on a drop with drop properties and Reynolds number, we consider the history force on a spherical drop for the motion described by (4.63). Using the same arguments as for the bubble, the history force for a drop using (4.57) is given to $O(\delta)$ by

$$\frac{F_{dh}^H(t)}{6\pi\mu a U} = \delta\, \mathrm{e}^{-\mathrm{i}\omega t}\left(\frac{1/3}{\lambda+1} + \frac{2/3\lambda + 5/9}{(\lambda+1)^2}\alpha - \frac{(1+\alpha)^2 f(\alpha\beta)}{\lambda\, g(\alpha\beta) + (3+\alpha) f(\alpha\beta)}\right) + Re\,\delta\, \mathrm{e}^{-\mathrm{i}\omega t}\left(\frac{\lambda + 2/3}{\lambda+1}\right)^2 \left(\frac{2^{\frac{1}{2}}(1-\mathrm{i})(\gamma_\omega + \mathrm{i})^{\frac{3}{2}} - 2\mathrm{i}}{4\gamma_\omega} - \frac{3}{4}\right). \qquad (4.71)$$

The history force coefficient can then be expressed as a function of α by

$$D_{1H}(\alpha) = \left(\frac{1/3}{\lambda+1} + \frac{2/3\lambda+5/9}{(\lambda+1)^2}\alpha - \frac{(1+\alpha)^2 f(\alpha\beta)}{\lambda g(\alpha\beta)+(3+\alpha)f(\alpha\beta)}\right)$$
$$+Re\left(\frac{\lambda+2/3}{\lambda+1}\right)^2\left(\frac{(4\alpha^2/Re^2+1)^{\frac{3}{2}}-1}{8\alpha^2/Re^2} - \frac{3}{4}\right), \qquad (4.72)$$

where we have used $\gamma_\omega = 4i\alpha^2/Re^2$. The high-frequency behavior of (4.72) is given by

$$D_{1H} \sim \alpha\left(1-\frac{\alpha}{3\lambda}\right), \qquad 1 \ll \alpha \ll \tfrac{1}{\beta}, \lambda \qquad (4.73a)$$
$$\sim \frac{3(2/3+\lambda)^2}{\lambda+1} - \tfrac{3}{4}Re\left(\frac{\lambda+2/3}{\lambda+1}\right)^2, \qquad 1, \lambda \ll \alpha \ll \tfrac{1}{\beta} \qquad (4.73b)$$
$$\sim \tfrac{4}{3} - \tfrac{1}{3}Re, \qquad 1, \tfrac{1}{\beta} \ll \alpha \ll \tfrac{1}{\lambda\beta} \qquad (4.73c)$$
$$\sim \frac{\lambda\beta}{\lambda\beta+1}\alpha, \qquad \alpha \gg 1, \tfrac{1}{\beta}, \tfrac{1}{\lambda\beta}, \qquad (4.73d)$$

while the low-frequency behavior is

$$D_{1H} \sim \frac{1/3}{\lambda+1} + \alpha - \frac{1}{\lambda\beta\alpha} - \tfrac{3}{4}Re\left(\frac{\lambda+2/3}{\lambda+1}\right)^2, \qquad Re, \tfrac{1}{\beta}, \tfrac{1}{\lambda\beta} \ll \alpha \ll 1 \qquad (4.74a)$$
$$\sim \tfrac{4}{9}\alpha + \frac{\lambda\beta\alpha}{9} - \tfrac{3}{4}Re\left(\frac{\lambda+2/3}{\lambda+1}\right)^2, \qquad Re, \tfrac{1}{\beta} \ll \alpha \ll 1, \tfrac{1}{\lambda\beta} \qquad (4.74b)$$
$$\sim \left(\frac{\lambda+2/3}{\lambda+1}\right)^2\alpha, \qquad Re \ll \alpha \ll 1, \tfrac{1}{\beta} \qquad (4.74c)$$
$$\sim \tfrac{3}{4}Re\frac{\alpha^2}{Re^2}\left(\frac{\lambda+2/3}{\lambda+1}\right)^2$$
$$-Re^2\frac{\alpha^2}{Re^2}\frac{28+84\lambda-3\beta^2\lambda+63\lambda^2-3\beta^2\lambda^2}{189(1+\lambda)^3}, \qquad \alpha \ll \tfrac{1}{\beta}, Re. \qquad (4.74d)$$

In Fig. 4.2 we show the dependence of D_{1H} on the viscosity ratio λ for $\lambda = 0$ to ∞. For low frequency, λ alters the behavior of D_{1H} by simply a numerical coefficient. At high frequency, there is a stronger dependence on λ, particularly for small λ. This is because the properties of the drop are tending toward that of a bubble which has

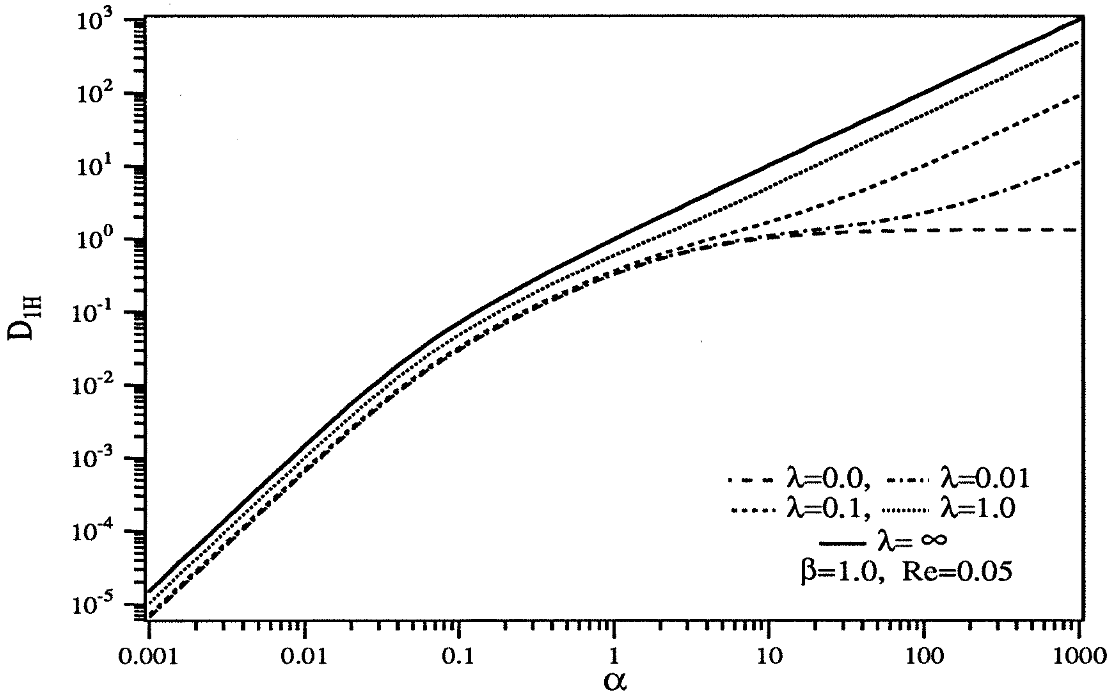


Figure 4.2: Dependence of the history force coefficient D_{1H} for a drop on the viscosity ratio λ as a function of the frequency parameter α.

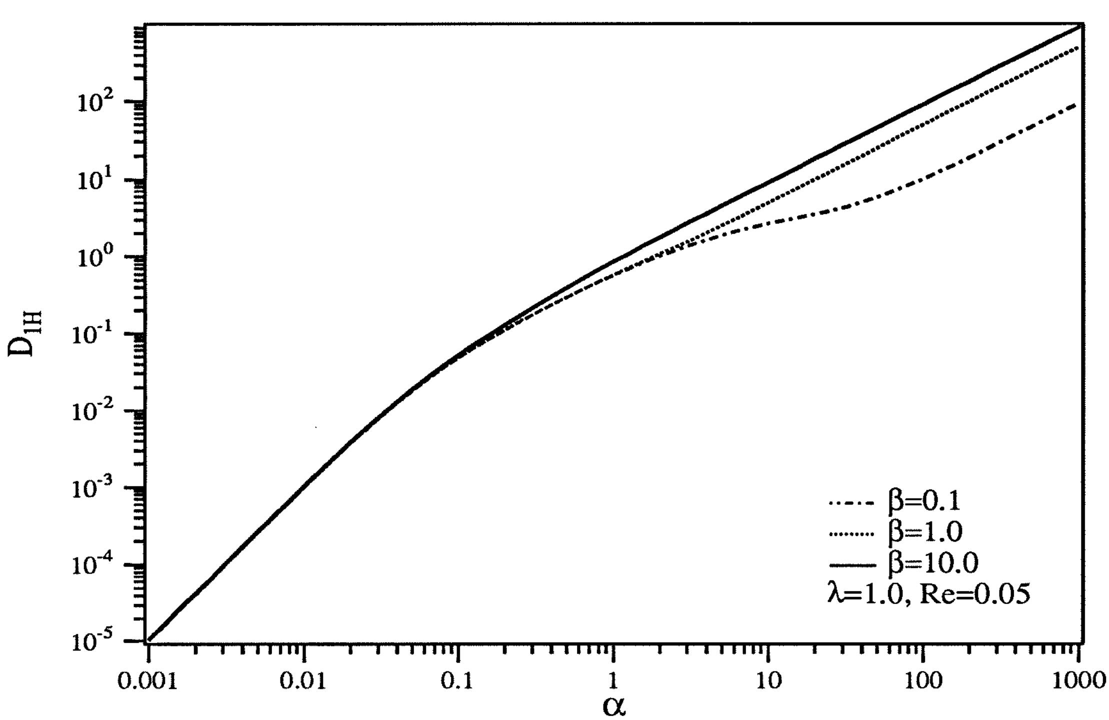


Figure 4.3: Dependence of the history force coefficient D_{1H} for a drop on the kinematic viscosity ratio β as a function of the frequency parameter α.

a very different high-frequency asymptote from that of a drop, $O(1)$ versus $O(\alpha)$. Figure 4.3 shows the dependence of D_{1H} on the kinematic viscosity ratio through the parameter β. In this case, since λ is fixed, it actually shows the effect of the density of the drop relative to the surrounding fluid, i.e., $\beta \sim (\rho^*/\rho)^{\frac{1}{2}}$. There is little variation in D_{1H} with β at low frequency. However, there is a stronger dependence, very similar to λ, at high frequency; higher/lower density drops behave similarly as higher/lower viscosity drops at high frequency.

In Fig. 4.4(a) the dependence of D_{1H} on the Reynolds number is shown. As the Reynolds number is increased the deviation of the history force from its corresponding unsteady Stokes solution increases, with the deviations becoming evident at higher frequencies. This deviation of the history force from the unsteady Stokes solution at low frequencies leads to a much different temporal behavior of the force on a bubble, drop, or particle at finite Reynolds number, particularly as steady state is approached. Finally, in Fig. 4.4(b) we plot the curves of Fig. 4.4(a) at finite Reynolds number in inertial coordinates rescaled by dividing by Re. It demonstrates that the results can be collapsed quite well on a single curve for given values of λ and β. The rescaling works provided the high-frequency asymptote varies linearly with α, e.g., we are not dealing with a bubble which shows no variation with α at high frequency, and it improves at intermediate α as λ and β are increased.

We now conclude by deriving the equation of motion for a bubble, drop, or particle in a fluid, appropriate for motion at small but finite Reynolds number. The general expression is given by

$$m_d\dot{\mathbf{U}} = \mathbf{F}_d^H + \mathbf{F}^{Ext}, \tag{4.75}$$

where m_d is the mass of the body. Here, we have assumed that the only forces acting on the body are hydrodynamic forces and external body forces, $\mathbf{F}^{Ext}$, such as the

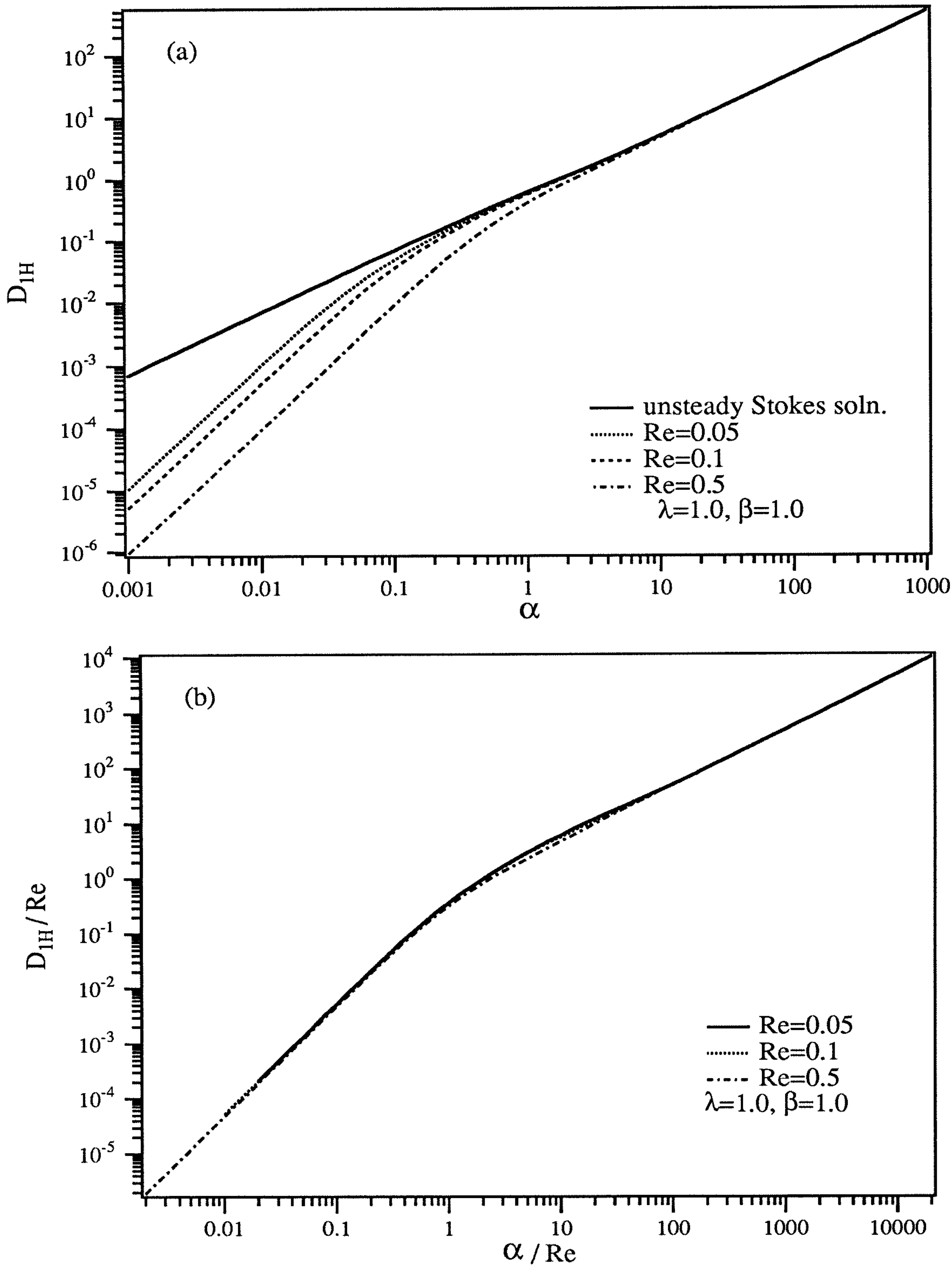


Figure 4.4: Dependence of the history force coefficient D_{1H} for a drop on the Reynolds number *Re* as a function of the frequency parameter α: (a) viscous scale and (b) inertial scale.

buoyancy force due to gravity.

In the absence of velocity gradients in the imposed flow,[g] the general expression for the hydrodynamic force is given by (4.51), where one can take into account the influence of the deformation of the body by including the last term of (4.26). The two volume integrals in (4.51) are absent for spherical bodies. For long time scale motion ($> a^2/\nu$) of nonspheres they can be approximated using the solution of the steady Stokes flow field for the translating body. Under this condition, it can be shown that they contribute only a side force perpendicular to the direction of motion (see Chapter 2.).

