Diameter Bounds on the Complex of Minimal Genus Seifert Surfaces for Hyperbolic Knots

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To Andraya Beth Gough

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Abstract

Given a link L in S^3, one can build simplicial complexes $MS(L)$ and $IS(L)$, called the Kakimizu complexes. These complexes have isotopy classes of minimal genus and incompressible Seifert surfaces for L as their vertex sets and have simplicial structures defined via a disjointness property. The Kakimizu complexes enjoy many topological properties and are conjectured to be contractible. Following the work of Gabai on sutured manifolds and Murasugi sums, $MS(L)$ and $IS(L)$ have been classified for various classes of links. This thesis focuses on hyperbolic knots; using minimal surface representatives and Kakimizu's formulation of the path-metric on $MS(K)$, we are able to bound the diameter of this complex in terms of only the genus of the knot. The techniques of this paper are also generalized to one-cusped manifolds with a preferred relative homology class.

Contents

Chapter 1

Introduction

1.1 Seifert Surfaces and Links

Seifert surfaces have immensely aided the understanding of link theory. At the most basic level, one may use well-known topological properties of surfaces (e.g., their genera and fundamental groups) to make the classification of links a more manageable program. Furthermore, many invariants of knots, like the genus and Alexander polynomial, may be computed using any Seifert surface of a knot. Operations defined at the level of a Seifert surface (e.g., Murasugi sum) many times seem to say more about the boundary links than the surfaces themselves.

Since minimal genus and incompressible Seifert surfaces are, in many regards, the surfaces of minimal complexity for a link, their classification is of the utmost importance. Understanding the space of possibilities for these surfaces as well as how they interact provides much insight into the topology of a link complement. Therefore, the Kakimizu complexes $MS(L)$ and $IS(L)$ provide a natural combinatorial representation that may have very deep ramifications in the decomposition of links.

1.2 Classifying Seifert Surfaces

The technology developed in the last few decades provide concrete tools for finding and describing all possible minimal genus and incompressible Seifert surfaces for a link. Beginning with Hatcher and Thurston's seminal work on the classification of incompressible surfaces in two-bridge knot complements, several low-dimensional topologists have subsequently determined incompressible Seifert surfaces for many combinatorially defined links. In particular, the work of Gabai has been a *sine qua non*; his developments in sutured manifold theory and its implications in the Murasugi sums of surfaces have thrust explicit calculations of $MS(L)$ and $IS(L)$ into the realm of possibility.

The motivating conjecture in studying these Kakimizu complexes insists that, as topological spaces, they are contractible. Since $MS(L)$ and $IS(L)$ represent a space of choices defined in a natural way, contractibility would imply that, in a variety of contexts, the choice of minimal genus or incompressible Seifert surface is immaterial. In particular, the contractibility of the complexes may have implications in classifying Murasugi sum structures on the level of links (instead of surfaces).

1.3 Minimal Surfaces and Hyperbolic Knots

A potential complication in studying the Kakimizu complexes lies in the definition of their vertex sets as isotopy classes of surfaces. To choose a canonical or natural representative in each isotopy class, one must invoke the geometry of the knot complement. Hyperbolic knots are, therefore, the most natural class of knots in which one benefits from requiring surfaces to have vanishing mean curvature. After obtaining existence results for minimal surfaces in hyperbolic knot complements, one then almost immediately obtains curvature, area, and distance bounds on these geometric Seifert surfaces.

1.4 The Bounded Diameter Theorem

By manipulating these surfaces as geometric objects and the ambient manifold as a negatively curved space, one is able to control the growth of these complexes relative to purely topological quantities via the Gauss-Bonnet Theorem. The Bounded Diameter theorem provides an explicit diameter bound on $MS(K)$ for a hyperbolic knot in terms of only the genus g of K. This bound indicates that, unlike satellite knots, these complexes are relatively small for low genus hyperbolic knots.

The hope is that this result will add to the growing body of knowledge on the Kakimizu complex. Understanding the topology and combinatorics of $MS(L)$ and $IS(L)$ is necessary to more fully incorporating link theory in the blossoming field of 3-dimensional geometric topology.

Chapter 2

Preliminaries

2.1 Topological Definitions

2.1.1 Manifolds

We will be working almost exclusively with manifolds. Recall that an *n-manifold* is a second-countable, Hausdorff topological space which is locally homeomorphic to $\mathbb{R}^n$ with its standard metric topology. Many times, we will be interested in *manifolds-with-boundary*; these are n-manifolds that may be partitioned into *interior points* (which are locally homeomorphic to $\mathbb{R}^n$) and *boundary points* (which are locally homeomorphic to the half-space $R^n_+ = \{(x_1, x_2, \ldots, x_n) \,|\, x_n \geq 0\}$). This thesis deals almost exclusively with manifolds of dimension 1 and 2 lying in 3-manifolds.

The most relevant 3-manifold we will be discussing is the 3-dimensional sphere S^3. We define S^3 to be the set of vectors in R^4 with unit length and endow it with the subspace topology. Thus,

$$S^3 = \{(x, y, z, w) \in \mathbb{R}^4 \,|\, x^2 + y^2 + z^2 + w^2 = 1\}.$$

Equivalently, we will many times consider S^3 to be the one-point compactification of R^3 obtained by adding a point at infinity. This latter interpretation is particularly useful when visualizing knots in $\mathbb{R}^3$ by keeping track of the extra point at infinity, usually using it as a basepoint.

2.1.2 Knots and Links

Note that the unique connected, closed 1-manifold is the circle. In knot theory, one considers how these circles can lie in $\mathbb{R}^3$ (or more practically in S^3). Thus, we define a *knot* to be a continuous embedding of the circle S^1 into S^3. We many times speak of a knot as this embedding or its image in S^3. In more generality, we may consider a link as a many-component knot. Formally, a link is given by a continuous embedding of a disjoint union of circles into S^3; the number of circles known as the number of components of the link. Intuitively, one considers knots or links to be equivalent if they can be moved around in 3-space without breaking or tearing the knot. This concepts is formalized by the notion of ambient isotopy: two knots $g, h : S^1 \to S^3$ are *ambient isotopic* if there is a homotopy $f_t : S^3 \to S^3$ such that $f_0 = \mathrm{id}$, each f_t is a homeomorphism, and $h = g \circ f_1$. When we speak of knots, we really mean the ambient isotopy class of the knot. Any two such knots that are in the same isotopy class are considered *equivalent*.

Thus, a knot is simply a 1-component link. As embeddings of circles into S^3 may be quite unwieldy, we restrict our discussion to *tame knots*, which are knots equivalent to finite-sided polygons in $\mathbb{R}^3$; a knot that is not tame is called *wild*. If a knot is tame, then it has a *regular knot diagram* on the plane. Thus, we may represent the tame knot K by a projection onto a plane P where only transverse double-points are allowed. At the double points, a choice of over or under-crossings must be made. This information defines a knot in S^3, though not uniquely. Reidemeister [43] classified three local moves (called Reidemeister moves) such that any two diagrams of an equivalent knot may be transformed from one to the other by a finite sequence of these moves. This classification has been immensely important in defining invariants for knots; many of these invariants are defined on the level of the knot diagram and then are shown to be invariant under these three Reidemeister moves.

In studying a link, it is many times useful to consider its complement in the 3-sphere; thus, we frequently investigate the exterior or link complement:

$$E(L) = S^3 - \overset{\circ}{N}(L) \ .$$

Here, we actually delete the interior of a tubular neighborhood of the knot, producing a compact 3-manifold with toral boundary components. When studying the geometry of these complements, we many times delete only the knot to obtain a non-compact 3-manifold. After much time as a conjecture, the following theorem, due to Cameron Gordon and John Luecke [20], states that knots are determined by their complements.

Theorem 2.1. *Two knots K_1 and K_2 are equivalent if and only their knot complements $E(K_i) = S^3 - \overset{\circ}{N}(K_i)$ are homeomorphic.*

Note that the above theorem is not true in the more general case of links; there are inequivalent 2-component links that have homeomorphic link complements.

2.1.3 Seifert Surfaces

A well-known classical topology theorem states that any closed, orientable surface is homeomorphic to a surface of genus g for some $g \geq 0$. The genus 0 surface is the *sphere* S^2, the genus 1 surface is the *torus* $T^2 = S^1 \times S^1$, and a higher genus surface is merely called a *surface of genus* g. If we allow that our surfaces have boundary, then for any surface S, its boundary ∂S will be a disjoint union of circles. If we are given a particular embedding $i : S \hookrightarrow S^3$, the restriction of i to the boundary $i|\partial S : \bigcup S_i^1 \to S^3$ will give an oriented link. Thus, we may ask the inverse question: Does every oriented link $L \subset S^3$ have a surface S such that $\partial S = L$. If we can find such a surface S with no closed components, then we say that S is a *Seifert surface* for L.

Seifert's algorithm is the most concrete way to show that any tame knot has a Seifert surface. His algorithm constructs, given a regular projection of the link L, a compact, orientable surface S with boundary $\partial S = L$ with no closed components; we review the construction here. Given a regular projection of the oriented link L, we may resolve each over and under-crossing by replacing them with *short-cut arcs*, the unique resolution of the crossing respecting the orientation of L. This gives a collection of *disjoint* simple closed oriented curves in the plane (called *Seifert circles*), which may be nested. Each of these curves bounds a disk in the plane; if the curves are nested, we may imagine pushing one off the other slightly. We may orient these individual disks by giving a positive orientation to the disks with boundary circle oriented counterclockwise and a negative orientation to those with clockwise oriented boundary. We may then reconnect these disks together via their old crossings with half-twists to form an *oriented* surface S whose boundary is the original link L. If L is a knot, then S is automatically connected. Otherwise, we may join the components my tubes to create a connected surface if we wish. Our definition of Seifert surface does not require that they be connected, but only have no closed components.

From this construction, it is a straightforward exercise to compute the genus g of the Seifert surface S. If s is the number of Seifert circles, n the number of components of L, and c the number of crossings in the diagram, we obtain the formula

$$g = 1 - \frac{s + n - c}{2}$$

by considering the Euler characteristic. Given a knot, the set of possible genera for Seifert surfaces are a subset of the natural numbers and are thus well-ordered. So, we define the *genus* of a knot to be

$$g(K) = \min\{g \mid K \text{ has a Sefiert surface of genus } g\}.$$

By its very definition, genus is a knot invariant; it hast has been useful in distinguishing certain knot pairs. One remarkable fact about the genus of a knot is that it completely classifies the unknot; that is, a tame knot has genus 0 if and only if it is the unknot. Clearly, the disk is Seifert surface for the unknot. Conversely, since the disk is the unique surface of genus 0 with 1 boundary component, the boundary knot can be contracted to a single point by following the disk radially inward, giving it the structure of an unknot.

2.2 Geometric Definitions

2.2.1 Riemannian Manifolds

Given a topological manifold M with a smooth structure, we may assign a smoothly varying choice of inner product at each tangent space T_pM called a *Riemannian metric*; this may be viewed as a symmetric, bilinear 2-form on the tangent bundle of the manifold. Given some local coordinate system $x_1, \ldots, x_n$ in a neighborhood of $p \in M$, this metric tensor takes the form of a matrix denoted by g, where the components are given by $g_{ij} = \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle$ with $\left\{ \frac{\partial}{\partial x_j} \right\}$ serving as a basis for T_pM. One may use this version of the metric to define geometries entities like *length, area, volume*, and *curvature.*

We now investigate some classic examples of Riemannian metrics. Euclidean n-space $\mathbb{E}^n$ with its standard Euclidean metric and the standard coordinate functions is represented by the matrix

$$g_{\mathbb{E}^n} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & \cdots & 1 \end{bmatrix},$$

the $n \times n$ identity matrix. When viewing this as a symmetric, bilinear form on two tangent vectors $\mathbf{v} = \langle v_1, v_2, \ldots, v_n \rangle, \mathbf{w} = \langle w_1, w_2, \ldots, w_n \rangle \in T_p\mathbb{E}^n$, we obtain the usual inner product:

$$\mathbf{v}^T g \mathbf{w} = \begin{bmatrix} v_1 & v_2 & \ldots & v_n \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & \cdots & 1 \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = v_1 w_1 + v_2 w_2 + \cdots v_n w_n.$$

Another critical Riemannian manifold in geometric topology is that of hyperbolic space. Topologically, $\mathbb{H}^n$ may be represented in its *upper-half space model* given by

$$\mathbb{H}^n = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_n > 0\}.$$

Using the given coordinates $x_1, \ldots, x_n$, the metric on hyperbolic space is given by the matrix

$$g_{\mathbb{H}^n} = \frac{1}{x_n^2} I_n,$$

where I_n is the $n \times n$ identity matrix (i.e., the Euclidean metric). It can be shown that hyperbolic space has constant sectional curvature -1.

When we restrict to hyperbolic 3-space, the group of isometries is isomorphic to $PSL(2,\mathbb{C})$ acting by Möbius transformations on the upper-half space $\mathbb{H}^3$. Thus, if we are given any finitely-generated torsion-free subgroup Γ of $PSL(2,\mathbb{C})$ acting discretely on $\mathbb{H}^3$, we will obtain a 3-manifold that is locally isometric to $\mathbb{H}^3$. Thus, these are hyperbolic manifolds in that they also have constant sectional curvature of -1 and have $\mathbb{H}^3$ as a universal cover with the covering map a local isometry.

2.3 Combinatorial Definitions

We will use simplicial complexes, which are combinatorial objects, to describe classes of Seifert surfaces and how they lie in S^3. The building blocks of a simplicial complex is the *n-simplex*. It may be defined linearly as follows:

$$\Delta^n = \left\{ (t_0, t_1, \dots, t_n) \in \mathbb{R}^{n+1} \;\middle|\; \sum_i t_i = 1 \text{ and } t_i \geq 0 \right\}.$$

Given this description, we endow Δ^n with the subspace topology coming from $\mathbb{R}^{n+1}$. These simplices are convex subsets of $\mathbb{R}^{n+1}$ and therefore contractible.

For any n-simplex, there are $n+1$ distinguished points corresponding to the unit coordinate vectors

$$v_i = (0, 0, \dots, 1, \dots, 0, 0).$$

Intuitively, these points correspond to the corners of Δ^n and are known as the *vertices*. In fact, we many times represent an n-simplex by its vertices:

$$\Delta^n = [v_0, v_1, \dots, v_n].$$

A *face* of a simplex Δ^n is a k-simplex (with $0 \leq k \leq n$) which is a subset of Δ^n defined by some subset of its vertices. Thus, any subset of the vertices of $\Delta^n = [v_0, v_1, \dots, v_n]$ will define a k-dimensional face, where $k+1$ is the cardinality of the subset.

A *simplicial complex* $\mathcal{K}$ is a collection of simplices subject to the following two conditions:

(1) If Δ is a simplex in $\mathcal{K}$, then any face of Δ is also a simplex in $\mathcal{K}$.

(2) If Δ_1 and Δ_2 are simplices in $\mathcal{K}$, then their intersection $\Delta_1 \cap \Delta_2$ is a face of both Δ_1 and Δ_2.

Given any simplicial complex $\mathcal{K}$, we can consider the subcomplex $\mathcal{K}^r$ consisting of all simplices Δ^n with n at most some r; this subcomplex is called the *r-skeleton* of $\mathcal{K}$. Thus, the 0-skeleton of $\mathcal{K}$ is the disjoint union of the vertices of the simplicial complex. Note that the 1-skeleton of a simplicial complex is a graph. The *dimension* of a simplicial complex $\mathcal{T}$ is the highest n for which $\mathcal{T}$ contains an n-simplex. If no such n exists, then the simplicial complex is *infinite-dimensional*.

In general, mathematicians are many times only interested in a simplicial complex at the level of its 1-skeleton. A simplicial complex is *connected* if it is connected as a topological space. Since the individual simplices are connected, a simplicial complex is connected if and only if its 1-skeleton is a connected graph. Furthermore, a simplicial complex is *locally finite* if its 1-skeleton has finite valence at every vertex.

Every simplicial complex $\mathcal{K}$ may also be endowed with a metric, where the n-simplices are given the standard Euclidean metric coming from their definition as a subspace of $\mathbb{R}^{n+1}$. More practically, one is many times only interested in the metric restricted to the 1-skeleton. In particular, given any two vertices $v, w \in \mathcal{K}^0$, we define the distance $d(v, w)$ to be the minimum number of edges one must traverse in $\mathcal{C}^1$ to join v and w. This is equivalent to the path-metric on $\mathcal{C}^1$ and is quasi-isometric to the path-metric on all of $\mathcal{K}$. We define the *diameter* $\operatorname{diam}(\mathcal{K})$ to be the maximum of $d(v, w)$, where v and w run over all vertices.

Chapter 3

The Kakimizu Complexes

3.1 Defining the Kakimizu Complexes

3.1.1 Definitions.

Given an oriented link L in S^3, we may build a simplicial complex $S(L)$ as follows. The vertex set of $S(L)$ consists of isotopy classes of oriented Seifert surfaces for L. A set of $k+1$ isotopy classes $[S_0], \ldots, [S_k]$ span a k-simplex in $S(L)$ if and only if there exist representatives S_i of each isotopy class such that $S_i \cap S_j = \emptyset$ in $S^3 - \overset{\circ}{N}(L)$ for $i < j$.

The complex $S(L)$ is commonly too cumbersome, so one frequently restricts attention to any one of a variety of subcomplexes. One of the most natural classes of subcomplexes is $MS(L) \subset S(L)$, where the vertices are *minimal genus* Seifert surfaces for the link L; that is, one considers Seifert surfaces with no closed components that maximize the Euler characteristic χ.

One may also consider a slightly larger subcomplex $IS(L)$, where the vertex set is now the isotopy classes of *incompressible* Seifert surfaces. These surfaces are defined as those whose inclusion map $i : S \hookrightarrow S^3 - N(L)$ induces a monomorphism of fundamental groups:

$$i_* : \pi_1(S) \hookrightarrow \pi_1(S^3 - \overset{\circ}{N}(L)).$$

The Loop Theorem tells us that incompressibility for a Seifert surface S is equivalent to the absence of compressing disks for S. So, if a minimal genus Seifert surface contained a compressing disk, cutting along this disk would produce a Seifert surface of strictly lesser genus. Thus, we have the following inclusions of subcomplexes:

$$MS(L) \subseteq IS(L) \subset S(L).$$

Though these two smaller subcomplexes $MS(L)$ and $IS(L)$ deal with Seifert surfaces of least complexity, their properties are quite distinct. Generally speaking, $MS(L)$ lends itself well to geometric analysis (by considering area bounds), while $IS(L)$ is tailor-made for combinatorial arguments. In this spirit, it is many times fruitful to consider the complex $S_g(L)$, where one considers Seifert surfaces of genus at most g for some $g \geq \text{genus}(L)$. Thus, we also have the following inclusions:

$$MS(L) \subset S_g(L) \subset S(L).$$

In general, there are links with incompressible Seifert surfaces of arbitrarily high genus, so we may not include $IS(L)$ in this inclusion for all g and an arbitrary link L.

3.2 Motivating the Kakimizu Complex

3.2.1 The Murasugi Sum

The Murasugi Sum is a topological operation defined on two compact oriented surfaces-with-boundary S_1, S_2 embedded in S^3; more relevantly, these surfaces should be viewed as Seifert surfaces for two oriented links $\partial S_i = L_i$. Intuitively, the Murasugi Sum is a way to glue our two Seifert surfaces to obtain a new Seifert surfaces for a different link.

