Homotopy and homology of p-subgroup complexes

Thesis by

Kaustuv Mukul Das

In Partial Fulfillment of the Requirements

for the Degree of Doctor of Philosophy

California Institute of Technology
Pasadena, California

1994
(Submitted April 12, 1994)

Acknowledgments

My heartfelt thanks go to my advisor, Professor Michael Aschbacher, for suggesting this thesis problem, for providing me with the tools and techniques required to tackle the problem, and for unstintingly sharing his knowledge and ideas with me. This thesis, as well as my entire graduate experience, has benefited greatly through Professor Aschbacher's guidance and generosity; and I thank him for that.

This work could not have been possible without the additional mathematical inspiration and encouragement that I received. My special thanks to Professors Ross Lawther, Dinakar Ramakrishnan, Yoav Segev, and David Wales for teaching me many algebraic and group theoretic techniques; to Professor Stephen Smith whose suggestions have greatly improved the exposition in this thesis; and to Professors John Feroe, John McCleary, and Charles Steinhorn for showing me the beauty of mathematics and teaching me the basics.

My years at Caltech were greatly enriched by all the wonderful people I met here. As they know who they are, I will omit a list. I would like to thank my mother for showing me what was possible and Elizabeth Kaftan for believing that I could do it. Finally, I would like to thank my kindest critic, best friend, and most valued confidant – my brother. Any achievement of mine is equally an achievement of his and so I dedicate this thesis to Samantak Das.

Abstract

In this thesis we analyzed the simple connectivity of the Quillen complex at p for the classical groups of Lie type. In light of the Solomon-Tits theorem, we focused on the case where p is not the characteristic prime. Given $(p,q)=1$ let $d_p(q)$ be the order of q in $\mathbb{Z}/p\mathbb{Z}$. In this thesis we proved the following result:

Main Theorem. *When $(p,q)=1$ we have the following results about the simple connectivity of the Quillen complex at p, $\mathcal{A}_p(G)$, for the classical groups of Lie type:*

1. *If $G=GL_n(q), d_p(q)\geq 2$ and $m_p(G)>2$, then $\mathcal{A}_p(G)$ is simply connected.*

2. *If $G=Sp_{2n}(q)$, then:*

(a) *$\mathcal{A}_p(G)$ is Cohen-Macaulay of dimension $n-1$ if $d_p(q)=1$.*

(b) *If $m_p(G)>2$ and $d_p(q)$ is odd, then $\mathcal{A}_p(G)$ is simply connected.*

3. *If $G=GU_n(q)$, then:*

(a) *$\mathcal{A}_p(G)$ is Cohen-Macaulay of dimension $n-1$ if $p\neq 2$ and $d_p(q)=1$.*

(b) *If $m_p(G)>2$ and $d_p(q)$ is odd, then $\mathcal{A}_p(G)$ is simply connected.*

(c) *If $n>3$, $q\geq 5$ is odd, and $d_p(q)=2$, then $\mathcal{A}_p(G)(>Z)$ is simply connected, where Z is the central subgroup of G of order p.* □

In the course of analyzing the p-subgroup complexes we developed new tools for studying relations between various simplicial complexes and generated results about the join of complexes and the p-subgroup complexes of products of groups. For example we proved:

Theorem A. *Let $f: X\to Y$ be a map of posets satisfying:*

(1) *f is strict; that is, $x\precneqq y \Rightarrow f(x)\precneqq f(y)$,*

(2) *$f^{-1}(Y(\leq y))$ is min $\{n, h(y)-1\}$-connected for all $y\in Y$, and*

(3) *$Y(>y)$ is $(n-h(y)-1)$-connected for all $y\in Y$ with $h(y)\leq n$.*

Then Y n-connected implies X is n-connected. □

Theorem A provides us with a tool for studying $\mathcal{A}_p(G)$ in terms of $\mathcal{A}_p(G/O_{p'}(G))$. For example, we used this method to prove:

Theorem 8.6. *Let $G = O_{p'}(q) \rtimes S$ where $O_{p'}(G)$ is solvable and S is a p-group of symplectic type. Then $\mathcal{A}_p(G)$ is $(m_p(G)-1)$-spherical.* □

In this thesis we also generated a library of results about geometric complexes which do not arise as p-subgroup complexes. This library includes, but is not restricted to, the following:

(1) the poset of proper nondegenerate subspaces of a $2n$-dimensional symplectic space – ordered by inclusion – is Cohen-Macaulay of dimension $n-2$.

(2) If q is an odd prime power and $n \geq 4$ (with $n \geq 5$ if $q = 3$), then the poset of proper nondegenerate subspaces of an n-dimensional unitary space over $\mathbb{F}_{q^2}$ is simply connected.

Lasciate ogni speranza, o voi ch'entrate

Contents

Introduction

The study of finite groups has always relied on analyzing group actions on certain "natural" objects. One class of objects studied extensively in the past few years is the class of simplicial complexes. Much of the activity in this area is inspired by papers of K. Brown [Br] and D. Quillen [Q], the latter being an excellent introduction to this area of research.

In this thesis we focus most of our attention on p-subgroup complexes – simplicial complexes naturally identified with p-subgroups of a finite group. In particular we study the homotopical and homological invariants of the p-subgroup complexes of finite groups, especially the classical groups of Lie type.

The applications of this area of research are far reaching. For example, G. Robinson and R. Staszewski [RS] have given a reformulation of Alperin's Conjecture in terms of the stabilizers of the orbit representatives of the group action on certain p-subgroup complexes. There are also applications to combinatorial topology and K-theory. A more complete list of the applications of the study of p-subgroup complexes is given in the paper by P.J. Webb [W].

The analysis of p-subgroup complexes has played, and continues to play, an important role in the study of finite simple groups, although early work in the area was not stated in the language of complexes. Namely, it is easy to see that the p-subgroup complexes for a finite group G of order divisible by p are disconnected if and only if G has a strongly p-embedded subgroup. In the classification of the finite simple groups, knowledge of the groups with strongly p-embedded subgroups plays an important role. A complete classification of groups with strongly 2-embedded subgroups was given by H. Bender in [Be], and is a crucial step in the classification. To date the corresponding theorem for odd primes has not been proven without an appeal to the classification of finite simple groups. This makes the proof of certain parts of the classification theorem unusually difficult.

However, along with the classification theorem, results of D. Gorenstein and R. Lyons ((24.1) on pg. 307 in [GL]), give a complete characterization of finite groups with strongly

p-embedded subgroups, and hence a complete characterization of finite groups for which the p-subgroups complexes are disconnected.

Connectivity of a complex is the crucial piece of information in 0-dimension. The 1-dimensional analogue is simple connectivity. The knowledge of the simple connectivity of p-subgroup complexes is also important in the study of finite simple groups. M. Aschbacher and Y. Segev used the simple connectivity of p-subgroup complexes to give the first computer free proof of the uniqueness of the Lyons group, Ly, and the Janko group, J_4, in [AS2] and [AS5] respectively. The methods developed in these and other papers give us a better understanding of some presentations of finite simple groups, as well as giving us a more unified method for proving the uniqueness of the sporadic groups.

A theory for reducing the question of simple connectivity of the p-subgroup complexes to a question of the simple connectivity of the corresponding p-subgroup complexes for the finite simple groups was developed by Aschbacher in [A2]. As part of this program we begin a systematic study, in this thesis, of the simple connectivity of the p-subgroup complexes for the most difficult class of finite simple groups: the classical groups of Lie type.

It is hoped that this program to study simple connectivity of the p-subgroup complexes will be the first step towards a complete classification of the topological invariants of the p-subgroup complexes of as large a class of finite groups as possible. With that in mind we also computed the higher homology groups and other topological properties of the classical groups of Lie type, whenever possible.

The research in this area has typically focused on one of two complementary problems. The first is the development of tools to study group actions on simplicial complexes. This includes developing tools to study p-subgroup complexes. The other area of research on which we concentrated is the analysis of the topological properties of the simplicial complexes upon which the groups act. Moreover, in analyzing the p-subgroup complexes of the classical groups, we develop new tools to study relations between various simplicial complexes, as well as proving results about geometric complexes which do not arise as p-subgroup complexes. We believe that this collection of tools, results, and relations between various simplicial complexes has applications to an area of research wider than the area upon which this thesis focuses; and thus is important in its own right.

RESULTS ABOUT p-SUBGROUP COMPLEXES

Given a poset P one can consider the simplicial complex whose vertex set consists of P, and whose simplices consist of all finite chains in P. This well known simplicial complex is called the *order complex of* P, and is denoted $\mathcal{O}(P)$. Associating a topological object with the poset P allows us to attribute topological properties to P [section 1]. This association allows us, for example, to talk about a poset being connected or n-connected, when in fact we mean that the corresponding order complex is connected or n-connected as a topological space.

Given a simplicial complex K, recall that the *dimension of* K is given by:

$$dim(K) = max\{dim(\sigma)|\sigma \text{ is a simplex of } K\},$$

where $dim(\sigma) = k$ if σ contains $k+1$ vertices. Also recall that an n-dimensional simplicial complex K is *n-spherical* if it is $(n-1)$-connected; that is, $\tilde{H}_i(K) = 0 \ \forall\, 0 \leq i \leq n-1$, and K is simply connected when $n \geq 2$. Note that an n-spherical complex has the homotopy of bouquet of n-spheres and in general the top dimension of the complex will be more than one dimensional. Furthermore, following Quillen, an n-spherical simplicial complex K is said to be *Cohen-Macaulay*, if, for every k-simplex $\sigma \in K$, $lk_K(\sigma)$ is $(n-k-1)$-spherical [see section 14 for details]. Finally, we say that a simplicial complex K is *contractible* if it is contractible as a topological space; that is, K is n-connected for all n.

An important class of simplicial complexes which arises as order complexes of posets is the *p-subgroup complexes.* An example of such a complex is the following:

> Given a finite group G and a prime p dividing the order of G, consider the poset of nontrivial elementary abelian p-subgroups of G – ordered by inclusion. The order complex of this poset is called the *Quillen complex of* G *at* p, and is denoted $\mathcal{A}_p(G)$.

In the literature, other p-subgroup complexes are considered, such as the Brown complex studied in [Br] and the Commuting complex in [A2]. However, well-known equivalences in the literature show that these complexes are homotopy equivalent to the Quillen complex; see for example (5.2–3) on pp. 14–15 in [A2]. Thus, for the purposes of studying topological invariants, it suffices to consider $\mathcal{A}_p(G)$.

We are interested in answering the following question:

Question: *Let G be a finite group and p a prime dividing the order of G. When is $\mathcal{A}_p(G)$ simply connected?*

Following the usual strategy the above question is tackled in two steps:

> **Reduction :** Reduce the question of simple connectivity of $\mathcal{A}_p(G)$ for arbitrary groups G to the simple connectivity of $\mathcal{A}_p(G)$ for some minimal class of groups, particularly simple groups. This reduction was begun by Aschbacher in [A2].
>
> **Analysis of the minimal case :** Given G in this minimal class of groups, determine when $\mathcal{A}_p(G)$ is simply connected.

The main focus of the thesis is the simple connectivity of $\mathcal{A}_p(G)$ for the classical groups of Lie type. The following facts – well known to researchers in this area – helped motivate and direct our research:

Fact 1. Given a finite group G and a prime p dividing the order of G, $\mathcal{A}_p(G)$ is contractible whenever $O_p(G) \neq 1$ (Proposition 2.4 on pg. 106 in [Q]).

Fact 2. If G is a finite simple group of Lie type in characteristic p, then $\mathcal{A}_p(G)$ has the homotopy type of the Tits building $\mathfrak{B}$ of G – Theorem 3.1 on pg. 108 in [Q]. Furthermore, by the Solomon-Tits Theorem [see section 1], $\mathfrak{B}$ is Cohen-Macaulay. Therefore, the homology and fundamental groups of $\mathcal{A}_p(G)$ are well understood when p is the characteristic prime.

Fact 3. By the above comment it suffices to consider the case when p is not the characteristic prime. We also know that when $p|q-1$ and $G = GL_n(q)$, $\mathcal{A}_p(G)$ is Cohen-Macaulay of dimension $n-1$ (Theorem 12.4 on pg. 126 in [Q]).

Given these facts and the overall program to classify all instances in which $\mathcal{A}_p(G)$ is simply connected, and more generally to describe the homology of $\mathcal{A}_p(G)$, we begin our systematic investigation by considering the Quillen complex for the classical groups.

Our main results are given below. Note that for a 1-dimensional simplicial complex, simple connectivity and contractibility are equivalent. Therefore, we addressed this question only for those groups G and primes p for which $\mathcal{A}_p(G)$ is at least two dimensional. Before stating the result we need to define two terms:

(1) If p is a prime and $G = GL_n(q), Sp_{2n}(q)$, or $GU_n(q)$ we let $d_p(q)$ be the order of q in $\mathbb{Z}/p\mathbb{Z}$; that is, the minimal integer with respect to $p|(q^{d_p(q)} - 1)$.

(2) We use the notation *C.M.* to indicate that a simplicial complex is Cohen-Macaulay.

Main Theorem. *When $(p, q) = 1$ we have the following results about the simple connectivity of $\mathcal{A}_p(G)$ for the classical groups of Lie type:*

1. *If $G = GL_n(q)$ and $m_p(G) > 2$, then $\mathcal{A}_p(G)$ is simply connected.*

2. *If $G = Sp_{2n}(q)$, then:*

(a) *$\mathcal{A}_p(G)$ is C.M. of dimension $n-1$ if $d_p(q) = 1$.*

(b) *If $m_p(G) > 2$ and $d_p(q)$ is odd, then $\mathcal{A}_p(G)$ is simply connected.*

3. *If $G = GU_n(q)$, then:*

(a) *$\mathcal{A}_p(G)$ is C.M. of dimension $n-1$ if $p \neq 2$ and $d_p(q) = 1$.*

(b) *If $m_p(G) > 2$ and $d_p(q)$ is odd, then $\mathcal{A}_p(G)$ is simply connected.*

(c) *If $n > 3$, $q \geq 5$ is odd, and $d_p(q) = 2$, then $\mathcal{A}_p(G)(> Z)$ is simply connected, where Z is the subgroup of order p in the center of G.* □

We note that $U_4(2) \cong PSp_4(3)$ implies that $\mathcal{A}_3(GU_4(2))(> Z)$ has the homotopy type of the building of $Sp_4(3)$. Hence $\mathcal{A}_3(GU_4(2))(> Z)$ is not simply connected. Therefore we need some restrictions on the values of q in **3**(c). The further restriction in **3**(c) that $n > 3$ is to ensure that $\mathcal{A}_p(G)(> Z)$ is at least two dimensional, and hence is also a necessary condition for simple connectivity.

It should be noted at this point that the analysis of the simple connectivity of the Quillen complex of the classical groups of Lie type is far from complete. When $G = Sp_{2n}(q)$ and $d_p(q)$ is even we have yet to show that $\mathcal{A}_p(G)$ is simply connected. However, we have reduced this case to showing that $\mathcal{A}_p(Sp_{2n}(q))$ is simply connected when $2n = 3d_p(q)$. This is similar to our proof that $\mathcal{A}_p(GL_n(q))$ is simply connected, where the case when $n = 3d_p(q)$ was addressed by special combinatorial methods. We expect that similar methods will prevail in the case of $\mathcal{A}_p(Sp_{3d_p(q)}(q))$. Furthermore, the simple connectivity of $\mathcal{A}_p(G)$ when $G = GU_n(q)$ and $d_p(q)$ is even has not been addressed yet, nor has the case when G is an orthogonal group. We hope to consider these and related questions in the near future.

A THEOREM ABOUT MAP OF POSETS AND EXAMPLES OF GEOMETRIC COMPLEXES

One of the important tools for studying relations between order complexes developed in the thesis is the following result about maps of posets:

Theorem A. *Let $f : X \to Y$ be a map of posets satisfying:*

(1) *f is strict; that is, $x \lneq y \Rightarrow f(x) \lneq f(y)$,*

(2) *$f^{-1}(Y(\leq y))$ is min $\{n, h(y) - 1\}$-connected for all $y \in Y$, and*

(3) *$Y(> y)$ is $(n - h(y) - 1)$-connected for all $y \in Y$ with $h(y) \leq n$.*

Then Y n-connected implies X is n-connected. □

The proof of this theorem uses the theory of *homology of posets with local coefficient systems*, and *spectral sequences* [section 2–3]. Note that this theorem is analogous to Theorem 9.1 on pg. 119 in [Q].

One application of Theorem A to the study of p-subgroup complexes is the following:

> The canonical homomorphism $\pi : G \to G/O_{p'}(G)$ induces a natural map of posets $f : \mathcal{A}_p(G) \to \mathcal{A}_p(G/O_{p'}(G))$. Note that given $x < y \in \mathcal{A}_p(G)$, we have $f(x) < f(y)$ as $x \cap O_{p'}(G) = 1 \; \forall \; x \in \mathcal{A}_p(G)$. Thus, if we can show that f and $\mathcal{A}_p(G/O_{p'}(G))$ satisfy the remaining criterion of Theorem A, then the analysis of $\mathcal{A}_p(G)$ reduces to analyzing the often simpler structure of $\mathcal{A}_p(G/O_{p'}(G))$.

This method was valuable to us not only in computing the higher homology groups of the p-subgroup complexes, but also in establishing simple connectivity. Quillen used this exact strategy, along with Theorem 9.1 on pg. 119 in [Q], to study $\mathcal{A}_p(G)$ in the following minimal case. Let $G = O_{p'}(G) \rtimes A$ where $O_{p'}(G)$ is solvable and A is an elementary abelian p-subgroup of *p-rank* $n+1$ [see section 1 for the definition of p-rank]. Quillen shows that under these assumptions $\mathcal{A}_p(G)$ is Cohen-Macaulay of dimension n (Theorem 11.2 on pg. 123 in [Q]). We use Theorem A and the above strategy to study another interesting minimal case. Let $G = O_{p'}(G) \rtimes S$ where $O_{p'}(G)$ is solvable and S is a p-group of symplectic type. Then we have:

Theorem 8.6. *$\mathcal{A}_p(G)$ is $(m_p(G)-1)$-spherical, where $m_p(G)$ is the p-rank of G.* □

We often prove results about $\mathcal{A}_p(G)$ by analyzing the action of G on simplicial complexes K which have appropriate homotopical and homological properties. The methods used to compare $\mathcal{A}_p(G)$ to K vary widely; they can be as simple as the argument given in Theorem 9.13 and Remark 9.14, or they can involve more complicated arguments such as the method of *n-approximations* developed in [AS4]. However, in every case we need to have the appropriate simplicial complex upon which G acts. Thus, a large part of this thesis was dedicated to finding these complexes and proving that they satisfied the correct n-connectedness properties. These proofs involved a variety of tools: the *Nerve Theorem* and *triangulability of graphs* to name but a couple. Some of the important results about these complexes generated during our investigations are the following:

(1) The poset of proper nondegenerate subspaces of a $2n$-dimensional symplectic space – ordered by inclusion – is Cohen-Macaulay of dimension $n-2$.

(2) Given an n-dimensional vector space V and its dual V^*, for each $U \leq V$ let $U^\perp$ be the *annihilator of* U. Then the set $\{0 \neq U \times U' \mid U \leq V, U' \leq U^\perp\}$ partially ordered by inclusion is $(n-1)$-spherical.

(3) If q is an odd prime power and $n \geq 4$ (with $n \geq 5$ if $q=3$), then the poset of proper nondegenerate subspaces of an n-dimensional unitary space over $\mathbb{F}_{q^2}$ is simply connected.

We also prove a number of results about the join of complexes and the p-subgroup complexes of the product of groups; these results are contained in section 9. In the course of our research we observe the following characterization of order complexes, which can greatly reduce the computation involved to show that the order complex of a poset is simply connected:

Theorem B. *A simplicial complex K is an order complex if and only if the following two conditions hold:*

(1) *K is a clique complex, and*

(2) *every odd cycle in $\Delta(K)$ has at least one triangular chord.* $\square$

ORGANIZATION OF THE MATERIAL

We now give a brief summary of the organization of the material in this thesis. Section 1 contains preliminary definitions and well-known results from algebraic topology. It also includes the Solomon-Tits Theorem. Sections 2–4 contain definitions and results about homology of posets with local coefficient systems and spectral sequences, which culminate in Theorem A in section 5.

Sections 6 and 7 contain results about the poset of nontrivial totally singular subspaces of hyperbolic orthogonal spaces and about $\mathcal{A}_p(G)$ for groups of symplectic type. In section 8 we use Theorem A and the strategy described earlier to prove Theorem 8.6.

Section 9 contains a number of results about the join of simplicial complexes and the p-subgroup complexes of the product of groups. Also, a lot of the notation used in succeeding sections is introduced in section 9. Section 10–13 contain examples of geometric complexes with appropriate n-connectedness properties. Section 10 also contains a proof of Theorem B.

Section 14 contains the definition and basic properties of Cohen-Macaulay complexes. These properties are used to study three different Cohen-Macaulay complexes in the following three sections: $\mathcal{A}_p(Sp_{2n}(q))$ when $p|q-1$; $\mathcal{A}_p(GU_n(q))$ when $p|q-1$ and $p \neq 2$; and the poset of proper nondegenerate subspaces of a symplectic space.

Sections 18 and 19 are analogous in that they contain an analysis of $C_G(A)$ when $A \in \mathcal{A}_p(G)$ and $G = Sp_{2n}(q)$ and $GU_n(q)$, respectively. These sections also contain information about the decomposition, under the action of A, into homogeneous components of the corresponding symplectic and unitary spaces. Section 20 uses results from the previous two sections to compute the p-rank of G when $G = GL_n(q)$, $Sp_{2n}(q)$, or $GU_n(q)$. Section 20 also contains a proof of the fact that $\mathcal{A}_p(G)$ is connected when $m_p(G) \geq 2$ for these groups. The results from section 20 are used extensively in chapter 6. Section 21 contains an analysis of the action of A on the Tits building of V – here $A \in \mathcal{A}_p(G)$ with $G = Sp_{2n}(q)$ or $GU_n(q)$; V is the corresponding space; and $d_p(q)$ is odd.

Sections 23–25 contain results about the simple connectivity of the p-subgroup complexes given in the Main Theorem.

Chapter 1

A theorem on the n-connectedness of order complexes

In Chapter 1 we will give an analogue of Quillen's result on the n-connectedness of spherical complexes – Theorem 9.1 on pg. 119 in [Q]. We say an analogue since we weaken our assumptions about the complexes to obtain a result which is weaker than Quillen's result. We adopt Quillen's techniques of using spectral sequences to study the homology of simplicial complexes.

In Section 1 we recall some basic concepts and state some well-known results from algebraic topology. For example, we see that to any poset one can associate a simplicial complex called the order complex of the poset; thereby allowing one to study such topological properties as homology and homotopy of posets. We also give a brief description of Tits buildings for groups with (B, N)-pairs and include a statement of the Solomon-Tits Theorem for completeness. Section 2 contains the definition of spectral sequences and some results about spectral sequences. In Section 3 some definitions and facts, required to use spectral sequences to study the homology of posets, are mentioned. We recall that every poset can be considered as a category. We also recall that given a functor from a poset – thought of as a category – to the category of abelian groups, one can study the homology of the poset with local coefficient system given by the functor. We define short exact sequences of functors, and use natural transformations between functors to define maps between homology complexes with local coefficient systems for different functors from a poset to the category of abelian groups. Finally, we state a result about the existence of a convergent spectral sequence for the homology of a poset, as given in Appendix II of [GZ]. In Section 4 we give some more results on the homology of posets. Section 5 contains a theorem about maps of posets inducing trivial homology under certain assumptions. As a corollary to which we obtain the main result of this chapter, Theorem A.

SECTION 1 : PRELIMINARY DEFINITIONS AND RESULTS FROM ALGEBRAIC TOPOLOGY

By a *simplicial complex* K we mean a set of *vertices* $\{v\}$ and a set $\{s\}$ of finite nonempty subsets of $\{v\}$ called *simplices* such that:

(1) any subset of $\{v\}$ consisting of exactly one vertex is a simplex, and

(2) any nonempty subset of a simplex is a simplex.

The *dimension of* K is given by $dim(K) = sup\{|s|-1 : s \text{ a simplex}\}$. A *subcomplex* L of a simplicial complex K is a subset of K that has the structure of a simplicial complex. A subcomplex L is a *full subcomplex* of K if every simplex of K which has all of its vertices in L is a simplex of L. A simplicial complex is a *pure simplicial complex* if every maximal simplex, that is, a simplex which is not contained properly in any other simplex, has the same dimension. Note that if K is a pure simplicial complex then $dim(K) = |s| - 1$ for any maximal simplex $s \in K$. As in section 1 of chapter 3 in [Sp] we identify a simplicial complex K with its topological-space realization $|K|$. This identification allows us to attribute topological properties to the simplicial complex K. We can therefore speak of a simplicial complex being connected or simply connected and so on.

Given a *poset* (partially ordered set) P, we can identify the following simplicial complex with P :

> the set of vertices is equal to the elements of P and the simplices are all finite chains contained in P.

The above complex is called the *order complex of* P and is denoted by $\mathcal{O}(P)$; or, when no chance of confusion exists, simply by P.

Let G be a finite group and p a prime dividing the order of G. We consider the following two complexes:

(1) The *Brown complex of* G *at* p, denoted $\mathcal{S}_p(G)$, is the order complex of the poset of all nontrivial p-subgroups of G – ordered by inclusion.

(2) The *Quillen complex of* G *at* p, denoted $\mathcal{A}_p(G)$, is the full subcomplex of $\mathcal{S}_p(G)$ on the vertex set consisting of all elementary abelian p-subgroups of G.

When p is fixed we refer to the above complexes as the Brown complex of G and the Quillen complex of G. It is a well-known fact that $\mathcal{S}_p(G)$ and $\mathcal{A}_p(G)$ are homotopy equivalent; for example, see Proposition 2.1 on pg. 105 in [Q].

Given G and p as above, note that each $A \in \mathcal{A}_p(G)$ can be considered as a vector space over the field $\mathbb{F} = \mathbb{Z}/p\mathbb{Z}$. This allows us to define the *p-rank of* A, denoted $m_p(A)$, as $m_p(A) = dim_{\mathbb{F}}(A)$. We can then define the *p-rank of* G as follows:

$$m_p(G) = max\{m_p(A) \mid A \in \mathcal{A}_p(G)\}.$$

Note that by the definition of $\mathcal{A}_p(G)$ and the dimension of a simplicial complex, $\mathcal{A}_p(G)$ is a complex of dimension $m_p(G) - 1$.

If X is a poset and $x \in X$, we denote by $X(\leq x)$ the full subcomplex of the order complex of X on the vertex set $\{y \in X | y \leq x\}$. The dimension of $X(\leq x)$ is called the *height of* x, and is denoted by $h(x)$. Note that when $X = \mathcal{A}_p(G)$, for some group G, then $h(A) = m_p(A) - 1 \ \forall \ A \in \mathcal{A}_p(G)$. The subcomplexes $X(\geq x)$, $X(> x)$, and $X(< x)$ are analogously defined.

Since we will be using techniques from algebraic topology to study these simplicial complexes, we next state some basic definitions and results from algebraic topology. The concepts of homology and reduced homolgy with integer coefficients (denoted $H_*(X)$ and $\tilde{H}_*(X)$ for some topological space X) are as given in chapter 4 of [Sp].

We define reduced homology on the empty set by setting:

$$\tilde{H}_i(\emptyset) = \begin{cases} \mathbb{Z}, & \text{if } i = -1, \\ 0, & \text{else.} \end{cases}$$

Furthermore, we extend homology to the negative indices for nonempty topological spaces X by setting $H_i(\mathrm{X}) = 0$ for all $i < 0$.

As per the discussion on pp. 181–184, 220–227 in [R1], we define $\mathrm{Tor}_n^R(-, B)$ as the left derived functors of the functor $T = \_ \otimes_R B$. More precisely, if

$$\cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} A \to 0$$

is a projective resolution of an R-module A, then

$$\mathrm{Tor}_n^R(A, B) = ker(d_n \otimes 1)/im(d_{n+1} \otimes 1).$$

Note that by the definition of Tor we have:

Lemma 1.1. *If P is projective, then $\operatorname{Tor}_n^R(P, B) = 0$ for all B and all $n \geq 1$; similarly in the other variable. Specifically we have:*

$$\operatorname{Tor}_n^{\mathbb{Z}}(\mathbb{Z}, A) = 0 \text{ and } \operatorname{Tor}_n^{\mathbb{Z}}(0, A) = 0$$

for all abelian groups A.

Proof. This is Theorem 8.4 in [R1]. □

Also by the definition of Tor we have:

Theorem 1.2 (Universal Coefficient Theorem). *If $\mathfrak{A}$ is a chain complex of torsion-free abelian groups and $H_n(\mathfrak{A})$ is homology with integral coefficients, then for every abelian group C and each n, there is a split exact sequence:*

$$0 \to H_n(\mathfrak{A}) \otimes C \to H_n(\mathfrak{A} \otimes C) \to \operatorname{Tor}_1^{\mathbb{Z}}(H_{n-1}(\mathfrak{A}), C) \to 0.$$

Thus $H_n(\mathfrak{A} \otimes C) \cong (H_n(\mathfrak{A}) \otimes C) \oplus \operatorname{Tor}_1^{\mathbb{Z}}(H_{n-1}(\mathfrak{A}), C)$.

Proof. This follows from the Künneth formula, for example, see pg. 169 in [R2]. □

Next we give a result about short exact sequences:

Lemma 1.3. *Consider abelian groups B, C and B_2 and maps i, j and k, with k an isomorphism, such that:*

$$\begin{tikzcd}
 & & 0 & & \\
 & & B_2 \arrow[u] & & \\
0 \arrow[r] & A \arrow[r, "\alpha"] & B \arrow[r, "j"] \arrow[u, "i"] & C \arrow[r] \arrow[ul, "k"'] & 0 \\
 & & B_1 \arrow[u, "\beta"] & & \\
 & & 0 \arrow[u] & &
\end{tikzcd}$$

commutes, and the row and column are short exact sequences. Then $A \cong B_1$.

Proof. As $k \circ j = i$ and k is an isomorphism, $ker(j) = ker(i)$. But this means that $\alpha(A) = ker(j) = ker(i) = \beta(B_1)$, which implies:

$$\beta^{-1} \circ \alpha : A \to B_1 \text{ is an isomorphism.}$$

Thus $A \cong B_1$ as claimed. $\square$

In conclusion we state two important results from the research on p-subgroup complexes and Tits buildings.

Let G be a finite group with BN-pair of rank l. And let $G_1, G_2, \dots, G_l$ be the maximal parabolic subgroups of G containing B. For each $1 \leq i \leq l$ let V_i be the set of cosets of the form gG_i, $g \in G$. Finally, let $V = V_1 \cup \dots \cup V_l$. The *Tits building of* G, denoted $\Delta(G; B, N)$ or simply as Δ, is the simplicial complex on the vertex set V with $V \supseteq \sigma$ a simplex if and only if $\bigcap_{v \in \sigma} v \neq \emptyset$.

Note that G acts on Δ and this induces a G-action on $H_*(\Delta)$ – the homology groups of Δ with integral coefficients. Recall that a d-dimensional simplicial complex K is *d-spherical* (or just *spherical*) if it is $(d-1)$-connected. If K is d-spherical and the link of each s-simplex is $(d-s-1)$-spherical then K is said to be *Cohen-Macaulay* (or *C.M.*) of dimension d[1]. Note that a d-spherical complex has the homotopy type of a bouquet of d-spheres and in general the top dimension of the complex will be more than one dimensional. We can now state the following result:

Theorem 1.4 (The Solomon-Tits Theorem). *Let Δ be the Tits building of a finite group G with BN-pair of rank $l \geq 2$. Then Δ is Cohen-Macaulay of dimension $l-1$. Furthermore, the action of G on $H_{l-1}(\Delta)$ affords the Steinberg representation of G.*

Proof. The proof of the fact that Δ is $(l-1)$-spherical and that $H_{l-1}(\Delta)$ affords the Steinberg representation of G is given in [So] and also in [CL]. The fact that Δ is C.M. is given in terms of *shellability* in section 4 in pp. 188–194 in [B1]. Note that the fact that

[1] For a more detailed description of Cohen-Macaulay complexes and their properties refer to section 14.

$H_{l-1}(\Delta)$ affords the Steinberg representation implicitly implies that Δ has the homotopy type of a bouquet of $(l-1)$-spheres as opposed to being a single $(l-1)$-sphere. $\square$

Finally, we include as a corollary to the Solomon-Tits Theorem a result of Quillen on $\mathcal{A}_p(G)$ where G is a classical group of Lie type and p is the characteristic prime. Recall that a classical group of Lie type posseses a BN-pair of rank, say l. Let $\Delta = \Delta(G; B, N)$ be the corresponding Tits building. Then we have:

Corollary 1.5. *$\mathcal{A}_p(G)$ is homotopy equivalent to the building Δ. Consequently, $\mathcal{A}_p(G)$ has the homotopy type of a bouquet of $(l-1)$-spheres.*

Proof. This is Theorem 3.1 on pg. 108 in [Q]. $\square$

SECTION 2 : INTRODUCTION TO SPECTRAL SEQUENCES

The discussion in this section closely follows the discussion in chapter 11 of [R1].

A *filtration* $\{F^pH_*\}$ of a graded module H_* is a sequence of subobjects for each n satisfying:

$$\cdots \subsetneq F^{p-1}H_n \subsetneq F^pH_n \subsetneq F^{p+1}H_n \subsetneq \cdots .$$

A filtration $\{F^pH_*\}$ is *bounded* if for each n there exist integers $s = s(n)$ and $t = t(n)$ such that:

$$0 = F^sH_n \subsetneq F^{s+1}H_n \subsetneq \cdots \subsetneq F^tH_n = H_n.$$

An E^k*-spectral sequence* is a sequence $\{E^r, d^r \ : \ r \geq k\}$ of bigraded modules E^r_{pq} and bidegree maps $d^r : E^r_{pq} \to E^r_{p-r,q+r-1}$ satisfying:

(1) $d^rd^r = 0$, and

(2) $E^{r+1}_{pq} = ker(d^r : E^r_{p,q} \to E^r_{p-r,q+r-1})/im(d^r : E^r_{p+r,q-r+1} \to E^r_{p,q})$.

By the definition of spectral sequences we have $H(E^r_{pq}, d^r) = E^{r+1}_{pq}$. An E^k-spectral sequence such that $E^k_{pq} = 0$ for all $p < 0$ and $q < 0$ is called a *first quadrant spectral sequence.*

Given an E^k-spectral sequence, let $Z^k_{pq} = ker(d^k : E^k_{p,q} \to E^k_{p-k,q+k-1})$ and $B^k_{pq} = im(d^k : E^k_{p+k,q-k+1} \to E^k_{p,q})$, then $E^{k+1}_{pq} = Z^k_{pq}/B^k_{pq}$. If $Z(E^{k+1}_{pq}) = ker(d^{k+1} : E^{k+1}_{p,q} \to$

$E^{k+1}_{p-k-1,q+k}$) and $B(E^{k+1}_{pq}) = im(d^{k+1} : E^{k+1}_{p+k+1,q-k} \to E^{k+1}_{p,q})$, then by the third isomorphism theorem for modules, there exist submodules:

$$B^k_{pq} \subset B^{k+1}_{pq} \subset Z^{k+1}_{pq} \subset Z^k_{pq} \quad \text{such that} \quad E^{k+2}_{pq} \cong Z^{k+1}_{pq}/B^{k+1}_{pq}.$$

By induction we get:

$$B^k_{pq} \subset B^{k+1}_{pq} \subset \cdots \subset B^{k+n}_{pq} \subset Z^{k+n}_{pq} \subset \cdots \subset Z^{k+1}_{pq} \subset Z^k_{pq} \text{ with } E^{k+n+1}_{pq} \cong Z^{k+n}_{pq}/B^{k+n}_{pq}.$$

We let $B^\infty_{pq} = \bigcup_n B^{k+n}_{pq}$ and $Z^\infty_{pq} = \bigcap_n Z^{k+1}_{pq}$ and define $E^\infty_{pq} = Z^\infty_{pq}/B^\infty_{pq}$.

Lemma 2.1. *If $E^r_{pq} = 0$ for some r, then $E^\infty_{pq} = 0$.*

Proof. By the definition of E^∞_{pq} we see that E^∞_{pq} is a section of E^r_{pq}. Since, by our assumption, $E^r_{pq} = 0$ we have $E^\infty_{pq} = 0$ as claimed. □

A spectral sequence $\{E^r, d^r\}$ is said to *converge* to a graded module H_* if there is a bounded filtration $\{\Phi^p H_*\}$ of H_* such that:

$$E^\infty_{pq} \cong \Phi^p H_{p+q}/\Phi^{p-1} H_{p+q}.$$

In that case we write $E^k_{pq} \Rightarrow H_{p+q}$.

Lemma 2.2. *Let $E^k_{pq} \Rightarrow H_{p+q}$ be a convergent spectral sequence. Assume there exists an integer m such that for all p, q with $p + q = m$ and $s(m) < p \leq t(m)$ there exists n with $E^n_{pq} = 0$. Then $H_m = 0$. Here $s(m)$ and $t(m)$ are as in the definition of bounded filtrations for H_{p+q}.*

Proof. As $E^k_{pq} \Rightarrow H_{p+q}$ we know that there exists a bounded filtration:

$$0 = \Phi^s H_m \subsetneq \Phi^{s+1} H_m \subsetneq \cdots \subsetneq \Phi^t H_m = H_m,$$

with $s = s(m)$ and $t = t(m)$, and $\Phi^{s+1} H_m = \Phi^{s+1} H_m/\Phi^s H_m \cong E^\infty_{(s+1)(m-s-1)}$. But by our assumption there exists an n with $E^n_{(s+1)(m-s-1)} = 0$; so by Lemma 2.1, $E^\infty_{(s+1)(m-s-1)} = 0$. Thus $\Phi^{s+1} H_m = 0$, which implies $t = s + 1$, and hence $H_m = \Phi^t H_m = 0$. □

Section 3 : Homology of a poset with local coefficient systems

Given categories $\mathfrak{B}$ and $\mathfrak{D}$ and covariant functors $E, F : \mathfrak{B} \to \mathfrak{D}$, a *natural transformation* $\phi : E \to F$ is a collection of morphisms $\phi_B : E(B) \to F(B)$ for each $B \in \mathfrak{B}$, giving commutativity of the following diagram:

$$\begin{array}{ccc} E(B) & \xrightarrow{E(f)} & E(B') \\ {\scriptstyle \phi_B}\downarrow & & \downarrow{\scriptstyle \phi_{B'}} \\ F(B) & \xrightarrow{F(f)} & F(B') \end{array} \qquad \text{for all } f : B \to B'.$$

A natural transformation ϕ is called *epic* (respectively *monic*) if for every $B \in \mathfrak{B}$, ϕ_B is epic (repectively monic).

Now given a poset X, we can consider it as a category by letting $\mathfrak{Ob}(X)$, the *objects of* X, equal $\{x \in \mathrm{X}\}$ and by defining:

$$Mor(x, y) = \begin{cases} \{id\}, & \text{if } x = y, \\ \{<\}, & \text{if } x < y, \\ \emptyset, & \text{otherwise}, \end{cases} \qquad \text{for all } x, y \in \mathfrak{Ob}(X).$$

So given a poset X, we can have functors $F : X \to \mathcal{A}b$, where $\mathcal{A}b$ is the category of abelian groups. Recall that we can define *homology on* X *with local coefficient system* F, written $H_*(X, F)$, as follows:

the p-chains are given by $C_p(X, F) = \bigoplus_{s \in \Sigma_p} F(s)$, where Σ_p is the set of p-simplices $s = (x_0 < \cdots < x_p)$ and $F(s) = F(x_0)$. The face maps $d_p^i : C_p \to C_{p-1}$ are defined by $d_p^i = \sum_{s \in \Sigma_p} d_{p,s}^i$ where:

$$d_{p,s}^i : F(s) \to F(s^i), \quad s^i = (x_0 < \cdots < x_{i-1} < x_{i+1} < \cdots < x_p),$$

and $d_{p,s}^i = id$ if $i \neq 0$ and $d_{p,s}^0 = F(<) : F(x_0) \to F(x_1)$ if $i = 0$. If $p = i = 0$ then $d_{0,s}^0 = 0$. If we let $d_p = \sum_{i=0}^p (-1)^i d_p^i$, then $(d_p)^2 = 0$, and so $(C_p(X, F), d_p)$ forms a chain complex and we define $H_p(X, F) = ker(d : C_p \to C_{p-1})/im(d : C_{p+1} \to C_p)$.

This definition of homology of a poset X with local coefficient system F is as given in section 3 of Appendix II in [GZ].

Given functors $F, G : X \to \mathcal{A}b$, we can define the functor $F \oplus G : X \to \mathcal{A}b$ by letting:

$$(F \oplus G)(x) = F(x) \oplus G(x) \qquad \text{for all } x \in X$$

$$(F \oplus G)(\phi) = F(\phi) \oplus G(\phi) \qquad \text{for all } \phi \in Mor(x, y).$$

By the definition of $C_p(X, F)$ we can see that:

$$C_p(X, F \oplus G) = C_p(X, F) \oplus C_p(X, G)$$

$$d(F \oplus G) = d(F) \oplus d(G).$$

So we have:

Lemma 3.1. *Given functors F and G, we have:* $H_p(X, F \oplus G) = H_p(X, F) \oplus H_p(X, G)$.□

If F is a constant functor, $F \equiv L$, we write $H_p(X, L)$ instead of $H_p(X, F)$.

Lemma 3.2. *Given a constant functor $F \equiv L$ from X to $\mathcal{A}b$, we have:*

$$H_p(X, L) = (H_p(X) \otimes L) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(H_{p-1}(X), L).$$

Proof. Since:

$$C_p(X, L) = \bigoplus_{s \in \Sigma_p} L = (\bigoplus_{s \in \Sigma_p} \mathbb{Z}) \otimes L = C_p(X, \mathbb{Z}) \otimes L,$$

we get the desired equality by Theorem 1.2. □

If a poset X is contractible, then $\tilde{H}_i(X) = 0$ for all i. So we get the following:

Lemma 3.3. *Given a constant functor $F \equiv L$ from a contractible space X into $\mathcal{A}b$, we have:*

$$H_p(X, L) = \begin{cases} L, & \text{if } p = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Since X is contractible, we have $H_p(X) = \mathbb{Z}$ or 0 for all p. By Lemma 3.2:

$$H_p(X, L) = (H_p(X) \otimes L) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(H_{p-1}(X), L).$$

But, by Lemma 1.1, we have $\mathrm{Tor}_1^{\mathbb{Z}}(H_{p-1}(X), L) = 0$; so $H_p(X, L) = H_p(X) \otimes L$. This gives us the desired equality. □

Given a poset X, functors F, $G \in Funct(X, \mathcal{A}b)$, and a natural transformation $\phi : F \to G$, we get a map:

$$\tilde{\phi} : C_p(X,F) \to C_p(X,G) \text{ given by } \tilde{\phi} = \sum_{s\in\Sigma_p} \tilde{\phi}_s \text{ where}$$
$$\tilde{\phi}_s = \phi_{x_0} : F(s) = F(\tilde{x_0}) \to G(x_0) = G(s).$$

By the definition of $\tilde{\phi}$ we get that:

$$\begin{aligned}
&\tilde{\phi} \circ d_p^0(\sum_{s\in\Sigma_p} F(s)) = \tilde{\phi}(\sum_{s\in\Sigma_p} F(<) : F(s) \to F(s^0)) \\
&= \sum_{s\in\Sigma_p} (\tilde{\phi}_{s^0} \circ F(<) : F(s) \to G(s^0)) = \sum_{s\in\Sigma_p} (G(<) \circ \tilde{\phi}_s : F(s) \to G(s^0)) \\
&= d_p^0(\sum_{s\in\Sigma_p} \tilde{\phi}_s : F(s) \to G(s)) = d_p^0 \circ \tilde{\phi}(\sum_{s\in\Sigma_p} F(s)).
\end{aligned}$$

Thus we have shown:

$$\tilde{\phi} \circ d_p^0 = d_p^0 \circ \tilde{\phi} \quad \text{for all } p.$$

Similarly:

$$\tilde{\phi} \circ d_p^j = \tilde{\phi} \circ id = id \circ \tilde{\phi} = d_p^j \circ \tilde{\phi} \quad \text{for all } 1 < j \leq p.$$

So we have a map $\tilde{\phi}$ which commutes with d and thus gives us a map:

$$\phi_* : H_*(X,F) \to H_*(X,G).$$

Notice that the definition of $\tilde{\phi}$ implies that if ϕ is epic (monic), then $\tilde{\phi}$ is epic (monic).

Given functors L, K, and $F \in Funct(\mathfrak{A}, \mathfrak{C})$, and a sequence of natural transformations:

$$0 \to L \xrightarrow{\phi} K \xrightarrow{\lambda} F \to 0,$$

we say it is a *short exact sequence of functors* if for every $A \in \mathfrak{A}$ the sequence:

$$0 \to L(A) \xrightarrow{\phi_A} K(A) \xrightarrow{\lambda_A} F(A) \to 0,$$

is exact.

From the discussion above we see that given a poset X, functors L, K, and $F \in Funct(X, \mathcal{A}b)$, and a short exact sequence of functors:

$$0 \to L \xrightarrow{\phi} K \xrightarrow{\lambda} F \to 0,$$

we get a short exact sequence of complexes given by:

$$(*) \qquad 0 \to C_p(X, L) \xrightarrow{\tilde{\phi}} C_p(X, K) \xrightarrow{\tilde{\lambda}} C_p(X, F) \to 0.$$

Thus we have:

Theorem 3.4. *Given a poset X, functors L, K, and $F \in Funct(X, \mathcal{A}b)$ and a short exact sequence of functors:*

$$0 \to L \xrightarrow{\phi} K \xrightarrow{\lambda} F \to 0,$$

we get a long exact sequence:

$$\cdots \to H_n(X, F) \xrightarrow{\partial} H_{n-1}(X, L) \xrightarrow{\phi_{n-1}} H_{n-1}(X, K) \xrightarrow{\lambda_{n-1}} H_{n-1}(X, F) \to \cdots$$

in homology.

Proof. This follows directly from $(*)$ and Theorem 6.3 in [R1]. □

Consider a poset X and a poset P consisting of a single element. A map $f : X \to P$ of posets induces a map $f_* : H_*(X) \to H_*(P)$ on homology with integral coefficients. And we have $\ker(f_0) = \tilde{H}_0(X)$, the reduced homology of X with integral coefficients. See the discussion on pp. 167–168 in [Sp] for further details. This leads to the following definition:

Definition. *Let X be a nonempty poset, P a poset consisiting of a single point, and $f : X \to P$ a map of posets. Given a fixed abelian group L, let $f_* : H_*(X, L) \to H_*(P, L)$ be the map induced on homology with local coefficient system L. Then the reduced homology of X with coefficients L, denoted $\tilde{H}_*(X, L)$, is given by:*

$$\tilde{H}_i(X, L) = \begin{cases} \ker(f_0), & \text{if } i = 0, \\ H_i(X, L), & \text{otherwise.} \end{cases}$$

In particular we have the short exact sequence:

$$0 \to \tilde{H}_0(X, L) \to H_0(X, L) \xrightarrow{f_*} H_0(P, L) \to 0.$$

Finally we come to a theorem on maps of posets and homology with local coefficient system:

Theorem 3.5. *Let $f : X \to Y$ be a map of posets and define $f|y = \{\, x \in X \mid f(x) \le y\} = f^{-1}(Y(\le y))$. Then for any functor $F : X \to \mathcal{A}b$, we have a convergent spectral sequence:*

$$E^2_{pq} = H_p(Y, y \mapsto H_q(f|y, F)) \Rightarrow H_{p+q}(X, F).$$

Proof. This is Theorem 3.6 in Appendix II of [GZ]. □

SECTION 4 : A RESULT ON THE RELATIVE HOMOLOGY OF POSETS

Throughout this section, let X, a poset, and $L \in \mathcal{A}b$ be fixed.

Given a subset $S \subseteq X$, we can define a functor:

$$\begin{aligned} L_S : X &\to \mathcal{A}b \\ x &\mapsto \begin{cases} L, & x \in S, \\ 0, & x \notin S. \end{cases} \end{aligned}$$

When $S = \{a\}$ we write L_a instead of $L_{\{a\}}$.

Lemma 4.1. *If $U \subseteq X$ such that $X(\ge u) \subseteq U$ for all $u \in U$ then:*

$$H_*(X, L_U) = H_*(U, L).$$

Proof. By the definition of $C_*(X, L_U)$ we have:

$$C_p(X, L_U) = \bigoplus_{s \in \Sigma_p} L_U(s) = \bigoplus_{\substack{s \in \Sigma_p \\ x_0 \in U}} L = \bigoplus_{\substack{s \in \Sigma_p \\ s \subseteq U}} L = C_p(U, L).$$

Since the face maps remain unchanged, we see that $H_*(X, L_U) = H_*(U, L)$ as desired. □

If $V \subseteq U \subseteq X$ then we can define:

$$\phi_{V,U} : L_V \to L_U \text{ and } \phi_{U,V} : L_U \to L_V \text{ as :}$$

$$\phi_{*,*}(x) = \begin{cases} id, & \text{if } x \in V, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 4.2. *If $V \subseteq U \subseteq X$ then:*

$$0 \to L_V \xrightarrow{\phi_{V,U}} L_U \xrightarrow{\phi_{U,U-V}} L_{U-V} \to 0$$

is exact.

Proof. We prove this by showing that the sequence:

$$(*) \qquad 0 \to L_V(x) \xrightarrow{\phi_{V,U}(x)} L_U(x) \xrightarrow{\phi_{U,U-V}(x)} L_{U-V}(x) \to 0$$

is exact in each of the following three cases:

CASE 1 : $x \notin U$. Then $(*)$ is the zero sequence and there is nothing to prove.

CASE 2 : $x \in V$. Then $(*)$ equals the sequence, $0 \to L \xrightarrow{id} L \xrightarrow{0} 0 \to 0$, which is exact.

CASE 3 : $x \in U - V$. Then $(*)$ equals the sequence, $0 \to 0 \xrightarrow{0} L \xrightarrow{id} L \to 0$, which is also exact.

Thus we have the lemma. □

We can now give the main result of this section:

Theorem 4.3. *Let X be a poset and fix $a \in X$. Let $V = X(> a)$, then:*

$$H_i(X, L_a) \cong \tilde{H}_{i-1}(V, L) \text{ for all } i.$$

Proof. Notice that both V and $U = X(\geq a)$ satisfy the criterion of Lemma 4.1. From the proof of Lemma 4.1 we have:

$$(4.4) \qquad C_p(X, L_V) = \bigoplus_{\substack{s \in \Sigma_p \\ x_0 \in V}} L_V(s) \qquad \text{and } C_p(X, U) = \bigoplus_{\substack{s \in \Sigma_p \\ x_0 \in U}} L_U(s).$$

As $U - V = \{a\}$, by Lemma 4.2, we have a short exact sequence of functors:

$$0 \to L_V \xrightarrow{\mu} L_U \xrightarrow{\eta} L_a \to 0,$$

where $\mu = \phi_{V,U}$ and $\eta = \phi_{U,a}$. By the discussion preceding Theorem 3.4, and (4.4) above, we have the following commutative diagram:

$$\begin{array}{ccccc}
0 \to C_p(X, L_V) & \xrightarrow{\tilde{\mu}} & C_p(X, L_U) & \xrightarrow{\tilde{\eta}} & C_p(X, L_a) \to 0 \\
\| & & \| & & \| \\
0 \to \bigoplus_{\substack{s\in\Sigma_p \\ x_0\in V}} L_V(s) & \xrightarrow{\tilde{\mu}} & \bigoplus_{\substack{s\in\Sigma_p \\ x_0\in U}} L_U(s) & \xrightarrow{\tilde{\eta}} & C_p(X, L_a) \to 0 \\
\| & & \| & & \| \\
0 \to C_p(V, L) & \xrightarrow{\tilde{\mu}} & C_p(U, L) & \xrightarrow{\tilde{\eta}} & C_p(X, L_a) \to 0,
\end{array} \tag{4.5}$$

where the horizontal rows are short exact sequences of chain complexes. By definition, $\tilde{\mu} = \sum_{s\in\Sigma_p} \tilde{\mu}_s : C_p(X, L_V) \to C_p(X, L_U)$, where $\tilde{\mu}_s = \tilde{\mu}_{x_0} = \phi_{V,U}(x_0) = \text{id}$ since $x_0 \in V \subset U$. Thus, $\tilde{\mu} = \tilde{i}$, the map induced by the canonical inclusion $i : V \hookrightarrow U$.

The bottom row of (4.5) along with Theorem 6.3 in [R] and the identification of $\tilde{\mu}$ with $\tilde{i}$ gives the following long exact sequence in homology:

$$\cdots \to H_{i+1}(X, L_a) \to H_i(V, L) \xrightarrow{i_*} H_i(U, L) \to H_i(X, L_a) \to \cdots, \tag{$**$}$$

where i_* is the map induced on homology by the inclusion of V into U.

As a is the unique minimal element of U, by the discussion on pg. 103 in [Q], the map $k : U \to \{a\}$ is a homotopy equivalence. Thus, by Lemma 3.3, we have:

$$H_i(U, L) \cong \begin{cases} L, & \text{if } i = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Thus for $i \neq 0, 1$, $(**)$ equals the exact sequence:

$$\cdots \to 0 \to H_i(X, L_a) \to H_{i-1}(V, L) \to 0 \to \cdots .$$

Hence $H_i(X, L_a) \cong H_{i-1}(V, L) \cong \tilde{H}_{i-1}(V, L)$ when $i \neq 0, 1$.

Also from $(**)$ we have:

$$\to H_1(U, L) = 0 \to H_1(X, L_a) \to H_0(V, L) \xrightarrow{i_*} H_0(U, L) \to H_0(X, L_a) \to \cdots . \tag{4.6}$$

Now $C_{-1}(X, L_a) = 0$ and $C_0(X, L_a) = \bigoplus_{x\in X} L_a(x) = L$. Also, if $V = \emptyset$ then $C_1(X, L_a) = \bigoplus_{(x_0<x_1)} L_a(x_0) = 0$, because $V = X(> a) = \emptyset$ implies there are no 1-simplices of the form

$(a < x_1)$. So $H_1(X, L_a) = 0$ and $H_0(X, L_a) = L$. But, by Theorem 1.2 and Lemma 1.1, we have:

$$\begin{aligned}\tilde{H}_i(\emptyset, L) &= (\tilde{H}_i(\emptyset) \otimes L) \oplus \mathrm{Tor}_1^{\mathbf{Z}}(\tilde{H}_{i-1}(\emptyset), \mathbf{Z}) \\ &\cong \begin{cases} L, & \text{if } i = -1 \\ 0, & \text{otherwise.} \end{cases}\end{aligned}$$

Thus, if $V = \emptyset$, $H_1(X, L_a) = \tilde{H}_0(\emptyset, L) = \tilde{H}_0(V, L) = 0$ and $H_0(X, L_a) = \tilde{H}_{-1}(\emptyset, L) = \tilde{H}_{-1}(V, L) = L$.

So assume that $V \neq \emptyset$. Then $C_1(X, L_a) = \bigoplus_{(x_0 < x_1)} L_a(x_0) \neq 0$. Therefore $H_0(X, L_a) = 0 = \tilde{H}_{-1}(V, L)$. If $k : U \to \{a\}$ is the canonical contraction and $j = k \circ i$, then the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "i"] \arrow[dr, "j"'] & U \arrow[d, "k"] \\ & \{a\}. \end{tikzcd}$$

By the functorial property of $_*$, the map $j_* : H_*(V, L) \to H_*(\{a\}, L)$ equals $k_* \circ i_*$, where k_* and i_* are the maps induced on homology by k and i respectively.

Thus we have the following commutative diagram:

$$\begin{tikzcd} H_0(V, L) \arrow[r, "i_*"] \arrow[dr, "j_*"'] & H_0(U, L) \arrow[d, "k_*"] \\ & H_0(\{a\}, L), \end{tikzcd} \tag{4.7}$$

where k_* is an isomorphism.

Now the fact that $\{a\}$ is a poset consisting of a single point and $j : V \to \{a\}$ is a map of posets, along with the definition of reduced homology relative to an abelian group L (in section 3), imply that we have the short exact sequence:

$$0 \to \tilde{H}_0(V, L) \to H_0(V, L) \xrightarrow{j_*} H_0(\{a\}, L) \to 0. \tag{4.8}$$

On the other hand, by (4.6) and the fact that $H_0(X, L_a) = 0$, we have the short exact sequence:

$$0 \to H_1(X, L_a) \to H_0(V, L) \xrightarrow{i_*} H_0(U, L) \to 0. \tag{4.9}$$

Now, considering the commutative diagram (4.7) and the short exact sequences (4.8) and (4.9), by Lemma 1.3, $H_1(X, L_a) \cong \tilde{H}_0(V, L)$. Thus, for $V \neq \emptyset$ we also have $H_0(X, L_a) \cong \tilde{H}_{-1}(V, L) = 0$ and $H_1(X, L_a) \cong \tilde{H}_0(V, L)$.

Thus for all i and independent of whether $V = X(> a)$ is empty or not we have shown that $H_i(X, L_a) = \tilde{H}_{i-1}(V, L)$. □

SECTION 5 : A THEOREM ON THE n-CONNECTEDNESS OF POSETS

In this section we prove an important theorem about maps of posets. A direct corollary to the theorem will give us an analogue to Quillen's theorem regarding the n-connectedness of spherical complexes – Theorem 9.1 on pg. 119 in [Q]. We shall use the results from sections 2, 3, and 4 as tools to prove a result about the vanishing of homology under certain assumptions – Theorem 5.2. The main result of chapter 1 – Theorem A – will follow immediately from the theorem and (1.4) on pg. 5 in [A2].
We first need the following definition:

Definition. *Given a map of posets $f : X \to Y$, we define:*

$$f|y = f^{-1}(Y(\leq y)) = \{x \in X \mid f(x) \leq y\}.$$

We have the following lemma:

Lemma 5.1. *Let $f : X \to Y$ be a map of posets satisfying:*

(1) *f is strict; that is, $x \lneq y \implies f(x) \lneq f(y)$, and*

(2) *$f|y$ is min $\{n, h(y) - 1\}$-connected.*

Then for each $y \in Y$ and $q \leq n$, $dim(f|y) \leq h(y)$ and:

$$\tilde{H}_q(f|y) = \begin{cases} \text{a free abelian group}, & \text{if } h(y) = dim(f|y) = q, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. If $h(y) > q$, then $\min\{n, h(y) - 1\} \geq q$, so by assumption (2) $\tilde{H}_q(f|y) = 0$.