For a spherical body the hydrodynamic force to $O(Re)$ simplifies to

$$\mathbf{F}_d^H = \frac{4\pi}{3} a^3 \rho \dot{\mathbf{U}}^\infty(t) + \mathbf{F}^H_{USt} - \sqrt{\frac{a^2}{\pi\nu}} \int_{-\infty}^{t} \frac{\dot{\mathbf{F}}^H_{St}(s)}{\sqrt{t-s}} ds \cdot \mathbf{\Phi} + \mathbf{F}^H_{Os}, \tag{4.76}$$

where the unsteady Oseen force $\mathbf{F}^H_{Os}$ is given by (4.55). A closed-form expression for the unsteady Stokes force $\mathbf{F}^H_{USt}$ is available only for a solid body or a bubble. For bodies of intermediate viscosity one must invert the expression given by (4.57) to the time domain. The solution can be written formally as

$$\mathbf{F}^H_{USt}(t) = -3\mu a \int_{-\infty}^{t} \bar{\mathbf{U}}_s(s) \int_{-\infty}^{\infty} \left(1 + \alpha + \frac{\alpha^2}{9} - \frac{(1+\alpha)^2 f(\alpha\beta)}{\lambda\, g(\alpha\beta) + (3+\alpha) f(\alpha\beta)}\right) \times e^{-i\omega(t-s)}\, d\omega\, d s. \tag{4.77}$$

For a solid sphere (a body of infinite viscosity relative to the exterior fluid), this expression reduces to the well-known result of Basset [3]:

$$\mathbf{F}^H_{USt}(t) = -6\pi\mu a \bar{\mathbf{U}}_s(t) - \frac{2}{3}\pi\rho a^3 \dot{\bar{\mathbf{U}}}_s(t) - 6\pi\mu a \sqrt{\frac{a^2}{\pi\nu}} \int_{-\infty}^{t} \frac{\dot{\bar{\mathbf{U}}}_s(s)}{\sqrt{t-s}} ds. \tag{4.78}$$

[g]The requirement here is that the Oseen time scale ν/U_c^2 is much less than the characteristic shear rate in the imposed flow.

For an inviscid sphere it becomes [52]:

$$\mathbf{F}^{H}_{USt}(t) = -4\pi\mu a\bar{\mathbf{U}}_s(t) - \frac{2}{3}\pi\rho a^3\dot{\bar{\mathbf{U}}}_s(t) - 8\pi\mu a\int_{-\infty}^{t} e^{9\nu(t-s)/a^2}\mathrm{erfc}\left(\sqrt{9\nu(t-s)/a^2}\right)\dot{\bar{\mathbf{U}}}_s(s)\,ds. \qquad (4.79)$$

It is interesting to point out that in both of these limits the force expression is independent of the density of the material inside of the body. However, it can be seen in (4.77) that through the parameter β the density of the drop can influence the force for general drop viscosities.

To compare the equation of motion for a solid sphere to that of an inviscid sphere (the distinction here from a bubble being that the body may have finite density), we combine the appropriate forms of $\mathbf{F}^{H}_{USt}$ from (4.78) and (4.79) with $\mathbf{F}^{H}_{Os}$ from (4.55) in (4.76) to obtain (4.75) for a solid sphere:

$$\left(m_d + \frac{1}{2}m_f\right)\dot{\bar{\mathbf{U}}}_s + 6\pi\mu a\bar{\mathbf{U}}_s + 6\pi\mu a\frac{3}{8}\sqrt{\frac{a^2}{\pi\nu}}\left\{\int_{-\infty}^{t}\left[\frac{2}{3}\bar{\mathbf{U}}_s^{||}(t) - \left\{\frac{1}{A^2}\left(\frac{\sqrt{\pi}}{2A}\mathrm{erf}(A) - \exp(-A^2)\right)\right\}\bar{\mathbf{U}}_s^{||}(s) + \frac{2}{3}\bar{\mathbf{U}}_s^{\perp}(t) - \left\{\exp(-A^2) - \frac{1}{2A^2}\left(\frac{\sqrt{\pi}}{2A}\mathrm{erf}(A) - \exp(-A^2)\right)\right\}\bar{\mathbf{U}}_s^{\perp}(s)\right]\frac{2\,ds}{(t-s)^{3/2}}\right\} = -(m_d - m_f)\dot{\mathbf{U}}^{\infty}(t) + (m_d - m_f)\,\mathbf{g}, \qquad (4.80)$$

and for an inviscid sphere:

$$
\begin{aligned}
&\left(m_d + \frac{1}{2}m_f\right)\dot{\bar{\mathbf{U}}}_s + 4\pi\mu a \bar{\mathbf{U}}_s \\
&\qquad + 8\pi\mu a \int_{-\infty}^{t} e^{9\nu(t-s)/a^2}\,\mathrm{erfc}\left(\sqrt{9\nu(t-s)/a^2}\right)\dot{\bar{\mathbf{U}}}_s(s)\,ds \\
&\qquad\qquad - \frac{8\pi\mu a}{\sqrt{\pi}}\int_{-\infty}^{t}\frac{\dot{\bar{\mathbf{U}}}_s(s)}{\sqrt{9\nu(t-s)/a^2}}ds \\
&+\pi\mu a\sqrt{\frac{a^2}{\pi\nu}}\left\{\int_{-\infty}^{t}\left[\ \frac{2}{3}\bar{\mathbf{U}}_s^{\|}(t) - \left\{\frac{1}{A^2}\left(\frac{\sqrt{\pi}}{2A}\mathrm{erf}(A) - \exp(-A^2)\right)\right\}\bar{\mathbf{U}}_s^{\|}(s)\right.\right. \\
&+\frac{2}{3}\bar{\mathbf{U}}_s^{\perp}(t) - \left\{\exp(-A^2) - \frac{1}{2A^2}\left(\frac{\sqrt{\pi}}{2A}\mathrm{erf}(A) - \exp(-A^2)\right)\right\}\bar{\mathbf{U}}_s^{\perp}(s)\Big] \\
&\qquad\qquad\qquad\qquad \left.\frac{2\,ds}{(t-s)^{3/2}}\right\} \\
&\qquad = -\left(m_d - m_f\right)\dot{\mathbf{U}}^{\infty}(t) + \left(m_d - m_f\right)\mathbf{g},
\end{aligned}
\tag{4.81}
$$

where m_f is the mass of the exterior fluid displaced by the sphere and $\mathbf{g}$ is the acceleration due to gravity. Here, we have assumed the only external force is the buoyancy force, $\mathbf{F}^{Ext} = (m_d - m_f)\,\mathbf{g}$.

Apart from the two additional history integrals in (4.81), the governing equations for the two bodies are essentially the same, in that they have the same terms with only a difference in numerical coefficients. For very short time scale motion ($< a^2/\nu$), the temporal behavior of the two bodies will be nearly the same since the dominant contribution from the LHS of their respective equations is the same first term. For time scales of $O(a^2/\nu)$, all terms of the governing equations will be important in the bodys' motion, and it is under this condition that the temporal behavior of the motion can possibly be different owing to the difference in the history dependence, a much weaker history dependence for the case of the inviscid drop.[h] For long time scale motion ($\geq \nu/U_c^2$), the temporal behavior of the bodys' motion will now be very

[h] We note this condition could have been observed from the results of unsteady Stokes flow without any of the Oseen-like considerations used here.

similar since it is largely dictated by the unsteady Oseen force given by the last term on the LHS of (4.80) and (4.81), which differ only by a numerical coefficient.

Chapter 5

The reciprocal theorem for an N-particle system

Summary

The dynamics of an N-particle system is expressed in terms of the hydrodynamic forces, torques and stresslets on each particle in an arbitrary imposed flow at finite Reynolds number. The results generalize the prior analyses of Kim and Mifflin (1985) and Jeffery and Onishi (1984) which are restricted to two spherical particles in Stokes flow (i.e., zero Reynolds number). The results for large separations are shown to agree with the mobility formulation by Durlofsky, Brady, and Bossis (1987) in a previous study of monodisperse spherical particles in Stokes flow.

5.1 Introduction

Considerable research has been carried out on the study of particle interactions under Stokes flow conditions [21, 26]. Here we consider the case of a cluster or collection of particles in an unbounded locally linear flow without restriction on the magnitude of the Reynolds number for the imposed flow or the particle disturbance flow. We are able to derive expressions that represent a mobility-like formulation for the particle interactions since, for spherical particles, they reduce to the far-field mobility formulation of the Stokesian dynamics approach to suspension flows.

In the following section we introduce the reciprocal theorem for fluid flow. Then in Section 5.3 we apply this expression to a general N-particle system by defining the appropriate velocity, pressure, and stress fields along with carefully chosen boundary conditions. In Section 5.4 we make simplifications of the expressions while maintaining the generality of their validity, while in Section 5.5 the simplifications are made under the assumption that the particles are far apart. The last section provides some concluding remarks on the application of the expressions to multiparticle dynamics and includes an outline of the modified approach to obtain a resistance-like formulation.

5.2 The reciprocal theorem expression

Consider a closed fluid domain V_f bounded by surfaces S. If the fluid is Newtonian and incompressible the reciprocal theorem applies and can be expressed as

$$\int_S (\mathbf{n}\cdot\boldsymbol{\sigma}')\cdot\hat{\mathbf{u}}\,dS + \int_{V_f}(\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}')\cdot\hat{\mathbf{u}}\,dV = \int_S(\mathbf{n}\cdot\hat{\boldsymbol{\sigma}})\cdot\mathbf{u}'\,dS + \int_{V_f}(\boldsymbol{\nabla}\cdot\hat{\boldsymbol{\sigma}})\cdot\mathbf{u}'\,dV, \quad (5.1)$$

where $\mathbf{n}$ is the normal to the surfaces S which points *into* the volume of fluid V_f. Here $\boldsymbol{\sigma}'$ and $\hat{\boldsymbol{\sigma}}$ represent two Newtonian fluid stress tensors defined in the domain V_f:

$$\boldsymbol{\sigma}' = -p'\mathbf{I} + \mu(\nabla\mathbf{u}' + \nabla\mathbf{u}'^T), \quad \hat{\boldsymbol{\sigma}} = -\hat{p}\mathbf{I} + \mu(\boldsymbol{\nabla}\hat{\mathbf{u}} + \boldsymbol{\nabla}\hat{\mathbf{u}}^T), \quad (5.2)$$

where $(p', \mathbf{u}')$ and $(\hat{p}, \hat{\mathbf{u}})$ are the corresponding pressure and velocity fields and μ is the suspending fluid viscosity. At this point the only restrictions on (5.1) are that the velocity fields satisfy

$$\nabla\cdot\mathbf{u}' = 0, \quad \nabla\cdot\hat{\mathbf{u}} = 0, \tag{5.3}$$

that the stress tensor fields represent a Newtonian fluid, and that all the fields are defined and well-behaved in V_f.

In what follows we shall find that by making appropriate choices for the 'primed' and the 'hatted' fields, we can obtain generalized expressions for particle dynamics in a suspending fluid. In the case of zero-Reynolds-number flow problems we obtain a mobility-like formulation for many-body hydrodynamic interactions.

5.3 Formulation of the problem for a general N-particle system

For the case of an N-particle system in an unbounded fluid, we begin by defining the domain V_f as the volume of fluid bounded by the surfaces of the N particles S_{p_i} and a large surface S^∞ which encloses both V_f and the N particles. We also define S^∞ large enough so that it is far from any particle surface and it cuts through only the suspending fluid phase. Thus, S in (5.1) is given by

$$S = \sum_{i=1}^{N} S_{p_i} + S^\infty. \tag{5.4}$$

Next we define the 'hatted' fields above as those corresponding to the disturbance steady Stokes problem for the motion of the particles each isolated in turn in a unbounded domain, say particle j having surface S_{p_j} and having the same location and orientation as the particle $i = j$ in the actual N-particle system at the current time. Recall that a restriction is that all fields be defined and well-behaved in V_f. This restriction will be clearly satisfied here since the hatted fields are defined *everywhere* except inside the particle j which has been excluded from V_f to begin with. The

Stokes fields are governed by:

$$\nabla \cdot \hat{\boldsymbol{\sigma}} = -\nabla \hat{p} + \mu \nabla^2 \hat{\mathbf{u}} = \mathbf{0}, \quad \nabla \cdot \hat{\mathbf{u}} = 0, \tag{5.5a}$$

with

$$\hat{\mathbf{u}} = \sum_{j=1}^{N} \hat{\mathbf{u}}_j, \quad \hat{p} = \sum_{j=1}^{N} \hat{p}_j, \quad \hat{\boldsymbol{\sigma}} = \sum_{j=1}^{N} \hat{\boldsymbol{\sigma}}_j \tag{5.5b}$$

and

$$\hat{\boldsymbol{\sigma}}_j = -\hat{p}_j \mathbf{I} + \mu(\nabla \hat{\mathbf{u}}_j + \nabla \hat{\mathbf{u}}_j^T) \tag{5.5c}$$

and with boundary conditions

$$\hat{\mathbf{u}}_j = \hat{\mathbf{U}}_j + \hat{\boldsymbol{\Omega}}_j \wedge \mathbf{x}_j + \hat{\mathbf{E}}_j \cdot \mathbf{x}_j \quad \text{for} \quad \mathbf{x}_j \in S_{p_j}, \tag{5.5d}$$

$$\hat{\mathbf{u}}_j \to \mathbf{0}, \hat{p}_j \to 0 \quad \text{as} \quad |\mathbf{x}_j| \to \infty, \tag{5.5e}$$

where $\mathbf{x}_j = \mathbf{x} - \mathbf{Y}_{p_j}$ and $\mathbf{Y}_{p_j}$ is the centroid of particle j, i.e., $\int_{V_{p_j}} \mathbf{x}_j \, dV = \mathbf{0}$ and V_{p_j} is the volume of particle j. Here $\hat{\mathbf{U}}_j$ and $\hat{\boldsymbol{\Omega}}_j$ are arbitrary vectors, while $\hat{\mathbf{E}}_j$ represents an arbitrary second-order symmetric traceless tensor.

It is very important to reiterate at this point that these isolated particle Stokes fields (those with an index j) represent the solution to the problem of a particle moving *by itself* in an unbounded domain. Thus, they actually satisfy the governing equations (5.5a) over the entire region of space outside only the particle to which their index j signifies. This property will be exploited in the step involving (5.8) below. Also, since the isolated particle Stokes flow problem contains no particle interaction effects, it is relatively easy to solve for these Stokes fields. We shall show in the last section how to redefine the Stokes fields to have the particle interaction effects embedded in them at the outset with the price being that these fields are much more difficult to evaluate. This, however, leads to expressions that represent a resistance formulation which at times may be more desirable.

Finally, the primed fields are defined as those corresponding to the full disturbance Navier–Stokes problem. The governing equations are

$$\begin{aligned}\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}' &= -\boldsymbol{\nabla}p' + \mu\nabla^2\mathbf{u}' \\ &= \rho\left(\frac{D\,\mathbf{u}'}{D\,t} + \mathbf{u}^{\infty}\cdot\boldsymbol{\nabla}\mathbf{u}' + \mathbf{u}'\cdot\boldsymbol{\nabla}\mathbf{u}^{\infty}\right) \\ &\equiv \mathbf{f}'(\mathbf{u}',\mathbf{u}^{\infty}), \qquad \boldsymbol{\nabla}\cdot\mathbf{u}' = 0,\end{aligned} \tag{5.6a}$$

where the substantial derivative is given by

$$\frac{D\,\mathbf{u}'}{D\,t} = \frac{\partial\,\mathbf{u}'}{\partial\,t} + \mathbf{u}'\cdot\boldsymbol{\nabla}\mathbf{u}' \tag{5.6b}$$

and ρ denotes the density of the suspending fluid. The corresponding boundary conditions are the specified motion of the particles in our real system relative to the imposed flow:

$$\mathbf{u}' = \mathbf{U}_{p_i} + \boldsymbol{\Omega}_{p_i}\wedge\mathbf{x}_i - \mathbf{u}^{\infty}(\mathbf{x}_i) \quad \text{for} \;\; \mathbf{x}_i \in S_{p_i}, \tag{5.6c}$$

$$\mathbf{u}' \to \mathbf{0}, p' \to 0 \quad \text{as} \;\; |\mathbf{x}_i| \to \infty \;\; \forall \;\; i \in [1, N]. \tag{5.6d}$$

Here we have assumed the imposed far-field flow $\mathbf{u}^{\infty}$ satifies the Navier–Stokes equations. The finite size of the grouping of N particles is necessary so that the last boundary condition makes sense. That is, the actual flow field $\mathbf{u}$ becomes $\mathbf{u}^{\infty}$ far from all of the N particles and $\mathbf{u}' = \mathbf{u} - \mathbf{u}^{\infty}$.

Having introduced the above fields, we now proceed to simplify the reciprocal theorem expression (5.1). By defining disturbance quantities that decay to zero in the far-field, it is assumed that the part of the surface integrals in (5.1) associated with S^{∞} can be neglected. Since we know the Stokes fields decay ($\hat{\mathbf{u}}_j$ as $O(r_j^{-1})$ and

$\hat{p}_j$ as $O(r_j^{-2})$ where $r_j = |\mathbf{x}_j|$), the requirement is

$$\mathbf{u}' \sim O(R^{-\alpha}), \qquad p' \sim O(R^{-1-\alpha}) \quad \text{for} \quad \alpha > 0, \tag{5.7}$$

where R is a representative distance from the collection of particles. It can also be seen that the last volume integral of (5.1) is identically zero because $\nabla\cdot\hat{\boldsymbol{\sigma}} = \mathbf{0}$.