The oriented surface $S \subset S^3$ is a $2n$*-Murasugi Sum* of two connected oriented surfaces $S_1, S_2 \subset S^3$ if we have a decomposition of the surfaces

$$S = S_1 \cup_D S_2 \text{ with } D = 2n - \text{gon}$$

along with a decomposition of S^3 into two 3-balls of the following form:

$$S^3 = B_1 \cup B_2$$

$$S_1 \subset B_1,\ S_2 \subset B_2 \text{ with } B_1 \cap B_2 = R \text{ a 2-sphere and}$$

$$S_1 \cap R = S_2 \cap R = D.$$

Note that when $n = 1$, a 2-Murasugi Sum S is the well-known *boundary-connected sum* of S_1 and S_2 and the boundary link $\partial S = K$ is a *link connected sum* of $\partial S_i = L_i$:

$$L = L_1 \# L_2.$$

There exists many classical results about link connected sums in Knot Theory literature; many of the motivating questions surrounding analysis of the Murasugi Sum operation are natural generalizations of these theorems.

When $n = 2$, the 4-Murasugi Sum is known as a *plumbing* of our two Seifert Surfaces. Plumbings are also relatively well-understood [obtain references for plumbings]. Unlike the more manageable operation of connected sum, a plumbing of two surfaces S_1 and S_2 depends critically on the choice of plumbing disk D and how the two 3-balls B_1 and B_2 are glued. Consider a Hopf link and its Seifert Surfaces of a ± 2 twisted unknotted annulus. Varying the above choices in the Murasugi Sums produces the Seifert surface for a trefoil, figure-8 knot, or a 3-component link. In this regard, Murasugi Sum is many times seen as a *generalized plumbing.*

Clearly, this definition of Murasugi Sum may be extended from surfaces in S^3 to those in a general closed 3-manifold M by replacing the decomposition

$$B_1 \cup B_2 = S^3$$

with

$$B_1 \cup B_2 = M,$$

a decomposition of M into two compact submanifolds B_i with homeomorphic boundaries. In this case, the boundaries of our surfaces can be seen as knots in our 3-manifold M; necessarily, these knots must be null-homologous, as they bound Seifert surfaces.

Gabai's seminal work [16] [18] on Murasugi Sums asserts that this operation preserves many of the best qualities of our surfaces and their boundary links. We will explore these in more detail in a later section.

3.3 Connected Sum Decomposition for Knots

Since Murasugi Sum is a natural generalization of the operation of connected sum of two knots in S^3, hopes of expanding classical results about this simpler operation motivate a great many open questions in Knot Theory.

The most important theorem about this connected sum operation is arguably the Prime Decomposition Theorem of Schubert. We say that a knot K is *prime* if K can never be written as a non-trivial connected sum. Explicitly, if K is given as the connected sum

$$K = K_1 \# K_2,$$

then at least one K_i must be the unknot. Any knot which is not prime is called *composite.*

Clearly, connected sum is a well-defined associative, commutative, binary operation on the space of oriented knots; moreover, the unknot obviously serves as the identity element in this monoid. So, a prime knot is one that has no non-trivial factors with respect to this operation. Using the equality

$$g(K_1 \# K_2) = g(K_1) + g(K_2)$$

and the fact that $g(K) = 0$ if and only if K is the unknot, we see that any non-trivial knot does not have an inverse in the monoid of knots. Further, any genus 1 knot must be prime.

The Prime Decomposition Theorem says that any non-trivial knot is the connected sum of a finite number of non-trivial prime knots.

Theorem 3.1 (Schubert [53]). *Let K be a non-trivial knot in S^3. Then K may be decomposed as*

$$K = K_1 \# K_2 \# \cdots \# K_n$$

of non-trivial prime knots. Furthermore, this decomposition is unique up to permuting summands.

Clearly, this theorem mirrors the prime decomposition of positive integers, with prime knots playing the role of prime numbers (thus justifying the terminology) and the unknot serving as the unit element $1 \in \mathbb{Z}$.

This theorem also allows us to calculate the genus of composite knots by using its prime summands. We may extend the additivity of the genus to obtain the following equality for the genus of K:

$$g(K) = \sum_{k=1}^{n} g(K_i),$$

where this sum ranges over the prime summands of K_i. Coincidentally, this also demonstrates that there exists knots of arbitrary genus g by taking the connected sum of g copies of a prime knot (e.g., the trefoil knot or the figure-8 knot).

3.4 Conjectures about the Kakimizu Complex

The main motivation of computing the Kakimizu complexes $MS(L)$ and $IS(L)$ is that they categorize the *least complexity* Seifert surfaces for L along with information on how these surfaces sit relative to each other in the link complement. Since many operations on knots are defined on minimal genus or incompressible Seifert surfaces, it would be very helpful to obtain qualitative information about this complex and how a choice of Seifert surface impacts these operations.

More concretely, we have seen that Murasugi sum is a very natural operation; in future sections, we will see that Murasugi sum also provides for many applications in knot theory. However, the definition of Murasugi sum is defined on the level on its Seifert surfaces. Since Murasugi sum is also a generalization of connected sum, it is reasonable to ask if the classical theorems can be extended to the Murasugi sum operation. Thus, Sakuma [47] raises the following question.

Question 3.2. *Is there a certain kind of uniqueness in the decomposition of links into Murasugi sums?*

One of the key components in beginning to answer this question is to classify the minimal genus or incompressible Seifert surfaces for links. A deep understanding of the simplicial structure of $MS(L)$ or $IS(L)$ may be required, as well, to classify Murasugi sum structures of Seifert surfaces.

Since the Kakimizu complexes give a combinatorial description of the space of choices for the Murasugi sum operation, various topological properties about these spaces become desirable in hopes that they will lead to an independence of choices. Thus, Kakimizu gives the following bold conjecture.

Conjecture 3.3. *$MS(L)$ and $IS(L)$ are contractible as topological spaces.*

As we shall see, this conjecture has been verified for several large classes of links, but no program is yet in place to prove contractibility in more generality.

Chapter 4

Properties of the Kakimizu Complexes

4.1 Local Properties

4.1.1 Finite Dimensional

One of the cornerstones of classic 3-manifold topology is the Haken-Kneser Finiteness Theorem. Essentially, this theorem bounds the number of non-parallel, 2-sided, pairwise disjoint incompressible surfaces in a compact, irreducible 3-manifold M using only the combinatorial and homological data of M.

While various different explicit quantities exist for the upper bound (Bachman [3], Hatcher [24], Haken [22]), they all arise from a variation of the same method. Given a triangulation τ of the compact manifold M, the surfaces S_i are put in normal form with respect to τ. In this form, the surfaces S_i must meet each simplex transversely in either triangles or rectangles and are subject the gluing conditions of the triangulation; furthermore, the incompressibility of the surfaces and the irreducibility of the manifold imply the intersection of a surface with a 2-simplex may be isotoped to contain no Jordan curves. From this, one obtains a linear bound on the number of components of $M - \bigcup S_i$ in terms of the t, the number of 2-simplices in τ and the number of non-trivial I-bundles. The latter is then bounded by the second homology of M with $\mathbb{Z}_2$ coefficients. For example, Hatcher's proof [24] of the Haken-Kneser Finiteness Theorem gives the upper bound as

$$4t + \dim H_2(M; \mathbb{Z}_2).$$

The importance of Haken-Kneser Finiteness in classic 3-manifold topology lies in the almost immediate corollaries of the Prime Decomposition Theorem and the JSJ Torus Decomposition Theorem of 3-manifolds. The Prime Decomposition Theorem states that any compact, orientable 3-manifold

M can be written as

$$M = P_1 \# P_2 \# \cdots \# P_n$$

where each P_i is a prime manifold; furthermore, this decomposition is unique up to a permutation of the factors P_i. The Torus Decomposition Theorem is the analogue when the splitting surfaces are tori; that is, any orientable, compact 3-manifold can be split along a finite collection of disjoint tori T_i such that every components of $M - \bigcup T_i$ is atoroidal. This result contrasts with the Prime Decomposition Theorem in that the atoroidal pieces are not unique.

In the context of the Kakimizu complexes, the Haken-Kneser Finiteness theorem says that the complex $IS(L)$ is finite-dimensional. For any link a non-split link L, the theorem gives an explicit bound on the number of disjoint, non-parallel incompressible Seifert surfaces. Since $MS(L)$ is a subcomplex, it is also finite-dimensional.

Note that the complex $S(L)$ – the complex of all Seifert surfaces – does not have such a property. Given any link L and any Seifert surface S of genus g, we can form a countable collection of disjoint Seifert surface S_k of genus $g + k$ by adding trivial handles and shrinking slightly. This highlights the importance of the subcomplexes $MS(L)$ and $IS(L)$:their finite-dimensional property makes the combinatorics of these complexes more manageable.

4.1.2 Local Finiteness

A simplicial complex is said to be *locally finite* if any vertex is contained in only finitely many 1-simplices. Thus, every vertex in a locally finite complex is adjacent to a finite number of other vertices. In terms of the Kakimizu complexes, local finiteness is equivalent to every isotopy class of minimal genus (resp. incompressible) Seifert surface being disjoint from only finitely minimal genus (resp. incompressible) Seifert surfaces classes.

We will, in a future section, present a class of knots which do not have locally infinite $IS(K)$. These knots have the property that there exists an infinite family of incompressible surfaces all disjoint from one minimal genus Seifert surfaces. These incompressible surfaces are, however, not minimal genus.

4.2 Global Properties

4.2.1 Connectedness

Given a topological space with sufficient structure (e.g., simplicial, cellular), one of the fundamental properties of appeal is that of connectedness. In the context of $MS(L)$ and $IS(L)$, to be connected is equivalent to the existence of a a path in the one-skeleton of the complexes from any vertex to any other. Phrased in the language of Seifert surfaces, this connectedness is equivalent to, given any two isotopy classes σ, $\sigma' \in MS(L)$ (resp. $IS(L)$), does there exist a sequence of minimal genus (resp. incompressible) Seifert surfaces $S_0, S_1, S_2, \ldots, S_n$ such that $\sigma = [S_0]$, $\sigma' = [S_n]$ and $S_i \cap S_{i+1} = \emptyset$ in the link complement for $0 \leq i \leq n-1$.

For a knot K, Scharlemann and Thompson [49] find such a path using the notion of *double-curve sum* and least area surfaces. Given two oriented Seifert surfaces S and T in general position for our knot K with non-trivial intersection $S \cap T$ in $S^3 - N(K)$, double-curve sum allows us to alter alter the topology of the two surfaces while leaving the sum of the Euler characteristics constant. Specifically, a neighborhood A of $S \cap T$ can be viewed as $(S \cap T) \times D^2$, where S and T intersect each component of A in the horizontal or vertical axes, respectively. If α is the line running from $(1,0)$ to $(0,-1)$ and β the line running from $(-1,0)$ to $(0,1)$ in D^2, we can obtain the double-curve sum $S \asymp T$ by replacing $S \cap T$ in A by $(S \cap T) \times (\alpha \cup \beta)$. This double-curve sum operation is supported in A and resolves the intersection of these surfaces. Since this operation simply removes and replaces neighborhoods of arcs and circles and glues them back differently, $\chi(S \asymp T) = \chi(S) + \chi(T)$.

Scharlemann and Thompson use this double-curve sum operation to obtain the following result, implying the connectedness of $MS(L)$:

Proposition 4.1. *If S and T are minimal genus Seifert surfaces for K, then three is a sequence of minimal genus Seifert surfaces $S = S_0, S_1, \ldots, S_n = T$ such that for each $0 \leq i \leq n-1$, $S \cap S_{i+1} = \emptyset$.*

Scharlemann and Thompson obtain this result by starting with S and T and iteratively taking double-curve sum until some complexity defined in terms of a lexigraphical order reaches zero. It is precisely the sequence corresponding to zero complexity that provides the appropriate sequence of Seifert surfaces (and thus a path in the one-skeleton of $MS(K)$). Least area surfaces and the usual cut and area reduction arguments are used to obtain control on the possible intersections.

Using vastly different techniques, Kakimizu generalizes this result in his seminal paper [28]:

Theorem 4.2. *Let L be a non-split link. Both $MS(L)$ and $IS(L)$ are connected.*

In fact, Kakimizu's interpretation of the path metric on $MS(L)$ and $IS(L)$ has as a crucial step the construction of a sequence connecting two isotopy classes.

4.2.2 Diameter Bounds

The question of trying to bound the diameter of the complexes $MS(L)$ and $IS(L)$ is certainly of great importance in understanding the topology of these complexes and the nature of Seifert surfaces of various classes of links. The main result of this paper will focus on obtaining distance bounds on the complex $MS(K)$ for hyperbolic knots K. As we shall see, there are many (non-hyperbolic) knots where the diameter of the Kakimizu complexes are infinite. Thus, in the trichotomy of knots, hyperbolic knots are a compromise (with respect to diam($MS(L)$)) between torus knots (with trivial $MS(L)$) and satellite knots (which allows for unbounded $MS(L)$)

Chapter 5

Tools Employed in Calculating the Complex

5.1 Sutured Manifolds and Thurston Norm

Gabai defined and has used sutured manifolds to answer very deep theorems in knot theory and the foliation theory of 3-manifolds [16],[18],[19],[17]. A sutured manifold is a pair (M, γ) with M a compact oriented 3-manifold and a set $\gamma \subset \partial M$ of pairwise disjoint annuli $A(\gamma)$ and tori $T(\gamma)$; in all of the relevant applications and examples, we will be in the situation where $T(\gamma) = \emptyset$. The interior of each annulus in $A(\gamma)$ contains a *suture*, a homologically non-trivial oriented simple closed curve; the set of sutures is denoted by $s(\gamma)$. Furthermore, every component of $\partial M - \overset{\circ}{\gamma}$ is oriented; define $R_+(\gamma)$ and $R_-(\gamma)$ to be the components of $\partial M - \overset{\circ}{\gamma}$ whose normal vector points out of or into M, respectively. The orientations on this complement $R(\gamma)$ must also be coherent with respect to the sutures $s(\gamma)$.

While examples of sutured manifold abound, we restrict ourselves to the relevant ones that aid in the understanding of minimal genus Seifert surfaces and foliations. First, note that if (N, γ) is a sutured manifold and M is a codimension 0 manifold, then $(M - N, \gamma)$ has a natural sutured manifold structure. Next, if we are given a compact surface F with boundary in S^3 with no closed components (i.e., a Seifert surface for some link ∂F), then $(F \times I, \partial F \times I)$ is a sutured manifold in S^3. We may view the sutures as the level sets $s(\gamma) = \partial F \times 1/2$, while $R_+(\gamma)$ and $R_-(\gamma)$ correspond to the two sides of F. The exact distinction between $R_+(\gamma)$ and $R_-(\gamma)$ depends on the orientation given to F.

Sutured manifolds that admit certain decompositions are of special interests to 3-dimensional topologists. A disc decomposition of a sutured manifold is a removal of embedded disks that preserve the suture manifold structure. Specifically, assume that we have a sutured manifold (M_0, γ_0) in S^3 with $D \times I$ embedded in $S^3 - \overset{\circ}{M}_0$, where D is a disc, $D \times \partial I$ is properly embedded in $S^3 - \overset{\circ}{M}_0$, and the boundary annulus $A = \partial D \times I$ is embedded in ∂M_0. If the annulus A is transverse to γ_0 and the corresponding sutures $s(\gamma_0)$ and each arc of $\partial A \cap \gamma_0$ intersects $s(\gamma_0)$ exactly once, then we may define the disc decomposition

$$(M_0, \gamma_0) \xrightarrow{D} (M_1, \gamma_1).$$

The new manifold M_1 is obtained by $M_1 = M_0 \cup D \times I$ and the new sutures $s(\gamma_1)$ are obtained from $s(\gamma_0)$ by modifying $s(\gamma_0)$ near $D \times I$. Depending on how many sutures $D \times I$ meet, there are potential choices in how to glue glue $s(\gamma_0)$ to produce $s(\gamma_1)$. If $D \times s(\gamma_0)$ is just two points, then there is only one way to connect $s(\gamma_0)$. If $D \times s(\gamma_1)$ consists of more than 2 points, there are two distinct ways of creating $s(\gamma_1)$ from $s(\gamma_0)$.

A sutured manifold (M_0, γ_0) is said to be *completely disc decomposable* if there is a sequence

$$(M_0, \gamma_0) \xrightarrow{D_1} (M_1, \gamma_1) \xrightarrow{D_2} \cdots \xrightarrow{D_n} (M_n, \gamma_n)$$

of disc decompositions such that

(1) M_n is connected;

(2) ∂M_n is a union of 2-spheres $S_1, S_2, \ldots, S_k$; and

(3) $S_i \cap S(\gamma_n)$ is a simple closed curve for $1 \leq i \leq k$.

Gabai shows that when the sutured manifold obtained from a Seifert surface F for a link L in S^3 has a complete disc decomposition, then the link L is fibred with fibre F.

The connection with detecting the genus of a link comes when one considers how foliations extend across these sutured manifold operations. Gabai's main result [17] states that complete disc decomposability of a sutured manifold allows for the existence of nice foliations.

Theorem 5.1. *Let R be an oriented surfaces in S^3 and let L be the oriented link $L = \partial R$. If $R \neq D^2$ is completely disc decomposable, then the following hold:*

(1) There exists a C^∞ transversely oriented foliation $\mathcal{F}$ of $S^3 - \mathring{N}(L)$ such that $\mathcal{F}$ is transverse to $\partial N(L)$, $\mathcal{F}|\partial N(L)$ has no Reeb components, and R is the unique compact leaf of the foliation.

(2) $p : S^3 - (\mathring{N}(L) \cup R) \to$ Space of leaves of $\mathcal{F}|(S^3 - (\mathring{N}(L) \cup R))$ is a fibration over S^1, where p contracts each leaf to a point.

(3) R is a surface of minimal genus (maximal Euler characteristic) for the oriented link L.

(4) If R has a decomposition such that for each term

$$(M_i, \gamma_i) \xrightarrow{D_{i+1}} (M_{i+1}, \gamma_{i+1})$$

of the decomposition, either D_{i+1} *separates* $S^3 - \overset{\circ}{M}_1$ *or* $D_{i+1} \cap s(\gamma_i)$ *is* 2 *points, then* L *is a fibred link with fibre* R*.*

The proof of this theorem is inductive on the length n of the decomposition. The base case of $n = 0$ is trivial since then the complementary sutured manifold will be $(S \times I, \partial S \times I)$, where S is a union of discs; thus, this space can be given the product foliation, which satisfies the conclusion. The content of the theorem is then a case-by-case analysis of how to extend the foliation over the decompositions in the inductive step.

In fact, Gabai's results allow him to give a geometric proof that applying Seifert's algorithm to an alternating knot diagram will produce a minimal surface. By treating the cases of nested or un-nested Seifert circles separately, Gabai produces the following result.

Theorem 5.2. *Let* L *be a non-split alternating link. If* R *is a Seifert surface obtained by applying Seifert's algorithm to an alternating projection, then* R *is disc decomposable.*

Of course, using his previous result, he is then able to conclude that this Seifert surface is indeed of minimal genus. This result had been obtained by Murasugi [35] and Crowell [13], but by more algebraic means.

The connection between sutured manifolds and genus comes when one consider the Thurston norm on the relative second homology of a 3-manifold M. Let S be a compact oriented surface $S = \bigcup_{i=1}^{n} S_i$, with each S_i a connected surface. The *norm* of the surface S is given by

$$x(S) = \sum_{i \mid \chi(S_i) < 0} |\chi(S_i)| \, .$$

If K is a codimension-0 submanifold of ∂M and $z \in H_2(M, K)$ is a relative second homology class, define the *norm*

$$x(z) = \min \left\{ x(S) \mid (S, \partial S) \right\},$$

where the set runs over all properly embedded surfaces $(S, \partial S)$ in (M, K) such that $[S] = z \in H_2(M, K)$. A properly embedded oriented surface S in M is *norm-minimizing* in $H_2(M, K)$ if $\partial S \subset K$, S is incompressible, and $x(S) = x([S])$ for $[S] \in H_2(M, K)$. Thurston [55] proved that x is actually a pseudo-norm and that, in the presence of a nice foliation, can give information about surfaces in M.