If s is a k-simplex of X, then by assumption (1), $f(s)$ is a k-simplex of Y. Thus, as $f(f|y) \subset Y(\leq y)$, $\dim(f|y) \leq \dim(Y(\leq y)) = h(y)$. If $h(y) < q$ or $h(y) =$

q but $\dim(f|y) < h(y)$, then $\dim(f|y) < q$; hence $\tilde{H}_q(f|y) = 0$. Thus, for all $q \leq n$ $\tilde{H}_q(f|y) \neq 0$ only when $h(y) = \dim(f|y) = q$. But, by the definition of $H_q(f|y)$, we have that $H_q(f|y)$ is a subgroup of the free group $C_q(f|y) = \bigoplus_{(x_o<\cdots<x_q)} \mathbb{Z}$, and thus is free abelian. In turn $\tilde{H}_q(f|y)$ is a subgroup of $H_q(f|y)$ and is also free abelian. Thus we have the lemma. □

Theorem 5.2. *Let $f : X \to Y$ be a map of posets satisfying:*

(1) *f is strict; that is, $x \lneq y \implies f(x) \lneq f(y)$,*

(2) *$f|y$ is min $\{n, h(y) - 1\}$ – connected for all $y \in Y$, and*

(3) *let $m \leq n$ be an integer such that $H_m(Y) = 0$, and for all $y \in Y$ with $h(y) = q \leq m$, $\tilde{H}_{m-q-1}(Y(> y)) = 0$.*

Then $H_m(X) = 0$.

Proof. Theorem 3.5 tells us that there is a convergent spectral sequence:

$$E^2_{pq} = H_p(Y, y \mapsto H_q(f|y)) \Rightarrow H_{p+q}(X).$$

By Lemma 2.2 we will be done if we show that for all p, q such that $p + q = m$, $E^2_{pq} = 0$. Remember that for a nonempty topological space, X, we extend homology to negative indices by setting $H_i(X) = 0$ for all $i < 0$. Thus $H_p(Y, y \mapsto H_q(f|y)) = 0$ if $p < 0$. Since assumption (2) tells us that $f|y \neq \emptyset$, we also have that if $q < 0$ then $H_q(f|y) = 0$. Thus, in this case too, $H_p(Y, y \mapsto H_q(f|y)) = 0$. Hence we see that for $p < 0$ or $q < 0$, we have $E^2_{pq} = 0$, so E^2_{pq} is a first quadrant spectral sequence. Thus we can restrict our attention to p and q such that $p + q = m$ and $0 \leq p, q \leq m$.

For $q \neq 0$, $H_q(f|y) = \tilde{H}_q(f|y)$ for all $y \in Y$, thus:

$$E^2_{pq} = H_p(Y, y \mapsto \tilde{H}_q(f|y)).$$

Given functors L, K, and $F \in Funct(Y, \mathcal{A}b)$, and natural transformations:

$$0 \to L \xrightarrow{\phi} K \xrightarrow{\lambda} F \to 0,$$

we defined the above sequence to be a short exact sequence of functors if for every $y \in Y$, the sequence:

$$0 \to L(y) \xrightarrow{\varphi_y} K(y) \xrightarrow{\lambda_y} F(y) \to 0$$

is exact. Given $y' \leq y$, we have the inclusion map $i : f|y' \hookrightarrow f|y$ which induces maps on homology. By the functorial property of $_*$ we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \tilde{H}_0(f|y') \arrow[r] \arrow[d, "i_*"] & H_0(f|y') \arrow[r] \arrow[d, "i_*"] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \tilde{H}_0(f|y) \arrow[r] & H_0(f|y) \arrow[r] & \mathbb{Z} \arrow[r] & 0,
\end{tikzcd}$$

where the rows are exact (by the definition of reduced homology).

Thus we have the short exact sequence of functors:

$$0 \to (y \mapsto \tilde{H}_0(f|y)) \to (y \mapsto H_0(f|y)) \to \mathbb{Z} \to 0.$$

Hence, by Theorem 3.4, we get a long exact sequence:

$$\cdots \to H_{m+1}(Y) \to H_m(Y, y \mapsto \tilde{H}_0(f|y)) \to E^2_{m0} \to H_m(Y) \to \cdots.$$

As $H_m(Y) = 0$, by assumption, replacing $H_m(Y)$ by zero in the above sequence gives us the following long exact sequence:

$$\cdots \to H_{m+1}(Y) \to H_m(Y, y \mapsto \tilde{H}_0(f|y)) \to E^2_{m0} \to 0 \to \cdots.$$

Thus, $H_m(Y, y \mapsto \tilde{H}_0(f|y))$ maps onto E^2_{m0}.

Therefore to prove that $E^2_{pq} = 0$ for all p, q such that $p + q = m$, it remains to show that:

$$H_p(Y, y \mapsto \tilde{H}_q(f|y)) = 0 \text{ for all } p + q = m.$$

To prove the above identity we need the following result:

Lemma 5.3. *If $h(y) = q$ and $p - 1 = m - q - 1$, then $\tilde{H}_{p-1}(Y(> y), \tilde{H}_q(f|y)) = 0$.*

Proof. By Lemma 3.2:

$$\tilde{H}_{p-1}(Y(> y), \tilde{H}_q(f|y)) = (\tilde{H}_{p-1}(Y(> y)) \otimes \tilde{H}_q(f|y)) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(\tilde{H}_{p-2}(Y(> y)), \tilde{H}_q(f|y)).$$

But $-1 \leq p-1 = m-q-1$ and assumption (3) gives $\tilde{H}_{p-1}(Y(>y)) = 0$. And, by Lemma 5.1, $\tilde{H}_q(f|y)$ is free abelian. So, by Lemma 1.1, $\mathrm{Tor}_1^{\mathbb{Z}}(\tilde{H}_{p-2}(Y(>y)), \tilde{H}_q(f|y)) = 0$. Thus, $\tilde{H}_{p-1}(Y(>y), \tilde{H}_q(f|y)) = 0$ if $h(y) = q$ and $p-1 = m-q-1$. □

We now complete the proof of Theorem 5.2. Let $F : Y \to \mathcal{A}b$ be the functor defined by $y \mapsto \tilde{H}_q(f|y)$ (here $q \leq m$). Any $f : X \to Y$ satisfying the criterion of the theorem satisfies the criterion of Lemma 5.1, so $F(y) = \tilde{H}_q(f|y) = 0$ unless $h(y) = q$. Since $\mathrm{Mor}_Y(y, y') = \emptyset$ unless y and y' are comparable, that is, $h(y) \neq h(y')$, we have:

$$F = \bigoplus_{h(y)=q} \tilde{H}_q(f|y)_y \text{ where } \tilde{H}_q(f|y)_y(x) = \begin{cases} \tilde{H}_q(f|y), & \text{if } x = y, \\ 0, & \text{otherwise.} \end{cases}$$

From Lemma 3.1 we know that for a functor $F = F_1 \oplus F_2$ $H_*(Y, F) = H_*(Y, F_1) \oplus H_*(Y, F_2)$, so we have:

$$\begin{aligned} H_p(Y, y \mapsto \tilde{H}_q(f|y)) &= \bigoplus_{h(y)=q} H_p(Y, \tilde{H}_q(f|y)_y) \\ &\cong \bigoplus_{h(y)=q} \tilde{H}_{p-1}(Y(>y), \tilde{H}_q(f|y)) \quad \text{(by Theorem 4.3)} \\ &= 0 \quad \text{if } p+q = m \quad \text{(by Lemma 5.3).} \end{aligned}$$

Thus we have $H_p(Y, y \mapsto \tilde{H}_q(f|y)) = 0$ if $p + q = m$. So, for all p and q such that $p + q = m$, we have $E^2_{pq} = 0$. Lemma 2.2 then tells us that $H_m(X) = 0$. □

As an immediate corollary to **Theorem 5.2** we get the following analogue to Quillen's Theorem 9.1 on pg. 119 in [Q]:

Theorem A. *Let $f : X \to Y$ be a map of posets satisfying:*

(1) *f is strict; that is, $x \lneq y \Rightarrow f(x) \lneq f(y)$,*

(2) *$f|y$ is min $\{n, h(y) - 1\}$ – connected for all $y \in Y$, and*

(3) *$Y(>y)$ is $(n - h(y) - 1)$-connected for all $y \in Y$ with $h(y) \leq n$.*

Then Y n-connected implies X is n-connected.

Proof. The proof for $n = 1$ is given by (1.4) on pg. 5 in [A2]. Now, if for $n > 1$ and any $f : X \to Y$, the criteria of this theorem hold and Y is n-connected; then (1), (2), and (3) hold for $n = 1$ and Y is simply connected. Thus, by the proof for $n = 1$, we know that X is simply connected. Hence, in the proof for $n > 1$, it suffices to show:

$$H_i(X) = 0 \qquad \text{for all } 1 \leq i \leq n.$$

But if for any $f : X \to Y$ and $n > 1$, critërion (1), (2), and (3) of this corollary are true and Y is n-connected then criterion (1), (2), and (3) of Theorem 5.2 are satisfied for all $m \leq n$ and $m \neq 0$. Thus, by Theorem 5.2, we have that for all $p \neq 0$ and $p \leq n$, $H_p(X) = 0$ as desired. □

Chapter 2

The Quillen complex of a group of symplectic type

Let $G = H \rtimes A$ be a semidirect product of a solvable p'-group H by an elementary abelian p-group A. In Theorem 11.2 on pg. 123 in [Q], Quillen uses Theorem 9.1 on pg. 119 in [Q] to prove that $\mathcal{A}_p(G)$ is Cohen-Macaulay of dimension $m_p(A)-1$ (recall the definition of a Cohen-Macaulay complex from section 1).

In this chapter we consider another interesting minimal case. For a fixed prime p remember that a p-group G is *extraspecial* if:

$$\Phi(G) = Z(G) = G^{(1)} \quad \text{is cyclic.}$$

Here $\Phi(G)$ is the Frattini subgroup of G, and $Z(G)$ is the center of G. Also remember that a p-group is of *symplectic type* if it has no noncyclic characteristic subgroups. We will consider a group S of symplectic type with $S = \mathbb{Z}/p^n\mathbb{Z} * T$, where T is extraspecial and of exponent p if p is odd. Recall that $G * H$ denotes the *central product* of the groups G and H.[1] Groups of symplectic type arise as interesting minimal objects in the study of finite groups, so we shall consider the Quillen complex of a semidirect product of a solvable p'-group by a group of symplectic type.

In Section 6 we include, for completeness, a proof of the well-known fact that the order complex of nontrivial totally singular subspaces of a hyperbolic orthogonal space ordered by inclusion, can be given the structure of a weak building. In Section 7 we consider the group $G = \mathbb{Z}/4\mathbb{Z} * T$ – T an extraspecial 2-group – and show that $\mathcal{A}_p(G)(> Z)$ is $(m_p(G)-3)$-connected; here Z is the center of T. Finally in Section 8, we use Theorem A and results from sections 6 and 7 to show that $\mathcal{A}_p(G)$ is $(m_p(G)-1)$-spherical when $G = O_{p'}(G)S$ with $O_{p'}(G)$ solvable and $S \in Syl_p(G)$ of symplectic type with $S = \mathbb{Z}/p^n\mathbb{Z} * T$.

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[1] Refer to section 23 in [A1] for further definitions and properties of extraspecial groups and groups of symplectic type.

SECTION 6 : THE ORDER COMPLEX OF NONTRIVIAL TOTALLY SINGULAR SUBSPACES OF A HYPERBOLIC ORTHOGONAL SPACE

In this and subsequent sections the concept of geometric complexes, chamber complexes, and buildings are in the sense of J. Tits. For the sake of completeness we include the definition of some specific complexes and also of weak buildings The notation is as given in sections 3 and 41 in [A1]. For further details on complexes and buildings, refer to sections 3, 41, and 42 in [A1].

Given a finite set I, a *geometry* over I is a triple $(\Gamma, \tau, *)$. Γ is a set of objects. $\tau : \Gamma \to I$ is a map called the *type map*; that is, $\tau(x)$ is the *type of* x for each $x \in \Gamma$. And $*$ is a symmetric incidence relation on Γ, referred to as *adjacency*, with the one constraint that objects of the same type are adjacent if and only if they are equal. We will often refer to the triple $(\Gamma, \tau, *)$ simply as Γ.

A *flag* of a geometry is a set T of objects such that given any two objects in T they are adjacent. Note that our constraint on τ implies that $\tau : T \to I$ is injective for each flag T. $\tau(T)$ is called the type of the flag T. The *rank* of a flag T is the cardinality of $\tau(T)$, whereas the *corank* of T refers to $|I - \tau(T)|$. A *geometric complex* (or simply a *complex*) is a pair $(\Gamma, \mathfrak{C})$ where Γ is a geometry and $\mathfrak{C}$ is a collection of flags of type I called *chambers*. We will often refer to a complex simply as $\mathfrak{C}$. Subflags of chambers are called *simplices*, and simplices of corank 1 are called *walls*. A complex is *thin* if every wall is contained in exactly two chambers.

The *chamber graph* of a complex $\mathfrak{C}$ is an undirected graph on the set of chambers $\mathfrak{C}$ obtained by joining chambers which share a common wall. A complex is *connected* if its chamber graph is connected. A connected complex $(\Gamma, \mathfrak{C})$ in which every flag of rank 1 and 2 are simplices is called a *chamber complex*.

A *morphism* $\alpha : (\Gamma, \mathfrak{C}) \to (\Delta, \mathfrak{D})$ of complexes is a morphism of geometries $\Gamma \to \Delta$ such that $\alpha(\mathfrak{C}) \subseteq \mathfrak{D}$. Given a thin chamber complex $\mathfrak{C}$, a *folding* of $\mathfrak{C}$ is an idempotent morphism of $\mathfrak{C}$ whose fiber on each chamber is of order exactly two. A thin chamber complex $\mathfrak{C}$ is a *Coxeter complex* if for every pair of adjacent chambers C and D, there is a folding mapping C to D.

A *weak building* is a pair $(\mathfrak{B}, \mathfrak{A})$ consisting of a complex $\mathfrak{B}$ and set $\mathfrak{A}$ of subcomplexes, called *apartments*, satisfying:

(B1) : the elements of $\mathfrak{A}$ are Coxeter complexes,

(B2) : any two chambers belong to a common apartment, and

(B3) : given two simplices S_1, S_2 and apartments Σ_1, $\Sigma_2 \in \mathfrak{A}$ containing them, there is an isomorphism from Σ_1 onto Σ_2, which is the identity on $S_1 \cup S_2$.

Note that this is what Tits calls a building , on pg. 524 in [T2].

Fix an integer m and let D^m be the $2m$-dimensional hyperbolic orthogonal space over some finite field $\mathbb{F}$. Remember that the Tits building of D^m is given by the *oriflamme geometry* on D^m (refer to pg. 99 in [A1] for the definition of the above geometry). For the sake of completeness we now provide a proof of the well-known fact that the order complex of nontrivial totally singular subspaces of D^m can be given the structure of a weak building of dimension $m-1$, and thus is $(m-2)$-connected. But first we need the following result:

Lemma 6.1. *If $U \subsetneq V$ are totally singular subspaces of D^m of dimensions $m-1$ and m respectively, then there is a unique totally singular subspace of maximal dimension, say W, distinct from V and containing U.*

Proof. By (19.2) on pg. 77 in [A1], we know that $\dim(U^\perp) = 2m - \dim(U) = m+1$. So if we write $U^\perp = U \oplus U'$, then by (19.3) on pg. 77 in [A1], U' is a 2-dimensional nondegenerate subspace.

Now $U \subsetneq V \subseteq U^\perp$ so $V \cap U' \neq \emptyset$. Pick $0 \neq v_0 \in (V \cap U')$. Then v_0 is a nontrivial singular element of U'; thus, by (19.12) on pg. 80 in [A1], there exists $w_0 \in U'$ such that $\{v_0, w_0\}$ is a hyperbolic pair for U'. Certainly $W = U \oplus < w_0 >$ satisfies the criteria of the lemma.

Now let Y be another totally singular subspace satisfying the criteria of the lemma. As for V, we can pick $0 \neq y_0 \in (Y \cap U')$. Then $y_0 = av_0 + bw_0$ for some scalars $a, b \in \mathbb{F}$.

Now $Y = U \oplus < y_0 >$ and $Y \neq V$ implies $b \neq 0$. Also:

$$0 = Q(y_0) = Q(av_0) + Q(bw_0) + abf(v_0, w_0) = ab \quad \Longrightarrow a = 0.$$

So $y_0 = bw_0$ and $Y = U \oplus < w_0 >= W$, as claimed. □

Define P to be the poset of nontrivial totally singular subspaces of D^m – ordered by inclusion. Note that P is equivalent to the following complex. Let:

$$\Gamma = P \qquad U * V \iff U \subseteq V \text{ or } V \subseteq U \qquad \begin{array}{l} \tau : \Gamma \to I = \{0, \ldots, m-1\} \\ U \mapsto dim(U) - 1. \end{array}$$

Then $(\Gamma, *, \tau)$ is a geometry. Let $\mathfrak{C} = \{$ flags of Γ of type I $\}$. Finally set $\mathfrak{B}$ to be the complex $(\Gamma, \mathfrak{C})$.

Now for any hyperbolic basis $X = \{x_i \mid 1 \leq x_i \leq 2m\}$ of D^m, let $Y = \{< x > \mid x \in X\}$. Then set Γ_Y to be the subgeometry of Γ generated by elements of Y, and $\mathfrak{C}_Y = \mathfrak{C} \cap \Gamma_Y$. Then $(\Gamma_Y, \mathfrak{C}_Y)$ is a subcomplex of $\mathfrak{B}$. Let:

$$\mathfrak{A} = \{(\Gamma_Y, \mathfrak{C}_Y) \mid X \text{ is some hyperbolic basis for } D^m\}.$$

Theorem 6.2. $(\mathfrak{B}, \mathfrak{A})$ *is a weak building.*

Proof. Note that any hyperbolic basis of D^m forms a hyperbolic basis for the underlying symplectic space (with symplectic form $f(x, y) = Q(x + y) - Q(x) - Q(y)$). Therefore any $(\Gamma_Y, \mathfrak{C}_Y) \in \mathfrak{A}$ is an apartment of the Tits building of the $2m$-dimensional symplectic space. Similarly, any simplex S of $\mathfrak{B}$ is a simplex of the building of the symplectic space. Thus, since the simplices and apartments of the building of the $2m$-dimensional symplectic space satisfy axioms (B1) and (B3), all $(\Gamma_Y, \mathfrak{C}_Y) \in \mathfrak{A}$ and $S \in \mathfrak{B}$ satisfy axioms (B1) and (B3). Thus it remains to show that $(\mathfrak{B}, \mathfrak{A})$ satisfy axiom (B2).

Let $V = (V_0 \subset \cdots \subset V_{m-1})$ and $W = (W_0 \subset \cdots \subset W_{m-1})$ be chambers of $\mathfrak{B}$. Then, if $\mathfrak{O}$ is the chamber complex of the oriflamme geometry on D^m, we have:

$$\overline{V} = (V_0 \subset \cdots \subset V_{m-3} \subset V_{m-1}) \text{ and } \overline{W} = (W_0 \subset \cdots \subset W_{m-3} \subset W_{m-1}),$$

are walls of $\mathfrak{D}$. Let $U(V)$ be the totally singular subspace of maximal dimension such that $V_{m-2} = U(V) \cap V_{m-1}$ as given by Lemma 6.1. Let $U(W)$, such that $W_{m-2} = U(W) \cap W_{m-1}$, be similarly defined. Then we have:

$$\tilde{V} = (V_0 \subset \cdots \subset V_{m-3} \subset U(V), V_{m-1}) \text{ and } \tilde{W} = (W_0 \subset \cdots \subset W_{m-3} \subset U(W), W_{m-1})$$

are chambers of $\mathfrak{D}$ containing $\overline{V}$ and $\overline{W}$, respectively. Since the oriflamme geometry gives the Tits building for D^m, there is a hyperbolic basis, $X = \{x_i \mid 1 \leq x_i \leq 2m\}$, such that:

$$V_j = \sum_{i=0}^{j} < v_i > \quad W_j = \sum_{i=0}^{j} < w_i > \quad j \in \{0, \cdots, m-3, m-1\},$$

and

$$U(V) = \sum_{i=0}^{m-3} < v_i > \oplus < u_1 > \oplus < u_2 > \quad U(W) = \sum_{i=0}^{m-3} < w_i > \oplus < u_1' > \oplus < u_2' >,$$

where $v_i, w_i, u_i, u_i' \in X$.

Now:

$$V_{m-2} = V_{m-3} \oplus < v > \qquad v \neq 0$$

$$\Longrightarrow v = a_1 v_{m-2} + a_2 v_{m-1} = b_1 u_1 + b_2 u_2 \neq 0$$

(assume without loss of generality that $a_1 \neq 0,\ b_1 \neq 0$)

$$\Longrightarrow 0 = a_1 v_{m-2} + a_2 v_{m-1} - b_1 u_1 - b_2 u_2 \text{ and } b_1 \neq 0$$

$$\Longrightarrow u_1 = v_{m-2} \text{ or } v_{m-1}.$$

If $u_1 = v_{m-1}$ then $b_1 = a_2$, which implies $a_1 v_{m-2} - b_2 u_2 = 0$. Hence, $u_1 = v_{m-1}$ and $u_2 = v_{m-2}$ (as $a_1 \neq 0$). Thus, $U(V) = V_{m-1}$, contradicting our choice of $U(V)$. Therefore, $u_1 = v_{m-2}$ and $a_2 = b_2 = 0$, which implies that $V_{m-2} = V_{m-3} \oplus < v_{m-2} >$, as desired. Similarly, $W_{m-2} = W_{m-3} \oplus < w_{m-2} >$. So, for this choice of basis, we have V and W are elements of $(\Gamma_Y, \mathfrak{C}_Y)$.

Thus we have shown that $(\mathfrak{B}, \mathfrak{A})$ is indeed a weak building. $\square$

Let p be a prime, and:

$$V = \begin{cases} \text{a 2m-dimensional symplectic space,} & \text{if } p \text{ is an odd prime,} \\ \text{a 2m-dimensional orthogonal space,} & \text{if } p = 2. \end{cases}$$

Set P to be the poset of nontrivial totally singular subspaces of V – ordered by inclusion. Finally, let n equal the Witt index(V).

Theorem 6.3. *P is $(n-2)$-connected.*

Proof. When p is an odd prime or $p = 2$ and V is not hyberbolic, then P is equivalent to the Tits building of V. In the case $V = D^m$, we have that P is equivalent to $\mathfrak{B}$ (as defined above). But, by Theorem 6.2, $(\mathfrak{B}, \mathfrak{A})$ is a weak building (for an appropriate choice of $\mathfrak{A}$). Thus, by the Solomon-Tits theorem P is Cohen-Macaulay of dimension $(n-1)$. In particular, P is $(n-2)$-connected as claimed. □

SECTION 7 : $\mathcal{A}_p(G)(> Z)$ WHERE $G = \mathbb{Z}/4\mathbb{Z} * T$, T IS EXTRASPECIAL, AND $Z = Z(T)$

In this section we let T be an extraspecial 2-group, with $|T| = 2^{2n+1}$ and $G = \mathbb{Z}/4\mathbb{Z} * T$. If $Z = Z(T)$ is the center of T, we show that $\mathcal{A}_p(G)(> Z)$ is $(m_p(G) - 3)$-connected.

We first need some results from linear algebra. Let (V, Q) be an orthogonal space over $\mathbb{Z}/2\mathbb{Z}$ with associated bilinear form f. Assume that $Rad(V)$ is a nonsingular 1-dimensional subspace and let $Rad(V) = < v_0 >$.

Note that given $v \in V$, $Q(v + v_0) = Q(v) + Q(v_0) \neq Q(v) \pmod 2$. So we have:

(∗) $v \in V$ is singular if and only if $v + v_0$ is nonsingular.

Let $\pi : V \to V/Rad(V)$ be the canonical homomorphism, and for each $v \in V$, $U \leq V$ let $\bar{v}$, $\overline{U}$ be their respective images under π. Define:

$$\bar{f} : \overline{V} \times \overline{V} \longrightarrow \mathbb{Z}/2\mathbb{Z}$$

$$(\overline{u}_1, \overline{u}_2) \mapsto f(u_1, u_2) \qquad \text{where } u_i \in \pi^{-1}(\overline{u}_i) \quad (\text{for } i = 1, 2).$$

Lemma 7.1. *$\bar{f}$ is a nondegenerate symplectic form on $\overline{V}$.*

Proof. $\bar{f}$ is well-defined since $\overline{u}_i = \overline{u}_i'$ implies that $u_i = u_i' + v_0$ (for $i = 1, 2$). So that:

$$\bar{f}(\overline{u}_1, \overline{u}_2) = f(u_1, u_2) = f(u_1' + v_0, u_2' + v_0) = f(u_1', u_2') = \bar{f}(\overline{u}_1', \overline{u}_2').$$

Suppose $\overline{v} \in \overline{V}$ satisfies $\overline{f}(\overline{v}, \overline{u}) = 0 \;\forall\; \overline{u} \in \overline{V}$. If $v \in \pi^{-1}(\overline{v})$, then:

$$f(v, u) = \overline{f}(\overline{v}, \overline{u}) = 0 \;\forall\; u \in V \implies v \in Rad(V) \implies \overline{v} = 0.$$

Thus $\overline{f}$ is nondegenerate.

Finally, the fact that f is symplectic, and the definition of $\overline{f}$, implies that $\overline{f}$ is also symplectic. □

Now, for each $\overline{u} \in \overline{V}$, $\pi^{-1}(\overline{u}) = \{v, w\}$, since we are working over $\mathbb{Z}/2\mathbb{Z}$. So $v = w + v_0$ and $w = v + v_0$. By $(*)$ above, for each $\overline{u} \in \overline{V}$ there is a unique $u \in V$, such that $\pi(u) = \overline{u}$ and $Q(u) = 0$. Define: $\phi : \overline{V} \to V$ by $\overline{u} \mapsto u$ where u is as above. Note that by the definition of ϕ, ϕ is $1 - 1$.

Lemma 7.2. *If $\overline{U} \leq \overline{V}$ is totally singular, then $\phi : \overline{U} \to V$ is a monomorphism. Furthermore, $\phi(\overline{U})$ (written as U) is a totally singular subspace of V.*

Proof. Let $\overline{u}_1, \overline{u}_2 \in \overline{U}$ with $\phi(\overline{u}_i) = u_i$. Then $\pi^{-1}(\overline{u_1 + u_2}) = \{u_1 + u_2, u_1 + u_2 + v_0\}$. Now:

$$Q(u_1 + u_2) = Q(u_1) + Q(u_2) + f(u_1, u_2) = f(u_1, u_2) = \overline{f}(\overline{u}_1, \overline{u}_2) = 0.$$

Thus $\phi(\overline{u_1 + u_2}) = u_1 + u_2 = \phi(\overline{u}_1) + \phi(\overline{u}_2)$. This, along with the fact that ϕ is $1 - 1$ as a map from $\overline{V}$ to V, implies that ϕ is a monomorphism, as claimed.

The choice of U along with the definition of $\overline{f}$ gives us that U is totally singular.

Also note that by the definition of ϕ, we have that $\pi^{-1}(\overline{U}) = U \oplus Rad(V)$. □

On the other hand, if $U \leq V$ is totally singular then, by the definition of $\overline{f}$, $\overline{U}$ is a totally singular subspace of $\overline{V}$. So in view of Lemma 7.2 and this remark we have:

Lemma 7.3. *The order complex of the totally singular subspaces of* V *and the order complex of the totally singular subspaces of* $\overline{V}$ *– both ordered by inclusion – are equivalent.*

Proof. We have a bijection of sets given by:

$$\{U \leq V |\ U \text{ totally singular }\} \underset{\phi}{\overset{\pi}{\rightleftarrows}} \{\overline{U} \leq \overline{V} |\ \overline{U} \text{ totally singular }\}.$$

But, by the definiton of π and ϕ, both maps are order preserving. Thus we have the desired equivalence. □

Let $v_1, \dots, v_n \in V^\natural$ such that $\{v_0, v_1, \dots, v_n\}$ forms a basis for V. Given $\overline{v} \in \overline{V}$, let $\pi^{-1}(\overline{v}) = \{u, w\}$. If $u = \sum_{i=0}^{n} \alpha_i v_i$, then $w = \sum_{i=0}^{n} \alpha_i v_i + v_0$. Thus for each $\overline{v} \in \overline{V}$, there is a unique $v \in < v_1, \dots, v_n >$ such that $\pi(v) = \overline{v}$.

Let $\overline{U} \leq \overline{V}$ be nondegenerate and choose a basis, $\{\overline{u}_1, \dots, \overline{u}_k\}$, for $\overline{U}$. For each $\overline{u}_i$ let $u_i \in < v_1, \dots, v_n >$ such that $\pi(u_i) = \overline{u}_i$. Then:

Lemma 7.4. $\{v_0, u_1, \dots, u_k\}$ *forms a basis for* $\pi^{-1}(\overline{U})$ *and* $U = < u_1, \dots, u_k >$ *is nondegenerate. In particular,* $\pi^{-1}(\overline{U}) = U \oplus Rad(V)$.

Proof. Assume $\alpha v_0 + \sum_{i=1}^{k} \beta_i u_i = 0$. Then $\alpha = 0$ (as $\{v_0, v_1, \dots, v_n\}$ is a basis for V). Also:

$$\pi(\sum \beta_i u_i) = \sum \beta_i \overline{u_i} = 0 \implies \beta_i = 0 \ \forall\ i = 1, \dots, k.$$

So $\{v_0, u_1, \dots, u_k\}$ is a linearly independent set with cardinality equal to the dimension of $\pi^{-1}(\overline{U})$. Thus, $\{v_0, u_1, \dots, u_k\}$ is a basis for $\pi^{-1}(\overline{U})$.

Now let $0 \neq u \in U$, and choose $w \in U$ such that $0 \neq \overline{f}(\pi(u), \pi(w)) = f(u, w)$. Such a w exists as $\overline{U}$ is nondegenerate by our assumption; and $0 \neq u \in U \leq < v_1, \dots, v_n >$, implies that $\pi(u) \neq 0 \in \overline{U}$. Thus U is nondegenerate, as claimed. Note that U and $\overline{U}$ have the same dimension. □

By the last statement in the proof of Lemma 7.2 and by Lemma 7.4, we have the following:

Lemma 7.5. *Let $\overline{U} \leq \overline{V}$ be totally singular and let $\overline{U}^{\perp} = \overline{U} \oplus \overline{W}$, then $\pi^{-1}(\overline{U}^{\perp}) = U \oplus W \oplus Rad(V)$ such that U is totally singular and W is nondegenerate. Here $\pi(U) = \overline{U}$ and $\pi(W) = \overline{W}$.*

Proof. $\overline{W}$ is nondegenerate by (19.3.3) on pg. 77 in [A1]. So, by Lemma 7.4, $\pi^{-1}(\overline{W}) = W \oplus Rad(V)$ with W nondegenerate. On the other hand, by the last sentence in the proof of Lemma 7.2, $\pi^{-1}(\overline{U}) = U \oplus Rad(V)$ with U totally singular. Thus the lemma. □

We now turn our attention to $G = \mathbb{Z}/4\mathbb{Z} * T$ where T is an extraspecial 2-group with $|T| = 2^{2n+1}$. Let Z be the center of T, $\widetilde{T} = T/Z$ and:

$$\pi : G \to G/Z \equiv \widetilde{G} \cong \mathbb{Z}/2\mathbb{Z} \times \widetilde{T},$$

be the canonical homomorphism. Given $g \in G$, let $\tilde{g}$ be its image in $\widetilde{G}$.

Define:

$$\begin{array}{lll} \tilde{f} : \widetilde{G} \times \widetilde{G} \to Z & \widetilde{Q} : \widetilde{G} \to Z & \\ & & \alpha, \alpha' \in \mathbb{Z}/4\mathbb{Z} \text{ and } t, t' \in T. \\ (\widetilde{\alpha t}, \widetilde{\alpha' t'}) \mapsto [t, t'] & \widetilde{\alpha t} \mapsto \alpha^2 t^2 & \end{array}$$

These maps are well-defined as $\mathbb{Z}/4\mathbb{Z}$ and T commute element-wise and $\mathbb{Z}/4\mathbb{Z}$ is abelian.

Lemma 7.6. *$\widetilde{Q}$ is a quadratic form on $\widetilde{G}$ and $\tilde{f}$ is the symplectic form associated to it. Furthermore, $Rad(\widetilde{G}) = \mathbb{Z}/2\mathbb{Z}$ is nonsingular.*

Proof. The proof of the first statement is identical to the proof of (23.10.3) on pg. 109 in [A1].

By the definition of $\tilde{f}$, $\mathbb{Z}/2\mathbb{Z} \subseteq Rad(\widetilde{G})$. Now let $\widetilde{\alpha t} \in \widetilde{G}$ such that $\tilde{f}(\widetilde{\alpha t}, \widetilde{\alpha' t'}) = 0 \ \forall \ \widetilde{\alpha' t'} \in \widetilde{G}$, then:

$$\tilde{f}(\widetilde{\alpha t}, \widetilde{\alpha' t'}) = [t, t'] = 1 \ \forall \ t' \in T$$

$$\implies t \in Z \implies \alpha t \in \mathbb{Z}/4\mathbb{Z} \implies \widetilde{\alpha t} \in \mathbb{Z}/2\mathbb{Z}.$$

So $Rad(\widetilde{G}) = \mathbb{Z}/2\mathbb{Z}$, as claimed. The nonsingularity of $Rad(\widetilde{G})$ follows from the fact that if $\mathbb{Z}/4\mathbb{Z} =< \eta >$, then $\widetilde{Q}(\tilde{\eta}) = \eta^2 \neq 1$. □

Lemma 7.7. *Let* $Z \lneq U \leq G$, *then* U *is extraspecial (respectively, elementary abelian) if and only if* $\widetilde{U}$ *is nondegenerate (respectively, totally singular).*

Proof. This follows from (23.10.4) on pg. 109 in [A1]. □

Now $(\widetilde{G}, \widetilde{Q})$ is an orthogonal space of dimension $2n+1$ over $\mathbb{Z}/2\mathbb{Z}$ such that $Rad(\widetilde{G})$ is a 1-dimensional nonsingular space. So let $\phi : \widetilde{G} \to \widetilde{G}/Rad(\widetilde{G}) \equiv \overline{G}$ be the canonical homomorphism and define:

$$\overline{f} : \overline{G} \times \overline{G} \longrightarrow Z$$

$$(\overline{u}_1, \overline{u}_2) \mapsto \tilde{f}(\widetilde{u_1}, \widetilde{u_2}) \qquad \text{where } \phi(\widetilde{u_i}) = \overline{u}_i.$$

By the definition of $\widetilde{G}$, $\overline{G}$, $\overline{f}$, they can be considered as V, $\overline{V}$, and $\overline{f}$ (respectively) in the discussion preceding Lemmas 7.1–5. Therefore, the above results apply to the map $\overline{f} : \overline{G} \times \overline{G} \to Z$. This fact, along with Lemma 7.7, gives us the following theorem:

Theorem 7.8. *Let* $G = \mathbb{Z}/4\mathbb{Z} * T$ *and* $Z = Z(T)$ *as above, then* $\mathcal{A}_p(G)(> Z)$ *is equivalent to the order complex of the totally singular subspaces of a* $2n$*-dimensional symplectic space over* $\mathbb{Z}/2\mathbb{Z}$ *– ordered by inclusion. In particular,* $\mathcal{A}_p(G)(> Z)$ *is* $(m_p(G) - 3)$*-connected.*

Proof. The first statement is a direct consequence of Lemmas 7.7 and 7.3. The second follows from the fact that the Witt index of $\overline{G} = n = m_p(G) - 1$, and the Solomon-Tits theorem which says that the order complex of the totally singular subspaces of a $2n$-dimensional symplectic space is $(n-2)$-connected. □

Finally, note that:

Lemma 7.9. *Let* $Z \lneq G_0 \leq G$ *be such that* $\overline{G_0}$ *(the image of* G_0 *in* $\overline{G}$*) is nondegenerate, then* $G_0 \cong \mathbb{Z}/4\mathbb{Z} * T_0$ *with* T_0 *extraspecial.*

Proof. By Lemma 7.4, $\overline{G_0}$ nondegenerate implies that there exists $\widetilde{G_0}$ nondegenerate in $\widetilde{G}$ such that $\phi^{-1}(\overline{G_0}) = \widetilde{G_0} \oplus Rad(\widetilde{G})$ (here ϕ is the canonical homomorphism from $\widetilde{G}$ onto $\overline{G}$). So by Lemma 7.7, the preimage of $\widetilde{G_0}$ in G, say T_0, is extraspecial and contains Z. Thus G_0, which by definition is the preimage of $\widetilde{G_0} \oplus Rad(\widetilde{G})$, is isomorphic to $\mathbb{Z}/4\mathbb{Z} * T_0$, as claimed. Note that if $dim(G_0) = k$, then $|T_0| = 2^{k+1}$. □

SECTION 8 : $\mathcal{A}_p(G)$ IS $(m_p(G)-2)$-CONNECTED WHEN $G = O_{p'}(G)S$ – WITH S OF SYMPLECTIC TYPE

Fix a prime p. Throughout this section, let $S = \mathbb{Z}/p^n\mathbb{Z} * T$ be of symplectic type (here $n \geq 0$), with T of exponent p if p is odd. Let Z be the center of T. Let $\mathcal{A} = \mathcal{A}_p(S)$ be the Quillen complex of the group S, and $Q = \mathcal{A}(> Z) = \{E \in \mathcal{A} | Z \lneq E\}$. Finally set $\tilde{S} \equiv S/Z$. Note that, as in (23.10) on pg. 109 in [A1], we can define a symplectic space structure on $\tilde{S}$ over $\mathbb{Z}/p\mathbb{Z}$ by defining $f : \tilde{S} \times \tilde{S} \to Z \cong \mathbb{Z}/p\mathbb{Z}$ by $f(\tilde{x}, \tilde{y}) = [x, y]$. When $p = 2$, we can define a quadratic form $Q : \tilde{S} \to Z$ by $Q(\tilde{x}) = x^2$, with corresponding symplectic form f as defined above.

Lemma 8.1. *When p is odd, $\Omega_1(S) = T$, and for $p = 2$, $\Omega_1(S) \leq \mathbb{Z}/4\mathbb{Z} * T$.*

Proof. Let p be odd, then by assumption $T = \Omega_1(T)$. So it suffices to show that $\Omega_1(S) = \Omega_1(T)$. $\Omega_1(T) \leq \Omega_1(S)$ by definition. Now if $\alpha t \in S$, with $\alpha \in \mathbb{Z}/p^n\mathbb{Z}$ and $t \in T$ such that $(\alpha t)^p = 1$, then:

$$(\alpha t)^p = \alpha^p t^p = \alpha^p = 1.$$

But $\alpha^p = 1$ implies $\alpha \in Z$, so $\alpha t \in T$. Thus, $\Omega_1(S) = T$.

Let $p = 2$. Let $\alpha t \in S$ with $\alpha \in \mathbb{Z}/2^n\mathbb{Z}$ and $t \in T$ such that $(\alpha t)^2 = 1$. Then:

$$(\alpha t)^2 = 1 \implies \alpha^2 = t^2 \in T^{(1)} = Z \implies \alpha^4 = 1.$$

Thus, $\Omega_1(S) \leq \mathbb{Z}/4\mathbb{Z} * T$. □

Now $E \in \mathcal{A} \implies E \leq \Omega_1(S)$. Also, $\mathbb{Z}/2\mathbb{Z} * T \cong T$. Thus, by Lemma 8.1, we can restrict our attention to $S = T$ when p is odd and $S = T$ or $\mathbb{Z}/4\mathbb{Z} * T$ when $p = 2$.

Lemma 8.2. *$Q \equiv \mathcal{A}(> Z)$ is $(m_p(S) - 3)$-connected.*

Proof. When $S = T$, Q is equivalent to the order complex of the poset of the nontrivial totally singular subspaces of $\tilde{S}$ by (23.10.4) on pg. 109 in [A1] and our assumption that T is of exponent p when p is odd. Then, by Theorem 6.3, Q is (Witt index$(\tilde{S}) - 2$) $= (m_p(S) - 3)$-connected.

When $S = \mathbb{Z}/4\mathbb{Z} * T$, the result follows directly from Theorem 7.8. □

By (23.10) on pg. 109 in [A1], we have that $|T| = p^{2k+1}$ for some integer k. We have four cases:

Table 1.

Prime(p)	S	**Witt index(S)**	$m_p(S)$
is odd	T	k	$k+1$
2	$\mathbb{Z}/4\mathbb{Z} * T$	k	$k+1$
2	T, $\tilde{S}$ is hyperbolic	k	$k+1$
2	T, $\tilde{S}$ not hyperbolic	$k-1$	k

Let $E \in \mathcal{A}$ with $|E| = p^{t+1}$, thus $h(E) = t$.

Lemma 8.3. *$\mathcal{A}(> E)$ is $(m_p(S) - h(E) - 3)$-connected.*

Proof. We want to show that $\mathcal{A}(> E)$ is $(m_p(S) - h(E) - 3) = (m_p(S) - t - 3)$-connected. If we let $\mathfrak{D} = \mathcal{A}_p(C_S(E))$, then $E' \in \mathcal{A}(> E) \iff E' \in \mathfrak{D}(> E)$. So it suffices to consider $\mathfrak{D}(> E)$.

<u>CASE A:</u> $Z \le E$. Then let $\tilde{E}$ be the image of E in $\tilde{S}$ and consider:

$$\tilde{E}^{\perp} = \tilde{E} \oplus \tilde{S}_0.$$

Then $\tilde{S}_0 = \tilde{S}_1 \oplus Rad(\tilde{S})$, with $\tilde{S}_1$ nondegenerate by (19.3) on pg. 77 in [A1] when $S = T$ (that is, $Rad(\tilde{S}) = 0$), and by Lemma 7.5 when $S = \mathbb{Z}/4\mathbb{Z} * T$. So when $S = T$, S_0 is extraspecial by (23.10) on pg. 109 in [A1]; and when $S = \mathbb{Z}/4\mathbb{Z} * T$, $S_0 = \mathbb{Z}/4\mathbb{Z} * T_0$ with T_0 extraspecial by Lemma 7.9. Thus we have:

$$\begin{aligned} dim(\tilde{S}_0) &= dim(\tilde{E}^{\perp}) - dim(\tilde{E}) = dim(\tilde{S}) - 2dim(\tilde{E}) \\ &= \begin{cases} 2(k-t), & \text{if } S = T, \\ 2(k-t)+1, & \text{if } S = \mathbb{Z}/4\mathbb{Z} * T. \end{cases} \end{aligned}$$

Also note that $\tilde{x} \in \tilde{E}^{\perp} \subseteq \tilde{S}$ if and only if $[x,u] = 1 \ \forall\ u \in E$ if and only if $\tilde{E}^{\perp} = \widetilde{C_S(E)}$. So we have:

$$(*) \qquad C_S(E) = E * S_0 \qquad \text{(central product).}$$

By $(*)$ we have $C_S(E)/E \cong S_0/Z$, which implies:

$$\mathfrak{D}(> E) \equiv \mathcal{A}_p(C_S(E))(> E) \cong \mathcal{A}_p(S_0)(> Z).$$

By Lemma 8.2 we know that $\mathcal{A}_p(S_0)(> Z)$ is $(m_p(S_0) - 3)$- connected. So we compute $m_p(S_0)$ in the different cases:

$$S = \mathbb{Z}/4\mathbb{Z} * T \implies dim(\tilde{S}_0) = 2(k-t)+1 \implies m_p(S_0) = k-t+1 = m_p(S) - t$$

$$S = T \implies dim(\tilde{S}_0) = 2(k-t).$$

For $S = T$ we can now compute the value of $m_p(S_0)$ in terms of $m_p(S)$ using Table 1. Bear in mind that if $\tilde{S}$ is hyperbolic, then $\tilde{S}_0$ is also hyperbolic. Thus we have:

Table 2.

p	$\tilde{S}$	$m_p(\tilde{S}_0)$	$m_p(S)$	$\tilde{S}_0$	$m_p(\tilde{S}_0)$
odd	–	–	$k-t+1$	$k+1$	$m_p(S)-t$
2	hyperbolic	hyperbolic	$k-t+1$	$k+1$	$m_p(S)-t$
2	not hyperbolic	hyperbolic	$k-t+1$	k	$m_p(S)-t+1$
2	not hyperbolic	not hyperbolic	$k-t$	k	$m_p(S)-t$

From Table 2 we see that $m_p(S_0) - 3 \geq m_p(S) - t - 3$ in all cases. So in all cases $\mathcal{A}_p(S_0)(> Z) \cong \mathfrak{D}(> E)$ is at least $(m_p(S) - t - 3)$-connected.

<u>CASE B:</u> $Z \nleq E$. Consider EZ, then as in Case A, $C_S(EZ) \cong EZ * S_0 = E \times S_0$ (as $S_0 \cap E = \{1\}$), here S_0 is as defined in Case A. Now $x \in C_S(EZ)$ if and only if $x(uz) = (uz)x\ \forall\ u \in E,\ \forall\ z \in Z$ if and only if $x \in C_S(E)$. Thus, $C_S(E) \cong E \times S_0$. Thus, $\mathfrak{D}(> E) \cong \mathcal{A}_p(S_0)$. But, by Lemma 2.2 on pg. 105 in [Q], $\mathcal{A}_p(S_0)$ is contractible. Thus, $\mathfrak{D}(> E)$ is contractible.

Therefore, we have that $\mathcal{A}(> E) \cong \mathfrak{D}(> E)$ is $(m_p(S) - h(E) - 3)$-connected. □

Now let $G = O_{p'}(G)S$ where S is as above with the added conditions that $S \in \mathrm{Syl}_p(G)$, and that $O_{p'}(G)$ is solvable. Note that by the definition of G and S, $m_p(G) = m_p(S)$.

Then the map:

$$f : G \to G/O_{p'}(G) \cong S$$

induces a map of posets:

$$f : \mathcal{A}_p(G) \to \mathcal{A}_p(S).$$

Lemma 8.4. *If* $x \lneq y$ *in* $\mathcal{A}_p(G)$, *then* $f(x) lneq f(y)$.

Proof. This follows from the fact that $x \cap O_{p'}(G) = y \cap O_{p'}(G) = \{1\}$. □

For each $x \in \mathcal{A}_p(G)$, let $\tilde{x} = f(x)$. Define $f|\tilde{a}$ to be $f^{-1}(\mathcal{A}_p(S)(\leq \tilde{a}))$.

Lemma 8.5. $f|\tilde{a}$ *is* $(h(\tilde{a})-1)$*-connected.*

Proof. $f^{-1}(\mathcal{A}_p(S)(\leq \tilde{a})) = \{b \in \mathcal{A}_p(G) \mid \tilde{b} \leq \tilde{a}\} = \mathcal{A}_p(O_{p'}(G) \cdot a)$. But, as $O_{p'}(G)$ is solvable, by Theorem 11.2 on pg. 123 in [Q], $\mathcal{A}_p(O_{p'}(G) \cdot a) = f|\tilde{a}$ is $(h(\tilde{a})-1)$-connected. □

Theorem 8.6. $\mathcal{A}_p(G)$ *is* $(m_p(G)-1)$*-spherical.*

Proof. We know that $\mathcal{A}_p(G)$ is $(m_p(G)-1)$-dimensional. Now, by Lemma 8.3, for each $\tilde{x} \in \mathcal{A}_p(S)$:

$$\mathcal{A}_p(S)(> \tilde{x}) \text{ is } (m_p(S) - h(\tilde{x}) - 3) = (m_p(G) - 2 - h(\tilde{x}) - 1)\text{-connected.}$$

This, along with Lemmas 8.4 and 8.5, shows that:

$$f : \mathcal{A}_p(G) \to \mathcal{A}_p(S)$$

satisfies all of the criteria of Theorem A. As $\mathcal{A}_p(S)$ is contractible by Lemma 2.2 on pg. 105 in [Q], it is certainly $(m_p(G)-2)$-connected. Therefore, by Theorem A, $\mathcal{A}_p(G)$ is $(m_p(G)-2)$-connected, as claimed. □

Chapter 3

Techniques for computing homotopy and homology of simplicial complexes

In Chapter 3 much of the terminology and tools for computing homotopy and homology of simplicial complexes – primarily order complexes and geometric complexes – is developed. Our primary goal in the following chapters will be to compute n-connectedness of the Quillen complex of some of the classical groups of Lie type. It is the nature of this area of study that the computation of n-connectedness of the Quillen complex of a group will often depend on our knowledge of the n-connectedness of some simplicial complex that admits action by the group. Therefore, in this chapter we will consider the homology and homotopy of various geometric complexes. Our knowledge about the n-connectedness of these complexes will then be utilized in subsequent chapters to compute n-connectedess of the Quillen complexes of the classical groups. There are various methods of computing homotopy and homology of simplicial complexes, and in this chapter some of these methods shall be demonstrated.

In Section 9 we recall some of the basic definitions and facts about simplicial complexes. We also include some results about the join of complexes and p-subgroup complexes of products of groups. In Section 10 we demonstrate the technique of computing simple connectivity of a geometric complex using closed subsets and triangulability of the graph of the complex, as developed in [AS1]. We also give a complete characterization of order complexes. In Section 11 we use the Nerve Theorem to identify an $(n-2)$-connected complex with an n-dimensional vector space. In Section 12 we prove a couple of technical results about subspaces of unitary spaces of dimension ≥ 3. Finally, in Section 13 we use Aschbacher and Segev's result on the fundamental group of a string geometry, along with the results from section 12, to show that the order complex of proper nondegenerate subspaces of an unitary space – ordered by inclusion – is simply connected.

SECTION 9 : FACTS ABOUT SIMPLICIAL COMPLEXES AND JOIN OF COMPLEXES

In this section we recall some basic notation and facts about simplicial complexes. We also prove certain results about the join of complexes and p-subgroup complexes of products of groups (refer to section 1 for the definition of an abstract simplicial complex and the definition of an order complex).

Given any finite simplicial complex K, we can consider the poset $sd(K)$ consisting of the simplices of K ordered by inclusion. The simplices of the order complex of $sd(K)$ consist of finite chains $\{s_0 < s_1 < \cdots < s_r\}$ where the s_i $(0 \leq i \leq r)$ are simplices of K. Identifying s_i $(0 \leq i \leq r)$ with the *barycenter* of s_i, defined on pg. 117 in [Sp], we see that the order complex of $sd(K)$ can be identified with the *first barycentric subdivision of* K, as defined on pg. 123 in [Sp].

Given posets X, Y and maps of posets $f, g : X \to Y$, we say that f and g are *homotopic*, denoted $f \simeq g$, if their geometric realizations are homotopic as maps of topological spaces. Furthermore, given $f : X \to Y$ and $g : Y \to X$ maps of posets, if $f \circ g \simeq id : Y \to Y$ and $g \circ f \simeq id : X \to X$, then we say that f and g are *homotopy inverses* of each other and that X and Y are *homotopy equivalent.*

Lemma 9.1. *Given a simplicial complex K, $K \simeq sd(K)$. Here $\simeq$ signifies that the two complexes are homotopy equivalent.*

Proof. This is Lemma (6.4.b) on pg. 126 in [Am]. □

Given posets X and Y, the *product of X and Y*, denoted $X \times Y$, is the poset on the ordered pairs $\{(x, y) \mid x \in X, y \in Y\}$ with $(x, y) \leq (x', y')$ if and only if $x \leq x'$ and $y \leq y'$. By (1.2) on pg. 102 in [Q]:

$$|X \times Y| \simeq |X| \times |Y|,$$

as topological spaces. Also, for posets X and Y the *join of X and Y*, denoted $X * Y$, is the poset on the disjoint union of X and Y equipped with an ordering which agrees with the ordering on X and the ordering on Y and such that every element of X is less than every element of Y.

By (1.3) on pg. 103 in [Q] if $f(x) \leq g(x)\ \forall\ x \in X$, then $f \simeq g$. In particular, if $x_0 \in X$ is the unique minimal (or maximal) element and $f_0 : X \to X$ is defined by $f(x) = x_0\ \forall\ x \in X$, then $f_0 \simeq id$ where id is the identity map on X. Thus a poset with a unique minimal (or maximal) element is homotopic to a point; that is, $X \simeq \{x_0\}$. So, following Quillen, we call such a poset *conically contractible.*

Given a poset X, let the *cone of* X, denoted CX, be the poset on the vertex set $X \cup \{0\}$ with partial order given by the partial order on X and $0 < x\ \forall\ x \in X$. The *double cone of* X, denoted $\overline{X}$, is the poset on the vertex set $X \cup \{0, \infty\}$ with partial order given by the partial order on X and $0 < x < \infty\ \forall\ x \in X$. Note that both CX and $\overline{X}$ are conically contractible.

Finally, given two simplicial complexes X and Y, the *join of* X *and* Y, denoted $X * Y$, is the simplicial complex with vertex set the disjoint union of the vertices of X and Y and k-simplices given by the disjoint union $\sigma * \tau$ of an i-simplex σ of X and a j-simplex τ of Y where $-1 \leq i, j$ and $k = i + j + 1$. Here we use the convention that the empty set is the unique -1-simplex of both X and Y.

Lemma 9.2. *Given posets* X *and* Y, *we have* $\mathfrak{O}(X * Y) \simeq \mathfrak{O}(X) * \mathfrak{O}(Y)$, *where* $\mathfrak{O}(X)$ *is the order complex of* X. *Furthermore:*

$$X * Y \simeq CX \times CY - \{(0,0)\}.$$

Proof. This is proposition 1.9 on pg. 105 in [Q]. □

Lemma 9.3. *Given posets* $X_1, X_2, \ldots, X_n$, *we have:*

$$X_1 * X_2 * \cdots * X_n \simeq CX_1 \times CX_2 \times \cdots \times CX_n - \{(0,0,\ldots,0)\}.$$

Proof. First of all, note that $C(X_1 * \cdots * X_n) \simeq CX_1 \times \cdots \times CX_n$ since both are conically contractible, $(0, \ldots, 0)$ being the unique minimal element of $CX_1 \times \cdots \times CX_n$. Now:

$$\begin{aligned} X_1 * \cdots * X_n &\simeq C(X_1 * \cdots * X_{n-1}) \times CX_n - \{(0,0)\} && \text{by Lemma 9.2,} \\ &\simeq CX_1 \times \cdots \times CX_{n-1} \times CX_n - \{(0,\ldots,0)\} && \text{by the comment above.} \end{aligned}$$

Thus we have the lemma. □

Lemma 9.4. *Given groups $G_1, G_2, \dots, G_n$:*

$$\mathcal{A}_p(G_1 \times G_2 \times \cdots \times G_n) \text{ and } \mathcal{A}_p(G_1) * \mathcal{A}_p(G_2) * \cdots * \mathcal{A}_p(G_n)$$

are homotopy equivalent.

Proof. Let $\mathcal{A}_p(G_1 \times \cdots \times G_n) = \mathcal{A} \supseteq T = \{1 \neq A_1 \times \cdots \times A_n \mid A_i \in \mathcal{A}_p(G_i) \cup \{1\}\ (1 \leq i \leq n)\}$. If $r : \mathcal{A} \to T$ is given by $r(A) = pr_1(A) \times \cdots \times pr_n(A)$ (where $pr_i : G_1 \times \cdots \times G_n \to G_i$ are the canonical projection), then for all $A \in \mathcal{A}$, $A \leq r(A)$. So r and $i : T \to \mathcal{A}$, the inclusion map, are homotopy inverses. Thus, T and $\mathcal{A}$ are homotopy equivalent. On the other hand, T can be identified with the poset $C\mathcal{A}_p(G_1) \times C\mathcal{A}_p(G_2) \times \cdots \times C\mathcal{A}_p(G_n) - \{(0, \dots, 0)\}$. The result then follows from Lemma 9.3. □

Lemma 9.5. *Given $X_1, \dots, X_n$ nonempty simplicial complexes, $X_1 * \cdots * X_n$ is simply connected if $n \geq 3$ or $n = 2$ and X_1 or X_2 is connected.*

Proof. This follows from (2.1) on pg. 6 in [A2] and induction on n. □

Lemma 9.6. *Let X and Y be nonempty posets, and set $P = CX \times CY - \{(0,0)\}$. Identify X with the subposet $\{(x, 0) \mid x \in X\}$ via $x \mapsto (x, 0)$, and fix $x \in X$. Then $P(> x)$ is connected if $X(> x)$ is nonempty or Y is connected.*

Proof. Let $X' = X(> x)$, then:

$$\begin{aligned} P(> x) &= X(\geq x) \times CY - \{(x, 0)\} \\ &\cong CX' \times CY - \{(0,0)\} \quad (\text{as } X(\geq x) \cong CX') \\ &\simeq X' * Y \quad \text{by Lemma 9.2.} \end{aligned}$$

But, by (2.1) on pg. 6 in [A2], $X' * Y$ is connected if X' is nonempty or Y is connected. Thus we have the lemma. □

Lemma 9.7. *Let $f : P \to Q$ be a surjective map of posets, satisfying $f^{-1}(Q(\leq q))$ is connected for all $q \in Q$. Then P is connected if Q is connected.*

Proof. Given $p, p' \in P$, let $f(p) = q$, $f(p') = q'$, and $q = q_0 q_1 q_2 \dots q_n = q'$ be a path joining q and q' in Q. Let $p_i \in f^{-1}(q_i)$ for $0 \leq i \leq n$. It suffices to show that p_i and p_{i+1}

are connected in P for all $0 \le i \le n-1$. But, $p_i \in f^{-1}(Q(\le q_{i+1}))$ or $p_{i+1} \in f^{-1}(Q(\le q_i))$ $\forall\, 0 \le i \le n-1$; and our assumption that $f^{-1}(Q(\le q))$ is connected for all $q \in Q$ implies that p_i and p_{i+1} are connected. $\square$

Theorem 9.8. *Let $G = G_1 \times G_2$ be a finite group with $m_p(G) \ge 3$ and $\mathcal{A}_p(G_i) \ne \emptyset$ $(i = 1, 2)$. Furthermore, assume that $\mathcal{A}_p(G_i)$ is connected if $m_p(G_i) \ge 2$ $(i = 1, 2)$. Then, if $A \in \mathcal{A}_p(G)$ with $h(A) = 0$, then $\mathcal{A}_p(G)(> A)$ is connected.*

Proof. Let $\mathcal{A}_p(G) \supseteq T = \{1 \ne A_1 \times A_2 \mid A_i \in \mathcal{A}_p(G_i) \cup \{1\}\ (i = 1, 2)\}$. Define:

$$\begin{aligned} r : \mathcal{A}_p(G) &\to T \\ B &\mapsto pr_1(B) \times pr_2(B), \end{aligned}$$

where $pr_i : G \to G_i$ $(i = 1, 2)$ are the canonical projections.