Next consider the second surface integral in (5.1) which is now the sum of the surface integrals over the N particles. By using the definition (5.5b) and applying the boundary condition (5.6c) we may write

$$\begin{aligned}\int_S (\mathbf{n}\cdot\hat{\boldsymbol{\sigma}})\cdot\mathbf{u}'\,dS &= \sum_{i=1}^{N}\sum_{j=1}^{N}\int_{S_{p_i}}(\mathbf{n}\cdot\hat{\boldsymbol{\sigma}}_j)\cdot(\mathbf{U}_{p_i}+\boldsymbol{\Omega}_{p_i}\wedge\mathbf{x}_i-\mathbf{u}^\infty)\,dS\\ &= \sum_{i=1}^{N}\int_{S_{p_i}}(\mathbf{n}\cdot\hat{\boldsymbol{\sigma}}_i)\cdot(\mathbf{U}_{p_i}+\boldsymbol{\Omega}_{p_i}\wedge\mathbf{x}_i)\,dS\\ &\quad -\sum_{i=1}^{N}\sum_{j=1}^{N}\int_{S_{p_i}}(\mathbf{n}\cdot\hat{\boldsymbol{\sigma}}_j)\cdot\mathbf{u}^\infty\,dS\end{aligned} \tag{5.8}$$

where we have used the fact that

$$\int_{S_{p_i}}(\mathbf{n}\cdot\hat{\boldsymbol{\sigma}}_j)\,dS = \mathbf{0} = \int_{S_{p_i}}\mathbf{x}_i\wedge(\mathbf{n}\cdot\hat{\boldsymbol{\sigma}}_j)\,dS \quad \text{for} \quad i\neq j \tag{5.9}$$

because $\hat{\boldsymbol{\sigma}}_j$ is divergence free over volumes V_{p_i} with $i \neq j$ and also because it is a symmetric tensor by definition. This shows that these Stokes fields give no force or torque on the particles for $i \neq j$ which can be seen to be a very advantageous property. The case of $i = j$ corresponds to the single summation remaining in (5.8).

Now in order to make a connection with the familiar quantities of the Stokes force, torque, and stresslet on a solid particle, we introduce the total Stokes stress $\hat{\boldsymbol{\sigma}}_{i_T}$ associated with the rigid body motion, $\hat{\mathbf{U}}_i+\hat{\boldsymbol{\Omega}}_i\wedge\mathbf{x}_i$, of particle i in a pure straining flow $(-\hat{\mathbf{E}}_i\cdot\mathbf{x}_i)$ so that

$$\hat{\boldsymbol{\sigma}}_i = \hat{\boldsymbol{\sigma}}_{i_T} + 2\mu\hat{\mathbf{E}}_i. \tag{5.10}$$

Then we may define the Stokes force, torque, and stresslet for the isolated particle i, $\hat{\mathbf{F}}_i$, $\hat{\mathbf{L}}_i$, and $\hat{\mathbf{S}}_i$ respectively, by

$$\hat{\mathbf{F}}_i = \int_{S_{p_i}} \mathbf{n} \cdot \hat{\boldsymbol{\sigma}}_{i_T} \, dS, \tag{5.11a}$$

$$\hat{\mathbf{L}}_i = \int_{S_{p_i}} \mathbf{x}_i \wedge (\mathbf{n} \cdot \hat{\boldsymbol{\sigma}}_{i_T}) \, dS, \tag{5.11b}$$

and

$$\hat{\mathbf{S}}_i = \int_{S_{p_i}} \tfrac{1}{2}\mathbf{x}_i(\mathbf{n} \cdot \hat{\boldsymbol{\sigma}}_{i_T}) + \tfrac{1}{2}(\mathbf{n} \cdot \hat{\boldsymbol{\sigma}}_{i_T})\mathbf{x}_i - \tfrac{1}{3}\mathbf{I}\mathbf{x}_i \cdot (\mathbf{n} \cdot \hat{\boldsymbol{\sigma}}_{i_T}) \, dS. \tag{5.11c}$$

Using (5.10) and (5.11), (5.8) becomes

$$\begin{aligned} \int_S (\mathbf{n} \cdot \hat{\boldsymbol{\sigma}}) \cdot \mathbf{u}' \, dS = &\sum_{i=1}^{N} (\hat{\mathbf{F}}_i \cdot \mathbf{U}_{p_i} + \hat{\mathbf{L}}_i \cdot \boldsymbol{\Omega}_{p_i}) \\ &- \sum_{i=1}^{N} \sum_{j=1}^{N} \int_{S_{p_i}} (\mathbf{n} \cdot \hat{\boldsymbol{\sigma}}_j) \cdot \mathbf{u}^\infty \, dS. \end{aligned} \tag{5.12}$$

The last term of (5.12) may be reexpressed for $i \neq j$ by noting

$$\begin{aligned} \int_{S_{p_i}} (\mathbf{n} \cdot \hat{\boldsymbol{\sigma}}_j) \cdot \mathbf{u}^\infty \, dS &= \int_{V_{p_i}} \hat{\boldsymbol{\sigma}}_j : \nabla \mathbf{u}^\infty \, dV \\ &= \int_{V_{p_i}} \mu(\nabla \hat{\mathbf{u}}_j + \nabla \hat{\mathbf{u}}_j^T) : \nabla \mathbf{u}^\infty \, dV \\ &= \int_{V_{p_i}} \mu \nabla \hat{\mathbf{u}}_j : (\nabla \mathbf{u}^\infty + \nabla \mathbf{u}^{\infty T}) \, dV \\ &= \int_{V_{p_i}} \nabla \hat{\mathbf{u}}_j : \left(-p^\infty \mathbf{I} + \mu(\nabla \mathbf{u}^\infty + \nabla \mathbf{u}^{\infty T}) \right) dV \\ &= \int_{V_{p_i}} \nabla \hat{\mathbf{u}}_j : \boldsymbol{\sigma}^\infty \, dV, \end{aligned} \tag{5.13}$$

where we applied the divergence theorem in the first step and made use of the condition that $\hat{\boldsymbol{\sigma}}_j$ and, in subsequent steps, $\hat{\mathbf{u}}_j$ and $\mathbf{u}^\infty$ are divergence free tensors and vectors, respectively. The third step represents an identity. For $i = j$ we can substitute (5.10) and write the last term of (5.12) as

$$\begin{aligned}\int_{S_{p_i}} (\mathbf{n}\cdot\hat{\boldsymbol{\sigma}}_i)\cdot\mathbf{u}^\infty\, dS &= \int_{S_{p_i}} (\mathbf{n}\cdot\hat{\boldsymbol{\sigma}}_{i_T})\cdot\mathbf{u}^\infty\, dS + \int_{S_{p_i}} (2\mu\mathbf{n}\cdot\hat{\mathbf{E}}_i)\cdot\mathbf{u}^\infty\, dS \\ &= \int_{S_{p_i}} (\mathbf{n}\cdot\hat{\boldsymbol{\sigma}}_{i_T})\cdot\mathbf{u}^\infty\, dS + \int_{V_{p_i}} 2\mu\nabla\mathbf{u}^\infty\, dV : \hat{\mathbf{E}}_i \\ &= \int_{S_{p_i}} (\mathbf{n}\cdot\hat{\boldsymbol{\sigma}}_{i_T})\cdot\mathbf{u}^\infty\, dS + \int_{V_{p_i}} \boldsymbol{\sigma}^\infty\, dV : \hat{\mathbf{E}}_i, \end{aligned} \tag{5.14}$$

where we have applied the divergence theorem and then exploited the symmetric and traceless properties of $\hat{\mathbf{E}}_i$. If we combine (5.13) and (5.14) in (5.12) we then have

$$\begin{aligned}\int_S (\mathbf{n}\cdot\hat{\boldsymbol{\sigma}})\cdot\mathbf{u}'\, dS = \sum_{i=1}^{N}\Big(&\hat{\mathbf{F}}_i\cdot\mathbf{U}_{p_i} + \hat{\mathbf{L}}_i\cdot\boldsymbol{\Omega}_{p_i} \\ &- \int_{S_{p_i}} (\mathbf{n}\cdot\hat{\boldsymbol{\sigma}}_{i_T})\cdot\mathbf{u}^\infty\, dS - \int_{V_{p_i}} \boldsymbol{\sigma}^\infty\, dV : \hat{\mathbf{E}}_i\Big) \\ &- \sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N} \int_{V_{p_i}} \nabla\hat{\mathbf{u}}_j : \boldsymbol{\sigma}^\infty\, dV. \end{aligned} \tag{5.15}$$

Now let us consider the first surface integral in (5.1):

$$\int_S (\mathbf{n}\cdot\boldsymbol{\sigma}')\cdot\hat{\mathbf{u}}\, dS = \sum_{i=1}^{N}\sum_{j=1}^{N} \int_{S_{p_i}} \mathbf{n}\cdot(\boldsymbol{\sigma}-\boldsymbol{\sigma}^\infty)\cdot\hat{\mathbf{u}}_j\, dS. \tag{5.16}$$

For $i = j$ we have

$$\begin{aligned}\int_{S_{p_i}} (\mathbf{n}\cdot\boldsymbol{\sigma})\cdot\hat{\mathbf{u}}_i\, dS &= \int_{S_{p_i}} (\mathbf{n}\cdot\boldsymbol{\sigma})\, dS\cdot\hat{\mathbf{U}}_i + \int_{S_{p_i}} \mathbf{x}_i\wedge(\mathbf{n}\cdot\boldsymbol{\sigma})\, dS\cdot\hat{\boldsymbol{\Omega}}_i \\ &\quad + \int_{S_{p_i}} (\mathbf{n}\cdot\boldsymbol{\sigma})\mathbf{x}_i\, dS : \hat{\mathbf{E}}_i \\ &= \mathbf{F}_i^H\cdot\hat{\mathbf{U}}_i + \mathbf{L}_i^H\cdot\hat{\boldsymbol{\Omega}}_i + \mathbf{S}_i^H : \hat{\mathbf{E}}_i, \end{aligned} \tag{5.17}$$

where the total hydrodynamic force, torque, and stresslet on particle i, $\mathbf{F}_i^H$, $\mathbf{L}_i^H$, and $\mathbf{S}_i^H$ respectively, are defined by analogous expressions to (5.11) with $\boldsymbol{\sigma}$ replacing $\hat{\boldsymbol{\sigma}}_{i_T}$.

Similarly for $i = j$ we may write

$$\begin{aligned}\int_{S_{p_i}} (\mathbf{n}\cdot\boldsymbol{\sigma}^\infty)\cdot\hat{\mathbf{u}}_i\,dS &= \int_{S_{p_i}} (\mathbf{n}\cdot\boldsymbol{\sigma}^\infty)\,dS\cdot\hat{\mathbf{U}}_i + \int_{S_{p_i}} \mathbf{x}_i\wedge(\mathbf{n}\cdot\boldsymbol{\sigma}^\infty)\,dS\cdot\hat{\boldsymbol{\Omega}}_i \\ &\quad + \int_{S_{p_i}} (\mathbf{n}\cdot\boldsymbol{\sigma}^\infty)\mathbf{x}_i\,dS : \hat{\mathbf{E}}_i \\ &= \int_{V_{p_i}} \rho\frac{D\,\mathbf{u}^\infty}{D\,t}\,dV\cdot\hat{\mathbf{U}}_i + \int_{V_{p_i}} \mathbf{x}_i\wedge(\rho\frac{D\,\mathbf{u}^\infty}{D\,t})\,dV\cdot\hat{\boldsymbol{\Omega}}_i \\ &\quad + \int_{V_{p_i}} (\rho\frac{D\,\mathbf{u}^\infty}{D\,t})\mathbf{x}_i\,dV : \hat{\mathbf{E}}_i + \int_{V_{p_i}} \boldsymbol{\sigma}^\infty\,dV : \hat{\mathbf{E}}_i, \end{aligned} \tag{5.18}$$

where we have used the condition that $\mathbf{u}^\infty$ satisfies the Navier–Stokes equations so that $\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}^\infty = \rho\,D\,\mathbf{u}^\infty/D\,t$. Lastly, for $i \neq j$ we we may apply the divergence theorem to the volume of particle i:

$$\int_{S_{p_i}} (\mathbf{n}\cdot\boldsymbol{\sigma}^\infty)\cdot\hat{\mathbf{u}}_j\,dS = \int_{V_{p_i}} (\rho\frac{D\,\mathbf{u}^\infty}{D\,t})\cdot\hat{\mathbf{u}}_j\,dV + \int_{V_{p_i}} \boldsymbol{\sigma}^\infty : \boldsymbol{\nabla}\hat{\mathbf{u}}_j\,dV. \tag{5.19}$$

Now if we combine (5.17), (5.18), and (5.19) in (5.16) we obtain

$$\begin{aligned}\int_S (\mathbf{n}\cdot\boldsymbol{\sigma}')\cdot\hat{\mathbf{u}}\,dS &= \sum_{i=1}^{N}\Big(\mathbf{F}_i^H\cdot\hat{\mathbf{U}}_i + \mathbf{L}_i^H\cdot\hat{\boldsymbol{\Omega}}_i + \mathbf{S}_i^H : \hat{\mathbf{E}}_i \\ &\quad - \int_{V_{p_i}} \rho\frac{D\,\mathbf{u}^\infty}{D\,t}\,dV\cdot\hat{\mathbf{U}}_i - \int_{V_{p_i}} \mathbf{x}_i\wedge(\rho\frac{D\,\mathbf{u}^\infty}{D\,t})\,dV\cdot\hat{\boldsymbol{\Omega}}_i \\ &\quad - \int_{V_{p_i}} (\rho\frac{D\,\mathbf{u}^\infty}{D\,t})\mathbf{x}_i\,dV : \hat{\mathbf{E}}_i - \int_{V_{p_i}} \boldsymbol{\sigma}^\infty\,dV : \hat{\mathbf{E}}_i\Big) \\ &\quad + \sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\Big(\int_{S_{p_i}} (\mathbf{n}\cdot\boldsymbol{\sigma})\cdot\hat{\mathbf{u}}_j\,dS \\ &\quad - \int_{V_{p_i}} (\rho\frac{D\,\mathbf{u}^\infty}{D\,t})\cdot\hat{\mathbf{u}}_j\,dV - \int_{V_{p_i}} \boldsymbol{\sigma}^\infty : \boldsymbol{\nabla}\hat{\mathbf{u}}_j\,dV\Big). \end{aligned} \tag{5.20}$$

Finally combining (5.15) and (5.20) in (5.1), while using the definition for $\mathbf{f}'$ in (5.6a) to represent the inertial terms in the disturbance Navier–Stokes equation, the reciprocal theorem expression becomes

$$
\begin{aligned}
\sum_{i=1}^{N}\Big(\hat{\mathbf{F}}_i \cdot \mathbf{U}_{p_i} + \hat{\mathbf{L}}_i \cdot \boldsymbol{\Omega}_{p_i} - \int_{S_{p_i}} (\mathbf{n}\cdot\hat{\boldsymbol{\sigma}}_{i_T})\cdot \mathbf{u}^{\infty}\, d\,S\Big) = \\
\sum_{i=1}^{N}\Big(\mathbf{F}_i^H \cdot \hat{\mathbf{U}}_i + \mathbf{L}_i^H \cdot \hat{\boldsymbol{\Omega}}_i + \mathbf{S}_i^H : \hat{\mathbf{E}}_i - \int_{V_{p_i}} \rho \frac{D\,\mathbf{u}^{\infty}}{D\,t}\, d\,V \cdot \hat{\mathbf{U}}_i \\
- \int_{V_{p_i}} \mathbf{x}_i \wedge (\rho \frac{D\,\mathbf{u}^{\infty}}{D\,t})\, d\,V \cdot \hat{\boldsymbol{\Omega}}_i - \int_{V_{p_i}} (\rho \frac{D\,\mathbf{u}^{\infty}}{D\,t}) \mathbf{x}_i\, d\,V : \hat{\mathbf{E}}_i\Big) \\
+ \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j\neq i}}^{N} \Big(\int_{S_{p_i}} (\mathbf{n}\cdot\boldsymbol{\sigma})\cdot \hat{\mathbf{u}}_j\, d\,S - \int_{V_{p_i}} (\rho \frac{D\,\mathbf{u}^{\infty}}{D\,t}) \cdot \hat{\mathbf{u}}_j\, d\,V\Big) \\
+ \sum_{i=1}^{N} \int_{V_f} \mathbf{f}' \cdot \hat{\mathbf{u}}_i\, d\,V. \qquad (5.21)
\end{aligned}
$$

While the terms inside the single sums exist even for an isolated particle (i.e., N=1), it is the terms in the double sum that are necessary take into account the multi-particle interactions. The last sum in (5.21), along with the terms containing the substantial derivative of the imposed far-field flow, represent contributions from the inertia of the suspending fluid and are finite Reynolds number effects.