Theorem 5.3 (Thurston)**.** *Let* M *be a compact, oriented* 3*-manifold. Let* $\mathcal{F}$ *be a codimension-1, transversely oriented foliation without Reeb components of* M *such that* $\mathcal{F}$ *is transverse to* ∂M*. If* R *is a compact leaf of* $\mathcal{F}$*, then* R *is norm-minimizing in* $H_2(M, \partial M)$*.*

Since the norm is stated in terms of the absolute value of the Euler characteristic of the connected components of a surface, R being norm-minimizing is equivalent to R having minimal genus among all Seifert surfaces (which represent the same relative homology class in a link complement). Thus, Gabai's result gives a concrete algorithm for computing the genus of a link.

5.2 Sutured Manifold Theory and Murasugi Sums

Murasugi sums lend themselves well to sutured manifold and foliation theory since the decomposition of a surface into Murasugi summands can be easily manipulated by operations on sutured manifolds.

Recall that an oriented surface $R \subset S^3$ is a Murasugi Sum of oriented surfaces R_1 and R_2 in S^3 if there is a decomposition of the following form:

(1) $R = R_1 \cup_D R_2$, where D is a $2n$-gon;

(2) $R_1 \subset B_1$, $R_2 \subset B_2$, where $B_1 \cap B_2 = S$, and S is a 2-sphere; and

(3) $B_1 \cup B_2 = S^3$ and $R_1 \cap S = R_2 \cap S = D$.

Following in the same vein as the above results of foliations of sutured manifolds, Gabai [16] [18] proves that the operation of Murasugi sum preserves the best properties of foliations.

Theorem 5.4 (Gabai). *Let S be the Murasugi Sum of S_1 and S_2 and let $L_i = \partial S_i$ and $L = \partial S$. If there exists a C^∞, transversely oriented foliation $\mathcal{F}_i$ on $S^3 - \overset{\circ}{N}(L_i)$ such that $\mathcal{F}_i$ is transverse to $\partial N(L_i)$, $\mathcal{F}_i$ has no Reeb components, $\mathcal{F}_i|\partial N(L_i)$ has no Reeb components, and S_i is a a compact leaf, then there exists a C^∞ foliation $\mathcal{F}$ on $S^3 - \overset{\circ}{N}(L)$ such that F is transverse to $\partial N(L)$, $\mathcal{F}$ has no Reeb components, $\mathcal{F}|\partial N(L)$ has no Reeb components, and $S$is a compact leaf. Furthermore,*

a) If each leaf of $\mathcal{F}_1$ and $\mathcal{F}_2$ is compact, then $\mathcal{F}$ can be constructed to only have compact leaves

b) If for each $i = 1, 2$, the quotient map to the leaf space sending each leaf of the restricted foliation $\mathcal{F}|(S^3 - (\overset{\circ}{N}(L_i) \cup S_i)$ is a fibration over S^1 or $(0, 1)$, then $\mathcal{F}$ can be constructed so that the quotient map to the leaf space of the restricted foliation $\mathcal{F}|S^3 - (\overset{\circ}{N}(L) \cup R)$ is a fibration over S^1.

Using this foliations result, Gabai then proves many useful combination-type theorems for the Murasugi sum operation. For example, Gabai shows that Murasugi sum preserves the minimal genus property. Gabai's result generalizes the classic theorem that the genus of a composite knot is the sum of the genera of its summands.

Theorem 5.5 (Gabai). *If S is a Murasugi Sum of S_1 and S_2 with $\partial S = L$ and $\partial S_i = L_i$, then S is a minimal genus Seifert surface for L if and only if S_i is a minimal genus Seifert Surface for L_i.*

In a similar spirit, Gabai shows that incompressibility is preserved under Murasugi sum.

Theorem 5.6 (Gabai)**.** *If S is incompressible in $S^3 - \overset{\circ}{N}(L_i)$ for $i = 1, 2$ and if S is any Murasugi Sum of S_1 and S_2, then S is incompressible in $S^3 - \overset{\circ}{N}(L)$.*

This theorem stands out since the converse is not true. A Seifert surface of a certain pretzel knot was shown by Parris [39] to be incompressible; however, this surface has a Murasugi decomposition in which one of the summands is clearly compressible.

Murasugi Sum also preserves fibredness of two Seifert surfaces and their boundary links.

Theorem 5.7 (Gabai)**.** *If S is a Murasugi Sum of S_1 and S_2, then $L = \partial S$ is a fibred link with fibre S if and only if $L_i = \partial S_i$ is a fibred link with fibre S_i for $i = 1, 2$.*

The above theorems are many times paraphrased heuristically: Murasugi sum preserves the best properties of the summands.

Chapter 6

Explicit Examples of the Kakimizu Complexes

6.1 Fibred Knots

Recall that a fibred knot is one whose knot complement fibres over S^1; that is, it has the topological description as the mapping torus of a surface and thus as a surface bundle over S^1. Specifically, if $X = S^3 - N(K)$, then K being fibred with fibre F means that there exists a continuous map $f : X \to S^1$ called a *fibration* such that for every point $\theta \in S^1$, there is a neighborhood U such that there exists a *trivializing homeomorphism* $h : f^{-1}(U) \to U \times F$ such that $f = \text{proj} \circ h$ on U.

From this definition, one may see that the complement of fibred knot (or fibred link, in more generality) has the structure of a mapping torus of a compact surface with boundary. Specifically, we may view the knot complement of a fibred link L as

$$S^3 - N(L) = \frac{F \times I}{(x, 0) \sim (\varphi(x), 1)},$$

where F is the fibre of the fibration and $\varphi : F \to F$ (known as the *monodromy*) is some orientation-preserving self-homeomorphism of F. F is forced to have n boundary components, where n is the number of components of the link L. Note that the topology of the knot complement is preserved by an isotopy of φ, thus one need only specify φ in the mapping class group $MCG(F)$ to define the link complement. In fact, more is true; any automorphism in $MCG(F)$ in the same conjugacy class will produce an identical link complement.

For a fibred knot K, we can consider $\widetilde{X}$, the cyclic cover of the knot complement X with covering map p corresponding to the commutator subgroup of the knot group. Using the exponential map $\exp : \mathbb{R}^1 \to S^1$, we may build the following commutative diagram:

$$\begin{tikzcd}
\widetilde{X} \arrow[r, "\widetilde{f}"] \arrow[d, "p"'] & \mathbb{R}^1 \arrow[d, "\exp"] \\
X \arrow[r, "f"] & S^1
\end{tikzcd}$$

In this diagram, $\widetilde{f} : \widetilde{X} \to \mathbb{R}^1$ is the lift of the map $f : X \to S^1$, which exists because $\mathbb{R}^1$ is the universal cover of S^1. Thus, we may view $\widetilde{X}$ as a fibration over $\mathbb{R}^1$; since $\mathbb{R}^1$ is contractible, elementary covering space theory says that the total space $\widetilde{X}$ has a global product structure. Since the fibre of f is a surface F, then $\widetilde{X}$ is homeomorphic to $F \times \mathbb{R}$. Since F is a compact surface with boundary K (being a Seifert surface of K), the fundamental group of this universal cover is a finitely-generated free group of rank twice the genus of F:

$$\pi_1(\widetilde{X}) = \pi_1(F \times \mathbb{R}) \cong \pi_1(F) = \mathrm{Free(2g)}.$$

Thus, finite generation of the commutator subgroup of the knot group is a necessary condition. Stallings [54] proved that it is also sufficient.

Theorem 6.1. *A knot K is fibred if and only if the commutator subgroup $[\pi_1(X), \pi_1(X)]$ of the knot group $\pi_1(X)$ is a finitely generated free group.*

In fact, the rank of the free group is precisely twice the genus of the knot (and not just the genus of F); that is, F is actually a minimal genus (and thus incompressible) Seifert surface. In fact, for fibred knots (and fibred links in general), there is a unique incompressible (and thus minimal genus) Seifert surface up to isotopy [9]; it is given by the fibre of the fibration over S^1 and thus its genus may be determined by computing its finitely generated commutator subgroup . In the language of the Kakimizu complexes, $MS(L)$ and $IS(L)$ are single vertices.

One necessary condition for a knot K to be fibred is for its Alexander polynomial to be monic [36]. Thus, a standard method for showing that certain knots are non-fibred is by computing their Alexander polynomials and noting that the lead coefficient is not ± 1.

Recall that Gabai's notion of the product decomposition provides an algorithm for deciding if a Seifert surface for a link is the fibre of a fibration. He obtains the following theorem.

Theorem 6.2. *Let F be an oriented surface with $\partial F = L$ its oriented link boundary. L is a fibred link with fibre F if and only if $(S^3 - (F \times I), \partial F \times I)$ has a product decomposition.*

6.2 Torus Knots

Given an unknotted torus T in S^3 with a longitude and meridian pair (l, m), the torus knot $T_{p,q}$ with $\gcd(p, q) = 1$ is the knot that wraps p-times around the meridian m and q times around the longitude l. The condition p and q be relatively prime guarantees that we do indeed obtain a knot; in general, if $\gcd(m, n) = d$, then we will will obtain d parallel copies of the torus knot $T_{p,q}$ with $p = m/d$ and $q = n/d$. Equivalently, viewing a torus as the quotient of $\mathbb{R}^2$ by the standard action of integer lattice $\mathbb{Z}^2$, $T_{p,q}$ the torus knot $T_{p,q}$ may be formed by taking the unique line of slope p/q through the origin and translating it around $\mathbb{R}^2$ by the action of $\mathbb{Z}^2$; the quotient $\mathbb{R}^2/\mathbb{Z}^2$ is the torus T^2 with a curve on its surface. Using the standard embedding of T^2 into S^3, the curve on the surface is the knot $T_{p,q}$.

Given two pairs of relatively prime integers (p, q) and (p', q'), one wishes to classify when the corresponding torus knots $T_{p,q}$ and $T_{p',q'}$ are isotopically equivalent in S^3. Using the first definition of torus knots, it is clear that the torus knots $T_{\pm 1,q}$ and $T_{p,\pm 1}$ are isotopic to the unknot. Also, by interchanging the chosen meridian and longitude in the first example, one can see that $T_{p,q}$ is equivalent to $T_{q,p}$. Furthermore, changing the sign of p or q will not affect the isotopy type of the torus knot. Aside from these obvious exceptions, all other torus knots are inequivalent. In fact, Schreier [52] proves that if $1 \leq p \leq q$, then the knot group of $T_{p,q}$ determines the (unordered) pair p, q. In fact, the torus knot group has a very straightforward presentation:

$$\pi_1(S^3 - N(T_{p,q})) = \langle x, y; x^p = y^q \rangle \,.$$

As shown by Burde and Zieschang [8], torus knots are unique in that they have the only knot groups with non-trivial center. In fact, the center $Z(\pi_1)$ of the knot group is the infinite cyclic group generated by the element $x^p = y^q$.

Torus knots also have interesting topological properties. If one takes the unknotted torus T^2 on which $T_{p,q}$ is embedded, and cuts along the knot, the resulting surface $T^2 - T_{p,q}$ is an incompressible, properly embedded annulus. This incompressible annulus is an obstruction to hyperbolizing the knot complement; that is, it does not admit a complete hyperbolic metric.

One important yet surprising knot-theoretic aspect of torus knots is that they are fibred. This is supported (but not proven) by the fact that the Alexander polynomial of $T_{p,q}$ is monic. In fact, it has the form:

$$A(t) = \frac{(1-t)(1-t^{pq})}{(1-t^p)(1-t^q)},$$

which is a monic polynomial of degree $(|p| - 1)(|q| - 1)$. In fact, the Alexander polynomial for the torus knots of type p, q and p', q' are distinct unless $\{|p|, |q|\} = \{|p'|, |q'|\}$, which gives us Schreier's classification theorem. The monodromy associated to the fibration of the complement of $T_{p,q}$ is hard to compute in general. Rolfsen demonstrates that such computations are difficult by computing the monodromy of the trefoil complement:

$$\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}.$$

Regardless, since $T_{p,q}$ are fibred knots, their Kakimizu complexes $MS(L)$ and $IS(L)$ are single vertices.

6.3 Murasugi Sums of Fibred Knots

Using the technology developed in [19], Gabai is able to prove that the connected sum operation preserves fibredness. In fact, he proves a much stronger theorem.

Theorem 6.3. *Let $F \subset S^3$ be a Murasugi sum of oriented surfaces $F_1, F_2 \subset S^3$. $L = \partial F$ is a fibred link with fibre F if and only if $L_i = \partial F_i$ is a fibred link with fibre F_i for $i = 1, 2$.*

Since connected sum is a 2-Murasugi sum, we see that the connected sum of fibred knots are fibred. In particular, since $MS(L_i) = IS(L_i)$ is a single vertex for fibred links, the same is true of $MS(L_1 \# L_2)$. Of course, the same is true for a general Murasugi sum of two fibred Seifert surfaces.

Since both L_i are fibred links with fibre F_i, they have their respective monodromies

$$\varphi_i : F_i \to F_i.$$

The monodromy of the Murasugi sum F of the two fibres can then be calculated in terms of φ. Gabai's following result says that the monodromy of F is a composition of the monodromies of the summands F_i given certain conditions.

Theorem 6.4. *Suppose that F is a Murasugi sum of F_1 and F_2, $\partial F = L$, $\partial F_i = L_i$, and L_i is a fibred link with monodromy $\varphi : F_i \to F_i$ such that $\varphi_i | \partial F_i = id$. Then L if a fibred link with fibre F and monodromy*

$$\varphi = \varphi_2' \circ \varphi_1' : F \to F$$

where $\varphi_i' | R_i = \varphi_i$ and $\varphi_i' | (R - R_i) = id$.

Notice that the two extensions φ_i' of φ_i commute since they restrict to the identity map on their complements. Of course, this reflect the fact that Murasugi sum is a commutative operation, and the monodromy is not affected by the order in which the sum is performed.

6.4 Murasugi Sums with Unique Seifert Surfaces

Using the technology of product decompositions in sutured manifold theory, Kobayashi [30] gives necessary and sufficient conditions for the link of the Murasugi sum of two minimal genus Seifert surfaces to have a unique minimal genus Seifert surface.

Theorem 6.5 (Kobayashi). *Let L_i be a link with minimal genus Seifert surface R_i for $i = 1, 2$ and R a Murasugi sum of R_1 and R_2. The minimal genus Seifert surfaces for $L = \partial R$ are unique if and only if one of L_1 and L_2 (say L_2) is fibred and the minimal genus Seifert surfaces for L_1 are unique.*

Note that when both L_1 and L_2 are fibred links, this gives precisely the fact that the boundary of any Murasugi Sum of the fibres of L_1 and L_2 will have unique minimal genus Seifert surface. Since the techniques employed rely heavily on sutured manifold theory, Kobayashi's results do not cover $IS(L)$.

6.5 Connected Sum of Non-fibred Knots

One of the first examples of a knot with an infinite collection of minimal genus (or incompressible) isotopy classes of Seifert surfaces was produced by Eisner [15] using the connected sum of non-fibred knots. Given two oriented non-trivial knots K_1 and K_2, one may form the connected sum $K_1 \# K_2$, which will have an incompressible torus in its knot complement; such *swallow-follow* tori place composite knots in the class of satellite knots and therefore have complements which never admit a complete hyperbolic structure. We will show that, with moderate conditions on the knots K_i, the diameter of both $MS(K_1 \# K_2)$ and $IS(K_1 \# K_2)$ are unbounded.

Eisner's construction of this infinite family of minimal genus Seifert surfaces was suggested by Haken and illustrates the exotic behavior of $MS(L)$ and $IS(L)$ in the presence of an incompressible torus.

Given two oriented knots K_1 and K_2 in S^3 with minimal genus Seifert surfaces S_1 and S_2. Let N be a regular neighborhood of a meridian m_1 of K_1, which meets S_1 in a disk and does not meet K_2 or S_2. Then $V_1 = \mathrm{cl}(S^3 - N)$ is an unknotted solid torus and has $l_1 = S_1 \cap \partial V_1$ as a longitude (since l_1 represents a meridian for N). Let V_2 be a regular neighborhood of K_2, meeting S_2 in an annulus having one boundary component K_2 and the other being $\partial V_2 \cap S_2$. Furthermore, V_2 has $l_2 = S_2 \cap \partial V$ as a longitude (since it is isotopic in V to K_2). Choosing a homeomorphism $f : V_1 \to V_2$ satisfying $f(l_1) = l_2$, we may set $\widetilde{S_1} = f(S_1 \cap V_1)$, $V' = \mathrm{cl}(S^3 - V_2)$ and $\widetilde{S_2} = S_2 \cap V'$. Then we see that the image $f(K_2)$ of $K_2 \subset V_1$ is precisely the composite knot $K_1 \# K_2$. Furthermore, we may form a Seifert surface $F = \widetilde{S_1} \cup \widetilde{S_2}$ for the composite knot; since F is built from minimal genus Seifert surfaces S_1 and S_2 by cutting and gluing along disks and circles, $g(F) = g(S_1) + g(S_2)$ and thus (by the additivity of knot genus), F is a minimal genus Seifert surface for $K_1 \# K_2$. Note as well that this construction of $K_1 \# K_2$ satisfies an alternate definition for a satellite knot, defined in terms of companionship.

The swallow-follow torus $T = \partial V_2$ is an incompressible torus which we will use to construct this infinite family of minimal genus Seifert surfaces. Choosing a meridian m for T, we may write this torus as the product $T = S^1 \times S^1 = m \times l$, oriented consistently with K and S^3. Given these coordinates, a *meridional roll* R is the following isotopic deformation of the torus:

$$R_t(\theta_1, \theta_2) = (\theta_1 + 2\pi t, \theta_2)\,.$$

Note that this isotopy has the special property that $R_0 = R_1$. We may now extend R to an isotopic deformation E_t of V_2 such that $E_t(K_1 \# K_2) = K_1 \# K_2$ pointwise. We may now extend E_1 to a homeomorphism e of S^3 by requiring that e be the identity on $V' = \mathrm{cl}(S^3 - V_2)$. For every integer n, we may now define new Seifert surfaces $F^n = e^n(F) = (E_1)^n(S_1) \cup S_2$. Topologically, the map e and its iterates serve to roll the Seifert surface F n times around the incompressible torus T to obtain a new Seifert surface that is also of minimal genus.

Given these Seifert surfaces F^n, one must now decide if they represent distinct isotopy classes. To this end, Eisner constructs the winding number w of an isotopy and a surface to distinguish these separate classes. Eisner shows that the obstruction to which this rolling homeomorphism produces a distinct isotopy class coincides precisely with the notion of a knot being fibred. If both K_1 and K_2 are non-fibred knots, Eisner uses the algebraic characterization of fibred knots (as those for which the commutator of the knot group is finitely generated) to show that there do indeed exist an infinite number of isotopy classes. In fact, the only manner in which $K_1 \# K_2$ may posses at most a finite number of Seifert surface isotopy classes is for at least one summands K_1 and K_2 to be fibred. The case that both are fibred, $K_1 \# K_2$ is also fibred and thus $MS(K_1 \# K_2) = IS(K_1 \# K_2)$ is a single vertex.

Kakimizu [28], using his characterization of the distance function on $MS(K)$ and $IS(K)$, explicitly computes his complexes when an extra assumption is placed on the summands.