Let $X = \mathcal{A}_p(G)(> A)$. Now one of two cases can occur:

<u>CASE I:</u> $A \notin T$. Then $r(A) > A$, set $Y = T(\ge r(A))$.

<u>CASE II:</u> $A \in T$. Then $r(A) = A$, set $Y = T(> r(A))$.

Restrict $r : X \to Y$ and note that it is a surjective map of posets, as $B \in Y$ implies $B \in X$. Also, for all $B \in Y$, we have:

$$\begin{aligned} r^{-1}(Y(\le B)) &= \{C \in X \mid r(C) \le B\} \\ &= \{C \in X \mid r(C) \le r(B)\} \quad (\text{as } r(B) = B) \\ &= X(\le B). \end{aligned}$$

Thus $r^{-1}(Y(\le B))$ is connected as it has a unique greatest element.

By Lemma 9.7, $\mathcal{A}_p(G)(> A)$ is connected if Y is connected. But in Case I, Y is conically contractible, so we are done. In Case II, $r(A) = A$ and $h(A) = 0$ imply $A \in \mathcal{A}_p(G_1)$ or $A \in \mathcal{A}_p(G_2)$. Assume without loss of generality that $A \in \mathcal{A}_p(G_1)$. Now if $m_p(G_1) > 1$, then $\mathcal{A}_p(G_1)(> A) \ne \emptyset$, and if $m_p(G_1) = 1$, then $m_p(G_2) \ge 2$ and $\mathcal{A}_p(G_2)$ is connected. Note that:

$$T \simeq C\mathcal{A}_p(G_1) \times C\mathcal{A}_p(G_2) - \{(0,0)\} \quad \text{and } Y = T(> A).$$

Then, by Lemma 9.6 and our observation about $\mathcal{A}_p(G_1)(> A)$ and $\mathcal{A}_p(G_2)$, $Y = T(> A)$ is connected. Thus $\mathcal{A}_p(G)(> A)$ is connected, as claimed. $\square$

Let P be a poset and $S \subseteq P$. We say that S is a *closed subposet* of P if $x \leq y$ and $y \in S$ implies that $x \in S$ for all $x, y \in P$.

Lemma 9.9. *Given a poset P and closed subposets R and Q if $P = R \cup Q$ as sets, then $P = R \cup Q$ as simplicial complexes.*

Proof. We need to show that given a simplex $\sigma = (x_0 < x_1 < \cdots < x_n)$, $\sigma \in R$ or $\sigma \in Q$. As $P = R \cup Q$ as sets, we can assume that $x_n \in R$. But then, as R is closed, $x_i \in R$ $\forall\ 0 \leq i \leq n$ and so $\sigma \in R$, as desired. $\square$

We fix the convention that the empty set is -2-connected. Note that this agrees with the definition of reduced homology of the empty set as given in section 1.

Theorem 9.10. *Let X_1 and X_2 be posets (possibly empty) which are n_1- and n_2-connected (respectively). Define $K = \overline{X}_1 \times \overline{X}_2 - \{(0,0),(\infty,\infty)\}$. Then K is (n_1+n_2+3)-connected.*

Proof. Let $K_1 = CX_1 \times \overline{X}_2 - \{(0,0)\}$ and $K_2 = \overline{X}_1 \times CX_2 - \{(0,0)\}$. By Lemma 9.2, $K_1 \cong X_1 * (X_2, \infty)$ and $K_2 \cong (X_1, \infty) * X_2$, where (X_i, ∞) is the (necessarily nonempty) poset on the vertex set $X_i \cup \{\infty\}$ with partial order given by the partial order on X_i and $x < \infty\ \forall\ x \in X_i$ $(i = 1, 2)$. (X_i, ∞) is conically contractible as it has a unique maximal element. Thus K_i is contractible (for $i = 1, 2$) by (2.6) on pg. 8 in [A2].

Note that $K = K_1 \cup K_2$ as sets, so if K_i $(i = 1, 2)$ are closed subposets then, by Lemma 9.9, $K = K_1 \cup K_2$ as simplicial complexes. By symmetry, it suffices to show that K_1 is a closed subposet of K. If $(x_1, x_2) \in K \setminus K_1$, then $x_1 = \infty$, which implies that $(x_1, x_2) \not\leq (y_1, y_2)$ for any $(y_1, y_2) \in K_1$ as $y_1 < \infty\ \forall\ y_1 \in CX_1$. Thus the K_i are closed subposets for $i = 1$ and 2.

By Lemma 9.2, $K_1 \cap K_2 = CX_1 \times CX_2 - \{(0,0)\} \cong X_1 * X_2$, which is $(n_1 + n_2 + 2)$-connected by (2.6) on pg. 8 in [A2]. K_1 and K_2 are open in K, so (K, K_1, K_2) is an exact triad by (17.1) on pg. 28 in [GH]. So we get an exact sequence (the Mayer-Vietoris sequence) as follows:

$$\cdots \to H_{q+1}(K) \to H_q(K_1 \cap K_2) \to H_q(K_1) \oplus H_q(K_2) \to H_q(K) \to \cdots .$$

Since $K_1 \cap K_2$ is nonempty, the sequence terminates in:

$$\cdots \to H_1(K) \to \tilde{H}_0(K_1 \cap K_2) \to \tilde{H}_0(K_1) \oplus \tilde{H}_0(K_2) \to \tilde{H}_0(K) \to 0,$$

by remark (17.9) on pg. 100 in [GH].

Now the contractibility of K_i implies $H_q(K_i) = 0 = \tilde{H}_0(K_i)\ \forall\ q \geq 1$ $(i = 1,2)$. Also, $K_1 \cap K_2$ $(n_1 + n_2 + 2)$-connected implies $H_q(K_1 \cap K_2) = 0 = \tilde{H}_0(K_1 \cap K_2)\ \forall\ 1 \leq q \leq (n_1 + n_2 + 2)$. Thus we have:

$$\cdots \to 0 \to \tilde{H}_0(K) \to 0 \ \text{ and} \cdots \to 0 \to H_q(K) \to 0 \quad \forall\ 1 \leq q \leq\ (n_1 + n_2 + 3).$$

So $\tilde{H}_0(K) = 0 = H_q(K)\ \forall\ 1 \leq q \leq\ (n_1 + n_2 + 3)$. Thus it remains to show that K is simply connected when $n_1 + n_2 + 3 \geq 1$.

So assume that $n_1 + n_2 + 3 \geq 1$. As above, $K_1 \cap K_2$ is $(n_1 + n_2 + 2)$-connected. Since $n_1 + n_2 + 2 \geq 0$, $K_1 \cap K_2$ is connected. Also, as the K_i are connected, K is connected. Therefore, by Van Kampen's Theorem – refer to pg. 138 in [Am] – and the fact that K_1 and K_2 are contractible, K is simply connected.

Thus we have shown that K is $(n_1 + n_2 + 3)$-connected, as desired. □

Corollary 9.11. *Given (possibly empty) posets $X_1, X_2, \ldots, X_n$, with X_i m_i-connected, define $K = \overline{X}_1 \times \cdots \times \overline{X}_n - \{(0,\ldots,0),(\infty,\ldots,\infty)\}$. Then K is $(\sum_{i=1}^{n} m_i + (n-1)3)$-connected.*

Proof. Let $Y_1 = \overline{X}_1 \times \cdots \overline{X}_{n-1} - \{(0,\ldots,0),(\infty,\ldots,\infty)\}$, and $Y_2 = X_n$. By induction on n, Y_1 is $(\sum_{i=1}^{n-1} m_i + (n-2)3)$-connected. Also, $K = \overline{Y}_1 \times \overline{Y}_2 - \{(0,0),(\infty,\infty)\}$. Then we have the desired result since, by Theorem 9.10, $\overline{Y}_1 \times \overline{Y}_2 - \{(0,0),(\infty,\infty)\}$ is $(\sum_{i=1}^{n-1} m_i + (n-2)3 + m_n + 3) = (\sum_{i=1}^{n} m_i + (n-1)3)$-connected, as claimed. □

Remark. *It is important to note that in Corollary 9.11 we allow some (or even all) of the posets in the collection under consideration to be empty. Later on we shall use the above result without explicit reference to this admission of empty posets.*

Recall the definition of a Cohen-Macaulay complex given in section 1.

Theorem 9.12. *Let $G = G_1 \times G_2$ be a finite group with $\mathcal{A}_p(G_1)$ Cohen-Macaulay of dimension n_1 and $\mathcal{A}_p(G_2)$ contractible. Furthermore, assume that $\mathcal{A}_p(G_2)(> a)$ is contractible for all $a \in \mathcal{A}_p(G_2)$, for which $h(a) \neq n_2 = dim(\mathcal{A}_p(G_2))$. Then $\forall\, a \in \mathcal{A}_p(G)$, $\mathcal{A}_p(G)(> a)$ is $(m_p(G) - 2 - h(a) - 1)$-connected.*

Proof. Let $T = \{a_1 \times a_2 \neq 1 \mid a_i \in \mathcal{A}_p(G_i) \cup \{1\}\ (i = 1,2)\}$. Let $pr_i : \mathcal{A}_p(G) \to \mathcal{A}_p(G_i)$ be the maps induced by the canonical projections. Finally, define the map $r : \mathcal{A}_p(G) \to T$ by $a \mapsto pr_1(a) \times pr_2(a)$. Note that for all $a \in \mathcal{A}_p(G), a \leq r(a)$.

Now fix an $a \in \mathcal{A}_p(G)$ and let $X = \mathcal{A}_p(G)(> a)$. If $a < r(a)$, let $Y = T(\geq r(a))$; otherwise, set $Y = T(> r(a))$. The inclusion map $i : Y \to X$ and $r : X \to Y$ are homotopy inverses of each other, since $\forall\, y \in Y\ r \circ i(y) = y$ and $\forall\, x \in X\ i \circ r(x) \geq x$. So it suffices to show Y is $(m_p(G) - 2 - h(a) - 1)$-connected.

If $a < r(a)$, then $Y = T(\geq r(a))$ has a unique minimal element and thus is conically contractible. So assume that $a = r(a) = a_1 \times a_2$. Now $n_1 + n_2 = m_p(G_1) - 1 + m_p(G_2) - 1 = m_p(G) - 2$, and $h(a_1) + h(a_2) = h(a) - 1$. We want Y to be $(m_p(G) - 2 - h(a) - 1) = (n_1 + n_2 - h(a_1) - h(a_2) - 2)$-connected. Fix the notation that if $a_i = \emptyset$ $(i = 1,2)$, then $\mathcal{A}_p(G_i)(> a_i) = \mathcal{A}_p(G_i)$ and $h(a_i) = -1$.

$$\begin{aligned} Y = T(> a_1 \times a_2) &= \mathcal{A}_p(G_1)(\geq a_1) \times \mathcal{A}_p(G_2)(\geq a_2) - \{a_1 \times a_2\} \\ &\cong C\mathcal{A}_p(G_1)(> a_1) \times C\mathcal{A}_p(G_2)(> a_2) - \{(0,0)\} \\ &\simeq \mathcal{A}_p(G_1)(> a_1) * \mathcal{A}_p(G_2)(> a_2) \quad \text{(by Lemma 9.2).} \end{aligned}$$

Two cases occur:

<u>Case I:</u> $h(a_2) = n_2$. Then $Y \cong \mathcal{A}_p(G_1)(> a_1)$, which is $(n_1 - h(a_1) - 2)$-connected by the assumption that $\mathcal{A}_p(G_1)$ is Cohen-Macaulay. But then Y is $(n_1 + n_2 - h(a_1) - h(a_2) - 2)$-connected as desired.

Case II: $h(a_2) < n_2$. If $h(a_1) = n_1$, then $Y \cong \mathcal{A}_p(G_2)(> a_2)$, which we assumed to be contractible. Otherwise, both $\mathcal{A}_p(G_1)(> a_1)$ and $\mathcal{A}_p(G_2)(> a_2)$ are nonempty. Furthermore $\mathcal{A}_p(G_2)(> a_2)$ is contractible. Thus, by (2.6) on pg. 8 in [A2] and Lemma 9.2, Y is contractible.

Therefore Y, and hence $X = \mathcal{A}_p(G)(> a)$, is $(m_p(G) - 2 - h(a) - 1)$-connected, as claimed. □

Next we include a result about a closed subposet of the product of posets. Given nonempty posets X and Y, recall that a subset $F \subseteq X \times Y$ is a closed subposet if and only if $(x, y) \leq (x', y')$ and $(x', y') \in F$ implies $(x, y) \in F$. This result generalizes Proposition 1.7 and Corollary 1.8 on pg. 104 in [Q], and was noted by Aschbacher on pg. 1 of [A3].

Theorem 9.13. *Let X and Y be nonempty posets and F a closed subposet of $X \times Y$. For each $x \in X$, let $F_x = \{y' \in Y \mid (x, y') \in F\}$. Similarly, for each $y \in Y$, let $F_y = \{x' \in X \mid (x', y) \in F\}$. If for all $x \in X$ and $y \in Y$, F_x and F_y are n-connected. Then:*

X is n-connected if and only if Y is n-connected.

Proof. It suffices to prove the following:

> If F_x is n-connected $\forall\ x \in X$, then F is n-connected if and only if X is n-connected.

Since then, under the assumptions of the theorem and by symmetry, we have: Y is n-connected if and only if F is n-connected, which in turn is n-connected if and only if X is n-connected.

Define $p : F \to X$, by $(x, y) \mapsto x$. For each $x \in X$, let:

$$p|x = \{(x', y) \in F \mid x' \leq x\} = p^{-1}(X(\leq x)).$$

Then, by Proposition 7.6 on pg. 115 in [Q], if $p|x$ is n-connected $\forall\ x \in X$, then F is n-connected if and only if X is n-connected. So we are reduced to showing:

$$(*) \qquad p|x \text{ is } n\text{-connected } \forall\ x \in X.$$

Fix $x \in X$ and consider the following maps: $u : F_x \to p|x$ defined by $(x, y) \mapsto (x, y)$, and $v : p|x \to F_x$ defined by $(x', y) \mapsto (x, y)$. Then, by the definition of $p|x$ and F_x, both u and v are well-defined maps of posets. Furthermore, given $(x, y) \in F_x$, we have: $v \circ u((x, y)) = (x, y)$; that is, $v \circ u = id : F_x \to F_x$. Also, given $(x', y) \in p|x$, $u \circ v((x', y)) = (x, y) \geq (x', y)$. Thus, by (1.3) on pg. 103 in [Q], $u \circ v \simeq id : p|x \to p|x$. Therefore, $F_x \simeq p|x$. Since we assumed F_x is n-connected for each $x \in X$, $p|x$ is n-connected.

Thus, by $(*)$ and the discussion preceding $(*)$, the theorem holds. □

The above theorem will be used frequently in chapters 4, 5, and 6 to compare the Quillen complex of a finite group G to a simplicial complex K on which G acts. This comparison will be carried out, by considering:

$$sd(K) \times \mathcal{A}_p(G) \supseteq F = \{(x, A) \mid x \in Fix(A)\},$$

where $sd(K)$ is the first barycentric subdivision of K. And $Fix(A)$ is the subcomplex of $sd(K)$ defined on those simplices of K on which A acts as the identity automorphism. In algebraic topology this complex is denoted by K^A.

Now given $x \in sd(K)$, $x = (v_i \mid i \in I)$ where I is an indexing set and v_i are vertices of K for all $i \in I$. Then $(x, A) \in F \iff Av_i = v_i \; \forall \; i \in I$. Thus we have:

Remark 9.14. *F is a closed subposet of $sd(K) \times \mathcal{A}_p(G)$ since given $(x, A) \leq (x', A') \in sd(K) \times \mathcal{A}_p(G)$ such that $(x', A') \in F$ we have:*

(1) *$x = (v_j \mid j \in J)$ and $x' = (v_i \mid i \in I)$ with $J \subseteq I$, and*

(2) *$A'v_i = v_i \; \forall \; i \in I \implies Av_i = v_i \; \forall \; i \in I$.*

Hence, $Av_j = v_j \; \forall \; j \in J$, which implies $(x, A) \in F$.

Finally we state a well-known result about the Quillen complex of certain factor groups:

Theorem 9.15. *Given a finite group G such that $O_{p'}(G) \leq Z(G)$ where p is a prime dividing $|G|$, the canonical homomorphism $\pi : G \to G/O_{p'}(G)$ induces an isomorphism between $\mathcal{A}_p(G)$ and $\mathcal{A}_p(G/O_{p'}(G))$ (in the category of posets).* □

SECTION 10 : TRIANGULABILITY OF GRAPHS AND SIMPLE CONNECTIVITY

In this section we demonstrate one of the more useful methods for computing simple connectivity of simplicial complexes. This method was introduced by Aschbacher and Segev in [AS1]; and though we include some of the basic notions from that paper, we urge the reader to refer to that paper for the details.

Given a graph Δ, let $P = P(\Delta)$ be the set of paths in Δ. For every path $p = x_0x_1 \dots x_r \in P$, let $org(p) = x_0$, the *origin* of p; $end(p) = x_r$, the *end* of p; and $p^{-1} = x_rx_{r-1} \dots x_0$. Given paths $p = x_0 \dots x_r$ and $q = y_0 \dots y_s$ if $x_r = y_0$, then we can *concatenate* p and q to get $pq = x_0 \dots x_{r-1}y_0 \dots y_s$. The *length of a path* $p = x_0 \dots x_r$, denoted $l(p)$, is r. A path $p = x_0 \dots x_r$ is a *cycle* if $x_0 = x_r$.

A set S of cycles of P is *closed* if S satisfies the following six properties:

(1) $rr^{-1} \in S$ for all $r \in P$.

(2) If $p \in S$, then $p^{-1} \in S$.

(3) If $p, q \in S$ with $org(p) = end(p) = org(q)$, then $pq \in S$.

(4) If $p \in S$, then $r^{-1}pr \in S$ for each $r \in P$ with $org(r) = end(p)$.

(5) If p is a cycle and $r \in P$ with $org(p) = end(r)$ and $r^{-1}rp \in S$, then $p \in S$.

(6) $xx \in S$ for all $x \in \Delta$ (so we allow a "loop" at each vertex of the graph).

The intersection of closed subsets is closed, so given a set of cycles T of P, we define the *closure of* T to be the intersection of all closed subsets containing T. For any positive integer n, define $\mathfrak{C}_n(\Delta)$ to be the closure of the set of all cycles of length at most n. Δ is said to be *triangulable* if all cycles of Δ are contained in $\mathfrak{C}_3(\Delta)$.

Given two points $x, y \in \Delta$, the *distance between* x *and* y, denoted $d_\Delta(x,y)$ (or simply $d(x,y)$), is given by:

$$d_\Delta(x,y) = \min_p\{l(p) \mid p \text{ is a path joining } x \text{ and } y\},$$

with $d_\Delta(x,y) = \infty$ if no such path exists. The *diameter* of Δ, denoted $diam(\Delta)$, is defined by:

$$diam(\Delta) = \max_{x,y\in\Delta}\{d(x,y)\}.$$

Note that Δ is connected if and only if $diam(\Delta) < \infty$. Given integers n, m with $n \geq 2$, define $|m|_n = r$ where $0 \leq r \leq \frac{n}{2}$ and $m \equiv \pm r (\text{mod } n)$. Define a cycle $p = x_0 \dots x_n$ to be an *n-gon* if:

$$d(x_i, x_j) = |i - j|_n \text{ for all } 0 \leq i, j \leq n.$$

Lemma 10.1. *Given a graph Δ with $diam(\Delta) = d$, Δ is triangulable if and only if all r-gons are in $\mathfrak{C}_3(\Delta)$ for all $r \leq 2d + 1$.*

Proof. This follows from (3.3) on pg. 303 in [AS1]. □

Given a graph Δ, a *clique* is a finite set of vertices of Δ such that given any pair of vertices in the set they are connected to one another. Given Δ the *clique complex of* Δ, denoted $K(\Delta)$, is the simplicial complex with vertex set equal to the vertices of Δ and simplices the cliques of Δ. Given a simplicial complex K the *graph of* K, denoted $\Delta(K)$, is the graph on the vertices of K with a pair of vertices adjacent if and only if they form a 1-simplex in K. Note that a simplicial complex K is a subcomplex of $K(\Delta(K))$ and $K = K(\Delta(K))$ if and only if K is a clique complex.

Lemma 10.2. *Given a clique complex K, K is simply connected if and only if $\Delta(K)$ is triangulable. In particular, if $diam(\Delta(K)) = d$ then K is simply connected if and only if every r-gon is in $\mathfrak{C}_3(\Delta)$ for all $r \leq 2d + 1$.*

Proof. The first statement follows from (4.3.3) and Remark 5 on pg. 307 in [AS1] along with our observation that $K = K(\Delta(K))$ if and only if K is a clique complex. The second statement is a restatement of the first in light of Lemma 10.1. □

Fix an integer $d \geq 1$, and let V be a $3d$-dimensional vector space over some finite field. In this section we associate a geometric complex with V and use Lemma 10.2 to show that the complex is simply connected. First we need a couple of results from linear algebra.

Lemma 10.3. *Given a $2d$-dimensional vector space W and two d-dimensional subspaces W_1, W_2, we can find a third d-dimensional subspace W_3, such that $W_3 \cap W_i = 0$ for i=1,2.*

Proof. Let $\{a_1, \dots, a_r\}$ be a basis for $W_1 \cap W_2$. Choose $\{u_1, \dots, u_{d-r}\} \subseteq W_1$, such that $\{a_i, u_j \mid 1 \leq i \leq r, 1 \leq j \leq d - r\}$ is a basis for W_1. Similarly, pick $\{v_1, \dots, v_{d-r}\}$ for W_2.

Finally, choose $\{b_1, \ldots, b_n\}$ such that $\{a_i, u_j, v_k, b_l \mid 1 \leq i \leq r; 1 \leq j, k \leq d-r; 1 \leq l \leq n\}$ is a basis for W. Note that $n = 2d - r - 2(d-r)$.

Notice that $X = \{b_i, u_j + v_j \mid 1 \leq i \leq n, 1 \leq j \leq d-r\}$ is linearly independent. Set $W_3 = < X >$; then clearly, $W_3 \cap W_i = 0$ $(i = 1, 2)$. So we only need to check that W_3 has the correct dimension. But that follows from the fact that $n + d - r = 2d - r - 2(d-r) + d - r = 2d - r - (d-r) = d$. Hence we have the claim. $\square$

Lemma 10.4. *Given V as above and three 2d-dimensional subspaces V_1, V_2 and V_3, we can find a d-dimensional subspace, U, of V such that $U \cap V_i = 0$ $(1 \leq i \leq 3)$.*

Proof. Let $V_0 = < V_i \mid 1 \leq i \leq 3 >$, and pick an ordered basis for V as follows:

Let $\{a_1, \ldots, a_n\}$ be a basis for $V_1 \cap V_2 \cap V_3$. Choose $\{b_1^{12}, \ldots, b_{n_{12}}^{12}\}$ such that $\{a_i, b_j^{12} \mid 1 \leq i \leq n, 1 \leq j \leq n_{12}\}$ is a basis for $V_1 \cap V_2$. Similarly, choose $\{b_1^{13}, \ldots, b_{n_{13}}^{13}\}$ for $V_1 \cap V_3$ and $\{b_1^{23}, \ldots, b_{n_{23}}^{23}\}$ for $V_2 \cap V_3$.

Next pick $\{c_1^1, \ldots, c_{m_1}^1\}$ such that:

$$\{a_i, b_j^{12}, b_k^{13}, c_l^1 \mid 1 \leq i \leq n, 1 \leq j \leq n_{12}, 1 \leq k \leq n_{13}, 1 \leq l \leq m_1\}$$

is a basis for V_1. Similarly, pick $\{c_1^2, \ldots, c_{m_2}^2\}$ for V_2 and $\{c_1^3, \ldots, c_{m_3}^3\}$ for V_3. Then $B = \{a_i, b_{j_{kp}}^{kp}, c_{j_l}^l \mid 1 \leq i \leq n, 1 \leq j_{kp} \leq n_{kp}, 1 \leq j_l \leq m_l\}$ is a basis for V_0. Finally, pick $E = \{e_1, \ldots, e_r\}$ in $V - V_0$, such that $B \cup E$ is an ordered basis for V. This notation is summarized in *Figure 1* below:

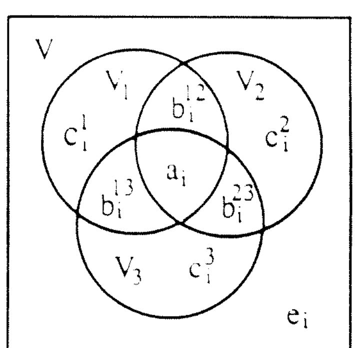

Figure 1

By definition we have:

$$\left.\begin{aligned} n+n_{12}+n_{13}+m_1&=2d\\ n+n_{12}+n_{23}+m_2&=2d\\ n+n_{13}+n_{23}+m_3&=2d \end{aligned}\right\} \implies \left.\begin{aligned} n_{13}+m_1&=n_{23}+m_2\\ n_{12}+m_1&=n_{23}+m_3\\ n_{12}+m_2&=n_{13}+m_3 \end{aligned}\right\} \qquad (*)$$

$$r=3d-(n+n_{12}+n_{13}+n_{23}+m_1+m_2+m_3).$$

Let $d_1=min\{n_{12},m_3\}, d_2=min\{n_{13},m_2\}$ and $d_3=min\{n_{23},m_1\}$.

<u>CASE I:</u> $d_1=n_{12}$. So $n_{12}\le m_3$, and by $(*)$, $n_{23}\le m_1$ and $n_{13}\le m_2$ so $d_2=n_{13}$ and $d_3=n_{23}$. Let:

$$B_1=\{b_i^{12}+c_i^3 \mid 1\le i\le n_{12}\}\cup\{b_j^{13}+c_j^2 \mid 1\le j\le n_{13}\}\cup\{b_k^{23}+c_k^1 \mid 1\le k\le n_{23}\}.$$

Note that by $(*)$, $m_2-n_{13}=m_1-n_{23}=m_3-n_{12}$, so let:

$$B_2=\{c^1_{n_{23}+j}+c^2_{n_{13}+j} \mid\le j\le m_1-n_{23}\}\cup\{c^1_{n_{23}+k}+c^3_{n_{12}+k} \mid 1\le k\le m_3-n_{12}\}.$$

Let $U=< E\cup B_1\cup B_2 >$.

Claim 1: $E\cup B_1\cup B_2$ is a basis for U.

As E and B are linearly independent it suffices to show that $B_1\cup B_2$ is linearly independent. Let $\alpha_1,\dots,\alpha_5\in k$ such that:

$$\alpha_1(b_i^{12}+c_i^3)+\alpha_2(b_j^{13}+c_j^2)+\alpha_3(b_k^{23}+c_k^1)+\alpha_4(c^1_{n_{23}+l}+c^2_{n_{13}+l})+\alpha_5(c^1_{n_{23}+m}+c^3_{n_{12}+m})=0.$$

But as $\{b^{kp}_{j_{kp}},c^l_{j_l}\}$ are linearly independent, $\alpha_1=\dots=\alpha_5=0$. $\square$

Thus, U is a subspace of dimension:

$$r+n_{12}+n_{13}+n_{23}+m_1-n_{23}+m_3-n_{12}=r+n_{13}+m_1+m_3$$
$$=3d-(n+n_{12}+n_{23}+m_2)=3d-2d=d \qquad \text{(equalities given by } (*)).$$

Claim 2: $U\cap V_i=0\ (1\le i\le 3)$.

I prove this claim for $i = 1$; the other two cases follow analogously. Let $v \in U \cap V_1$. Then:

$$v = \sum_{i=1}^{r} \alpha_i e_i + \sum_{j=1}^{n_{12}} \beta_j (b_j^{12} + c_j^3) + \sum_{k=1}^{n_{13}} \gamma_k (b_k^{13} + c_k^2) + \sum_{l=1}^{n_{23}} \delta_l (b_l^{23} + c_l^1) +$$

$$\sum_{m=1}^{m_1 - n_{23}} \eta_m (c^1_{n_{23}+m} + c^2_{n_{13}+m}) + \sum_{p=1}^{m_1+n_{23}} \rho_p (c^1_{n_{23}+p} + c^3_{n_{12}+p})$$

$$= \sum_{i=1}^{n} \alpha_i' a_i + \sum_{j=1}^{n_{12}} \beta_j' b_j^{12} + \sum_{k=1}^{n_{13}} \gamma_k' b_k^{13} + \sum_{l=1}^{m_1} \delta_l' c_l^1 .$$

But equating coefficients gives $v = 0$. □

Thus when $d_1 = n_{12}$, we have proven the claim.

<u>CASE II:</u> $d_1 = m_3$. Then by $(*)$, $d_2 = m_2$, and $d_3 = m_1$. But then setting:

$B_1 = \{b_i^{12} + c_i^3 \mid 1 \le i \le m_3\} \cup \{b_j^{13} + c_j^2 \mid 1 \le j \le m_2\} \cup \{b_k^{23} + c_k^1 \mid 1 \le k \le m_1\}$ and

$B_2 = \{b^{12}_{m_3+j} + b^{13}_{m_2+j} \mid 1 \le j \le n_{12} - m_3\} \cup \{b^{12}_{m_3+k} + b^{23}_{m_1+k} \mid 1 \le k \le n_{23} - m_1\}$;

we see that, as in Case I, $U =< E \cup B_1 \cup B_2 >$ satisfies the claim of this lemma. □

Associate the following geometric complex with V (refer to section 6 for the definition of geometric complexes):

Let $\Gamma_1 = \{U \le V \mid dim(U) = d\}, \Gamma_2 = \{U \le V \mid dim(U) = 2d\}$ and let Λ be the set of triples of the form $\{U_1, U_2, U_3\}$, such that $dim(U_i) = d$ $(1 \le i \le 3)$ and $V = U_1 \oplus U_2 \oplus U_3$. We write a typical element of Λ as $[U_1|U_2|U_3]$. Finally, set $\Gamma = \Gamma_1 \cup \Gamma_2$ and $\Delta = \Gamma \cup \Lambda$.

The type function on Δ is given by:

$$\tau^{-1}(1) = \Gamma_1,\ \tau^{-1}(2) = \Gamma_2,\ \text{and } \tau^{-1}(3) = \Lambda.$$

$*$: the incidence relation is given as follows:

If $x \ne y \in \Gamma$, then $x * y$ iff $x < y$ or $x > y$, that is, by inclusion.

If $x = [U_1|U_2|U_3] \in \Lambda$, then $x * y$ iff $y = U_i$ or $U_i \oplus U_j$ $(1 \le i \ne j \le 3)$.

Then $(\Delta, \tau, *)$ is a geometry. Let $B(V)$ be the clique complex of the graph $(\Delta, *)$.

If $\pi : V \to V^*$ is an isomorphism between V and its dual space, then for each $W \leq V$ we have a subspace $W^{\perp} = \pi^{-1}(\{\phi \in V^* \text{ such that } \phi|_W \equiv 0\})$. Note that $dim(W^{\perp}) = codim(W)$. If S is the set of all subspaces of V, then let $perp : S \to S$ be the map taking each $W \in S$ to $W^{\perp}$. Note that if $W_1 \leq W_2$, then $perp(W_2) \leq perp(W_1)$ and $dim(W) + dim(perp(W)) = dim(V)$. Furthermore, the *perp* map preserves direct sums. So we have:

Remark 1. *There is a well-defined automorphism on the complex $B(V)$, given by the perp map, which interchanges Γ_1 and Γ_2 and preserves Λ. The action of the perp map on Λ is given by $[U_1|U_2|U_3] \mapsto [(U_1 \oplus U_2)^{\perp}|(U_1 \oplus U_3)^{\perp}|(U_2 \oplus U_3)^{\perp}]$. Thus if $\lambda_0\lambda_1 \cdots \lambda_{n-1}\lambda_0$ is an n-gon in Δ such that $S_1 = \{\lambda_{i_1}, \ldots, \lambda_{i_r}\} \subseteq \Gamma_1$ and $S_2 = \{\lambda_{j_1}, \ldots, \lambda_{j_s}\} \subseteq \Gamma_2$, we can assume without loss of generality that $S_1 \subseteq \Gamma_2$ and $S_2 \subseteq \Gamma_1$.*

Lemma 10.5. *$diam(\Delta) \leq 4$ and $diam(\Gamma) = 3$. Furthermore if $x \neq y \in \Gamma$ then:*

$$d_\Gamma(x, y) = \begin{cases} 2, & \text{if } \tau(x) = \tau(y), \\ 3, & \text{if } \tau(x) \neq \tau(y) \text{ and } x \notin \Delta(y). \end{cases}$$

Here $\Delta(y) = \{x \in \Delta \mid x \text{ is adjacent to } y\}$.

Proof. Let $x, y \in \Delta$. We do a case-by-case analysis, keeping in mind Remark 1:

(1) $\tau(x) = \tau(y) = 1$. Then there exists $z \in \Gamma_2$ such that $x, y < z$. $\tau(x) = \tau(y) = 2$ follows by duality.

(2) $\tau(x) = 1$, $\tau(y) = 2$. Let $z \leq y$ with $\tau(z) = 1$ and $x \cap z = 0$. Then $x(x \oplus z)zy$ is a path joining x and y.

(3) $\tau(x) = 3$, $\tau(y) = 1$. Let $z * x$ with $\tau(z) = 1$. Then there exists a path zwy joining z and y. So $xzwy$ joins x and y. $\tau(y) = 2$ follows by duality.

(4) $\tau(x) = \tau(y) = 3$. Let $z * x$ and $u * y$ with $\tau(z) = \tau(u)$. Then $d_\Delta(x, y) \leq 4$, as $d_\Gamma(z, u) \leq 2$.

This case-by-case analysis proves all three assertions of the lemma. □

Fix the following notation:

(1) Let $x_i \in \Gamma_1$, $y_i \in \Gamma_2$, and $z_i \in \Lambda$.

(2) For any path $p \in \Delta$, let $\lambda(p) =$ number of elements of Λ occurring in p.

(3) If $y \in \Gamma(x)$ and $z \in \Lambda$, then we write $y \in \Gamma(x) \cap z$ if $x, y \in \Delta(z)$.

(4) For $X \subseteq \Delta$, let $\Delta(X) = \bigcap_{x \in X} \Delta(x)$; $\Gamma(X)$ is similarly defined.

Note the following facts:

<u>Fact 1:</u> If $\lambda_1, \lambda_2 \in \Gamma$ with $d \nmid dim(\lambda_1 \cap \lambda_2)$, then $\Delta(\lambda_1, \lambda_2) = \Gamma(\lambda_1, \lambda_2)$.

<u>Fact 2:</u> If $x_1, x_2 \in \Gamma_1$ and $x_1 \cap x_2 = 0$, then there exists a unique $y \in \Gamma(\lambda_1, \lambda_2)$. A dual statement for $y_1, y_2 \in \Gamma_2$ also holds.

We would like to show that $B(V)$ is simply connected. In light of Lemmas 10.2 and 10.5, it suffices to show that for $r \leq 9$, each r-gon (as defined earlier) is in $\mathfrak{C}_3(\Delta)$. We say that a cycle p is *trivial* if $p \in \mathfrak{C}_3(\Delta)$, and write $p \sim 1$.

Lemma 10.6. Δ *has no* 9*-gons.*

Proof. Assume that $p = \lambda_0 \cdots \lambda_8 \lambda_0$ is a 9-gon in Δ. Then we can assume that $\lambda_0 \in \Gamma$. But as $d_\Delta(\lambda_0, \lambda_i) = 4$ for $i = 4$ or 5, by Lemma 10.5, $\lambda_4, \lambda_5 \in \Lambda$. This gives us a contradiction as distinct elements of Λ are not adjacent in Δ. $\square$

We now proceed to show that all r-gons $(4 \leq r \leq 8)$ are trivial.

Lemma 10.7. *All squares in* Δ *are trivial.*

Proof. Let $p = \lambda_0 \ldots \lambda_3 \lambda_0$ be a square in Δ. In view of Remark 1, we can assume without loss of generality (awlog) that $\lambda_0 = x_1$.

<u>CASE I:</u> $\lambda_2 = x_2$ and $x_1 \cap x_2 = 0$. Consider $y_1 = x_1 \oplus x_2$. If $\lambda_i \in \Gamma$ for $i = 1$ or 3, then, by Fact 2, $\lambda_i = y_1$. And if $\lambda_i \in \Lambda$, then $\lambda_i = [x_1|x_2|x_3]$ (for some $x_3 \in \Gamma_1$). So, $y_1 \in \lambda_i^\perp$ for $i = 1,3$; and $p \sim 1$. Recall that $x^\perp = \Delta(x) \cup \{x\}$ for all $x \in \Delta$.

<u>CASE II:</u> $\lambda_2 = y_1$ and $x_1 \cap y_1 = 0$. Then $\lambda_1, \lambda_3 \in \Lambda$; so let $\lambda_1 = z_1 = [x_1|x_{11}|x_{12}]$ and $\lambda_3 = z_2 = [x_1|x_{21}|x_{22}]$ with $y_1 = x_{i1} \oplus x_{i2}$ for $i = 1,2$. Now one of two cases occurs:

<u>IIA:</u> For some $1 \leq i,j \leq 2$ $x_{1i} \cap x_{2j} = 0$. Then we can awlog that $x_{11} \cap x_{21} = 0$. So let $y_2 = x_1 \oplus x_{11}$, $y_3 = x_1 \oplus x_{21}$, and $z_3 = [x_1|x_{11}|x_{21}]$. Consider the

path $z_1y_2x_{11}z_3x_{21}y_3z_2$. Then p is in the closure of the triangles $x_1z_iy_{i+1}x_1$, $y_1z_ix_{i1}y_1$, and $z_iy_{i+1}x_{i1}z_i$ $(i = 1,2)$ and the squares $x_1y_{i+1}x_{i1}z_3x_1$ $(i = 1,2)$ and $y_1x_{11}z_3x_{21}y_1$ (see *Fig. 2* below). But these squares are trivial by Case I, so $p \sim 1$.

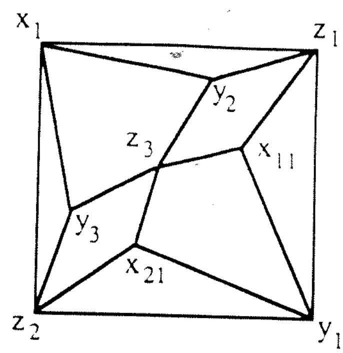

Figure 2

IIB: For all $1 \le i,j \le 2$ $x_{1i} \cap x_{2j} \neq 0$. Now by Lemma 10.3, there exists $x_3 \le y_1$ such that $x_3 \cap x_{i1} = 0$ for $i = 1,2$. Let $y_1 = x_3 \oplus x_3'$ and set $z_3 = [x_1|x_3|x_3']$. Then p is in the closure of the squares $x_1z_1y_1z_3x_1$ and $x_1z_3y_1z_2x_1$, both of which are trivial by Case IIa.

So $p \sim 1$.

CASE III: $\lambda_2 = x_2$ and $x_1 \cap x_2 \neq 0$. Then, by Fact 1, $p = x_1y_1x_2y_2x_1$. By Lemma 10.4, we can find x_3 such that $x_3 \cap y_i = 0$ for $i = 1,2$. Let:

$$y_i = x_j \oplus x_{ij} \ (1 \le i,j \le 2) \quad \text{and set } z_{ij} = [x_i|x_{ij}|x_3].$$

Then p is in the closure of the triangles $x_iz_{ij}y_jx_i$ $(1 \le i,j \le 2)$ and squares $x_iz_{i1}x_3z_{i2}x_i$ and $y_iz_{1i}x_3z_{2i}y_i$ $(1 \le i \le 2)$ (see *Fig. 3* below). But, by Cases I and II above, these squares are trivial, so $p \sim 1$.

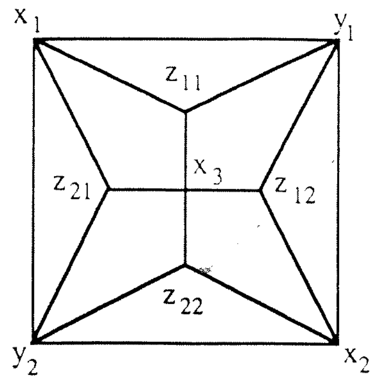

Figure 3

Thus all squares in Δ are trivial. □

Lemma 10.8. *All pentagons in Δ are trivial.*

Proof. Let $p = \lambda_0 \cdots \lambda_4 \lambda_0$ be the pentagon. If $p \subseteq \Gamma$ then, as Γ is bipartite, $\tau(\lambda_0) = \tau(\lambda_2) = \tau(\lambda_4)$, a contradiction. So $1 \leq \lambda(p) \leq 2$. Assume that $\lambda_0 = x_1$, and $\lambda_1 = z_1$.

<u>CASE I:</u> $\lambda(p) = 1$. Then $\tau(x_1) = \tau(\lambda_3) \neq \tau(\lambda_2) = \tau(\lambda_4)$, so $p = x_1 z_1 y_1 x_2 y_2 x_1$. As $x_1, y_1 \in \Delta(z_1)$, $x_1 \cap y_1 = 0$. Thus, if $y_1 = x_2 \oplus x_3$, setting $z_2 = [x_1|x_2|x_3]$, we see that $z_2 \in x_1^\perp \cap y_1^\perp \cap x_2^\perp$. So by (1.5) on pg. 77 in [AS2], $p \sim 1$.

<u>CASE II:</u> $\lambda(p) = 2$. We can awlog that $p = x_1 z_1 \lambda_2 z_2 \lambda_4 x_1$, then $\tau(x_1) \neq \tau(\lambda_4)$, so we have $p = x_1 z_1 \lambda_2 z_2 y_1 x_1$. If $\lambda_2 = x_2$, then by Lemma 10.5 there exists $y_2 \in \Gamma(x_1, x_2)$. But then p is in the closure of $x_1 z_1 x_2 y_2 x_1$ and $x_1 y_2 x_2 z_2 y_1 x_1$. The latter is trivial by Case I, so $p \sim 1$.

Otherwise, $\lambda_2 = y_2$; but then, keeping Remark 1 in mind, $p \sim 1$ by the argument in the previous paragraph applied to y_1 and y_2.

So all pentagons are trivial. □

Lemma 10.9. *For all $6 \leq r \leq 8$, all r-gons in Δ are trivial.*

Proof. Let $p = \lambda_0 \ldots \lambda_{r-1} \lambda_0$ be a nontrivial r-gon. We first show that this implies that $\lambda(p) = 0$ or $r = 6$. For a contradiction, assume that p is a counterexample with minimal $\lambda(p) > 0$. Now, we can awlog that $\lambda_0 = x_1$, and $\lambda_1 = z_1$.

If $\lambda_2 = x_2$, then $x_1 \cap x_2 = 0$, since $x_1, x_2 \in \Delta(z_1)$. Setting $y_1 = x_1 \oplus x_2$, we find that p is in the closure of $x_1 z_1 x_2 y_1 x_1$ and $p' = x_1 y_1 x_2 \lambda_3 \ldots \lambda_{r-1} x_1$. But $\lambda(p') = \lambda(p) - 1$, so by the minimality of $\lambda(p)$ we have that p' is trivial. This contradicts the fact that p is nontrivial.

Thus, $\lambda_2 = y_1$, but $\lambda_3 \neq x_2$ as $d(x_1, x_2) = 2$, thus $\lambda_3 = z_2$. But by applying the above argument to y_1, and z_2, we see that $\lambda_4 = x_2$. Thus, we have $p = x_1 z_1 y_1 z_2 x_2 \lambda_5 \ldots \lambda_{r-1} x_1$. But as $d(x_1, x_2) = 2$, this is possible only if $r = 6$.

Thus if $r \neq 6$, then the only nontrivial r-gons are contained in Γ. But when $r = 7$, this contradicts the fact that Γ is a bipartite graph. So there are no nontrivial heptagons. Now when $r = 8$, $\tau(\lambda_0) = \tau(\lambda_2) = \tau(\lambda_4)$. But by Lemma 10.5, $d_\Gamma(\lambda_0, \lambda_4) = 2$, which contradicts our assumption that $d_\Delta(\lambda_0, \lambda_4) = 4$. Thus, all octagons are trivial too.

So let $r = 6$ and let $p = \lambda_0 \ldots \lambda_5 \lambda_0$ be a nontrivial hexagon with $\lambda(p)$ minimal. If $\lambda(p) \neq 0$, then by the argument above, $p = x_1 z_1 y_1 z_2 x_2 \lambda_5 x_1$. Now $\lambda_5 \neq y_2$ as $d_\Gamma(y_1, y_2) = 2$. On the other hand, if $\lambda_5 = z_3$, then as above, $y_2 = x_1 \oplus x_2$ exists and p is in the closure of a square and a trivial hexagon (by minimality of $\lambda(p)$). Thus the only nontrivial hexagons are the ones contained in Γ.

We can awlog that $p = x_1 y_1 x_2 y_2 x_3 y_3 x_1$. By Lemma 10.4, there exists x_4 such that $x_4 \cap y_i = 0$ $(1 \leq i \leq 3)$. If $y_i = x_i \oplus x_i'$, then set $z_i = [x_i | x_i' | x_4]$. Then p is in the closure of the triangles $x_i z_i y_i x_i$ $(1 \leq i \leq 3)$ and pentagons $x_1 z_1 x_4 z_3 y_3 x_1$, $x_3 z_3 x_4 z_2 y_2 x_3$, and $x_2 z_2 x_4 z_1 y_1 x_2$. Thus, by Lemma 10.8, $p \sim 1$.

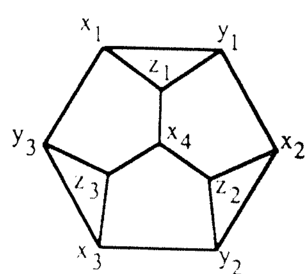

Figure 4

Therefore for $6 \leq r \leq 8$, there are no nontrivial r-gons in Δ. □

Thus we have proven the following:

Theorem 10.10. *Let $d \geq 1$ and V be a $3d$-dimensional vector space over some finite field* *and $B(V)$ as defined above. Then $B(V)$ is simply connected.* □

In the proof of the lemmas above we were often able to use the fact that Γ is a bipartite graph to reduce the computation. It turns out that if we are considering an order complex of a poset then the amount of computation required to show that the complex is simply connected is significantly reduced using the following result – Theorem B. Given a poset P, we can define the *graph of* P, denoted $\Delta(P)$, to be the graph with vertex set P and two vertices $a, b \in \Delta(P)$ incident if $a < b$, or $b < a$. Such a graph is called a *comparability graph.* The order complex of P is then the clique complex of $\Delta(P)$. By Lemma 10.2, the question of simple connectivity of P reduces to a question about the triangulability of $\Delta(P)$. We now give a characterization of order complexes, which significantly reduces the computation required to show that $\Delta(P)$ is triangulable. The result is based on a result of Gilmore and Hoffman in [GiH].

Given a cycle $p = x_0 \dots x_n$, we call p an *odd* or *even cycle* depending on whether n is odd or even. We restrict our attention to cycles $x_0 \dots x_n$ such that given $a, b \in \Delta(P)$ and $i, j \leq n-1$, we do not have $a = x_i = x_j$ and $b = x_{i+1} = x_{j+1}$; that is, $x_0 \dots x_n$ is not equal to the path $x_0 \dots x_{i-1} a b x_{i+2} \dots x_{j-1} a b x_{j+2} \dots x_n$. In other words, if we travel from x_0 to x_n, going from x_i to x_{i+1}, we do not traverse any edge in the same direction twice. In particular, all n-gons are included in this set of cycles. Given a cycle $x_0 \dots x_n$, a *triangular chord* is an edge of the form (x_i, x_{i+2}) $0 \leq i \leq n-2$ and the edge (x_{n-1}, x_1). With this notation in mind we have the following characterization of order complexes:

Theorem B. *A simplicial complex K is an order complex if and only if the following two conditions hold:*

(1) *K is a clique complex, and*

(2) *every odd cycle in $\Delta(K)$ has at least one triangular chord – here we restrict our attention to the collection of cycles defined above.*

Proof. Let K be a simplicial complex, then by the definition of an order complex:

K is an order complex if and only if K is the clique complex of a comparability graph.

Now by Theorem 1 on pg. 540 in [GiH] a graph G is a comparability graph if and only if every odd cycle has at least one triangular chord. Thus:

K is an order complex if and only if K is a clique complex and every odd cycle in $\Delta(K)$ has at least one triangular chord. Thus we have the theorem. $\square$

As an immediate corollary we obtain:

Corollary 10.11. *An order complex K is simply connected if and only if every $2k$-gon of $\Delta(K)$ is contained in $\mathfrak{C}_3(\Delta(K))$ for all $k \leq diam(\Delta(K))$.*

Proof. This follows from Lemma 10.2 and the fact that an order complex has no $(2k+1)$-gon for $k \geq 2$ by Theorem B. $\square$

SECTION 11 : A $(n-1)$-SPHERICAL COMPLEX ASSOCIATED WITH AN n-DIMENSIONAL VECTOR SPACE

In section 10 we demonstrated a method of computing simple connectivity of a clique complex. In this section we demonstrate a method of computing n-connectedness of a simplicial complex. We use a result from algebraic topology, called the Nerve Theorem and Lemma 9.1.

Recall that the *nerve* of a family of sets $(A_i)_{i \in I}$ is the simplicial complex $\mathcal{N}(A_i)$ on the vertex set I such that a finite subset $\sigma \subseteq I$ is a simplex if and only if $\bigcap_{i \in \sigma} A_i \neq \emptyset$. We have the following result:

Lemma 11.1 (Nerve Theorem). *Let Δ be a simplicial complex and $\{\Delta_i\}_{i \in I}$ a family of subcomplexes such that $\Delta = \bigcup_{i \in I} \Delta_i$. Suppose that every nonempty finite intersection, $\Delta_{i_1} \cap \Delta_{i_2} \cap \ldots \cap \Delta_{i_t}$, is $(k-t+1)$-connected; then Δ is k-connected if and only if the nerve $\mathcal{N}(\Delta_i)$ is k-connected. If we have the stronger condition that every nonempty finite intersection, $\Delta_{i_1} \cap \ldots \cap \Delta_{i_t}$, is contractible, then Δ and $\mathcal{N}(\Delta_i)$ are homotopy equivalent.*

Proof. This is (10.6) on pg. 23 in [B1]. $\square$

Given a simplicial complex K, recall the definition of $sd(K)$, and Lemma 9.1 from section 9.

Corollary 11.2. *Given a finite simplicial complex C, let C^* be the complex whose vertices are the maximal simplices of C, with σ a simplex of C^* if and only if $\bigcap_{M \in \sigma} M \neq \emptyset$. Then:*

$$C \simeq C^*.$$

Proof. Note that C^* is the nerve of the cover of C by maximal simplices. Also, whenever $\bigcap_{M \in \sigma} M \neq \emptyset$, $\bigcap_{M \in \sigma} M$ is contractible. Thus $C^* \simeq C$ by Lemma 11.1. □

Let V^* be the dual space of V and for each $U \leq V$ let $U^{\perp} = \{f \in V^* \mid f|_U \equiv 0\}$ be the *annihilator* of U. Let K be the order complex of the poset on the set of vertices:

$$\{0 \neq U \times U' \mid U \leq V \text{ and } U' \leq U^{\perp}\},$$

partially ordered by inclusion. Note that by the definition of K, K is a $(n-1)$-dimensional complex.

For each $U \leq V$, let $K_U = K(\leq U \times U^{\perp})$. Then note that $\{K_U \mid U \leq V\}$ is a cover of K. Let $N = \mathcal{N}(K_U)$, the nerve of the cover. Given a finite subset $\{K_{U_i} \mid 1 \leq i \leq r\}$ of N, note that:

$$\bigcap_{1 \leq i \leq r} K_{U_i} \neq \emptyset \iff 0 \neq (\bigcap_{1 \leq i \leq r} U_i) \times (\bigcap_{1 \leq i \leq r} U_i^{\perp})$$

is the unique maximal element of $\cap K_{U_i}$. Thus, given a simplex $\sigma \in N$, $\bigcap_{K_U \in \sigma} K_U$ is conically contractible. Hence, by the Nerve Theorem, $K \simeq N$.

Next let L be the simplicial complex whose vertices are all the subspaces of V (including V and 0). A subset $\sigma \subseteq L$ is a simplex if and only if $\bigcap_{U \in \sigma} U \neq 0$ or $< U | U \in \sigma > \lneq V$. We can identify L with N, via the isomorphism $U \mapsto K_U$.

Let $\mathcal{P}(V) = \{U \leq V \mid dim(U) = 1\}$ and $\mathcal{H}(V) = \{U \leq V \mid dim(U) = n-1\}$. For each $p \in \mathcal{P}(V)$, let $\sigma_p = \{U \leq V | p \leq U\}$ and for each $h \in \mathcal{H}(V)$, let $\tau_h = \{U \leq V | U \leq h\}$. Define a simplicial complex M as follows:

> Let the vertex set of M equal $\mathcal{P}(V) \cup \mathcal{H}(V)$. σ is a simplex if and only if $< p | p \in \sigma > \leq \bigcap_{h \in \sigma} h$. Here we use the convention that $\bigcap_{h \in \sigma} h = V$ if $\sigma \cap \mathcal{H}(V) = \emptyset$ and $< p | p \in \sigma > = 0$ if $\sigma \cap \mathcal{P}(V) = \emptyset$. Now the maximal simplices of L are

σ_p, τ_h where $p \in \mathcal{P}, h \in \mathcal{H}$; and $s = \{\sigma_{p_i}, \tau_{h_j}\}$ is a simplex of L^* if and only if $< p_i > \leq \cap h_j$ (here L^* is as defined in Corollary 11.2). Thus M and L^* are isomorphic.

Theorem 11.3. *$K, L,$ and M all have the same homotopy type.*

Proof. By the preceding discussion, $M \cong L^*$ and L can be identified with N. Also, by Corollary 11.2, $L^* \simeq L$ and by the Nerve Theorem, $K \simeq N$. Thus $M \simeq N \simeq K$, as claimed. $\square$

So our question regarding the $(n-2)$-connectedness of K is reduced to showing that M is $(n-2)$-connected. Now let P and H be the subcomplexes generated by $\mathcal{P}(V)$ and $\mathcal{H}(V)$, respectively. Let $P_1 = \{\sigma \in P \,|< p|p \in \sigma >= V\}$ and $H_1 = \{\tau \in H \mid \bigcap_{h \in \tau} h = 0\}$. Finally, let M' be the subcomplex generated by $M \setminus \{P_1 \cup H_1\}$. Now $M = P \cup H \cup M'$, and the nerve of this cover is given by $P < \{P, M'\} > M' < \{M', H\} > H$, and hence is contractible.

Theorem 11.4. *M is $(n-2)$-connected if M' is $(n-2)$-connected.*

Proof. By Lemma 11.1 it suffices to show that $P, H,$ and M' are $(n-2)$-connected; and $P \cap M'$ and $H \cap M'$ are $(n-3)$-connected.

As we are working over a finite field, both $\mathcal{P}(V)$ and $\mathcal{H}(V)$ are finite and thus are the unique maximal elements of P and H, respectively. Thus, P and H are conically contractible.

Lemma 11.5. *$P \cap M'$ and $H \cap M'$ are $(n-3)$-connected.*

Proof. As above, let V^* be the dual space of V, then $h \mapsto h^\perp$ (the annihilator of h) gives a map $\pi : \mathcal{H}(V) \to \mathcal{P}(V^*)$. So we have a canonical identification of $H \cap M'$ with $P^* \cap M'^*$ (the corresponding complexes for V^*). But $P^* \cap M'^*$ is canonically homeomorphic to $P \cap M'$. Thus it suffices to show that $P \cap M'$ is $(n-3)$-connected.

Define a map $f : P \cap M' \to \mathfrak{B}$ the Tits building of V by $\sigma \mapsto < p \mid p \in \sigma >$. Now if $X = sd(P \cap M')$ and $Y = sd(\mathfrak{B})$, and $\sigma < \tau \in X$, then $< p \mid p \in \sigma > \leq < p \mid p \in \tau >$. So f induces a well-defined map of posets from X to Y.

For each $y = (U_0 < \cdots < U_s) \in Y$, let $\sigma = \mathcal{P}(U_s)$. Given any $(\sigma_0 < \cdots < \sigma_r) \in f^{-1}(Y(\leq y))$, certainly $< p \mid p \in \sigma_r > \leq U_s$, so $\sigma_r \leq \sigma$. Thus σ is the unique maximal element of $f^{-1}(Y(\leq y))$, and thus $f^{-1}(Y(\leq y))$ is conically contractible.