To proceed further, either additional detailed information concerning the Stokes flow quantities and $\mathbf{u}^{\infty}$ are required or we may employ Taylor series expansions of the velocity fields about the centers of the particles. In order to simplify the expression (5.21) in a general way, we shall carry on with the latter in the next section.

5.4 Simplifications of the reciprocal theorem for a general N-particle system

We shall first consider the first surface integral in (5.21). By assuming $\mathbf{u}^{\infty}$ has locally linear flow behavior over the volume of each particle i (or, rather, the length scale over which the velocity gradient varies is large compared to the size of the particle), we are justified in using a Taylor series expansion to write:

$$\begin{aligned}\mathbf{u}^\infty(\mathbf{x}_i) &= \mathbf{u}^\infty(\mathbf{0}) + \mathbf{x}_i \cdot \nabla \mathbf{u}^\infty(\mathbf{0}) + O(\mathbf{x}_i^2) \\ &= \mathbf{U}_i^\infty + \boldsymbol{\Omega}_i^\infty \wedge \mathbf{x}_i + \mathbf{E}_i^\infty \cdot \mathbf{x}_i + O(\mathbf{x}_i^2),\end{aligned} \tag{5.22}$$

where $\mathbf{u}^\infty(\mathbf{x}_i = \mathbf{0}) = \mathbf{U}_i^\infty$ and $\boldsymbol{\Omega}_i^\infty$ and $\mathbf{E}_i^\infty$ represent the local rotational and pure straining flow, respectively, at the center of particle i. Substituting (5.22) in the surface integral we obtain

$$\int_{S_{p_i}} (\mathbf{n} \cdot \hat{\boldsymbol{\sigma}}_{i_T}) \cdot \mathbf{u}^\infty \, dS = \hat{\mathbf{F}}_i \cdot \mathbf{U}_i^\infty + \hat{\mathbf{L}}_i \cdot \boldsymbol{\Omega}_i^\infty + \hat{\mathbf{S}}_i : \mathbf{E}_i^\infty, \tag{5.23}$$

where we note $\hat{\mathbf{S}}_i$ is the Stokes stresslet acting on the isolated particle i and is given by (5.11c). Next we can express the Stokes force, torque, and stresslet in terms of the familiar resistance tensor formulation of, for example, Happel and Brenner [18]:

$$\hat{\mathbf{F}}_i = -\hat{\mathbf{R}}_{FU_i} \cdot \hat{\mathbf{U}}_i - \hat{\mathbf{R}}_{F\Omega_i} \cdot \hat{\boldsymbol{\Omega}}_i + \hat{\mathbf{R}}_{FE_i} : (-\hat{\mathbf{E}}_i), \tag{5.24a}$$

$$\hat{\mathbf{L}}_i = -\hat{\mathbf{R}}_{LU_i} \cdot \hat{\mathbf{U}}_i - \hat{\mathbf{R}}_{L\Omega_i} \cdot \hat{\boldsymbol{\Omega}}_i + \hat{\mathbf{R}}_{LE_i} : (-\hat{\mathbf{E}}_i), \tag{5.24b}$$

$$\hat{\mathbf{S}}_i = -\hat{\mathbf{R}}_{SU_i} \cdot \hat{\mathbf{U}}_i - \hat{\mathbf{R}}_{S\Omega_i} \cdot \hat{\boldsymbol{\Omega}}_i + \hat{\mathbf{R}}_{SE_i} : (-\hat{\mathbf{E}}_i). \tag{5.24c}$$

Here $\hat{\mathbf{R}}_{FU_i}$, $\hat{\mathbf{R}}_{F\Omega_i}$, $\hat{\mathbf{R}}_{LU_i}$, and $\hat{\mathbf{R}}_{L\Omega_i}$ are second-order, $\hat{\mathbf{R}}_{FE_i}$, $\hat{\mathbf{R}}_{LE_i}$, $\hat{\mathbf{R}}_{SU_i}$, and $\hat{\mathbf{R}}_{S\Omega_i}$ are third-order, and $\hat{\mathbf{R}}_{SE_i}$ is a fourth-order resistance tensor for an isolated particle i in an unbounded domain. Now if we exploit the symmetry properties of the resistance tensors, where, for example, $\hat{\mathbf{R}}^T_{FU_i} = \hat{\mathbf{R}}_{FU_i}$ and $\hat{\mathbf{R}}^T_{FE_i} = \hat{\mathbf{R}}_{SU_i}$, we can express the right-hand side of (5.21) by substituting (5.23) and (5.24) to find

$$\sum_{i=1}^{N}\Big(\hat{\mathbf{F}}_i \cdot \mathbf{U}_{p_i} + \hat{\mathbf{L}}_i \cdot \boldsymbol{\Omega}_{p_i} - \int_{S_{p_i}} (\mathbf{n} \cdot \hat{\boldsymbol{\sigma}}_{i_T}) \cdot \mathbf{u}^{\infty}\, d\,S\Big) = \sum_{i=1}^{N}\Big($$
$$\Big(\hat{\mathbf{R}}_{FU_i} \cdot (\mathbf{U}_i^{\infty} - \mathbf{U}_{p_i}) + \hat{\mathbf{R}}_{F\Omega_i} \cdot (\boldsymbol{\Omega}_i^{\infty} - \boldsymbol{\Omega}_{p_i}) + \hat{\mathbf{R}}_{FE_i} : \mathbf{E}_i^{\infty}\Big) \cdot \hat{\mathbf{U}}_i$$
$$+\Big(\hat{\mathbf{R}}_{LU_i} \cdot (\mathbf{U}_i^{\infty} - \mathbf{U}_{p_i}) + \hat{\mathbf{R}}_{L\Omega_i} \cdot (\boldsymbol{\Omega}_i^{\infty} - \boldsymbol{\Omega}_{p_i}) + \hat{\mathbf{R}}_{LE_i} : \mathbf{E}_i^{\infty}\Big) \cdot \hat{\boldsymbol{\Omega}}_i$$
$$+\Big(\hat{\mathbf{R}}_{SU_i} \cdot (\mathbf{U}_i^{\infty} - \mathbf{U}_{p_i}) + \hat{\mathbf{R}}_{S\Omega_i} \cdot (\boldsymbol{\Omega}_i^{\infty} - \boldsymbol{\Omega}_{p_i}) + \hat{\mathbf{R}}_{SE_i} : \mathbf{E}_i^{\infty}\Big) : \hat{\mathbf{E}}_i\Big). \qquad (5.25)$$

Based on the linearity of the governing equations for the Stokes velocity fields and the linear dependence of the velocity fields on $\hat{\mathbf{U}}_i$, $\hat{\boldsymbol{\Omega}}_i$, and $\hat{\mathbf{E}}_i$, the velocity fields may be expressed as

$$\hat{\mathbf{u}}_i = \hat{\mathcal{U}}_{U_i} \cdot \hat{\mathbf{U}}_i + \hat{\mathcal{U}}_{\Omega_i} \cdot \hat{\boldsymbol{\Omega}}_i + \hat{\mathcal{U}}_{E_i} : \hat{\mathbf{E}}_i, \qquad (5.26)$$

where the second-order tensors $\hat{\mathcal{U}}_{U_i}$ and $\hat{\mathcal{U}}_{\Omega_i}$ and the third-order tensor $\hat{\mathcal{U}}_{E_i}$ are functions of position and depend on the current orientation of particle i. Note that $\hat{\mathcal{U}}_{U_i}$ is equivalent to the $\hat{\mathbf{M}}$-field described in Chapter 2. Note also that i is just a particle index so that $\hat{\mathbf{u}}_i$ satisfies the same boundary conditions as in (5.5d) and (5.5e) with i replacing j. Now since the $\hat{\mathbf{U}}_i$, $\hat{\boldsymbol{\Omega}}_i$, and $\hat{\mathbf{E}}_i$ are arbitrary (except the symmetric/traceless requirement of $\hat{\mathbf{E}}_i$) and independent we can write a separate equation for each of them and for each particle i. Thus we can reduce (5.21) to three vector equations for each particle by using (5.25) and (5.26) to obtain

$$\hat{\mathbf{R}}_{FU_i} \cdot (\mathbf{U}_i^{\infty} - \mathbf{U}_{p_i}) + \hat{\mathbf{R}}_{F\Omega_i} \cdot (\boldsymbol{\Omega}_i^{\infty} - \boldsymbol{\Omega}_{p_i}) + \hat{\mathbf{R}}_{FE_i} : \mathbf{E}_i^{\infty} =$$
$$\mathbf{F}_i^H - \int_{V_{p_i}} \rho \frac{D\,\mathbf{u}^{\infty}}{D\,t}\, d\,V$$
$$+\sum_{j=1}^{N,\ j\neq i}\Big(\int_{S_{p_j}} (\mathbf{n} \cdot \boldsymbol{\sigma}) \cdot \hat{\mathcal{U}}_{U_i}\, d\,S - \int_{V_{p_j}} (\rho \frac{D\,\mathbf{u}^{\infty}}{D\,t}) \cdot \hat{\mathcal{U}}_{U_i}\, d\,V\Big)$$
$$+\int_{V_f} \mathbf{f}' \cdot \hat{\mathcal{U}}_{U_i}\, d\,V \qquad (5.27a)$$

$$
\begin{aligned}
&\hat{\mathbf{R}}_{LU_i} \cdot (\mathbf{U}_i^\infty - \mathbf{U}_{p_i}) + \hat{\mathbf{R}}_{L\Omega_i} \cdot (\boldsymbol{\Omega}_i^\infty - \boldsymbol{\Omega}_{p_i}) + \hat{\mathbf{R}}_{LE_i} : \mathbf{E}_i^\infty = \\
&\mathbf{L}_i^H - \int_{V_{p_i}} \mathbf{x}_i \wedge (\rho \frac{D\,\mathbf{u}^\infty}{D\,t})\, d\,V \\
&+ \sum_{j=1}^{\substack{j \neq i \\ N}} \left(\int_{S_{p_j}} (\mathbf{n} \cdot \boldsymbol{\sigma}) \cdot \hat{\mathcal{U}}_{\Omega_i}\, d\,S - \int_{V_{p_j}} (\rho \frac{D\,\mathbf{u}^\infty}{D\,t}) \cdot \hat{\mathcal{U}}_{\Omega_i}\, d\,V \right) \\
&+ \int_{V_f} \mathbf{f}' \cdot \hat{\mathcal{U}}_{\Omega_i}\, d\,V
\end{aligned}
\tag{5.27b}
$$

$$
\begin{aligned}
&\hat{\mathbf{R}}_{SU_i} \cdot (\mathbf{U}_i^\infty - \mathbf{U}_{p_i}) + \hat{\mathbf{R}}_{S\Omega_i} \cdot (\boldsymbol{\Omega}_i^\infty - \boldsymbol{\Omega}_{p_i}) + \hat{\mathbf{R}}_{SE_i} : \mathbf{E}_i^\infty = \\
&\mathbf{S}_i^H - symtr \int_{V_{p_i}} (\rho \frac{D\,\mathbf{u}^\infty}{D\,t}) \mathbf{x}_i\, d\,V \\
&+ symtr \sum_{j=1}^{\substack{j \neq i \\ N}} \left(\int_{S_{p_j}} (\mathbf{n} \cdot \boldsymbol{\sigma}) \cdot \hat{\mathcal{U}}_{E_i}\, d\,S - \int_{V_{p_j}} (\rho \frac{D\,\mathbf{u}^\infty}{D\,t}) \cdot \hat{\mathcal{U}}_{E_i}\, d\,V \right) \\
&+ symtr \int_{V_f} \mathbf{f}' \cdot \hat{\mathcal{U}}_{E_i}\, d\,V,
\end{aligned}
\tag{5.27c}
$$

where *symtr* indicates "the symmetric and traceless part of."

Ignoring for the moment the last volume integral in the above expressions and assuming the Stokes quantities are known, we have obtained relationships between the motion of particle i, the imposed flow and the surface stresses on all the particles. In their present form the above expressions are of little utility. In the next section, however, we will evaluate the particle interaction contributions from the terms under the j summation which is appropriate when the particles are far from each other. This will result in a mobility-like formulation for the relationship between the particle motion and the hydrodynamic forces, torques, and stresslets acting on them. The volume integral over the fluid domain represents the contribution from the inertia of the fluid due to the disturbance flow. It is difficult to evaluate in general since it contains all the nonlinearities and history dependence.

5.5 Evaluation of the far-field particle interactions

We can make use of Taylor series expansions of the $\hat{\mathcal{U}}_i$'s about the center of particle $j \neq i$ to compute the far-field particle interactions:

$$\begin{aligned}\hat{\mathcal{U}}_{U_i}(\mathbf{x}_j) &= \hat{\mathcal{U}}_{U_i}(\mathbf{0}) + \mathbf{x}_j \cdot \nabla \hat{\mathcal{U}}_{U_i}(\mathbf{0}) + \frac{\mathbf{x}_j\mathbf{x}_j}{2} : \nabla\nabla \hat{\mathcal{U}}_{U_i}(\mathbf{0}) + \dots \\ &= \hat{\mathcal{U}}^j_{U_i} + \mathbf{x}_j \cdot \nabla \hat{\mathcal{U}}^j_{U_i} + \frac{\mathbf{x}_j\mathbf{x}_j}{2} : \nabla\nabla \hat{\mathcal{U}}^j_{U_i} + \dots,\end{aligned} \tag{5.28a}$$

$$\begin{aligned}\hat{\mathcal{U}}_{\Omega_i}(\mathbf{x}_j) &= \hat{\mathcal{U}}_{\Omega_i}(\mathbf{0}) + \mathbf{x}_j \cdot \nabla \hat{\mathcal{U}}_{\Omega_i}(\mathbf{0}) + \frac{\mathbf{x}_j\mathbf{x}_j}{2} : \nabla\nabla \hat{\mathcal{U}}_{\Omega_i}(\mathbf{0}) + \dots \\ &= \hat{\mathcal{U}}^j_{\Omega_i} + \mathbf{x}_j \cdot \nabla \hat{\mathcal{U}}^j_{\Omega_i} + \frac{\mathbf{x}_j\mathbf{x}_j}{2} : \nabla\nabla \hat{\mathcal{U}}^j_{\Omega_i} + \dots,\end{aligned} \tag{5.28b}$$

$$\begin{aligned}\hat{\mathcal{U}}_{E_i}(\mathbf{x}_j) &= \hat{\mathcal{U}}_{E_i}(\mathbf{0}) + \mathbf{x}_j \cdot \nabla \hat{\mathcal{U}}_{E_i}(\mathbf{0}) + \frac{\mathbf{x}_j\mathbf{x}_j}{2} : \nabla\nabla \hat{\mathcal{U}}_{E_i}(\mathbf{0}) + \dots \\ &= \hat{\mathcal{U}}^j_{E_i} + \mathbf{x}_j \cdot \nabla \hat{\mathcal{U}}^j_{E_i} + \frac{\mathbf{x}_j\mathbf{x}_j}{2} : \nabla\nabla \hat{\mathcal{U}}^j_{E_i} + \dots,\end{aligned} \tag{5.28c}$$

where the superscript j on the above terms indicate the terms are evaluated at the center of particle j. Note that each successive higher order derivative above is $O(r_{ij}^{-1})$ smaller where r_{ij} is the center-to-center distance between particles i and j. Now if we substitute these expansions into the surface integrals of (5.27) we find, for example,

$$\begin{aligned}\int_{S_{p_j}} (\mathbf{n} \cdot \boldsymbol{\sigma}) \cdot \hat{\mathcal{U}}_{U_i}\, dS &= \mathbf{F}^H_j \cdot \hat{\mathcal{U}}^j_{U_i} \\ &+ \mathbf{L}^H_j \cdot \frac{1}{2}\nabla \wedge \hat{\mathcal{U}}^j_{U_i} + \mathbf{S}^H_j : \frac{1}{2}\left(\nabla \hat{\mathcal{U}}^j_{U_i} + {}^T\nabla \hat{\mathcal{U}}^j_{U_i}\right) + O(\nabla\nabla \hat{\mathcal{U}}^j_{U_i}),\end{aligned} \tag{5.29a}$$

$$\int_{S_{P_j}} (\mathbf{n}\cdot\boldsymbol{\sigma})\cdot\hat{\mathcal{U}}_{\Omega_i}\, dS = \mathbf{F}_j^H\cdot\hat{\mathcal{U}}^j_{\Omega_i}$$
$$+\mathbf{L}_j^H\cdot\frac{1}{2}\boldsymbol{\nabla}\wedge\hat{\mathcal{U}}^j_{\Omega_i} + \mathbf{S}_j^H : \frac{1}{2}\left(\boldsymbol{\nabla}\hat{\mathcal{U}}^j_{\Omega_i} + {}^T\boldsymbol{\nabla}\hat{\mathcal{U}}^j_{\Omega_i}\right) + O(\boldsymbol{\nabla}\boldsymbol{\nabla}\hat{\mathcal{U}}^j_{\Omega_i}), \qquad (5.29b)$$

$$\int_{S_{P_j}} (\mathbf{n}\cdot\boldsymbol{\sigma})\cdot\hat{\mathcal{U}}_{E_i}\, dS = \mathbf{F}_j^H\cdot\hat{\mathcal{U}}^j_{E_i}$$
$$+\mathbf{L}_j^H\cdot\frac{1}{2}\boldsymbol{\nabla}\wedge\hat{\mathcal{U}}^j_{E_i} + \mathbf{S}_j^H : \frac{1}{2}\left(\boldsymbol{\nabla}\hat{\mathcal{U}}^j_{E_i} + {}^T\boldsymbol{\nabla}\hat{\mathcal{U}}^j_{E_i}\right) + O(\boldsymbol{\nabla}\boldsymbol{\nabla}\hat{\mathcal{U}}^j_{E_i}). \qquad (5.29c)$$

Since the tensor fields $\hat{\mathcal{U}}^j_{U_i}$ and $\hat{\mathcal{U}}^j_{\Omega_i}$ or $\hat{\mathcal{U}}^j_{E_i}$ are order $O(r_{ij}^{-1})$ and $O(r_{ij}^{-2})$ respectively, the error in truncating the series at the force dipole is order $O(r_{ij}^{-3})$ for the force/velocity expression (5.27a) and $O(r_{ij}^{-4})$ for both the torque/rotation and stresslet/rate-of-strain expressions (5.27b) and (5.27c). A higher level of accuracy for spherical particles can be achieved without additional moments. This will be discussed in the next section.