Theorem 6.6 (Kakimizu)**.** *Let $K = K_1 \# K_2$ be a composite knot. Suppose that for $i = 1, 2$, K_i is not fibred and the incompressible Seifert surfaces for K_i are unique. Then $IS(K) = MS(K)$ and they have the topology of $\mathbb{Z} \subset \mathbb{R}$. In particular, this infinite class coincides precisely with those constructed by Eisner.*

Certainly, for the composite knot $K = K_1 \# K_2$ to have infinitely many minimal genus Seifert surfaces, both summands must be non-fibred by Eisner's results. In fact, using Kobayashi's result, we see the weaker result of having plural minimal genus Seifert surfaces requires both knots to be non-fibred. Using this extra (relatively moderate) condition that $IS(K_1) = IS(K_2)$ be a single point, Kakimizu is able to precisely pin down the topology of the complexes for K. Clearly, $\mathrm{diam}(MS(K)) = \mathrm{diam}(IS(K)) = \infty$.

6.6 Knots with 10 or fewer crossings

Using the vast technology and classification theorems established by the turn of the century, Kakimizu [29] computes the complexes for prime knots of 10 or fewer crossings. In fact, Kakimizu proves that most knots of 10 or fewer crossings have unique incompressible Seifert surfaces. For the 11 prime knots that have plural incompressible Seifert surfaces, $IS(K) = MS(K)$ and each K has at most 4 isotopy classes. Of course, Kakimizu does not include composite knots in this tabulation, as the connected sum of the non-fibred knot 5_2 with itself has infinitely many minimal genus Seifert surfaces by Eisner's result. Actually, there are many prime knots with 11 crossing with infinitely many incompressible Seifert surfaces.

The content of Kakimizu's paper actually rested in computing the complexes $IS(K)$ for exactly four knots: $10_{53}, 10_{67}, 10_{68}$, and 10_{74}. These knots had complexes that were particularly difficult to calculate because they are not 2-bridge knots and thus not subject to the Hatcher-Thurston [25] classification theorem, described below.

6.7 Two-Bridge Knots

Hatcher and Thurston's seminal paper [25] classifying incompressible surfaces in 2-bridge knot complements has served as a model for classifying incompressible and minimal genus Seifert surfaces for other combinatorially defined knots and links in S^3. In particular, the Hatcher-Thurston paper reinforces the technique of encoding isotopy classes of incompressible Seifert surfaces in terms of other topological gadgets, which take the form of a branched surface in this context.

Given a rational number p/q with q odd, one can associate a 2-bridge knot by bridging a line of slope p/q in a meaningful knot-theoretic manner to be defined later. In general, every knot is a bridge knot in the following way. Given a knot K and a projection of K onto a plane P, an *overpass* of a projection is a subarc of the knot that goes over at least one crossing but never under a crossing. A *maximal overpass* is an overpass which cannot be extended further. The bridge number of a projection is the number of these maximal overpasses (also known as *bridges* for obvious visual reasons). This can be interpreted in a different manner by viewing these bridges as lying *above* the projection plane P as individual unknotted arcs penetrating P in $2b$ distinct points. This becomes an invariant for K when we define the bridge number $b(K)$ of a knot K to be the least bridge number of all possible projections for the knot K. Thus, the only 1-bridge knot is the unknot.

Two-bridge knots lend themselves well to classification problems because of this relatively simple combinatorial description. In fact, after pulling one of the strands straight, a 2-bridge knot can just be thought of as defined by a 3-braid. This description of $K_{p/q}$ can now be interpreted as an element of the braid group B_3 by considering *continued fraction expansion* of p/q. Recall that any rational number has a continued fraction expansion given by

$$\frac{p}{q} = r + [b_1, b_2, \ldots, b_k] = r + \cfrac{1}{b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots - \cfrac{1}{b_k}}}},$$

where $r, b_i \in \mathbb{Z}$ with $b_k \neq 0$. Conway [12] showed that the knot $K_{p/q}$ knot was precisely the boundary of the surface obtained by plumbing together k bands in a row, with the i-th band having b_i half-twists (being right-handed twists if $b_i > 0$ and left-handed if $b_i < 0$). As with most plumbing operations, there are two distinct ways of plumbing these bands together; this becomes clear when one recalls that the Conway sphere S^2 for the plumbing has a choice of a square D or its complement $S^2 - D$.

The classification of incompressible surfaces in 2-bridge knot complements is best phrased in the language of branched surfaces, which generalize the plumbing choices for Seifert surfaces. Given a continued fraction expansion $p/q = r + [b_1, b_2, \ldots, b_k]$, we may stack the boundary of k bands with b_i half-twists. We construct a branched surface $\Sigma[b_1, \ldots, b_k]$ by including these bands as well as the sphere S^2 that constitutes the two choices at each of the $k-1$ horizontal band intersections. This branched surface can carry a large number of (possibly disconnected) surfaces labelled $S_n(n_1, \ldots n_{k-1})$ where $n \geq 1$ and $0 \leq n_i \leq n$. This surface is given by taking n parallel sheets running closed to the vertical portion of the band of $\Sigma[b_1, \ldots b_k]$, which bifurcate into n_i parallel copies of the i-th inner plumbing square and $n - n_i$ parallel copies to the corresponding outer plumbing square. Note that when $n = 1$, this corresponds precisely to the 2^{k-1} plumbing choices of the k bands.

Hatcher and Thurston use these branched surfaces to classify all incompressible, ∂-incompressible, possibly disconnected surfaces in the 2-bridge knot complement $S^3 - K_{p/q}$.

Theorem 6.7. *(a) A closed incompressible surface in $S^3 - K_{p/q}$ is a torus isotopic to the boundary of a tubular neighborhood of $K_{p/q}$.*

(b) A non-closed incompressible, ∂-incompressible surfaces in $S^3 - K_{p/q}$ is isotopic to one of the surfaces $S_n(n_1, \ldots, n_{k-1})$ carried by $\Sigma[b_1, \ldots, b_k]$ for some continued fraction expansion $p/q = r + [b_1, \ldots b_k]$ with $|b_i| \geq 2$ for each i.

(c) The surfaces $S_n(n_1, \ldots, n_{k-1})$ carried by $\Sigma[b_1, \ldots, b_k]$ is incompressible and ∂-incompressible if and only if $|b_i| \geq 2$ for each i.

(d) *Surfaces* $S_n(n_1, \ldots, n_{k-1})$ *carried by distinct* $\Sigma[b_1, \ldots b_k]$ *with* $|b_i| \geq 2$ *for each* i *are not isotopic.*

(e) *The relation of isotopy among the surfaces* $S_n(n_1, \ldots, n_{k-1})$ *carried by a given branched surface* $\Sigma[b_1, \ldots b_k]$ *with* $|b_i| \geq 2$ *for each* i *is generated by the following:* $S_n(n_1, \ldots, n_{i-1}, n_i, n_{i+1}, \ldots n_{k-1})$ *is isotopic to* $S_n(n_1, \ldots, n_{i-1}+1, n_i+1, \ldots, n_{k-1})$ *if* $b_i = \pm 2$.

Since these surfaces are not guaranteed to be connected, a further analysis is needed:

Theorem 6.8. *Consider the surfaces* $S_n(n_1, \ldots, n_{k-1})$ *carried by a given* $\Sigma[b_1, \ldots, b_k]$. *The following is true:*

(1) *If all the* b_i*'s re even,* $S_n(n_1, \ldots, n_{k-1})$ *is connected only when* $n = 1$.

(2) *If at least one* b_i *is odd, each two-sheeted surface* $S_2(n_1, \ldots, n_{k-1})$ *is connected.*

We note that the 1-sheeted surfaces are orientable if and only if all of the b_i's are even. Since there is exactly one such fraction expansion $p/q = r + [b_1, \ldots, b_k]$ with each b_i even, we obtain the following classification for incompressible Seifert surfaces:

Corollary 6.9. *The orientable incompressible Seifert surfaces for* $K_{p/q}$ *all have the same genus and are all isotopic if and only if at most one of the* b_i*'s in the unique expansion* $p/q = r + [b_1, \ldots, b_k]$ *with all* b_i *even is not* ± 2.

Note that the band with ± 2 half-twists corresponds to a fibred link (the Hopf link which has complement $T^2 \times I$). Since fibred links have unique incompressible Seifert surface and the Conway sum (or Murasugi sum in more generality) of fibred links is fibred, then $K_{p/q}$ where all $b_i = \pm 2$ in the even expansion will be fibred as well and also have a unique incompressible Seifert surface. In fact, if m is the number of b_k with $b_k \neq \pm 2$, then there are precisely 2^{m-1} isotopy classes of minimal genus (equivalently, incompressible) Seifert surfaces.

We require the terminology of the next section to completely describe the complex $MS(K_{p/q})$. As we shall see, 2-bridge knots are a proper subclass of Arborescent Links, for which Sakuma has classified the minimal genus Seifert surfaces and calculated the complexes $MS(L)$. Clearly, the work of Hatcher and Thurston motivated this classification, but Sakuma's enumeration in terms of orientations on a tree are much more natural combinatorial objects.

6.8 Special Arborescent Links

Arborescent links, as a generalization of 2-bridge knots, are defined combinatorially and thus lend themselves well to classification. Arborescent links are defined via a finite, weighted plane tree, and their minimal genus Seifert surfaces correspond to edge orientations on this tree.

Let T be a finite plane tree and

$$w : V(T) \to \mathbb{Z}\text{–}\{0\}$$

be a weight function from the vertex set of T to the set of non-zero integers. To each $v \in V(T)$, associate an unknotted annulus $F(v)$ in S^3 with $w(v)$ right-handed twists (and $w(v) < 0$ corresponding to $|w(v)|$ left-handed twists). To produce a core orientation on $F(v)$, we draw the unknotted annulus with its $w(v)$ twists on the top portion of the band and no twists in the bottom portion. Then, let c_v be the counterclockwise orientation given by this realization and n_v be the normal orientation which points to the top of the page on the bottom (flat) part of the unknotted annulus. Let $e_1, e_2, \dots e_k$ be the edges of T which have v as a vertex and suppose they lie around v in a counterclockwise order (which is well-defined since T is a plane tree). We now specify k squares $D(e_1, v), D(e_2, v), \dots D(e_k, v)$ lying on the bottom (flat) part of the unknotted annulus ordered according to their cyclic order coming from the plane tree.

If two vertices v_1 and v_2 are incident in T by an edge e, then we plumb the bands $F(v_1)$ and $F(v_2)$ by gluing the squares $D(e, v_1)$ and $D(e, v_2)$ together using the orientation rule that c_{v_1} matches with n_{v_2} and c_{v_2} matches with n_{v_1}. This plumbing produces an orientable surfaces whose isotopy class depends crucially on the way this plumbing occurs; however, the boundary of any surface obtained by these plumbings are well-defined up to simultaneous change in orientation of the link components (i.e., up to semi-orientation). Denote this semi-oriented *special arborescent link* by $L(T, w)$, reinforcing the dependence of the link on only the weighted plane tree.

Fixing a base vertex v^*, we may orient the unknotted annulus $F(v)$ according to $c_v \wedge n_v$ or $-c_v \wedge n_v$ if v is an even or odd distance from v^*. Notice that this convention forces the gluing maps utilized in the plumbings to be orientation-preserving and that the resulting surface is thus oriented; furthermore, the choice of base vertex v^* will not affect the semi-orientation class of the surfaces and thus only affect the orientation of the surfaces by a simultaneous switch in orientation. We may further use this orientation to produce well-defined method of plumbing the unknotted annuli. Specifically, let ρ be an edge orientation on the plane tree T; that is, ρ is a designation at each edge e of T of which of its two vertices is the initial and terminal vertex. If v_1 and v_2 are the initial and terminal vertices of an oriented edge e according to the orientation ρ, then plumb $F(v_1)$ and $F(v_2)$ so that $F(v_2)$ lies above $F(v_1)$ with respect to the normal vector. Notice that the orientation on the edge e then corresponds to the choice of square on the Conway sphere corresponding to this plumbing.

The above construction of the special arborescent links and these Seifert surfaces is significant because it signals that topological classifications may be described combinatorially. A weighted plane tree T gives a well-defined semi-orientation class of a special arborescent link L; the further structure of an edge orientation ρ on T describes many (potentially distinct) isotopy classes of Seifert surfaces for L. Using the powerful tool of sutured manifolds, Sakuma [47] computes the minimal genus Seifert surface using exactly these combinatorial descriptions:

Theorem 6.10. *Let L be the special arborescent link defined by the plane tree T.*

(1) Any minimal genus Seifert surface for a special arborescent link L is equivalent to one obtained by plumbing according to some edge-orientation ρ on T.

(2) Two Seifert surfaces S and S' for L given by the orientations ρ and ρ', respectively, are isotopic if and only if ρ and ρ' are related by an iteration of a finite number of elementary operations.

Sakuma's result thus completely classifies the minimal genus Seifert surfaces for L in terms of orientations on T and relations amongst them. Thus, for any special arborescent link L coming from a weighted plane tree T with n vertices (and thus $n-1$ edges), there are at most 2^{n-1} isotopy classes of minimal genus Seifert surfaces (since there are precisely 2^{n-1} edge orientations on T).

To study the minimal genus Kakimizu complex $MS(L)$, we must define a complex whose vertex set is the set of orientations on the T, since these combinatorial objects define all possible minimal genus Seifert surfaces for the special arborescent link L. To understand the simplicial structure of $MS(L)$ with these orientation as vertices, more combinatorial terminology related to the space of orientations is required. A vertex $v \in T$ is *positive* with respect to an orientation ρ on T if every edge containing v has v as its terminal vertex. Let $v(\rho)$ be the new orientation obtained by switching all of the orientation of the edges containing a positive vertex v; thus, v is now the initial vertex of all edges containing v in the new orientation $v(\rho)$. A *cycle* is a sequence of orientations

$$\rho_0 \xrightarrow{v_0} \rho_1 \xrightarrow{v_1} \rho_2 \cdots \rho_n \xrightarrow{v_n} \rho_0$$

with $\rho_{k+1} = v_k(\rho_k)$ for a positive vertex v_k with respect to ρ_k. Now, we define simplicial complex $K(T)$ as follows. The vertex set of $K(T)$ is precisely this set of orientations, and a set of vertices $\{\rho_0, \rho_1, \ldots, \rho_k\}$ span a k-simplex if there is a *cycle* containing them. We make a further refinement by collapsing edges corresponding to vertices of weight $w(v) = \pm 1$. Specifically, collapse each edge in $K(T)$ of the form $\overline{\rho \cdot v(\rho)}$ to a point if $|w(v)| = 1$ which is positive with respect to ρ; call this new simplicial complex $K(T, w)$. Sakuma then makes the insight that this simplicial complex is precisely the Kakimizu complex.

Theorem 6.11. *For a special arborescent link L with plane tree T and weight w, $MS(L)$ is simplicially isomorphic to $K(T, w)$.*

The combinatorial type of the simplicial complex $K(T, w)$ is then computed to give a triangulation of the $(n-1)$-dimensional cube (where, again, n is the number of vertices of T) whose vertex set consists of its corners. Of course, since the unit cube is contractible, this verifies Kakimizu's conjecture for special arborescent links.

The purpose of passing to the quotient simplicial complex $K(T,w)$ from $K(T)$ by contracting edges arising from vertices of weight ± 1 comes from the fact that an annulus with ± 1 right-handed twists is a fibred link. Thus, any two potentially different choices plumbings of this annulus are isotopic via the fibre structure. Furthermore, note that even though two different plane trees T_1 and T_2 with n vertices give the same topological space of the $(n-1)$-dimensional unit cube, the underlying triangulation of this cube varies with the tree structures of T_1 and T_2.

We note that the set of 2-bridge links form a proper subclass of special arborescent links. In particular, if $K_{p/q}$ has $p/q = r + [b_1, b_2, \ldots, b_k]$ as its unique continued fraction expansion with b_i even, then $K_{p/q}$ is the special arborescent link with T an interval with k vertices weighted by $b_i/2$ at the i-th vertex. Note that both Hatcher-Thurston and Sakuma do take into consideration the plumbing of bands with ± 1 right-handed twists since these correspond to plumbing fibred links.

6.9 Special Alternating Knots

Special alternating knots form a subclass of oriented alternating knots that have deep relations to Seifert's algorithm for finding a Seifert surface for a link given a projection. Recall that an *alternating link* is a link L which possesses a link diagram that is alternating; that is, as we run around each component of L, the crossing alternate between over and under-crossings. Seifert's algorithm has as an input a link diagram and produces a Seifert surface by resolving each crossing so that a set of disjoint circles (called *Seifert circles*), which may or may not be nested; depending on the orientation on the boundary of these circles, a disk with positive or negative side is produced with these circles as their boundaries. Then, by adding twists at the crossings, an oriented surface S with $\partial S = L$. If, when resolving the crossings in Seifert's algorithm, only unnested Seifert circles are produced, the diagram for the alternating link is called a *special alternating diagram.* A link which possesses a special alternating diagram is known as a *special alternating link.* Note that since we are running Seifert's algorithm on alternating diagrams, the Seifert surfaces obtained must be minimal genus [13],[17].

To manipulate these ideas, we develop some notation developed in [26]. Given a special alternating diagram D on the plane, it gives rise to an oriented link $L(D)$. Running Seifert's algorithm on the diagrams D will produce a Seifert surface $F(D)$. Given a diagram D, there is a natural transformation which produces a new diagram D' called a *flype.* While changing the diagram from D to D', this flype move preserves the link type: $L(D) = L(D')$; thus, there exists a natural homeomorphism

$$\varphi : (S^3, L(D)) \to (S^3, L(D'))$$

called a *flype homeomorphism*. Since the diagrams are different, the Seifert surfaces $F(D)$ and $F(D')$ generated by D and D' may be inequivalent. In fact, if $F(D)$ and $\varphi^{-1}(F(D'))$ are not isotopic, we say that the diagram D' is obtained from D by an *essential flype*.

Hirasawa and Sakuma [26] find that for a prime, special alternating link, all of its minimal genus Seifert surfaces may be obtained from each other by a sequence of essential flypes.

Theorem 6.12. *Let L be an oriented, prime, special alternating link L and let D be an oriented, reduced, special alternating diagram representing L. Then for any minimal genus Seifert surface F for L, there is a finite sequence of oriented, reduced, special alternating diagrams $D_1, D_2, \ldots, D_n$ such that the following hold:*

(1) $D = D_1$ and $D' = D_n$,

(2) D_{i+1} is obtained from D_i by an essential flype (for $1 \leq i \leq n-1$), and

(3) F is equivalent to the pullback of $F(D')$ by $\varphi_{n-1} \circ \cdots \circ \varphi_2 \circ \varphi_1$.

Of crucial importance in this theorem is that the diagrams in question be special alternating. In fact, Hirasawa and Sakuma find that this theorem is far from true when considering the larger class of prime alternating link diagrams.

Theorem 6.13. *There are infinitely many prime alternating links with minimal genus Seifert surfaces which do not arise from alternating link diagrams. More precisely,*

(1) For any positive integer d, there are infinitely many prime alternating links with the following property: There is a minimal genus Seifert surface such that the minimal distance in $MS(L)$ from it to the minimal genus Seifert surfaces arising from running Seifert's algorithm on alternating diagrams is equal to d.

(2) There are infinitely many prime alternating links with arbitrarily many minimal genus Seifert surfaces which do not arise from alternating diagrams, and that such surfaces are disjoint in the link exteriors from one another and the minimal genus Seifert surfaces which arise from alternating diagrams.

Thus, there are many alternating links with arbitrarily many Seifert surfaces that do not arise from running Seifert's algorithm on alternating link diagrams.