Also, as $\mathfrak{B}$ is Cohen-Macaulay of dimension $(n-2)$, by the Solomon-Tits theorem, Y is also Cohen-Macaulay of dimension $(n-2)$. Hence $Y(> y)$ is $(n - h(y) - 3)$-connected. Thus, by Theorem 9.1 on pg. 119 in [Q], X is $(n-3)$-connected since Y is $(n-3)$-connected. By Lemma 9.1, $X \simeq P \cap M'$ so $P \cap M'$ is $(n-3)$-connected, as claimed. $\square$

So we have shown that $P \cap M'$ and $H \cap M'$ are $(n-3)$-connected; P and H are contractible. Thus M is $(n-2)$-connected if M' is $(n-2)$-connected, as claimed. $\square$

Let $X = sd(M')$, then $G = GL(V)$ acts on X. Now if p is the characteristic of $\mathbb{F}_q$, then by Corollary 1.5, $\mathcal{A}_p(G)$ is $(n-2)$-connected. We can consider $X \times \mathcal{A}_p(G) \supseteq F = \{(x, A) | x \in Fix(A)\}$, where $Fix(A)$ is the subcomplex of X defined on the simplices fixed under the action of A. By Theorem 9.13, if for each $x \in X$ and $A \in \mathcal{A}_p(G)$ we can show that $F_x = \{A' \in \mathcal{A}_p(G) \mid (x, A') \in F\}$ and $F_A = \{x' \in X \mid (x', A) \in F\}$ are contractible, then X is $(n-2)$-connected since $\mathcal{A}_p(G)$ is $(n-2)$-connected.

Lemma 11.6. *F_x is contractible for each $x = (\sigma_0 < \cdots < \sigma_s) \in X$.*

Proof. Note that $F_x = \mathcal{A}_p(G_x)$. Two cases occur:

<u>Case I</u> : $W = < p \mid p \in \sigma_s > \neq 0$. Let P be the parabolic subgroup of G fixing W and let $U \leq P$ be the unipotent radical of P. Then U centralizes W and V/W. Since $\forall\ p \in \sigma_s,\ p \leq W$, U fixes each $p \in \sigma_s$. On the other hand, given $h \in \sigma_s$, $W \leq h$; thus U normalizes each $h \in \sigma_s$. So U fixes $\sigma_i\ \forall\ 0 \leq i \leq s$. Thus $U \trianglelefteq G_x$; but the fact that $W \neq 0$ implies that $U \neq 1$. By definition, U is a p-subgroup of G_x. Hence we have $O_p(G_x) \neq 1$; thus, by Proposition 2.4 on pg. 106 in [Q], $\mathcal{A}_p(G_x)$ is contractible.

<u>Case II</u> : $< p \mid p \in \sigma_s > = 0$ and $W = \bigcap_{h \in \sigma_s} h \neq 0$. Again, let U be the unipotent radical of the parabolic subgroup fixing W. By the definition of M', we know $0 \neq W \lneq V$ so $U \neq 1$. But as above, $Uh = h\ \forall\ h \in \sigma_s$, and hence U centralizes each $\sigma_i\ \forall\ 0 \leq i \leq s$. But then $O_p(G_x)$ is nontrivial and $\mathcal{A}_p(G_x)$ is contractible as above.

Thus we have shown that F_x is contractible, as claimed. □

Now fix an $A \in A_p(G)$ and let $C_V(A)$ be the subspace fixed pointwise by A. Let $[V,A] = < av - v \mid a \in A, v \in V >$ be the *commutator of* A *with* V. Now A is a p-subgroup of G with $p = char(\mathbb{F}_q)$; hence A is unipotent on V, so $[V,A] \cap C_V(A) = W_1 \neq 0$.

Lemma 11.7. *If* $p \in \mathcal{P}(V) \cap Fix(A)$, *then* $p \leq C_V(A)$. *And if* $h \in \mathcal{H}(V) \cap Fix(A)$, *then* $[V,A] \leq h$.

Proof. This follows as A is unipotent on V. □

Thus, by Lemma 11.7, $x = (\sigma_0 < \sigma_1 < \ldots < \sigma_r) \in Fix(A)$ if and only if $\forall\ 0 \leq i \leq r$, $\forall\ p,h \in \sigma_i$, $p \leq C_V(A)$, $h \geq [V,A]$, and $< p \mid p \in \sigma_i > \leq \bigcap_{h \in \sigma_i} h$. Now let $\hat{\sigma} = W_1 \cap \mathcal{P}(V) \in Fix(A)$, where $W_1 = [V,A] \cap C_V(A) \neq 0$.

Lemma 11.8. *For all* $A \in A_p(G)$, F_A *is contractible.*

Proof. Fix an $A \in A_p(G)$ and let $C_V(A)$, $[V,A]$, $Fix(A) = F_A$, and $\hat{\sigma}$, as defined prior to this lemma. Now let $\sigma = \{p_i, h_j \mid i \in I, j \in J\} \in Fix(A)$. Then notice that $< p \mid p \in \hat{\sigma} > = W_1 \leq [V,A] \leq \bigcap_{j \in J} h_j$ so that $< p_i, p \mid i \in I, p \in \hat{\sigma} > \leq \bigcap_{j \in J} h_j$; thus $\sigma \cup \hat{\sigma} \in Fix(A)$. Also note that if $\sigma \leq \tau$, then $\sigma \cup \hat{\sigma} \leq \tau \cup \hat{\sigma}$. Thus we can define a map of posets $f : Fix(A) \to Fix(A)$ given by $\sigma \mapsto \sigma \cup \hat{\sigma}$. Note that $\forall\ \sigma \in Fix(A)$, $\sigma \leq f(\sigma) \geq \hat{\sigma}$. Hence $id \simeq f_{\hat{\sigma}}$ the map taking each $\sigma \in Fix(A)$ to $\hat{\sigma}$. Thus, by (1.3) on pg. 103 in [Q], $Fix(A)$ is conically contractible. □

In view of Theorem 9.13 and Lemmas 11.6 and 11.8, $sd(M')$ is $(n-2)$-connected.

Theorem 11.9. K *is* $(n-1)$-*spherical.*

Proof. By the definition of K we have that K is $(n-1)$-dimensional. Now, as $sd(M')$ is $(n-2)$-connected, by Lemma 9.1, so is M'. By Theorems 11.3 and 11.4, $K \simeq M$ and M is $(n-2)$-connected. Hence K is $(n-2)$-connected. And thus, K is $(n-1)$-spherical, as claimed. □

Section 12 : Some results about subspaces of a unitary space of dimension ≥ 3

Let q be an odd prime power and (V, θ) an n-dimensional unitary space over $\mathbb{F}_{q^2}$. In section 13 we show that if $n \geq 4$, then the order complex of proper nondegenerate subspaces of V – ordered by inclusion – is simply connected using a result of Aschbacher and Segev's on collinearity graphs of string geometries [AS3]. In this section we prove some preliminary results about the nondegenerate subspaces of V.

Remember that $\theta(v, w) = \tau(\theta(w, v)) \ \forall \ v, w \in V$, where τ is the involutory field automorphism on $\mathbb{F}_{q^2}$. Vectors $v \in V$ which satisfy $\theta(v, v) = 1$ are called *unit vectors.* We shall use the following fact about unitary spaces implicitly:

<u>Fact 1:</u> In an n-dimensional unitary space over $\mathbb{F}_{q^2}$, there are $q^{n-1}(q^n - (-1)^n)$ unit vectors.

Theorem 12.1. *Assume that $q \geq 5$ and $n \geq 4$. Let v_1, v_2, v_3 be unit vectors of V satisfying:*

(1) $< v_i, v_j > (i \neq j)$ *are 2-dimensional nondegenerate subspaces of V, and*

(2) $< v_1, v_2, v_3 >$ *is a degenerate 3-dimensional subspace of V.*

Then there exists a unit vector $v_4 \in V$ such that $< v_1, v_4 >, < v_2, v_4 >$, and $< v_3, v_4 >$ are 2-dimensional nondegenerate subspaces, and $< v_1, v_2, v_4 >, < v_1, v_3, v_4 >$ and $< v_2, v_3, v_4 >$ are 3-dimensional nondegenerate subspaces of V.

Proof. The fact that $< v_1, v_2, v_3 >$ is degenerate and $< v_1, v_2 >$ is nondegenerate implies that $< v_1, v_2, v_3 >=< v_1, v_2, s >$, where $s \neq 0$ is a totally singular element of $< v_1, v_2 >^{\perp}$ with $v_3 = a_1 v_1 + a_2 v_2 + s$. As $< v_1, v_2 >^{\perp}$ is nondegenerate of dimension $n - 2 \geq 2$, by (19.14) on pg. 80 in [A1], we can find $t \in < v_1, v_2 >^{\perp}$ such that $\{s, t\}$ is a hyperbolic pair.

We will show that there exists $v_4 = \beta_1 s + \beta_2 t \in < s, t >$ satisfying the claims of this theorem. First of all, for v_4 to be a unit vector we require:

$$(1) \qquad \theta(v_4, v_4) = \theta(\beta_1 s + \beta_2 t, \beta_1 s + \beta_2 t) = \beta_1 \tau(\beta_2) + \beta_2 \tau(\beta_1) = 1.$$

Now if v_4 is a unit vector, then the fact that $v_4 \in < v_1, v_2 >^{\perp}$ implies that $< v_1, v_4 >$, $< v_2, v_4 >$, and $< v_1, v_2, v_4 >$ are nondegenerate of the correct dimensions. So we need to consider the requirements for $< v_3, v_4 >, < v_1, v_3, v_4 >$, and $< v_2, v_3, v_4 >$ to be nondegenerate.

Consider $< v_3, v_4 >$, we have:

$$\theta(v_3, v_4) = \theta(a_1 v_1 + a_2 v_2 + s, \beta_1 s + \beta_2 t) = \tau(\beta_2).$$

So if $\alpha_1 v_3 + \alpha_2 v_4 \in \ Rad(< v_3, v_4 >)$, then we have:

$$\theta(\alpha_1 v_3 + \alpha_2 v_4, v_3) = \alpha_1 + \alpha_2 \beta_2 = 0$$

$$\theta(\alpha_1 v_3 + \alpha_2 v_4, v_4) = \alpha_1 \tau(\beta_2) + \alpha_2 = 0.$$

Therefore, $Rad(< v_3, v_4 >) = 0$ if and only if:

$$\begin{pmatrix} 1 & \beta_2 \\ \tau(\beta_2) & 1 \end{pmatrix} \quad \text{is nonsingular.}$$

This is true if and only if:

(2) $$\beta_2 \tau(\beta_2) \neq 1.$$

Consider $< v_1, v_3, v_4 > = < v_1, a_2 v_2 + s, v_4 >$. Let $\theta(v_1, v_2) = \eta$. And let $\alpha_1 v_1 + \alpha_2 a_2 v_2 + \alpha_2 s + \alpha_3 v_4 \in \ Rad(< v_1, v_3, v_4 >)$, then:

$$\theta(\alpha_1 v_1 + \alpha_2 a_2 v_2 + \alpha_2 s + \alpha_3 v_4, v_1) = \alpha_1 + \alpha_2 a_2 \tau(\eta) = 0$$

$$\theta(\alpha_1 v_1 + \alpha_2 a_2 v_2 + \alpha_2 s + \alpha_3 v_4, a_2 v_2 + s) = \alpha_1 \tau(a_2)\eta + \alpha_2 a_2 \tau(a_2) + \alpha_3 \beta_2 = 0$$

$$\theta(\alpha_1 v_1 + \alpha_2 a_2 v_2 + \alpha_2 s + \alpha_3 v_4, v_4) = \alpha_2 \tau(\beta_2) + \alpha_3 = 0.$$

So $Rad(< v_1, v_3, v_4 >) = 0$ if and only if:

$$\begin{pmatrix} 1 & a_2 \tau(\eta) & 0 \\ \tau(a_2)\eta & a_2 \tau(a_2) & \beta_2 \\ 0 & \tau(\beta_2) & 1 \end{pmatrix} \quad \text{is nonsingular.}$$

This is true if and only if:

$$
\begin{aligned}
& a_2\tau(a_2) - \beta_2\tau(\beta_2) - a_2\tau(a_2)\eta\tau(\eta) \neq 0 \\
\textbf{(3)} \qquad & \iff \beta_2\tau(\beta_2) \neq a_2\tau(a_2)(1 - \eta\tau(\eta)).
\end{aligned}
$$

By symmetry, $< v_2, v_3, v_4 >$ is nondegenerate if and only if:

(4) $$\beta_2\tau(\beta_2) \neq a_1\tau(a_1)(1 - \eta\tau(\eta)).$$

The number of elements $\alpha \in \mathbb{F}_{q^2}$ satisfying:

$$\alpha\tau(\alpha) = 1 \text{ or } \alpha\tau(\alpha) = a_2\tau(a_2)(1 - \eta\tau(\eta)) \text{ or } \alpha\tau(\alpha) = a_1\tau(a_1)(1 - \eta\tau(\eta)),$$

is at most $3(q+1)$. For each such $\alpha \in \mathbb{F}_{q^2}$, there are q $\alpha' \in \mathbb{F}_{q^2}$ such that:

$$\alpha\tau(\alpha') + \alpha'\tau(\alpha) = 1.$$

So there are at most $3q(q+1)$ vectors $w \in < s, t >$ such that **(1)** holds for w but one of **(2)**, **(3)**, or **(4)** fails. The total number of unit vectors in $< s, t >$ is $q(q^2 - 1)$. Our assumption that $q \geq 5$ implies that $q(q^2 - 1) = q(q+1)(q-1) > 3q(q+1)$. Hence, there is a unit vector $v_4 = \beta_1 s + \beta_2 t \in < s, t >$ for which all **(1)**–**(4)** hold; in other words, a vector satisfying the claim of the theorem. □

Suppose that $n \geq 4$, and let Δ be a graph whose vertex set is the set of all 1-dimensional nondegenerate subspaces of V, with x adjacent to y (denoted $x * y$) if and only if $x + y$ is a 2-dimensional nondegenerate subspace of V. Remember that given $x, y \in \Delta$, $\Delta(x, y) = \{z \in \Delta \mid x * z \text{ and } y * z\}$. We show that:

Theorem 12.2. *If $x, y \in \Delta$ and x and y are not adjacent, then $\Delta(x, y)$ is connected.*

The proof of this theorem will follow from a couple of lemmas. First let $x =< v >$ with v a unit vector, and note that x and y not adjacent implies that there is a nonzero totally singular element $s \in v^\perp$ such that $y =< v + s >$. Let $\Delta(x, y) \supseteq \Gamma = \{z \in \Delta(x, y) | z \leq s^\perp\}$. We will prove the following two lemmas:

Lemma 12.3. Γ *is a connected subgraph of* $\Delta(x,y)$.

Lemma 12.4. *Given* $z \in \Delta(x,y) \setminus \Gamma$, *there exist* $z' \in \Gamma$ *adjacent to* z.

Note that Theorem 12.2 follows immediately from Lemmas 12.3 and 12.4.

Now $s^{\perp} =< s > \oplus W$, where W is a nondegenerate subspace of V of dimension $n-2 \geq 2$. Therefore, by (21.5) on pg. 87 in [A1], we can find $w \in W$ such that $\{v, w\}$ is an orthogonal pair; and we have:

$$s^{\perp} =< s > \oplus < v, w > \oplus U,$$

where $U =< v, w >^{\perp} \cap W$ is nondegenerate of dimension $n-4 \geq 0$. Now $U^{\perp} \cap < v, w >^{\perp}$ is a nondegenerate 2-dimensional subspace of V containing s. So there exists $t \in U^{\perp} \cap < v, w >^{\perp}$ such that $\{s, t\}$ form a hyperbolic pair and:

$$(*) \qquad V =< s, t > \perp < v, w > \perp U.$$

We are now ready to prove:

Lemma 12.3. Γ *is a connected subgraph of* $\Delta(x,y)$.

Proof. Since $s^{\perp} =< s > \oplus < v, w > \oplus U$, $z \in \Gamma$ if and only if $z =< a_1 v + a_2 w + a_3 s + u >$ (with $u \in U$). Now z is nondegenerate if and only if:

$$\theta(a_1 v + a_2 w + a_3 s + u, a_1 v + a_2 w + a_3 s + u) = a_1 \tau(a_1) + a_2 \tau(a_2) + \theta(u, u) \neq 0.$$

Also note $z + x =< v, a_2 w + a_3 s + u >$ is nondegenerate if and only if $a_2 w + a_3 s + u$ is nonsingular since $a_2 w + a_3 s + u \in v^{\perp}$; that is, if and only if:

$$\theta(a_2 w + a_3 s + u, a_2 w + a_3 s + u) = a_2 \tau(a_2) + \theta(u, u) \neq 0.$$

Similarly, $z + y =< v + s, a_2 w + (a_3 - a_1)s + u >$ is nondegenerate if and only if $a_2 w + (a_3 - a_1)s + u$ is nonsingular since $a_2 w + (a_3 - a_1)s + u \in v^{\perp} \cap s^{\perp}$; that is, if and only if:

$$\theta(a_2 w + (a_3 - a_1)s + u, a_2 w + (a_3 - a_1)s + u) = a_2 \tau(a_2) + \theta(u, u) \neq 0.$$

Thus we see that:

$$\Gamma = \{< a_1v + a_2w + a_3s + u > | a_1\tau(a_1) + a_2\tau(a_2) + \theta(u,u) \neq 0, \text{ and } a_2\tau(a_2) + \theta(u,u) \neq 0\}.$$

Note that $< w > \in \Gamma$.

Now $< a_1v + a_2w + a_3s + u > + < w > = < a_1v + a_3s + u, w >$ is nondegenerate if and only if $a_1v + a_3s + u$ is nonsingular. So all:

$$\Gamma \ni < a_1v_1 + a_2w + a_3s + u > \text{ with } a_1\tau(a_1) + \theta(u,u) \neq 0$$

are connected to $< w >$. So let $z = < a_1v + a_2w + a_3s + u > \in \Gamma$ with $a_1\tau(a_1) + \theta(u,u) = 0$. We want to find $z' \in \Gamma$ such that z' is adjacent to both z and w.

If $< b_1v + a_2w + u >$ is to belong to $\Gamma(< w >)$, then we must have:

(1) $$b_1\tau(b_1) + a_2\tau(a_2) + \theta(u,u) \neq 0, \text{ and}$$

(2) $$b_1\tau(b_1) + \theta(u,u) \neq 0.$$

Such a b_1 exists in $\mathbb{F}_{q^2}$, since there are at most $2(q+1)$ $\alpha \in \mathbb{F}_{q^2}$ for which:

$$\alpha\tau(\alpha) + a_2\tau(a_2) + \theta(u,u) = 0 \text{ or } \alpha\tau(\alpha) + \theta(u,u) = 0,$$

and $2(q+1) < q^2$ since $q \geq 3$. Note that (2) implies that $a_1 \neq b_1$. Also, $z \in \Gamma \Rightarrow a_2\tau(a_2) + \theta(u,u) \neq 0$. Let $b_1 \in \mathbb{F}_{q^2}$ satisfy (1) and (2) and set $z' = < b_1v + a_2w + u >$. Now $z + z' = < (a_1 - b_1)v + a_3s, b_1v + a_2w + u >$. Let:

$$r = (\alpha_1(a_1 - b_1) + \alpha_2 b_1)v + \alpha_1 a_3 s + \alpha_2 a_2 w + \alpha_2 u \in \text{ Rad}(z + z'),$$

then:

(3) $$\theta(r, (a_1 - b_1)v + a_3s) = (\alpha_1(a_1 - b_1) + \alpha_2 b_1)\tau(a_1 - b_1) = 0$$

$$\Rightarrow (\alpha_1(a_1 - b_1) + \alpha_2 b_1) = 0 \text{ since } a_1 \neq b_1$$

$$\Rightarrow r = \alpha_1 a_3 s + \alpha_2 a_2 w + \alpha_2 u.$$

Thus:

$$\begin{aligned}\theta(r, b_1 v + a_2 w + u) &= \alpha_2(a_2\tau(a_2) + \theta(u,u)) = 0 \\ &\Rightarrow \alpha_2 = 0 \text{ since } a_2\tau(a_2) + \theta(u,u) \neq 0 \\ &\Rightarrow \alpha_1 = 0 \text{ from (3) above and } a_1 \neq b_1.\end{aligned}$$

Thus for any b_1 satisfying (1) and (2) above, we have $< b_1 v + a_2 w + u > \in \Gamma(z, < w >)$ as required. Therefore, $\mathrm{diam}(\Gamma) = 2$ and Γ is connected, as claimed. □

Lemma 12.4. *Given $z \in \Delta(x,y) \setminus \Gamma$, there exist $z' \in \Gamma$ adjacent to z.*

Proof. Remember that by $(*)$, $V =< s,t > \perp < v,w > \perp U$. Given $z =< \hat{v} > \in \Delta(x,y) \setminus \Gamma$, we have:

$$\hat{v} = a_1 v + a_2 w + a_3 s + a_4 t + u \quad \text{with } u \in U, \text{ and } a_4 \neq 0.$$

Nondegeneracy of z implies:

$$\theta(\hat{v},\hat{v}) = a_1\tau(a_1) + a_2\tau(a_2) + a_3\tau(a_4) + a_4\tau(a_3) + \theta(u,u) \neq 0. \tag{4}$$

Two cases arise:

<u>Case I:</u> $a_1 v + a_3 s + a_4 t + u$ is nonsingular.

Then since $z+ < w > =< a_1 v + a_3 s + a_4 t + u, w >$, with $a_1 v + a_3 s + a_4 t + u$ a nonsingular element of $w^{\perp}$, we have z adjacent to $< w > \in \Gamma$ and are done.

<u>Case II:</u> $a_1 v + a_3 s + a_4 t + u$ is singular; that is, $a_1\tau(a_1) + a_3\tau(a_4) + a_4\tau(a_3) + \theta(u,u) = 0$.

Then (4) implies $a_2\tau(a_2) \neq 0$. Now if $< b_1 v + a_2 w >$ is to belong to Γ, then from the characterization of Γ in the proof of Lemma 12.3, we would require that:

$$b_1\tau(b_1) + a_2\tau(a_2) \neq 0 \text{ and } a_2\tau(a_2) \neq 0.$$

We already have the latter, so must choose b_1 such that:

$$b_1\tau(b_1) + a_2\tau(a_2) \neq 0. \tag{5}$$

Consider $z+ < b_1 v + a_2 w > = < (a_1 - b_1)v + a_3 s + a_4 t + u, b_1 v + b_2 w >$. And let:

$$r = (\alpha_1(a_1 - b_1) + \alpha_2 b_1)v + \alpha_1 a_3 s + \alpha_1 a_4 t + \alpha_1 u + \alpha_2 a_2 w \in \text{Rad}(z+ < b_1 v + a_2 w >),$$

then:

$$\begin{aligned}
&\theta(r, (a_1 - b_1)v + a_3 s + a_4 t + u) \\
&= (\alpha_1(a_1 - b_1) + \alpha_2 b_1)\tau(a_1 - b_1) + \alpha_1 a_3 \tau(a_4) + \alpha_1 a_4 \tau(a_3) + \alpha_1 \theta(u, u) = 0 \\
&\Rightarrow \alpha_1(a_1 - b_1)\tau(a_1 - b_1) - \alpha_1 a_1 \tau(a_1) + \alpha_2 b_1 \tau(a_1 - b_1) = 0 \\
&(\text{since } a_1 \tau(a_1) + a_3 \tau(a_4) + a_4 \tau(a_3) + \theta(u, u) = 0) \\
&\Rightarrow \alpha_1(-b_1 \tau(a_1 - b_1) - a_1 \tau(b_1)) + \alpha_2 b_1 \tau(a_1 - b_1) = 0, \text{ and} \\
&\theta(r, b_1 v + a_2 w) = \alpha_1(a_1 - b_1)\tau(b_1) + \alpha_2(b_1 \tau(b_1) + a_2 \tau(a_2)) = 0.
\end{aligned}$$

Therefore $Rad(z+ < b_1 v + a_2 w >) = 0$ if and only if:

$$(*) \qquad \begin{pmatrix} -a_1 \tau(b_1) - b_1 \tau(a_1 - b_1) & b_1 \tau(a_1 - b_1) \\ \tau(b_1)(a_1 - b_1) & b_1 \tau(b_1) + a_2 \tau(a_2) \end{pmatrix} \quad \text{is nonsingular.}$$

We choose b_1 as follows:

<u>**II A:**</u> $a_1 = 0$. $(*)$ holds if:

$$\begin{pmatrix} -b_1 \tau(-b_1) & b_1 \tau(-b_1) \\ -b_1 \tau(b_1) & b_1 \tau(b_1) + a_2 \tau(a_2) \end{pmatrix} \quad \text{is nonsingular.}$$

This is true if and only if:

$$b_1 \tau(b_1) a_2 \tau(a_2) \neq 0.$$

Since $a_2 \tau(a_2) \neq 0$, if we choose $b_1 \in \mathbb{F}^{\sharp}_{q^2}$ such that $b_1 \tau(b_1) + a_2 \tau(a_2) \neq 0$, then $< b_1 v + a_2 w > \in \Gamma \cap \Delta(z)$ as desired.

<u>**II B:**</u> $a_1 \neq 0$ and $a_1 \tau(a_1) + a_2 \tau(a_2) \neq 0$. Let $b_1 = a_1$, then $b_1 \tau(b_1) + a_2 \tau(a_2) \neq 0$. Furthermore, $(*)$ holds if:

$$\begin{pmatrix} -a_1 \tau(a_1) & 0 \\ 0 & a_1 \tau(a_1) + a_2 \tau(a_2) \end{pmatrix} \quad \text{is nonsingular.}$$

But this is true since $a_1 \neq 0 \neq a_1\tau(a_1) + a_2\tau(a_2)$.

<u>**II C:**</u> $a_1 \neq 0$ but $a_1\tau(a_1) + a_2\tau(a_2) = 0$. Let $\delta \in \mathbb{F}_{q^2}$ with $\delta\tau(\delta) = 2$, and $\delta + \tau(\delta) \neq 4$. Such a δ exists since q is odd implies that there are $(q+1)$ $\delta \in \mathbb{F}_{q^2}$ with $\delta\tau(\delta) = 2$, of which only two also satisfy $\delta + \tau(\delta) = 4$. Set $b_1 = \delta a_1$, then:

$$b_1\tau(b_1) + a_2\tau(a_2) = \delta\tau(\delta)a_1\tau(a_1) + a_2\tau(a_2) = 2a_1\tau(a_1) + a_2\tau(a_2) = a_1\tau(a_1) \neq 0.$$

In this case, $(*)$ holds if:

$$\begin{pmatrix} -a_1\tau(\delta a_1) - \delta a_1\tau(a_1 - \delta a_1) & \delta a_1\tau(a_1 - \delta a_1) \\ \tau(\delta a_1)(a_1 - \delta a_1) & 2a_1\tau(a_1) - a_1\tau(a_1) \end{pmatrix}$$

$$= \begin{pmatrix} a_1\tau(a_1)(2 - \delta - \tau(\delta)) & a_1\tau(a_1)(\delta - 2) \\ a_1\tau(a_1)(\tau(\delta) - 2) & a_1\tau(a_1) \end{pmatrix} \text{ is nonsingular.}$$

But that holds if and only if:

$$\begin{pmatrix} 2 - \delta - \tau(\delta) & \delta - 2 \\ \tau(\delta) - 2 & 1 \end{pmatrix} \text{ is nonsingular.}$$

Or equivalently:

$$2 - \delta - \tau(\delta) - (\delta - 2)(\tau(\delta) - 2) \neq 0$$

$$\Longleftrightarrow 2 - \delta - \tau(\delta) - (2 - 2(\delta + \tau(\delta)) + 4) \neq 0$$

$$\Longleftrightarrow \delta + \tau(\delta) - 4 \neq 0 \Longleftrightarrow \delta + \tau(\delta) \neq 4.$$

We had assumed that $\delta + \tau(\delta) \neq 4$, so the last inequality holds and $z + < b_1v + a_2w >$ is nondegenerate.

Thus for each possible choices of a_1, we have found b_1 such that $< b_1v + a_2w > \in \Gamma \cap \Delta(z)$, completing our proof in the case that $a_1v + a_3s + a_4t + u$ is singular.

This proves our assertion that given $z \in \Delta(x, y) \setminus \Gamma$, there exists $z' \in \Gamma$ adjacent to z. Thus this lemma (and hence Theorem 12.2) is true. □

Finally, assume that $n \geq 3$, and let K be the order complex of the poset of proper nondegenerate subspaces of V – ordered by inclusion. Then we have:

Lemma 12.5. *K is connected.*

Proof. Given any $x \in K$, there exists a $y \in K$ such that $dim(y) = 1$ and $y \leq x$. So it suffices to show that given any $x, y \in K$ with $dim(x) = dim(y) = 1$, there exists a path in K joining x and y.

If $x \oplus y$ is nondegenerate then $x * x \oplus y * y$ is the desired path. So assume that $x \oplus y$ is degenerate. Let $x =< v >$ with v a unit vector, then $x \oplus y$ degenerate implies that $y =< v + s >$ with s a totally singular element of $v^{\perp}$. Now $v^{\perp}$ is nondegenerate of dimension $n - 1 \geq 2$, so there exists a $t \in v^{\perp}$ such that $\{s, t\}$ is a hyperbolic pair. We will show that there is a unit vector $w = \alpha_1 s + \alpha_2 t \in< s, t >$ such that $< w >$, $x \oplus < w >$, and $y \oplus < w >$ are nondegenerate. Now $w \in< s, t >$ and a unit vector implies that $< w >$ and $x \oplus < w >$ are nondegenerate and:

$$\theta(w, w) = \alpha_1 \tau(\alpha_2) + \alpha_2 \tau(\alpha_1) = 1. \tag{1}$$

Now consider $y \oplus < w > =< v + s, \alpha_1 s + \alpha_2 t >$. If $r = \beta(v + s) + \gamma(\alpha_1 s + \alpha_2 t)$ is in Rad($y \oplus < w >$), then we have:

$$\theta(r, v + s) = \theta(\beta v + (\beta + \gamma\alpha_1)s + \gamma\alpha_2 t, v + s) = \beta + \gamma\alpha_2 = 0,$$

$$\theta(r, \alpha_1 s + \alpha_2 t) = \theta(\beta v + (\beta + \gamma\alpha_1)s + \gamma\alpha_2 t, \alpha_1 s + \alpha_2 t) =$$

$$\gamma\alpha_2\tau(\alpha_1) + (\beta + \gamma\alpha_1)\tau(\alpha_2) = 0,$$

$$\Rightarrow \beta\tau(\alpha_2) + \gamma(\alpha_2\tau(\alpha_1) + \alpha_1\tau(\alpha_2)) = 0$$

$$\Rightarrow \beta\tau(\alpha_2) + \gamma = 0 \quad \text{since } \alpha_1\tau(\alpha_2) + \alpha_2\tau(\alpha_1) = 1 \text{ by (1) above.}$$

Thus Rad($y \oplus < w >$) $= 0$ if and only if:

$$\begin{pmatrix} 1 & \alpha_2 \\ \tau(a_2) & 1 \end{pmatrix} \quad \text{is nonsingular.}$$

Equivalently, $y \oplus < w >$ is nondegenerate if and only if $\alpha_2\tau(\alpha_2) \neq 1$. Now there are $(q + 1)$ $\alpha \in \mathbb{F}_{q^2}$ such that $\alpha\tau(\alpha) = 1$ and for each such α there are q $\alpha' \in \mathbb{F}_{q^2}$ such that $\alpha\tau(\alpha') + \alpha'\tau(\alpha) = 1$. So there are $q(q + 1)$ $w = \alpha_1 s + \alpha_2 t \in< s, t >$ such that

w is a unit vector and $y \oplus < w >$ is degenerate. However, since $q \geq 3$, there are $q(q^2-1) > q(q+1)$ $w \in < s,t >$ with w a unit vector. Thus we can find $w \in < s,t >$ a unit vector, such that $y \oplus < w >$ is nondegenerate.

Thus, we have found $z \in K$ such that $x * x \oplus z * z * z \oplus y * y$ is a path in K joining x and y, as desired. □

SECTION 13 : THE COMPLEX OF PROPER NONDEGENERATE SUBSPACES OF A UNITARY SPACE OF DIMENSION $n \geq 4$ IS SIMPLY CONNECTED

Before we can prove this result we need to recall some basic terminology. Recall the definition of a geometry and flags from section 6. Given a flag T of a geometry Γ the *residue of* T in Γ is the subcomplex of Γ on the vertex set $\{v \in \Gamma \backslash T \mid v*t \text{ for all } t \in T\}$. A geometry is *residually connected* if the residue of all flags of corank at least 2 are connected and the residue of all flags of corank 1 are nonempty. A rank 2 geometry Γ on $\{1,2\}$ is a *generalized digon* if $x * y\ \forall\ x \in \Gamma_1$ and $y \in \Gamma_2$, where $\Gamma_i = \tau^{-1}(i)$ for $(i = 1,2)$.

Given a geometry Γ over I, the *diagram of* Γ is a graph on I such that given $i \neq j \in I$, $i * j$ if and only if there is a flag F of type $I - \{i,j\}$ such that the residue F is not a generalized digon.

A graph on a set $I = \{1, 2, \ldots, n\}$ is a *string* if there is an ordering on I such that the edges of the graph are precisely of the form $(i, i+1)\ \forall\ 1 \leq i \leq n-1$. A geometry Γ over I is a *string geometry* if the diagram of Γ is a string.

Let q be an odd prime power and (V, θ) an n-dimensional unitary space over $\mathbb{F}_{q^2}$, with $n \geq 4$. Assume that if $q = 3$, then $n \geq 5$. Let τ be the involutory field automorphism of $\mathbb{F}_{q^2}$.

Let K be the order complex of the poset of proper nondegenerate subspaces of V – ordered by inclusion. Let the 1,2, and $n-1$ dimensional subspaces in K be called *points, lines,* and *hyperplanes,* respectively. K can be thought of as a geometry over $I = \{1, 2, \ldots, n-1\}$ with the type function given by $\tau(x) = dim(x)\ \forall\ x \in K$.

Lemma 13.1. *K is a string geometry.*

Proof. Fix $i < j \in I$, and note that the residue of any flag of type $I - \{i,j\}$ is contained in $K_i \cup K_j$, where $K_l = \tau^{-1}(l)$ for $(l = i,j)$.

Two cases arise:

<u>CASE I</u> $j \neq i+1$.

Let $(x_1 < \ldots < \hat{x}_i < \ldots < \hat{x}_j < \ldots < x_{n-1})$ be a flag of type $I - \{i,j\}$ (here $\hat{x}_i$ denotes "exclude x_i"). If R is the residue of the above flag and $x, y \in R$ with $\tau(x) = i$ and $\tau(y) = j$, then $x < x_{i+1} < y$. Thus R is a generalized digon and hence (i,j) is not an edge of the diagram of K.

<u>CASE II</u> $j = i+1$.

Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis for V and set $U_k =< e_1, e_2, \ldots, e_k >$ for all $1 \leq k \leq n$. Consider the flag: $F = (U_1 < U_2 < \ldots < U_{i-1} < U_{i+2} < \ldots < U_{n-1})$ which is of type $I - \{i, i+1\}$. Then U_i and $W =< e_k \mid 1 \leq k \leq i+2 \text{ and } k \neq i >$ are elements of the residue of F of type i and $i+1$ respectively. However $U_i \nleq W$, hence the residue of F is not a generalized digon. Thus $(i, i+1)$ is an edge in the diagram of K.

Thus K is a string geometry, as claimed. □

Lemma 13.2. *K is residually connected.*

Proof. Since K is a pure complex (i.e., the maximal flags are all of type I), the residue of all flags of corank 1 are nonempty. So we need to show that the residues of flags of corank at least 2 are connected. Given $x < y \in K$, we let $K(< x) = \{z \in K \mid z < x\}$, $K(> x) = \{z \in K \mid x < z\}$, and $(x,y) = K(> x) \cap K(< y)$.

Given a flag $F = (x_1 < x_2 < \ldots x_r)$ of corank at least 2, note that the residue of F is given by:

$$K(< x_1) * (x_1, x_2) * \cdots * (x_{r-1}, x_r) * K(> x_r),$$

where the join of complexes is as defined in section 9. If more than one of the above complexes are nonempty, by (2.1) on pg. 6 in [A2], the residue is connected. So assume that the residue is in one of the following forms:

(1) $K(< x)$. Then $dim(x) \geq 3$ since the flag is of corank at least 2. Also, $K(< x)$

can be identified with the order complex of proper nondegenerate subspaces of x.

(2) (x, y). Then $dim(y) - dim(x) \geq 3$ since the flag is of corank at least 2. And (x, y) can be identified with the order complex of proper nondegenerate subspaces of y/x considered as a $(dim(y) - dim(x))$-dimensional unitary space.

(3) $K(> x)$ Then $dim(x) \leq n - 3$ since the flag is of corank at least 2. Also, $K(> x)$ can be identified with the order complex of proper nondegenerate subspaces of V/x considered as a $(n - dim(x))$-dimensional unitary space.

In all three cases, the residue can be identified with the order complex of proper nondegenerate subspaces of a unitary space of dimension ≥ 3; and thus, by Lemma 12.5, is connected. Thus K is residually connected, as claimed. □

We have shown that K is a *residually connected string geometry.* Also note that given points $x, y \in K$, they are incident to at most one line l.

Let Δ be the collinearity graph of K; that is, the graph with vertex set the points of K with distinct points x, y adjacent (denoted $x * y$) if $x + y$ is a line in K. Let $P(\Delta)$ be the set of paths of Δ and let $\sim$ be the invariant relation on $P(\Delta)$ such that $ker(\sim)$ is the closure of all cycles in the collinearity graphs of residues of hyperplanes in K. Refer to [AS1] and [AS3] for the definition of the terminology. Then we have the following result:

Lemma 13.3. $\pi_1(K) \equiv \tilde{\pi}(\Delta)$.

Proof. This follows from the remark about points in K being incident to at most one line, and Theorem B on pg. 18 in [AS3]. □

Remark. *Note that if a cycle is in the collinearity graph of the residue of a nondegenerate subspace, say U, of dimension $m \leq n - 1$, then the cycle is in the collinearity graph of the residue of any hyperplane containing U, and thus is null-homotopic. So to show a cycle is null-homotopic, it suffices to show that it is in the collinearity graph of the residue of some proper nondegenerate subspace of V.*

Lemma 13.4. *Δ is connected with diam(Δ) = 2. Furthermore, if x and y are not adjacent and U is a 3-dimensional subspace of V containing both, then there exists*

$z \leq U$ with $z \in \Delta(x, y)$.

Proof. It suffices to prove the second statement; so let x, y, and U as in the statement of the lemma. However, the second statement follows from the proof of Lemma 12.5. □

Lemma 13.5. *Let x, y and $z \in \Delta$ be distinct, with x not adjacent to either y or z, and y and z adjacent. Then $x + y + z$ is nondegenerate of dimension 3.*

Proof. $x + y + z$ is 3-dimensional, since otherwise $x + y = x + y + z = y + z$ contradicting the choice of x, y, and z. Let $x =< v_1 >$, $y =< v_2 >$, and $z =< v_3 >$ with v_i unit vectors $\forall\ 1 \leq i \leq 3$.

Consider $r = \alpha v_2 + \beta v_3 \in$ Rad$(y + z)$. Then:

$$\theta(\alpha v_2 + \beta v_3, v_2) = \alpha + \beta\theta(v_3, v_2) = 0, \text{ and}$$

$$\theta(\alpha v_2 + \beta v_3, v_3) = \alpha\theta(v_2, v_3) + \beta = 0.$$

Therefore, $Rad(y + z) = 0$ implies:

$$(*) \qquad \begin{pmatrix} 1 & \theta(v_3, v_2) \\ \theta(v_2, v_3) & 1 \end{pmatrix} \text{ is nonsingular.}$$

Suppose $x+y+z$ is degenerate. Then the fact that $x+y+z$ is 3-dimensional implies that there is a nonzero totally singular element $s \in< v_2, v_3 >^{\perp}$ such that $v_1 = a_2v_2 + a_3v_3 + s$. Note that $\theta(v_1, v_1) = 1$ implies that $a_2 \neq 0$ or $a_3 \neq 0$. Assume without loss of generality that $a_3 \neq 0$.

Consider $x + y =< a_2v_2 + a_3v_3 + s, v_2 >=< v_3 + a_3^{-1}s, v_2 >=< v_3 + t, v_2 >$ where $t = a_3^{-1}s$. Then $r = \alpha v_3 + \alpha t + \beta v_2 \in Rad(x + y)$ implies:

$$\theta(\alpha v_3 + \alpha t + \beta v_2, v_2) = \alpha\theta(v_3, v_2) + \beta = 0$$

$$\theta(\alpha v_3 + \alpha t + \beta v_2, v_3 + t) = \alpha + \beta\theta(v_2, v_3) = 0.$$

But then $Rad(x + y) = 0$, since:

$$\begin{pmatrix} \theta(v_3, v_2) & 1 \\ 1 & \theta(v_2, v_3) \end{pmatrix}$$

is nonsingular by $(*)$. This contradicts our assumption that $x + y$ is degenerate.

Therefore, $x + y + z$ must be nondegenerate of dimension 3, as claimed. □

In view of Lemma 13.4 and Lemma 10.1 it suffices to show that triangles, squares, and pentagons in Δ are null-homotopic.

Lemma 13.6. *Triangles in Δ are null-homotopic.*

Proof. Let $x_1x_2x_3x_1$ be a triangle in Δ. Then one of the following occurs:

<u>Case I</u> $x_1+x_2+x_3$ is 2-dimensional. Let U be any nondegenerate hyperplane containing $x_1 + x_2 = x_1 + x_2 + x_3$. Then the triangle is in the collinearity graph of the residue of U and is null-homotopic.

<u>Case II</u> $x_1 + x_2 + x_3$ is nondegenerate of dimension 3. Then the triangle is in the collinearity graph of $x_1 + x_2 + x_3$ and by the Remark is null-homotopic.

<u>Case III</u> $x_1 + x_2 + x_3$ is degenerate and 3-dimensional. We assumed that if $q = 3$ then $n \geq 5$. So let U be a 4-dimensional nondegerate subspace containing $x_1 + x_2 + x_3$ – such a subspace exists by Witt's Lemma. Then, by the Remark, the triangle is null-homotopic. So assume that $q \geq 5$ and note that if $x_i =< v_i >$ with $\theta(v_i, v_i) = 1$ for $1 \leq i \leq 3$, then v_1, v_2, v_3 satisfy the conditions of Theorem 12.1. So there exists $x_4 \in \Delta$ such that $x_1x_2x_3x_1$ is in the closure of the triangles $x_1x_2x_4x_1$, $x_1x_4x_3x_1$, and $x_2x_3x_4x_2$ with the last three null-homotopic by Theorem 12.1 and the Remark. So $x_1x_2x_3x_1$ is null-homotopic as desired.

Thus we have the lemma. □

Lemma 13.7. *Squares in Δ are null-homotopic.*

Proof. Let $x_1x_2x_3x_4x_1$ be a square in Δ. If $x_1 * x_3$, then $x_1x_2x_3x_4x_1$ is in the closure of $x_1x_2x_3x_1$ and $x_1x_3x_4x_1$, and by Lemma 13.6 is null-homotopic.

So assume that x_1 and x_3 are disjoint; then Theorem 12.2 says that there is a path $\lambda_0\lambda_1 \cdots \lambda_n$ with $\lambda_0 = x_2$, $\lambda_n = x_4$ and $\lambda_i \in \Delta(x_1, x_3) \, \forall \, 0 \leq i \leq n$. Therefore, $x_1x_2x_3x_4x_1$ is in the closure of the triangles:

$$x_1\lambda_i\lambda_{i+1}x_1 \text{ and } x_3\lambda_i\lambda_{i+1}x_3 \;\forall\; 0 \leq i \leq n-1;$$

and thus, by Lemma 13.6, is null-homotopic. □

Lemma 13.8. *Pentagons in Δ are null-homotopic.*

Proof. Let $x_1 \cdots x_5 x_1$ be a pentagon in Δ. We can assume that x_1 is not adjacent to either x_3 or x_4; otherwise the pentagon is in the closure of a square and a triangle and thus is null-homotopic. But then, by Lemma 13.5, $U = x_1 + x_3 + x_4$ is a nondegenerate 3-dimensional subspace of V. By Lemma 13.4, we can find $x_6 \in \Delta(x_1, x_3) \cap U$ and $x_7 \in \Delta(x_1, x_4) \cap U$. Then the pentagon $x_1x_6x_3x_4x_7x_1$ is in the collinearity graph of the residue of U and thus by the Remark is null-homotopic.

The original pentagon, $x_1x_2x_3x_4x_5x_1$, is then in the closure of $x_1x_6x_3x_4x_7x_1$ (null-homotopic), and the squares $x_1x_2x_3x_6x_1$ and $x_1x_7x_4x_5x_1$ null-homotopic by Lemma 13.7. Thus $x_1x_2x_3x_4x_5x_1$ is null-homotopic, as claimed. □

Lemmas 13.6–8 show that $\tilde{\pi}(\Delta)$ is null-homotopic, so in view of Lemma 13.3 we have:

Theorem 13.9. *Let q be an odd prime power and V an n-dimensional unitary space over $\mathbb{F}_{q^2}$, with $n \geq 4$. Assume that if $q = 3$, then $n \geq 5$. Then the order complex of the poset of proper nondegenerate subspace of V – ordered by inclusion – is simply connected.*

Chapter 4

Cohen-Macaulay complexes

In Chapter 4 we consider Cohen-Macaulay complexes. A well-known class of such complexes is the class of buildings. We consider three other Cohen-Macaulay complexes in this chapter.

In Section 14 we include some basic properties and well-known results about Cohen-Macaulay complexes. In Section 15 we show that if a prime p divides $q-1$, then $\mathcal{A}_p(Sp_{2n}(q))$ is Cohen-Macaulay of dimension $n-1$. In Section 16 we show that an analogous result holds for $\mathcal{A}_p(GU_n(q))$ under the added assumption that $p \neq 2$. Finally, in Section 17 we show that the order complex of proper nondegenerate subspaces of a $2n$-dimensional symplectic space is Cohen-Macaulay of dimension $n-2$.

It is worth noting that the base field $\mathbb{F}$ for the groups and complexes considered in this chapter is the splitting field for the action of a p-element; that is, $\mathbb{F}$ contains p^{th} roots of unity. This is the main difference between the complexes considered in this chapter and those studied in chapters 5 and 6 where q has order $d > 1$ in $\mathbb{Z}/p\mathbb{Z}$.

SECTION 14 : BASIC PROPERTIES OF COHEN-MACAULAY COMPLEXES

Given a simplicial complex K and a simplex $\sigma \in K$, the *star of* σ, denoted $st_K(\sigma)$, is the subcomplex of K defined on the simplices τ such that $\sigma \cup \tau$ is a simplex of K. And the *link of* σ, denoted $lk_K(\sigma)$, is the subcomplex of K defined on those $\tau \in st_K(\sigma)$ such that $\sigma \cap \tau = \emptyset$.

A d-dimensional complex K is *d-spherical* (or simply *spherical*) if it is $(d-1)$-connected. And, using Quillen's terminology from [Q], a d-spherical complex K is *Cohen-Macaulay* (or simply *C.M.*) of dimension d if $lk_K(\sigma)$ is $(d-r-1)$-spherical for each r-simplex $\sigma \in K$. Throughout this chapter we shall fix the following terminology:

> Let K be a simplicial complex and G be a group acting on K. Then for any $A \leq G$, we can consider the action of A on $sd(K)$, the first barycentric subdivision of K as defined in section 9. Then $Fix(A)$ will be the subcomplex of

$sd(K)$ generated by those simplices of K on which A acts as the identity. Note that this agrees with the notion of $Fix(A)$ defined in the discussion following Theorem 9.13. In algebraic topology this complex is often denoted K^A.

We shall be using the term $Fix(A)$ without explicitly defining it throughout this and the following sections.

Lemma 14.1. *Given simplicial complexes $K_1, \dots, K_r$ with K_i n_i-spherical $\forall\ 1 \leq i \leq r$, $K_1 * \cdots * K_r$ is $(\sum_{i=1}^{r} n_i + r - 1)$-spherical.*

Proof. When $r = 1$ the statement holds trivially. Assume that the result holds for $r - 1 \geq 0$. Now, by the definition of $K * K'$, $dim(K * K') = dim(K) + dim(K') + 1$. Thus:

$$\begin{aligned} dim(K_1 * \cdots * K_r) &= dim(K_1 * \cdots * K_{r-1}) + dim(K_r) + 1 \\ &= \sum_{i=1}^{r-1} n_i + r - 2 + n_r + 1 = \sum_{i=1}^{r} n_i + r - 1. \end{aligned}$$

By the inductive hypothesis, $K_1 * \cdots * K_{r-1}$ is $\sum_{i=1}^{r-1} n_i + r - 3$-connected, and by our assumption K_r is $n_r - 1$-connected. Thus, by (2.6) on pg. 8 in [A2], $K_1 * \cdots * K_r$ is:

$$\sum_{i=1}^{r-1} n_i + r - 3 + n_r - 1 + 2 = \sum_{i=1}^{r} n_i + r - 2\text{-connected.}$$

Thus we have the lemma. □

Given a poset P and $x \in P$, recall the definition of $P(\leq x)$, $P(< x)$, and $P(> x)$ given in section 1. Also recall the height of x equals the dimension of $P(\leq x)$. A poset P is called C.M. if the order complex of the poset is C.M. Note that if $\sigma = (x_0 < x_1 < \cdots < x_r)$ is an r-simplex in P, then:

$$(*) \qquad lk_P(\sigma) = P(< x_0) * (x_0, x_1) * \cdots * (x_{r-1}, x_r) * P(> x_r).$$

This fact gives us the following lemma.

Lemma 14.2. *A poset P is C.M. of dimension n if and only if the following four conditions are satisfied:*

(1) *P is n-spherical,*

(2) $P(> x)$ *is* $(n - h(x) - 1)$*-spherical,*

(3) $P(< x)$ *is* $(h(x) - 1)$*-spherical, and*

(4) (x, y) *is* $(h(y) - h(x) - 2)$*-spherical, for* $x < y$.

Furthermore, if P *is C.M. of dimension* n *then the above four conditions hold with spherical replaced by C.M. Finally, in a C.M. poset every maximal element has height* n*; that is, the order complex of C.M. posets are pure complexes.*

Proof. This is Proposition 8.6 on pg. 118 in [Q]; however, for the sake of completeness, we include a proof providing the details.

Assume that P is C.M. of dimension n. Then (1) holds trivially. Let $x \in P$, choose $\sigma = (x_0 < x_1 < \cdots < x_r)$ a maximal chain in $P(\leq x)$; that is, $r = h(x)$ and $x_r = x$. By $(*)$, $lk_P(\sigma) = P(< x_0) * (x_0, x_1) * \cdots * P(> x_r) = P(> x)$ since σ is a maximal chain of $P(\leq x)$. Thus, by the definition of C.M., we have $P(> x)$ is $(n - r - 1) = (n - h(x) - 1)$-spherical. Thus (2) holds.

Let $\sigma = (x_0 < x_1 < \cdots < x_r)$ be a maximal chain of $P(\geq x)$; that is, $x_0 = x$ and $r = n - h(x)$. Again by $(*)$, $lk_P(\sigma) = P(< x_0) * \cdots * P(> x_r) = P(< x)$. And so $P(< x)$ is $(n - r - 1) = (h(x) - 1)$-spherical. Thus (3) holds.

Statement (4) follows from (2) and the definiton of $h(y)$, by identifying (x, y) with $P(< y)(> x)$. Thus, when P is C.M., (1)–(4) hold.

Now assume that (1)–(4) hold and let $\sigma = (x_0 < \cdots < x_r)$ be a chain in P. As P is n-spherical, it remains to show $lk_P(\sigma)$ is $(n - r - 1)$-spherical.

By $(*)$ $lk_P(\sigma) = P(< x_0) * (x_0, x_1) * \cdots * P(> x_r)$ and since (1)–(4) hold we have: $P(< x_0)$ is $(h(x_0) - 1)$-spherical, (x_i, x_{i+1}) is $(h(x_{i+1}) - h(x_i) - 2)$-spherical for $0 \leq i \leq r-1$ and $P(> x_r)$ is $(n - h(x_r) - 1)$-spherical. Thus, by Lemma 14.1, we have $lk_P(\sigma)$ is spherical of dimension:

$$(h(x_0) - 1) + \sum_{i=0}^{r-1} (\ (h(x_{i+1}) - h(x_i) - 2\ ) + (n - h(x_r) - 1) + r + 2 - 1$$
$$= -1 + h(x_r) - 2r + n - h(x_r) - 1 + r + 1 = n - r - 1.$$

Thus P is C.M. as claimed.

The fact that spherical can be replaced by C.M. in the four statements follow from this characterization of C.M. posets. Assume that P is C.M. of dimension n. Note that replacing spherical by C.M. in (1) gives us a tautology. I will prove that $P(> x)$ is C.M. of dimension $(n - h(x) - 1)$; the proofs for $P(< x)$ and (x, y) are analogous. Given $x \in P$, it suffices to show that (1)–(4) hold for $P(> x)$. The dimension of $P(> x)$ is $(n - h(x) - 1) = n'$. Since P is C.M., $P(> x)$ is n'-spherical and (1) holds for $P(> x)$. Also given $y \in P(> x)$, note that $h_{P(>x)}(y) \equiv h'(y) = h(y) - h(x) - 1$.
Now $P(> x)(> y) = P(> y)$ which is spherical of dimension:

$$n - h(y) - 1 = n' + h(x) + 1 - h'(y) - h(x) - 1 - 1 = n' - h'(y) - 1.$$

So (2) holds for $P(> x)$.
Similarly, $P(> x)(< y) = (x, y)$ which is spherical of dimension: $h(y) - h(x) - 2 = h'(y) - 1$.
Thus (3) holds for $P(> x)$.
Since (4) follows from (2), (4) also holds for $P(> x)$, and $P(> x)$ is C.M. of dimension $(n - h(x) - 1)$.

Finally, given a maximal element $x \in P$, we have $P(> x) = \emptyset$ which is -1-spherical. On the other hand, $P(> x)$ is $(n - h(x) - 1)$-spherical by (2) above. Thus $n - h(x) - 1 = -1$ or $h(x) = n$.

Thus we have the lemma. □

Consider two C.M. simplicial complexes K, L of dimension n, m, respectively. Note that, by Lemma 14.1, $K * L$ is $(n + m + 1)$-spherical. Let $\sigma * \tau$ be an r-simplex in $K * L$ with $r = i + j + 1$, and σ is an i-simplex of K, and τ is a j-simplex of L. As K and L are C.M., note that $lk_K(\sigma)$ and $lk_L(\tau)$ are $(n - i - 1)$ and $(m - j - 1)$-spherical, respectively. Consider $\sigma' * \tau' \in K * L$. $\sigma * \tau \cup \sigma' * \tau' = \sigma \cup \sigma' * \tau \cup \tau'$ is a simplex of $K * L$ if and only if $\sigma \cup \sigma'$ and $\tau \cup \tau'$ are simplices of K and L, repectively. Thus, $lk_{K*L}(\sigma * \tau) = lk_K(\sigma) * lk_L(\tau)$. By Lemma 14.1 this implies that $lk_{K*L}(\sigma * \tau)$ is spherical of dimension:

$$n - i - 1 + m - j - 1 + 1 = n + m + 1 - (i + j + 1) - 1 = dim(K * L) - r - 1.$$

Therefore $K * L$ is C.M. of dimension $n+m+1$, and we have the following result:

Lemma 14.3. *Given C.M. complexes K and L of dimensions n and m respectively, $K * L$ is C.M. of dimension $n+m+1$. Furthermore if $G_1, G_2, \ldots, G_s$ are finite groups such that $\mathcal{A}_p(G_i)$ is C.M. of dimension n_i for $1 \le i \le s$, then $\mathcal{A}_p(G_1 \times \cdots \times G_s)$ is C.M. of dimension $\sum_{i=1}^{s} n_i + s - 1$.*

Proof. We have already proved the first statement. The second statement follows from the first statement and the fact that $\mathcal{A}_p(G_1 \times \cdots \times G_s) \simeq \mathcal{A}_p(G_1) * \cdots * \mathcal{A}_p(G_s)$ (Lemma 9.4), by induction on s. □

Section 15 : If $p|q-1$, then $\mathcal{A}_p(Sp_{2n}(q))$ is Cohen-Macaulay of dimension $n-1$

In Theorem 12.4 on pg. 126 in [Q] Quillen shows that if p is a prime dividing $q-1$, then $\mathcal{A}_p(GL_n(q))$ is C.M. of dimension $n-1$. In this section we show that, under the same assumptions for p and q, $\mathcal{A}_p(Sp_{2n}(q))$ is also C.M. of dimension $n-1$.

Let (V, f) be a $2n$-dimensional symplectic space over $\mathbb{F}_q$, $G = Sp_{2n}(q)$, and p a prime dividing $q-1$. We prove that $\mathcal{A}_p(G)$ is C.M. of dimension $n-1$ by induction on n.

We first need to consider the decomposition of V under the action of A, where $A \in \mathcal{A}_p(G)$. Let $V = \bigoplus_{\lambda} V_\lambda$ be a decomposition into homogeneous components under the action of A. And let $\Phi = \{\lambda : A \to \mathbb{F}_q^\sharp \mid V_\lambda \neq 0\}$, then we have:

Lemma 15.1. *If p is odd, then given $\lambda \in \Phi$, $\lambda^{-1} \in \Phi$ and $V_\lambda + V_{\lambda^{-1}}$ is symplectic. Furthermore, if $1 \neq \lambda \in \Phi$ then V_λ is totally singular and $V_\lambda + V_{\lambda^{-1}} = V_\lambda \oplus V_{\lambda^{-1}}$. If $p = 2$, then for all $\lambda \in \Phi$ V_λ is nondegenerate. Also, $V_\lambda \perp V_\gamma$ $\forall\ \lambda \neq \gamma \in \Phi$.*

Proof. First assume that $p \neq 2$. We show that if $\lambda \neq 1$, then V_λ is totally singular. Note that given $\lambda \in \Phi$ and $a \in A$, $\lambda(a)^p = \lambda(a^p) = 1$. Let $x, y \in V_\lambda$ and $a \in A$, then:

$$f(x,y) = f(ax, ay) = \lambda^2(a) f(x,y) \Rightarrow (1 - \lambda^2(a)) f(x,y) = 0.$$

Now $p \neq 2$ and $\lambda \neq 1$ implies that there exists $a \in A$ such that $\lambda^2(a) \neq 1$. Then, as the above equation is satisfied for all $a \in A$, $f(x,y) = 0$.