A similar level of accuracy can be obtained for the volume integrals over particle j in (5.27). For example,

$$\int_{V_{P_j}} (\rho\frac{D\,\mathbf{u}^\infty}{D\,t})\cdot\hat{\mathcal{U}}_{U_i}\, dV = \int_{V_{P_j}} \rho\frac{D\,\mathbf{u}^\infty}{D\,t}\, dV\cdot\hat{\mathcal{U}}^j_{U_i}$$
$$+\int_{V_{P_j}} \mathbf{x}_j\wedge(\rho\frac{D\,\mathbf{u}^\infty}{D\,t})\, dV\cdot\frac{1}{2}\boldsymbol{\nabla}\wedge\hat{\mathcal{U}}^j_{U_i}$$
$$+symtr\int_{V_{P_j}} \mathbf{x}_j(\rho\frac{D\,\mathbf{u}^\infty}{D\,t})\, dV : \frac{1}{2}\left(\boldsymbol{\nabla}\hat{\mathcal{U}}^j_{U_i} + {}^T\boldsymbol{\nabla}\hat{\mathcal{U}}^j_{U_i}\right)$$
$$+O(\boldsymbol{\nabla}\boldsymbol{\nabla}\hat{\mathcal{U}}^j_{U_i}). \qquad (5.30)$$

If we use the results derived here (5.29) and (5.30) in the expressions in the previous section (5.27), one can see that we have a closed-form expression for the particle dynamics once the fluid volume integral involving $\mathbf{f}'$ is evaluated or approximated in terms of the other quantities present. That is, for example, if we have specified the imposed flow and the forces and torques acting on the particles, we then have three expressions to evaluate the three quantities: the particle velocity, rotation, and stresslet. This assumes, of course, that the Stokes fields are known.

5.6 Conclusions

If we introduce the appropriate quantities for monodisperse spherical particles and neglect all inertial terms in the above expressions we recover the mobility formulation for multi-particle dynamics under Stokes flow conditions which agree with those used for the Stokesian dynamics calculations of Durlofsky, Brady, and Bossis [16]. For a spherical particle of radius a_i the Stokes quantities are given by

$$\hat{\mathcal{U}}_{U_i} = a_i\frac{3}{4}\left(\frac{\mathbf{I}}{r_i} + \frac{\mathbf{x}_i\mathbf{x}_i}{r_i^3}\right) + a_i^3\frac{1}{4}\left(\frac{\mathbf{I}}{r_i^3} - \frac{3\mathbf{x}_i\mathbf{x}_i}{r_i^5}\right),$$
$$\hat{\mathcal{U}}_{\Omega_i} = a_i^3\frac{\mathbf{x}_i\cdot\boldsymbol{\epsilon}}{r_i^3}, \quad \hat{\mathcal{U}}_{E_i} = a_i^3\frac{5}{2}\frac{\mathbf{x}_i\mathbf{x}_i\mathbf{x}_i}{r_i^5} + a_i^5\left(\frac{\mathbf{I}\mathbf{x}_i}{r_i^5} - \frac{5}{2}\frac{\mathbf{x}_i\mathbf{x}_i\mathbf{x}_i}{r_i^7}\right), \tag{5.31a}$$

and

$$\hat{\mathbf{R}}_{FU_i} = 6\pi\mu a_i\mathbf{I}, \qquad \hat{\mathbf{R}}_{L\Omega_i} = 8\pi\mu a_i^3\mathbf{I}, \quad (\hat{\mathbf{R}}_{SE_i})_{klmn} = \frac{20\pi}{3}\mu a_i^3\delta_{km}\delta_{ln},$$
$$\hat{\mathbf{R}}_{F\Omega_i} = \hat{\mathbf{R}}_{FE_i} = \hat{\mathbf{R}}_{LU_i} = \hat{\mathbf{R}}_{LE_i} = \hat{\mathbf{R}}_{SU_i} = \hat{\mathbf{R}}_{S\Omega_i} = \mathbf{0}, \tag{5.31b}$$

where $a_i = |\mathbf{x}_i|$ and $\boldsymbol{\epsilon}$ is a third-order permutation tensor called the alternating tensor. As explained by Durlofsky *et al.* [16], we may achieve a higher level of accuracy for spherical particles while maintaining the retention of force moments up to the force dipole only. This is accomplished by including the reducible force moments from the quadrupoles and octupoles, the so-called finite-size quadrupoles and octupoles. Thus, for example, (5.29a) would become

$$\int_{S_{p_j}} (\mathbf{n}\cdot\boldsymbol{\sigma})\cdot\hat{\mathcal{U}}_{U_i}\,dS = \mathbf{F}_j^H\cdot(1+\tfrac{1}{6}a_j^2\nabla^2)\hat{\mathcal{U}}^j_{U_i}$$
$$+\mathbf{L}_j^H\cdot\frac{1}{2}\boldsymbol{\nabla}\wedge\hat{\mathcal{U}}^j_{U_i}+\mathbf{S}_j^H:(1+\tfrac{1}{10}a_j^2\nabla^2)\frac{1}{2}\left(\boldsymbol{\nabla}\hat{\mathcal{U}}^j_{U_i}+{}^T\boldsymbol{\nabla}\hat{\mathcal{U}}^j_{U_i}\right)$$
$$+O(\boldsymbol{\nabla}\boldsymbol{\nabla}\hat{\mathcal{U}}^j_{U_i}). \tag{5.32}$$

This modification reduces the error in the translational velocity expression (5.27a) to $O(r_{ij}^{-6})$ and that in the rotational velocity and rate-of-strain expressions to $O(r_{ij}^{-7})$ because the remaining parts of the force quadrupoles and octupoles must be induced by the spherical particle interactions. In other words, the coefficient of the $O(\boldsymbol{\nabla}\boldsymbol{\nabla}\hat{\mathcal{U}}^j_{U_i})$-term representing the error in (5.32), for example, is the induced force quadrupole, which has magnitude $O(r_{ij}^{-3})$.

In a more general manner, higher order accuracy can be achieved by introducing higher order boundary conditions in the Stokes fields than simply linear variations and also by retaining terms beyond the force dipoles, e.g., the force quadrupole, octupole, etc. One must take care to be consistent in doing this, so that all terms are collected to the proper level of accuracy and that one comes out in the end with the proper number equations for a well-posed problem. Here, we have three vector equations. Thus, in principle, they can be used to solve for the particle translational and rotational velocities and the particle stresslets if given the hydrodynamic forces and torques from the equations of motion for the particles and the imposed flow. This assumes, of course, that the volume integrals involving $\mathbf{f}'$ can be approximated in some fashion.

Except the volume integrals involving $\mathbf{f}'$, all terms in the above expressions are instantaneous quantities evaluated at the current time in the current configuration of particle positions and orientations. In other words, history dependence must come from these volume integrals over the fluid domain. These contributions are difficult to evaluate in general. However, one can make approximations that take advantage of the far-field condition of the particle configuration and of the smallness of the Reynolds number, for example. This is left for future work. We have at least reduced the computation of the inertial effects to the evaluation of a volume integral which may lend itself to simpler (or less computationally intensive) numerical computations that rely on a "coarse-grid"-approximation of the flow field.

5.6.1 The Resistance Formulation

To obtain a resistance formulation for the particle dynamics, the Stokes fields' boundary conditions must be redefined. The new boundary conditions are

$$\hat{\mathbf{u}}_j = \hat{\mathbf{U}}_j + \hat{\mathbf{\Omega}}_j \wedge \mathbf{x}_j + \hat{\mathbf{E}}_j \cdot \mathbf{x}_j \quad \text{for} \quad \mathbf{x}_j \in S_{p_j}, \tag{5.33a}$$

$$\hat{\mathbf{u}}_j = \mathbf{0} \quad \text{for} \quad \mathbf{x}_j \in S_{p_i}, \quad i \neq j, \tag{5.33b}$$

$$\hat{\mathbf{u}}_j \to \mathbf{0}, \hat{p}_j \to 0 \quad \text{as} \quad |\mathbf{x}_j| \to \infty, \tag{5.33c}$$

which means the individual particle Stokes fields now represent the solution to the problem of the motion of a given particle j in a fixed bed of all the other particles present. Thus, these Stokes fields have multiparticle effects since they must recognize the existence of the other particles in their actual locations at the current time. We note also that the individual particle Stokes fields are now only defined in the real fluid domain of our system, and thus their governing equations are only valid outside of all the particles in the system.

If we ignore for the moment the existence of an imposed flow, the reciprocal theorem expression becomes

$$\sum_{i=1}^{N}\left(\mathbf{F}_i^H\cdot\hat{\mathbf{U}}_i+\mathbf{L}_i^H\cdot\hat{\boldsymbol{\Omega}}_i+\mathbf{S}_i^H:\hat{\mathbf{E}}_i\right)+\int_{V_f}(\boldsymbol{\nabla}\cdot\boldsymbol{\sigma})\cdot\hat{\mathbf{u}}\,dV=$$
$$\sum_{i=1}^{N}\sum_{j=1}^{N}\left(\hat{\mathbf{F}}_i^j\cdot\mathbf{U}_{p_i}+\hat{\mathbf{L}}_i^j\cdot\boldsymbol{\Omega}_{p_i}\right), \tag{5.34}$$

where $\hat{\mathbf{F}}_i^j$ and $\hat{\mathbf{L}}_i^j$ are the Stokes flow force and torque on particle i due to the motion of particle j (given by (5.33a)) with all particles held fixed except j. This represents a resistance formulation in the sense that the force, torque, and stresslet on a single particle in our system can be related to the motion of all the particles in the system. Further, since the Stokes quantities must be linear in the $\hat{\mathbf{U}}_i$, $\hat{\boldsymbol{\Omega}}_i$, and $\hat{\mathbf{E}}_i$ which are arbitrary and independent, we can write a separate equation for the hydrodynamic force on each particle, the torque on each particle, and stresslet on each particle.

It is also worth noting that the form of the above expression is amenable to matrix manipulations which can fit nicely into the Stokesian dynamics framework. Since Stokesian dynamics can efficiently compute the Stokes quantities of $\hat{\mathbf{F}}_i^j$ and $\hat{\mathbf{L}}_i^j$ for arbitrary interparticle separations, the ability to approximate the above fluid volume integral represents a means by which to model multiparticle dynamics that include the inertial effects of the suspending fluid.

Chapter 6

Future directions

Summary

Here we will discuss some of the future directions and applications of the work presented in the previous chapters. The areas that are addressed in greater detail involve the evaluation of the inertial forces acting on isolated particles in linear flows and multiple particles in periodic suspensions. In the first area, for small-but-finite Reynolds number, we evaluate the force acting on a sphere isolated in a general linear flow and in a simple shear flow near a planar wall. In the second area, we solve for the inertial correction to the hydrodynamic force (the "Oseen" force) acting on a particle in a dilute, sedimenting, cubic array of identical particles.

6.1 Introduction

One of the purposes of the previous chapters was to lay the groundwork for modeling suspension flows at finite Reynolds number with the ultimate goal of incorporating inertial effects into the Stokesian dynamics simulation method. Currently, Stokesian dynamics is capable of modeling multiparticle suspensions undergoing linear shear flows where the particles can describe independent curvilinear paths. The method can also treat unbounded suspensions by periodically replicating domains containing several particles. Thus, in order to carry on further toward the above stated goal, several issues need to be investigated. These topics include: the role of inertial effects in (1) curvilinear particle motion, (2) higher order force moments such as the torque or particle stresslet, (3) imposed flows which vary linearly with position, and (4) periodically replicated, multiparticle systems. The first two subjects will be discussed below. The latter two subjects will be addressed in the following two sections.

The issue of curvilinear particle motion can be dealt with quite easily by using the force expressions derived in Chapter 2. The difficulty in applying these force expressions to time-dependent particle motion along a curvilinear path is that a given vector component of the force depends, in general, on all the other components of the velocity. This complication simply reflects the nonlinearities of the force/velocity relationship caused by convective inertial effects. Particular applications of the expressions would include, for example, a particle describing a circular path, a particle making a 90 degree turn between two rectilinear motions, or a particle experiencing small fluctuations in its velocity normal to its average rectilinear path. The first case was studied by Davis and Brenner [15] and Davis [14] in the context of the steady rotation of a tethered sphere. The case of a particle making a right angle turn in its motion has not been investigated and is an inherently unsteady problem which would show the temporal behavior (or decay) of the two force components in the old and new velocity directions. The last problem is interesting in that it would provide the history dependence of the force on a particle experiencing a fluctuating motion perpendicular to its average straight-line velocity. This last problem also completes the investigation begun by the study in Chapter 3, which may have important implications for particle dispersion in turbulent flow.

The background and formulas for computing higher order force moments are embedded in the analyses of Chapter 5. They may be extracted by setting $N = 1$ to eliminate particle interaction effects. The stresslet, for example, is important in suspension mechanics as it allows the calculation of the effective viscosity of a suspension (see Batchelor [4] for in introduction to the stress in a suspension; Lin, Peery, and Schowalter [32] consider inertial effects on suspension rheology). The torque/velocity relationship is of significance for nonspherical particles or for rotating spherical particles at finite Reynolds number because of the coupling between the hydrodynamic force and the angular velocity of a particle. For example, using the notation from Chapter 5, the hydrodynamic torque and stresslet expression for an isolated spherical particle i translating and rotating in a quiescent fluid takes the form:

$$\mathbf{L}_i^H(t) = -8\pi\mu a^3 \boldsymbol{\Omega}_{p_i}(t) - \int_{V_f} \mathbf{f}(\mathbf{u}) \cdot \hat{\boldsymbol{\mathcal{U}}}_{\Omega_i} \, dV \tag{6.1a}$$

and

$$\mathbf{S}_i^H(t) = -symtr \int_{V_f} \mathbf{f}(\mathbf{u}) \cdot \hat{\boldsymbol{\mathcal{U}}}_{E_i} \, dV, \tag{6.1b}$$

where *symtr* indicates "the symmetric and traceless part of." Provided that the time scale for the variation of the particle's motion is greater than the vorticity diffusion time scale a^2/ν, and the Reynolds numbers for translation and rotation are small, we can compute inertial contributions to the torque and stresslet by simply replacing in enofl the exact Navier–Stokes solution for the velocity field $\mathbf{u}$ with the steady Stokes solution. Both volume integrals are convergent and the results are

$$\mathbf{L}_i^H(t) = -8\pi\mu a^3 \boldsymbol{\Omega}_{p_i}(t) - \frac{8\pi}{3}\rho a^4 \dot{\boldsymbol{\Omega}}_{p_i}(t) \tag{6.2a}$$

and

$$\begin{aligned} \mathbf{S}_i^H(t) = &-\frac{\pi}{3}\rho a^4 \left[\boldsymbol{\Omega}_{p_i}(t)\boldsymbol{\Omega}_{p_i}(t) - \tfrac{1}{3}\boldsymbol{\Omega}_{p_i}(t) \cdot \boldsymbol{\Omega}_{p_i}(t)\mathbf{I}\right] \\ &-\frac{21\pi}{20}\rho a^2 \left[\mathbf{U}_{p_i}(t)\mathbf{U}_{p_i}(t) - \tfrac{1}{3}\mathbf{U}_{p_i}(t) \cdot \mathbf{U}_{p_i}(t)\mathbf{I}\right]. \end{aligned} \tag{6.2b}$$

The result given by (6.2a) agrees with the low-frequency limit of the solution for the torque on a sphere undergoing oscillatory rotation (see, for example, Kim and Karrila [25] section 6.2.3). One can show by simple scaling arguments that the inertial contributions from the outer far-field region are smaller than $O(Re)$. These outer contributions can be computed, however, using the same methods developed in Chapter 2.