The calculation of $MS(L)$ for special alternating links, as with that of the special arborescent links, takes advantage of the combinatorial naturel of L. Since L is special alternating, the decomposition in Seifert's algorithm into unnested Seifert circles gives rise naturally to a planar graph G, which has as its vertices the Seifert circles and edges the half-twists joining them to form the surface S. If we embed G in S, this graph forms a *spine* are carries the homotopy information of the Seifert surface. In fact, G carries more structure; since the orientations of the disks that point up in Seifert's algorithm are given by the boundary orientations on the Seifert circles, the vertices are marked with $+$ or $-$ depending this distinction.

Using this class of graphs with marked vertices, Hirasawa and Sakuma give a complete classification of minimal genus Seifert surfaces for special alternating links. The authors show that $MS(L)$, as a simplicial complex, is PL-homeomorphic to a finite product of n_i-simplices (where the n_i are given by the combinatorial data). Thus, as a topological space, $MS(L)$ is homeomorphic to a finite-dimensional ball and thus contractible. The proof of this statement uses the technology developed by Gabai in [17]: a sutured manifold decomposition corresponding to Murasugi sums of Seifert surfaces.

6.10 Doubled Knots with Non-Fibred Companions

The incompressible torus produced in the construction of satellite knots like doubles of non-trivial knots have been used in studying the Kakimizu complexes to demonstrate that $IS(K)$ need not be locally finite; that is, this complex need not have finite valence at a vertex.

Given a knot K_1, we can build its untwisted doubled knot K by essentially joining two parallel flat copies of K_1 and joining them in a canonical. More specifically, Let V_0 be an unknotted solid torus and J the standard doubling simple closed curve in V_0. Let V_1 be a tubular neighborhood of K_1 and $f : V_0 \to V_1$ a faithful homeomorphism; that is, we require that $f(l_0) = l_i$ and $f(m_0) = m_1$, where $(m_i, l_i) \subset \partial V_i$ is a meridian-longitude pair for the two solid tori. The image $K = f(J)$ is called the *untwisted doubled knot* of K_1. We say that the doubled knot K has K_1 as its *companion.* Note that by weakening the meridian-longitude assumption of f, we can form a general (twisted) doubled knot.

General statements may be made about companionship when we don't require J to be the standard doubling knot, but any *geometrically essential* knot in the standardly embedded solid torus. In this broader sense, we still say that $f(J)$ has K_1 as its companion. It is easy to see that companionship is a reflexive and transitive relation. In fact, companionship forms a partial order on the set of knots. Note that the unknot is a companion for every other knot, but it itself has no other companion. Thus, the unknot is the unique minimal element in this partial order. More can be said about the relationship between companions.

Theorem 6.14 (Rolfsen [44]). *If K_1 is a companion of K_2, then the knot group of K_2 contains a subgroup isomorphic to the knot group of K_1.*

Since the knot group of a tame knot is infinite cyclic if and only if the knot is trivial, we can deduce that any knot with a non-trivial companion is itself non-trivial.

Doubled knots are of interest to knot-theorists because their construction produced an incompressible torus ∂V_1 assuming that the companion V_0 is non-trivial. For such a doubled knot, the image of the standard genus 1 surface for J in V_0 is always exists a genus 1 Seifert surface for K. Assuming that K_2 is non-trivial, K will itself be non-trivial and thus have genus 1. Doubled knots, however, may have many incompressible (non-minimal genus) Seifert surfaces.

Theorem 6.15 (Kakimizu [27]). *Suppose that K_1 is a non-fibred knot of genus $g \geq 1$. Then the untwisted doubled knot K of K_1 has infinitely many non-equivalent incompressible Seifert surfaces, each of which is of genus $2g > 1$. Moreover, they are all disjoint from the canonical minimal genus Seifert surface S of genus 1 for K.*

The construction of this infinite class relies on spinning the standard incompressible genus $2g$ surface about the incompressible torus. More concretely, if we take two parallel copies of a Seifert surface F for K_1 outside of V_1 with opposite orientations, then we can connect the two surfaces inside of V_1 to the doubled knot. Note that at the twist in the double surface, we add a band that necessitates the two copies of the surface having opposite orientations. This Seifert surface is incompressible and is disjoint from the standard genus 1 surface S described above. Now, using either branched surface theory or Haken sum, we can construct an infinite family of incompressible Seifert surfaces by spinning F around the incompressible torus ∂V_1. By taking more and more copies of ∂V_i, we obtain incompressible Seifert surfaces of genus $2g$ that are potentially inequivalent; the inequivalence of this family relies heavily on the fact that K_1 is non-fibred. Clearly, every surface in this family is disjoint from S and thus the Kakimizu complex $IS(K)$ is not locally finite at $[S]$.

Chapter 7

Realizing the Path Metric

7.1 Kakimizu's formulation of Distance

For any simplicial complex $\mathcal{K}$, one can define the path-metric on the 1-skeleton $\mathcal{K}^1$ by giving every edge a Euclidean distance of length 1. Thus, the distance between any two vertices $v, w \in C^1$ is given by

$$d(x, y) = \min\{\# \text{ of edges traversed}\},$$

where the set runs over all possible paths from v to w in C^1. Kakimizu's great insight came from noting that the path metric on $IS(K)$ and $MS(K)$ has a geometric formulation [28]. The result is true for links, but we state it here for knots for simplicity.

Consider the knot group $\pi_1(S^3 - K)$ and its commutator subgroup

$$A = [\pi_1, \pi_1] = \{ghg^{-1}h^{-1} | g, h \in \pi_1\}.$$

For any group G, its commutator subgroup $[G, G]$ is normal in G and we can thus form the quotient group $G/[G, G]$, called the abelianization of G. When G is the fundamental group of some topological space X, the Hurewicz Theorem tells us that this abelianization gives us the first singular homology of X:

$$\pi_1(X)/[\pi_1(X), \pi_1(X)] \cong H_1(X; \mathbb{Z}).$$

By Alexander Duality, the homology group of the link complement is just

$$H_1(S^3 - K; \mathbb{Z}) = \mathbb{Z}.$$

Since the commutator subgroup $A \triangleleft \pi_1(S^3 - L)$ is a normal subgroup, the cover it corresponds to (called the universal abelian cover) is a regular cover, and its group of Deck transformations is given as the quotient group $\mathbb{Z}$. For notational ease, we will let

$$E = S^3 - N(K)$$

and $\widetilde{E}$ to this universal abelian cover. Thus, we have a covering

$$p : (\widetilde{E}, a_0) \to (E, a)$$

such that $p_*\pi_1(\widetilde{E}, a_0)$ is the commutator subgroup of $\pi_1(E, a)$. Let τ be a generator of the Deck group. Let $S \subset E$ be a Seifert surface for K, and consider E_0, the closure of a lift of $E - S$ to $\widetilde{E}$; since Seifert surfaces have no closed components, $E - S$ is connected, and thus its lift E_0 is also connected. Let $E_j = \tau^j(E_0)$ be the translated of E_0 under the Deck group and let their overlaps be $S_j = E_{j-1} \cap E_j$. Since τ generates the Deck group of this covering, we see that

$$\widetilde{E} = \bigcup_{j \in \mathbb{Z}} E_j, \;\; p^{-1}(S) = \bigcup S_j$$

and that the restricted covering map

$$p|S_j : S_j \to S$$

is a homeomorphism.

Clearly, we may perform this construction for any other Seifert surface S' for the knot K. Giving similar notation for the complement of the surface and its lifts, we have $E' = E - S'$, E_0' the closure of a lift in $\widetilde{E}$, and $E_j' = \tau^j(E_0')$. This gives an alternate description of the universal abelian cover in terms of S':

$$\widetilde{E} = \bigcup_{k \in \mathbb{Z}} E_k', \;\; E_{k-1}' \cap E_k' = S_k', \;\; p^{-1}(S') = \bigcup_{k \in \mathbb{Z}} S_k'.$$

Kakimizu's insight is that comparing these two descriptions of $\tilde{E}$ will give a manifestation of the path metric. This occurs by first considering a pseudo-distance function on the space of Seifert surfaces, then modifying this to a bona fide distance well-defined on equivalence classes.

To this end, define

$$m = \min\{k \in \mathbb{Z} \,|\, E_0 \cap E_k' \neq \emptyset\}, \;\; r = \max\{k \in \mathbb{Z} \,|\, E_0 \cap E_k' \neq \emptyset\}.$$

We can thus measure how far E_0' extends in $\tilde{E}$ relative to E_0 by taking the difference; thus, set

$$d(S, S') = r - m.$$

This function d has some basic properties useful in defining the distance:

(1) $d(S, S') \geq 1$

(2) $d(S, S') = 1$ if and only if $S \cap S' = \emptyset$.

(3) $E_j \cap E'_k \neq \emptyset$ if and only if $m \leq k - j \leq r$, and

(4) $E_0 \subset \bigcup_{m \leq k \leq r} E'_k$, and $S_j \subset \bigcup_{m+1 \leq k \leq r} E'_k$.

Using this function defined on the space of Seifert surfaces, we modify d to have a well-defined function on the space of equivalence classes $S(L)$. Let σ, σ' denote two isotopy classes of Seifert surfaces; define d on $S(L)$ as follows:

$$d(\sigma, \sigma') = \begin{cases} 0 & \text{if } \sigma = \sigma' \\ \min\{d(S, S') \mid S \in \sigma, S' \in \sigma'\} & \text{if } \sigma \neq \sigma' \end{cases}$$

Notice that if we use the d defined on Seifert surfaces (and not their isotopy classes) and consider $d(S, S)$ for the same surface S, then $d(S, S) = 2$ since the two closed lifts coincide. Further, any parallel copy S' of S will also have $d(S, S') = 1$. Since $d(S, S') \geq 1$ for all Seifert surfaces, it is necessary to require that $d(\sigma, \sigma) = 0$ in the distance function on $S(L)$.

Using the above properties, one can show that this function $d : S(L) \times S(L) \to \mathbb{Z}_+$ satisfies the distance axioms.

Proposition 7.1. *For any isotopy classes of Seifert Surfaces $\sigma, \sigma', \sigma'' \in S(L)$, the function d satisfies the following:*

(1) $d(\sigma, \sigma') = 0$ *if and only if* $\sigma = \sigma'$.

(2) $d(\sigma, \sigma') = d(\sigma', \sigma)$, *and*

(3) $d(\sigma, \sigma'') \leq d(\sigma, \sigma') + d(\sigma', \sigma'')$.

So, the function d is a distance function on $S(L)$. However, when we restrict attention to the subcomplexes of incompressible and minimal genus Seifert surfaces $IS(L)$ and $MS(L)$, the distance function d is exactly the path metric. Kakimizu summarizes his main result as follows:

Theorem 7.2. *Let $L \subset S^3$ be a non-split link and $S, S' \subset E$ be two incompressible (resp. minimal genus) Seifert surfaces for L. Suppose that $n = d([S], [S']) \geq 1$. Then there is a sequence of incompressible (resp. minimal genus) Seifert surfaces $S = F_0, F_1, \ldots, F_n$ such that*

(1) $[F_n] = [S']$,

(2) $F_{i-1} \cap F_i = \emptyset$ *for each* $1 \leq i \leq n$, *and*

(3) $d([S], [F_i]) = i$ *for each* $0 \leq i \leq n$.

Note that this proof strengthens a previous theorem of Scharlemann and Thompson, which stated that $IS(K)$ and $MS(K)$ are connected complexes for a *knot* K. Kakimizu's result applies to non-split links. Prior to Scharlemann-Thompson, it was only known that the complex of all Seifert surfaces $S(L)$ is connected.

Let l_I and l_M be the path-metrics on $IS(L)$ and $MS(L)$, respectively. This theorem does not directly state that this distance function d agrees with the path-metrics on the Kakimizu complexes. However, we note that if we consider the same isotopy class σ, then it is clear from the definition of d that

$$d(\sigma,\sigma) = 0 = l_I(\sigma,\sigma) = l_M(\sigma\sigma).$$

Further, $l_I(\sigma,\sigma') = 1$ (resp,. $l_M(\sigma,\sigma') = 1$) if and only if $\sigma,\sigma' \in IS(L)$ (resp., in $MS(L)$) have disjoint representatives. Using the property that $d(S,S') = 1$ if and only if S and S' are disjoint, we see that $d(\sigma,\sigma') = 1$ is equivalent to $l_i(\sigma,\sigma') = 1$ (and equivalent to $l_M(\sigma,\sigma') = 1$ when the classes are minimal genus). Using this, we can prove the following proposition.

Proposition 7.3. *(1)* $l_I(\sigma,\sigma') = d(\sigma,\sigma')$ *for* $\sigma,\sigma' \in IS(L)$.

(2) $l_M(\sigma,\sigma') = d(\sigma,\sigma')$ *for* $\sigma,\sigma' \in MS(L)$.

Proof. We give the proof for $IS(L)$ since the $MS(L)$ case is almost identical. Given two isotopy classes of incompressible Seifert surfaces σ and σ', the above theorem tells us that there exists a path in $IS(L)$ traversing vertices $\sigma = [S_0], [S_2], [S_3], \ldots, [S_{n-1}], [S_n] = \sigma'$ such that S_i is disjoint from S_{i+1} for $0 \leq i \leq n-1$ and $d(\sigma,\sigma') = n$. Of course, this gives a path of length n in $IS(L)$. Thus, we have the inequality

$$l_I(\sigma,\sigma') \leq n = d(\sigma,\sigma').$$

For the other direction, if $l_I(\sigma,\sigma') = n$, then by the definition of the path-metric, there is a sequence of vertices $\sigma = \sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_n = \sigma' \in IS(L)$ with $l_I(\sigma_i,\sigma_{i+1}) = 1$ for $0 \leq 1 \leq n-1$ (since they are connected by an edge). Since $l_i(\sigma,\sigma') = 1$ is equivalent to $d(\sigma,\sigma') = 1$, we have

$$l_I(\sigma,\sigma') = n = l_I(\sigma_0,\sigma_1) + l_I(\sigma_1,\sigma_2) + \cdots + l_I(\sigma_{n-1},\sigma_n) =$$

$$d(\sigma_0,\sigma_1) + d(\sigma_1,\sigma_2) + \cdots + d(\sigma_{n-1},\sigma_n)$$

$$\geq d(\sigma_0,\sigma_n) = d(\sigma,\sigma').$$

These two inequalities give us the desired result $l_I(\sigma,\sigma') = d(\sigma,\sigma')$.

Chapter 8

Hyperbolic Geometry in Knot Theory

8.1 Hyperbolic Manifolds

A hyperbolic 3-manifold $M$is a Riemannian manifold with a complete metric of constant sectional curvature -1. This is equivalent to M being the quotient of $\mathbb{H}^3$ by a subgroup Γ of Isom($\mathbb{H}^3$) acting freely, properly-discontinuously. Of course, any subgroup Γ' conjugate to Γ will produce an isometric manifold. The covering map $p : \mathbb{H}^3 \to M$ is a local isometry, and many local arguments in M may be made in hyperbolic space. Since $\mathbb{H}^3$ is the universal cover of M, we have the identification $\pi_1(M) \cong \Gamma$.

As we shall see, hyperbolic manifolds have many properties that make this subclass of Riemannian 3-manifolds a very active area of research. Even more striking is the fact that, in a certain sense, most topological 3-manifolds may be hyperbolized; that is, they may be given complete hyperbolic metrics of finite volume.

8.2 Mostow Rigidity

It is a fruitful question to ask if one can quantify or parameterize the space of complete hyperbolic metrics on a given topological space. Clearly, many compact manifolds have various obstructions to having being hyperbolic: non-trivial $\pi_2(M)$, essential tori, finite $\pi_1(M)$ to name a few). Mostow's insight was that for dimension $n \geq 3$, these complete hyperbolic structures are unique. This contrasts starkly with the dimension 2 case, in which the space of hyperbolic structures, called Teichmuller space, is homeomorphic to a finite-dimensional ball.

In contrast to dimension 2, any 3-manifold which admits a complete hyperbolic metric of finite-volume admits *only one* such metric [34]. Since such manifolds are described in terms of the subgroup of isometries $\Gamma \subset \mathrm{Isom}^+(\mathbb{H}^n)$ such that

$$M = \mathbb{H}^n/\Gamma.$$

Of course, any conjugate subgroup $g\Gamma g^{-1}$ will yield an isometric hyperbolic manifold. Thus, we may state Mostow Rigidity in its following algebraic form:

Theorem 8.1 (Mostow)**.** *Suppose Γ_1 and Γ_2 are two discrete subgroups of the group of isometries of $\mathbb{H}^n$ for $n \geq 3$ such that $\mathbb{H}^n/\Gamma_i$ has finite volume and suppose that $\varphi : \Gamma_1 \to \Gamma_2$ is a group isomorphism. Then Γ_1 and Γ_2 are conjugate subgroups.*

The geometric implications of this algebraic theorem come when we realize that a hyperbolic manifold always has $\pi_1(M)$ acting on $\mathbb{H}^n$ as a discrete subgroup of isometries.

Theorem 8.2 (Mostow)**.** *If M_1 and M_2 are two complete hyperbolic n-manifolds with finite total volume and $n \geq 3$, any isomorphism of the fundamental groups $\varphi : \pi_1(M_1) \to \pi_1(M_2)$ is realized by a unique isometry.*

Note that there is no *a priori* condition that φ be realized by a continuous map; the conclusion of Mostow's theorem, though, shows that φ is realized by not only a continuous map, but an isometry. Thus, if any two complete hyperbolic manifolds of finite volume have isomorphic fundamental groups, then they are isometric as Riemannian manifolds (and therefore homeomorphic as well). This means that for this class of manifolds, any geometric invariant (e.g., hyperbolic volume, length of shortest geodesic, first eigenvalue of the Laplacian) are actually *topological invariants*. In the case of knots, for example, differing volumes of hyperbolic knot complements immediately indicate that the knots inequivalent.

Since the universal cover of a hyperbolic manifold is contractible, it is an Eilenberg MacLane $K(\pi, 1)$ space. Thus, basic algebraic topology says that any two $K(\pi, 1)$ spaces with isomorphic fundamental groups must be *homotopic*. Thus, we obtain the following corollary.

Corollary 8.3. *If M_1 and M_2 are complete hyperbolic manifolds of finite volume, then they are homeomorphic if and only if they are homotopy equivalent.*

Of course, for dimension 2, this is straightforward since the homeomorphism class of a hyperbolic surface is given uniquely by its fundamental group (in fact, the *rank* of the fundamental group).

8.3 Margulis Thick-Thin Decomposition

Margulis' Theorem provides a decomposition of a hyperbolic 3-manifold into components of bounded geometry and controlled topology. This partition into submanifolds is derived from an an algebraic statement about subgroups of the fundamental group of the manifold.

8.3.1 The Margulis Lemma

Given a Riemannian manifold M and any piecewise differentiable path α in M, we can make sense of the length $L(\alpha)$ in terms of the given metric. Since any homotopy class of loops in M based at some point x contains such a piecewise differentiable representative based at x, we may define the fundamental group $\pi_1(M, x)$ to be the set of homotopy classes of piecewise differentiable loops based at x.

Given any positive constant $\varepsilon > 0$, we consider the following subsets of M:

$$M_{(0,\varepsilon]} = \{x| \text{ there exists some } \alpha \in \pi_1(M, x) \setminus 1 \text{ such that } l(\alpha) \leq \varepsilon\}$$

$$M_{[\varepsilon,\infty)} = \{x| \text{ there exists some } \alpha \in \pi_1(M, x) \setminus 1 \text{ such that } l(\alpha) \geq \varepsilon\}\,.$$

The submanifold $M_{(0,\varepsilon]}$ is known as the *ε-thin part* of M and $M_{[\varepsilon,\infty)}$ is the *ε-thick part* of M; in contexts in which ε is understood, we will simply refer to the regions as the thin and thick parts of M.