Now let $x \in V_\lambda^\sharp$. If $\forall\ \gamma \in \Phi$, $V_\gamma \leq x^\perp$, then $x \in \mathrm{Rad}(V)$, a contradiction. So let $\gamma \in \Phi$ such that $V_\gamma \not\leq x^\perp$, and choose $y \in V_\gamma$ such that $f(x,y) = 1$. Then:

$$1 = f(x,y) = f(ax,ay) = \lambda(a)\gamma(a)f(x,y) = \lambda(a)\gamma(a)\ \forall\ a \in A.$$

So $\gamma = \lambda^{-1}$, thus $\lambda^{-1} \in \Phi$. Also note that $U =< x, y >$ is a hyperbolic plane of V. Thus, by (19.3.1) on pg. 77 in [A1], $V = U \oplus U^\perp$.

As $A \leq Sp_{2n}(q)$, A acts on $U^\perp$. Let $W_1 = (U^\perp)_\lambda$ and $W_2 = (U^\perp)_{\lambda^{-1}}$; then $dim(V_\lambda) = dim(W_1) + 1$ and $dim(V_{\lambda^{-1}}) = dim(W_2) + 1$, so by induction $dim(V_\lambda) = dim(V_{\lambda^{-1}})$. Furthermore, as $V_\lambda = W_1 \oplus < x >$, $V_{\lambda^{-1}} = W_2 \oplus < y >$, and $V_\lambda + V_{\lambda^{-1}} = W_1 + W_2 \oplus U$; by induction on the dimension of V_λ we see that $V_\lambda + V_{\lambda^{-1}}$ is symplectic as claimed.

Now let $p = 2$. It suffices to show that for $\lambda \neq \gamma \in \Phi$, $V_\lambda \perp V_\gamma$, since then the nondegeneracy of f implies that $f|_{V_\lambda}$ is nondegenerate for each $\lambda \in \Phi$.
Let $\lambda \neq \gamma \in \Phi$ and $a \in A$ such that $\lambda(a) \neq \gamma(a)$. If $u \in V_\lambda$ and $v \in V_\gamma$, then we have:

$$f(u,v) = f(au,av) = \lambda(a)\gamma(a)f(u,v) \quad \Rightarrow (1 - \lambda(a)\gamma(a))f(u,v) = 0.$$

But since $\lambda(a) \neq \gamma(a)$ and $\lambda(a)^2 = 1$, this implies that $f(u,v) = 0$. Since u, v were arbitrary, we have $V_\lambda \perp V_\gamma$ if $\lambda \neq \gamma \in \Phi$. $\square$

Note that since $p|q-1$, $\mathcal{A}_p(Sp_2(q)) \neq \emptyset$. Thus we have:

Lemma 15.2. *Given $G = Sp_{2n}(q)$ and a prime $p|q-1$, $m_p(G) = n$.*

Proof. Consider an orthogonal decomposition of $V = V_1 \perp V_2 \perp \cdots \perp V_n$ into two dimensional symplectic subspaces. Then, since $\mathcal{A}_p(Sp_2(q))$ is nonempty, we can choose an element $A \in \mathcal{A}_p(G)$ given by $A = A_1 \times A_2 \times \cdots \times A_n$ where $A_i \in Sp(V_i)\ \forall\ 1 \leq i \leq n$. Thus, $m_p(G) \geq n$.

Fix $A \in \mathcal{A}_p(G)$. When $p \neq 2$, by Lemma 15.1, we have a decomposition of $V = V_0 \perp V_{\lambda_1} \oplus V_{\lambda_1^{-1}} \perp \cdots \perp V_{\lambda_s} \oplus V_{\lambda_s^{-1}}$, where $V_0 = C_V(A) = V_\lambda$ when $\lambda = 1$. Note that $s \leq n$. Again by Lemma 15.1, when $p = 2$ we have an orthogonal decomposition $V = V_0 \perp V_{\lambda_1} \perp \oplus \perp V_{\lambda_s}$, with $s \leq n$.

Now, for all $1 \leq i \leq s$, let $A_i = C_A(V_{\lambda_i} \oplus V_{\lambda_i^{-1}})$ when $p \neq 2$, and $A_i = C_A(V_{\lambda_i})$ when $p = 2$. Note that when $p \neq 2$, $A_i = C_A(V_{\lambda_i}) = C_A(V_{\lambda_i^{-1}})$. If W_i is a simple $\mathbb{F}_q[A]$-submodule of V_{λ_i}, then $V_{\lambda_i} = < W \mid W \cong W_i >$; thus, $C_A(W_i) = C_A(V_{\lambda_i}) = A_i$. But $A/C_A(W_i)$ acts faithfully irreducibly on W_i, and since $p|q-1$ we have $A/A_i = A/C_A(W_i) \cong \mathbb{Z}/p\mathbb{Z}$. Since $\bigcap_{1 \leq i \leq s} A_i = 1$, we have the normal series:

$$A > A_1 \geq A_1 \cap A_2 \geq \cdots \geq \bigcap_{1 \leq i \leq s-1} A_i \geq 1,$$

where each factor group is isomorphic to 1 or $\mathbb{Z}/p\mathbb{Z}$. Thus $m_p(A) \leq s \leq n$. Since this holds for each $A \in \mathcal{A}_p(G)$ $m_p(G) \leq n$.

Thus we have shown that $m_p(G) = n$, as claimed. □

Note that in particular $dim(\mathcal{A}_p(Sp_2(q))) = m_p(Sp_2(q)) - 1 = 0$. Since $\mathcal{A}_p(Sp_2(q))$ is nonempty, that is, -1-connected, we have our claim in the base case $n = 1$. By Proposition 10.1 on pg. 122 in [Q], in the general case we need to show that $\forall\ A \in \mathcal{A}_p(G) \cup \{1\}$ $\mathcal{A}_p(G)(> A)$ is $(m_p(G) - m_p(A) - 1)$-spherical. Now the fact that $dim(\mathcal{A}_p(G)) = m_p(G) - 1$ and $dim(\mathcal{A}_p(G)(\leq A)) = m_p(A) - 1$ implies that $\mathcal{A}_p(G)(> A)$ is $(m_p(G) - m_p(A) - 1)$-dimensional. Thus it suffices to show that $\mathcal{A}_p(G)$ is $(n-2)$-connected and that $\forall\ A \in \mathcal{A}_p(G)$ $\mathcal{A}_p(G)(> A)$ is $(n - m_p(A) - 2) = (n - h(A) - 3)$-connected.

Assume that $p \neq 2$ and $n > 1$. We have:

Lemma 15.3. *$\mathcal{A}_p(G)(> A)$ is $(n - h(A) - 3)$-connected for all $A \in \mathcal{A}_p(G)$.*

Proof. Since $\mathcal{A}_p(G)(> A) = \mathcal{A}_p(C_G(A))(> A)\ \forall\ A \in \mathcal{A}_p(G)$, it suffices to show that the latter is $(n - h(A) - 3)$-connected. Let $\Phi \supseteq \Phi^+$ be a subset such that $\lambda \neq \gamma \in \Phi^+$ implies $\lambda\gamma \neq 1$ and $\Phi = \{\lambda, \lambda^{-1} | \lambda \in \Phi^+\}$. Then note that $V = \bigoplus_{\lambda \in \Phi^+} (V_\lambda + V_{\lambda^{-1}})$ with $V_\lambda + V_{\lambda^{-1}} = V_\lambda \oplus V_{\lambda^{-1}}$ if $\lambda \neq 1$.

We have $C_G(A) = \bigcap_{\lambda \in \Phi} N_G(V_\lambda) = C_G(V_0^\perp) \times \prod_{\lambda \in \Phi^+ - \{1\}} H_\lambda$, where $V_0 = C_V(A) = V_\lambda$ when $\lambda = 1$, and $H_\lambda = C_G((V_\lambda \oplus V_{\lambda^{-1}})^\perp) \cap N_G(N_\lambda) \cap N_G(N_{\lambda^{-1}})$. For any nondegenerate subspace $U \leq V$, we have $C_G(U^\perp) = N_G(U)/C_G(U) \cong Sp(U)$. Therefore, $C_G(V_0^\perp) \cong Sp(V_0)$, and $C_G((V_\lambda \oplus V_{\lambda^{-1}})^\perp) = Sp(V_\lambda \oplus V_{\lambda^{-1}})$. For each $\lambda \in \Phi^+ - \{1\}$, let $G_\lambda = Sp(V_\lambda \oplus V_{\lambda^{-1}})$, then $H_\lambda \cong N_{G_\lambda}(V_\lambda) \cap N_{G_\lambda}(V_{\lambda^{-1}}) \cong GL(V_\lambda)$. Thus we have:

$$C_G(A) \cong Sp(V_0) \times \prod_{\lambda \in \Phi^+ - \{1\}} GL(V_\lambda).$$

By Theorem 12.4 on pg. 126 in [Q] and induction on the dimension of the symplectic spaces, we have $\mathcal{A}_p(Sp(V_0))$ and $\mathcal{A}_p(GL(V_\lambda))$ are C.M. of dimensions $n_1 - 1$ and $n_\lambda - 1$, respectively, where $n_1 = \frac{dim(V_0)}{2}$, $n_\lambda = dim(V_\lambda)$, and $n = \sum_{\lambda \in \Phi^+} n_\lambda$. By Lemma 14.3 we thus have $\mathcal{A}_p(C_G(A))$ is C.M. of dimension $n-1$.

Thus, by Lemma 14.2, $\mathcal{A}_p(C_G(A))(> A)$ is $(n - h(A) - 3)$-connected as desired. □

It remains to show that $\mathcal{A}_p(G)$ is $(n-2)$-connected. Let $\mathfrak{B}$ be the Tits building of (V, f), and note that it is $(n-2)$-connected. Let $B = sd(\mathfrak{B})$ and note that by Lemma 9.1, B is also $(n-2)$-connected. Define:

$$B \times \mathcal{A}_p(G) \supseteq F = \{(x, A) | x \in Fix(A)\}.$$

Note that F is the closed subposet of $\mathfrak{B} \times \mathcal{A}_p(G)$ mentioned just prior to Remark 9.14. Thus, by Theorem 9.13, if $\forall\ x \in B$ and $A \in \mathcal{A}_p(G)$, F_x and F_A are $(n-2)$-connected, then $\mathcal{A}_p(G)$ is $(n-2)$-connected; here F_x and F_A are as defined in Theorem 9.13.

Lemma 15.4. *Let $x \in B$ then F_x is $(n-2)$-connected.*

Proof. Note that $F_x = \mathcal{A}_p(G_x)$, where G_x stabilizes x. Now G_x is a standard parabolic subgroup P. Let U be the unipotent radical of P and $\pi : P \to P/U \cong L$ the canonical homomorphism (where L is the Levi factor). Let $x = (U_0 < U_1 < \cdots < U_s)$, $n_0 = dim(U_0)$ and $\forall\ 1 \leq i \leq s$ let $n_i = dim(U_i) - \sum_{j=0}^{i-1} n_j$. Finally, let $n_{s+1} = n - \sum_{i=0}^{s} n_i$, then $n_{s+1} < n$, since $U_s \neq 0$. By the definition of the Levi factor we have:

$$L \cong GL_{n_0}(q) \times GL_{n_1}(q) \times \cdots \times GL_{n_s}(q) \times Sp_{2n_{s+1}}(q).$$

By induction on the dimension of the symplectic spaces and Theorem 12.4 on pg. 126 in [Q], we have:

(1) $\mathcal{A}_p(Sp_{2n_{s+1}}(q))$ is C.M. of dimension $n_{s+1} - 1$, and

(2) $\mathcal{A}_p(GL_{n_i}(q))$ is C.M. of dimension $n_i - 1$ $\ \forall\ 0 \leq i \leq s$.

Thus, by Lemma 14.3, $\mathcal{A}_p(L)$ is C.M. of dimension:

$$\sum_{i=0}^{s+1}(n_i - 1) + (s+2) - 1 = \sum_{i=0}^{s+1} n_i - 1 = n - 1.$$

[**Note:** The fact that $\mathcal{A}_p(L)$ is C.M. of dimension $n-1$ is independent of $p \neq 2$.]

Let $f : \mathcal{A}_p(P) \to \mathcal{A}_p(L)$ defined by $x \mapsto \bar{x}$ be the map induced by π. Then f satisfies the following properties:

(1) If $x < y \in \mathcal{A}_p(P)$, then $f(x) < f(y)$. This follows from the fact that $U \leq O_{p'}(P)$ so $x \cap U = y \cap U = \{1\}$.

(2) $f|y \equiv \{x \in \mathcal{A}_p(P) \mid \bar{x} \leq y\} = f^{-1}(\mathcal{A}_p(L)(\leq y))$ is C.M. of dimension $h(y)$ for all $y \in \mathcal{A}_p(L)$. This follows from the fact that $f|y = \mathcal{A}_p(U \cdot y)$ and Theorem 11.2 on pg. 123 in [Q], since U is a solvable p'-subgroup of P.

Thus f satisfies the criterion of Corollary 9.7 on pg. 121 in [Q]. Hence, since $\mathcal{A}_p(L)$ is C.M. of dimension $n-1$, $\mathcal{A}_p(P)$ is also C.M. of dimension $n-1$. In particular, $\mathcal{A}_p(P) = \mathcal{A}_p(G_x) = F_x$ is $(n-2)$-connected as claimed. □

Remark: The proof of the next lemma depends on the fact that the complex constructed in section 11 is $(n-1)$-spherical.

Lemma 15.5. *Let $A \in \mathcal{A}_p(G)$, then F_A is $(n-2)$-connected.*

Proof. Fix $A \in \mathcal{A}_p(G)$, and let $V = \bigoplus_{\lambda \in \Phi^+} (V_\lambda + V_{\lambda^{-1}})$ be a decomposition into homogenous components under the action of A (as earlier in this section). Remember that $F_A = Fix(A)$. If $Fix(A)'$ is the full subcomplex of the Tits builiding defined on those simplices fixed under the action of A, then $Fix(A) = sd(Fix(A)')$. Thus, by Lemma 9.1, it suffices to show that $Fix(A)'$ is $(n-2)$-connected. Now one of two cases occurs:

<u>CASE I</u>: $|\Phi^+| = 1$. As $A \neq 1$ and $\lambda \neq 1$, we have $\dim(V_\lambda) = n$. Consider:

$$Fix(A)' = \{x = (s_0 < s_1 < \ldots < s_r) \in \mathfrak{B} \mid As_i = s_i \ \forall\ 0 \leq i \leq r\}.$$

Now each $s_i = U_i \times W_i$ where $U_i \leq V_\lambda, W_i \leq V_{\lambda^{-1}}$, and $f(u,w) = 0 \ \forall\ u \in U_i, w \in W_i$. Now the map $\pi : V_{\lambda^{-1}} \to V_\lambda^*$ defined by $v \mapsto f(\cdot, v)$ is an isomorphism. Thus, identifying

W_i with $\pi(W_i)$, we see that $s_i = U_i \times W_i$ where $W_i \leq U_i^{\perp} =< \phi \in V_\lambda^* \mid \phi|_{U_i} = 0 >$ the annihilator of U_i $\forall\ 0 \leq i \leq r$. Thus $Fix(A)' \cong K$ the $(n-1)$-spherical complex constructed in section 11 (where V_λ is the n-dimensional vector space). So in this case $Fix(A)'$ is $(n-1)$-spherical, and in particular is $(n-2)$-connected as desired.

<u>CASE II</u> : $|\Phi^+| \geq 2$. Now let $n_\lambda = dim(V_\lambda)$ when $\lambda \neq 1$ and $n_\lambda = \frac{dim(V_1)}{2}$ when $\lambda = 1$. Then $n_\lambda < n\ \forall\ \lambda \in \Phi^+$ and $\sum_{\lambda \in \Phi^+} n_\lambda = n$. For each $\lambda \in \Phi^+ - \{1\}$, let $B_\lambda = Fix(A|_{V_\lambda})'$, where by $Fix(A|_{V_\lambda})'$ we mean the full subcomplex of the Tits building of $V_\lambda + V_{\lambda^{-1}}$ generated by those simplices on which A, restricted to $V_\lambda + V_{\lambda^{-1}}$, acts as the identity. Then, by Case I, B_λ is $(n_\lambda - 1)$-spherical. For $\lambda = 1$, let B_1 be the Tits building of the symplectic space V_1, which is (n_1-1)-spherical. For all $\lambda \in \Phi^+$, let $D_\lambda = B_\lambda \cup \{0\}$, ordered by inclusion. Then D_λ is isomorphic to CB_λ, the cone on B_λ as defined in section 9. Also let $pr_\lambda : V \to V_\lambda + V_{\lambda^{-1}}$ be the canonical projection. Then $pr_\lambda(Fix(A)') = D_\lambda$ for each $\lambda \in \Phi^+$ and $Fix(A)' = \prod_{\lambda \in \Phi^+} D_\lambda - \{(0,\dots,0)\}$. Thus, by Lemma 9.3, $Fix(A)' \cong \mathop{*}_{\lambda \in \Phi^+} B_\lambda$. So, by Lemma 14.1, $Fix(A)'$ is spherical of dimension:

$$\sum_{\lambda \in \Phi^+} (n_\lambda - 1) + |\Phi^+| - 1 = \sum_{\lambda \in \Phi^+} n_\lambda - 1 = n - 1.$$

In particular, $Fix(A)'$ is $(n-2)$-connected in this case too.

Thus, by considering the two possible cases, we have shown that $F_A = Fix(A) = sd(Fix(A)')$ is $(n-2)$-connected as claimed. □

Theorem 15.6. *If $p \neq 2$ and $p|q-1$, then $\mathcal{A}_p(Sp_{2n}(q))$ is C.M. of dimension $n-1$.*

Proof. By Lemmas 15.4–5 and Theorem 9.13, we have $\mathcal{A}_p(Sp_{2n}(q))$ is $(n-2)$-connected and thus is $(n-1)$-spherical. And by Lemma 15.3 we know that $\mathcal{A}_p(Sp_{2n}(q))(> A)$ is $(n-h(A)-3)$-connected. Thus, by the discussion just prior to Lemma 15.3, $\mathcal{A}_p(Sp_{2n}(q))$ is C.M. of dimension $n-1$ as claimed. □

Now consider the case when $2|q-1$. By the discussion preceding Lemma 15.3, it suffices to show that $\mathcal{A}_2(G)$ is $(n-2)$-connected and that for all $A \in \mathcal{A}_2(G)$, $\mathcal{A}_2(G)(> A)$ is $(n-h(A)-3)$-connected. Now, since there is a central element of G of order 2, $O_2(G)$

is nontrivial; and, by Proposition 2.4 on pg. 106 in [Q], $\mathcal{A}_2(G)$ is contractible. So we are reduced to showing that:

$$(*) \qquad \mathcal{A}_2(G)(> A) \text{ is } (n-h(A)-3)\text{-connected } \forall\, A \in \mathcal{A}_2(G).$$

Let $A \in \mathcal{A}_2(G)$ and let $V = \underset{\lambda}{\oplus} V_\lambda$ be a decomposition of V into homogeneous components under the action of A. Let $\Phi = \{\lambda : A \to \mathbb{F}_q^\sharp \mid V_\lambda \neq 0\}$. Note that by Lemma 15.1 we know that $V = \underset{\lambda\in\Phi}{\perp} V_\lambda$ is an orthogonal decomposition of V into nondegenerate subspaces.

Lemma 15.7. *If $A \in \mathcal{A}_2(G)$ and $A \nleq Z(G)$, the center of G, then $\mathcal{A}_2(G)(> A)$ is $(n-h(A)-3)$-connected.*

Proof. We have an orthogonal decomposition of $V = \underset{\lambda\in\Phi}{\perp} V_\lambda$ into nondegenerate homogeneous components under the action of A; then the fact that $A \nleq Z(G)$ implies $|\Phi| \geq 2$. Therefore, for all $\lambda \in \Phi$, $V_\lambda \lneq V$.

Now $\mathcal{A}_2(G)(> A) = \mathcal{A}_2(C_G(A))(> A)$, and $C_G(A) = \underset{\lambda\in\Phi}{\cap} N_G(V_\lambda)$. By the proof of Lemma 15.3 and the fact that $V = \underset{\lambda\in\Phi}{\oplus} V_\lambda$ is an orthogonal decomposition, we have:

$$\underset{\lambda\in\Phi}{\cap} N_G(V_\lambda) = \prod_{\lambda\in\Phi} C_G(V_\lambda^\perp) = \prod_{\lambda\in\Phi} Sp(V_\lambda),$$

where $dim(V_\lambda) < dim(V)\ \forall\, \lambda \in \Phi$. Thus, by induction on dimension of the symplectic spaces and Lemma 14.3, $C_G(A)$ is C.M. of dimension $n-1$. Therefore $\mathcal{A}_2(G)(> A)$ is $(n-h(A)-3)$-connected as claimed. □

Let $Z \in \mathcal{A}_2(G) \cap Z(G)$, the central elementary abelian 2-subgroup. It remains to show that $\mathfrak{C} = \mathcal{A}_2(G)(> Z)$ is $(n-3)$-connected. Let $\mathfrak{B}$ be the Tits building of (V, f) and consider $B = sd(\mathfrak{B})$. Then let:

$$B \times \mathfrak{C} \supseteq F = \{(x, A) \mid x \in Fix(A)\}.$$

Then F is closed, and by Theorem 9.13, it suffices to show that for all $x \in B$ and $A \in \mathfrak{C}$, F_x and F_A, as defined in Theorem 9.13, are $(n-3)$-connected.

Lemma 15.8. *If $x \in B$ then F_x is $(n-3)$-connected.*

Proof. Let $x = (U_0 < U_1 < \cdots < U_s) \in B$. Note that $F_x = \mathcal{A}_2(G_x)(> Z)$, and G_x is a standard parabolic subgroup P. Let U be the unipotent radical of P, a solvable $2'$-subgroup of P, and $\pi : P \to P/U \cong L$, where L is the Levi factor of P.

Note that $Z \in \mathcal{A}_2(L)$. If $g : \mathcal{A}_2(P) \to \mathcal{A}_2(L)$ is the map induced by π then:

$$x < y \in \mathcal{A}_2(P) \Rightarrow g(x) < g(y) \in \mathcal{A}_2(L),$$

since $x \cap U = \{1\}$ for all $x \in \mathcal{A}_2(P)$. Thus we have a map of posets $g : \mathcal{A}_2(P)(> Z) \to \mathcal{A}_2(L)(> Z)$ such that:

$$(1) \qquad x < y \in \mathcal{A}_2(P)(> Z) \Rightarrow g(x) < g(y) \in \mathcal{A}_2(L)(> Z).$$

In the proof of Lemma 15.4 we showed that $\mathcal{A}_p(L)$ is C.M. of dimension $n-1$. Since the proof did not depend on the fact that $p \neq 2$, we have $\mathcal{A}_2(L)$ is C.M. of dimension $n-1$. Therefore, by Lemma 14.2, $\mathcal{A}_2(L)(> Z)$ is C.M. of dimension $n-2$.

Now given $y \in \mathcal{A}_2(L)(> Z)$, $h(y) = m_2(y) - 2$. Also, the fact that U is a solvable $2'$-subgroup of P, together with Theorem 11.2 on pg. 123 in [Q]; implies that $\mathcal{A}_2(y \cdot U)$ is C.M. of dimension $m_2(y) - 1$. Therefore, again by Lemma 14.2, we have:

$$g^{-1}(\mathcal{A}_2(L)(> Z)(\leq y)) = \mathcal{A}_2(y \cdot U)(> Z),$$

is C.M. of dimension $m_2(y) - 2 = h(y)$.

Therefore $g : \mathcal{A}_2(P)(> Z) \to \mathcal{A}_2(L)(> Z)$ is a strict map of posets with $g^{-1}(\mathcal{A}_2(L)(> Z)(\leq y))$ C.M. of dimension $h(y)$. Therefore, by Corollary 9.7 on pg. 121 in [Q]; since $\mathcal{A}_2(L)(> Z)$ is C.M. of dimension $n-2$, we have that $\mathcal{A}_2(P)(> Z) = \mathcal{A}_2(G_x)(> Z)$ is C.M. of dimension $n-2$. In particular, we have $F_x = \mathcal{A}_2(G_x)(> Z)$ is $(n-3)$-connected as claimed. □

Lemma 15.9. *Given $A \in \mathfrak{C}$, F_A is $(n-3)$-connected.*

Proof. As in Lemma 15.5 it suffices to show that $Fix(A)'$ – the full subcomplex of the Tits building defined on those simplices fixed under the action of A – is $(n-3)$-connected. Let

$V = \underset{\lambda\in\Phi}{\oplus} V_\lambda$ be a decomposition of V into homogeneous components under the action of A. Note that $A \in \mathfrak{C}$ implies that $|\Phi| \geq 2$. For each $\lambda \in \Phi$, let $2n_\lambda = dim(V_\lambda)$, $\mathfrak{B}_\lambda$ be the Tits building of the symplectic space V_λ and $pr_\lambda : V \to V_\lambda$ be the canonical projection. Then, by the Solomon-Tits theorem, for all $\lambda \in \Phi$, $\mathfrak{B}_\lambda$ is $(n_\lambda - 1)$-spherical and $pr_\lambda(Fix(A)') = CB_\lambda$, the cone on $\mathfrak{B}_\lambda$ as defined in section 9.

Thus $Fix(A)' = \prod_{\lambda\in\Phi} C\mathfrak{B}_\lambda - \{(0,\dots,0)\} \simeq \underset{\lambda\in\Phi}{*} \mathfrak{B}_\lambda$ by Lemma 9.3. Therefore, by Lemma 14.1, $Fix(A)'$ is spherical of dimension:

$$\sum_{\lambda\in\Phi}(n_\lambda - 1) + |\Phi| - 1 = \sum_{\lambda\in\Phi} n_\lambda - 1 = n - 1.$$

Thus $F_A = sd(Fix(A)')$ is $(n-2)$, and hence $(n-3)$-connected as claimed. $\square$

By Lemmas 15.8 and 15.9 and Theorem 9.13 we have the following:

Lemma 15.10. *Let $Z \in \mathcal{A}_2(G) \cap Z(G)$ then $\mathcal{A}_2(G)(> Z)$ is $(n-3)$-connected.*

Thus, by $(*)$ and Lemmas 15.7 and 15.10, we have shown that if $2|q-1$, then $\mathcal{A}_2(G)$ is C.M. of dimension $n-1$. Therefore, in light of Theorem 15.6, we have the following result:

Theorem 15.11. *If p is prime and $p|q-1$, then $\mathcal{A}_p(Sp_{2n}(q))$ is Cohen-Macaulay of dimension $n-1$.*

As a direct corollary of Theorems 9.15 and 15.11 we obtain:

Corollary 15.12. *Let $G = PSp_{2n}(q)$ and let $p \neq 2$ be a prime dividing $q-1$. Then $\mathcal{A}_p(G)$ is Cohen-Macaulay of dimension $n-1$.*

Proof. Since a prime $p \neq 2$ divides $q-1$, $Sp_{2n}(q) \neq Sp_2(2)$, $Sp_2(3)$, or $Sp_4(2)$. Hence $PSp_{2n}(q)$ is simple.

When q is even, $G = PSp_{2n}(q) = Sp_{2n}(q)$ and the result follows directly from Theorem 15.11. Otherwise, $p \neq 2$ implies that $O_{p'}(Sp_{2n}(q)) = Z(Sp_{2n}(q))$. Therefore, by Theorem 9.15, $\mathcal{A}_p(G) \cong \mathcal{A}_p(Sp_{2n}(q))$. Once again the result follows from Theorem 15.11. $\square$

SECTION 16 : IF $p \neq 2$ AND $p|q-1$, THEN $\mathcal{A}_p(GU_n(q))$ IS COHEN-MACAULAY

In section 15 we showed that given a prime $p|q-1$, $\mathcal{A}_p(Sp_{2n}(q))$ is C.M. of dimension $n-1$. In this section we show that an analogous result holds for $GU_n(q)$ under the added constraint that $p \neq 2$.

Let $G = GU_n(q)$ and let (V, f) be the corresponding n-dimensional unitary space over $\mathbb{F}_{q^2}$. Fix a prime $p \neq 2$ dividing $q-1$. We show that $\mathcal{A}_p(G)$ is C.M. by induction on n.

Given a fixed $A \in \mathcal{A}_p(G)$, consider a decomposition of $V = \bigoplus_{\lambda} V_\lambda$ into homogeneous components under the action of A. Let $\Phi = \{\lambda : A \to \mathbb{F}^{\#}_{q^2} \mid V_\lambda \neq 0\}$, and note that for all $\lambda \in \Phi$ and all $a \in A$, we have $\lambda(a)^p = \lambda(a^p) = 1$. This gives us the following result:

Lemma 16.1. *If $1 \neq \lambda \in \Phi$ then $\lambda^{-1} \in \Phi$. Furthermore, if $1 \neq \lambda$ then V_λ is totally singular and $V_\lambda \oplus V_{\lambda^{-1}}$ is a hyperbolic subspace of V. If $V_0 = V_\lambda$ when $\lambda = 1$, then V_0 is nondegenerate and $dim(V_0) \equiv n \pmod 2$.*

Proof. Let $\lambda, \gamma \in \Phi$, $u \in V_\lambda$, and $v \in V_\gamma$. Then, for all $a \in A$, we have:

$$(*) \qquad \begin{aligned} f(u,v) = f(au, av) = f(\lambda(a)u, \gamma(a)v) = \lambda(a)\gamma(a)^q f(u,v) \\ \implies (1 - \lambda(a)\gamma(a)^q) f(u,v) = 0. \end{aligned}$$

Now $p|q-1$ implies $\gamma(a)^{q-1} = 1 \implies \gamma(a)^q = \gamma(a)$ $\forall\, \gamma \in \Phi$, and $\forall\, a \in A$. Thus $1 - \lambda(a)\gamma(a)^q = 0 \iff \lambda(a) = \gamma(a)^{-1}$ $\forall\, a \in A$. Hence, by $(*)$, if $\gamma \neq \lambda^{-1}$, then $V_\lambda \perp V_\gamma$. Thus, V_λ is totally singular for $\lambda \neq 1$, and V_0 is nondegenerate. Also, by the nondegeneracy of f, if $\lambda \neq 1$, then $\lambda^{-1} \in \Phi$ and $V_\lambda \oplus V_{\lambda^{-1}}$ is nondegenerate as claimed.

Now let $\lambda \neq 1$ and let $0 \neq u \in V_\lambda$. Then, as V_λ is totally singular and $V_\lambda \oplus V_{\lambda^{-1}}$ is nondegenerate, by (19.12) on pg. 80 in [A1], we can find $v \in V_{\lambda^{-1}}$ such that $U = \langle u, v \rangle$ is a hyperbolic plane. Now $V = U \oplus U^\perp$ with $U^\perp$ nondegenerate of dimension $n-2$. So passing to $U^\perp$ and by induction on dimension, as in the proof of Lemma 15.1, we have $V_\lambda \oplus V_{\lambda^{-1}}$ is a hyperbolic subspace of V.

Fix a set of "positive weights" $\Phi \supseteq \Phi^+$, such that $\lambda \neq \gamma \in \Phi^+ \implies \lambda\gamma \neq 1$, and $\Phi = \{\lambda, \lambda^{-1} | \lambda \in \Phi^+\}$. Then, for all $1 \neq \lambda \in \Phi^+$, we have $dim(V_\lambda \oplus V_{\lambda^{-1}}) = 2dim(V_\lambda)$. Also, $V = V_0 \perp \underset{\lambda \in \Phi^+ - \{1\}}{\perp} V_\lambda \oplus V_{\lambda^{-1}}$, thus $dim(V_0) \equiv n \pmod 2$ as claimed.

Thus we have the lemma. □

Let $A \in \mathcal{A}_p(G)$ and Φ^+ be as defined above and assume that $\Phi^+ = \{\lambda_0, \lambda_1, \ldots, \lambda_d\}$, where $\lambda_0 = 1$. Let $V_{\lambda_0} = V_0$ and $\forall\ 1 \leq i \leq d$; let V_i and V_i' equal V_{λ_i} and $V_{\lambda_i^{-1}}$, respectively. Finally, for $0 \leq i \leq d$, let $n_i = dim(V_i)$. Then, by the proof of Lemma 16.1, we have:

$$V = V_0 \perp V_1 \oplus V_1' \perp \cdots \perp V_d \oplus V_d', \tag{6.2}$$

where V_0 is nondegenerate and $V_i \oplus V_i'$ is a hyperbolic subspace of dimension $2n_i$ with V_i, and V_i' totally singular $\forall\ 1 \leq i \leq d$. Hence we have:

$$C_G(A) \cong GU_{n_0}(q) \times \prod_{1 \leq i \leq d} GL_{n_i}(q^2). \tag{6.3}$$

Note that since $p|q-1$, $\mathcal{A}_p(GU_2(q))$ is nonempty; thus we have:

Theorem 16.4. *If $k = [\frac{n}{2}]$, the greatest integer less than or equal to $\frac{n}{2}$, and $p \neq 2$ is a prime dividing $q-1$, then $m_p(GU_n(q)) = k$.*

Proof. Consider an orthogonal decomposition of $V = \perp_{1 \leq i \leq k} V_i \perp V'$, where for $1 \leq i \leq k$ V_i is a 2-dimensional nondegenerate subspace, and V' is an r-dimensional nondegenerate subspace. Since $\mathcal{A}_p(GU_2(q)) \neq \emptyset$, we can find $A = A_1 \times \cdots \times A_k \in \mathcal{A}_p(G)$ where $A_i \in \mathcal{A}_p(GU(V_i))$ for each $1 \leq i \leq k$. Hence, $m_p(G) \geq k$.

Now given $A \in \mathcal{A}_p(G)$, by (16.2), we have a decomposition of $V = V_0 \perp \cdots \perp V_d \oplus V_d'$. For each $1 \leq i \leq d$, let $A_i = C_A(V_i \oplus V_i')$, then $A_i = C_A(V_i) = C_A(V_i')$. Now, if W_i is an simple $F[A]$-submodule of V_i, then $V_i = < W | W \cong W_i >$; thus $C_A(W_i) = C_A(V_i) = A_i$. But, $A/C_A(W_i)$ acts faithfully irreducibly on W_i and since $p|q-1$, we have $A/A_i = A/C_A(W_i) \cong \mathbb{Z}/p\mathbb{Z}$ by (5.21) on pg. 159 in [Su]. Since $\bigcap_{1 \leq i \leq d} A_i = 1$, we have the normal series:

$$A > A_1 \geq A_1 \cap A_2 \geq \cdots \geq \bigcap_{1 \leq i \leq d-1} A_i \geq 1,$$

with factor groups equal to 1 or $\mathbb{Z}/p\mathbb{Z}$. Thus, $m_p(A) \leq d$ and since $d \leq k$, we have $m_p(G) \leq k$.

Thus, we have shown that $m_p(G) = k$, as claimed. □

By the above result, $m_p(GU_2(q)) = 1$, so the fact that $\mathcal{A}_p(GU_2(q))$ is nonempty implies that it is indeed C.M. So we can now prove the main result by induction. Let $n = 2k + r$, then $m_p(G) = k$ and by Lemma 14.2 it suffices to show that:

(1) $\forall\ A \in \mathcal{A}_p(G)\ \mathcal{A}_p(G)(> A)$ is $(k - h(A) - 2)$-spherical, and

(2) $\mathcal{A}_p(G)$ is $(k - 1)$-spherical.

Now $\mathcal{A}_p(G)$ is $(k - 1)$-dimensional since $m_p(G) = k$. Also $dim(\mathcal{A}_p(G)(\leq A)) = h(A)$, $\forall\ A \in \mathcal{A}_p(G)$; thus $\mathcal{A}_p(G)(> A)$ is $(k - h(A) - 2)$-dimensional. So we are reduced to showing that:

$$\begin{aligned} &\forall\ A \in \mathcal{A}_p(G),\ \mathcal{A}_p(G)(> A) \text{ is } (k - h(A) - 3)\text{-connected, and} \\ &\mathcal{A}_p(G) \text{ is } (k-2)\text{-connected.} \end{aligned} \tag{16.5}$$

The proof of these two facts will closely mimic the proof of the fact that $\mathcal{A}_p(Sp_{2n}(q))$ is C.M. when $p|q - 1$.

Lemma 16.6. *Given $A \in \mathcal{A}_p(G)$, $\mathcal{A}_p(G)(> A)$ is $(k - h(A) - 3)$-connected.*

Proof. $\mathcal{A}_p(G)(> A) = \mathcal{A}_p(C_G(A))(> A)$ and by (16.3) we have:

$$C_G(A) \cong GU_{n_0}(q) \times \prod_{1 \leq i \leq d} GL_{n_i}(q^2),$$

where $n_0 < n$, since $A \in \mathcal{A}_p(G)$ implies that A is not central. Let $n_0 = 2k_0 + r_0$, then by Lemma 16.1 we have $r_0 = r$ and $k_0 + \sum_{i=1}^{d} n_i = k$. Now $p|q - 1$ and Theorem 12.4 on pg. 126 in [Q] imply that $\mathcal{A}_p(GL_{n_i}(q^2))$ is C.M. of dimension $n_i - 1\ \forall\ 1 \leq i \leq d$. And, by induction on dimensions, $\mathcal{A}_p(GU_{n_0}(q))$ is C.M. of dimension $k_0 - 1$. Therefore, by Lemma 14.3, $\mathcal{A}_p(C_G(A))$ is C.M. of dimension:

$$k_0 - 1 + \sum_{i=0}^{d}(n_i - 1) + (d + 2) - 1 = k_0 - 1 + \sum_{i=0}^{d} n_i = k - 1.$$

In particular, by Lemma 14.2, $\mathcal{A}_p(G)(> A) = \mathcal{A}_p(C_G(A))(> A)$ is $(k - h(A) - 3)$-connected as claimed. □

By (16.5), we are reduced to showing that $\mathcal{A}_p(G)$ is $(k-2)$-connected. Let $\mathfrak{B}$ be the Tits building of (V, f) and set $B = sd(\mathfrak{B})$, the first barycentric subdivision of $\mathfrak{B}$. Then, by the Solomon-Tits theorem and Lemma 9.1, B is $(k-2)$-connected. Consider:

$$B \times \mathcal{A}_p(G) \supseteq F = \{(x, A) \mid x \in Fix(A)\}.$$

Then, by Remark 9.14, F is closed; and by Theorem 9.13 it suffices to show that F_x, and F_A – as defined in Theorem 9.13 – are $(k-2)$-connected for all $x \in B$, $A \in \mathcal{A}_p(G)$.

Lemma 16.7. *Let $x \in B$ then F_x is $(k-2)$-connected.*

Proof. Let $x = (U_0 < U_1 < \cdots < U_s) \in B$, $n_0 = dim(U_0)$, and $\forall\ 1 \leq i \leq s$; let $n_i = dim(U_i) - \sum_{j=0}^{i-1} n_j$. Finally, let $n_{s+1} = n - \sum_{i=0}^{s} n_i$. By Lemma 16.1, if $n_{s+1} = 2k' + r'$ $(0 \leq r' \leq 1)$, then $r' = r$ and $\sum_{i=0}^{s} n_i + k' = k$.

Now $F_x = \mathcal{A}_p(G_x)$ where G_x stabilizes x and is equal to a parabolic subgroup P of G. Let U be the unipotent radical of P and $\pi : P \to P/U \cong L$ be the canonical homomorphism; here L is the Levi factor of P. By definition of the Levi factor:

$$L \cong GL_{n_0}(q^2) \times GL_{n_1}(q^2) \times \cdots \times GL_{n_s}(q^2) \times GU_{n_{s+1}}(q).$$

Now $U_s \neq 0$ and the definition of n_{s+1} imply that $n_{s+1} < n$. Then, as in the proof of Lemma 16.6, by induction on dimension and Theorem 12.4 on pg. 126 in [Q], we have $\mathcal{A}_p(L)$ is C.M. of dimension $k-1$.

Let $f : \mathcal{A}_p(P) \to \mathcal{A}_p(L)$ be the map induced by π. Then, since U is a solvable p'-subgroup of P, as in Lemma 15.4, f satisfies the criterion of Corollary 9.7 on pg. 121 in [Q]. Thus, as $\mathcal{A}_p(L)$ is C.M. of dimension $k-1$, $\mathcal{A}_p(P)$ is also C.M. of dimension $k-1$.

In particular, $F_x = \mathcal{A}_p(G_x) = \mathcal{A}_p(P)$ is $(k-2)$-connected as claimed. $\square$

Lemma 16.8. *Given $A \in \mathcal{A}_p(G)$, F_A is $(k-2)$-connected.*

Proof. By (16.2) we have a decomposition of $V = V_0 \perp \cdots \perp V_d \oplus V_d'$. Now, $F_A = Fix(A)$ the subcomplex of B defined on those simplices of $\mathfrak{B}$ fixed vertex-wise under the action of A. As in the proof of Lemma 15.5, let $Fix(A)'$ be the full subcomplex of $\mathfrak{B}$ defined on those simplices fixed under the action of A. Then, $Fix(A) = sd(Fix(A)')$ and, by Lemma 9.1, it suffices to show that $Fix(A)'$ is $(k-2)$-connected. One of two possible cases arises:

<u>CASE I :</u> $V_0 = 0$ and $d = 1$. Then, by Lemma 16.1, $dim(V_1) = k$, and $V = V_1 \oplus V_1'$ is a hyperbolic space. By definition:

$$Fix(A)' = \{x = (s_0 < s_1 < \cdots < s_r) \in \mathfrak{B} \mid As_i = s_i \ \forall\ 0 \le i \le r\},$$

where each $s_i = U_i \times W_i$ with $U_i \le V_1$ and $W_i \le V_1' \cap U_i^{\perp}$ $\forall\ 0 \le i \le r$. Identifying V_1' with the dual space of V_1, considered as a k-dimensional space over $\mathbb{F}_{q^2}$, we see that $Fix(A)' = K$ the $(k-1)$-spherical complex constructed in section 11. Thus, $F_A = sd(Fix(A)')$ is $(k-1)$-spherical and in particular, $(k-2)$-connected as claimed.

<u>CASE II :</u> $|\Phi^+| \ge 2$. Let $n_0 = 2k_0 + r$ and $dim(V_i) = n_i$ $\forall\ 1 \le i \le d$, then $k_0 + \sum_{i=0}^{d} n_i = k$. Now for each $1 \le i \le d$, let $B_i = Fix(A|_{V_i})'$, where $Fix(A|_{V_i})'$ is the full subcomplex of the Tits building of $V_i \oplus V_i'$ defined on those simplices on which A acts as the identity. Then $\forall\ 1 \le i \le d$, B_i is $(n_i - 1)$-spherical by Case I. Let B_0 be the Tits building of V_0. Then, by the Solomon-Tits theorem, B_0 is $(k_0 - 1)$-spherical. If for all $1 \le i \le d$, $pr_i : V \to V_i \oplus V_i'$ and $pr_0 : V \to V_0$ are the canonical projections, then:

$$pr_i(Fix(A)') \cong B_i \cup \{0\} \cong CB_i \ \forall\ 0 \le i \le d,$$

where $B_i \cup \{0\}$ is ordered by inclusion, and CB_i is the cone of B_i, as defined in section 9. Thus, by Lemma 9.3, we have:

$$Fix(A)' \cong \prod_{0 \le i \le d} CB_i - \{(0, \ldots, 0)\} \simeq B_0 * B_1 * \cdots * B_d.$$

Thus, by Lemma 14.1, $Fix(A)'$ is spherical of dimension:

$$k_0 - 1 + \sum_{i=0}^{d}(n_i - 1) + (d+2) - 1 = k_0 - 1 + \sum_{i=0}^{d} n_i = k - 1.$$

In particular, $F_A = sd(Fix(A)')$ is $(k-2)$-connected as claimed. □

By Lemmas 16.7–8 and Theorem 9.13, we have shown that:

Lemma 16.9. *$\mathcal{A}_p(GU_n(q))$ is $(k-2)$-connected.*

Thus, in light of (16.5) and Lemmas 16.6 and 16.9, we have:

Theorem 16.10. *Let $p \neq 2$ be a prime dividing $q-1$, then $\mathcal{A}_p(GU_n(q))$ is C.M. of dimension $m_p(GU_n(q))-1$.* □

We obtain the following result as a corollary to Theorems 9.15 and 16.10:

Corollary 16.11. *Let $G = SU_n(q)$ or $U_n(q)$ and let $p \neq 2$ be a prime dividing $q-1$. Then $\mathcal{A}_p(G)$ is Cohen-Macaulay of dimension $m_p(GU_n(q))-1$.*

Proof. Since $p \neq 2$ and $p|q-1$, $p \nmid (q+1)$. Thus $\mathcal{A}_p(GU_n(q)) = \mathcal{A}_p(SU_n(q))$ as posets. Thus when $G = SU_n(q)$, the result follows directly from Theorem 16.10.

Now let $G = U_n(q)$. When $n = 2$, by Theorem 16.4, $m_p(GU_n(q)) = 1$. Since $\mathcal{A}_p(G) \neq \emptyset$, $\mathcal{A}_p(G)$ is indeed C.M. of dimension $m_p(GU_n(q))-1$. So assume that $n \geq 3$. Since we have a prime dividing $q-1$, $GU_n(q) \neq GU_3(2)$ and so $U_n(q)$ is simple. From the previous paragraph we know that $\mathcal{A}_p(SU_n(q))$ is C.M. of dimension $m_p(GU_n(q))-1$. Also, $p \nmid (q+1)$ implies that $O_{p'}(SU_n(q)) = Z(SU_n(q))$. Thus, by Theorem 9.15, $\mathcal{A}_p(G) \cong \mathcal{A}_p(SU_n(q))$ and we have the desired result. □

Section 17 : The order complex of proper nondegenerate subspaces of a symplectic space is Cohen-Macaulay

Let (V, f) be a $2n$-dimensional symplectic space over $\mathbb{F}_q$. It is well known that the order complex of nontrivial totally singular subspaces of V – ordered by inclusion – that is, the Tits building of V, is C.M. of dimension $n-1$. Consider the order complex, $\mathfrak{C}$, of proper nondegenerate subspaces of V – ordered by inclusion. In this section we show that $\mathfrak{C}$ is C.M. of dimension $n-2$. However, we first need to consider a different complex.

Let W be an n-dimensional vector space over $\mathbb{F}_q$ and W^* its dual space. For each $U \leq W$, let $U^\perp = < f \in W^* \mid f|_U \equiv 0 >$, the annihilator of U. Define K to be the order complex on $\{U \times U' \leq W \times W^* \mid 0 \neq U \neq W; W^* = U^\perp \oplus U'\}$ ordered by inclusion.

Lemma 17.1. *The complex K defined above is $(n-3)$-connected.*

Proof. We prove this by induction on n. When $n = 2$, K is nonempty and thus is -1-connected. So assume that $n \geq 3$. Note that $GL_n(q)$ acts on K by $\rho(g)(U \times U') = g(U) \times g^*(U')$ where $g^* = {}^tg^{-1}$, $\forall\ g \in GL_n(q)$ and all $U \times U' \in K$. Let $p|q-1$ and ζ a nontrivial p^{th} root of unity in $\mathbb{F}_q$. Then $Z = diag(\zeta, \dots, \zeta) \in \mathcal{A}_p(GL_n(q))$, so $C = \mathcal{A}_p(GL_n(q))(> Z)$ is $(n-3)$-connected by Theorem 12.4 on pg. 126 in [Q] and Lemma 14.2. Consider the closed subposet:

$$sd(K) \times C \supseteq F = \{(x, A) \mid x \in \ Fix(A)\}.$$

Then F is as defined just prior to Remark 9.14. By Theorem 9.13, if for all $x \in sd(K)$ and $A \in C$, F_x and F_A, as defined in Theorem 9.13, are $(n-3)$-connected, then $sd(K)$ is $(n-3)$-connected. And hence, by Lemma 9.1, K is $(n-3)$-connected.

Given $x = (U_0 \times U_0' < U_1 \times U_1' < \cdots < U_s \times U_s') \in sd(K)$, let $n_0 = dim(U)$ and $\forall\ 1 \leq i \leq s$, let $n_i = dim(U_i) - \sum_{j=0}^{i-1} n_j$. Since $\forall\ U \times U' \in K$, $g^*(U^{\perp}) = (gU)^{\perp}$ and $W = U \oplus (U')^{\perp}$ and $W^* = U^{\perp} \oplus U'$, we have:

$$GL_n(q)_x \cong GL_{n_0}(q) \times GL_{n_1}(q) \times \cdots \times GL_{n_s}(q) \times GL_{n_{s+1}}(q),$$

where $n = \sum_{i=0}^{s+1} n_i$. Now $F_x = \mathcal{A}_p(GL_n(q)_x)(> Z)$, where, by Lemma 9.4, $\mathcal{A}_p(GL_n(q)_x) \cong \mathcal{A}_p(GL_{n_0}(q)) * \cdots * \mathcal{A}_p(GL_{n_{s+1}}(q))$. By Theorem 12.4 on pg. 126 in [Q] and Lemma 14.3, $\mathcal{A}_p(GL_n(q)_x)$ is C.M. of dimension $n-1$. Thus $F_x = \mathcal{A}_p(GL_n(q)_x)(> Z)$ is $(n-3)$-connected.

Now given $A \in C$, let $W = W_{\lambda_0} \oplus W_{\lambda_1} \oplus \cdots \oplus W_{\lambda_s}$ be a decomposition of W into weight spaces of A. Then, since $A > Z$, $s \geq 1$. Given $1 \leq i \leq s$ let $(W_{\lambda_i})^*$ be the annihilator in W^* of all W_{λ_j} such that $i \neq j$. Note that $W_{\lambda_i} \times (W_{\lambda_i})^* \in K$.

Consider a decomposition of W^* into $W^*_{\gamma_0} \oplus \cdots \oplus W^*_{\gamma_r}$ under the action of A. Then given $a \in A$, $w_i \in W_{\lambda_i}$, and $w_j^* \in W^*_{\gamma_j}$, we have:

$$w_j^*(w_i) = \rho(a)w_j^*(\rho(a)w_i) = \gamma_j(a)\lambda_i(a)w_j^*(w_i).$$

Thus, if $\gamma_j \neq \lambda_i^{-1}$, then $W^*_{\gamma_j} \leq W^{\perp}_{\lambda_i}$. Therefore, since W^* is the dual space of W for each $1 \leq i \leq s$, there exists $1 \leq j \leq r$ such that $\gamma_j = \lambda_i^{-1}$. By the symmetry between W^* and W, we have a decomposition of $W^* = W^*_{\lambda_0{}^{-1}} \oplus \cdots \oplus W^*_{\lambda_s{}^{-1}}$ under the action of A, where $W^*_{\lambda_i{}^{-1}} = (W_{\lambda_i})^*$ (as defined above) $\forall\ 0 \leq i \leq s$.

If $pr_i : W \to W_{\lambda_i}$ and $pr_i^* : W^* \to W^*_{\lambda_i{}^{-1}}$ are the canonical projections, then we have $pr_i^*(U^{\perp}) = pr_i(U)^{\perp}$ $\forall\ U \leq W$. Thus, if $U \times U' \in K$ then, since $W^* = U^{\perp} \oplus U'$, we have:

$$(*) \qquad (W_{\lambda_i})^* = W^*_{\lambda_i{}^{-1}} = pr_i^*(U^{\perp}) \oplus pr_i^*(U') = (pr_i(U))^{\perp} \oplus pr_i^*(U').$$

Now let K_i be the order complex of the poset:

$$\{U \times U' \leq W_{\lambda_i} \times W^*_{\lambda_i{}^{-1}} \mid 0 \neq U \neq W_{\lambda_i}; W^*_{\lambda_i{}^{-1}} = (U^{\perp} \cap W^*_{\lambda_i^{-1}}) \oplus U'\},$$

ordered by inclusion. And let $D_i = \overline{sd(K_i)}$, the double cone of $sd(K_i)$ as defined in section 9, for all $0 \leq i \leq s$. Then, the fact that $s \geq 1$ along with $(*)$, implies:

$$pr_i \times pr_i^*(Fix(A)) \cong D_i \quad \forall\ 0 \leq i \leq s.$$

Note that $Fix(A)$ equals F_A. Hence, $F_A \cong D_0 * \cdots * D_s - \{(0,\ldots,0),(\infty,\ldots,\infty)\}$. If $n_i = dim(W_{\lambda_i})$, then, by induction, K_i is (n_i-3)-connected for $0 \leq i \leq s$, and $n = \sum_{i=0}^{s} n_i$. Thus, by Corollary 9.11, F_A is: $\sum_{i=0}^{s}(n_i-3)+3s = \sum_{i=0}^{s} n_i - 3(s+1)+3s = (n-3)$-connected.

Hence, by Theorem 9.13, K is $(n-3)$-connected as claimed. □

We are now ready to show that $\mathfrak{C}$ is C.M. of dimension $n-2$. When $n = 2$, $\mathfrak{C}$ is nonempty and we are done. So assume that $n \geq 3$. By (19.16) on pg. 81 in [A1], we know that $\mathfrak{C}$ is $(n-2)$-dimensional. Thus it remains to show that $\mathfrak{C}$ satisfies the four conditions of Lemma 14.2. First we show that $\mathfrak{C}$ is $(n-3)$-connected.

Let p be a prime dividing $q-1$ and $G = Sp_{2n}(q)$. Then, by Theorem 15.11, $\mathcal{A}_p(G)$ is C.M. of dimension $n-1$. So consider the closed subposet:

$$C \times \mathcal{A}_p(G) \supseteq F = \{(x, A) \mid x \in Fix(A)\},$$

where $C = sd(\mathfrak{C})$ and $Fix(A)$ is as previously defined. By Theorem 9.13, if for all $x \in C$ and $A \in \mathcal{A}_p(G)$, F_x and F_A are $(n-3)$-connected, then C is $(n-3)$-connected.

Lemma 17.2. *Given $x \in C$, F_x is $(n-2)$-connected.*

Proof. Let $x = (U_0 < U_1 < \cdots < U_s)$ with $dim(U) = 2n_0$, and $\forall\ 1 \leq i \leq s\ \ 2n_i = dim(U_i) - \sum_{j=0}^{i-1} 2n_j$. Now $F_x = \mathcal{A}_p(G_x)$ with:

$$G_x \cong Sp_{2n_0}(q) \times Sp_{2n_1}(q) \times \cdots \times Sp_{2n_{s+1}}(q),$$

where $n = \sum_{i=0}^{s+1} n_i$.

Now, by Theorem 15.11, $\mathcal{A}_p(Sp_{2n_i}(q))$ is C.M. of dimension $(n_i - 1)\ \forall\ 0 \leq i \leq s+1$. Thus, by Lemma 14.3, $\mathcal{A}_p(G_x)$ is C.M. of dimension:

$$\sum_{i=0}^{s+1}(n_i - 1) + (s+2) - 1 = \sum_{i=0}^{s+1} n_i - 1 = n - 1.$$

In particular, $F_x = \mathcal{A}_p(G_x)$ is $(n-2)$-connected. □

Lemma 17.3. *Given $A \in \mathcal{A}_p(G)$, F_A is $(n-3)$-connected.*

Proof. Let $V = V_{\lambda_0} \perp V_{\lambda_1} \oplus V_{\lambda_1^{-1}} \perp \ldots \perp V_{\lambda_s} \oplus V_{\lambda_s^{-1}}$ be orthogonal direct sum decomposition of V into weight spaces under the action of A (with $\lambda_0 = 1$). Then, as in section 15, $V_{\lambda_i} + V_{\lambda_i^{-1}}$ is a symplectic subspace of $V\ \forall\ 0 \leq i \leq s$. Two cases arise:

<u>Case I :</u> $V_{\lambda_0} = 0$ and $s = 1$. In that case $V = V_\lambda \oplus V_{\lambda^{-1}}$ and each nondegenerate subspace of V is of the form $U_\lambda \oplus U_{\lambda^{-1}}$ with $U_\lambda \leq V_\lambda$ and $U_{\lambda^{-1}} \leq V_{\lambda^{-1}}$. The map $\pi : V_{\lambda^{-1}} \to V_\lambda^*$ given by $v \mapsto f(\cdot, v)$ is an isomorphism. So identifying $V_{\lambda^{-1}}$ with V_λ^*, we see $U_\lambda \oplus U_{\lambda^{-1}}$ is nondegenerate if and only if $V_\lambda^* = U_\lambda^\perp \oplus U_{\lambda^{-1}}$. So we can identify F_A with the complex K defined at the beginning of this section. Thus, by Lemma 17.1, F_A is $(n-3)$-connected.

<u>Case II :</u> $V_{\lambda_0} \neq 0$ or $s > 1$. Let $pr_i : V \to V_{\lambda_i} + V_{\lambda_i^{-1}}$ $(0 \leq i \leq s)$ be the canonical projections. And for each $0 \leq i \leq s$, let C_i be the first barycentric subdivision of the order complex of proper nontrivial subspaces of $V_{\lambda_i} + V_{\lambda_i^{-1}}$. If $n_i = dim(V_{\lambda_i})$, then $n = \sum_{i=0}^{s} n_i$, and by induction C_i is $(n_i - 3)$-connected for all $0 \leq i \leq s$. If $D_i = \overline{C}_i$ is the double cone on C_i, as defined in section 9, we have:

$$pr_i(Fix(A)') \cong C_i \cup \{0, V_{\lambda_i} + V_{\lambda_i^{-1}}\} \cong D_i, \quad \forall\ 0 \leq i \leq s,$$

where $C_i \cup \{0, V_{\lambda_i} + V_{\lambda_i^{-1}}\}$ is ordered by inclusion. And $Fix(A)'$ is the full subcomplex of $\mathfrak{C}$ defined on those simplices fixed under the action of A; that is, $F_A = Fix(A) = sd(Fix(A)')$. Thus, $Fix(A)' \cong D_0 * D_1 * \cdots * D_s - \{(0, \ldots, 0), (\infty, \ldots, \infty)\}$. As in the proof of Lemma 17.1, we thus have $Fix(A)'$; and hence F_A is $n-3$-connected as claimed.

