# Cooperation and Social Choice:
# How foresight can induce fairness

Thesis by

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# Abstract

I present three models of dynamic agenda formation and policy selection, and demonstrate that in each, outcomes emerge which are in keeping with those predicted by cooperative solution concepts such as the von Neumann-Morgenstern stable set and the core. These outcomes are a consequence of players "thinking ahead," or conditioning how they bargain on the notion that policies selected today should stand up to tomorrow's agenda. Players are induced into taking the payoffs of others into account when voting over and proposing policies, not because of a behavioral assumption such as altruism or inequality aversion, but because they know that the behavior of others in large part determines which policies are enacted in the future. In this sense, fairness is induced through the foresight of the players involved.

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# Chapter 1 Introduction

Social choice theory is the study of how decisions are made collectively. It examines the idea that, for a given society, the preferences of individual members can be directly aggregated to reflect a quintessential "social preference." An example of such a preference aggregation method is simple plurality rule, in which citizens vote for a candidate and the candidate winning the most votes is deemed society's most-preferred. In general, social choice theory does not aim to analyze or predict behavior, but rather to compare and evaluate different means of aggregating preferences. However, for this technique to have any relevance, it must be capable of making predictions. In this vein, Austen-Smith and Banks (1998) write, "for the direct preference aggregation approach to work as a general theory of politics, we need to determine the extent to which different aggregation methods insure the existence and characterization of best alternatives."

One of the most compelling social choice-theoretic notions is the idea of a *core*, or alternative that can defeat all other alternatives via majority rule. Most would agree that in the presence of such an alternative, we could expect it to be chosen as a policy outcome. However, social choice theory is rife with nonexistence results. The well-known Plott conditions prove that only in extremely rare circumstances does a core alternative exist. Arrow's Theorem tells us that the requirement that social choices be made on normatively appealing criteria cannot lead to normatively appealing outcomes. And McKelvey's "chaos" Theorem says that if individuals vote sincerely, virtually any alternative can defeat any other via a finite amendment agenda. And yet there is reason to remain hopeful; many other social choice-theoretic concepts are capable of refining the set of "best alternatives" in many environments, even in the absence of a core. Two of the most commonly cited such concepts are the *uncovered set* and the *Banks set*.

An uncovered policy is an alternative that can defeat any other alternative by majority rule in two or fewer steps; that is, if policy $x$ is uncovered and there exists a policy $y$ which defeats $x$, then there also exists a third policy $z$, such that $x$ defeats $z$ and $z$ defeats $y$. Miller

(1980) was the first to define the uncovered set (or set of all uncovered policies) as a general solution concept, useful particularly in the absence of a core. Shepsle and Weingast (1984) examine finite agendas in a spatial setting and show that no outcome can be a sophisticated voting outcome if an alternative that covers it is also included in the agenda. Banks (1985) returns to Miller's framework of strict preferences over a finite alternative space, and shows that, for a specific definition of the uncovered set, all sophisticated voting outcomes are uncovered yet all uncovered points are not sophisticated voting outcomes. He then provides a method for calculating both the uncovered set and the set of sophisticated voting outcomes in this setting. And McKelvey (1986) and Cox (1987) show that under general conditions the uncovered set exists, shrinks to the core when a core exists, and becomes smaller the closer individual preferences are to admitting a core. Thus, these authors demonstrate that generally, and under many different institutional arrangements, strategic behavior by voters leads to outcomes in the uncovered set.

However, there exists a particular institutional arrangement under which the predictions yielded by the uncovered set are entirely useless: that of distributive politics. A distributive setting is one in which there exists a fixed pie that players seek to divide among themselves. Players' preferences are solely a function of how much they get. Distributive games are of particular interest to social scientists because strategic interaction between individuals often involves money. Epstein (1998) and Penn (2001) compute the uncovered set (under different definitions) for this class of game, and both find that the uncovered set has full measure on the space of policies, under every definition of "covering". Thus the entire policy space, possibly minus a set of measure zero, is uncovered, and this concept leaves us with with no way of characterizing the set of best alternatives.

In Chapter 2, I focus on the concept of the Banks set. The Banks set is an important social choice-theoretic concept because it was one of the first to incorporate strategy into a method of preference aggregation. The Shepsle-Weingast algorithm gives us a means of finding the "sophisticated voting outcome" of an agenda; given an amendment procedure, this outcome is defined to be the last item on the agenda that defeats all of its predecessors given a fixed voting rule. If individuals are sophisticated and know which items will be voted upon when, this algorithm computes the best alternative any person can procure for himself. The Banks set equals the set of policies that can be supported as sophisticated voting outcomes of a finite-length, externally stable agenda, where external stability is simply the requirement that an agenda admits no further profitable amendments. Thus, while McKelvey tells us that any alternative can be the outcome of an amendment procedure over some finite agenda when people vote sincerely, Banks tells us that this is not really the case, because the assumption that players vote sincerely does not always make sense.