Margulis' Lemma is an algebraic statement about a particular fundamental group of a complete hyperbolic n-manifold. Given any complete hyperbolic n-manifold M, it may be obtained from a subgroup Γ of the group of isometries $\mathrm{Isom}(\mathbb{H}^n)$ of hyperbolic n-space acting properly discontinuously; thus, we view M as the quotient by this subgroup:

$$M = \mathbb{H}^n/\Gamma.$$

Since $\mathbb{H}^n$ is simply-connected, the fundamental group of M is isomorphic to this proper discontinuous subgroup:

$$\pi_1(M) = \Gamma.$$

Let $d(x, y)$ be the hyperbolic distance of two points $x, y \in \mathbb{H}^n$.

Theorem 8.4 (Margulis' Lemma)**.** *For every $n \in \mathbb{N}$, there exists an $\varepsilon_n > 0$ so that for any properly discontinuous subgroup $\Gamma <$ Isom$(\mathbb{H}^n)$ and for any $x \in \mathbb{H}^n$, the group Γ_{ε_n} generated by the set*

$$F_{\varepsilon_n} = \{\gamma \in \Gamma \,|\, d(x, \gamma(x)) \leq \varepsilon\}$$

is almost-nilpotent.

In general, a group G is said to be almost-nilpotent if it contains a finite-index subgroup H which is nilpotent. The striking feature of this theorem is its universality: the constant $\varepsilon_n > 0$ is dependent solely on the dimension n. Thus, when we restrict to the class of complete hyperbolic

3-manifolds, the Margulis constant ε_3 is completely universal.

8.3.2 Thick-Thin Decomposition

Using the Margulis Lemma [5], one is able to obtain surprising control on the topology of both the ε_n-thin and thick parts of M. For example, one can obtain that the thick-part of any finite-volume hyperbolic n-manifold is a *compact* submanifold.

Isometries of hyperbolic space come in 3 flavors—hyperbolic, parabolic, and elliptic—depending on their fixed point set in $\mathbb{H}^n \cup S^{n-1}_\infty$, hyperbolic space together with its visual boundary. Elliptic isometries are ruled out in our context since they correspond to a fixed point in the interior $\mathbb{H}^n$ and thus do not act properly discontinuously. Hyperbolic isometries have two fixed points x, y on the visual boundary S^{n-1}_∞ and preserve the unique geodesic with asymptotic endpoints x and y. Parabolic isometries fix a unique point z on S^{n-1}_∞ and preserve $(n-1)$-dimensional horospheres tangent to z; one may conjugate a parabolic isometry to have z correspond to ∞. Margulis' Lemma then gives completely classifies the possibilities for the subgroup Γ_{ε_n}, viewed as isometries of $\mathbb{H}^n$.

Theorem 8.5. *The following three mutually exclusive possibilities are given for* Γ_{ε_n}*:*

1) $\Gamma_{\varepsilon_n} = \{id\}$

2) $\Gamma_{\varepsilon_n} = \mathbb{Z}$ *generated by a hyperbolic isometry.*

3) Γ_{ε_n} *consists of parabolic isometries with the same fixed point* z *at infinity; every horosphere centered at* z *is* Γ_{ε_n}*-invariant and the action of* Γ_{ε_n} *restricted to these horospheres is isometric with respect to their Euclidean structures. Thus,* Γ_{ε_n} *may be viewed as a discrete subgroup of* $Isom(\mathbb{R}^{n-1})$.

Using this algebraic classification of the subgroup Γ_{ε_n} of $\pi_1(M)$ generated by ε_n-short loops, one can now obtain strong topological control for the possibilities of the thin part of M.

Theorem 8.6. *For any finite-volume hyperbolic manifold* M*, the thin part* $M_{(0,\varepsilon_n]}$ *is the union of pieces homeomorphic to one of the following types:*

1) $\overline{D^{n-1}} \times S^1$*, or*

2) $V \times [0, \infty)$ *where* V *is a differentiable oriented closed* $(n-1)$*-manifold supporting a Euclidean structure.*

Moreover, these pieces have positive hyperbolic distance from each other and are finitely many.

In particular, we can also say more about the geometry of the thin part of M. The interior of a Margulis tube C is diffeomorphic to $D^{n-1} \times S^1$ and contains a unique geodesic loop β. Given a

$\delta > 0$ small enough, the δ-neighborhood of β is contained in C and is isometric to the Riemannian manifold $B_r \times S_\rho$ (for $r, \rho > 0$ small enough), where B_r is a closed ball of radius r in $\mathbb{R}^{n-1}$ and S_ρ is a circle of length ρ. If $y \in B_r, z \in S_\rho, v \in \mathbb{R}^{n-1}$, and $l \in \mathbb{R}$, then the metric on $B_r \times S_\rho$ is given by

$$ds^2_{(y,z)}(v, l) = ||v + ly||^2 + l^2.$$

In the case of a cusp component $D$of the thin part, there is also much control on the geometric topology. The interior of D is diffeomorphic to to $B \times \mathbb{R}$, where B is a closed orientable $(n-1)$-manifold supporting a Euclidean structure. As above, we gain some geometric control of a subset $D' \subset D$ such that D' is *isometric* to the Riemannian manifold $B \times [0, \infty)$, where B has the following metric:

$$ds^2_{(x,t)}(v, l) = e^{-2t} de^2_x(v) + l^2.$$

In this expression of the metric, de^2_x is the Euclidean metric on B.

If we restrict to our relevant case of $n = 3$, then we gain a tremendous amount of rigidity in the topological type for the thin pieces of a finite-volume hyperbolic 3-manifold. In particular, the Margulis tubes have the topological type of a solid torus $D^2 \times S^1$ containing a geodesic. Since the unique orientable surface admitting a Euclidean structure is the torus T^2, the non-compact components manifest themselves as a product $T^2 \times [0, \infty)$.

In the even more specific case of a hyperbolic link complement $S^3 - L$, the cusp pieces correspond exactly to a neighborhood of the deleted link. Since the metric is assumed to be unique, we imagine these cusp pieces to exit the manifold and extend infinitely far away from the manifold. Along with these cusp regions, the Margulis tube components of the thin part precisely consist of the regions of the manifold with small injectivity radius.

8.4 Geometrization and the Trichotomy of Knots

In an attempt to topologically classify 3-manifolds, mathematicians have searched for geometric structures that, in a very strong sense, constrain the possible topologies of the underlying topological space. Motivated the Uniformization Theorem for surfaces and the insight that topological spaces admitting a hyperbolic geometry were more the rule than the exception, Thurston conjectured that every 3-manifold can be canonically decomposed into pieces, each of which may be given one of eight geometries:

Conjecture 8.7 (Thurston's Geometrization Conjecture)**.** *Any compact, connected* 3*-manifold can be cut along essential spheres and tori so that the interior of the resulting pieces admits one of* 8 *geometric structures with finite volume.*

The eight possible model geometries were previously classified by Thurston. They include the three geometries with constant curvature $S^3, \mathbb{E}^3, \mathbb{H}^3$, the product geometries $S^2 \times \mathbb{R}$, $\mathbb{H}^2 \times R$, $\widetilde{SL(2, \mathbb{R})}$, and two exceptional geometries *Nil* and *Solv*. These *model geometries* come from considering all simply-connected smooth 3-manifolds M with a transitive action of a Lie group G on M with compact stabilizers; these geometries are also required to be *maximal* in the sense that the Lie group G acting on M is maximal, and that there exist at least one compact manifold with that geometry.

Assuming the Geometrization Conjecture (which may now be a theorem due to Perelman [40]), other several deep conjectures become true. The most historically important is that of the *Poincaré Conjecture*, which states that any closed, simply-connected 3-manifold is homeomorphic to S^3; this follows since the only compact geometry is S^3 and thus any closed manifold with finite π_1 must have a spherical geometry. Furthermore, the Geometrization Conjecture also implies that any closed irreducible, atoroidal 3-manifold admits a (unique) complete hyperbolic metric; this is many times paraphrased by saying that *most* 3-manifolds are hyperbolic.

While the status of Perelman's proof of Geometrization remains unclear (yet optimistic), this conjecture has been known to be true for several decades for a proper subclass of 3-manifolds. An compact, orientable 3-manifold is called *Haken* if it is irreducible and contains an orientable, 2-sided incompressible surface. Since all links have an incompressible Seifert surface, link complements fall into the class of Haken manifolds. In general, compact, irreducible 3-manifolds with positive first betti number are also Haken for homological reasons.

Restricting to the subclass of knot complements, Thurston's Geometrization Theorem gives us the following *trichotomy of knots*: torus, satellite, and hyperbolic knots. This mutually exclusive classification may be stated in terms of topological data. The presence of a properly embedded, essential annulus give torus knots; clearly, if one has a torus knot on the standardly embedded torus in S^3, then cutting along the knot will produce such an annulus. Second, the presence of an essential (i.e., incompressible and non-boundary parallel) torus corresponds to satellite knots; note that all composite knots having an incompressible swallow-follow torus and are thus satellite (according to this definition). The noteworthy part of Thurston's theorem is that the absence of such an annulus or torus are the only requirement for a knot complement to admit a complete hyperbolic structure with finite volume. Thus, any anannuluar, atoroidal knot complement is hyperbolic.

8.5 Hyperbolic Knots

In many senses, hyperbolic knots are the most abundant among the class of prime knots. In fact, of the 2977 non-trivial prime knots with 12 or fewer crossings, the only non-hyperbolic knots are seven torus knots [1]. Given that the only topological obstructions to a complete hyperbolic metric is an essential annulus or torus, the fact that hyperbolic knots are so ubiquitous is not surprising.

Many geometers have tried to identify subclasses of knots with hyperbolic structures. One such class are the *arithmetic links*; such links have link groups isomorphic with a *Bianchi Group* $PSL(2, \mathcal{O}_d)$, where d is a square-free integer and $\mathcal{O}_d$ is the ring of integers in the imaginary quadratic field $\mathbb{Q}\left(\sqrt{-d}\right)$. Baker [4] proved that every link in S^3 is the sublink of an arithmetic link. For knots, Reid [42] proved that the only arithmetic knot is the figure-eight knot. A more combinatorial class of hyperbolic links are a certain subcollection of *Montesinos links*. These are generalizations of pretzel links and are obtained by arranging rational tangles in a cyclic fashion. These links are denoted by $K\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots, \frac{p_n}{q_n}\right)$, where $\frac{p_i}{q_i}$ denotes the i-th rational tangle. Oertel [37] proved that a Montesinos link is hyperbolic if is not a torus link and not equivalent to a finite collection of exception Montesinos links.

One important operation on knots that preserves hyperbolicity is that of mutation. Given a knot or link projection and a circle on the plane that intersects the link in 4 points and separates it into two tangles, one can perform a *mutation* of the knot using this decomposition. This mutations takes the form of flipping the interior tangle about a vertical or horizontal axis or rotating the tangle by 180°. Ruberman [45] showed that mutation on a hyperbolic link produces another hyperbolic link *with the same hyperbolic volume.* Mutants are many times difficult to distinguish and cause headaches in the search for complete knot invariants.

Chapter 9

Minimal Surface Theory

9.1 Motivating Minimal Surfaces

9.1.1 Plateau's Problem

The motivation for studying minimal surfaces is derived from analyzing Plateau's problem for finding an area or energy minimizing surface in $\mathbb{R}^3$ with certain boundary conditions. This question was first posed by the nineteenth-century Belgian biologist Joseph Antoine Ferdinand Plateau after attempting to describe the geometry of intersecting soap bubbles. Plateau's problem asks, given a Jordan curve $\gamma \subset \mathbb{R}^3$, is there a surface Σ with $\partial\Sigma = \gamma$ that minimizes among all such surfaces with boundary γ; this naive definition of a minimal surface as area-minimizing will soon be generalized. As mathematicians varied the conditions on γ (e.g., rectifiable, piecewise-smooth) and Σ (e.g., immersed, regular, embedded), different solutions and techniques have manifested themselves historically to give rise to a crucial area of geometry and the calculus of variations with far-reaching applications.

One of the first major contributions to a solution of Plateau's problem came from Jesse Douglas, who pioneered the use of the *Dirichlet Integral* to solve such variational problems.

Theorem 9.1 (Douglas)**.** *Let Γ be an arbitrary Jordan curve in $\mathbb{R}^n$. Then there exists a simply-connected generalized minimal surfaced bounded by Γ.*

Here, a generalized minimal surface is a non-constant map of a surface with a conformal structure with various restrictions on the transition maps. As indicated above, his technique for finding such a surface was to minimize the following Dirichlet integral:

$$\mathcal{D}(x) = \iint_D \sum_{k=1}^{n} \left[\left(\frac{\partial x_k}{\partial u_1} \right)^2 + \left(\frac{\partial x_k}{\partial u_2} \right)^2 \right] du_1\, du_2.$$

Here, the surface is given by the a the map $x : D \to \mathbb{R}^n$ where the disk D has coordinates u_1, u_2 the map $x : D \to \mathbb{R}^n$ is given by $x(u_1, u_2) = (x_1(u_1, u_2), \ldots, x_n(u_1, u_2))$.

In subsequent decades, refinements of Douglas' result have been obtained by Osserman [38] and Gulliver [21], who found a *regular* simply-connected surface Σ minimizing area with $\partial\Sigma = \Gamma$; regular surfaces are defined as those whose Jacobian has full rank. If one adds the condition that the Jordan curve Γ lie in the boundary of a convex body, then Meeks and Yau [33] prove that an area-minimizing surface is indeed embedded.

The mathematical significance of Plateau's problem stretches far beyond the intuitive soap-film description of minimal surface. In fact, it is when one generalizes the definition of minimal surfaces (beyond those which only minimize area) and allows such surfaces to lie in arbitrary Riemannian manifolds that a rich theory with deep ramifications in geometric topology (and many other subfields) begins to emerge.

9.2 Minimal Surfaces

9.2.1 Connections on a Riemannian Manifold

Given a smooth manifold M, a *connection* on M is a linear map

$$\nabla : \Gamma(TM) \otimes_{\mathbb{R}} \Gamma(TM) \to \Gamma(TM)$$

$$X \otimes Y \mapsto \nabla_X Y$$

which is tensorial in the first factor

$$\nabla_{fX} Y = f \nabla_X Y$$

and follows a Liebniz-type rule in the second factor:

$$\nabla_X fY = X(f)Y + f\nabla_X Y.$$

Intuitively, a connection gives one a way of differentiating one vector field against another. With respect to a given connection, one may define a path $c : [0,1] \to M$ to be *geodesic* if

$$\nabla_{c'} c' \equiv 0.$$

In the context of a Riemannian manifold with inner product $\langle \cdot, \cdot \rangle$ on TM, we may ask that the connection ∇ be *metric*:

$$X \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle .$$

We may also ask that the connection ∇ be *torsion-free*; that is, we ask that it be symmetric up to

Lie bracket:

$$\nabla_X Y - \nabla_Y X = [X, Y] .$$

Requiring that a connection be both metric and torsion-free defines it uniquely; this connection is known as the *Levi-Civita connection* and is given explicitly by the following formula:

$$\langle \nabla_X Y, Z \rangle = \frac{1}{2} \{ X \langle Y, Z \rangle - Z \langle X, Y \rangle - \langle X, [Y, Z] \rangle + \langle Z, [X, Y] \rangle + \langle Y, [Z, X] \rangle \} .$$

9.2.2 The Riemannian Curvature Tensor

Given a Riemannian manifold and its uniquely defined Levi-Civita connection ∇, we may define the *Riemannian curvature tensor* $R(\cdot, \cdot)$ as a map

$$R(\cdot, \cdot) : \Gamma(TM) \otimes \Gamma(TM) \to \Gamma(\mathrm{End(TM)})$$

given by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Note that this map is both $C^\infty(M)$-linear and tensorial.

Using the Riemannian curvature tensor, one may define the various flavors of curvature on a Riemannian manifold. In particular, one can, given any two linearly-independent vectors $X, Y \in T_p M$ spanning the 2-plane $X \wedge Y$ in $T_p M$, one may define the *sectional curvature* of $X \wedge Y$ by

$$K(X \wedge Y) = \frac{\langle R(X, Y)Y, X \rangle}{||X \wedge Y||}$$

with $||X \wedge Y||$ denoting the area of the parallelogram spanned by X and Y in $T_p M$. It is straightforward to check that $K(X \wedge Y) = K(Z \wedge W)$ if X and Y span the same plane as Y and W. On a surface, the tangent spaces $T_p M$ are 2-dimensional and thus the sectional curvature K is a function $K : M \to \mathbb{R}$.

Another important curvature used frequently in differential geometry is Ricci curvature. Viewing the Riemannian curvature tensor $R(X, Y)$ as an endomorphism of the tangent bundle, the trace of this map gives the Ricci curvature. Formally, if $e_1, \ldots, e_n \in T_p M$ form an orthonormal basis for the tangent space at $p \in M$, then the Ricci curvature is given by

$$\mathrm{Ric}(X, Y) = \sum_{i=1}^{n} \langle R(e_i, X)Y, e_i \rangle = \sum_{i=1}^{n} \langle R(X, e_i)e_i, Y \rangle .$$

Since Ric is a symmetric bilinear form, it is many times written as a $(1, 1)$-tensor:

$$\mathrm{Ric}(X) = \sum_{i=1}^{n} R(X, e_i)e_i.$$

Ricci curvature should be viewed as a generalization of the Laplacian of the the metric. In fact, this insight motivated Hamilton's construction of Ricci flow [23] on a manifold to parallel the heat flow equation (which is stated in terms of the Laplacian of the temperature function).

Riemannian manifolds with restrictions on their Ricci curvature frequently have very strong geometric and topological constraints. In particular, some existence theorems for minimal surfaces require restrictions on the sign of the Ricci curvature; this should be interpreted in the following way. A Riemannian manifold is said to have $\mathrm{Ric} \geq k$ if and only if all the eigenvalues of $\mathrm{Ric}(X)$ are greater than or equal to k; indeed, since $\mathrm{Ric}(X)$ is symmetric, all of its eigenvalues must be real. A particularly special Riemannian metric is one which is a scalar multiple of its Ricci curvature:

$$\mathrm{Ric}(X, Y) = \lambda \cdot \langle X, Y \rangle .$$

These *Einstein manifolds* are of crucial importance in both Differential Geometry and Physics. Note that if a Riemannian manifolds has constant sectional curvature k, then its metric is Einstein of with Einstein coefficient $(n - 1) \cdot k$. In dimension 3, the converse is true; an Einstein metric is also a constant sectional curvature metric.

9.2.3 Mean Curvature

Intuitively, the mean curvature vector field on a surface S in a Riemannian manifold M should serve as a type of gradient vector for the area functional on the space of (embedded, immersed, generalized immersed) surfaces in M. Thus, given a co-orientation on a surface S, its mean curvature vector field will be a normal vector field that points in the direction one should perturb S so as to reduce area. If this vector field is everywhere zero, then the surface at hand is a critical point for this area functional.

To make this precise, given some isometric immersion of S in a 3-manifold M, let e_1, e_2 be an orthonormal basis for T_pS and consider the average curvature given by $\mu(p) = \nabla_{e_1} e_1 + \nabla_{e_2} e_2$. Since ∇ is the Levi-Civita connection and e_1, e_2 form an orthonormal basis for the tangent space of the surface, μ gives a normal vector field. Given the co-orientation of S in M, the vector field μ is equivalent to a real-value function $g : S \mapsto \mathbb{R}$ since the codimension of S in M is 1. Note that this definition of the mean curvature vector field is independent of the choice of orthonormal basis and is thus well-defined on S.