□

Thus we have shown that:

Lemma 17.4. *$\mathfrak{C}$ is $(n-2)$-spherical.*

Proof. By Lemmas 17.2–3 and Theorem 9.13 we have shown that $C = sd(\mathfrak{C})$ is $(n-3)$-connected. Thus, by Lemma 9.1, $\mathfrak{C}$ is also $(n-3)$-connected. Since, by definition, $\mathfrak{C}$ is $(n-2)$-dimensional, we have the result. □

Now given $x = U \in \mathfrak{C}$, $h(x) = \frac{dim(U)}{2} - 1$. For any symplectic space W, let $\mathfrak{C}(W)$ be the order complex of nondegenerate proper subspace of W. Then, by Lemma 17.4, $\mathfrak{C}(W)$ is $(\frac{dim(W)}{2} - 2)$-spherical.

Theorem 17.5. *$\mathfrak{C}$ is C.M. of dimension $n-2$.*

Proof. Let $x' = U' < U = x \in \mathfrak{C}$. Note that $\mathfrak{C}$ satisfies:

(1) $\mathfrak{C}$ is $(n-2)$-spherical. This is Lemma 17.4.

(2) $\mathfrak{C}(< x)$ is $(h(x)-1)$-spherical. We know that $h(x) = \frac{dim(U)}{2} - 1$ and $\mathfrak{C}(U)$ is $(\frac{dim(U)}{2} - 2) = (h(x)-1)$-spherical. Since $\mathfrak{C}(< x)$ can be identified with $\mathfrak{C}(U)$ the claim holds.

(3) $\mathfrak{C}(> x)$ is $(n - h(x) - 3)$-spherical. We know that $\mathfrak{C}(> x)$ can be identified with $\mathfrak{C}(V/U)$. Now $\mathfrak{C}(V/U)$ is $(\frac{dim(V)-dim(U)}{2} - 2) = (n - \frac{dim(U)}{2} - 2)$-spherical. But $(n - h(x) - 3) = (n - (\frac{dim(U)}{2} - 1) - 3) = (n - \frac{dim(U)}{2} - 2)$, so the claim holds.

(4) $(x', x) = \mathfrak{C}(> x') \cap \mathfrak{C}(< x)$ is $(h(x) - h(x') - 2)$-spherical. Now (x', x) can be identified with $\mathfrak{C}(U)(> x')$ which is $(\frac{dim(U)}{2} - h(x') - 3)$-spherical by (3). But $(\frac{dim(U)}{2} - h(x') - 3) = (h(x) - h(x') - 2)$, so this claim holds.

Thus by Lemma 14.2 and facts (1)–(4) above, we have $\mathfrak{C}$ is C.M. of dimension $n-2$ as claimed. □

Chapter 5

Results about spaces with forms

In chapter 6 we shall consider $\mathcal{A}_p(G)$ where G is one of $GL_n(q)$, $Sp_{2n}(q)$, and $GU_n(q)$ with $(p,q) = 1$. These groups arise as groups of isometries of spaces with forms, and in Chapter 5 we record some results about such spaces. The results pertain to the decomposition of these spaces, and geometries identified with these spaces, under the action of elementary abelian p-subgroups. These results will play an important role in our analysis in chapter 6.

Let p be a prime and q a prime power such that $(p,q) = 1$. In Section 18 we let $G = Sp_{2n}(q)$ and $A \in \mathcal{A}_p(G)$ and consider the structure of $C_G(A)$. We also study the decomposition of the corresponding symplectic space into $\mathbb{F}_q[A]$-submodules. In Section 19 an analogous computation for $G = GU_n(q)$ is carried out. Some of the results in this section are identical to those in section 18 and have been included only to make section 19 complete. In Section 20 we use results from the previous two sections to obtain some preliminary results about $m_p(G)$ and $\mathcal{A}_p(G)$, where G is one of $GL_n(q), Sp_{2n}(q)$, and $GU_n(q)$. With p and q as above, let d be the order of q in $\mathbb{Z}/p\mathbb{Z}$, that is, the minimal integer with respect to $p|q^d - 1$. Our analysis in sections 18 and 19 will show that the structure of $C_G(A)$ – where $G = Sp_{2n}(q)$ or $GU_n(q)$ – and the decomposition of the corresponding spaces with forms depend greatly on whether d is even or odd. In Section 21 we let $G = Sp_{2n}(q)$ or $GU_n(q)$ and assume that d is odd. Given $A \in \mathcal{A}_p(G)$ and B, the Tits building of the corresponding space with form, we study the structure of $Fix(A)$, the full subcomplex of B defined on the simplices fixed under the action of A. Finally, in Section 22 we consider $Fix(A)$, when $G = Sp_{2n}(q)$, $A \in \mathcal{A}_p(G)$ and d is even.

At this point we should remark on the strategy used in sections 18 and 19:

Strategy: We want to compute $C_G(A)$ where $A \in \mathcal{A}_p(G)$ and $G = Sp_{2n}(q)$ or $GU_n(q)$. We also want to study the decomposition of (V, f), the corresponding symplectic or unitary space, into homogeneous components under the action of A. Given $A \in \mathcal{A}_p(G)$, let K

be the *field of definition of* A (see section 26 in [A1] for the definition and properties of K), and let $(\widehat{V},\hat{f})$ be the "appropriate" space with form over K. Let $\hat{G}$ be the group of isometries of $(\widehat{V},\hat{f})$ and σ an "appropriate" automorphism of $\widehat{V}$ such that $g^\sigma v^\sigma = (gv)^\sigma$ $\forall\, g \in \hat{G},\ v \in \widehat{V}$. By "appropriate" we mean that $C_{\widehat{V}}(\sigma) = V$ and $C_{\hat{G}}(\sigma) = G$.

Using results from section 15 and 16 (since $p \mid |K^\sharp|$) and facts about fields of definition, we get a decomposition of $\widehat{V}$ into homogeneous components, as follows: $\widehat{V} = \widehat{V}_1 \oplus \cdots \oplus \widehat{V}_m$. And $C_{\hat{G}}(A) = \cap N_{\hat{G}}(\widehat{V}_i) = \hat{L}_1 \times \cdots \times \hat{L}_m$, where each $\hat{L}_i$ corresponds in a natural way to $\widehat{V}_i$; for example, if $\hat{G}$ is a symplectic group and each $\widehat{V}_i$ is nondegenerate, then $\hat{L}_i = C_{\hat{G}}(\widehat{V}_i^{\perp})$.

Note that σ acts on $\{\widehat{V}_i \mid 1 \le i \le m\}$ and since $g^\sigma v^\sigma = (gv)^\sigma$, this defines an action of σ on $\{\hat{L}_i\}$. Given a set of orbit representatives $\widehat{V}_{i_1}, \widehat{V}_{i_2}, \ldots, \widehat{V}_{i_s}$ for the action of σ on $\{\widehat{V}_i\}$, $\hat{L}_{i_1}, \ldots, \hat{L}_{i_s}$ is a set of orbit representatives for the action of σ on $\{\hat{L}_i\}$. Also, given $1 \le j \le s$, σ acts on $\widehat{V}_{i_j}$ (and thus on $\hat{L}_{i_j}$) in one of two ways:

(1) Either $\sigma\widehat{V}_{i_j} = \widehat{V}_{i_j}$, which implies $\sigma\hat{L}_{i_j} = \hat{L}_{i_j}$. In this case, let $W_j = \widehat{V}_{i_j}$ and $H_j = \hat{L}_{i_j}$, and note that σ induces a *field automorphism* on W_j and H_j. Thus $C_{W_j}(\sigma) \le V$ and $C_{H_j}(\sigma) \le G$.

(2) Or σ acts regularly on $\{\widehat{V}_{i_j}, \sigma\widehat{V}_{i_j}, \ldots, \sigma^{d-1}\widehat{V}_{i_j}\}$ and thus acts regularly on $\{\hat{L}_{i_j}, \sigma\hat{L}_{i_j}, \ldots, \sigma^{d-1}\hat{L}_{i_j}\}$. Then let $W_j = \widehat{V}_{i_j} \oplus \sigma\widehat{V}_{i_j} \oplus \cdots \oplus \sigma^{d-1}\widehat{V}_{i_j}$ and $H_j = \hat{L}_{i_j} \times \cdots \times \sigma^{d-1}\hat{L}_{i_j}$. Then we have $C_{W_j}(\sigma) = < v + \sigma v + \cdots + \sigma^{d-1}v \mid v \in \widehat{V}_{i_j} > \cong \widehat{V}_{i_j}$ as K-subspaces and $C_{H_j}(\sigma) = < g\sigma g \cdots \sigma^{d-1}g \mid g \in \hat{L}_{i_j} > \cong \hat{L}_{i_j} \le G$.

Note that by the choice of orbit representatives and the definition of W_j and H_j for all $1 \le j \le s$, we have $\widehat{V} = W_1 \oplus \cdots \oplus W_s$ and $C_{\hat{G}}(A) = H_1 \times \cdots \times H_s$, with $< \sigma > W_j = W_j$ and $< \sigma > H_j = H_j$ $\forall\ 1 \le j \le s$, thus:

$$V = C_{\widehat{V}}(\sigma) = C_{W_1}(\sigma) \oplus \cdots \oplus C_{W_s}(\sigma), \text{ and}$$

$$C_G(A) = C_{\hat{G}}(A) \cap C_{\hat{G}}(\sigma) = C_{H_1}(\sigma) \times \cdots \times C_{H_s}(\sigma).$$

We thus say that "$C_{W_j}(\sigma)$ contributes $C_{H_j}(\sigma)$ towards $C_G(A)$." Notice if σ acts on $\widehat{V}_{i_j}$ as (2), then $C_{H_j}(\sigma) \cong \hat{L}_{i_j}$. □

We shall use the facts and notation contained in Strategy in the following two sections without explicit reference to Strategy.

SECTION 18 : $C_G(A)$ WHERE $G = Sp_{2n}(q)$ AND $A \in \mathcal{A}_p(G)$

Let $G = Sp_{2n}(q)$, and (V, f) be the corresponding $2n$-dimensional symplectic space over $F = \mathbb{F}_q$. In chapter 6 we will consider the simple connectivity of $\mathcal{A}_p(G)$. By Theorem 15.11 we know that if $p|q-1$, then $\mathcal{A}_p(G)$ is Cohen-Macaulay – so we may assume that $p \neq 2$ is a prime with $(p, q) = 1$.

Let d be minimal with respect to $p|q^d - 1$ and in light of Theorem 15.11 assume that $d > 1$. Let $K = \mathbb{F}_{q^d}$ and $\Gamma = Gal(K/F) = <\sigma>$. Let $\widehat{V} = V \otimes_F K$ and $\hat{f} : \widehat{V} \times \widehat{V} \to K$ be defined by $\hat{f}(v \otimes a, w \otimes b) = abf(v, w)$ $\forall\ a, b \in K,\ v, w \in V$ (extended by linearity). Then $(\widehat{V}, \hat{f})$ is a $2n$-dimensional symplectic space over K with $\hat{f}|_V = f$. Note that σ acts on $(\widehat{V}, \hat{f})$ by $(\Sigma v_i \otimes a_i)^\sigma = \Sigma v_i \otimes \sigma(a_i)$.

Now $G = Sp_{2n}(q) \leq Sp_{2n}(q^d) = \hat{G}$, and σ acts on $\hat{G}$ as a *field automorphism* with $C_{\hat{G}}(\sigma) = G$ and $g^\sigma v^\sigma = (gv)^\sigma\ \forall\ g \in \hat{G}, v \in \widehat{V}$. Given $A \in \mathcal{A}_p(G) \subseteq \mathcal{A}_p(\hat{G})$, we have $C_G(A) = C_{\hat{G}}(A) \cap C_{\hat{G}}(\sigma)$. So we first have to compute $C_{\hat{G}}(A)$. Fix $A \in \mathcal{A}_p(G)$, and as in section 15, consider a decomposition of $\widehat{V} = \bigoplus\limits_{\lambda} \widehat{V}_\lambda$ into homogeneous components under the action of A, and set $\Phi = \{\lambda : A \to K^\sharp \mid \widehat{V}_\lambda \neq 0\}$. Note that given $\lambda \in \Phi$, $\forall\ a \in A$ and $\forall\ v \in \widehat{V}_\lambda$ we have:

$$a^\sigma v^\sigma = (av)^\sigma = (\lambda(a)v)^\sigma = \lambda(a)^\sigma v^\sigma = \lambda(a)^q v^\sigma.$$

Thus $\sigma(\widehat{V}_\lambda) = \widehat{V}_{\lambda^q}$, and Γ acts on Φ by mapping $\lambda \mapsto \lambda^q$. Now two distinct cases arise:

Assume that d is odd. Then $|\Gamma| = d$ implies that there are no involutions in Γ. So $\forall\ 1 \neq \lambda \in \Phi$, $\lambda^{-1} \notin \Gamma\lambda$. Thus it is possible to find a set of *positive weights* $\Phi \supseteq \Phi^+$ preserved by Γ which satisfies the following conditions:

$$\text{if } \lambda \neq \gamma \in \Phi^+, \text{ then } \lambda\gamma \neq 1; \text{ and } \Phi = \{\lambda, \lambda^{-1} \mid \lambda \in \Phi^+\}.$$

By Theorem 15.11 we have a decomposition:

$$\widehat{V} = \widehat{V}_0 \underset{\lambda \in \Phi^+ - \{1\}}{\perp} (\widehat{V}_\lambda \oplus \widehat{V}_{\lambda^{-1}}), \tag{18.1}$$

where V_0, which equals $\widehat{V}_\lambda$ when $\lambda = 1$, is nondegenerate; and for all $\lambda \in \Phi^+ - \{1\}$ $\widehat{V}_\lambda$ and $\hat{V}_{\lambda^{-1}}$ are totally singular, and $\widehat{V}_\lambda \oplus \hat{V}_{\lambda^{-1}}$ is a hyperbolic subspace of V. Thus we have:

$$C_{\hat{G}}(A) \cong Sp_{2n_0}(q^d) \times \prod_{\lambda \in \Phi^+ - \{1\}} GL_{n_\lambda}(q^d),$$

where $2n_0 = dim(\widehat{V}_0)$, and $n_\lambda = dim(\widehat{V}_\lambda)\ \forall\ \lambda \in \Phi^+ - \{1\}$.

If we choose orbit representatives for the action of Γ on Φ^+, say $\lambda_0, \lambda_1, \ldots, \lambda_m$ with $\lambda_0 = 1$, then note that: $\Gamma\lambda_i \cap \Gamma\lambda_j^{-1} = \emptyset\ \forall\ 0 \le i, j \le m$. For each $0 \le i \le m$, let $\widehat{V}_i^+ = \bigoplus_{\gamma \in \Gamma\lambda_i} \widehat{V}_\gamma$ and $\widehat{V}_i^- = \bigoplus_{\gamma \in \Gamma\lambda_i^{-1}} \widehat{V}_\gamma$; then both $\widehat{V}_i^+$ and $\widehat{V}_i^-$ are normalized by Γ. Also, by (18.1) and Lemma 15.1, for each $1 \le i \le m$, $\widehat{V}_i^+$ and $\widehat{V}_i^-$ are totally singular and $\widehat{V}_i = \widehat{V}_i^+ \oplus \widehat{V}_i^-$ is a hyperbolic subspace of V.

Fix $1 \le i \le m$, and note that $\forall\ a \in A\ \lambda_i(a)^p = \lambda_i(a^p) = 1 \implies \lambda_i^p = 1$. So if $|\Gamma\lambda_i| \ne d$, then there exists $r < d$ such that $\lambda_i^{q^r - 1} = 1 \implies p | q^r - 1$, contradicting the minimality of d. So we have:

$$\widehat{V}_i^+ = \widehat{V}_{\lambda_i} \oplus \widehat{V}_{\sigma(\lambda_i)} \oplus \cdots \oplus \widehat{V}_{\sigma^{d-1}(\lambda_i)}.$$

Since $\widehat{V}_i^+$ is a $K[A]$-submodule normalized by Γ, by (25.7.2) on pg. 120 in [A1], there exists an $F[A]$-submodule $V_i^+ \le V$ such that $\widehat{V}_i^+ = V_i^+ \otimes_F K$. Since $\hat{f}|_V = f$ and $\widehat{V}_i^+$ is totally singular, V_i^+ is a totally singular subspace of V. Also, if $dim_K(\widehat{V}_{\lambda_i}) = n_i$ then $dim_F(V_i^+) = dim_K(\widehat{V}_i^+) = dn_i$. Similarly, for $\widehat{V}_i^-$ there is a dn_i-dimensional totally singular subspace $V_i^- \le V$ such that $\widehat{V}_i^- = V_i^- \otimes_F K$. Similarly, $\widehat{V}_i^+ \oplus \widehat{V}_i^-$ being hyperbolic implies $V_i = V_i^+ \oplus V_i^-$ is a $2dn_i$-dimensional hyperbolic subspace of V.

Note that σ acts regularly on $\{\widehat{V}_{\lambda_i}, \sigma\widehat{V}_{\lambda_i}, \ldots, \sigma^{d-1}\widehat{V}_{\lambda_i}\}$. Thus, as K-spaces $V_i^+ = C_{\widehat{V}_i^+}(\sigma) =< v + \sigma(v) + \cdots \sigma^{d-1}(v) \mid v \in \widehat{V}_{\lambda_i} >\cong \widehat{V}_{\lambda_i}$. Similarly, $V_i^- \cong \widehat{V}_{\lambda_i^{-1}}$. Thus, by the argument outlined at the beginning of this chapter, V_i contributes $GL_{n_i}(q^d)$ towards $C_G(A)$.

For $\widehat{V}_{\lambda_0}$, also by (25.7.2) on pg. 120 in [A1], we have $V_0 \le V$ such that $\widehat{V}_{\lambda_0} = \bigoplus_{\gamma \in \Gamma\lambda_0} \widehat{V}_\gamma = V_0 \otimes_F K$. So V_0 is a nondegenerate subspace of V with $dim_F(V_0) = dim_K(\widehat{V}_{\lambda_0})$. Now σ induces a field automorphism on $\widehat{V}_0$, and thus on $\widehat{V}_0^\perp$. Thus, V_0 contributes $C_{Sp_{2n_0}(q^d)}(\sigma) = Sp_{2n_0}(q)$ towards $C_G(A) = C_{\hat{G}}(A) \cap C_{\hat{G}}(\sigma)$. Thus we have the following result:

Theorem 18.2. *Let $p \neq 2$ be a prime and $d > 1$ be minimal with respect to $p|q^d - 1$. Let $G = Sp_{2n}(q)$ and V be the corresponding symplectic space, and assume that d is odd. Fix $A \in \mathcal{A}_p(G)$. Then we have an orthogonal decomposition:*

$$V = V_0 \perp V_1^+ \oplus V_1^- \perp \cdots \perp V_m^+ \oplus V_m^-$$

into homogeneous $F[A]$-submodules, where:

(1) *$V_0 = C_V(A)$ is nondegenerate with $dim(V_0) = 2n_0$ where $n_0 \equiv n (mod\ d)$.*

(2) *$\forall\ 1 \leq i \leq m$ V_i^+ and V_i^- are totally singular with $V_i^+ \oplus V_i^-$ hyperbolic of dimension $2dn_i$.*

And we have:

$$C_G(A) \cong Sp_{2n_0}(q) \times \prod_{1 \leq i \leq m} GL_{n_i}(q^d).$$

The decomposition of V and the structure of $C_G(A)$ differ significantly when d is even. In this case we get a unitary structure on $\widehat{V}$, but the proof of certain results about this structure depends on the fact that 2 is invertible; so we have to assume that q is odd.

As in the case when d is odd we can choose a set of positive weights Φ^+ such that:

$$\widehat{V} = \widehat{V}_0 \underset{\lambda \in \Phi^+ - \{1\}}{\perp} (\widehat{V}_\lambda \oplus \widehat{V}_{\lambda^{-1}}) \text{ and } C_{\hat{G}}(A) = Sp_{2n_0}(q^d) \times \prod_{\lambda \in \Phi^+ - \{1\}} GL_{n_\lambda}(q^d),$$

where $\widehat{V}_0 = \widehat{V}_\lambda$ for $\lambda = 1$, $2n_0 = dim(\widehat{V}_0)$, and $n_\lambda = dim(\widehat{V}_\lambda)\ \forall\ \lambda \in \Phi^+ - \{1\}$. Let $\tau = \sigma^{\frac{d}{2}}$ be the unique involution of Γ. Given $\lambda \in \Phi^+ - \{1\}$, consider $\tau(\lambda) = \lambda^x$. Then $\lambda = \tau^2(\lambda) = \lambda^{x^2}$ and $\lambda^p = 1 \implies x^2 \equiv 1 \pmod p$. Since K was chosen to be the minimal extension of F containing p^{th} roots of unity, $x \not\equiv 1 \pmod p$ so $\tau(\lambda) = \lambda^{-1}$. Thus we cannot choose a set of positive weights which are preserved by Γ. Thus we must analyze the action of Γ on Φ.

First note that as τ is a field automorphism, we have the following result (which will be used frequently without explicit reference):

Lemma 18.3. *For all $v, w \in \widehat{V}$ we have:*

$$\hat{f}(\tau v, \tau w) = \tau(\hat{f}(v, w)).$$

Fix $1 \neq \lambda \in \Phi$ and let $\hat{W} = \underset{\gamma \in \Gamma\lambda}{\oplus} \hat{V}_\gamma$. Then, by Lemma 15.1 and the discussion above, we have an orthogonal decomposition of $\hat{W}$ into symplectic subspaces as follows:

$$\begin{aligned}\hat{W} &= \hat{V}_\lambda \oplus \hat{V}_{\tau(\lambda)} \perp \cdots \perp \hat{V}_{\sigma^{\frac{d}{2}-1}(\lambda)} \oplus \hat{V}_{\sigma^{d-1}(\lambda)} \\ &= \hat{V}_\lambda \oplus \hat{V}_{\lambda^{-1}} \perp \cdots \perp \hat{V}_{\sigma^{\frac{d}{2}-1}(\lambda)} \oplus \hat{V}_{\sigma^{\frac{d}{2}-1}(\lambda)^{-1}}.\end{aligned}$$

Thus the contibution of $\hat{W}$ to $C_{\hat{G}}(A)$ is isomorphic to $GL_{n_\lambda}(q^d) \times GL_{n_\lambda}(q^d) \times \cdots \times GL_{n_\lambda}(q^d)$, which we denote by $L = L_\lambda \times L_\lambda \times \cdots \times L_\lambda$. We want to understand the structure of:

$$C_L(\sigma) = < g \times \sigma(g) \times \cdots \times \sigma^{\frac{d}{2}-1}(g) | g \in C_{L_\lambda}(\tau) > \cong C_{L_\lambda}(\tau).$$

Note that $C_{\hat{W}}(\sigma)$ is isomorphic to $\hat{U} = C_{\hat{V}_\lambda \oplus \hat{V}_{\lambda^{-1}}}(\tau) = < v + \tau v \mid in \hat{V}_\lambda >$. And $C_{L_\lambda}(\tau) = GL(\hat{U}) \cap L_\lambda$.

Now remember that L_λ is $C_{Sp(\hat{V}_\lambda \oplus \hat{V}_{\lambda^{-1}})}(A)$. Since $\hat{V}_{\lambda^{-1}}$ is L_λ-isomorphic to the dual of $\hat{V}_\lambda$ – via the map τ – $L_\lambda = \{\begin{pmatrix} g & 0 \\ 0 & g^* \end{pmatrix} \mid g \in GL(\hat{V}_\lambda); g^* = {}^t g^{-1}\}$ with respect to a basis $\{v_i, \tau v_i \mid 1 \leq i \leq n_\lambda\}$ for $\hat{V}_\lambda \oplus \hat{V}_{\lambda^{-1}}$. Thus:

$$C_{L_\lambda}(\tau) = \{\begin{pmatrix} g & 0 \\ 0 & g^* \end{pmatrix} \mid g \in GL(\hat{V}_\lambda); g^* = {}^t g^{-1}\} \text{ satisfying}$$

$$\begin{pmatrix} g & 0 \\ 0 & g^* \end{pmatrix}\begin{pmatrix} v \\ \tau v \end{pmatrix} = gv + g^*\tau v \in \hat{U} \iff g^*\tau v = \tau g v \iff \tau g^* \tau v = gv \ \forall\ v \in \hat{V}_\lambda.$$

Hence we have to classify all $g \in GL(\hat{V}_\lambda)$ such that $g = \tau g^* \tau$. We have the following result:

Lemma 18.4. *For* $v, v' \in \hat{V}_\lambda$, $\hat{f}(\tau v, v') = \hat{f}(g^*\tau v, g v')\ \forall\ g \in GL(\hat{V}_\lambda)$.

Proof. This result follows from the following computation:

$$\begin{aligned}2\hat{f}(\tau v, v') &= \hat{f}(v, \tau v') + \hat{f}(\tau v, v') - \hat{f}(v, \tau v') + \hat{f}(\tau v, v') \\ &= \hat{f}(v + \tau v, v' + \tau v') + \hat{f}(-v + \tau v, v' + \tau v') \\ &= \hat{f}(gv + g^*\tau v, gv' + g^*\tau v') + \hat{f}(-gv + g^*\tau v, gv' + g^*\tau v') \\ &\text{since } \begin{pmatrix} g & 0 \\ 0 & g^* \end{pmatrix} \in Sp(\hat{V}_\lambda \oplus \hat{V}_{\lambda^{-1}}) \\ &= \hat{f}(gv, g^*\tau v') + \hat{f}(g^*\tau v, gv') - \hat{f}(gv, g^*\tau v') + \hat{f}(g^*\tau v, gv') \\ &= 2\hat{f}(g^*\tau v, gv').\end{aligned}$$

Since q is odd we have $\hat{f}(\tau v, v') = \hat{f}(g^*\tau v, gv')$ as claimed. $\square$

Now let $k = K^\tau$. Then by Hilbert's Theorem 90 we know that there exists $\delta \in K$ such that $\tau(\delta) = -\delta$ and $K = k(\delta)$. Define:

$$\theta : \widehat{V}_\lambda \times \widehat{V}_\lambda \to K \text{ by } \theta(v, v') = \delta\hat{f}(v, \tau v').$$

Lemma 18.5. *θ is a unitary form on $\widehat{V}_\lambda$.*

Proof. Suppose $v \in \widehat{V}_\lambda$ such that $\theta(v, w) = 0 \ \forall \ w \in \widehat{V}_\lambda$. Then $\theta(v, \tau v') = 0 \ \forall \ v' \in \hat{V}_{\lambda^{-1}}$. Hence $\delta\hat{f}(v, v') = 0 \ \forall \ v' \in \hat{V}_{\lambda^{-1}}$. But since $\hat{f}$ is nondegenerate on $\widehat{V}_\lambda \oplus \hat{V}_{\lambda^{-1}}$, this implies $v = 0$.

$\theta(v+w, v') = \theta(v, v') + \theta(w, v')$; $\theta(v, v'+w) = \theta(v, v') + \theta(v, w)$; $\theta(av, v') = a\theta(v, v')$ and $\theta(v, av') = \tau(a)\theta(v, v')$ follow from the definition of θ and the linearity of $\hat{f}$. So consider:

$$\begin{aligned}\theta(v', v) &= \delta\hat{f}(v', \tau v) = \delta\tau(\hat{f}(\tau v', v)) = \delta\tau(-\hat{f}(v, \tau v'))\\ &= -\delta\tau(\hat{f}(v, \tau v') = \tau(\delta\hat{f}(v, \tau v')) = \tau(\theta(v, v')).\end{aligned}$$

Therefore θ is a unitary form as claimed. $\square$

Lemma 18.6. $C_{L_\lambda}(\tau) \cong GU_{n_\lambda}(q^{\frac{d}{2}})$.

Proof. We already know that $C_{L_\lambda}(\tau) \cong G' = \{g \in GL(\widehat{V}_\lambda) \mid g = \tau g^* \tau\}$ Since θ is a unitary form, it suffices to show that G' is the group of isometries of $(\widehat{V}_\lambda, \theta)$.

Let $g \in G'$. We want to show that $\theta(gv, gv') = \theta(v, v')$. So consider:

$$\begin{aligned}\theta(gv, gv') &= \delta\hat{f}(gv, \tau gv') = \delta\hat{f}(\tau g^*\tau v, \tau gv') = \delta\tau\hat{f}(g^*\tau v, gv')\\ (\text{by Lemma 18.4}) &= \delta\tau\hat{f}(\tau v, v') = \delta\hat{f}(v, \tau v') = \theta(v, v').\end{aligned}$$

Thus $G' \subseteq GU_{n_\lambda}(q^{\frac{d}{2}})$.

Now let $g \in GU_{n_\lambda}(q^{\frac{d}{2}})$ and let $\{v_j \mid 1 \leq j \leq n_\lambda\}$ be an orthonormal basis for $\widehat{V}_\lambda$ with respect to θ (such a basis exists by (21.5) on pg. 87 in [A1]). We want to show that $gv = \tau g^*\tau v \ \forall \ v \in \widehat{V}_\lambda$; so it suffices to show that $gv_i = \tau g^*\tau v_i \ \forall \ 1 \leq i \leq n_\lambda$. But as $\{v_j\}$ forms an orthonormal basis for $(\widehat{V}_\lambda, \theta)$, it is equivalent to showing $\theta(gv_i, v_j) = \theta(\tau g^* \tau v_i, v_j)\ \forall\ 1 \le i, j \le n_\lambda$. So consider:

$$\begin{aligned}
\theta(gv_i, v_j) &= \theta(gv_i, gg^{-1}v_j) = \theta(v_i, g^{-1}v_j) \text{ (since } g \in GU_{n_\lambda}(q^{\frac{d}{2}})\text{)}\\
&\Rightarrow \delta\hat{f}(gv_i, \tau v_j) = \delta\hat{f}(v_i, \tau g^{-1}v_j)\\
&\Rightarrow \tau\hat{f}(\tau gv_i, v_j) = \tau\hat{f}(\tau v_i, g^{-1}v_j)\\
&\Rightarrow \tau\hat{f}(\tau gv_i, v_j) = \tau\hat{f}(g^*\tau v_i, gg^{-1}v_j) \quad \text{(by Lemma 18.4)}\\
&\Rightarrow \hat{f}(gv_i, \tau v_j) = \hat{f}(\tau g^*\tau v_i, \tau v_j)\\
&\Rightarrow \theta(gv_i, v_j) = \theta(\tau g^* \tau v_i, v_j).
\end{aligned}$$

Thus $GU_{n_\lambda}(q^{\frac{d}{2}}) \subseteq G' \implies GU_{n_\lambda}(q^{\frac{d}{2}}) = G'$ and the lemma holds. □

Thus we have shown that $C_L(\sigma) \cong C_{L_\lambda}(\tau) \cong GU_{n_\lambda}(q^{\frac{d}{2}})$ where $dim(\widehat{V}_\lambda) = n_\lambda$.

Now choose orbit representatives $\lambda_0, \lambda_1, \ldots, \lambda_m$ for the action of G on Φ with $\lambda_0 = 1$. Let $\widehat{V}_i = \bigoplus_{\gamma \in \Gamma\lambda_i} \widehat{V}_\gamma\ \forall\ 0 \le i \le m$. Then, by the preceding discussion, for $1 \le i \le m$ we have $\widehat{V}_i$ contributing $GU_{n_i}(q^{\frac{d}{2}})$ towards $C_G(A)$, where $dim(\widehat{V}_{\lambda_i}) = n_i$. And, by (25.7.2) on pg. 120 in [A1], for all $1 \le i \le m$ there exists $V_i \le V$ which is nondegenerate of dimension dn_i such that $\widehat{V}_i = V_i \otimes_F K$. For λ_0 we have the same situation as in the case when d was odd. $\widehat{V}_0$ contributes $Sp_{2n_0}(q)$ towards $C_G(A)$ and $\widehat{V}_0 = V_0 \otimes_F K$ where V_0 is a nondegenerate subspace of V with $dim_F(V_0) = dim_K(\widehat{V}_0)$. Thus we have:

Theorem 18.7. *Let $p \neq 2$ be a prime and q be an odd prime power such that $(p, q) = 1$. Let $d > 1$ be minimal with respect to $p | q^d - 1$, and assume that d is even. Let $G = Sp_{2n}(q)$ and V the corresponding symplectic space. Fix $A \in \mathcal{A}_p(G)$, then we have an orthogonal decomposition:*

$$V = V_0 \perp V_1 \perp \cdots \perp V_m,$$

where:

(1) $V_0 = C_V(A)$ *is nondegenerate of dimension* $2n_0$ *where* $n_0 \equiv n \pmod{\frac{d}{2}}$.

(2) $\forall\ 1 \le i \le m$ V_i *is nondegenerate of dimension* dn_i.

And we have:

$$C_G(A) \cong Sp_{2n_0}(q) \times \prod_{1 \le i \le m} GU_{n_i}(q^{\frac{d}{2}}).$$

SECTION 19 : $C_G(A)$ WHERE $G = GU_n(q)$ AND $A \in \mathcal{A}_p(G)$

In this section we consider $C_G(A)$ where $G = GU_n(q)$ and $A \in \mathcal{A}_p(G)$ with $(p, q) = 1$. We also consider the decomposition of the corresponding unitary space under the action of A. Note that in section 16 we showed that if $p \neq 2$ is prime with $p|q-1$, then $\mathcal{A}_p(GU_n(q))$ is Cohen-Macaulay of dimension $n-1$. Also note that if $p|q+1$, then there is a central elementary abelian p-subgroup $Z \le GU_n(q)$. Thus, by Proposition 2.4 on pg. 106 in [Q], $\mathcal{A}_p(GU_n(q))$ is contractible. Hence in this section we will mostly be interested in the case when $p \nmid q^2 - 1$. However, we first need to analyze the case when $p|q+1$ in slightly greater detail.

Let p be a prime such that $(p,q) = 1$, $G = GU_n(q)$, and $A \in \mathcal{A}_p(G)$. Let (V, f) be the corresponding unitary space over $F = \mathbb{F}_{q^2}$. Assume that $p|q+1$, and consider a decomposition of $V = \bigoplus_{\lambda} V_\lambda$ into homogeneous components under the action of A. Let $\Phi = \{\lambda : A \to F^\sharp \mid V_\lambda \neq 0\}$ and note that for all $\lambda \in \Phi$ and $a \in A$ we have:

$$\lambda(a)^p = \lambda(a^p) = 1 \implies \lambda^p = 1 \implies \lambda^{q+1} = 1 \quad \text{as } p|q+1. \tag{19.1}$$

Now given $\lambda, \gamma \in \Phi$, let $u \in V_\lambda$, $v \in V_\gamma$, $a \in A$ and consider:

$$f(u,v) = f(au, av) = f(\lambda(a)u, \gamma(a)v) = \lambda(a)\gamma(a)^q f(u,v).$$

Thus we have:

$$(1 - \lambda(a)\gamma(a)^q)f(u,v) = 0 \quad \forall\ \lambda, \gamma \in \Phi,\ a \in A,\ u \in V_\lambda, \text{ and } v \in V_\gamma. \tag{19.2}$$

But by (19.1) we have:

$$1 - \lambda(a)\gamma(a)^q = 0 \iff \gamma(a)^{q+1} - \lambda(a)\gamma(a)^q = 0\ \forall\ \alpha \in A \iff \lambda = \gamma. \tag{19.3}$$

Thus, taking into consideration (19.2) and (19.3), we see that if $\lambda \neq \gamma \in \Phi$ then $f(u,v) = 0$ $\forall\ u \in V_\lambda$, $v \in V_\gamma$; that is, $V_\lambda \perp V_\gamma$ if $\lambda \neq \gamma \in \Phi$. By the nondegeneracy of f, we thus have V_λ is nondegenerate $\forall\ \lambda \in \Phi$. So $V = \underset{\lambda\in\Phi}{\perp} V_\lambda$ is an orthogonal decomposition of V into nondegenerate $F[A]$-submodules. If $dim(V_\lambda) = n_\lambda\ \forall\ \lambda \in \Phi$ then we have:

$$C_G(A) \cong \prod_{\lambda\in\Phi} GU_{n_\lambda}(q).$$

Now assume that F does not contain any p^{th} roots of unity, and let d be minimal with respect to $p|q^d - 1$. Note that $d \geq 3$ implies that $p \neq 2$. Given $A \in \mathcal{A}_p(G)$, let K be the *field of definition* for the action of A on (V, f), that is, the smallest extension field of F containing p^{th} roots of unity. Let $\Gamma = Gal(K/F) =< \sigma >$. Three possibilities exist:

<u>Case I</u> d is odd. Then $K = \mathbb{F}_{q^{2d}}$ and $|\Gamma| = d$.

This case is very similar to the case when d is odd in section 18. Let $\widehat{V} = V \otimes_F K$ and $\hat{f}$ be the unitary form on $\widehat{V}$ which restricts to f on V. $(\widehat{V}, \hat{f})$ is an n-dimensional unitary space over K, so set $\hat{G} = GU_n(q^d)$, then $G \leq \hat{G}$ Consider a decomposition of $\widehat{V} = \underset{\lambda}{\oplus}\widehat{V}_\lambda$ into homogeneous components under the action of A. Let $\Phi = \{\lambda : A \to K^\sharp \mid \widehat{V}_\lambda \neq 0\}$. If $1 \neq \lambda \in \Phi$, then the minimality of d with respect to $p|q^d - 1$ implies that $|\Gamma\lambda| = d$. And since d is odd, $1 \neq \lambda \in \Phi$ implies that $\lambda^{-1} \notin \Gamma\lambda$. Thus, as in section 18, we can choose a set of positive weights $\Phi^+ \subseteq \Phi$ which are stable under Γ.

When we consider the action of A, considered as a subgroup of $\hat{G}$, on $\widehat{V}$ then the fact that $p|q^d - 1$ and $p \neq 2$ implies that we are in the same situation as in section 16. Thus we know that for each $1 \neq \lambda \in \Phi$, $\widehat{V}_\lambda$ is totally singular and $\widehat{V}_\lambda + \widehat{V}_{\lambda^{-1}}$ is nondegenerate. Pick orbit representatives $\lambda_0, \lambda_1, \ldots, \lambda_m$, of the action of Γ on Φ^+, with $\lambda_0 = 1$. For all $1 \leq i \leq m$, let:

$$\widehat{V}_i = \underset{\gamma\in\Gamma\lambda_i}{\oplus} \widehat{V}_\gamma = \widehat{V}_{\lambda_i} \perp \widehat{V}_{\sigma(\lambda_i)} \perp \cdots \perp \widehat{V}_{\sigma^{d-1}(\lambda_i)}.$$

Then $\widehat{V}_i$ is a totally singular $K[A]$-submodule of $\widehat{V}$ which is stable under the action of Γ. Thus, by (25.7.2) on pg. 120 in [A1], there exists $V_i \leq V$ such that $\widehat{V}_i = V_i \otimes_F K$. Thus, if $dim_K(\widehat{V}_{\lambda_i}) = n_i$ then $dim_F(V_i) = dn_i\ \forall\ 1 \leq i \leq m$. Since $\hat{f}$ restricted to V is f and $\widehat{V}_i$ is totally singular, V_i is also totally singular for all $1 \leq i \leq m$, and $V_i \oplus V_i^-$ (where $V_i^- \otimes K = \underset{\gamma\in\Gamma\lambda^{-1}}{\oplus} \widehat{V}_\gamma$) is a hyperbolic subspace of V. Also:

$$V_i = C_{\widehat{V}_i}(\sigma) =< v + \sigma(v) + \cdots + \sigma^{d-1}(v) \mid v \in \widehat{V}_{\lambda_i} >\cong \widehat{V}_{\lambda_i},$$

with Γ regular on $\Gamma\widehat{V}_{\lambda_i}$. Thus, for all $1 \leq i \leq m$, $V_i \oplus V_i^-$ contributes $GL_{n_i}(q^{2d})$ towards $C_G(A)$ – by the discussion at the beginning of the chapter.

For $i = 0, \widehat{V}_0 = \underset{\gamma\in\Gamma\lambda_0}{\oplus} \widehat{V}_\gamma = \widehat{V}_{\lambda_0}$, and $C_{\widehat{V}_{\lambda_0}}(\sigma) = V_0$ is an n_0-dimensional nondegenerate subspace of V, and σ induces a field automorphism on $C_{\hat{G}}(\widehat{V}_0^{\perp}) \cong GU_{n_0}(q^d)$.

Thus we have:

$$C_G(A) \cong GU_{n_0}(q) \times \prod_{1\leq i\leq m} GL_{n_i}(q^{2d}).$$

And we have an orthogonal decomposition of $V = V_0 \perp V_1 \oplus V_1^- \perp \cdots \perp V_m \oplus V_m^-$ into $F[A]$-submodules, where:

(1) $V_0 = C_V(A)$ is nondegenerate of dimension $n_0 \equiv n \pmod{2d}$, and

(2) $V_i \oplus V_i^-$ is a nondegenerate hyperbolic subspace of dimension $2dn_i$ $\forall\ 1 \leq i \leq m$.

<u>Case II</u> $d \equiv 2 \pmod 4$. Then $K = \mathbb{F}_{q^d}$ and $|\Gamma| = \frac{d}{2}$ is odd.

As in Case I, let $(\widehat{V}, \hat{f})$ be an n-dimensional unitary space over K with $\widehat{V} = V \otimes_F K$ and $\hat{f}|_V \equiv f$. Let $\widehat{V} = \underset{\lambda}{\oplus}\widehat{V}_\lambda$ be a decomposition of $\widehat{V}$ into homogeneous components under the action of A. Let $\hat{G} = GU_n(q^{\frac{d}{2}})$ and $\Phi = \{\lambda : A \to K^\sharp | \widehat{V}_\lambda \neq 0\}$ as before. Since $p \nmid q^{\frac{d}{2}} - 1$, our analysis of the case when $p|q+1$ tells us that $\widehat{V}_\lambda$ is nondegenerate for each $\lambda \in \Phi$.

Let $\lambda_0, \lambda_1, \ldots, \lambda_m$ be orbit representatives of the action of Γ on Φ, with $\lambda_0 = 1$. For $1 \leq i \leq m$, let $\widehat{V}_i = \underset{\gamma\in\Gamma\lambda_i}{\oplus} \widehat{V}_\gamma$. Then $\widehat{V}_i$ is a nondegenerate $K[A]$-submodule of $\widehat{V}$ stable under Γ. Thus, again by (25.7.2) on pg. 120 in [A1], there exists an $F[A]$-submodule $V_i \leq V$ with $\widehat{V}_i = V_i \otimes_F K$. Hence, if $dim_K(\widehat{V}_{\lambda_i}) = n_i$, then $dim_F(V_i) = \frac{d}{2}n_i$ $\forall\ 1 \leq i \leq m$. Arguing as in Case I, V_i is nondegenerate. Also:

$$V_i = C_{\widehat{V}_i}(\sigma) =< v + \sigma(v) + \cdots + \sigma^{\frac{d}{2}-1}(v) | v \in \widehat{V}_{\lambda_i} > \cong \widehat{V}_{\lambda_i},$$

and so V_i contibutes $GU_{n_i}(q^{\frac{d}{2}})$ towards $C_G(A) = C_{\hat{G}}(A) \cap C_{\hat{G}}(\sigma)$.

For $i = 0$, we have $\widehat{V}_0 = \underset{\gamma\in\Gamma\lambda_0}{\oplus} \widehat{V}_\gamma = \widehat{V}_{\lambda_0}$. Thus $C_{\widehat{V}_0}(\sigma) = V_0$, a nondegenerate subspace of V with $dim_K(\widehat{V}_0) = dim_F(V_0)$, and σ induces a field automorphism on $C_{\hat{G}}(\widehat{V}_0^{\perp}) \cong GL_{n_0}(q^{\frac{d}{2}})$. So, V_0 contributes $GU_{n_0}(q)$ towards $C_G(A)$.

So we have:

$$C_G(A) \cong GU_{n_0}(q) \times \prod_{1 \le i \le m} GU_{n_i}(q^{\frac{d}{2}}).$$

And we have an orthogonal decomposition of $V = V_0 \perp V_1 \perp \cdots \perp V_m$ into $F[A]$-submodules, where:

(1) $V_0 = C_V(A)$ is nondegenerate and $dim(V_0) \equiv n \pmod{\frac{d}{2}}$.

(2) V_i is nondegenerate of dimension $\frac{d}{2}n_i$ for each $1 \le i \le m$.

<u>Case III</u> $d \equiv 0 \pmod 4$. Then $K = \mathbb{F}_{q^d}$ and $|\Gamma| = \frac{d}{2}$ is even.

As K is the field of definition for A, and $|K : F|$ is even, by (7.6.1) on pg. 493 in [A5], it is not possible to define a unitary structure $\hat{f}$ on $\widehat{V} = V \otimes_F K$ which restricts to f on V. Thus we have to use a different method to compute $C_G(A)$.

Throughout the remainder of this section, let $\{e_i \mid 1 \le i \le n\}$ be an orthonormal basis for V, which exists by (21.5) on pg. 87 in [A1]. Consider V as a vector space over F and let $\widehat{V} = V \otimes_F K$ be the corresponding space over K, with basis $\{e_i \otimes 1 \mid 1 \le i \le n\}$. By abuse of notation, we refer to $e_i \otimes 1$ also as e_i $\forall\ 1 \le i \le n$. If $\hat{G} = GL(\widehat{V}) = GL_n(q^d)$, then given the choice of basis above we have a representation of $\hat{G}$ as a subset of $M_n(q^d)$ – the $n \times n$ matrices over K. And we can define the following two maps:

> Let $\sigma : K \to K$, defined by $a \mapsto a^q$, be the map such that $\Gamma = <\sigma^2>$. Then σ acts on $\hat{G}$ as a *field automorphism*; that is, $(a_{ij})^\sigma = (a_{ij}^q)$. We also have the *transpose inverse* map $\theta : \hat{G} \to \hat{G}$ defined by $g^\theta = {}^t g^{-1}$. Define $\tau = \sigma\theta : \hat{G} \to \hat{G}$ and note that $\sigma\theta = \theta\sigma$ and $\theta^2 = 1$ imply that $\Gamma = <\sigma^2> = <\tau^2>$.

Also note that since $\{e_i \mid 1 \le i \le n\}$ was chosen to be an orthonormal basis for V we have:

$$G = GU_n(q) = C_{\hat{G}}(\tau).$$

Let $\widehat{V}^*$ be the dual space to $\widehat{V}$ and $\{f_i \mid 1 \le i \le n\}$ the basis dual to $\{e_i\}$. Note that given $g \in \hat{G}$, $v \in \widehat{V}$, and $w \in \widehat{V}^*$ we have:

(19.4) $$(g^\theta w)(gv) = w(v) \text{ which implies } (g^\theta w)(v) = w(g^{-1}v).$$

If we let $U = \widehat{V} \oplus \widehat{V}^*$, then we have a linear representation $\rho : \hat{G} \to GL(U)$ given by:

$$\rho(g)(v+w) = gv + g^\theta w \quad \forall\, g \in \hat{G},\ v \in \widehat{V},\ w \in \widehat{V}^*.$$

And with a basis for U given by $\{e_i, f_i \mid 1 \le i \le n\}$, we have the following maps defined on U :

> $\sigma : U \to U$ is defined by $(\Sigma\alpha_i e_i + \Sigma\beta_i f_i)^\sigma = (\Sigma\alpha_i^q e_i + \Sigma\beta_i^q f_i)$. And $\theta : U \to U$ is defined by $(\Sigma\alpha_i e_i + \Sigma\beta_i f_i)^\theta = \Sigma\alpha_i f_i + \Sigma\beta_i e_i$. Let $\tau = \sigma\theta$, and note that $\sigma\theta = \theta\sigma$ and $\theta^2 = 1$ imply that $\tau^2 = \sigma^2$. Also note that $\theta(\widehat{V}) = \widehat{V}^*$, hence $\tau(\widehat{V}) = \widehat{V}^*$ – with both $\widehat{V}$ and $\widehat{V}^*$ considered as subspaces of U.

The definition of σ and θ on $\hat{G}$ and U implies that for all $g \in \hat{G}$ and $u \in U$ we have:

$$\text{(19.5)} \qquad \rho(g^\sigma)(u^\sigma) = (\rho(g)u)^\sigma \text{ and } \rho(g^\theta)(u^\theta) = (\rho(g)u)^\theta \implies \rho(g^\tau)(u^\tau) = (\rho(g)u)^\tau.$$

Consider $C_U(\tau) =< v + v^\tau \mid v \in C_U(\tau^2) >$. Now $v = \sum \alpha_i e_i + \sum \beta_i f_i \in C_U(\tau^2)$ if and only if:

$$v^{\tau^2} = \sum \alpha_i^{q^2} e_i + \sum \beta_i^{q^2} f_i = \sum \alpha_i e_i + \sum \beta_i f_i \iff \alpha_i, \beta_i \in F\ \forall\, 1 \le i \le n.$$

Furthermore, we have:

$$\begin{aligned} v + \tau v &= \sum(\alpha_i + \beta_i^q)e_i + \sum(\alpha_i^q + \beta_i)f_i \\ &= \sum(\alpha_i + \beta_i^q)e_i + \sum(\alpha_i^q + \beta_i^{q^2})f_i \quad (\text{since } \beta_i \in F\ \forall\, 1 \le i \le n) \\ &= \sum(a_i + \beta_i^q)e_i + \sum(\alpha_i + \beta_i^q)^q f_i = \sum \gamma_i e_i + \gamma_i^q f_i \text{ with } \gamma_i \in F\ \forall\, 1 \le i \le n. \end{aligned}$$

So we have:

$$C_U(\tau) =< v + \tau v \mid v \in C_{\widehat{V}}(\tau^2) > .$$

Since $\tau(\widehat{V}) = \widehat{V}^*$, we can define $f' : C_U(\tau) \times C_U(\tau) \to F$ by $f'(v + \tau v, w + \tau w) = \tau w(v)$.

Lemma 19.6. *$(C_U(\tau), f')$ is a unitary space over F, with group of isometries $G = C_{\hat{G}}(\tau)$.*

Proof. We first show that $(C_U(\tau), f')$ is a unitary space, and then show that $C_{\hat{G}}(\tau)$ is the group of isometries of $(C_U(\tau), f')$.

Let $v + \tau v = \Sigma\alpha_i e_i + \alpha_i^q f_i$, $w + \tau w = \Sigma\beta_i e_i + \beta_i^q f_i$, $u + \tau u \in C_U(\tau)$ and $\alpha \in K$. Then:

$$f'(v + \tau v, w + \tau w + u + \tau u) = \tau(w + u)(v) =$$

$$\tau w(v) + \tau u(v) = f'(v + \tau v, w + \tau w) + f'(v + \tau v, u + \tau u).$$

The fact that:

$$f'(v + \tau v + w + \tau w, u + \tau u) = f'(v + \tau v, u + \tau u) + f'(w + \tau w, u + \tau u), \text{ and}$$

$$f'(\alpha(v + \tau v), w + \tau w) = \alpha f'(v + \tau v, w + \tau w)$$

are similarly proven. Also, note that:

$$f'(w + \tau w, v + \tau v) = \tau v(w) = \Sigma\alpha_i^q\beta_i = \Sigma\alpha_i^q\beta_i^{q^2} = (\Sigma\alpha_i\beta_i^q)^q = (f'(v + \tau v, w + \tau w))^\sigma.$$

Finally, given $0 \neq v+\tau v \in C_U(\tau)$, let $w+\tau w \in U$ such that $\tau w = v^*$. Since $v \in C_{\widehat{V}}(\tau^2)$ we have $v^* = \tau w \in C_{\widehat{V}^*}(\tau^2)$; that is, $w + \tau w \in C_U(\tau)$. Then we have:

$$f'(v + \tau v, w + \tau w) = \tau w(v) = v^*(v) = 1.$$

Thus f' is nondegenerate; and by the preceding discussion, $(C_U(\tau), f')$ is a unitary space over F.

To show that $C_{\widehat{G}}(\tau)$ is the group of isometries of $(C_U(\tau), f')$, it suffices to show containment since both groups are isomorphic to $GU_n(q)$. Now given $g \in C_{\widehat{G}}(\tau)$, and $v + \tau v$, $w + \tau w \in C_U(\tau)$ we have:

$$\begin{aligned}
&f'(\rho(g)(v + \tau v), \rho(g)(w + \tau w)) \\
&= f'(\rho(g)v + \rho(g)\tau v, \rho(g)w + \rho(g)\tau w) = f'(\rho(g)v + \rho(g^\tau)\tau v, \rho(g)w + \rho(g^\tau)\tau w) \\
&= f'(\rho(g)v + (\rho(g)v)^\tau, \rho(g)w + (\rho(g)w)^\tau) \quad \text{(by (19.5))} \\
&= (\rho(g)w)^\tau(\rho(g)v) = \rho(g^\tau)(\tau w)(\rho(g)v) = \rho(g)(\tau w)(\rho(g)v) \\
&= g^\theta(\tau w)(gv) = \tau w(v) \quad \text{(by (19.4))} \\
&= f'(v + \tau v, w + \tau w).
\end{aligned}$$

So $C_{\widehat{G}}(\tau)$ is contained in, and thus equals, the group of isometries of $(C_U(\tau), f')$. $\square$

Note that $V = C_{\widehat{V}}(\tau^2)$ by the definition of $\widehat{V}$ and the fact that $\Gamma = <\tau^2>$. Thus we have:

Lemma 19.7. *The map $\pi : V \to C_U(\tau)$ induces a G-invariant isomorphism between unitary spaces.*

Proof. Since $V = C_{\hat{V}}(\tau^2)$ and $C_U(\tau) =< v + \tau v \mid v \in C_{\hat{V}}(\tau^2) >$, π is certainly an isomorphism of vector spaces.

Also given $g \in G$ and $v \in V$ we have:

$$\pi(\rho(g)v) = \rho(g)v + \tau(\rho(g)v) = \rho(g)v + \rho(g^\tau)\tau(v) \quad (\text{by } (19.4))$$
$$= \rho(g)v + \rho(g)\tau(v) \quad (\text{since } G = C_{\hat{G}}(\tau)) = \rho(g)(v + \tau(v)) = \rho(g)\pi(v).$$

Thus it remains to show that π is a homomorphism of unitary spaces. Recall that the basis for $\widehat{V}$ $\{e_i \mid 1 \leq i \leq n\}$ is induced up from an orthonormal basis $\{e_i\}$ for V. Thus we have $\tau(e_i) = f_i$, the vector dual to e_i, for all $1 \leq i \leq n$. To show that π is a homomorphism of unitary spaces, it suffices to show that $\{\pi(e_i)\}$ forms an orthonormal basis for $(C_U(\tau), f')$. So for $1 \leq i,j \leq n$, consider:

$$f'(\pi(e_i), \pi(e_j)) = f'(e_i + f_i, e_j + f_j) = f_j(e_i) = \delta_{ij} = f(e_i, e_j).$$

Thus π does give a G-invariant isomorphism of unitary spaces as claimed. $\square$

In view of Lemmas 19.6 and 19.7 we will identify $(C_U(\tau), f')$ with (V, f) both having group of isometries equal to $G = C_{\hat{G}}(\tau)$. We are thus interested in studying the decomposition of $C_U(\tau)$ into homogeneous components under the action of A, and the contribution of each homogeneous component towards $C_G(A) = C_{\hat{G}}(A) \cap C_{\hat{G}}(\tau)$.

Consider a decomposition of $\widehat{V} = \bigoplus_\lambda \widehat{V}_\lambda$ into homogeneous components under the action of A. Let $\Phi = \{\lambda : A \to K^\sharp \mid \widehat{V}_\lambda \neq 0\}$, and note that for all $\lambda \in \Phi, a \in A$ we have $\lambda(a)^p = \lambda(a^p) = 1$. Similarly, let $\widehat{V}^* = \bigoplus_\phi \widehat{V}^*_\phi$ be a decomposition into homogeneous components under the action of A and $\Delta = \{\phi : A \to K^\sharp \mid \widehat{V}^*_\phi \neq 0\}$. Then note that we have a decomposition of $U = \bigoplus_{\lambda \in \Phi} \widehat{V}_\lambda \oplus \bigoplus_{\phi \in \Delta} \widehat{V}^*_\phi$ into homogeneous components under the action of A. We fix the following notation:

(1) Given $\lambda \in \Phi$, let $(\widehat{V}_\lambda)^*$ be the annihilator if $\widehat{V}^*$ of all $\widehat{V}_\gamma$ for $\gamma \neq \lambda$. Note that $(\widehat{V}_\lambda)^*$ can be identified with the dual space of $\widehat{V}_\lambda$.

(2) Given $\lambda \in \Phi$, let $\Gamma\widehat{V}_\lambda = \underset{\gamma\in\Gamma\lambda}{\oplus} \widehat{V}_\gamma$; and $(\Gamma\widehat{V}_\lambda)^*$ is the annihilator in $\widehat{V}^*$ of all $\widehat{V}_\beta$ for $\beta \notin \Gamma\lambda$. Note that $(\Gamma\widehat{V}_\lambda)^* = \underset{\gamma\in\Gamma\lambda}{\oplus}(\widehat{V}_\gamma)^*$. Given $\phi \in \Delta$, $\Gamma\widehat{V}^*_\phi$ and $(\Gamma\widehat{V}^*_\phi)^*$ are similarly defined.