6.2 Inertial effects on particle dynamics in linear flows

Consider a spherical particle translating and rotating in a general linear flow

$$\mathbf{u}^{\infty}(\mathbf{x},t) = \mathbf{U}^{\infty}(t) + \mathbf{\Gamma}(t)\cdot\mathbf{x} \tag{6.3}$$

where $\mathbf{x}$ is a position vector in a fixed coordinate system, $\mathbf{\Gamma}(t)$ is a traceless, second-order tensor and both $\mathbf{U}^{\infty}(t)$ and $\mathbf{\Gamma}(t)$ may only be functions of time. The appropriate governing equations for performing dynamic calculations are the disturbance Navier–Stokes equations,

$$\begin{aligned} -\nabla p' + \mu\nabla^2\mathbf{u}' &= \rho\left[\frac{D\,\mathbf{u}'}{D\,t} + \mathbf{\Gamma}(t)\cdot\mathbf{u}' + (\mathbf{\Gamma}(t)\cdot\mathbf{r})\cdot\nabla\mathbf{u}' - \mathbf{U}_s(t)\cdot\nabla\mathbf{u}'\right] \\ &= \mathbf{f}'(\mathbf{u}',\mathbf{\Gamma}(t)), \qquad \nabla\cdot\mathbf{u}' = 0, \end{aligned} \tag{6.4}$$

where the translating coordinate system has origin at the instantaneous center of the spherical particle $\mathbf{Y}_p(t)$, so that $\mathbf{r} = \mathbf{x} - \mathbf{Y}_p(t)$. The particle velocity $\mathbf{U}_p(t)$ relative to the imposed flow is defined through the slip velocity, $\mathbf{U}_s(t)$:

$$\mathbf{U}_s(t) = \mathbf{U}_p(t) - \mathbf{u}^{\infty}(\mathbf{Y}_p(t),t). \tag{6.5}$$

The corresponding boundary conditions are

$$\mathbf{u}' = \mathbf{U}_s(t) + \boldsymbol{\Omega}_p(t) \wedge \mathbf{r} - \boldsymbol{\Gamma}(t) \cdot \mathbf{r} \quad \text{on} \quad S_p, \tag{6.6a}$$

$$\mathbf{u}' = \mathbf{0} \quad \text{on} \quad S_w, \tag{6.6b}$$

and

$$\mathbf{u}' \to \mathbf{0} \quad \text{as} \quad |\mathbf{r}| \to \infty, \tag{6.6c}$$

where S_p is the surface of the particle and S_w represents any bounding walls. It is significant to indicate that the time scale over which both the unsteady and convective fluid inertia are important is given by $\dot{\gamma}^{-1}$, where $\dot{\gamma}$ is the characteristic strain rate of the imposed linear flow. In sheared suspensions this is the relevant time scale for the variation in the particle velocities.

The hydrodynamic force, torque, and stresslet on the sphere are given by the reciprocal theorem as

$$\begin{aligned}\mathbf{F}^H(t) = \mathbf{F}^H_{St}(t) + \frac{4\pi}{3}\rho a^3 \left[\dot{\mathbf{U}}^\infty(t) + \boldsymbol{\Gamma}(t) \cdot \mathbf{u}^\infty(\mathbf{Y}_p(t), t)\right] \\ - \int_{V_f} \mathbf{f}' \cdot \hat{\boldsymbol{\mathcal{U}}}_U \, dV\end{aligned} \tag{6.7a}$$

$$\begin{aligned}\mathbf{L}^H(t) = \mathbf{L}^H_{St}(t) + \frac{8\pi}{15}\rho a^5 \left[\dot{\boldsymbol{\Omega}}^\infty(t) - \mathbf{E}^\infty(t) \cdot \boldsymbol{\Omega}^\infty(t)\right] \\ - \int_{V_f} \mathbf{f}' \cdot \hat{\boldsymbol{\mathcal{U}}}_\Omega \, dV\end{aligned} \tag{6.7b}$$

$$\begin{aligned}\mathbf{S}^H(t) = \mathbf{S}^H_{St}(t) + \frac{4\pi}{15}\rho a^5 symtr\left[\dot{\mathbf{E}}^\infty(t) + \mathbf{E}^\infty(t) \cdot \mathbf{E}^\infty(t) + \boldsymbol{\Omega}^\infty(t)\boldsymbol{\Omega}^\infty(t)\right] \\ - symtr \int_{V_f} \mathbf{f}' \cdot \hat{\boldsymbol{\mathcal{U}}}_E \, dV.\end{aligned} \tag{6.7c}$$

Here we have decomposed $\boldsymbol{\Gamma}$ into two parts so that $\boldsymbol{\Gamma} \cdot \mathbf{r} = \boldsymbol{\Omega}^\infty \wedge \mathbf{r} + \mathbf{E}^\infty \cdot \mathbf{r}$ and

$\mathbf{E}^{\infty}$ is a symmetric rate-of-strain tensor. The first term on the RHS of each of these expressions represents the steady Stokes solution for the quantity on the LHS. Note that there are no integrals over any of the wall surfaces because all fluid velocity fields are zero there. Thus, the Stokes fields such as $\hat{\mathcal{U}}_U$ must account for the effects of any existing walls.

6.2.1 Force on a sphere in an unbounded domain

The Stokes quantities in (6.7) for a sphere in an unbounded domain are

$$\mathbf{F}^H_{St}(t) = -6\pi\mu a \mathbf{U}_s(t), \tag{6.8a}$$

$$\mathbf{L}^H_{St}(t) = -8\pi\mu a^3(\mathbf{\Omega}_p(t) - \mathbf{\Omega}^{\infty}(t)), \tag{6.8b}$$

and

$$\mathbf{S}^H_{St}(t) = \frac{20\pi}{3}\mu a^3 \mathbf{E}^{\infty}(t). \tag{6.8c}$$

In what follows we will focus on the hydrodynamic force and thus evaluate the lift force on a spherical particle caused by an unbounded linear flow. Analogous considerations can be applied to the torque and stresslet.

In an analogous fashion to the derivation of (6.2), we can attempt to compute inertial contributions to the hydrodynamic force by substituting in the integral in (6.7a) the known steady Stokes solution $\mathbf{u}_{St}$ for a translating and rotating sphere in an unbounded general linear flow:

$$\begin{aligned}\mathbf{u}_{St} = &\frac{3}{4}\mathbf{U}_s(t)\cdot\left[a\left(\frac{\mathbf{I}}{r}+\frac{\mathbf{rr}}{r^3}\right)+a^3\left(\frac{\mathbf{I}}{3r^3}-\frac{\mathbf{rr}}{r^5}\right)\right]\\ &+(\mathbf{\Omega}_p(t)-\mathbf{\Omega}^{\infty}(t))\wedge\mathbf{r}\frac{a^3}{r^3}\\ &-(\mathbf{E}^{\infty}(t)\cdot\mathbf{r})\cdot\left[a^3\frac{5}{2}\frac{\mathbf{rr}}{r^5}+a^5\left(\frac{\mathbf{I}}{r^5}-\frac{5}{2}\frac{\mathbf{rr}}{r^7}\right)\right].\end{aligned} \tag{6.9}$$

It is reasonable to expect that this procedure is applicable when the time scale for the motion is greater than a^2/ν and both the shear and slip Reynolds numbers are small but finite. One finds, however, that the point-forced contribution to the Stokes velocity field leads to nonconvergent volume integrals in (6.7a) when integration is carried out to infinity. The appropriate point-forced contribution to the velocity field under these conditions must include inertial effects directly to correctly describe the far-field flow and in order for the volume integral to be convergent. Anticipating that this will be accomplished with the use of spatial Fourier transforms over the entire fluid domain (including the volume of the sphere), as in Chapter 2, the integral of the divergent point-forced Stokes contributions over the volume of the particle must be subtracted from the force expression. When this corrected procedure is performed, and after a considerable amount of algebra, the result for the hydrodynamic force is

$$\begin{aligned}\mathbf{F}^H(t) = &-6\pi\mu a\mathbf{U}_s(t) + \frac{4\pi}{3}\rho a^3\left[\dot{\mathbf{U}}^\infty(t) + \mathbf{\Gamma}(t)\cdot\mathbf{u}^\infty(\mathbf{Y}_p(t),t)\right]\\ &+\pi\rho a^3\left[\frac{37}{12}\mathbf{E}^\infty(t)\cdot\mathbf{U}_s(t) + \frac{1}{3}\mathbf{\Omega}^\infty(t)\wedge\mathbf{U}_s(t) + \mathbf{\Omega}_p(t)\wedge\mathbf{U}_s(t) + \frac{16}{3}\dot{\mathbf{U}}_s(t)\right]\\ &-\int_{V_\infty}\tilde{\mathbf{f}}'\cdot\tilde{\mathcal{U}}_U\,d\mathbf{k},\end{aligned} \tag{6.10}$$

where the tilde on the terms in the volume integral indicates the Fourier transform of the point-forced contribution, and $\mathbf{k}$ is the transform variable. The Fourier transformed velocity field $\tilde{\mathbf{u}}$ satisfies

$$\begin{aligned}(\mathbf{n}_k\mathbf{n}_k - \mathbf{I})\cdot\mathbf{F}_1 = &\left[4\pi^2k^2\mu\mathbf{I} + \rho\left((\mathbf{I}-2\mathbf{n}_k\mathbf{n}_k)\cdot\mathbf{\Gamma}(t) - 2\pi i\mathbf{U}_s(t)\cdot\mathbf{k}\mathbf{I}\right)\right]\cdot\tilde{\mathbf{u}}\\ &-\rho\mathbf{k}\cdot\mathbf{\Gamma}(t)\cdot\nabla_k\tilde{\mathbf{u}} + \rho\frac{\partial\tilde{\mathbf{u}}}{\partial t},\end{aligned} \tag{6.11}$$

where $\mathbf{F}_1$ is the magnitude of the point force and $\mathbf{n}_k = \mathbf{k}/k$. We have ignored the nonlinear term $\mathbf{u}'\cdot\nabla\mathbf{u}'$, since here, as in all cases of small Reynolds numbers, it makes a smaller order correction to the inertial force than any of the other terms retained.

Also we note that $\tilde{\mathbf{f}}'$ is given by

$$\tilde{\mathbf{f}}' = \rho\left(\boldsymbol{\Gamma}(t)\cdot\tilde{\mathbf{u}} - \mathbf{k}\cdot\boldsymbol{\Gamma}(t)\cdot\boldsymbol{\nabla}_k\tilde{\mathbf{u}} - 2\pi i\mathbf{U}_s(t)\cdot\mathbf{k}\tilde{\mathbf{u}} + \frac{\partial\,\tilde{\mathbf{u}}}{\partial\,t}\right). \tag{6.12}$$

For steady simple shear flow, the "outer" contribution to the hydrodynamic force, which follows from the volume integral in (6.10), was evaluated by Harper and Chang [19] as

$$-\int_{V_\infty} \tilde{\mathbf{f}}'\cdot\hat{\tilde{\mathcal{U}}}_U\,d\,\mathbf{k} = 6\pi Re_{\dot\gamma}^{\frac{1}{2}}\mathbf{L}\cdot(-6\pi\mu a\mathbf{U}_s) + o(Re_{\dot\gamma}^{\frac{1}{2}}), \tag{6.13}$$

where $Re_{\dot\gamma} = a^2\dot\gamma/\nu$ and $\mathbf{L}$ is a dimensionless lift tensor (not to be confused with the torque). This contribution is produced by setting $\mathbf{F}_1 = -6\pi\mu a\mathbf{U}_s$ in (6.11). The next order correction from the outer region is $O(Re_{\dot\gamma})$ and is produced by simply replacing $\mathbf{F}_1$ with the contribution to the force given by (6.13). Thus, for a sphere moving steadily in a simple shear flow that is in the 1-direction and varies in the 3-direction, the expression for the hydrodynamic force from (6.10) is

$$\begin{aligned}\mathbf{F}^H &= -6\pi\mu a\mathbf{U}_s + \pi\rho a^3\left[\dot\gamma\left(\frac{11}{8}U_{s_1}\mathbf{e}_3 + \frac{41}{24}U_{s_3}\mathbf{e}_1\right) - \mathbf{U}_s\wedge\boldsymbol{\Omega}_p\right] \\ &\quad +6\pi Re_{\dot\gamma}^{\frac{1}{2}}\mathbf{L}\cdot(-6\pi\mu a\mathbf{U}_s) + (6\pi)^2 Re_{\dot\gamma}\mathbf{L}\cdot\mathbf{L}\cdot(-6\pi\mu a\mathbf{U}_s) + o(Re_{\dot\gamma})\end{aligned} \tag{6.14}$$

where $\mathbf{e}_i$ is a unit vector in the i-direction. This expression is accurate to $O(Re_{\dot\gamma})$.

6.2.2 The lift force on a sphere in a wall-bounded shear flow

Using existing results for a sphere translating near a single planar wall in a shear flow at low Reynolds number, we shall derive the lift force on a sphere of radius a when

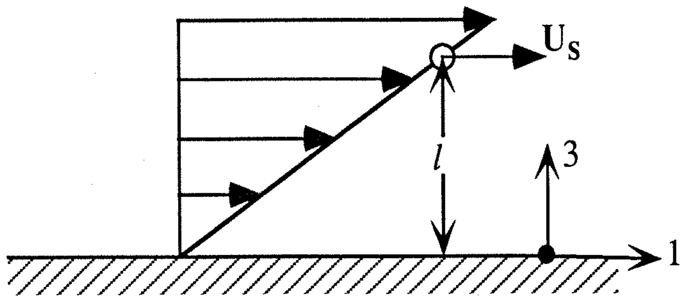


Figure 6.1: A sphere with velocity $\mathbf{U}_s$ relative to a planar simple shear flow and with location at a distance l from a planar wall.

its distance from the wall l satisfies

$$a \ll l \ll min\left(\frac{\nu}{U_s}, \left(\frac{\nu}{\dot{\gamma}}\right)^{\frac{1}{2}}\right), \tag{6.15}$$

where ν is the kinematic viscosity of the fluid, U_s is the magnitude of the sphere's slip velocity relative to the fluid, and $\dot{\gamma}$ is the shear rate of the imposed flow. The lower bound in the above inequality allows the sphere's weak interaction with the wall to be adequately accounted for by a point-force plus force-dipole description of the sphere. The upper bound provides for the condition that the wall is in the inner region of expansion and allows for the use of regular perturbation techniques. Our goal is to collect all terms up to $O(1)$ in inverse powers of the particle distance from the wall l while neglecting those of higher order since they decay as the sphere moves farther from the wall.

For a steady, planar, simple shear flow in the 1-direction which varies in the 3-direction, the disturbance flow $\mathbf{u}'$ created by the particle translating with velocity of magnitude U_s in the 1-direction relative to the shear flow evaluated at the sphere center (see Figure 6.1) is described by the Navier–Stokes equations as

$$-\nabla p' + \mu\nabla^2\mathbf{u}' = \rho\left(-U_s\frac{\partial \mathbf{u}'}{\partial r_1} + \mathbf{u}'\cdot\nabla\mathbf{u}' + \dot{\gamma}u_3'\mathbf{e}_1 + \dot{\gamma}r_3\frac{\partial \mathbf{u}'}{\partial r_1}\right) \tag{6.16a}$$

$$\equiv \mathbf{f}(\mathbf{u}') \tag{6.16b}$$

$$\nabla\cdot\mathbf{u}' = 0, \tag{6.16c}$$

where μ and ρ are the viscosity and density of the fluid, and the coordinate system has its origin at the center of the sphere. Here, the wall surface is at $r_3 = -l$, on which the fluid satisfies the no-slip boundary condition.