We will now demonstrate that μ does produce precisely the gradient vector field on the space of smooth embeddings of S into M. Let S_t be a 1-parameter variation of the surface $S = S_0$ satisfying the ordinary differential equation

$$\left.\frac{dS_t(p)}{dt}\right|_{t=0} = f(p)\nu,$$

where ν is the unit normal vector field defined by the co-orientation on S and $t \in (-\varepsilon, \varepsilon)$. We can view this differential equation as defining a family of surfaces flowing along $f(p)\nu$.

Differentiation the area function on this 1-parameter variation yields the *first variational formula*:

$$\left.\frac{d\mathrm{Area}(S_t)}{dt}\right|_{t=0} = -\int_S \langle f\nu, \mu\rangle \, dA,$$

where dA is the area form on S and μ is the mean curvature vector field. Since the metric $\langle\cdot,\cdot\rangle$ is positive-definite, $S = S_0$ is a critical point for the area functional restricted to this perturbation if and only if the mean curvature vector field μ vanishes identically. Since this is true for an arbitrary perturbation $f(p)$, this holds in general and μ is indeed the gradient vector field for the area functional.

Thus, the formal definition of a minimal surface is given precisely in this language. An isometric immersion S in a Riemannian 3-manifold M is a *minimal surface* if its mean curvature vector field μ vanishes identically.

If we consider a surface S that minimizes area in its isotopy class, then the above arguments demonstrates that S will be a minimal surface (since a minimum for the area functional is certainly a critical point).

9.3 Properties of Minimal Surfaces

9.3.1 The Second Fundamental Form and Curvature Bounds

The second fundamental form for an isometrically immersed hypersurface in a Riemannian manifold gives information on how the intrinsic and extrinsic metric properties of the submanifold differ. In the context of a co-oriented surface in a 3-manifold, we may take any two vector fields $e_1, e_2 \in \Gamma(TS)$ and a unit normal vector ν to define

$$A(e_i, e_j) = \langle \nu, \nabla_{e_i} e_j \rangle .$$

This *second fundamental form* A for the surface S is tensorial and symmetric in e_1 and e_2; thus, A defines a symmetric bilinear form on the tangent space T_pS for each $p \in M$.

Gauss' Lemma gives an explicit relationship between the curvature of the sectional curvature of the surface and that of the ambient manifold M in terms of the second fundamental form. If K_S and K_M are respectively the sectional curvatures of S (as a function) and M (in terms of the subspace of TM corresponding to TS), then the Gauss equation is given as follows.

Theorem 9.2 (The Gauss Lemma)**.** *If A denotes the second fundamental form of S, then*

$$K_S = K_M + \det(A).$$

Note that if we are given an orthonormal basis $e_1, e_2 \in T_pS$, then the trace of the second fundamental form A is precisely the mean curvature for S:

$$\operatorname{tr}(A) = \nabla_{e_1} e_1 + \nabla_{e_2} e_2.$$

By definition of a minimal surface, this trace will vanish; using this with the fact that A is symmetric, we have the identity that

$$\det(A) = -\frac{1}{2} \sum_{i,j} |A(e_i, e_j)|^2 .$$

Substituting into the Gauss equation, we see that

$$K_S = K_M - \frac{1}{2} \sum_{i,j} |A(e_i, e_j)|^2 .$$

Since the determinant part is non-negative, we obtain the pointwise curvature bound for K_S in terms of the sectional curvature of the ambient manifold:

$$K_S \leq K_M.$$

In the context of constant sectional curvature metrics, this provides moderate topological control on the classes of minimal closed surfaces immersed in M. Recall that the Gauss-Bonnet Theorem gives the following average curvature expression in terms of purely topological information:

$$\int_S K_S \, dA = 2\pi\chi(S),$$

where $\chi(S)$ is the Euler characteristic of S. Using the well-known relation $\chi(S) = 2 - 2g$ between the genus g of the surface and its Euler characteristics, we see in certain constant curvature manifolds, certain immersed minimal surfaces are not possible. For example, if M is any quotient of $\mathbb{E}^3$ with the quotient metric of constant 0 sectional curvature, then $K_S \leq 0$ and

$$0 \leq \int_S K_S \, dA = 2\pi\chi(S) = 2\pi(2 - 2g).$$

Thus, it must follow that $g \geq 1$ there are no immersed minimal spheres in flat 3-manifolds. Note, however there are minimal (actually geodesic) embedded tori in T^3, so the bound $g \geq 1$ is certainly sharp.

In hyperbolic manifolds, there a further constraint on the types of minimal immersed surfaces. Using the above bound, we see that when $M = \mathbb{H}^3/\Gamma$ is a hyperbolic manifold, then the curvature bound gives a pointwise bound on the intrinsic curvature of an immersed minimal surface S: $K_S \leq -1$. Using Gauss-Bonnet and the curvature bound, we obtain an area bound on the entire surfaces:

$$\text{Area}(S) = \int_S 1\, dA \leq \int_S K_S\, dA = 2\pi\chi(S) = 2\pi(2-2g).$$

Since the area of an immersed surface is always strictly positive, an immersed minimal surface must have $g \geq 2$. Thus, there are no immersed minimal spheres or tori in hyperbolic manifolds.

9.3.2 Totally Geodesic Surfaces

The special case where the second fundamental form A for a hypersurface S is identically zero is called a *totally geodesic surface*; such an S is trivially a minimal surface as well. Such surfaces are called totally geodesic because the vanishing of the second fundamental form is equivalent to the condition that all geodesics on S remain geodesics in the ambient manifold M. In general, totally geodesic surfaces are difficult to produce and form a very small subclass of minimal surfaces.

We restrict our attention to the more concrete case where the ambient manifold M is a hyperbolic link complement; thus, this non-compact manifold is $S^3 - L$ admitting a complete finite-volume metric of constant negative curvature. It is currently unknown whether *closed* totally geodesic surfaces exist in link complements; necessarily, such surfaces must have negative Euler characteristics (since they are minimal surfaces). Adams and Schoenfeld [2] were the first to produce totally geodesic Seifert surfaces for several subclasses of hyperbolic links; they constructed both free (i.e., their complements were handlebodies) and non-free Seifert surfaces. Such surfaces are not necessarily minimal genus Seifert surfaces, but must be incompressible in the link complement.

Their work relies heavily on previous theorems by Thurston that classify the three types of Seifert surfaces one can have for a hyperbolic link. The first type is that of an *accidental surface*. For such a surface, there is a non-trivial simple closed curve that is not boundary parallel but does correspond to a parabolic isometry; topologically, this implies that the curve is isotopic into a neighborhood of the missing knot. The second type occurs when K is a fibred knot and the surface S lifts to a fibre in some finite cover; this corresponds to the lift of S in $\mathbb{H}^3$ having the entire sphere at infinity S^2_∞ as its limit set. The last class of Seifert surfaces, called *Quasi-Fuchsian Seifert surfaces*, are those where the limit set on the sphere at infinity are quasi-circles. In the special case that the limit set is a union of geometric circles, S is a totally geodesic Seifert surface and lifts to a union of geodesic planes in $\mathbb{H}^3$.

The techniques in Adams and Schoenfeld's papers rely on finding totally geodesic surfaces in hyperbolic orbifolds; then they lift these surfaces to Seifert surfaces for special classes of links. One such class are *balanced pretzel knots*, which are pretzel knots $K(n, n, \ldots, n)$ with equivalent integral tangles. They further use Hatcher-Thurston classification of incompressible Seifert surfaces to show that any hyperbolic 2-bridge knot (i.e., non-torus 2-bridge knots) never admit totally geodesic Seifert surfaces).

9.3.3 The Monotonicity Principle

The Monotonicity Principle for minimal surfaces gives monotonic control for the growth of the intersection of metric balls with the surface. Consider the map

$$r \mapsto \frac{\text{Area}(S \cap B_r(p))}{\text{Area}(G_r(p) \cap B_r(p))},$$

where $G_r(p)$ is a geodesic surface passing through p and $B_r(p)$ is a metric ball in M with origin p. The Monotonicity Principle states that this map is non-increasing. Note that there is no restriction on the topology of the intersection of S with the ball $B_r(p)$. Certainly, this is true (using Gauss-Bonnet) when the intersection is a disk in S, but the Monotonicity principle places no such requirement. In fact, using Monotonicity, we see that the growth of the area of a ball of radius r on a minimal surface is bounded below by the growth of the area of a ball in a geodesic surface. Such a statement should not seem surprising as geodesic surfaces are a strict subclass of minimal surfaces.

When we apply this principle to the context of constant sectional curvature Riemannian manifolds, we obtain explicit quantitative lower bounds for disks in a minimal surface. In discrete, free quotients of $\mathbb{E}^3$ with the induced metric, disks of radius r in geodesic surfaces have area $2\pi r$. Thus, the Monotonicity Principle states that any corresponding disk (or other subsurface of a minimal surface will have area at least $2\pi r$.

In the context of minimal surfaces in hyperbolic manifolds, we obtain similar geometric control of the area. Since a geodesic plane in hyperbolic space has area $2\pi(\cosh r - 1) = 4\pi \sinh^2\left(\frac{r}{2}\right)$, this gives a lower bound on the intersection of a metric ball $B_r(p)$ of radius r with a minimal surface S, regardless of the topology inside $B_r(p)$.

In quantitative assessments of the area of minimal surfaces in constant curvature Riemannian manifolds (where the area of geodesic disks are known explicitly), this form of the Monotonicity Principle frequently provides an appropriate context from which one may produce lower bounds. Coupled with an upper bound on the area coming from Gauss-Bonnet, these area constraints greatly limit the topology of these minimal surfaces.

9.3.4 The Barrier Principle

As a complete analog to the maximum principle in harmonic analysis, minimal surfaces enjoy a *barrier principle* heuristically stating that minimal surfaces cannot have local maxima or minima. Stated for two minimal surfaces S_1 and S_2 in a Riemannian manifold, this says that if $x \in S_1 \cap S_2$ and there exists some neighborhood U of x such that S_1 lies on one side of S_2 in U, then $S_1 = S_2$. Thus, one minimal surface acts as a barrier for another, and their intersections must be sufficiently complicated.

In fact, we may generalize the above barrier principle further by introducing the notion of convexity. A surface is said to be *mean convex* if it second fundamental form is definite. In the same way that one minimal surface serves as a barrier for another, a mean-convex surface is a barrier for any minimal surface. Thus, the above concepts also works when one of the surfaces satisfies the weaker hypothesis of mean-convexity.

The importance of this lies in the ability to generalize statements about minimal surfaces in closed manifolds to those with boundary. If M is a Riemannian manifold with ∂M mean-convex, its mean-curvature vector field will all be pointing into the manifold. Since these boundaries serve as barriers for minimal surfaces in the interior, many crucial existence theorems for closed manifolds generalize to those with mean-convex boundary.

9.3.5 Convexity Property

One of the major constraining properties of minimal surfaces is that they must lie in convex sets. Recall that a subset X of a metric space is *convex* if every geodesic between every two points in X lies completely in X. The *convex hull* of a subset X is the smallest convex set containing X. In the special case of subsets in $\mathbb{R}^n$, the convex hull is the intersection of all halfspaces containing X.

Minimal surfaces must lie in the convex hull of their boundaries. Thus, in trying to solve Plateau's problem, one only needs to search for minimal surfaces in the convex hull of the boundary condition. Remarkably, this holds for general Riemannian manifolds as well.

9.4 Minimal Surfaces in $\mathbb{R}^3$

The study if minimal surfaces in $\mathbb{R}^3$ is not only historically important, it also serves as a concrete arena in which many properties of these surfaces that generalize to arbitrary Riemannian 3-manifolds may manifest themselves more visually.

9.4.1 The Minimal Surface Equation in $\mathbb{R}^n$

If we are given a non-parametric surfaces S in $\mathbb{R}^n$ given by coordinates $x_1 = u_1, x_2 = u_2$, and $x_k = f(u_1, u_2)$ for $3 \leq k \leq n$, then partially differentiating we obtain the vectors

$$\frac{\partial x}{\partial u_1} = \left(1, 0, \frac{\partial f_3}{\partial u_1}, \ldots, \frac{\partial f_n}{\partial u_1}\right), \quad \frac{\partial x}{\partial u_1} = \left(0, 1, \frac{\partial f_3}{\partial u_2}, \ldots, \frac{\partial f_n}{\partial u_2}\right).$$

Thus, we may compute the metric g_{ij} by taking the Euclidean dot products of the vectors:

$$g_{11} = 1 + \sum_{k=3}^{n} \left(\frac{\partial f_k}{\partial u_1}\right)^2; \quad g_{12} = g_{21} = \sum_{k=3}^{n} \frac{\partial f_k}{\partial u_1}\frac{\partial f_k}{\partial u_2}; \quad g_{22} = 1 + \sum_{k=3}^{n} \left(\frac{\partial f_k}{\partial u_2}\right)^2.$$

If our surface S is a C^2 surface (i.e., the f_k functions have continuous second derivatives), then we may further compute the second fundamental form with respect to any normal vector $N = (N_1, N_2, \ldots, N_n)$. Using the explicit form for our vectors above, we differentiate to see that

$$\frac{\partial^2 x}{\partial u_i \partial u_j} = \left(0, 0, \frac{\partial^2 f_3}{\partial u_i \partial u_j}, \ldots, \frac{\partial^2 f_n}{\partial u_i \partial u_j}\right).$$

Using the definition of $A = a_{ij}$, we obtain the equation

$$a_{ij} = \sum_{k=3}^{n} N_k \frac{\partial^2 f_k}{\partial u_i \partial u_j}.$$

After suitable cancellations and suitable choice for N and the introduction the vector notation $f(u_1, u_2) = (f_3(u_1, u_2), \ldots, f(u_1, u_2))$, the minimal surface equation becomes the following vector equation

$$\left(1 + \left|\frac{\partial f}{\partial u_2}\right|^2\right) \frac{\partial^2 f}{\partial u_1^2} - 2\left(\frac{\partial f}{\partial u_1} \cdot \frac{\partial f}{\partial u_2}\right) \frac{\partial^2 f}{\partial u_1 \partial u_2} + \left(1 + \left|\frac{\partial f}{\partial u_1}\right|^2\right) \frac{\partial^2 f}{\partial u_2^2} = 0.$$

This formulation of the minimal surface equation indicates that many of the techniques to finding solutions of elliptic second-order partial differential equations may be employed successful to find such minimal surfaces.

9.4.2 Examples of Minimal Surfaces in $\mathbb{R}^3$

In the case of surfaces in $\mathbb{R}^3$, the vector $f(u_1, u_2)$ in the above formulation becomes the scalar $f_3(u_1, u_2)$ and the minimal surface equation is a single partial differential equation. Thus, it is more straightforward to check that a given f does indeed supply a minimal surfaces.

The first example is the most trivial one. Namely, if we are given any plane P in $\mathbb{R}^3$, it is defined via a linear function and thus all second degree derivatives vanish identically. So, any plane is a minimal surface (actually, a geodesic surface). In fact, Bernstein [6] proves that the plane is the only minimal surface defined as $f : D \to \mathbb{R}$ where the domain D the all of $\mathbb{R}^2$.

The *helicoid* another classic example of a minimal surface. It is give non-parametrically by the function

$$f_3(u_1, u_2) = \arctan\left(\frac{u_2}{u_1}\right),$$

which is defined on $\mathbb{R}^2$ minus the x-axis. The helicoid is also given implicitly by the equation

$$u_2 = u_1 \tan u_3.$$

This, along with the plane, are the only *ruled surfaces*, those minimal surfaces which can be generated by straight lines or rulings [10].

The other classic minimal surface example is the *catenoid*, defined non-parametrically by

$$f_3(u_1, u_2) = \cosh^{-1}\sqrt{u_1^2 + u_2^2};$$

it is defined on $\mathbb{R}^2 - \{(0,0\}$. Implicitly, this surface is the solution to

$$u_1^2 + u_2^2 = (\cosh u_3)^2.$$

The minimal surface known as *Scherk's surface* is of particular interest because of its periodic nature. It is defined non-parametrically by

$$f_3(u_1, u_2) = \log\left(\frac{\cos u_2}{\cos u_1}\right),$$

and is thus defined on a checkerboard of squares in the u_1u_2-plane where $\frac{\cos u_2}{\cos u_1} > 0$. We note that by properties of the logarithm, the defining function f_3 can be written as

$$f_3(u_1, u_2) = \log(\cos u_2) - \log(\cos u_1)$$

and thus is a *minimal surface of translation*; that is, its function can be written in the form $f_3(u_1, u_2) = g(u_1) + h(u_2)$. This surface is *doubly periodic* since both functions g and h are periodic of period 2π. The implicit equation for Scherk's surface is given by

$$e^{u_3} \cos u_1 - \cos u_2 = 0.$$

9.4.3 The Weierstrass Representation

The use of complex analysis in the search for minimal surfaces in $\mathbb{R}^3$ has proven to be particularly fruitful. One of the most noteworthy ramifications of such a consideration was the formulation of minimal surfaces in terms of the *Enneper-Weierstrass representation.* The representation data is given by two complex-valued functions f and g defined on the unit disk or complex plane with f analytic and g meromorphic. We further require that wherever g has a pole or order m, f has a zero of order $2m$ (so that fg^2 is holomorphic). Then, the surface defined parametrically by

$$x_1(\zeta) = \mathrm{Re}\left(\frac{1}{2}\int_0^\zeta f(z)(1 - g(z)^2)\,dz\right) + c_1,$$

$$x_2(\zeta) = \mathrm{Re}\left(\frac{i}{2}\int_0^\zeta f(z)(1 + g(z)^2)\,dz\right) + c_2$$

$$\text{and } x_3(\zeta) = \mathrm{Re}\left(\int_0^\zeta f(z)g(z)\,dz\right) + c_3$$

is a minimal surface. Surprisingly, the converse is true as well; that is, any regular minimal surface admits such a representation. The functions f and g are referred to as the Weierstrass data associated to the minimal surface.

This representation is particularly useful because it allows one to more easily generate minimal surfaces in $\mathbb{R}^3$. In particular, *Enneper's surface* may be easily represented in terms of its Weierstrass parametrization. Its Weierstrass data is given by $f(z) = 1$ and $g(z) = z$. Enneper's surface is a complete immersion and has the property that it contains two perpendicular straight lines.

9.4.4 Curvature

Given the Weierstrass representation of a minimal surface, the calculation of the Gaussian curvature is straightforward. For a minimal surface S with Weierstrass data f and g, its curvature at any point is given by

$$K = -\left(\frac{4|g'|}{|f|(1 + |g|^2)^2}\right)^2.$$

Note this quantity is always non-positive, reinforcing the fact that the curvature of a minimal surface is pointwise bounded above by the curvature of the manifold, which is zero in the case of $\mathbb{R}^3$. We note that S is a plane if and only if $K = 0$ if and only if $g' \equiv 0$. Furthermore, since g is analytic, g' is analytic and must be either be identically zero (in which case S is a plane) or g' must have isolated zeroes. Thus, the Gaussian curvature of any minimal surface must be identically zero or have only isolated zeroes.

9.5 Minimal Surfaces in 3-manifolds

While the study of minimal surfaces in $\mathbb{R}^3$ has led to the formulation of a very important and thriving field of differential geometry, much of its importance rests in the fact that many results can be generalized to Riemannian 3-manifolds as well as higher dimensional manifolds.

9.5.1 Existence Results for 3-manifolds

The general philosophy of existence results in 3-manifolds is that there exists *least area* surfaces in the homotopy class of each incompressible surface. For compressible surfaces, homotopic minimal surfaces do tend to exist, but they are no longer guaranteed to be least area.

The main existence results for incompressible surfaces come from two vastly different techniques. The first set offers analytic and geometric measure-theoretic approach to minimal surfaces. Schoen and Yau [51] prove the existence of a minimal immersion in the homotopy class of each incompressible surface.