Fix $\lambda \in \Phi$ and consider $\tau(\widehat{V}_\lambda)$. Given $v \in \widehat{V}_\lambda$, $a \in A$, remember that $a \in C_{\hat{G}}(\tau)$, so by (19.5) we have:

$$\rho(a)v^\tau = \rho(a^\tau)v^\tau = (\rho(a)v)^t = (\lambda(a)v)^t = \lambda(a)^q v^\tau.$$

So we have: $\tau(\widehat{V}_\lambda) = \widehat{V}^*_{\lambda^q}$. Given the definition of $\Gamma\widehat{V}_\lambda$, we thus have:

$$\tau(\Gamma\widehat{V}_l) = \tau(\underset{\gamma\in\Gamma\lambda}{\oplus}\widehat{V}_\gamma) = \underset{\gamma\in\Gamma\lambda}{\oplus}\tau(\widehat{V}_\gamma) = \underset{\gamma\in\Gamma\lambda}{\oplus}\widehat{V}^*_{\gamma^q} = \underset{\gamma\in\Gamma\lambda^q}{\oplus}\widehat{V}^*_\gamma = \Gamma\widehat{V}^*_{\lambda^q}. \tag{19.8}$$

If we let $T =< \tau >$, then we have:

$$T\widehat{V}_\lambda = \underset{\gamma\in\Gamma\lambda}{\oplus}\widehat{V}_\gamma \oplus \underset{\phi\in\Gamma\lambda^q}{\oplus}\widehat{V}^*_\phi \quad (\text{since we have } \Gamma =< \tau^2 >). \tag{19.9}$$

We also have the following result:

Lemma 19.10. *Given* $1 \neq \lambda \in \Phi$, *we have:*

$$(\widehat{V}_\lambda)^* = \widehat{V}^*_{\lambda^{-1}}.$$

Proof. Pick a basis $X = \{x_i \mid 1 \le i \le n\}$ of $\widehat{V}$ consisting of "weight" vectors; that is, for each $1 \le i \le n$, there exists $\lambda \in \Phi$ such that $x_i \in \widehat{V}_\lambda$. Let $X^* = \{x_j^*\}$ be the dual basis. It suffices to show that given $x_i \in \widehat{V}_\lambda$, we have $x_i^* \in \widehat{V}^*_{\lambda^{-1}}$. Let $x_i \in \widehat{V}_\lambda$, and $x_j^* \in X^*$; then for all $a \in A$ we have:

$$\begin{aligned}(\rho(a)x_j^*)(x_i) &= (a^\theta x_j^*)(x_i) = x_j^*(a^{-1}x_i) \quad (\text{by } (19.4)\ )\\ &= x_j^*(\lambda(a)^{-1}x_i) = \lambda(a)^{-1}x_j^*(x_i) = \lambda(a)^{-1}\delta_{ij} \quad (\text{Kronecker delta}).\end{aligned}$$

Thus we see that $(\widehat{V}_\lambda)^* = \widehat{V}^*_{\lambda^{-1}}$ as claimed. □

In view of Lemma 19.10 and the definition of $(\Gamma\widehat{V}_\lambda)^*$, we have:

$$(\Gamma\widehat{V}_\lambda)^* = \underset{\gamma\in\Gamma\lambda}{\oplus}(\widehat{V}_\gamma)^* = \underset{\gamma\in\Gamma\lambda}{\oplus}\widehat{V}^*_{\gamma^{-1}} = \underset{\gamma\in\Gamma\lambda^{-1}}{\oplus}\widehat{V}^*_\gamma = \Gamma\widehat{V}^*_{\lambda^{-1}}. \tag{19.11}$$

Now if $1 \neq \lambda \in \Phi$ with $\lambda^{-1} \in \Gamma\lambda^{q}$, then $\lambda^{-1} = \lambda^{q^{2r+1}}$ for some $0 \leq r < \frac{d}{2}$. But we assumed that $d \equiv 0 \pmod 4$ is minimal with respect to $p|q^{d}-1$, therefore $p|q^{\frac{d}{2}}+1$. Since $\lambda^{p}=1$, we thus have $\lambda^{-1} = \lambda^{q^{\frac{d}{2}}}$. And thus we have $\lambda^{q^{\frac{d}{2}}} = \lambda^{q^{2r+1}}$. But that implies:

$$d \equiv 4r+2 \pmod d \iff 4r+2 \equiv 0 \pmod d,$$

a contradiction since $d \equiv 0 \pmod 4$. Hence, if $1 \neq \lambda \in \Phi$, then:

$$(\widehat{V}_{\lambda})^{*} = \widehat{V}^{*}_{\lambda^{-1}} \neq \widehat{V}^{*}_{\phi} \quad \text{for any } \phi \in \Gamma\lambda^{q}. \tag{19.12}$$

Now let $\gamma \in \Phi$ such that $\tau(\widehat{V}_{\gamma}) = \widehat{V}^{*}_{\lambda^{-1}}$; that is, $\widehat{V}^{*}_{\gamma^{q}} = \widehat{V}^{*}_{\lambda^{-1}}$, or equivalently $\gamma^{q} = \lambda^{q^{\frac{d}{2}}}$. Then we have:

$$\gamma^{-1} = \gamma^{q^{\frac{d}{2}}} = (\lambda^{q^{(\frac{d}{2}-1)}})^{q^{\frac{d}{2}}} = \lambda^{q^{(d-1)}} = (\lambda^{q})^{q^{(d-2)}} \in \Gamma\lambda^{q}.$$

Thus, given $1 \neq \lambda \in \Phi$, if we choose $\gamma \in \Phi$ such that $\tau(\widehat{V}_{\gamma}) = \widehat{V}^{*}_{\lambda^{-1}}$, then we have:

$$(\widehat{V}_{\gamma})^{*} = \widehat{V}^{*}_{\gamma^{-1}} = \widehat{V}^{*}_{\phi} \quad \text{for some } \phi \in \Gamma\lambda^{q}.$$

Note that $\gamma \notin \Gamma\lambda$, since otherwise $\gamma^{q} = \lambda^{-1} \in \Gamma\lambda^{q}$, a contradiction to (19.12).

Thus given $1 \neq \lambda \in \Phi$, we can find $\gamma \in \Phi$ such that:

$$\begin{aligned}
&T\widehat{V}_{\lambda} = \bigoplus_{\alpha\in\Gamma\lambda} \widehat{V}_{\alpha} \oplus \bigoplus_{\beta\in\Gamma\lambda^{q}} \widehat{V}^{*}_{\beta} = \bigoplus_{\alpha\in\Gamma\lambda} \widehat{V}_{\alpha} \oplus \bigoplus_{\beta\in\Gamma\gamma^{-1}} \widehat{V}^{*}_{\beta} = \Gamma\widehat{V}_{\lambda} \oplus \Gamma(\widehat{V}^{*}_{\gamma^{-1}}),\\
&T\widehat{V}_{\gamma} = \bigoplus_{\eta\in\Gamma\gamma} \widehat{V}_{\eta} \oplus \bigoplus_{\mu\in\Gamma\gamma^{q}} \widehat{V}^{*}_{\mu} = \bigoplus_{\eta\in\Gamma\gamma} \widehat{V}_{\eta} \oplus \bigoplus_{\mu\in\Gamma\lambda^{-1}} \widehat{V}^{*}_{\mu} = \Gamma\widehat{V}_{\gamma} \oplus \Gamma(\widehat{V}^{*}_{\lambda^{-1}}), \text{ and}\\
&T\widehat{V}_{\lambda} + T\widehat{V}_{\gamma} = T\widehat{V}_{\lambda} \oplus T\widehat{V}_{\gamma}.
\end{aligned} \tag{19.13}$$

By the remark immediately following Lemma 19.7 we want to understand the decomposition of $C_{U}(\tau)$ into homogeneous components under the action of A, and thus need to understand the structure of $C_{T\widehat{V}_{\lambda}}(\tau)$ for all $\lambda \in \Phi$. Let $1 \neq \lambda \in \Phi$ and $\gamma \in \Phi$ such that $\tau(\widehat{V}_{\gamma}) = \widehat{V}^{*}_{\lambda^{-1}}$. Let $n_{\lambda} = dim_{K}(\widehat{V}_{\lambda})$, then since $\tau(\widehat{V}_{\gamma}) = (\widehat{V}_{\lambda})^{*}$, $dim(\widehat{V}_{\gamma})$ is also equal to n_{λ}. By (19.11) we have: $(\Gamma\widehat{V}_{\lambda})^{*} = \Gamma\widehat{V}^{*}_{\lambda^{-1}}$. And by (19.8) we have:

$$\tau(\Gamma\widehat{V}_{\lambda}) = \Gamma\widehat{V}^{*}_{\lambda^{q}} = \Gamma\widehat{V}^{*}_{\gamma^{-1}} \text{ (as } \gamma^{-1} \in \Gamma\lambda^{q}) = \Gamma(\widehat{V}_{\gamma})^{*} \text{ (by Lemma 19.10).}$$

To understand the structure of $C_{T\widehat{V}_\lambda}(\tau)$, we want to prove the following three facts:

(1) $C_{T\widehat{V}_\lambda}(\tau)$ is totally singular under the action of f' for $\lambda \neq 1$.

(2) $C_{T\widehat{V}_\lambda}(\tau) \oplus C_{T\widehat{V}_\gamma}(\tau)$ is a hyperbolic subspace of V under the action of f' when $\gamma^q = \lambda^{-1}$.

(3) If $\phi \in \Phi$ such that $\phi^q \notin \Gamma\lambda^{-1}$, then $C_{T\widehat{V}_\lambda}(\tau) \oplus C_{T\widehat{V}_\phi}(\tau)$ is totally singular.

Now $T\widehat{V}_\lambda = \Gamma\widehat{V}_\lambda \oplus (\Gamma\widehat{V}_\lambda)^*$ by (19.13) and (19.11); thus:

$$C_{T\widehat{V}_\lambda}(\tau) =< v + \tau v|\ v \in C_{\Gamma\widehat{V}_\lambda}(\tau^2) > .$$

If we pick a basis $\{x_1, \dots, x_{n_\lambda}\}$ for $\Gamma\widehat{V}_\lambda$, then $\{\tau x_1, \dots, \tau x_{n_\lambda}\}$ is a basis for $\Gamma\widehat{V}^*_{\lambda^q} = \Gamma\widehat{V}^*_{\gamma^{-1}}$. Now, $C_{\Gamma\widehat{V}_\lambda}(\tau^2) =< \sum_{i=1}^{n_\lambda} \alpha_i x_i \mid a_i \in F >$, so we have:

$$\tau(C_{\Gamma\widehat{V}_\lambda}(\tau^2)) =< \Sigma\alpha_i^q \tau x_i \mid \alpha_i \in F >= C_{\Gamma\widehat{V}^*_{\gamma-1}}(\tau^2) = (C_{\Gamma\widehat{V}_\gamma}(\tau^2))^*.$$

Thus given $v \in C_{\Gamma\widehat{V}_\lambda}(\tau^2)$, $\tau v \in (C_{\Gamma\widehat{V}_\gamma}(\tau^2))^*$ which is contained in the annihilator of $C_{\Gamma\widehat{V}_\lambda}(\tau^2)$ since $\Gamma\widehat{V}_\lambda \cap \Gamma\widehat{V}_\gamma = 0$. Since the action of f' is defined by $f'(v + \tau v, w + \tau w) = \tau w(v)$, and by the preceding discussion we see that for $v, w \in C_{\Gamma\widehat{V}_\lambda}(\tau^2)$, $f'(v+\tau v, w+\tau w) = \tau w(v) = 0$. Thus:

f' restricted to $C_{T\widehat{V}_\lambda}(\tau)$ is identically 0.

Since $dim_K(\Gamma\widehat{V}_\lambda) = \frac{d}{2}n_\lambda$, we have $dim_F(C_{T\widehat{V}_\lambda}(\tau)) = \frac{d}{2}n_\lambda$. Thus we have shown:

(19.14) $C_{T\widehat{V}_\lambda}(\tau)$ is a $(\frac{d}{2}n_\lambda)$-dimensional totally singular $F[A]$-submodule of V.

Now consider $C_{T\widehat{V}_\lambda}(\tau) \oplus C_{T\widehat{V}_\gamma}(\tau)$ where $\gamma^q = \lambda^{-1}$. By (19.13) and (19.11) we have:

$$C_{T\widehat{V}_\lambda}(\tau) \oplus C_{T\widehat{V}_\gamma}(\tau) =< v + \tau v \mid v \in C_{\Gamma\widehat{V}_\lambda}(\tau^2) \oplus C_{\Gamma\widehat{V}_\gamma}(\tau^2) > .$$

But we have:

$$\begin{aligned} &\tau(C_{\Gamma\widehat{V}_\lambda}(\tau^2) \oplus C_{\Gamma\widehat{V}_\gamma}(\tau^2)) = \tau(C_{\Gamma\widehat{V}_\lambda}(\tau^2)) \oplus \tau(C_{\Gamma\widehat{V}_\gamma}(\tau^2)) \\ &(C_{\Gamma\widehat{V}_\gamma}(\tau^2))^* \oplus (C_{\Gamma\widehat{V}_\lambda}(\tau^2))^* = (C_{\Gamma\widehat{V}_\lambda}(\tau^2) \oplus C_{\Gamma\widehat{V}_\gamma}(\tau^2))^*. \end{aligned} \tag{19.15}$$

If we let $W = C_{\Gamma\widehat{V}_\lambda}(\tau^2) \oplus C_{\Gamma\widehat{V}_\gamma}(\tau^2)$, then we have $C_{T\widehat{V}_\lambda}(\tau) \oplus C_{T\widehat{V}_\gamma}(\tau) = < v + \tau v \mid v \in W >$, where $\tau(W) = (W)^*$ which can be identified with the dual space of W. Since $f'(v + \tau v, w + \tau w) = \tau w(v)$, we see that:

$$f' \text{ restricted to } C_{T\widehat{V}_\lambda}(\tau) \oplus C_{T\widehat{V}_g}(\tau) \text{ is nondegenerate.}$$

Since $dim_F(C_{T\widehat{V}_\lambda}(\tau)) = dim_F(C_{T\widehat{V}_\gamma}(\tau)) = \frac{d}{2} n_\lambda$, we have:

(19.16) Given $\lambda \neq 1$ and γ such that $\gamma^q = \lambda^{-1}$, $C_{T\widehat{V}_\lambda}(\tau) \oplus C_{T\widehat{V}_\gamma}(\tau)$ is a dn_λ-dimensional hyperbolic F[A]-submodule of V.

Finally, consider $\phi \in \Phi$ such that $\phi^q \notin \Gamma\lambda^{-1}$, then:

$$\tau(C_{\Gamma\widehat{V}_\phi}(\tau^2)) \cap (C_{\Gamma\widehat{V}_\lambda}(\tau^2))^* = C_{\Gamma\widehat{V}^*_{\phi^q}}(\tau^2) \cap C_{\Gamma\widehat{V}^*_{\lambda^{-1}}}(\tau^2) = 0,$$

since $\phi^q \notin \Gamma\lambda^{-1}$. Thus, arguing as in the discussion immediately preceding (19.14), we have:

(19.17) If $\lambda \neq 1$ and $\phi \in \Phi$ such that $\phi^q \notin \Gamma\lambda^{-1}$, then $C_{T\widehat{V}_\lambda}(\tau) \oplus C_{T\widehat{V}_\phi}(\tau)$ is a totally singular subspace of V.

Since for each $\phi \in \Delta$, there exists $\lambda \in \Phi$ such that $\lambda^q = \phi$, it is possible to choose a set of *positive* orbit representatives, say Φ^+, of the action of Γ on Φ, with the following properties:

Let $\Phi^+ = \{\lambda_0, \lambda_1, \ldots, \lambda_m\}$ with $\lambda_0 = 1$. Then we have:

(1) If $0 \leq i \neq j \leq m$, then $\lambda_i \notin \Gamma\lambda_j$ and $\lambda_i^q \notin \Gamma\lambda_j^{-1}$.

(2) If for each $1 \leq i \leq m$ we set $\lambda_i' \in \Phi$ such that $\tau(\widehat{V}_{\lambda_i'}) = \widehat{V}^*_{\lambda_i^{-1}}$, then we have $\Phi = \Gamma\{\lambda_0, \lambda_i, \lambda_i' \mid 1 \leq i \leq m\}$ and $\Delta = \Gamma\{\lambda_0, \lambda_i^q, (\lambda_i')^q \mid 1 \leq i \leq m\}$.

By the definition of Φ^+ and (19.13) we have:

$$U = T\widehat{V}_{\lambda_0} \oplus \bigoplus_{1 \leq i \leq m} (T\widehat{V}_{\lambda_i} \oplus T\widehat{V}_{\lambda_i'}).$$

For each $0 \leq i \leq m$, let $dim_K(\widehat{V}_{\lambda_i}) = n_i$. Then, by the discussion preceding the definition of Φ^+, for $1 \leq i \neq j \leq m$ we have:

(a) $C_{T\widehat{V}_{\lambda_i}}(\tau)$ is a $(\frac{d}{2}n_i)$-dimensional totally singular $F[A]$-submodule of V.

(b) $C_{T\widehat{V}_{\lambda_i}}(\tau) \oplus C_{T\widehat{V}_{\lambda'_i}}(\tau)$ is a (dn_i)-dimensional $F[A]$-submodule of V.

(c) $C_{T\widehat{V}_{\lambda_i}}(\tau)$ is orthogonal to both $C_{T\widehat{V}_{\lambda_j}}(\tau)$ and $C_{T\widehat{V}_{\lambda'_j}}(\tau)$.

Now for $1 \leq i \leq m$, τ acts regularly on $T\widehat{V}_{\lambda_i}$, thus $C_{T\widehat{V}_{\lambda_i}}(\tau) = < v + \tau v + \cdots + \tau^{d-1} v \mid v \in \widehat{V}_{\lambda_i} >$ is isomorphic to $\widehat{V}_{\lambda_i}$ as K-subspaces. And by (19.14) and (19.16) $C_{T\widehat{V}_{\lambda_i}}(\tau) \oplus C_{T\widehat{V}_{\lambda'_i}}(\tau)$ is a hyperbolic subspace of V and thus contributes $GL(\widehat{V}_{\lambda_i}) = GL_{n_i}(q^d)$ towards $C_G(A)$.

For λ_0, we have $T\widehat{V}_{\lambda_0} = \widehat{V}_{\lambda_0} \oplus (\widehat{V}_{\lambda_0})^*$, so we have:

(a) $C_{T\widehat{V}_{\lambda_0}}(\tau)$ is a nondegenerate $F[A]$-submodule of V with $dim(C_{T\widehat{V}_{\lambda_0}}(\tau)) = n_0$.

(b) $C_{T\widehat{V}_{\lambda_0}}(\tau)$ is orthogonal to $C_{T\widehat{V}_{\lambda_i}}(\tau)$ and $C_{T\widehat{V}_{\lambda'_i}}(\tau)$ for all $1 \leq i \leq m$.

Now τ induces a field automorphism on $T\widehat{V}_{\lambda_0}$, thus $C_{T\widehat{V}_{\lambda_0}}(\tau)$ contributes $GU_{n_0}(q)$ towards $C_G(A)$. And we have the following result:

$$C_G(A) \cong GU_{n_0}(q) \times \prod_{1 \leq i \leq m} GL_{n_i}(q^d).$$

We also have an orthogonal decomposition of $V = V_0 \perp V_1 \oplus V_1^- \perp \cdots \perp V_m \oplus V_m^-$ into $F[A]$-submodules, where:

(1) $V_0 = C_V(A)$ is nondegenerate of dimension $n_0 \equiv n \pmod{d}$.

(2) And for all $1 \leq i \leq m$, $V_i \oplus V_i^-$ is hyperbolic of dimension dn_i and V_i, and V_i^- are totally singular.

This case-by-case analysis gives us the following result:

Theorem 19.18. *Let* $G = GU_n(q)$, p *be a prime such that* $(p, q) = 1$, *and* $A \in \mathcal{A}_p(G)$. *Let* $d \geq 3$ *be minimal with respect to* $p | q^d - 1$; *then we have:*

$$C_G(A) \cong \begin{cases} GU_{n_0}(q) \times \prod_{1 \leq i \leq m} GL_{n_i}(q^d), & \textit{if } d \textit{ is odd}, \\ GU_{n_0}(q) \times \prod_{1 \leq i \leq m} GU_{n_i}(q^{\frac{d}{2}}), & \textit{if } d \equiv 2 \ (mod\ 4), \\ GU_{n_0}(q) \times \prod_{1 \leq i \leq m} GL_{n_i}(q^d), & \textit{if } d \equiv 0 \ (mod\ 4), \end{cases}$$

where n_i (for $0 \le i \le m$) is as defined in the preceding discussion. $\square$

Section 20 : Basic results about $\mathcal{A}_p(G)$ where $G = GL_n(q)$, $Sp_{2n}(q)$, or $GU_n(q)$

In this section we state some results about the Quillen complex of $GL_n(q)$, $Sp_{2n}(q)$, and $GU_n(q)$. These results are well known and are stated mainly for completeness. We shall use the results of the previous two sections along with results of Aschbacher, Gorenstein, and Lyons to establish the p-rank of G and the connectivity of $\mathcal{A}_p(G)$ – where G is one of $GL_n(q), Sp_{2n}(q)$, or $GU_n(q)$.

Let $G = GL_n(q)$, $Sp_{2n}(q)$, or $GU_n(q)$; p be a prime such that $(p, q) = 1$, and d be minimal with respect to $p | q^d - 1$. Note that $GL_n(q)$ can be considered as the group of isometries of (V, f) where f is the *trivial form*; that is, $f(u, v) = 0 \ \forall \ u, v \in V$. We use this convention for notational convenience. Before we can compute $m_p(G)$, we need the following two results:

Lemma 20.1. *Suppose that a finite abelian group A acts irreducibly and faithfully on an additive group V. Then A is cyclic.*

Proof. This is (5.21) on pg. 159 in [Su]. $\square$

Lemma 20.2. *Let $G = GL_n(q)$ and V an n-dimensional vector space over $F = \mathbb{F}_q$. Let p be a prime with $(p, q) = 1$, and let d be minimal with repect to $p | q^d - 1$. Given $A \in \mathcal{A}_p(G)$, we have a decomposition of $V = \bigoplus_{0 \le i \le m} V_i$ into homogeneous components where $V_0 = C_V(A)$ and $\forall \ 1 \le i \le m$ $dim(V_i) \equiv 0 \ (mod \ d)$. We also have:*

$$C_G(A) \cong GL_{n_0}(q) \times \prod_{1 \le i \le m} GL_{n_i}(q^d),$$

where $n_i = dim(V_i) \ \forall \ 0 \le i \le m$.

Proof. Let $K = \mathbb{F}_{q^d}$ and $\Gamma = Gal(K/F) = <\sigma>$. Let $\widehat{V} = V \otimes_F K$ and $\hat{G} = GL_n(q^d) \ge G$, and consider A as an element of $\mathcal{A}_p(\hat{G})$. As in section 15 and 16 we can consider a decomposition of $\widehat{V} = \bigoplus_{\lambda} \widehat{V}_\lambda$ into homogeneous components under the action of A.

Let $\Phi = \{\lambda : A \to K^\sharp | \widehat{V}_\lambda \neq 0\}$, and note that Γ acts on Φ since $\sigma(\widehat{V}_\lambda) = \widehat{V}_{\sigma(\lambda)}$. Choose orbit representatives, $\lambda_0, \lambda_1, \ldots, \lambda_m$, for this action, with $\lambda_0 = 1$. For each $0 \leq i \leq m$, let $n_i = dim_K(\widehat{V}_{\lambda_i})$ and $\widehat{V}_i = \bigoplus_{\gamma \in \Gamma\lambda_i} \widehat{V}_\gamma$. Since $\forall\ 0 \leq i \leq m$, $\widehat{V}_i$ is normalized by Γ; by (25.7.2) on pg. 120 in [A2], there exists $F[A]$-submodules $V_i \leq V$ such that $\widehat{V}_i = V_i \otimes_F K$.

If $1 \leq i \leq m$ then, arguing as in the discussion prior to Theorem 18.2, the minimality of d implies that $|\Gamma\lambda_i| = d$. Thus, $dim_F(V_i) = dim_K(\widehat{V}_i) = dn_i$. Also, as K-vector spaces $V_i \cong \widehat{V}_{\lambda_i}$, so V_i contributes $GL_{n_i}(q^d)$ towards $C_G(A)$.

For $i = 0$, $\widehat{V}_0 = \widehat{V}_{\lambda_0}$ implies that $dim_F(V_0) = dim_K(\widehat{V}_{\lambda_0}) = n_0$. Also, since $V_0 = C_{\widehat{V}_0}(\Gamma)$, V_0 contributes $GL_{n_0}(q)$ towards $C_G(A)$.

Thus, we have a decomposition of $V = V_0 \oplus V_1 \oplus \cdots \oplus V_m$ where $dim(V_i) = dn_i$ $\forall\ 1 \leq i \leq m$. Also:

$$C_G(A) \cong GL_{n_0}(q) \prod_{1 \leq i \leq m} GL_{n_i}(q^d),$$

as claimed. □

We are now ready to compute $m_p(G)$. Note that in the case when $G = Sp_{2n}(q)$ or $GU_n(q)$, the value of $m_p(G)$ will depend on the parity of d. This is not surprising since the decomposition of the corresponding symplectic and unitary form depended on the value of d, and $m_p(G)$ depends on this decomposition.

Theorem 20.3. *Let $G = GL_n(q)$, $Sp_{2n}(q)$ or $GU_n(q)$, and (V, f) be the corresponding space with form. Let p be a prime with $(p, q) = 1$, and let d be minimal with respect to $p | q^d - 1$. Then we have $m_p(G) = k$ where k is given as follows:*

Table 3.

	Group(G)	**Prime(p)**	$\boldsymbol{d}$	$\boldsymbol{n}$[a]
(1)	$GL_n(q)$	$\neq 2$	> 1	$n = kd + r$
(2)	$Sp_{2n}(q)$	$\neq 2$	*Odd* (> 1)	$n = kd + r$
(3)		*Any*	*Even*[b]	$n = k\frac{d}{2} + r$
(4)	$GU_n(q)$	$\neq 2$	*Odd* (> 1)	$n = 2kd + r$
(5)		*Any*	$d \equiv 2 \ (mod\ 4)$	$n = k\frac{d}{2} + r$
(6)		*Any*	$d \equiv 0 \ (mod\ 4)$	$n = kd + r$

a. *When we write* $n = kx + r$, *we mean* $0 \leq r < x$ *where* $x = 2d$, d, *or* $\frac{d}{2}$.

b. *When* $G = Sp_{2n}(q)$ *and* d *is even, in view of Theorem 18.7, we have to restrict our attention to the case when* q *is odd.*

Proof. Fix $A \in \mathcal{A}_p(G)$. From Theorems 18.2 and 18.7, the case-by-case analysis in section 19 and Lemma 20.2 we have the following decomposition of (V, f) into $F[A]$-submodules, where the numbers correspond to those in the table above:

(1) $V = V_0 \oplus V_1 \oplus \cdots \oplus V_m$ where $dim(V_i) = dn_i \ \forall\ 1 \leq i \leq m$.

(2) $V = V_0 \perp V_1^+ \oplus V_1^- \perp \cdots \perp V_m^+ \oplus V_m^-$ with $dim(V_i^+) = dim(V_i^-) = dn_i$ $\forall\ 1 \leq i \leq m$.

(3) $V = V_0 \perp V_1 \perp \cdots \perp V_m$ with $dim(V_i) = dn_i \ \forall\ 1 \leq i \leq m$.

(4) $V = V_0 \perp V_1^+ \oplus V_1^- \perp \cdots \perp V_m^+ \oplus V_m^-$ with $dim(V_i^+) = dim(V_i^-) = dn_i$ $\forall\ 1 \leq i \leq m$.

(5) $V = V_0 \perp V_1 \perp \cdots \perp V_m$ with $dim(V_i) = \frac{d}{2}n_i \ \forall\ 1 \leq i \leq m$.

(6) $V = V_0 \perp V_1^+ \oplus V_1^- \perp \cdots \perp V_m^+ \oplus V_m^-$ with $dim(V_i^+) = dim(V_i^-) = \frac{d}{2}n_i$ $\forall\ 1 \leq i \leq m$.

Note that by the definition of k, $m \leq k$ in all cases. In all of the cases for $1 \leq i \leq m$, let W_i be a simple $F[A]$-submodule such that $V_i =< W \mid W \cong W_i >$ in cases 1, 3, and 5; and $V_i^+ \cong< W \mid W \cong W_i >$ in the other cases. For $1 \leq i \leq m$, let $A_i = C_A(V_i)$ in cases 1, 3, and 5; and let $A_i = C_A(V_i^+ \oplus V_i^-)$ in the other cases. Note that by the definition of W_i and the action of A on V, in all cases $A_i = C_A(W_i)$. Also note that in all cases $\bigcap_{1 \leq i \leq m} A_i = 1$ since A is nontrivial and $V_0 = C_V(A)$. Since $A_i = C_A(W_i)$, A is elementary abelian, and W_i is simple, by Lemma 20.1 in all cases $A/A_i \cong \mathbb{Z}/p\mathbb{Z}$. Thus we have a normal series:

$$A \geq A_1 \geq A_1 \cap A_2 \geq \cdots \geq \bigcap_{1 \leq i \leq m-1} A_i \geq 1,$$

with factor groups isomorphic to 1 or $\mathbb{Z}/p\mathbb{Z}$. Therefore $m_p(A) \leq m \leq k$. Since this result holds for all $A \in \mathcal{A}_p(G)$, by the definition of $m_p(G)$, we have $m_p(G) \leq k$.

Now, in all of the cases above, setting $n_i = 1 \; \forall \; 1 \leq i \leq m$, by the definition of k, we have $m = k$. Thus we can find $A \cong A_1 \times A_2 \times \cdots \times A_k$ where $\forall \; 1 \leq i \leq k \;\; A_i \in \mathcal{A}_p(G_i)$, where G_i is the isometry group of V_i or $V_i^+ \oplus V_i^-$ depending on the case under consideration. Thus $m_p(G) \geq k$.

Hence $m_p(G) = k$ as claimed. □

To any finite group G and prime divisor, p, of the order of G, we have associated two simplicial complexes – the Quillen and the Brown complexes. We now associate a third complex to G at p. Recall the definition of the clique complex, $K(\Delta)$, of a graph, Δ, from section 10.

Given a finite group G and a prime p dividing $|G|$, let $\Lambda_p(G)$ be the graph on the vertex set $\{A \leq G| \; |A| = p\}$ with $A, A' \in \Lambda_p(G)$ adjacent if and only if $[A, A'] = 1$. $\Lambda_p(G)$ is called the *commuting graph of G at p*. The clique complex $\mathcal{K}_p(G) \equiv K(\Lambda_p(G))$ is called the *commuting complex of G at p*. It is well known that given G and p as above:

$$\mathcal{S}_p(G) \simeq \mathcal{A}_p(G) \simeq \mathcal{K}_p(G),$$

(where "$\simeq$" indicates "homotopic"). For a proof see (5.2–3) on pp. 14–15 in [A2].

Since $\mathcal{A}_p(G) \simeq \mathcal{K}_p(G) = K(\Lambda_p(G))$, $\mathcal{A}_p(G)$ is connected if and only if $\Lambda_p(G)$ is connected. However, by (46.6) on pg. 247 in [A1], we know that $\Lambda_p(G)$ is disconnected if and only if G contains a *strongly p-embedded subgroup* (see pp. 246–247 in [A1] for a definition of strongly p-embedded subgroups). The above fact along with the classification theorem and a result of Gorenstein and Lyons on strongly p-embedded subgroups – 24.1 on pg. 307 in [GL] – gives us the following lemma:

Lemma 20.4. *Given a finite group G and a prime $p| \; |G|$, $\Lambda_p(G)$ is disconnected if and only if $O_p(G) = 1$ and $m_p(G) = 1$ or $< \Lambda_p(G) > / O_{p'}(< \Lambda_p(G) >)$ is one of the following:*

(1) *Simple of Lie type of Lie rank 1 and characteristic p.*

(2) *A_{2p} with $p \geq 5$.*

(3) *${}^2G_2(3)$, $L_3(4)$, or M_{11} with $p = 3$.*

(4) *$Aut(Sz(32))$, ${}^2F_4(2)'$, Mc, or $M(22)$ with $p = 5$.*

(5) *J_4 with $p = 11$.*

Proof. This is (6.2) on pg. 16 in [A2]. □

Now let $G = GL_n(q)$ or $Sp_{2n}(q)$ and assume that $p|q-1$, then by Theorem 12.4 on pg. 126 in [Q] and Theorem 15.12, $\mathcal{A}_p(G)$ is connected if $m_p(G) \geq 2$. If $G = GU_n(q)$ and $p|q-1$, then two either $p = 2$ in which case G contains a central 2-subgroup and $\mathcal{A}_p(G)$ is contractible by Proposition 2.4 on pg. 106 in [Q]; or $p \neq 2$ in which case $\mathcal{A}_p(G)$ is connected when $m_p(G) \geq 2$ by Theorem 16.10. We will show that if $G = GL_n(q), Sp_{2n}(q)$, or $GU_n(q)$ and p is a prime such that $(p, q) = 1$ and $m_p(G) \geq 2$, then $\mathcal{A}_p(G)$ is almost always connected. By the above discussion we can restrict our attention to $p \nmid q-1$. We can now state the main result about connectivity of $\mathcal{A}_p(G)$:

Theorem 20.5. *Let $G = GL_n(q), Sp_{2n}(q)$, or $GU_n(q)$ and let p be a prime such that $(p, q) = 1$. If $m_p(G) \geq 2$, then $\mathcal{A}_p(G)$ is connected.*

Proof. Let $N =< \Lambda_p(G) >$. If $p| \; |Z(G)|$, then $O_p(G) \neq 1$; thus, by Proposition 2.4 on pg. 106 in [Q], $\mathcal{A}_p(G)$ is contractible and we are done.

So assume that $p \nmid |Z(G)|$. This assumption, along with the fact that $m_p(G) \geq 2$ and Theorem 20.3, implies that $G \notin \{GL_2(2), GL_2(3), Sp_2(2), Sp_2(3), GU_3(4)\}$. Therefore $Z(G) = O_{p'}(G)$. By Theorem 9.15, the canonical homomorphism $\pi : G \to G/O_{p'}(G) \equiv \bar{G}$

induces an isomorphism between $\mathcal{A}_p(G)$ and $\mathcal{A}_p(\bar{G})$, since $O_{p'}(G)$ is central. Note that $m_p(\bar{G}) \geq 2$, and if $\pi(N) \equiv \bar{N}$, then $m_p(\bar{N}) \geq 2$. Assume that $\mathcal{A}_p(\bar{G})$ is disconnected; then, by Lemma 20.4, $\bar{N}$ is in the list given in Lemma 20.4. Since $\bar{N}$ is classical and $m_p(\bar{N}) > 1$, $\bar{N} = L_3(4)$ with $p = 3$. Equivalently, $G = GL_3(4)$ with $p = 3$, contradicting our assumption that $p \nmid |Z(G)|$. Thus $\mathcal{A}_p(\bar{G})$ – hence $\mathcal{A}_p(G)$ – is connected. $\square$

The results from this section will play a crucial role in our computations of the simple connectivity of the Quillen complexes for $GL_n(q), Sp_{2n}(q)$, and $GU_n(q)$ carried out in chapter 6.

SECTION 21 : THE ACTION OF A ON THE TITS BUILDING OF V. THE d : ODD CASE

Let $G = Sp_{2n}(q)$ or $GU_n(q)$ and let p be a prime such that $(p, q) = 1$. Let d be the order of q in $\mathbb{Z}/p\mathbb{Z}$, that is, the minimal integer with respect to $p|(q^d - 1)$. In this section we consider the case when $d \geq 3$ is odd. Let (V, f) be the $2n$-dimensional symplectic space over $F = \mathbb{F}_q$ when $G = Sp_{2n}(q)$; or let (V, f) be the n-dimensional unitary space over $F = \mathbb{F}_{q^2}$ when $G = GU_n(q)$. Let $K = \mathbb{F}_{q^d}$ when $G = Sp_{2n}(q)$, and $K = \mathbb{F}_{q^{2d}}$ when $G = GU_n(q)$. Then we have $\Gamma = Gal(K/F) =< \sigma >$, with $|\Gamma| = d$. Finally, let $\widehat{V} = V \otimes_F K$ with $\hat{f}$ the symplectic or unitary form on $\widehat{V}$ such that $f|_V \equiv f$.

Given $A \in \mathcal{A}_p(G)$, we want to analyze the action of A on the Tits building of V. So fix an $A \in \mathcal{A}_p(G)$. By Theorem 18.2 and Case I in section 19, we have an orthogonal decomposition of:

$$V = V_0 \perp V_1^+ \oplus V_1^- \perp \cdots \perp V_m^+ \oplus V_m^-$$

into homogeneous components under the action of A. Here $V_0 = C_V(A)$ is nondegenerate and $\forall\ 1 \leq i \leq m$; we have V_i^+ and V_i^- are totally singular with $V_i^+ \oplus V_i^-$ a hyperbolic subspace of dimension $2dn_i$. Also remember that if $\widehat{V} = \bigoplus_{\lambda \in \Phi} \widehat{V}_\lambda$ is a decomposition of $\widehat{V}$ into homogeneous components under the action of A, then Γ acts of Φ and we have:

(1) $\Phi \supseteq \Phi^+$ a set of positive weights fixed by Γ.

(2) $\lambda_0, \lambda_1, \ldots, \lambda_m$, the orbit representatives of the Γ-action of Φ^+, with $\lambda_0 = 1$.

(3) $\forall\ 1 \le i \le m$, $\widehat{V}_i^+ \equiv \bigoplus_{\gamma \in \Gamma\lambda_i} \widehat{V}_\gamma = V_i^+ \otimes_F K$, $\widehat{V}_i^- \equiv \bigoplus_{\gamma \in \Gamma\lambda_i^{-1}} \widehat{V}_\gamma = V_i^- \otimes_F K$; and $V_i^+ = C_{\widehat{V}_i^+}(\sigma) = < v + \sigma(v) + \cdots + \sigma^{d-1}(v) | v \in \widehat{V}_{\lambda_i} >$ as F-vector spaces and $V_i^+ \cong \widehat{V}_{\lambda_i}$ as K-vector spaces.

[All of this information is worked out in complete detail in sections 18 and 19 for the symplectic and unitary spaces, respectively.]

Now fix $1 \le i \le m$, and note that by (3) above we have a K-structure on V_i^+ as follows: Given $\alpha \in K$ and $w \in V_i^+$, we define $\alpha w = \alpha v + \sigma(\alpha v) + \cdots + \sigma^{d-1}(\alpha v)$ where $w = v + \sigma(v) + \cdots + \sigma^{d-1}(v)$. Thus given a nontrivial p^{th} root of unity $\zeta \in K$, it makes sense to talk about ζw for $w \in V_i^+$. Given a subspace $U \le V_i^+$, we say that U is a *K-subspace of V_i^+* if and only if $U = KU$. We have the following result:

Lemma 21.1. *Given a nontrivial p^{th} root of unity $\zeta \in K$ and $0 \ne w \in V_i^+$, the set $\{w, \zeta w, \ldots, \zeta^{d-1} w\}$ is linearly independent over F.*

Proof. Assume to the contrary that there exist $a_0, \ldots, a_{d-1} \in F$ such that

$$0 = a_0 w + a_1 \zeta w + \cdots + a_{d-1}\zeta^{d-1} w = (a_0 + a_1\zeta + \cdots a_{d-1}\zeta^{d-1})w$$

$$\Longrightarrow\ a_0 + a_1\zeta + \cdots + a_{d-1}\zeta^{d-1} = 0 \quad \text{since } w \ne 0.$$

But then ζ is a root of a polynomial of degree $< d$ over F, contradicting the minimality of d with respect to $p | q^d - 1$. Thus we have $\{w, \zeta w, \ldots, \zeta^{d-1} w\}$ is linearly independent over F as claimed. □

Now given $a \in A$ and $0 \ne w \in V_i^+$, we have:

$$aw = a(v + \sigma(v) + \cdots + \sigma^{d-1}(v)) = \lambda_i(a)v + \lambda_i(a)^q \sigma(v) + \cdots + \lambda_i^{q^{d-1}}(a)\sigma^{d-1}(v)$$
$$= \lambda_i(a)v + \sigma(\lambda_i(a)v) + \cdots + \sigma^{d-1}(\lambda_i(a)v) = \lambda_i(a)w.$$

Thus for any $0 \ne w \in V_i^+$, we have $Aw = \{w, \zeta w, \ldots, \zeta^{d-1} w\}$ where ζ is a nontrivial p^{th} of unity. This gives us the following result:

Lemma 21.2. *$U \le V_i^+$ is A-invariant if and only if U is a K-subspace of V_i^+.*

Proof. If W is a K-subspace of V_i^+ and $w \in W$, then, by the remark immediately preceding this lemma, we have:

$$Aw = \{w, \zeta w, \ldots, \zeta^{d-1} w\} \leq W.$$

Thus W is A-invariant.

On the other hand if U is A-invariant for some $U \leq V_i^+$, let $0 \neq u \in U$. Then, by the remark immediately preceding this lemma, we have $Au = \{u, \zeta u, \ldots, \zeta^{d-1} u\}$. Hence $U \geq Ku$ and thus U is a K-subspace of V. □

Note that by Lemmas 21.1–2 if $AU = U \leq V_i^+$ then $dim(U) \equiv 0 \pmod{d}$.

Let B be the Tits building of V, and $Fix(A)$ the full subcomplex of B defined on the set of simplices $\{\sigma \in B \mid A\sigma = \sigma\}$. Since the decomposition of $V = V_0 \perp \cdots \perp V_m^+ \oplus V_m^-$ is a decomposition into homogeneous components under the action of A, given a totally singular subspace $U \leq V$ fixed by the action of A, we have $U = U_0 \perp U_1 \perp \cdots \perp U_m$ where $U_0 \leq V_0$, $U_i \leq V_i^+ \oplus V_i^-$ $\forall\ 1 \leq i \leq m$, and $\forall\ 0 \leq i \leq m$, $AU_i = U_i$. This decomposition leads us to a consideration of the following simplicial complexes:

> Define $Fix(A)_0$ to be the full subcomplex of the Tits building of V_0 defined on the set of simplices fixed under the action of A restricted to V_0. And similarly for $1 \leq i \leq m$, define $Fix(A)_i$ to be the full subcomplex of the Tits building of $V_i^+ \oplus V_i^-$ defined on the set of simplices fixed by the action of A restricted to $V_i^+ \oplus V_i^-$.

By the statement preceding the definition of $Fix(A)_i$, we see that the structure of $Fix(A)$ can be understood by studying the structure of $Fix(A)_i$ $\forall\ 0 \leq i \leq m$.

When $G = Sp_{2n}(q)$, by Theorem 20.3, we have $n = m_p(G)d + r$ where $0 \leq r < d$. Let $dim(V_0) = 2n_0$; then note that since $\forall\ 1 \leq i \leq m$, $dim(V_i^+ \oplus V_i^-) = 2dn_i$, and $d \geq 3$, we have:

$$\sum_{i=0}^{m} n_i \geq m_p(G). \tag{21.3}$$

Similarly when $G = GU_n(q)$, we have $n = 2m_p(G)d + r$ where $0 \leq r < 2d$. Let $dim(V_0) = 2n_0 + \epsilon$ where $\epsilon \equiv n \pmod{2}$. Then since $\forall\ 1 \leq i \leq m$, $dim(V_i^+ \oplus V_i^-) = 2dn_i$, and $d \geq 3$, we have:

$$\sum_{i=0}^{m} n_i \geq m_p(G). \tag{21.4}$$

Note that since $A|_{V_0} \equiv 1$, we have $Fix(A)_0 = B_0$ the Tits building of V_0. Thus by the Solomon-Tits theorem we have $Fix(A)_0$ is $(n_0 - 1)$-spherical.

Fix $1 \leq i \leq m$, and let B_i be the Tits building of $V_i^+ \oplus V_i^-$. Since V_i^+ and V_i^- are homogeneous under the action of A, given $U \in Fix(A)_i$, we have $U = U_1 \oplus U_2$ where $U_1 \leq V_i^+$ and $U_2 \leq V_i^- \cap U_1^\perp$; and $AU_j = U_j$ for $j = 1, 2$. By Lemma 21.2, U_1 and U_2 are thus K-subspaces of V_i^+ and V_i^-. Thus, if $\widehat{B}_i$ is the Tits building of $\widehat{V}_{\lambda_i} \oplus \widehat{V}_{\lambda_i^{-1}}$, then $Fix(A)_i$ can be identified with the full subcomplex of $\widehat{B}_i$ defined on those simplices fixed by the action of A restricted to $\widehat{V}_{\lambda_i} \oplus \widehat{V}_{\lambda_i^{-1}}$. Now $p|q^d - 1$, and $d \geq 3$ odd (i.e., $p \neq 2$) implies that this last complex is the one analyzed in Lemma 15.5 in the case when $\widehat{V}_{\lambda_i} \oplus \widehat{V}_{\lambda_i^{-1}}$ is symplectic, and in Lemma 16.8 in the case that it is unitary. By the proof of these lemmas we know that the full subcomplex of $\widehat{B}_i$ defined on those simplices fixed under the action of A is spherical of dimension $n_i - 1$. Thus we have shown that:

$$Fix(A)_i \text{ is } (n_i - 1)\text{-spherical} \quad \forall\, 1 \leq i \leq m. \tag{21.5}$$

Now one of two cases may occur: $|\Phi| = 2d$ or $|\Phi| > 2d$. If $|\Phi| = 2d$, then the fact that $A \neq 1$ implies that $\Phi = \{\Gamma\lambda_1, \Gamma\lambda_1^{-1}\}$ for some positive weight $\lambda_1 \in \Phi^+$. Thus, $V = V_1^+ \oplus V_1^-$, and by (21.5) we have $Fix(A) = Fix(A)_1$ is spherical of dimension $n_1 - 1 = m_p(G) - 1$. When $|\Phi| > 2d$, let $pr_0 : V \to V_0$ and $\forall\, 1 \leq i \leq m$, $pr_i : V \to V_i^+ \oplus V_i^-$ be the canonical projections. If $\forall\, 0 \leq i \leq m$, we set $D_i = CFix(A)_i$, the cone of $Fix(A)_i$ as defined in section 9, then we have:

$$pr_i(Fix(A)) = Fix(A)_i \cup \{0\} \cong D_i,$$

where $Fix(A)_i \cup \{0\}$ is ordered by inclusion. Thus we have:

$$\begin{aligned} Fix(A) &\cong D_0 \times D_1 \times \cdots \times D_m - \{(0, \ldots, 0)\} \\ &\cong Fix(A)_0 * Fix(A)_1 * \cdots * Fix(A)_m. \end{aligned}$$

Since $\forall\, 0 \le i \le m$, we have $Fix(A)_i$ is $(n_i - 1)$-spherical; by Lemma 14.1, we have $Fix(A)$ is spherical of dimension $\sum_{i=0}^{m}(n_i - 1) + (m + 1 - 1) = \sum_{i=0}^{m} n_i - 1$. But, by (21.3) and (21.4), we have $\sum_{i=0}^{m} n_i \ge m_p(G)$ both when $G = Sp_{2n}(q)$ and when $G = GU_n(q)$. Thus we have the following result:

Theorem 21.6. *Let $G = Sp_{2n}(q)$ or $GU_n(q)$ and let p be prime such that $(p, q) = 1$. Assume that if d is minimal with respect to $p|q^d - 1$, then $d \ge 3$ is odd. Let B be the Tits building of the corresponding symplectic or unitary space. Let $A \in \mathcal{A}_p(G)$ be fixed. If $Fix(A)$ is the full subcomplex of B defined on the set of simplices $\{\sigma \in B \mid A\sigma = \sigma\}$, then we have $Fix(A)$ is $(m_p(G) - 1)$-spherical.* $\square$

Recall the definition of the graph, $\Delta(D)$, of a simplicial complex D given in section 10. Also recall the definition of a geometric complex given in section 6. Given a geometric complex K (with type function τ) over an indexing set I, and a subset $J \subseteq I$, the *truncation of K at J* is the full subcomplex of K defined on the vertex set $\{x \in \Delta(K) \mid \tau(x) \in J\}$. It is well known that if K is a C.M. geometric complex then any truncation of K is also C.M, see for example (3.5) on pg. 11 in [A2].

Now note that when $G = Sp_{2n}(q)$, then B is C.M. of dimension $n-1$. And we can define a type function $\tau : \Delta(B) \to \{0, \dots, n-1\}$ given by $\tau(x) = dim(x) - 1\ \forall\, x \in \Delta(B)$. Then $(\Delta(B), *, \tau)$ is a geometry with B the corresponding geometric complex. Similarly, when $G = GU_n(q)$ then B is C.M. of dimension $[\frac{n}{2}] - 1$ where $[\frac{n}{2}]$ is the greatest integer less than or equal to $\frac{n}{2}$. As above, $\tau : \Delta(B) \to \{0, \dots, [\frac{n}{2}] - 1\}$ defined by $\tau(x) = dim(x) - 1$ gives B the structure of a geometric complex. And in both cases we can define B_d to be the full subcomplex of B defined on the vertex set $\{x \in \Delta(B) | \tau(x) \equiv -1 \pmod d\} = \{x \in \Delta(B) \mid d | dim(x)\}$. Note that B_d is a truncation of B.

Lemma 21.7. *B_d is C.M. of dimension $(m_p(G) - 1)$.*

Proof. As B_d is a truncation and B is C.M. – by the Solomon-Tits Theorem – B_d is C.M. by the comment following the definition of truncations. Now, by Theorem 20.3,

$m_p(G) = k$, where:

(1) $n = kd + r$ $(0 \leq r < d)$ when $G = Sp_{2n}(q)$, and

(2) $n = 2kd + r$ $(0 \leq r < 2d)$ when $G = GU_n(q)$.

Thus, by the definition of B and the Witt index of the spaces, B does contain totally singular subspaces of dimensions: $d, 2d, \ldots, kd$. Thus, B_d is $(k-1) = (m_p(G)-1)$-dimensional as desired. □

Now let $V = V_0 \perp \cdots \perp V_m^+ \oplus V_m^-$ be the decomposition into homogeneous components as considered earlier. Let $dim(V_0) = 2n_0d + t$ $(0 \leq t < 2d)$. We know that for $1 \leq i \leq m$, $dim(V_i^+ \oplus V_i^-) = 2dn_i$. Thus we have:

(21.8) $2n = \sum_{i=0}^{m} 2dn_i + t = 2kd + 2r$ $(0 \leq t < 2d)$ when V is symplectic; that is, $\sum_{i=0}^{m} n_i = k = m_p(G)$.

(21.9) $n = \sum_{i=0}^{m} 2dn_i + t = 2kd + r$ $(0 \leq t < 2d)$ when V is unitary; that is, $\sum_{i=0}^{m} n_i = k = m_p(G)$.

Consider $Fix(A)'$ the full subcomplex of B_d defined on those simplices fixed under the action of A. Let $Fix(A)'_0 = Fix(A)' \cap V_0$ and for all $1 \leq i \leq m$, let $Fix(A)'_i = Fix(A)' \cap (V_i^+ \oplus V_i^-)$. If we let B_{0_d} be the truncation of the Tits building of V_0, then note that $Fix(A)'_0 = B_{0_d}$. We have the following two results:

Lemma 21.10. *$Fix(A)'_0$ is $(n_0 - 1)$-spherical, where $dim(V_0) = 2dn_0 + t$ $(0 \leq t < 2d)$.*

Proof. We have $Fix(A)'_0 = B_{0_d}$, thus the lemma follows from the proof of Lemma 21.7, considering V_0 as a $(2dn_0 + t)$-dimensional space with $(0 \leq t < 2d)$. □

Lemma 21.11. *$Fix(A)'_i = Fix(A) \cap (V_i^+ \oplus V_i^-) = Fix(A)_i$ $\forall\ 1 \leq i \leq m$.*

Proof. This follows from the fact that if $U \leq V_i^+ \oplus V_i^-$ such that $AU = U$, then, by Lemmas 21.1–2, $d|dim(U)$ $\forall\ 1 \leq i \leq m$. □

We know that $\forall\ 1 \leq i \leq m$, $Fix(A)_i$ is $(n_i - 1)$-spherical. So in view of Lemmas 21.10–11 we have shown that $Fix(A)'_i$ is $(n_i - 1)$-spherical for all $0 \leq i \leq m$. Then arguing as in the discussion prior to Theorem 21.6, we have $Fix(A)'$ is spherical of dimension $\sum_{i=0}^{m}(n_i - 1) + m = \sum_{i=0}^{m} n_i - 1$. Thus we have:

Theorem 21.12. *Let $G = GU_n(q)$ or $Sp_{2n}(q)$, p a prime such that $(p,q) = 1$, and B be the Tits building of the corresponding unitary or symplectic space. Assume that if d is minimal with respect to $p|q^d - 1$, then $d \geq 3$ is odd. Set B_d to be the full subcomplex of B on the set of vertices $\{x \in \Delta(B) \mid d|dim(x)\}$, and let $A \in \mathcal{A}_p(G)$ be fixed. If $Fix(A)'$ is the full subcomplex of B_d defined on those simplices fixed under the action of A, then we have:*

$$Fix(A)' \text{ is } (m_p(G) - 1)\text{-spherical.}$$

Proof. This result follows trivially from the fact that $Fix(A)'$ is spherical of dimension $\sum_{i=0}^{m} n_i - 1$ and the fact that, by (21.8–9), $\sum_{i=0}^{m} n_i = m_p(G)$. □

Chapter 6

Simple connectivity of p-subgroup complexes

In Chapter 6 we analyze the simple connectivity of $\mathcal{A}_p(G)$, where G is one of $GL_n(q)$, $Sp_{2n}(q)$, or $GU_n(q)$ and $(p,q) = 1$. We shall rely heavily on the linear algebraic results obtained in chapter 5, along with the n-connectivity of some of the complexes constructed in chapter 3.

The proof of the simple connectivity of $\mathcal{A}_p(GL_n(q))$ will be by induction on n and the p-rank of $GL_n(q)$. In Section 22 we show that in the minimal case $\mathcal{A}_p(GL_n(q))$ is indeed simply connected. In Section 23 we show the simple connectivity of $\mathcal{A}_p(GL_n(q))$ in the general case. With p and q relatively prime, choose d minimal with respect to $p|q^d - 1$. In Section 24 we show that $\mathcal{A}_p(G)$ is simply connected when d is odd, $G = Sp_{2n}(q)$ or $GU_n(q)$, and $m_p(G) \geq 3$. We shall rely on our analysis of $Fix(A)'$ in section 21 for this result. Recall that when $p|q+1$ and $G = GU_n(q)$, there is a central elementary abelian p-subgroup Z of G. Hence, by Proposition 2.4 on pg. 106 in [Q], $\mathcal{A}_p(G)$ is contractible. In Section 25 we show that when $p|q+1$, $q \geq 5$ is odd and $m_p(GU_n(q)) \geq 4$, $\mathcal{A}_p(GU_n(q))(> Z)$ is simply connected.

Section 22 : Simple connectivity of $\mathcal{A}_p(GL_n(q))$ in the minimal case

Let $G = GL_n(q)$ and V be the corresponding vector space over $\mathbb{F}_q$. Choose a prime p such that $(p,q) = 1$ and $d \geq 2$ is minimal with respect to $p|q^d - 1$. Assume that $n = 3d$. Then, by Theorem 20.3, $m_p(G) = 3$. We will show that $\mathcal{A}_p(G)$ is simply connected by using the method of *n-approximation* developed in [AS4].

Given a simplicial complex D, recall the definition of the graph of D, $\Delta(D)$ given in section 10. Also remember that given simplicial complexes L and D :

> a 1-*approximation of L by D* is a surjective map $\Theta : \Delta(D) \rightarrow \mathcal{F}(\Theta)$ onto a family of subcomplexes of L such that each 0 and 1 simplex of L is contained in a member of $\mathcal{F}(\Theta)$ and:

(1-Approx) $\bigcap_{a\in\sigma}\Theta(a)$ is a $(1-k)$-connected subcomplex of L for each k-simplex σ of D, for $0 \le k \le 2$. Recall that -1-connected is equivalent to nonempty.

Lemma 22.1. *Let L, D be simplicial complexes and assume that Θ is a 1-approximation of L by D. For each $x \in \Delta(L)$, let $\mathcal{C}(x) = \{a \in \Delta(D) \mid x \in \Theta(a)\}$ regarded as a subgraph of $\Delta(D)$. Assume that whenever $a, b \in \Delta(D)$ and $x \in \Theta(a) \cap \Theta(b)$, there is a path in $\mathcal{C}(x)$ joining a and b. Then L is simply connected if D is simply connected.*

Proof. This follows directly from Theorem 3 on pg. 286 in [AS4]. □

Now V being $3d$-dimensional allows us to associate with V the simply connected complex $B = B(V)$ which was analyzed in section 10. Let τ be the type function on B, also as defined in section 10. Since B is simply connected, $\mathcal{A}_p(G)$ will be simply connected if we can find a 1-approximation of $\mathcal{A}_p(G)$ by B which satisfies the criterion of Lemma 22.1.

Lemma 22.2. *Let $\sigma = (x_0 * x_1 * \ldots * x_s)$ be a simplex of B with $\tau(x_i) < \tau(x_{i+1})$ $\forall\, 0 \le i < s$. Then we have the following:*

(1) *If $\tau(x_s) = 3$, then $G_\sigma = K \rtimes T$, where $T \le S$, and K and S are as follows:*

$$K = \left\{ \begin{pmatrix} g_1 & 0 & 0 \\ 0 & g_2 & 0 \\ 0 & 0 & g_3 \end{pmatrix} \in G \mid g_i \in GL_d(q)\ \forall\ 1 \le i \le 3 \right\}$$
$$\cong GL_d(q) \times GL_d(q) \times GL_d(q),$$

$$S = \{\text{permutation matrices of } G \text{ with three } d \times d \text{ identity blocks}\} \cong \mathcal{S}_3.$$

Furthermore, $\mathcal{A}_p(K)$ is simply connected.

(2) *If $\tau(x_s) \neq 3$, then $\mathcal{A}_p(G_\sigma)$ is simply connected.*

Proof. If $\tau(x_s) = 3$, then $x_s = [U_1|U_2|U_3]$ is a partition of V, and G_σ preserves this partition. Let K be the "pointwise" stabilizer of this partition; that is, $KU_i = U_i\ \forall\ 1 \le i \le 3$. Then K is as claimed above. S permutes the U_i, and T is the subgroup generated by the admissible permutations. The admissible permutations depend of course on the structure of the x_i for $0 \leq i \leq s-1$. Thus $G_\sigma = K \rtimes T$ as claimed. The fact that $\mathcal{A}_p(K)$ is simply connected follows from Lemmas 9.4–5, and the fact that $\mathcal{A}_p(GL_d(q)) \neq \emptyset$ (by Theorem 20.3).