Now since a reversibility argument demonstrates that the Stokes equations result in no lift force perpendicular to the wall, the existence of a lift force must be due solely to inertial effects. The general reciprocal theorem (6.7a) thus provides the following expression for the lift force F_l in the 3-direction:

$$F_l = -\int_{V_f} \mathbf{f}'(\mathbf{u}')\cdot\mathbf{v}_3\, dV, \tag{6.17}$$

where V_f represents the entire volume of fluid surrounding the sphere and bounded by the wall. Here, $\mathbf{v}_3$ is the Stokes velocity field produced by the sphere, a distance l from the wall, translating with unit velocity in the 3-direction in a quiescent fluid (in the absence of a shear field).

Due to the condition (6.15), we can approximate F_l by replacing $\mathbf{u}'$ with the corresponding Stokes solution for the disturbance flow, $\mathbf{v}$, as a regular perturbation approach:

$$F_l \sim -\int_{V_f} \mathbf{f}'(\mathbf{v})\cdot\mathbf{v}_3\, dV. \tag{6.18}$$

The errors in the above expression are of higher order in Reynolds number as can be seen from both a regular and singular perturbation analysis. The integral will remain convergent because far from the particle-wall system the disturbance appears as that due to a dipole and thus decays as $O(r^{-2})$ making the integrand $O(r^{-4})$.

The contributions to the lift force can be divided into two sources: that due to the presence of the wall and that due to the finite size of the sphere which would exist in the absence of the wall. The former lift force contribution was evaluated by Cox and Hsu [13], while the latter was evaluated by Saffman [46]. The results from Cox and Hsu were obtained by using a point-force plus force-dipole description of the sphere's disturbance velocity, a representation which is valid under the assumption that the sphere is far from the wall. In their analysis the the nonlinear term of $\mathbf{f}'$, $\mathbf{v} \cdot \nabla \mathbf{v}$, is neglected since it produces a contribution to the lift which can be shown to decay as $O(l^{-1})$. Thus, $\mathbf{f}'$ can be treated as being linear in $\mathbf{v}$ and the point-force and force-dipole can be accounted for separately. If the point-forced velocity field is used in (6.18) for both $\mathbf{v}$ and $\mathbf{v}_3$, two terms are obtained. From the work of Cox and Hsu, the first is due to the first term on the RHS of (6.16a):

$$\frac{18\pi}{32}\mu a U_s \left(\frac{aU_s}{\nu}\right), \tag{6.19}$$

and the second from the last two terms of (6.16a):

$$-\frac{66\pi}{64}\mu a U_s \left(\frac{al\dot{\gamma}}{\nu}\right). \tag{6.20}$$

It is important to note that these two terms were computed using a point-force of magnitude equal to the Stokes drag on a sphere in an unbounded domain, $-6\pi\mu a U_s$. The term given by (6.20) must be corrected to obtain all terms to $O(1)$ for large l. This is accomplished by including a modification of the point-force due to the presence of the wall. For motion parallel to the wall the magnitude of the point force should have a multiplicative factor of $(1+9a/16l)$ (see Happel and Brenner [18] p.327), while for motion perpendicular to the wall it should have a factor of $(1+9a/8l)$ (see Happel and Brenner [18] p.330). If these factors are used in (6.18), the corrected term of

(6.20) becomes

$$-\frac{66\pi}{64}\mu a U_s \left(\frac{al\dot{\gamma}}{\nu}\right)\left(1+\frac{27}{16}\frac{a}{l}\right). \tag{6.21}$$

When the force-dipole description of the sphere is used for $\mathbf{v}$ in the last two terms of (6.16a), while the point-force description is used for $\mathbf{v}_3$, Cox and Hsu found contributions to the lift force from (6.18) given by

$$\frac{6\pi(61)}{144\cdot 4}\mu a^2\dot{\gamma}\left(\frac{a^2\dot{\gamma}}{\nu}\right), \tag{6.22}$$

for a sphere prevented from rotating, and

$$\frac{6\pi(55)}{144\cdot 4}\mu a^2\dot{\gamma}\left(\frac{a^2\dot{\gamma}}{\nu}\right), \tag{6.23}$$

for a sphere free to rotate.

In obtaining the above results (6.19, 6.20, 6.22, and 6.23), Cox and Hsu performed the integration in (6.18) by extending the volume of integration to the entire volume of space, ignoring the finite size of the sphere. The error made in doing this yields contributions to the lift force which are $O(1)$ for large l. These contributions may be evaluated by neglecting the presence of the wall and using the disturbance Stokes flow fields for the motion of the sphere in an unbounded domain. This is carried out by first evaluating the integral (6.18) over an unbounded fluid domain *outside* the sphere with the velocity fields replaced by those for the sphere motion in an unbounded domain (these fields are well-known), while taking care to not include the point-force or force-dipole contributions from the above four terms already evaluated by Cox and Hsu. Then, in order to correct the error in the Cox and Hsu analysis, one must subtract the integral over the volume of the *sphere* of these excluded point-force and force-dipole contributions. The result yields the second-order Saffman lift force [46], which is determined from a consideration of the inner expansion problem:

$$\pi\mu a U_s\left(\frac{11}{8}\frac{a^2\dot{\gamma}}{\nu}-\frac{a^2\Omega_p}{\nu}\right), \tag{6.24}$$

where Ω_p is the magnitude of the angular velocity of the sphere in the 2-direction. Note that this result agrees with the second term of (6.14) derived in the previous section.

If we now combine all these contributions we obtain an expression for the lift force to leading order in Reynolds number and appropriate when $a/l \ll 1$:

$$\begin{aligned} F_l = & \frac{18\pi}{32}\mu a U_s\left(\frac{aU_s}{\nu}\right)-\frac{66\pi}{64}\mu a U_s\left(\frac{al\dot{\gamma}}{\nu}\right)\left(1+\frac{27}{16}\frac{a}{l}\right) \\ & +\frac{6\pi(61)}{144\cdot 4}\mu a^2\dot{\gamma}\left(\frac{a^2\dot{\gamma}}{\nu}\right)+\frac{11}{8}\pi\mu a U_s\left(\frac{a^2\dot{\gamma}}{\nu}\right)+O(a/l), \end{aligned} \tag{6.25}$$

for a sphere prevented from rotating, and for a sphere free to rotate

$$\begin{aligned} F_l = & \frac{18\pi}{32}\mu a U_s\left(\frac{aU_s}{\nu}\right)-\frac{66\pi}{64}\mu a U_s\left(\frac{al\dot{\gamma}}{\nu}\right)\left(1+\frac{27}{16}\frac{a}{l}\right) \\ & +\frac{6\pi(55)}{144\cdot 4}\mu a^2\dot{\gamma}\left(\frac{a^2\dot{\gamma}}{\nu}\right)+\frac{7}{8}\pi\mu a U_s\left(\frac{a^2\dot{\gamma}}{\nu}\right)+O(a/l), \end{aligned} \tag{6.26}$$

where we have set $\Omega_p = \dot{\gamma}/2$ for a freely rotating sphere.[a] As an added note, the expression for a reversal in the direction of the shear flow or the slip velocity can be obtained by simply changing the sign of $\dot{\gamma}$ or U_s in the above two expressions.

[a]This value of the angular velocity is correct for zero Reynolds number in the absence of any bounding walls. The corrections for finite Reynolds number or for walls will bring only a smaller order correction to the lift force than those already provided.

6.3 Sedimenting cubic arrays

The purpose of this study is two-fold: first, we wish to show the application of the reciprocal theorem to periodic domains; and second, we want to investigate the effect of particle interactions on inertial effects. Sedimenting cubic arrays allow us to address both of these issues.

In order to use the reciprocal theorem for this system we simply apply the reciprocal theorem to a representative periodic box or cubic cell. For further simplicity, we shall consider the case of only one particle per cubic cell with the particle located at the center of the cell. The reciprocal theorem for this case is

$$\int_{\sum S_c} (\mathbf{n}\cdot\boldsymbol{\sigma})\cdot\hat{\mathbf{u}}\,dS + \int_{S_p} (\mathbf{n}\cdot\boldsymbol{\sigma})\cdot\hat{\mathbf{u}}\,dS + \int_{V_f^c} \mathbf{f}\cdot\hat{\mathbf{u}}\,dV = \int_{\sum S_c} (\mathbf{n}\cdot\hat{\boldsymbol{\sigma}})\cdot\mathbf{u}\,dS + \int_{S_p} (\mathbf{n}\cdot\hat{\boldsymbol{\sigma}})\cdot\mathbf{u}\,dS \qquad (6.27)$$

where S_c represents the outer bounding surfaces of the cubic cell, and V_f^c is the fluid volume within the cell. Here the hat $\hat{}$ indicates the Stokes flow (zero Reynolds number) solution for the velocity and stress fields, while the unhatted fields represent the finite Reynolds number solutions. Also note that the unit normal vector $\mathbf{n}$ points into the volume of the fluid within the cell.

Because we have a periodic system, all velocity fields will be periodic on the cell. The pressure, however, will have a linearly varying part as well as a periodic part. This condition is due to the fact that a particle exerts a force on the fluid and a pressure gradient across the cell, equal to this force per unit cell volume, is necessary to balance it. If, in addition, we specify that there is no net flux of material across any cross section of the cell (which is an obvious specification if we assume there are impermeable container walls far from the cell), then the linearly varying pressure gradient will result in a backflow of fluid to maintain the condition of no net flux. Under these conditions the contributions from the surface integrals over the boundaries of the cell are identically zero. This finding can be illustrated by considering the surface integral $\int_{\sum S_c} (\mathbf{n} \cdot \boldsymbol{\sigma}) \cdot \hat{\mathbf{u}} \, dS$. We assume that $\mathbf{u}$ and $\hat{\mathbf{u}}$ are periodic on the cubic lattice and that we can divide the pressure into a periodic and a constant gradient part:

$$p = p' + \mathbf{G} \cdot \mathbf{x}, \tag{6.28}$$

where p' is periodic on the cell and

$$\mathbf{G} = -\frac{\mathbf{F}^H}{L^3}, \tag{6.29}$$

where $\mathbf{F}^H$ is the hydrodynamic force acting on the particle in the cell and L is the lattice spacing. Then the stress $\boldsymbol{\sigma}$ can be expressed as

$$\boldsymbol{\sigma} = -(p' + \mathbf{G} \cdot \mathbf{x})\mathbf{I} + (\nabla \mathbf{u} + \nabla \mathbf{u}^T). \tag{6.30}$$

The above surface integral involving the periodic parts of $\boldsymbol{\sigma}$ will vanish due to the cancellation of contributions from opposite sides of the cell. Thus we are left with

$$\int_{\sum S_c} (\mathbf{n} \cdot \boldsymbol{\sigma}) \cdot \hat{\mathbf{u}} \, dS = \int_{\sum S_c} \mathbf{G} \cdot \mathbf{x} \hat{\mathbf{u}} \cdot \mathbf{n} \, dS. \tag{6.31}$$

Now consider two opposing, parallel sides of the cell, say S_1 and S_2, such that the normal for each ($\mathbf{n}_1$ and $\mathbf{n}_2$) points in the opposite direction:

$$\begin{aligned} \int_{S_1+S_2} \mathbf{G} \cdot \mathbf{x} \hat{\mathbf{u}} \cdot \mathbf{n} \, dS &= -\mathbf{G} \cdot (L\mathbf{n}_1) \int_{S_1} \hat{\mathbf{u}} \cdot \mathbf{n}_1 \, dS \\ &= 0, \end{aligned} \tag{6.32}$$

where in the last step we have used the condition that there is no net flux of fluid across the periodic boundaries. An analogous procedure can be used to show that the integral $\int_{\sum S_c} (\mathbf{n} \cdot \hat{\boldsymbol{\sigma}}) \cdot \mathbf{u} \, dS$ is also identically zero. The reciprocal theorem, therefore, will reduce to the familiar form

$$\mathbf{F}^H = \mathbf{F}_s^H - \int_{V_f^c} \mathbf{f} \cdot \hat{\mathcal{U}}_U^c \, dV, \tag{6.33}$$

where $\mathbf{F}_s^H$ represents the Stokes drag on the particle in the cubic lattice, and $\hat{\mathcal{U}}_U^c$ represents the Stokes velocity field for the translating particle in the cubic lattice. Note that for the arrangement where the cubic lattice of particles is fixed and fluid is moving passed this fixed bed, the surface integrals over the particle would be zero, while the surface integrals over the cell boundaries would be nonzero and would provide for a similar expression for the hydrodynamic force.

To compute the inertial correction to the hydrodynamic force on the particle, we make the simplifications that the Reynolds number is small and that the system is sufficiently dilute that the outer Oseen region is contained within the periodic cell. This condition requires that $\phi^{\frac{1}{3}} \ll Re \ll 1$ where ϕ is the volume fraction of particles and the Reynolds number, $Re = a|\mathbf{U}_p|/\nu$, is based on the characteristic particle size a and the sedimentation velocity $\mathbf{U}_p$. This further implies that the Stokes drag can be approximated by the drag on an isolated particle and the interaction effects can be accounted for by considering the point-forced *spatially periodic* Oseen equations:

$$\begin{aligned} -\boldsymbol{\nabla} p' - \mathbf{G} + \mu \nabla^2 \mathbf{u} &= -\rho \mathbf{U}_p \cdot \boldsymbol{\nabla} \mathbf{u} + \mathbf{F}_s^H \delta(\mathbf{x}) \\ \boldsymbol{\nabla} \cdot \mathbf{u} &= 0. \end{aligned} \tag{6.34}$$

The Oseen equations are then solved for in Fourier space by applying Fourier transforms which are periodic on the cubic cell:

$$\begin{aligned} -\frac{i2\pi}{L}\mathbf{k} p'_{\mathbf{k}} - \mathbf{G}\delta(\mathbf{k}) - \frac{4\pi^2}{L^2}\mathbf{k}\cdot\mathbf{k}\mathbf{u}_{\mathbf{k}} &= -\rho \mathbf{U}_p \cdot \left(\frac{i2\pi}{L}\mathbf{k}\mathbf{u}_{\mathbf{k}}\right) + \frac{\mathbf{F}_s^H}{L^3} \\ \mathbf{k}\cdot\mathbf{u}_{\mathbf{k}} &= 0, \end{aligned} \tag{6.35}$$

where

$$\delta(\mathbf{0}) = 0$$
$$\delta(\mathbf{k}) = 1, \quad \text{for} \ \mathbf{k} \neq \mathbf{0}. \tag{6.36}$$

A vector field $\mathbf{u}$, for example, that is periodic on the cubic lattice of dimension L can be expressed in its Fourier modes as

$$\mathbf{u}(\mathbf{x}) = \sum_{\mathbf{k}} \left[\mathbf{u}_{\mathbf{k}} \mathrm{e}^{\mathrm{i}2\pi\mathbf{k}\cdot\mathbf{x}/L} \right], \tag{6.37}$$

where $\mathbf{k}$ represents all three-dimensional vectors with components of integer magnitude. The Fourier modes are given by

$$\mathbf{u}_{\mathbf{k}} = \frac{1}{L^3} \int_{V^c} \mathbf{u}(\mathbf{x}) \mathrm{e}^{-\mathrm{i}2\pi\mathbf{k}\cdot\mathbf{x}/L} \, dV, \tag{6.38}$$

where V^c is the entire volume of the cell. The convolution theorem is then applied to the volume integral over the cell which leads to an inertial correction that is given by a three dimensional infinite sum:

$$\mathbf{F}^H = \mathbf{F}_s^H - L^3 \sum_{\mathbf{k}} \mathbf{f}_{\mathbf{k}} \cdot \left[\hat{\mathcal{U}}_{\scriptscriptstyle U}^{c} \right]_{\mathbf{k}}, \tag{6.39}$$

where

$$\mathbf{f}_{\mathbf{k}} = -\rho \mathbf{U}_p \cdot \mathbf{k} \frac{\mathrm{i}2\pi}{L} \mathbf{u}_{\mathbf{k}} \tag{6.40}$$

and

$$\left[\hat{\mathcal{U}}_{\scriptscriptstyle U}^{c} \right]_{\mathbf{k}} = \frac{3/L(\mathbf{I} - \mathbf{n}_k \mathbf{n}_k) \cdot \mathbf{\Phi}}{2\pi \mathbf{k} \cdot \mathbf{k}} (1 - \delta(\mathbf{k})). \tag{6.41}$$