Theorem 9.3. *Let M be a compact Riemannian manifold and S a surface of genus $g \geq 1$. Let $f : S \to M$ be a continuous map inducing an injection $f_* : \pi_1(S) \hookrightarrow \pi_1(M)$. Then there is a minimal immersion $h : S \to M$ so that $h_* = f_*$ as maps on π_1. If $\pi_2(M) = 0$, then h may be chosen to be homotopic to f.*

Using more topological techniques, Freedman, Hass, and Scott [31] find a similar result .

Theorem 9.4. *Let M be a closed, orientable Riemannian 3-manifold and let S be a closed surface of genus $g \geq 1$. Let $f : S \to M$ be a least area immersion such that f_* is injective and such that f is homotopic to a two-sided embedding h. Then either*

(1) f is an embedding, or

(2) f double covers a one-sided surface K embedded in M and $h(S)$ bounds a submanifold of M which is a twisted I-bundle over a surface isotopic to K.

When one considers the genus $g = 0$ case and searches for minimal surface representatives of non-trivial maps of spheres into 3-manifolds, the topic of the famous *Sphere Theorem* begins to emerge. The topological Sphere Theorem states that if M is a 3-manifold with $\pi_2(M) \neq 0$, then there exists a non-trivial element of $\pi_2(M)$ that has an embeddings $S^2 \hookrightarrow M$ as a representative. Meeks and Yau [32] prove this topological theorem with the added result that the embedding is actually a minimal surface.

Theorem 9.5. *Let M be a closed 3-manifold with non-trivial $\pi_2(M)$. Amongst the set of all smooth maps from S^2 to M representing non-trivial elements of $\pi_2(M)$, there is a map f of least area. Furthermore, f is either a smooth embeddings or a double cover a smoothly embedded projective plane.*

To obtain stronger embededness results, researchers have utilized geometric measure theory. In particular, the results of Meeks, Simon, and Yau [56] state that there is a least area surface in the isotopy class of any incompressible surface.

Theorem 9.6. *Let M be a closed, orientable, irreducible 3-manifold. Then every incompressible surface S is isotopic to a globally least area minimal surface.*

The same paper provides a similar statement for spheres.

Theorem 9.7. *Let M be a closed, orientable, reducible 3-manifold. Then there is a globally least area essential embedded sphere.*

The above theorems are stated for closed 3-manifolds. However, wince mean convex surface serve as barriers for minimal surfaces, many of these results generalize to the case with boundary when we require that the boundary itself be mean-convex. In particular, if we choose that ∂M be a totally geodesic boundary, then the above existence results go through. Intuitively, since these mean-convex boundaries have inward-pointing mean-curvature vector fields, a surface may be pushed into the manifold to reduce its area.

9.6 Applications of Minimal Surfaces to 3-dimensional geometric topology

As we will see below, minimal surfaces seem to have much potential in obtaining topological results about surfaces in 3-manifolds.

The above theorems deal with incompressible surfaces; when one turns attention to highly compressible surfaces like Heegaard surfaces for a 3-manifold, such existence theorems for minimal surfaces do exist, but these surfaces are not least area in general.

In particular, Rubinstein [46] has recently used minimal surfaces in a variety of topological and geometric settings. One of his most striking applications is in attempting to quantify strongly irreducible Heegaard splittings for closed 3-manifolds. Rubinstein, building off earlier work by Pitts and Rubinstein [41], proves that in an arbitrary Riemannian metric, one may find a minimal surface representative in the isotopy class of a strongly irreducible Heegaard surface S. Such a minimal surface, though, may be of index 1 and thus not a minima for the area functional. Using this existence theorem, he is able to prove the following topological result.

Theorem 9.8. *Let M be a complete, finite-volume hyperbolic 3-manifold. Then there are finitely many irreducible Heegaard splittings of bounded genus up to isotopy.*

There has also been substantial interest in using geometric information to produce lower bounds on the genus of a manifold (defined as the minimum genus of all Heegaard surfaces). In the same paper, Rubinstein provides the following bound.

Theorem 9.9. *Let M be a closed or complete finite-volume hyperbolic 3-manifold. Let ρ' be the injectivity radius of the thick part of M. Then the Heegaard genus of M satisfies*

$$g \geq \frac{\cosh \rho' + 1}{2}.$$

Using ideas from this paper along with an isoperimetric inequality for minimal surfaces in hyperbolic metric balls, Bachman, Cooper, and White [14] prove a similar theorem using the radius of an embedded metric ball.

As a different application, Brittenham and Rieck [7] use minimal surfaces to study the structure of Heegaard surfaces in hyperbolic bundles over S^1. If M is a surface bundle over S^1 with pseudo-Anosov monodromy φ, it admits a hyperbolic structure by Thurston's Geometrization of Haken manifolds. Let M_d be the cyclic cover of M with monodromy φ^d. A Heegaard surface for a bundle is called *standard* if it may be constructed by tubing together disjoint fibers of the fibration. Brittenham and Rieck's result is that for high enough covers, every Heegaard splitting of low genus is standard.

Theorem 9.10. *Let M be a bundle over S^1 with pseudo-Anosov monodromy φ. Then for any integer $h \geq 0$, there exists a constant n so that for any $d \geq n$, any Heegaard surface of genus $\leq h$ is standard.*

Chapter 10

The Bounded Diameter Theorem

10.1 Statement and Proof

Kakimizu's realization of the path-metric on $MS(L)$ and $IS(L)$ in terms of relative length in the universal abelian cover of a link complement hints at the potential of geometric methods to obtain diameter bounds on these complexes. The various properties of minimal surfaces (especially the intrinsic curvature bounds) lend themselves well for use in geometries of bounded curvature. Given Thurston's Geometrization Theorem for Haken manifolds, the appropriate class of links for which minimal surfaces could prove fruitful is hyperbolic links.

We obtain an upper bound on the diameter of the minimal genus complex $MS(K)$ for hyperbolic knots exactly using these arguments. This upper bound is quadratic in the genus g of the knot K.

Theorem 10.1. *Let K be a hyperbolic knot of genus g. Then, the diameter of $MS(K)$ is bounded by a constant $C(g)$ depending only on the genus of the knot. Furthermore, this bound is quadratic in the genus g.*

Proof. To satisfy the hypotheses for the existence of a minimal surface representative for a minimal genus Seifert surface, we must first deform the geometry of the knot complement at the cusp to obtain a Riemannian metric with inward-pointing mean curvature vector at the boundary torus. This is obtained simply by smoothly transitioning to a Euclidean metric (which is conformally equivalent to the original hyperbolic metric) as we approach the cusp. Thus, by deleting an open neighborhood of the knot, we obtain a manifold with totally geodesic toral boundary; since this torus is geodesic, its mean curvature vector field is equivalently zero and thus mean-convex.

Now, by Meeks-Simon-Yau [56], for any minimal genus Seifert surface S, there exists an area minimizing (and thus minimal) Seifert surface Σ in its isotopy class. We may also ensure that the boundary $\partial\Sigma = \Sigma \cap \partial M$ has geodesic curvature k_g identically zero. For if we consider the Riemannian manifold with totally geodesic boundary described above and double it, the doubled surface remains incompressible (since Σ was incompressible) and the boundary curve is invariant under the reflection about this totally geodesic torus. Thus, it will have geodesic curvature $k_g \equiv 0$.

Recall that by minimality, Σ has an intrinsic curvature bound $K_\Sigma \leq K_M = -1$. Thus, $1 \leq -K_\Sigma$; integrating this inequality, using the Gauss-Bonnet Theorem, and recalling that $k_g \equiv 0$, we obtain the area bound in terms of the genus g:

$$\mathrm{Area}(\Sigma) = \int_\Sigma 1\, dA \leq \int_\Sigma -K_\Sigma\, dA = -\int_\Sigma K_\Sigma\, dA =$$

$$-2\pi\chi(\Sigma) + \int_{\partial\Sigma} k_g\, ds = -2\pi(1-2g) = 4\pi g - 2\pi.$$

Next, we obtain a lower bound on the diameter of the minimal surface in terms of the 3-dimensional Margulis constant ρ. For the moment, we assume that the knot complement has no Margulis tubes in its thick-thin decomposition. Since the knot complement is equal to the thick part of the manifold, we have a lower bound on the injectivity radius $i(x)$ at each x:

$$\rho \leq i(x).$$

Thus, at every point $x \in \Sigma$, $\Sigma \cap B(i(x), x))$ has a global lower bound on its area in terms of the area of a totally geodesic disk G of radius ρ containing x by the monotonicity principle for minimal surfaces:

$$\mathrm{Area}(G \cap B(\rho, x)) \leq \mathrm{Area}(G \cap B(i(x), x)) \leq \mathrm{Area}(\Sigma \cap B(i(x), x)).$$

Totally geodesic disks of radius r in hyperbolic spaces are known to have area $2\pi(\cosh r - 1)$. Applying this to our above bound, we have

$$2\pi(\cosh\rho - 1) \leq \mathrm{Area}(\Sigma \cap B(i(x), x)).$$

From this area bound, we will obtain a bound on the diameter of Σ via a covering argument. Let $x_1, \ldots, x_m$ be a minimal collection of points on Σ such that $\Sigma \cap B(\rho, x_k)$ cover Σ. By compactness of the surface, such a finite m exists. Since these the x_k's form a minimal collection, the regions $\Sigma \cap B(\rho/2, x_k)$ are disjoint. Thus, we obtain the following lower bound on the area of Σ:

$$\sum_{k=1}^{m} \mathrm{Area}(\Sigma \cap B(\rho/2, x_k)) \leq \mathrm{Area}(\Sigma).$$

At each x_k, we can consider a geodesic surface of radius $\rho/2$ passing through x_k. Using monotonicity on each of these m regions gives

$$2\pi m(\cosh\rho/2 - 1) \leq \sum_{k=1}^{m} \mathrm{Area}(\Sigma \cap B(\rho/2, x_k)) \leq \mathrm{Area}(\Sigma).$$

Combining this with the upper bound on area

$$\mathrm{Area}(\Sigma) \leq 4\pi g - 2\pi,$$

we obtain

$$2\pi m(\cosh \rho/2 - 1) \leq \mathrm{Area}(\Sigma) \leq 4\pi g - 2\pi.$$

Thus, we obtain a bound linear in g on the number of regions needed to cover Σ:

$$m \leq \frac{4\pi g - 2\pi}{2\pi(\cosh \rho/2 - 1)} = \frac{2g-1}{\cosh \rho/2 - 1}.$$

We will now use the bound on m to obtain a diameter bound. Since Σ is a *least area surface*, it is a *stable* minimal surface a subject to a lower bound on its curvature as well [50]:

$$-2 \leq K_\Sigma \leq -1.$$

Thus, in any metric ball of radius ρ, the surface has a uniform upper bound $d(\rho) \geq \mathrm{Diam}\,(\Sigma \cap B(\rho/2, x_k))$, where $d(\rho)$ depends only on the Margulis constant ρ and is thus independent of the topology of the intersection. Since these patches of surface cover Σ,, we obtain the diameter bound by using our bound on m:

$$\mathrm{Diam}(\Sigma) \leq md(\rho) \leq \frac{d(\rho)(2g-1)}{\cosh \rho/2 - 1},$$

which is linear in g.

Isometrically lifting this minimal Seifert surface to a lift $\widetilde{\Sigma}$ in the universal abelian cover $\widetilde{E}$ will give the same diameter bound. Recall that distance in the Kakimizu complex $MS(L)$ is formulated in terms of relative distance in the universal abelian cover. So far, we only have a numeric bound on $\mathrm{Diam}(\widetilde{\Sigma})$. We need to reformulate this diameter in terms of the width of the lift of the complement of our other Seifert surface S'. This pseudo-distance is given by how many translates of this lift the surface Σ will meet. We will obtain an overestimate instead by counting the number of times Σ intersects the boundary of the lift.

To this end, let us consider a minimal surface representative Σ' in the isotopy class of a Seifert surface S'. We consider the knot complement $E = S^3 - \overset{\circ}{N}(K)$ and build the universal abelian cover by translating the closure of an isometric lift $E_0' = E - \Sigma'$ by the natural action of the Deck group $\mathbb{Z}$ generated by τ. This domain E_0 has two isometric copies of a lift of Σ' in its boundary; call one of these $\widetilde{\Sigma}'$. The generator τ takes the minimal surface $\widetilde{\Sigma}'$ isometrically to its translates, which are also minimal surfaces.

Now, we obtain a bound on the number of translates of $\widetilde{\Sigma}'$ that intersect $\widetilde{\Sigma}$ in terms of the diameter $\mathrm{Diam}(\widetilde{\Sigma})$. For any two points x and y in $\widetilde{\Sigma}$, consider a geodesic γ joining x and y. Furthermore, let

C be a neighborhood of γ of radius ρ, the Margulis constant. Our goal is now to divide this cylinder C into t subcylinders C_i of height ρ; since the height of the subcylinders is uniform, we obtain the bound $t \leq \frac{d(x,y)}{\rho} + 1$. Since each subcylinder C_i has radius ρ and height ρ, we may consider C to lie either in the universal abelian cover or the knot complement since it would lift isometrically by Margulis' theorem. Thus, in the knot complement, we can measure how many times n the minimal surface Σ' intersects the geodesic $\gamma \subset \Sigma$. By the Convexity property of minimal surfaces, since each component of $C \cap \Sigma'$ meets the geodesic γ, it will have area at least $2\pi(\cosh\rho - 1)$. Since the Gauss-Bonnet theorem also applies to Σ', we obtain the area bound

$$\text{Area}(\Sigma') \leq 2\pi(2g-1).$$

Thus, the intersection of the minimal surface Σ' with the cylinder C will at most have n components, where n satisfies

$$n \cdot 2\pi(\cosh\rho - 1) \leq \text{Area}(\Sigma') \leq 2\pi(2g-1).$$

Manipulating, this gives us a bound on n:

$$n \leq \frac{2\pi(2g-1)}{2\pi(\cosh\rho - 1)} = \frac{2g-1}{\cosh\rho - 1}.$$

Since the length of the geodesic γ is bounded above by the diameter of Σ, we will can obtain a bound on the number t of isometric copies of C that are needed to cover this path:

$$t \leq \frac{\text{Diam}(\Sigma)}{\rho} + 1 \leq \frac{d(\rho)(2g-1)}{\rho(\cosh\rho/2 - 1)} + 1 =: e(g,p).$$

Note that $e(g,\rho)$ is linear in g.

We can now combine our bound on t with the number of maximal times n that Σ' can intersect γ to produce an upper bound on the number of translates of Σ' that the lift of γ will meet in the universal abelian cover. By the Kakimizu formulation of distance in the Seifert surface complex, this thus gives a bound on the diameter of the complex in terms of the genus g of the knot:

$$\text{Diam}(MS(K)) \leq n \cdot t \leq \frac{2g-1}{\cosh\rho - 1} \cdot e(g,\rho)$$

which is quadratic in g since both terms are linear in g.

We now consider the case where the knot complement does have a thin part. Thus, we must analyze how the two minimal surfaces Σ and Σ' intersect inside the finitely many Margulis tubes. Recall that these Margulis tubes are topologically solid tori containing closed geodesics as cores. Since both Σ and Σ' are incompressible surfaces, any intersection with the Margulis tubes must be as meridional disks or as annuli; of course, any intersection of higher genus would contradict the incompressibility of the Seifert surfaces.

First, we consider the case when one of the surfaces, say Σ meets a Margulis tube X in a disk. Then, if Σ' intersects this disk, the intersection must be essential in both surfaces; if not, we may perform a disk swap and reduce area (see, for example, [31] or [11]). In this case, we can follow this intersection out to the boundary ∂X, where it will correspond to an intersection in the thick part of the manifold. Thus, if either of the surfaces meet X in a disk, we obtain a bound by the thick-part argument above.

Thus, we are left with the case when the intersection of both Σ and Σ' with a Margulis tube X is an annulus. Note that the number of times that Σ may meet the union of all Margulis tubes in an annulus is bounded above by the topology of Σ. These annular intersections are disjoint and essential on the surface; furthermore, they are non-parallel since then any product region between two parallel annuli can be used to decrease area. Thus, by Euler characteristic considerations, there may be at most $3g-2$ Margulis tubes intersecting any given minimal Seifert surface. Furthermore, any pair of local sheets may intersect only once or we can do a local exchange to decrease this local intersection number. Thus, the total number of times they intersect annularly in Margulis tubes is bounded above by the product, $(3g-2)^2$, which is, once again, quadratic in the genus. This intersection bound in the knot complement gives a corresponding bound in the universal abelian cover.

If the restriction of the surface to the thick part is separated by annuli intersecting the Margulis tubes, the diameter bound obtained for the thick part is true for each component of the surface in the thick part. By Euler characteristic considerations, there are only $2g-1$ such components. Thus, combining the corresponding bounds in the the thick and thin parts of the knot complement, we obtain a bound that is at most quadratic and written in terms of only the genus.

10.2 Generalizations of the Bounded Diameter Theorem

10.2.1 One-cusped Hyperbolic 3-manifolds

The proof of Kakimizu's realization of the distance on $MS(L)$ and $IS(L)$ follows when the ambient manifold M is any one-cusped manifold. Thus, we may define a similar complex $MS(M,\alpha)$ and $IS(M,\alpha)$ for a manifold M with one toral boundary component by defining its vertices to be isotopy classes of minimal genus or incompressible surfaces with no closed components in a fixed homology class $\alpha \in H_2(M, \partial M)$. The simplicial structure, of course, may be defined via the same disjointness property. Since Kakimizu's proof goes through for this broader complex, the above theorem remains valid in a broader sense.

Theorem 10.2. *Let M be a one-cusped hyperbolic 3-manifold and $\alpha \in H_2(M, \partial M)$. There is a* *constant $C(g)$ such that*

$$Diam(MS(M,\alpha)) \leq C(g),$$

where $C(g)$ depends only on g and is quadratic in g.

10.2.2 Non-boundary Links

One of the crucial elements in the Bounded Diameter Theorem is that the Seifert surfaces in question are connected, thus this theorem cannot be easily extended to links if we cannot guarantee that all Seifert surfaces must be connected. To this end, we consider non-boundary links.

A link L with $n \geq 2$ components is called a *boundary link* if its components bound disjoint Seifert surfaces in S^3. Thus, if a link is not a boundary link, all of its Seifert surfaces must be connected. One sufficient condition for a link L to not be a boundary link is that it its components have non-zero linking number.

Proposition 10.3 (Rolfsen [44])**.** *If any two components of $L \subset S^3$ have non-zero linking number, then L is* not *a boundary link.*

Thus, non-zero linking of any two components ensures that the Seifert surfaces for L are connected. Some simple examples of non-boundary links are the Whitehead link and the Borromean rings.

We obtain the following generalization of the Bounded Diameter Theorem.

Theorem 10.4. *Let K be any non-boundary link of genus g with n components. There exists a constant $C(g)$ such that*

$$Diam(MS(K)) \leq C(g,n),$$

where C depends only on g and n. Furthermore, C is quadratic in both g and n.

10.3 Other Diameter Bounds

Very recently, Sakuma and Shackleton [48] have posted a similar quadratic bound on the diameter of $MS(K)$ for atoroidal knots using vastly different combinatorial techniques.

Theorem 10.5. *Suppose that a genus g knot K is atoroidal in S^3. Then $MS(K)$ has diameter at most $2g(3g-2)+1$.*

Note that atoroidal knots are exactly torus and hyperbolic knots. Since torus knots are fibred, $MS(K)$ is a single vertex an the result trivially follows. Thus, Sakuma and Shackleton's result is precisely for hyperbolic knots even though no geometry is used in their arguments.

The two theorems are similar only in that they both utilize Kakimizu's formulation of the distance in terms of the universal abelian cover.

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