So assume that $\tau(x_s) \neq 3$. Now G_σ is a parabolic subgroup of G. Let $U \leq G_\sigma$ be the unipotent radical – a solvable p'-subgroup of G_σ. The canonical homomorphism $\pi : G_\sigma \to G_\sigma/U \cong L$, the Levi factor, induces a map of posets $f : \mathcal{A}_p(G_\sigma) \to \mathcal{A}_p(L)$.

By the definition of L, $L \cong GL_d(q) \times GL_d(q) \times GL_d(q)$ or $GL_{2d}(q) \times GL_d(q)$. By Theorems 20.3 and 20.5 we know that $\mathcal{A}_p(GL_d(q)) \neq \emptyset$ and $\mathcal{A}_p(GL_{2d}(q))$ is connected. Thus, by Lemmas 9.4–5, $\mathcal{A}_p(L)$ is simply connected.

Furthermore, by Theorem 20.3, $m_p(L) = 3$ and since $\mathcal{A}_p(L)$ is a pure complex (refer to section 1) if $h(A) = 1$ for any $A \in \mathcal{A}_p(L)$, then $\mathcal{A}_p(L)(> A) \neq \emptyset$.

When $L \cong GL_d(q) \times GL_d(q) \times GL_d(q)$, let $L_1 = GL_d(q) \times GL_d(q)$ and $L_2 = GL_d(q)$. And when $L \cong GL_{2d}(q) \times GL_d(q)$, let $L_1 = GL_{2d}(q)$ and $L_2 = GL_d(q)$. Then note that $L \cong L_1 \times L_2$ and for $i = 1$ or 2 if $m_p(L_i) \geq 2$ then, by Lemmas 9.4–5 and Theorem 20.3 and 20.5, $\mathcal{A}_p(L_i)$ is connected. Also, by Theorem 20.3, $\mathcal{A}_p(L_i) \neq \emptyset$ for $i = 1$ and 2. Thus, L, L_1 and L_2 satisfy the criterion of Theorem 9.8. Hence if $A \in \mathcal{A}_p(L)$ with $h(A) = 0$, then $\mathcal{A}_p(L)(> A)$ is connected.

Note that given $A \in \mathcal{A}_p(L)$, $f^{-1}(\mathcal{A}_p(L)(\leq A)) = \mathcal{A}_p(A \cdot U)$ which, since U is a solvable p'-subgroup of G_σ, is $(h(A)-1)$-connected by Theorem 11.2 on pg. 123 in [Q].

We have thus shown that $\mathcal{A}_p(L)$ and $f : \mathcal{A}_p(G_\sigma) \to \mathcal{A}_p(L)$ satisfy the criterion of (1.4) on pg. 5 in [A2]. Therefore, $\mathcal{A}_p(L)$ simply connected implies that $\mathcal{A}_p(G_\sigma)$ is simply connected. Hence statement (2), and thus the lemma, holds. □

We will use the following fact often (without explicitly stating it):

Remark. *Given $A \in \mathcal{A}_p(G)$, by the proof of Lemma 20.2, every nontrivial irreducible $\mathbf{F}_q[A]$-submodule of V has dimension d. Note that since $A \in \mathcal{A}_p(G)$ $A \neq 1$, which implies that A has a nontrivial irreducible $\mathbf{F}_q[A]$-submodule.*

Let $\Delta(B)$ be the graph of B. Given $x \in \Delta(B)$, note that by Lemma 22.2, if $\tau(x) \neq 3$

then $\mathcal{A}_p(G_x)$ is simply connected. If $\tau(x) = 3$, let K be the "pointwise" stabilizer defined in the proof of Lemma 22.2; that is, $K \cong GL_d(q) \times GL_d(q) \times GL_d(q)$, then $\mathcal{A}_p(K)$ is simply connected.

So we can define a map $\Theta : \Delta(B) \to \mathcal{F}(\Theta)$, where $\mathcal{F}(\Theta)$ is a family of subcomplexes of $\mathcal{A}_p(G)$ defined as follows:

$$\Theta(x) = \begin{cases} \mathcal{A}_p(G_x), & \text{if } \tau(x) \neq 3, \\ \mathcal{A}_p(K), & \text{if } \tau(x) = 3, \end{cases} \qquad \text{for all } x \in \Delta(B).$$

Lemma 22.3. *Given a 0- or 1-simplex σ of $\mathcal{A}_p(G)$, $\sigma \in \Theta(x)$ for some $x \in \Delta(B)$.*

Proof. Given a 1-simplex, $\sigma = (A_1 < A_2)$ if $A_2 \in \Theta(x)$ for some $x \in \Delta(B)$, then $\sigma \in \Theta(x)$. So it suffices to show that given any $A \in \mathcal{A}_p(G)$, there exists $x \in \Delta(B)$ such that $A \in \Theta(x)$.

Fix $A \in \mathcal{A}_p(G)$ and let U be a nontrivial irreducible $\mathbb{F}_q[A]$-submodule of V. Then $x = U \in \Delta(B)$ with $\tau(x) = 1$. By the choice of x, $\Theta(x) = \mathcal{A}_p(G_\sigma)$; so we have $A \in \Theta(x)$. Hence we have the claim. □

Lemma 22.4. *Given any simplex σ in B, $\bigcap_{x\in\sigma} \Theta(x)$ is simply connected.*

Proof. If σ is a 0-simplex then the claim follows by the discussion preceding the definition of Θ.

So let $\sigma = (x_0 * \cdots * x_s)$ with $\tau(x_i) < \tau(x_{i+1})$ $\forall\ 0 \le i \le s-1$. If $\tau(x_s) = 3$, then $\bigcap_{x\in\sigma} \Theta(x) = \Theta(x_s) = \mathcal{A}_p(K)$ which is simply connected. So assume that $\sigma = (x_0 * x_1)$ with $\tau(x_0) \neq 3 \neq \tau(x_1)$. Then $\Theta(x_0) \cap \Theta(x_1) = \mathcal{A}_p(G_{(x_0 * x_1)})$ which is simply connected by Lemma 22.2.

So in all cases $\bigcap_{x\in\sigma} \Theta(x)$ is simply connected as claimed. □

Lemma 22.5. $\Theta : \Delta(B) \to \mathcal{F}(\Theta)$ *is a 1-approximation of $\mathcal{A}_p(G)$ by B.*

Proof. This follows from Lemmas 22.3–4 and the definition of a 1-approximation. □

Given $A \in \mathcal{A}_p(G)$, define $\mathcal{C}(A) = \{\ a \in \Delta(B) \mid A \in \Theta(a)\ \}$.

Lemma 22.6. *Let $a, b \in \Delta(B)$ and assume that there is an $A \in \mathcal{A}_p(G)$ such that $A \in \Theta(a) \cap \Theta(b)$, then there is a path in $\mathcal{C}(A)$ joining a and b.*

Proof. We prove this result by a case-by-case analysis of $\tau(a)$ and $\tau(b)$.

<u>CASE 1:</u> $\tau(a) = \tau(b) = \epsilon$, with $\epsilon = 1$ or 2. If $\epsilon = 1$, let $c = < a, b >$, and if $\epsilon = 2$, let $c = a \cap b$. Since $A \in \Theta(a) \cap \Theta(b)$, c is a $\mathbb{F}_q[A]$-submodule. Therefore $c \in \Delta(B)$, $A \in \Theta(c)$ and $a * c * b$ is a path joining a and b in $\mathcal{C}(A)$.

<u>CASE 2:</u> $\tau(a) = 1$ and $\tau(b) = 2$. A stabilizes $a \cap b$, so $a \cap b = 0$ or a. If $a \cap b = a$, then $a * b$ and we are done, so assume that $a \cap b = 0$. Let $b = b_1 \oplus b_2$ be a decomposition of b into irreducible $\mathbb{F}_q[A]$-submodules. Then $b_1, b_2 \in \Delta(B)$ and $c = [a|b_1|b_2] \in \Delta(B)$ with $\tau(c) = 3$. By the definition of b_1 and b_2, $c \in \mathcal{C}(A)$ and $a * c * b$ is a path joining a and b in $\mathcal{C}(A)$.

<u>CASE 3:</u> $\tau(a) = \epsilon$ and $\tau(b) = 3$, with $\epsilon = 1$ or 2. If $a * b$, we are done. Otherwise, let $c * b$ with $\tau(c) = \epsilon$ and $c \in \mathcal{C}(A)$. By Case 1 we know that there is a path p in $\mathcal{C}(A)$ from a to c, so $p * b$ is a path joining a and b in $\mathcal{C}(A)$.

<u>CASE 4:</u> $\tau(a) = \tau(b) = 3$. Let $c * a$ with $\tau(c) = 1$ and $c \in \mathcal{C}(A)$. Then by Case 3 we know that there is a path p in $\mathcal{C}(A)$ from c to b, so $a * p$ is a path in $\mathcal{C}(A)$ joining a and b.

This case-by-case analysis shows us that if a, b, and A satisfy the criterion of this claim, then there is a path in $\mathcal{C}(A)$ joining a and b as claimed. □

Theorem 22.7. *Let $G = GL_n(q)$ and p a prime such that $(p, q) = 1$. If $d \geq 2$ is minimal with respect to $p|(q^d - 1)$, assume that $n = 3d$. Then $\mathcal{A}_p(G)$ is simply connected.*

Proof. By Theorem 10.8, B is simply connected. The map $\Theta : \Delta(B) \to \mathcal{F}(\Theta)$ is a 1-approximation (by Lemma 22.5) which satisfies the criterion of Lemma 22.1 (by Lemma 22.6). Therefore the result follows from Lemma 22.1. □

SECTION 23 : SIMPLE CONNECTIVITY OF $\mathcal{A}_p(GL_n(q))$ WHEN $m_p(GL_n(q)) \geq 3$

Let $G = GL_n(q)$, and V be the corresponding vector space over $\mathbb{F}_q$. Choose a prime p such that $(p, q) = 1$, and assume that $d \geq 2$ is minimal with respect to $p|q^d - 1$. In section 22 we showed that if $n = 3d$, then $\mathcal{A}_p(G)$ is simply connected. In this section we generalize the result to all $n \geq 3d$ by induction on n and $m_p(G)$. We follow the structure of Quillen's proof of Theorem 12.4 on pg. 126 in [Q].

Assume that $k = [\frac{n}{d}] \geq 3$. By Theorem 20.3, $m_p(G) = k \geq 3$. In view of Theorem 22.7 we can assume without loss of generality that when $k = 3$, $r \neq 0$; that is, $n > 3d$. Let B be the Tits building of V, that is, the order complex of the proper nontrivial subspaces of V - ordered by inclusion. By the Solomon-Tits Theorem B is C.M. of dimension $(n-2)$.

If $\Delta(B)$ is the graph of B, then $\tau : \Delta(B) \to \{0, \ldots, n-2\}$ defined by $\tau(x) = dim(x) - 1$ gives us a type function of B. Consider the subcomplex B_d, of B, defined as the full subcomplex on the set of simplices:

$$\{(x_0 < x_1 < \cdots < x_s) \in B \mid dim(x_i) \equiv 0 \ (\text{mod } d) \ \forall\ 0 \leq i \leq s\}.$$

Note that B_d is a truncation of B, as defined in section 21. By the discussion on truncations, since B is C.M., B_d is also C.M. Since $n = kd + r$ with $r < d$, and by the definition of B_d, B_d is $(k-1)$-dimensional if $d \nmid n$ and it is $(k-2)$-dimensional if $d|n$. Thus, B_d is $(k-2)$-connected if $d \nmid n$ and $(k-3)$-connected if $d|n$. Since we assumed that $n > 3d$:

(23.1) $\qquad B_d$ is simply connected.

Consider $sd(B_d) \times \mathcal{A}_p(G) \supseteq F = \{(\sigma, A) | \sigma \in Fix(A)\}$, where $sd(B_d)$ is the first barycentric subdivision of B_d, and $Fix(A)$ is the full subcomplex of $sd(B_d)$ defined on the set of simplices fixed under the action of A. This is the subset of $sd(B_d) \times \mathcal{A}_p(G)$ discussed prior to Remark 9.14, and hence is a closed subposet of $sd(B_d) \times \mathcal{A}_p(G)$. For each $\sigma \in sd(B_d)$ and $A \in \mathcal{A}_p(G)$, define:

$$F_\sigma = \{A' \in \mathcal{A}_p(G) \mid (\sigma, A') \in F\} \text{ and } F_A = \{\sigma' \in sd(B_d) \mid (\sigma', A) \in F\}.$$

By Lemma 9.1 and (23.1) we know that $sd(B_d)$ is simply connected. Hence, in view of Theorem 9.13, $\mathcal{A}_p(G)$ is simply connected if:

(23.2) $\qquad F_\sigma$ and F_A are simply connected $\forall\ \sigma \in sd(B_d)$, $A \in \mathcal{A}_p(G)$.

We first need the following result:

Lemma 23.3. *Let $G = GL_n(q)$ and consider a prime p with $(p,q) = 1$. Assume that $d \geq 2$ is minimal with respect to $p|q^d - 1$ and that $n = kd$ with $k \geq 3$. If $A \in \mathcal{A}_p(G)$ with $h(A) = 0$, then $\mathcal{A}_p(G)(> A)$ is connected.*

Proof. By Theorem 20.3 $m_p(G) = k \geq 3$. And by Lemma 20.2 we have:

$$C_G(A) \cong GL_{n_0'}(q) \times GL_{n_1}(q^d) \times \cdots \times GL_{n_s}(q^d),$$

where $n = n_0' + \sum_{i=1}^{s} dn_i \implies n_0' = dn_0$.

Note that $\mathcal{A}_p(G)(> A) = \mathcal{A}_p(C_G(A))(> A)$. Now if $n_0 = 0$ and $s = 1$, then $C_G(A) \cong GL_k(q^d)$. Now $p|q^d - 1$, so by Theorem 12.4 on pg. 126 in [Q], $\mathcal{A}_p(C_G(A))$ is C.M. of dimension $k - 1$. Thus, by Lemma 14.2, $\mathcal{A}_p(C_G(A))(> A)$ is $(k - h(A) - 3)$-connected; that is, $\mathcal{A}_p(C_G(A))(> A)$ is connected since $k \geq 3$, and $h(A) = 0$.

So assume that $n_0 \neq 0$ or $s > 1$, and let $K_1 = GL_{dn_0}(q) \times \cdots \times GL_{n_{s-1}}(q^d)$ and $K_2 = GL_{n_s}(q^d)$. Then, by Theorem 20.3 and Theorem 12.4 on pg. 126 in [Q], $\mathcal{A}_p(K_i) \neq \emptyset$ for $i = 1, 2$. Also if $m_p(K_i) \geq 2$, then $\mathcal{A}_p(K_i)$ is connected by Theorem 12.4 on pg. 126 in [Q], Theorem 20.5 and Lemmas 9.4–5. Thus $C_G(A) \cong K_1 \times K_2$, K_1, and K_2 satisfy the criterion of Theorem 9.8. Hence $\mathcal{A}_p(C_G(A))(> A)$ is connected as claimed.

Thus we have shown that $\mathcal{A}_p(G)(> A) = \mathcal{A}_p(C_G(A))(> A)$ is connected when $m_p(G) \geq 3$ and $h(A) = 0$. □

Lemma 23.4. *Given $\sigma = (x_0 < x_1 < \cdots < x_s) \in sd(B_d)$, F_σ is simply connected.*

Proof. Let $dim(x_i) = dk_i \ \forall\ 0 \leq i \leq s$. Note that $F_\sigma = \mathcal{A}_p(G_\sigma)$ where G_σ is a standard parabolic subgroup. Let $U \leq G_\sigma$ be the unipotent radical, a solvable q-subgroup of G_σ. Then the canonical homomorphism $\pi : G_\sigma \to G_\sigma/U \cong L$, the Levi factor, induces a map of posets $f : \mathcal{A}_p(G_\sigma) \to \mathcal{A}_p(L)$.

Note that $\forall\ A \in \mathcal{A}_p(L)$, $f^{-1}(\mathcal{A}_p(L)(\leq A)) = \mathcal{A}_p(A \cdot U)$; thus, as in the proof of Lemma 22.2, it is $(h(A) - 1)$-connected by Theorem 11.2 on pg. 123 in [Q].

By the definition of the Levi factor we have:

$$L \cong GL_{dk_0}(q) \times GL_{dk_1}(q) \times \cdots \times GL_{dk_s}(q) \times GL_t(q),$$

where $t = n - \sum_{i=0}^{s} dk_i = dk_{s+1} + r$ (and k_{s+1} may equal 0). By Theorem 20.3, $m_p(L) = \sum_{i=0}^{s+1} k_i = k \geq 3$. Thus if $h(A) = 1$, then $\mathcal{A}_p(L)(> A) \neq \emptyset$ since $\mathcal{A}_p(L)$ is a pure complex.

Now let $A \in \mathcal{A}_p(L)$ with $h(A) = 0$. If $\sigma = (x_0)$ with $dim(x_0) = kd$ – occurs only when $d \nmid n$ – then $L \cong GL_{dk}(q)$ with $k \geq 3$. So, by Lemma 23.3, $\mathcal{A}_p(L)(> A)$ is connected. Otherwise, let $L_1 = GL_{dk_0}(q) \times \cdots \times GL_{dk_s}(q)$ and $L_2 = GL_t(q)$ when $k_{s+1} \neq 0$; or $L_1 = GL_{dk_0}(q) \times \cdots \times GL_{dk_{s-1}}(q)$ and $L_2 = GL_{dk_s}(q) \times GL_t(q)$ when $k_{s+1} = 0$. Then, by Theorem 20.3, $\mathcal{A}_p(L_i) \neq \emptyset$ for $i = 1, 2$. Also, by Theorem 20.5 and Lemmas 9.4–5, if $m_p(L_i) \geq 2$, then $\mathcal{A}_p(L_i)$ is connected for $i = 1$ or 2. Since $L \cong L_1 \times L_2$, L_1, and L_2 satisfy the criterion of Theorem 9.8, $\mathcal{A}_p(L)(> A)$ is connected.

Thus, $f : \mathcal{A}_p(G_\sigma) \to \mathcal{A}_p(L)$ satisfies the criterion of (1.4) on pg. 5 in [A2]. Therefore, $\mathcal{A}_p(G_\sigma)$ is simply connected if $\mathcal{A}_p(L)$ is simply connected. By the structure of L and Lemma 9.4 we know that:

$$\text{(23.5)} \qquad \mathcal{A}_p(L) \cong \mathcal{A}_p(GL_{dk_0}(q)) * \mathcal{A}_p(GL_{dk_1}(q)) * \cdots * \mathcal{A}_p(GL_{dk_s}(q)) * \mathcal{A}_p(GL_t(q)),$$

where, as above, $t = dk_{s+1} + r$ with k_{s+1} possibly equal to 0. We show that $\mathcal{A}_p(L)$ is simply connected by considering the different cases that arise:

<u>CASE I :</u> $\sigma = (x_0)$. If $dim(x_0) = dk$ – occurs only when $d \nmid n$ – then $\mathcal{A}_p(L) \cong \mathcal{A}_p(GL_{dk}(q))$ where $dk < n$. Thus, by induction on n, $\mathcal{A}_p(L)$ is simply connected. Otherwise $\mathcal{A}_p(L) \cong \mathcal{A}_p(GL_d(q)) * \mathcal{A}_p(GL_t(q))$ where $t = dk_1 + r$ and $k = k_0 + k_1 \geq 3$. So $k_i \geq 2$ for $i = 1$ or 2, which implies by Theorems 20.3 and 20.5 that either $\mathcal{A}_p(GL_{dk_0}(q))$ or $\mathcal{A}_p(GL_t(q))$ is connected. Therefore, by Lemma 9.5, $\mathcal{A}_p(L)$ is simply connected as claimed.

<u>CASE II :</u> $\sigma = (x_0 < x_1)$. If $k_2 \neq 0$, where $t = dk_2 + r$, then, by Lemma 9.5, $\mathcal{A}_p(L)$ is simply connected. Otherwise, $k = k_0 + k_1 \geq 3$. Thus, arguing as in Case I, either $\mathcal{A}_p(GL_{dk_0}(q))$ or $\mathcal{A}_p(GL_{dk_1}(q))$ is connected. Therefore, by Lemma 9.5, $\mathcal{A}_p(L)$ is simply connected.

<u>CASE III :</u> $\sigma = (x_0 < \cdots < x_s)$ with $s > 1$. In this case $\mathcal{A}_p(L)$ is simply connected by Lemma 9.5.

Thus we have shown that $\mathcal{A}_p(L)$ is simply connected. Hence $F_\sigma = \mathcal{A}_p(G_\sigma)$ is simply connected as claimed. $\square$

Let $A \in \mathcal{A}_p(G)$ be fixed. We want to show that $F_A = Fix(A)$ is simply connected. By Lemma 20.2, we have a decomposition of $V = V_0 \oplus V_1 \oplus \cdots \oplus V_m$ into homogeneous components under the action of A, where $V_0 = C_V(A)$. Note that $\forall\ 1 \le i \le m$, $dim(V_i) = dn_i$ and so $dim(V_0) = dn_0 + r$. Now $Fix(A)$ is the subcomplex of $sd(B_d)$ defined on the set of vertices $\{\sigma \in B_d \mid A\sigma = \sigma\}$. So if we let $Fix(A)'$ to be the full subcomplex of B_d defined on the set of simplices $\{\sigma \in B_d \mid A\sigma = \sigma\}$, then $Fix(A) = sd(Fix(A)')$. And hence, by Lemma 9.1, the question of simplex connectivity of $F_A = Fix(A)$ is reduced to showing that $Fix(A)'$ is simply connected.

For $1 \le i \le m$, let W_i be the n_i-dimensional vector space over $\mathbb{F}_{q^d}$. Let B_i be the Tits building of W_i, and $pr_i : V \to V_i$ be the canonical projections. Let $D_i = B_i \cup \{O, W_i\}$ ordered by inclusion be the double cone of B_i as defined in section 9. By the proof of Lemma 20.2 and the definition of a homogeneous component, we have:

$$pr_i(Fix(A)') \cong D_i. \tag{23.6}$$

For $i = 0$, let B_0 be the Tits building of V_0, and $\widehat{B}_0$ be the full subcomplex of B_0 defined on the vertex set $\{x \in \Delta(B_0) \mid dim(x) \equiv 0 \pmod{d}\}$. Note that $\widehat{B}_0$ is a truncation of B_0. Since $dim(V_0) = dn_0 + r$ with $0 \le r < d$, $\widehat{B}_0$ is C.M. of dimension $(n_0 - 1)$ when $d \nmid n$, and dimension $(n_0 - 2)$ when $d|n$. Let $pr_0 : V \to V_0$ be the canonical projection. And let D_0 be defined as follows:

$$D_0 \cong \begin{cases} \widehat{B}_0 \cup \{0\}, \text{ if } d \nmid n, \\ \widehat{B}_0 \cup \{0, V_0\}, \text{ if } d|n, \end{cases}$$

where D_0 is ordered by inclusion. Then we have:

$$pr_0(Fix(A)') \cong D_0. \tag{23.7}$$

By (23.6) and (23.7), and since $Fix(A)' = \{0 \ne U_0 \oplus U_1 \oplus \cdots \oplus U_m \ne V \mid U_i \le V_i\}$, we have:

<u>Case I :</u> $d \nmid n$. $Fix(A)' \cong D_0 \times D_1 \times \cdots \times D_m - \{(0,\ldots,0)\} \cong \widehat{B}_0 * (B_1,\infty) * \cdots * (B_m,\infty)$ by Lemma 9.2. Here, $\forall\ 1 \leq i \leq m$, (B_i,∞) is the poset $B_i \cup \{W_i\}$ ordered by inclusion. Thus $Fix(A)'$ is simply connected by Lemma 9.5, since each (B_i,∞) is conically contractible for $1 \leq i \leq m$.

<u>Case II :</u> $d|n$. Then $Fix(A)' \cong D_0 \times D_1 \times \cdots \times D_m - \{(0,\ldots,0),(\infty,\ldots,\infty)\}$. Now, by the Solomon-Tits Theorem, $\forall\ 1 \leq i \leq m$ B_i is (n_i-3)-connected. And by definition, $\widehat{B}_0$ is (n_0-3)-connected. So, by Corollary 9.11, $Fix(A)'$ is:

$$\sum_{i=0}^{m}(n_i-3)+(m+1-1)3=\sum_{i=0}^{m}n_i-3=(k-3)\text{-connected.}$$

Since we assumed that $n > 3d$ and $d|n$, we have $k \geq 4$. Thus, in this case too, $Fix(A)'$ is simply connected.

Hence we have shown:

Lemma 23.8. *Given $A \in \mathcal{A}_p(G)$, F_A is simply connected.* □

In view of (23.2) and Lemmas 23.4 and 23.8 we have the main result of this section:

Theorem 23.9. *$\mathcal{A}_p(GL_n(q))$ is simply connected when $m_p(GL_n(q)) \geq 3$.*

Proof. We can assume that $(p,q)=1$; otherwise, $\mathcal{A}_p(GL_n(q))$ is spherical by the Solomon-Tits Theorem. When $p|q-1$, then the result follows from Theorem 12.4 on pg. 126 in [Q]. Finally the result was shown to be true for the case $p \nmid q-1$ in this section. □

As a direct corollary to Theorems 9.15 and 23.9 we obtain the following result:

Corollary 23.10. *Let $G = SL_n(q)$ or $L_n(q)$ and p be a prime dividing the order of G. Assume that $p \nmid (q-1)$ and $m_p(G) \geq 3$, then $\mathcal{A}_p(G)$ is simply connected.*

Proof. First assume that $G = SL_n(q)$, then the fact that $p \nmid (q-1)$ implies that $A \in \mathcal{A}_p(GL_n(q))$ if and only if $A \in \mathcal{A}_p(G)$. Thus $\mathcal{A}_p(GL_n(q)) = \mathcal{A}_p(G)$ as posets and the result holds for $G = SL_n(q)$ directly from Theorem 23.9.

Now $p \nmid (q-1)$ implies $Z(SL_n(q)) = O_{p'}(SL_n(q))$. Thus, by Theorem 9.15, the canonical homomorphism $\pi : SL_n(q) \to SL_n(q)/Z(SL_n(q)) \cong L_n(q)$ induces an isomorphism betweeen $\mathcal{A}_p(SL_n(q))$ and $\mathcal{A}_p(L_n(q))$. Thus $\mathcal{A}_p(L_n(q))$ is simply connected by the discussion of the preceding paragraph. □

SECTION 24 : SIMPLE CONNECTIVITY OF $\mathcal{A}_p(G)$ WHEN d IS ODD AND $G = Sp_{2n}(q)$ OR $GU_n(q)$

In this section we consider the simple connectivity of $\mathcal{A}_p(G)$ when $G = Sp_{2n}(q)$ or $GU_n(q)$ and p is a prime such that:

(1) $(p,q) = 1$, and

(2) if d is minimal with respect to $p|q^d - 1$, then $d \geq 3$ is odd.

Let (V,f) be the corresponding $2n$-dimensional symplectic space over $F = \mathbf{F}_q$ when $G = Sp_{2n}(q)$; or let (V,f) be the n-dimensional unitary space over $F = \mathbf{F}_{q^2}$ when $G = GU_n(q)$.

Let B be the Tits building of V, and B_d the full subcomplex of B defined on the set of vertices $\{x \in \Delta(B) \mid d|dim(x)\}$ the truncation of B discussed in secton 21. Consider:

$$sd(B_d) \times \mathcal{A}_p(G) \supseteq F = \{(\sigma, A) \mid \sigma \in Fix(A)\},$$

where $sd(B_d)$ is the first barycentric subdivision of B_d, and $Fix(A)$ is the full subcomplex of $sd(B_d)$ defined on the simplices fixed under the action of A. We wish to show that if $m_p(G) \geq 3$, then $\mathcal{A}_p(G)$ is simply connected. Now, by Lemma 21.7, B_d is $(m_p(G) - 2)$-connected; so when $m_p(G) \geq 3$, $sd(B_d)$ is simply connected by Lemma 9.1. Now, by Remark 9.14, we know that F is a closed subposet of $sd(B_d) \times \mathcal{A}_p(G)$. So, $\forall\, \sigma \in sd(B_d)$ and $\forall\, A \in \mathcal{A}_p(G)$, define:

$$F_\sigma = \{A' \in \mathcal{A}_p(G) \mid (\sigma, A') \in F\} \text{ and } F_A = \{\sigma' \in sd(B_d) \mid (\sigma', A) \in F\}.$$

Then, by Lemma 9.13, $\mathcal{A}_p(G)$ is simply connected if, $\forall\, \sigma \in sd(B_d)$ and $\forall\, A \in \mathcal{A}_p(G)$, F_σ and F_A are simply connected.

Lemma 24.1. *If $A \in \mathcal{A}_p(G)$, then F_A is simply connected.*

Proof. $F_A = Fix(A)$. If $Fix(A)'$ is the full subcomplex of B_d defined on those simplices fixed under the action of A, then $Fix(A)'$ is the complex analyzed in section 21. Hence, by Theorem 21.12, $Fix(A)'$ is $(m_p(G)-2)$-connected. Also, by the definition of $sd(B_d)$ and $Fix(A)$, we have $Fix(A) = sd(Fix(A)')$. Thus, by Lemma 9.1, $Fix(A)$ is also $(m_p(G)-2)$-connected. Since $m_p(G) \geq 3$, we have our result. □

Let $\sigma = (x_0 < x_1 < \cdots < x_s) \in sd(B_d)$. Now $F_\sigma = \mathcal{A}_p(G_\sigma)$ where G_σ is a standard parabolic subgroup of G. Let U be the unipotent radical, a solvable p'-subgroup of G_σ. Then the canonical homomorphism $\pi : G_\sigma \to G_\sigma/U \cong L$, the Levi factor, induces a map of posets $f : \mathcal{A}_p(G_\sigma) \to \mathcal{A}_p(L)$. We have the following results about $\mathcal{A}_p(L)$ and $f : \mathcal{A}_p(G_\sigma) \to \mathcal{A}_p(L)$:

Lemma 24.2. *$\mathcal{A}_p(L)$ is simply connected.*

Proof. Let $x'_0 = x_0$, and $\forall\ 1 \leq i \leq s$ let $x'_i \leq x_i$ such that $x_i = x'_0 \oplus x'_1 \oplus \cdots \oplus x'_i$. Then, by (19.14) on pg. 80 in [A1], we have $\forall\ 0 \leq i \leq s$, y'_i totally singular subspaces of V such that $x'_i \oplus y'_i$ are hyperbolic subspaces of V. Furthermore we have:

$$V = x'_0 \oplus y'_0 \perp x'_1 \oplus y'_1 \perp \cdots \perp x'_s \oplus y'_s \perp W,$$

where $W = (x_s \oplus y_s)^\perp$ when $y_s = \bigoplus_{0 \leq i \leq s} y'_i$. Since, by the definition of B_d, $\forall\ 0 \leq i \leq s$ $d | dim(x_i)$, $dim(x'_i) = dn_i$ $\forall\ 0 \leq i \leq s$. Let $dim(W) = m$. Then, by the definition of the Levi factor, we have:

$$L \cong \begin{cases} GL_{dn_0}(q) \times \cdots \times GL_{dn_s}(q) \times Sp_m(q), & \text{when } G = Sp_{2n}(q), \\ GL_{dn_0}(q^2) \times \cdots \times GL_{dn_s}(q^2) \times GU_m(q), & \text{when } G = GU_n(q). \end{cases} \tag{24.3}$$

Considering q^2 as q', we note that since d is odd, d is minimal with respect to $p | q'^d - 1$. Thus, all of our analysis in sections 20, 22, and 23 carry over to $GL_{dn_i}(q^2)$. So we have, by Theorem 20.3, $m_p(GL_{dn_i}(q)) = n_i = m_p(GL_{dn_i}(q^2))$. This fact, along with induction on the dimension of $Sp_{2n}(q)$ and $GU_n(q)$, implies that:

$$m_p(L) = m_p(G) \geq 3. \tag{24.4}$$

Also, as G ranges over $GL_{dn_i}(q)$, $GL_{dn_i}(q^2)$, $Sp_m(q)$, and $GU_n(q)$, by Theorem 20.5, $\mathcal{A}_p(G)$ is connected when $m_p(G) \geq 2$. Finally, by Theorem 23.9, when $G = GL_{dn_i}(q)$ or $GL_{dn_i}(q^2)$, $\mathcal{A}_p(G)$ is simply connected if $m_p(G) \geq 3$.

By (24.3) and Lemma 9.4 we have:

$$(24.5)\quad \mathcal{A}_p(L) \cong \begin{cases} \mathcal{A}_p(GL_{dn_0}(q)) * \cdots * \mathcal{A}_p(GL_{dn_s}(q)) * \mathcal{A}_p(Sp_m(q)), & \text{if } G = Sp_{2n}(q) \\ \mathcal{A}_p(GL_{dn_0}(q^2)) * \cdots * \mathcal{A}_p(GL_{dn_s}(q^2)) * \mathcal{A}_p(GU_n(q)), & \text{if } G = GU_n(q). \end{cases}$$

Thus, by Lemma 9.5 and Theorem 20.5, if $s > 1$ or $s = 1$ and $\mathcal{A}_p(Sp_m(q)) \neq \emptyset$, or $\mathcal{A}_p(GU_n(q)) \neq \emptyset$ (as appropriate), then $\mathcal{A}_p(L)$ is simply connected. So we have to consider a couple of special cases:

<u>CASE I :</u> $\sigma = (x_0)$. Let $m_p(Sp_m(q))$ or $m_p(GU_n(q))$ equal n_1 (as appropriate). Then $n_0 + n_1 = m_p(L) \geq 3$. Either $n_1 = 0$, in which case $n_0 = m_p(L) \geq 3$; and, by (24.5), $\mathcal{A}_p(L) \cong \mathcal{A}_p(GL_{dn_0}(q^\epsilon))$ is simply connected by Theorem 23.9 (here $\epsilon = 1$ or 2). Or $n_i \geq 2$ for $i = 1$ or 2. Then, by Theorem 20.5, the corresponding complex is connected. Hence, by Lemma 9.5, $\mathcal{A}_p(L)$ is simply connected.

<u>CASE II :</u> $\sigma = (x_0 < x_1)$. As in Case I, let $m_p(Sp_m(q))$ or $m_p(GU_m(q))$ equal n_2. Then, if $n_2 \neq 0$, by Lemma 9.5, $\mathcal{A}_p(L)$ is simply connected. So assume $n_2 = 0$: then $n_0 + n_1 = m_p(L) \geq 3$ implies that $n_i \geq 2$ for $i = 1$ or 2. The corresponding complex is connected by Theorem 20.5. Thus, Lemma 9.5, $\mathcal{A}_p(L)$ is simply connected.

This analysis of the two special cases, along with the simple connectivity of $\mathcal{A}_p(L)$ in the general case, proves that $\mathcal{A}_p(L)$ is simply connected when $m_p(G) \geq 3$. □

Lemma 24.6. *Given $A \in \mathcal{A}_p(L)$, $f^{-1}(\mathcal{A}_p(L)(\leq A))$ is $(h(A) - 1)$-connected.*

Proof. Given $A \in \mathcal{A}_p(L)$, $f^{-1}(\mathcal{A}_p(L)(\leq A)) = \mathcal{A}_p(A \cdot U)$, which, since U is a solvable p'-group, is $(h(a) - 1)$-connected by Theorem 11.2 on pg. 123 in [Q]. □

Lemma 24.7. *Given $A \in \mathcal{A}_p(L)$, $\mathcal{A}_p(L)(> A)$ is nonempty when $h(A) = 1$, and connected when $h(A) = 0$.*

Proof. By (24.4) we know that $m_p(L) \geq 3$. Since $\mathcal{A}_p(L)$ is a pure complex (refer to section 1) if $A \in \mathcal{A}_p(L)$ with $h(A) = 1$, then $\mathcal{A}_p(L)(> A)$ is nonempty. So assume that $A \in \mathcal{A}_p(L)$ with $h(A) = 0$. By (24.3) we know that:

$$L \cong \begin{cases} GL_{dn_0}(q) \times \cdots \times GL_{dn_s}(q) \times Sp_m(q), & \text{when } G = Sp_{2n}(q), \\ GL_{dn_0}(q^2) \times \cdots \times GL_{dn_s}(q^2) \times GU_m(q), & \text{when } G = GU_n(q). \end{cases}$$

Let $m_p(Sp_m(q))$ or $m_p(GU_m(q))$ (as appropriate) equal n_{s+1}. Note that if $\sigma = (x_0)$ and $n_{s+1} = 0$, then $\mathcal{A}_p(L) = \mathcal{A}_p(GL_{dn_0}(q))$ or $\mathcal{A}_p(L) = \mathcal{A}_p(GL_{dn_0}(q^2))$, where $n_0 = m_p(L) \geq 3$. Thus, by Lemma 23.3, $\mathcal{A}_p(L)(> A)$ is connected if $h(A) = 0$. So assume that $s \geq 1$ or $n_{s+1} \neq 0$.

If $n_{s+1} \neq 0$, let:

$$L_1 = \begin{cases} GL_{dn_0}(q) \times \cdots \times GL_{dn_s}(q), \text{ when } G = Sp_{2n}(q) \\ GL_{dn_0}(q^2) \times \cdots \times GL_{dn_s}(q^2), \text{ when } G = GU_n(q), \end{cases}$$

and

$$L_2 = \begin{cases} Sp_m(q), \text{ when } G = Sp_{2n}(q) \\ GU_m(q), \text{ when } G = GU_n(q). \end{cases}$$

And if $n_{s+1} = 0$, which implies $s \geq 1$, let:

$$L_1 = \begin{cases} GL_{dn_0}(q) \times \cdots \times GL_{dn_{s-1}}(q), \text{ when } G = Sp_{2n}(q) \\ GL_{dn_0}(q^2) \times \cdots \times GL_{dn_{s-1}}(q^2), \text{ when } G = GU_n(q), \end{cases}$$

and

$$L_2 = \begin{cases} GL_{dn_s}(q) \times Sp_m(q), \text{ when } G = Sp_{2n}(q) \\ GL_{dn_s}(q^2) \times GU_m(q), \text{ when } G = GU_n(q). \end{cases}$$

Then, by our assumption that $s \geq 1$ or $n_{s+1} \neq 0$, $m_p(L_i) \neq 0$ for $i = 1$ and 2. Also, if $m_p(L_i) \geq 2$ for $i = 1$ or 2, then, by Theorem 20.5 and Lemmas 9.4–5, $\mathcal{A}_p(L_i)$ is connected. So, $L \cong L_1 \times L_2$, L_1, and L_2 satisfy the criterion of Theorem 9.8. Thus if $A \in \mathcal{A}_p(L)$ with $h(A) = 0$, then $\mathcal{A}_p(L)(> A)$ is connected. Therefore we have the lemma.

□

We thus have the following result:

Lemma 24.8. *Let $\sigma \in sd(B_d)$, then F_σ is simply connected.*

Proof. $F_\sigma = \mathcal{A}_p(G_\sigma)$. By Lemmas 24.6–7, $\mathcal{A}_p(L)$ and $f : \mathcal{A}_p(G_\sigma) - \mathcal{A}_p(L)$ satisfy the criterion of (1.4) on pg. 5 in [A2]. Hence $\mathcal{A}_p(G_\sigma)$ is simply connected, since, by Lemma 24.2, $\mathcal{A}_p(L)$ is simply connected. □

By Lemmas 24.1 and 24.8 and the discussion immediately preceding Lemma 24.1, we have the main result of this section:

Theorem 24.9. *Let $G = Sp_{2n}(q)$ or $GU_n(q)$ and p be a prime such that $(p,q) = 1$. If d is minimal with respect to $p|(q^d - 1)$, then assume that $d \geq 3$ is odd. Then $\mathcal{A}_p(G)$ is simply connected whenever $m_p(G) \geq 3$.* □

Similar to the final result in section 23, we obtain the following corollary to Theorems 9.15 and 24.9:

Corollary 24.10. *Let $G = PSp_{2n}(q)$, $SU_n(q)$ or $U_n(q)$ and let p be a prime such that $(p,q) = 1$. If d is minimal with respect to $p|(q^d - 1)$, then assume that $d \geq 3$ is odd. Then $\mathcal{A}_p(G)$ is simply connected whenever $m_p(G) \geq 3$.*

Proof. When $G = SU_n(q)$, $d \geq 3$ implies that $\mathcal{A}_p(G) = \mathcal{A}_p(GU_n(q))$ and the result follows from Theorem 24.9.

For $G = PSp_{2n}(q)$ or $U_n(q)$, the fact that $d \geq 3$ and $m_p(G) \geq 3$ implies that G is simple. Thus $O_{p'}(Sp_{2n}(q))$ and $O_{p'}(SU_n(q))$ are central, which implies, by Theorem 9.15, that $\mathcal{A}_p(G)$ is isomorphic to either $\mathcal{A}_p(Sp_{2n}(q))$ or $\mathcal{A}_p(SU_n(q))$. The result then follows directly from Theorem 24.9 or from the preceding paragraph. □

Section 25 : Simple connectivity of $\mathcal{A}_p(GU_n(q))(> Z)$ when $p|q+1$ and Z is central

Let $G = GU_n(q)$ where $n \geq 4$ and $q \geq 5$ is odd. Let p be a prime dividing $q+1$, and fix Z to be the central elementary abelian p-subgroup of G. Since $O_p(G)$ is nontrivial, by Proposition 2.4 on pg. 106 in [Q], $\mathcal{A}_p(G)$ is contractible. However, unlike the case when $p \neq 2$ divides $q-1$ (section 16) we do not show $\mathcal{A}_p(G)$ is C.M. We do show, however, that $\mathcal{A}_p(G)(> Z)$ is simply connected.

It is worth remarking on that showing that $\mathcal{A}_p(GU_n(q))$ is C.M. dimensional $n-1$ when $p|q+1$ would be a useful result with implications towards the **Aschbacher-Smith Conjecture** Conjecture 4.1 on pg. 34 in [ASm]. However, though we have contractibility for $\mathcal{A}_p(GU_n(q))$, it is not clear that we have the appropriate connectedness for links of simplices. Some computation with regard to the question of C.M. behavior of $\mathcal{A}_p(GU_n(q))$ is included at the end of this section.

Let (V, f) be the n-dimensional unitary space over $K = \mathbb{F}_{q^2}$. Given $A \in \mathcal{A}_p(G)$, consider the decomposition of $V = \underset{\lambda \in \Phi}{+} V_\lambda$ into homogeneous components under the action of A. Then, from the discussion at the beginning of section 19, we have the following:

Lemma 25.1. *Let V, A, and Φ be as given above. Then we have:*

(1) $\forall \lambda \in \Phi$, V_λ *is nondegenerate. Also,* $V = \underset{\lambda \in \Phi}{\perp} V_\lambda$ *is an orthogonal decomposition of* V.

(2) $C_G(A) \cong \prod_{\lambda \in \Phi} GU_{n_\lambda}(q)$, *where* $dim(V_\lambda) = n_\lambda \ \forall \lambda \in \Phi$.

Let $\mathfrak{B}$ be the order complex of proper nondegenerate subspaces of V ordered by inclusion. By Theorem 13.9, $\mathfrak{B}$ is simply connected. Let $B = sd(\mathfrak{B})$, which is also simply connected by Lemma 9.1, and consider:

$$B \times \mathcal{A}_p(G)(> Z) \supseteq F = \{(\sigma, A) \mid \sigma \in Fix(A)\}.$$

Note that F is the closed subposet mentioned just prior to Remark 9.14. Thus, by Theorem 9.13, it suffices to show that for all $\sigma \in B$ and $A \in \mathcal{A}_p(G)(> Z)$, F_σ, F_A as defined in Theorem 9.13 are simply connected.

Lemma 25.2. *Given $\sigma \in B$, F_σ is simply connected.*

Proof. Let $\sigma = (U_0 < U_1 < \cdots < U_s) \in B$, $n_0 = dim(U_0)$, and for $1 \le i \le s$, let $n_i = dim(U_i) - \sum_{j=0}^{i-1} n_j$. Let $n_{s+1} = n - \sum_{i=0}^{s} n_s$, and note that $n_{s+1} \ge 1$ since U_s is a proper subspace of V.

Now $F_\sigma = \mathcal{A}_p(G_\sigma)(> Z)$, where $G_\sigma \cong GU_{n_0}(q) \times GU_{n_1}(q) \times \cdots \times GU_{n_{s+1}}(q)$. Now for each $0 \le i \le s+1$, let Z_i be the central elementary abelian p-subgroup of $GU_{n_i}(q)$. Set $W = Z_0 \times Z_1 \times \cdots \times Z_{s+1}$, and note that $W \in \mathcal{A}_p(G_\sigma)$. Since $n_{s+1} \ge 1$, $W > Z$; hence $W \in \mathcal{A}_p(G_\sigma)(> Z)$.

Now $g : \mathcal{A}_p(G_\sigma)(> Z) \to \mathcal{A}_p(G_\sigma)(> Z)$ defined by $A \mapsto AW$ is well defined and satisfies $A \le g(A)\ \forall A \in \mathcal{A}_p(G_\sigma)(> Z)$. Thus, by (1.3) on pg. 103 in [Q], $g \simeq id$, the identity map on $\mathcal{A}_p(G_\sigma)(> Z)$. Also, $\forall A \in \mathcal{A}_p(G_\sigma)(> Z)$, $W \le AW$. Hence $g \simeq r$, where $r : \mathcal{A}_p(G_\sigma)(> Z) \to \mathcal{A}_p(G_\sigma)(> Z)$ is the map defined by $A \mapsto W$. Therefore, $id \simeq r$ and

$\mathcal{A}_p(G_\sigma)(> Z)$ is contractible. Hence, $F_\sigma = \mathcal{A}_p(G_\sigma)(> Z)$ is simply connected, as claimed. □

Lemma 25.3. *Given $A \in \mathcal{A}_p(G)(> Z)$, F_A is simply connected.*

Proof. Let $A \in \mathcal{A}_p(G)(> Z)$, and consider the orthogonal decomposition of $V = \perp_{\lambda\in\Phi} V_\lambda$. As $A \neq Z$, $V_\lambda \lneq V$, $\forall\ \lambda \in \Phi$; in particular, $|\Phi| \geq 2$. Note that $F_A = Fix(A)$. If $Fix(A)'$ is the full subcomplex of $\mathfrak{B}$ defined on the set of simplices fixed under the action of A, then, by the definition of B and $Fix(A)$, $Fix(A) = sd(Fix(A)')$. Thus, by Lemma 9.1, it suffices to show that $Fix(A)'$ is simply connected. For each $\lambda \in \Phi$, let B_λ be the order complex of proper nondegenerate subspaces of V_λ ordered by inclusion. Let $pr_\lambda : V - V_\lambda$ be the canonical projection. Then we have:

$$pr_\lambda(Fix(A)) \cong B_\lambda \cup \{0, V_\lambda\} \cong \overline{B}_\lambda \quad (\text{since } V_\lambda \lneq V).$$

where $\overline{B}_\lambda$ is the double cone of B_λ as defined in section 9. Assume that B_λ is m_λ-connected for each $\lambda \in \Phi$ (where m_λ is defined later). Then, by Corollary 9.11:

$$Fix(A)' \cong \prod_{\lambda\in\Phi} \overline{B}_\lambda - \{(0, \ldots, 0), (\infty, \ldots, \infty)\}$$

is $(\sum_{\lambda\in\Phi} m_\lambda + (|\Phi| - 1)3)$-connected. Three possible cases arise (since $|\Phi| \geq 2$):

<u>Case I :</u> $\Phi = \{\lambda_1, \lambda_2\}$. Let $n_i = dim(V_{\lambda_i})$ and $B_i = B_{\lambda_i}$ for $i = 1, 2$. Note that $n_1 + n_2 = n \geq 4$. And if $n_i \geq 3$ for $i = 1$ or 2, then B_i is connected by Lemma 12.5. If $n_1 = n_2 = 2$, then B_i is -1-connected for $i = 1$ and 2. Then, $m_1 + m_2 + 3 = 1$ and so, $Fix(A)'$ is simply connected. Otherwise, $m_i \geq 0$ for $i = 1$ or 2, and $m_{3-i} \geq -2$, so $m_1 + m_2 + 3 \geq 1$. Thus, $Fix(A)'$ is still simply connected.

<u>Case II :</u> $3 \leq |\Phi| < n$. Then there exists $\lambda \in \Phi$ such that $dim(V_\lambda) \geq 2$; hence $m_\lambda \geq -1$. Thus we have:

$$\left(\sum_{\lambda\in\Phi} m_\lambda + (|\Phi| - 1)3\right) \geq \left(\sum_{|\Phi|-1} -2\right) - 1 + (|\Phi| - 1)3 = |\Phi| - 2 \geq 1.$$

Thus, $Fix(A)'$ is simply connected.

<u>CASE III :</u> $|\Phi| = n \geq 4$. Then we have:

$$\left(\sum_{\lambda\in\Phi} m_\lambda + (|\Phi|-1)3\right) = \left(\sum_{|\Phi|} -2\right) + (|\Phi|-1)3 = |\Phi| - 3 \geq 1.$$

Hence, $Fix(A)'$ is simply connected.

This case-by-case analysis shows that $Fix(A)'$ simply connected. Therefore, $F_A = Fix(A) = sd(Fix(A)')$ is simply connected as claimed. □

So, by Lemmas 25.2 3 and Theorem 9.13, we have:

Theorem 25.4. *Let $n \geq 4$, and $q \geq 5$ odd. If $p|q+1$ and Z is the central elementary abelian p-subgroup of $GU_n(q)$, then $\mathcal{A}_p(GU_n(q))(> Z)$ is simply connected.* □

We now address the question of whether $\mathcal{A}_p(GU_n(q))$ is C.M. when $p|q+1$. Let Z be the central elementary abelian p-subgroup of $G = GU_n(q)$ as before. Note that $\mathcal{A}_p(GU_1(q))$ is nonempty and hence C.M. of dimension 0 as required.

Conjecture. *Given q odd, and $p|q+1$, $\mathcal{A}_p(GU_n(q))$ is C.M. of dimension $n-1$.*

We have already shown that this conjecture holds for $n = 1$. So assume that $n \geq 2$, and the conjecture holds for all $k \leq n-1$.

Since $O_p(G)$ is nontrivial, we have $\mathcal{A}_p(G)$ is contractible. Thus, by Lemma 14.2, it suffices to show that for each $A \in \mathcal{A}_p(G)$, $\mathcal{A}_p(G)(> A)$ is $(n - h(A) - 2)$-spherical. Now, $dim(\mathcal{A}_p(G))$ is $n-1$ and $dim(\mathcal{A}_p(G)(\leq A))$ is $h(A)$; thus $\mathcal{A}_p(G)(> A)$ is indeed $(n - h(A) - 2)$-dimensional. So we are reduced to showing that $\mathcal{A}_p(G)(> A)$ is $(n - h(A) - 3)$-connected for all $A \in \mathcal{A}_p(G)$. The following result is immediate in light of Lemma 25.1:

Lemma 25.5. *Let G be a minimal counterexample. Given $A \in \mathcal{A}_p(G)$ with $A \neq Z$, $\mathcal{A}_p(G)(> A)$ is $(n - h(A) - 3)$-connected.*

Proof. By Lemma 25.1, we know that $C_G(A) \cong \prod_{\lambda\in\Phi} GU_{n_\lambda}(q)$. Note that since $A \neq Z$, $|\Phi| \geq 2$, and $n_\lambda < n$, $\forall\ \lambda \in \Phi$. Also, $\sum_{\lambda\in\Phi} n_\lambda = n$.

Now $\mathcal{A}_p(G)(> A) = \mathcal{A}_p(C_G(A))(> A)$, where $\mathcal{A}_p(C_G(A)) \simeq \underset{\lambda\in\Phi}{*} \mathcal{A}_p(GU_{n_\lambda}(q))$ by Lemma 9.4 and 25.1. Since $n_\lambda < n \ \forall \ \lambda \in \Phi$, by induction on dimensions, $\mathcal{A}_p(GU_{n_\lambda}(q))$ is C.M. of dimension $n_\lambda - 1 \ \ \forall \ \lambda \in \Phi$. Thus, by Lemma 14.3, $\mathcal{A}_p(C_G(A))$ is C.M. of dimension:

$$\sum_{\lambda\in\Phi}(n_\lambda - 1) + (|\Phi| - 1) = \left(\sum_{\lambda\in\Phi} n_\lambda\right) - 1 = n - 1.$$

Therefore, by the definition of C.M., $\mathcal{A}_p(G)(> A) = \mathcal{A}_p(C_G(A))(> A)$ is $(n - h(A) - 3)$-connected as claimed. □

So we have the Conjecture if we can prove the following result:

Conjecture 25.6. *$\mathcal{A}_p(G)(> Z)$ is $(n - 3)$-connected.* □

The proof of this conjecture is complicated by the fact that $\mathcal{A}_p(G)(> Z)$ is $(n - 2)$-dimensional while the dimension of the underlying building is $[\frac{n}{2}] - 1$. However, note that we have $\mathcal{A}_p(G)(> Z)$ is simply connected when $n \geq 4$, under the added assumption that $q \geq 5$ and odd. So Conjecture 25.6 reduces to a conjecture about the homology of $\mathcal{A}_p(G)(> Z)$ which may possibly be addressed using tools such as spectral sequences.

Bibliography

[A1] M. Aschbacher, *Finite Group Theory*, Cambridge studies in advanced mathematics, 10, Cambridge University Press, 1986.

[A2] ———, *Simple connectivity of p-group complexes*, preprint (1992).

[A3] ———, *Closed families of poset complexes*, preprint (1992).

[A4] ———, *Flag structures of Tits geometries*, Geom. Dedicata **14** (1983), 21–32.

[A5] ———, *On the maximal subgroups of the finite classical groups*, Invent. Math. **76** (1984), 469–514.

[AS1] M. Aschbacher and Y. Segev, *Extending morphisms of groups and graphs*, Ann. of Math. **135** (1992), 297–323.

[AS2] ———, *The uniqueness of groups of Lyons type*, J. Amer. Math. Soc. **5** (1992), no. 1, 75–98.

[AS3] ———, *A fixed point theorem for groups acting on finite 2-dimensional acyclic simplicial complexes*, preprint (1992).

[AS4] ———, *Locally connected simplicial maps*, Israel J. Math. **77** (1992), 285–303.

[AS5] ———, *The uniqueness of groups of type J_4*, Invent. Math. **105** (1991), 589–607.

[ASm] M. Aschbacher and S.D. Smith, *On Quillen's conjecture for the p-groups complex*, Ann. of Math. **137** (1993), 473–529.

[Am] M.A. Armstrong, *Basic Topology*, Undergraduate Texts in Mathematics, Springer-Verlag, New York, 1983.

[B1] A. Börner, *Some combinatorial and algebraic properties of Coxeter complexes and Tits buildings*, Ad. in Math. **52** (1984), 173–212.

[B2] ———, *Homotopy type of posets and lattice complementation*, J. Combin. Theory Ser. A **30** (1981), 90–100.

[Be] H. Bender, *Transitive Gruppen gerarder Ordnung, in denen jede Involution genau einen Punkt festläßt*, J. Algebra **17** (1971), 527–554.

[Br] K. Brown, *Euler characteristic of groups: The p-fractional part*, Invent. Math. **29**

(1975), 1–5.

[CL] C.W. Curtis and G.I. Lehrer, *A new proof of a theorem of Solomon-Tits*, Proc. Amer. Math. Soc. **85** (1982), no. 3, 154–156.

[GH] M.J. Greenberg and J.R. Harper, *Algebraic Topology : A First Course*, Addison-Wesley Publishing Co., Inc., New York, 1981.

[GiH] P.C. Gilmore and A.J. Hoffman, *A characterization of comparability graphs and of interval graphs*, Canad. J. Math. **16** (1967), 539–548.

[GL] D. Gorenstein and R. Lyons, *The local structure of finite groups of characteristic 2-type*, Mem. Amer. Math. Soc. **276** (1983), 1–731.

[GZ] P. Gabriel and M. Zisman, *Calculus of Fractions and Homotopy Theory*, Springer-Verlag, New York, 1967.

[Q] D. Quillen, *Homotopy properties of the poset of nontrivial p-subgroups of a group*, Ad. in Math. **28** (1978), 101–128.

[R1] J.J. Rotman, *An Introduction to Homological Algebra*, Academic Press, New York, 1979.

[R2] ———, *Notes on Homological Algebras*, Van Nostrand Reinhold Mathematical Studies 26, Van Nostrand Reinhold Company, New York, 1970.

[RS] G. Robinson and R. Staszewski, *More on Alperin's conjecture*, Astérisque **181–182** (1990), 237–255.

[So] L. Solomon, *The Steinberg character of a finite group with BN-pair*, Theory of Finite Groups, Benjamin, New York, 1966, pp. 213–221.

[Sp] E.H. Spanier, *Algebraic Topology*, Tata-McGraw-Hill Publishing Company Ltd., Bombay, 1966.

[St] R.P. Stanley, *Balanced Cohen-Macaulay complexes*, Trans. Amer. Math. Soc. **249** (1979), no. 1, 139–157.

[Su] M. Suzuki, *Group Theory I*, Grundlehren der mathematischen Wissenschaften 247, Springer-Verlag, Berlin, 1982.

[T1] J. Tits, *Buildings of Spherical type and finite BN-pairs*, Lecture Notes in Mathematics, 386, Springer-Verlag, Berlin, 1974.

[T2] ———, *A Local Approach to Buildings*, The Geometric Vein (C. Davis, B. Grünbaum, F.A. Sherk, eds.), Springer-Verlag, Berlin, 1981, pp. 519–547.

[W] P.J. Webb, *Subgroup complexes*, Proc. Symposia in Pure Math. **47** (1987), 349–365.