The result is

$$\mathbf{F}^H = \mathbf{F}_s^H + Re \sum_{\mathbf{r}}^{\mathbf{r}\neq\mathbf{0}} \left\{ \frac{\frac{3}{2\pi r^2}\mathbf{F}_s^H \cdot (\mathbf{I} - \mathbf{n}_k\mathbf{n}_k) \cdot \mathbf{\Phi}}{1 + \left(\frac{2\pi}{\mathbf{p}\cdot\mathbf{n}_k}\right)^2 r^2} \right\} \left(\frac{a}{LRe}\right)^3, \tag{6.42}$$

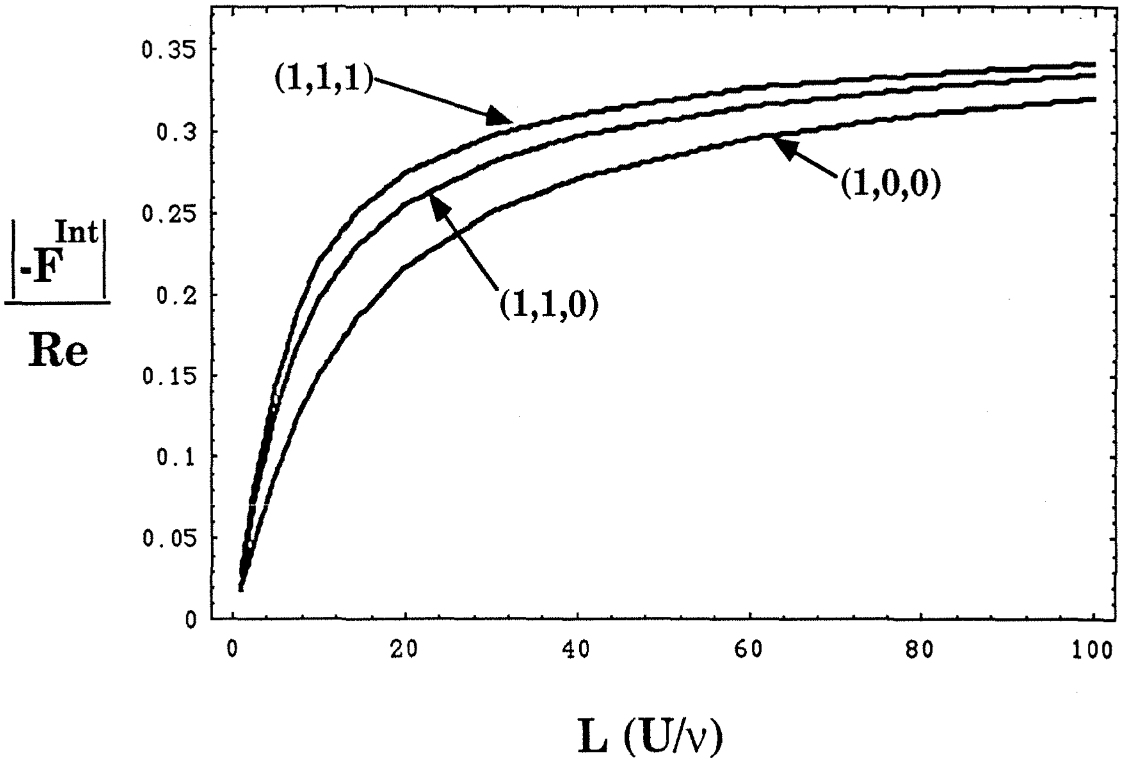
where $\mathbf{r}$ represents all three-dimensional vectors with components that are integer multiples of a/LRe and where $r = |\mathbf{r}|$ and $\mathbf{n}_k = \mathbf{r}/r$. Here $\mathbf{p}$ represents a unit vector in the direction of sedimentation and $\mathbf{\Phi}$ is the dimensionless Stokes resistance tensor for the particle which is equal to $\mathbf{I}$ for a sphere. The $\mathbf{r} = \mathbf{0}$ case is removed by the nonperiodic part of the pressure.

This expression was applied to sedimenting spheres so that the Stokes drag must be parallel to the sedimentation velocity independent of the orientation of the lattice. If we define the inertial contribution to the hydrodynamic force nondimensionalized by the Stokes drag as $\mathbf{F}^{Int}$, we have

$$\frac{-\mathbf{F}^{Int}}{Re} = \sum_{\mathbf{r}}^{\mathbf{r}\neq\mathbf{0}} \left\{ \frac{\frac{3}{2\pi r^2}\mathbf{p} \cdot (\mathbf{I} - \mathbf{n}_k\mathbf{n}_k)}{1 + \left(\frac{2\pi}{\mathbf{p}\cdot\mathbf{n}_k}\right)^2 r^2} \right\} \left(\frac{a}{LRe}\right)^3, \tag{6.43}$$

where a is the radius of the sphere. The magnitude of this quantity is plotted as a function of the lattice spacing L in Figure 6.2 for the three symmetric sedimentation orientations of the lattice. For each of these cases the force and velocity are parallel. We first observe that the inertial force approaches its "Oseen" value of 3/8 as the lattice spacing is increased. Secondly, we find that the inertial force drops of rapidly as the lattice spacing approaches the Oseen distance ν/U, which shows drag reduction due to wake interference effects. And lastly, we see that as the orientation of the lattice is adjusted so that there is a greater distance between nearest neighbors along the sedimentation direction, the drag increases since the wake interference effects are being reduced. In fact, if one plots the inertial force versus the nearest neighbor spacing (e.g., $2^{\frac{1}{2}}L$ for the $(1,1,0)$-direction and $3^{\frac{1}{2}}L$ for the $(1,1,1)$-direction) the curves collapse nearly on top of each other.

For nonsymmetric sedimentation orientations, the direction of the inertial force is not parallel to the sedimentation direction. This is illustrated by the lower portion Figure 6.2, which is for a cubic lattice acted upon by gravity. We note that the component of the velocity perpendicular to the force is due solely to inertial effects and leads to lattice drift.



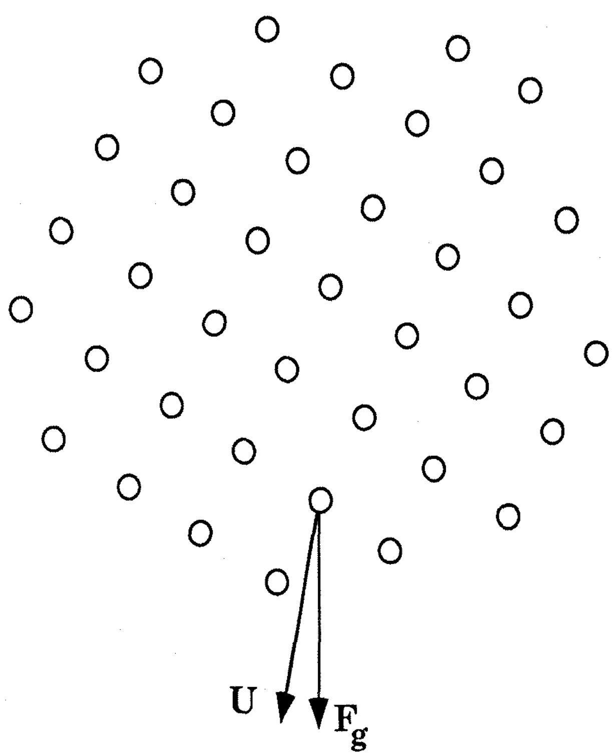


Figure 6.2: The inertial force on a sphere in a dilute sedimenting cubic lattice as a function of the lattice spacing L for sedimentation in the three symmetry orientations of the lattice (upper figure) and the qualitative sedimentation direction for the cubic lattice settling under gravity in a nonsymmetric orientation (lower figure).

Bibliography

[1] M. Abramowitz and I. A. Stegun. *Handbook of Mathematical Functions*, pages 504–515. Dover, New York, 1972.

[2] R. J. Adrian. Particle-imaging techniques for experimental fluid mechanics. *Annu. Rev. Fluid Mech.*, 23:261–304, 1991.

[3] A. B. Basset. *A Treatise on Hydrodynamics*, volume 2. Cambridge: Deighton Bell, 1888.

[4] G. K. Batchelor. The stress system in a suspension of force-free particles. *J. Fluid Mech.*, 41:545–570, 1970.

[5] M. Bentwich and T. Miloh. The unsteady matched Stokes-Oseen solution for the flow past a sphere. *J. Fluid Mech.*, 88:17–32, 1978.

[6] J. F. Brady and G. Bossis. Stokesian dynamics. *Annu. Rev. Fluid Mech.*, 20:111–157, 1988.

[7] H. Brenner. The Oseen resistance of a particle of arbitrary shape. *J. Fluid Mech.*, 11:604–610, 1961.

[8] H. Brenner and R. G. Cox. The resistance to a particle of arbitrary shape in translational motion at small Reynolds numbers. *J. Fluid Mech.*, 17:561–595, 1963.

[9] S. Childress. The slow motion of a sphere in a rotating, viscous fluid. *J. Fluid Mech.*, 20:305–314, 1964.

[10] R. F. Chisnell. The unsteady motion of a drop moving vertically under gravity. *J. Fluid Mech.*, 176:443–464, 1987.

[11] R. G. Cox. The steady motion of a particle of arbitrary shape at small Reynolds numbers. *J. Fluid Mech.*, 23:625–643, 1965.

[12] R. G. Cox and H. Brenner. The lateral migration of solid particles in Poiseuille flow – I. Theory. *Chem. Eng. Sci.*, 23:147–173, 1968.

[13] R. G. Cox and S. K. Hsu. The lateral migration of solid particles in a laminar flow near a plane. *Int. J. Multiphase Flow*, 3:201–222, 1977.

[14] A. M. J. Davis. Drag modifications for a sphere in a rotational motion at small, non-zero Reynolds and Taylor numbers: wake interference and possibly Coriolis effects. *J. Fluid Mech.*, 237:13–22, 1992.

[15] A. M. J. Davis and H. Brenner. Steady rotation of a tethered sphere at small, non-zero Reynolds and Taylor numbers: wake interference effects on drag. *J. Fluid Mech.*, 168:151–167, 1986.

[16] L. Durlofsky, J. F. Brady, and G. Bossis. Dynamic simulation of hydrodynamically interacting particles. *J. Fluid Mech.*, 180:21–49, 1987.

[17] E. Gavze. The accelerated motion of rigid bodies in non-steady Stokes flow. *Int. J. Multiphase Flow*, 16:153–166, 1990.

[18] J. Happel and H. Brenner. *Low Reynolds Number Hydrodynamics*. Martinus-Nijhoff, Dordrecht, The Netherlands, 1986.

[19] E. Y. Harper and I. Chang. Maximum dissipation resulting from lift in a slow viscous shear flow. *J. Fluid Mech.*, 33:209–225, 1968.

[20] B. P. Ho and L. G. Leal. Inertial migration of rigid spheres in two-dimensional unidirectional flows. *J. Fluid Mech.*, 65:365–400, 1974.

[21] D. J. Jeffery and Y. Onishi. Calculation of the resistance and mobility functions for two unequal rigid spheres in low-Reynolds-number flow. *J. Fluid Mech.*, 139:261–290, 1984.

[22] J. T. Jenkins. Balance laws and constitutive relations for rapid flows of granular materials. In J. Chandra and R. P. Srivastav, editors, *Constitutive Models of Deformation*, pages 109–119, Philadelphia, 1987. SIAM.

[23] J. T. Jenkins and S. B. Savage. A theory for the rapid flow of identical, smooth, nearly elastic, spherical particles. *J. Fluid Mech.*, 130:187–202, 1983.

[24] S. Kim and S. J. Karrila. *Microhydrodynamics: Principles and Selected Applications*. Butterworth-Heinemann, Boston, 1991.

[25] S. Kim and S. J. Karrila. *Microhydrodynamics: Principles and Selected Applications*, pages 154–162. Butterworth-Heinemann, Boston, 1991.

[26] S. Kim and R. T. Mifflin. The resistance and mobility functions of two equal spheres in low-Reynolds-number flow. *Phys. Fluids*, 28:2033–2045, 1985.

[27] D. L. Koch. Kinetic theory for a monodisperse gas-solid suspension. *Phys. Fluids* A, 2:1711–1723, 1990.

[28] C. J. Lawrence and S. Weinbaum. The force on an axisymmetric body in linearized, time-dependent motion: a new memory term. *J. Fluid Mech.*, 171:209–218, 1986.

[29] C. J. Lawrence and S. Weinbaum. The unsteady force on a body at low Reynolds number; the axisymmetric motion of a spheroid. *J. Fluid Mech.*, 189:463–489, 1988.

[30] L. G. Leal. Particle motions in a viscous fluid. *Annu. Rev. Fluid Mech.*, 12:435–476, 1980.

[31] L. G. Leal. *Laminar Flow and Convective Transport Processes*, pages 197–229. Butterworth-Heinemann, Boston, 1992.

[32] C. Lin, J. H. Peery, and W. R. Schowalter. Simple shear flow round a rigid sphere: inertial effects and suspension rheology. *J. Fluid Mech.*, 44:1–17, 1970.

[33] M. R. Maxey and J. J. Riley. Equation of motion for a small rigid sphere in a nonuniform flow. *Phys. Fluids*, 26:883–889, 1983.

[34] T. Maxworthy. Accurate measurements of sphere drag at low Reynolds numbers. *J. Fluid Mech.*, 23:369–372, 1965.

[35] R. Mei and R. J. Adrian. Flow past a sphere with an oscillation in the free-stream velocity and unsteady drag at finite Reynolds number. *J. Fluid Mech.*, 237:323–341, 1992.

[36] R. Mei, R. J. Adrian, and T. J. Hanratty. Particle dispersion in isotropic turbulence under Stokes drag and Basset force with gravitational settling. *J. Fluid Mech.*, 225:481–495, 1991.

[37] R. Mei, C. J. Lawrence, and R. J. Adrian. Unsteady drag on a sphere at finite Reynolds number with small fluctuations in the free-stream velocity. *J. Fluid Mech.*, 233:613–631, 1991.

[38] Renwei Mei and James F. Klausner. Unsteady force on a spherical bubble at finite Reynolds number with small fluctuations in the free-stream velocity. *Phys. Fluids* A, 4:63–70, 1992.

[39] J. R. Ockendon. The unsteady motion of a small sphere in a viscous liquid. *J. Fluid Mech.*, 34:229–239, 1968.

[40] C. W. Oseen. Über die Stokes'sche Formel und über eine verwandte Aufgabe in der hydrodynamik. *Ark. f. Mat. Astr. och Fys.*, 6(29), 1910.

[41] C. W. Oseen. Über den Gültigkeitsbereich der Stokesschen Widerstandsformel. *Ark. f. Mat. Astr. och Fys.*, 9(16), 1913.

[42] C. Pozrikidis. A study of linearized oscillatory flow past particles by the boundary-integral method. *J. Fluid Mech.*, 202:17–41, 1989.

[43] I. Proudman and J. R. A. Pearson. Expansion at small Reynolds numbers for the flow past a sphere and a circular cylinder. *J. Fluid Mech.*, 2:237–262, 1957.

[44] M. W. Reeks and S. McKee. The dispersive effects of Basset history forces on the particle motion in a turbulent flow. *Phys. Fluids*, 27:1573–1582, 1984.

[45] S. I. Rubinow and J. B. Keller. The transverse force on a spinning sphere moving in a viscous fluid. *J. Fluid Mech.*, 11:447–459, 1961.

[46] P. G. Saffman. The lift on a small sphere in a slow shear flow. *J. Fluid Mech.*, 22:385–400, 1965.

[47] T. Sano. Unsteady flow past a sphere at low Reynolds number. *J. Fluid Mech.*, 112:433–441, 1981.

[48] G. Segré and A. Silberberg. Behavior of macroscopic rigid spheres in Poiseuille flow: Part 2. Experimental results and interpretation. *J. Fluid Mech.*, 14:136–157, 1962.

[49] H. A. Stone and J. F. Brady. Inertial effects on the rheology of a suspension and on the motion of isolated rigid particles. *J. Fluid Mech.*, 1993. (to be submitted).

[50] T. D. Taylor and A. Acrivos. On the deformation and drag of a falling viscous drop at low Reynolds number. *J. Fluid Mech.*, 18:466–476, 1964.

[51] W. E. Williams. A note on the slow vibrations in a viscous fluid. *J. Fluid Mech.*, 25:589–590, 1966.

[52] S. Yang and L. G. Leal. A note on memory-integral contributions to the force on an accelerating spherical drop at low Reynolds number. *Phys. Fluids* A, 3:1822–1824, 1991.

[53] J. B. Young and T. J. Hanratty. Optical studies on the turbulent motion of solid particles in pipe flow. *J. Fluid Mech.*, 231:665–688, 1991.