In this chapter I characterize the Banks set for the class of 3-player distributive games and find that we are again struck down, because, like the uncovered set, the Banks set also has full measure on this policy space. Using this result as a starting point, the chapters that follow argue that the frequent inability of social choice theoretic concepts such as these to yield predictions arises not solely because the majority preference relation is unstable, but also because social choice theory implicitly assumes a static environment. Yes, the Shepsle-Weingast algorithm assumes the dynamics of the amendment procedure, but only with respect to a predetermined, static agenda. How does this agenda get chosen? And how can uncertainty about a future agenda effect individual behavior? Schattschneider (1960) writes, “The definition of the alternatives is the supreme instrument of power,” and given McKelvey’s chaos result, this claim becomes even more believable.

Dutta, Jackson, and LeBreton (2001) look at precisely the question of how agendas are formed, and develop an elegant definition of equilibrium agenda formation under very general conditions. Under an amendment procedure, the set of equilibrium outcomes generated by their definition coincides with the Banks set, but they also demonstrate that in many instances their definition can yield even sharper predictions. However, this definition can also predict indiscriminately, as is the case with the Banks set. In Chapter 2, I take a less general approach to the question of how agendas are formed, and look at an extension of a game of endogenous agenda formation by Banks and Gasmi (1987). The authors characterize the minimax-Stackelberg equilibrium of a three-player game in which each player gets to propose one item to an agenda. The constructed agenda is then voted upon via an amendment procedure. The agenda that is constructed in equilibrium, however, is not externally stable; there exist policies which defeat every item on the constructed agenda, and it would be in the best interest of a player to propose one of these policies. Thus, a natural question to ask is “when do players want to stop proposing items to an agenda?” In this chapter I extend their game by allowing players to make as many proposals as they wish, until there exists no policy that defeats every item on the constructed agenda. Allowing for an unspecified number of amendment proposals by players is appealing because it mimics the setting of an informal negotiation between three people. Simply put, individuals are randomly chosen to make proposals until no one wishes to make another proposal. In many environments this game form is more natural than one in which the order of the players is specified, or in which players are only allowed to make a certain number of proposals.

I find that such uncertainty about the future length of the agenda drives the first two proposers to collude. Thus, while Banks and Gasmi find the outcome of their game to be the universalistic allocation $M^* = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, I find that the outcome to the same game with no restrictions on agenda length is a policy in the simple von Neumann-Morgenstern stable set, $\{(\frac{1}{2}, \frac{1}{2}, 0), (0, \frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, 0, \frac{1}{2})\}$. Yet in both games, only three alternatives are ever proposed. Thus, allowing for an arbitrarily large number of amendments does not change the number of amendments actually made, but dramatically changes *which* amendments are made. In particular, it motivates the first two proposers to work as quickly as possible to ensure that the outcomes which leave them worst-off are infeasible. A consequence of this is that the last player is completely disenfranchised, and in this way the outcome generated here is less normatively “fair” than the outcome generated by Banks and Gasmi. But interestingly, this noncooperative game generates the same outcome as that generated by the cooperative solution concept of the von Neumann-Morgenstern stable set, which in this setting predicts the emergence of minimal-winning coalitions that divide the dollar evenly among every member. This cooperative result will be paralleled in the chapters that follow.

When we consider the concept of “strategic” behavior, we often think of it as referring to a situation in which players are willing to endure short-term losses for long-term gains. In a setting where current policies have an effect on future legislation, legislators may often forgo some satisfaction with respect to a minor policy in the short term to get a more important policy passed in the future. In the last two chapters, I argue that social choice theory fails to account for this type of dynamic because the theory is concerned with predicting a single social outcome. It fails to recognize the fact that societies collectively choose many different outcomes over time, and that the choice of policy in one round may effect which policies are feasible in the next. In these chapters I demonstrate that predictions can be generated in institution-free environments, by allowing current policies to effect future policies, and allowing players to have preferences over these future events. These predictions can even be generated in the seemingly inextricable environment of the divide-the-dollar, or distributive, game.

The notion of short- versus long-term gain is captured in these chapters by adding a new dimension to the standard social choice-theoretic framework. Here, individuals rank policies not only on the basis of the utility they yield, but also with respect to the types of alternatives they can and cannot defeat. The types of alternatives a policy can defeat are conditioned upon a probabilistic future agenda. In Chapter 3 this agenda is assumed to be exogenous and static; some alternatives are more likely than others to be brought to the floor in the future, regardless of the status quo at hand, and players know this. In Chapter 4 I allow the agenda setting process to be endogenous, so that players propose alternatives themselves. In this case, players can condition upon the current status quo policy to propose an alternative that defeats it and leaves them better-off.

Formal models to date have not been able to make compelling predictions in the setting of a continuing program. Baron (1996) shows that in this setting, policy selection eventually converges to the median voter's ideal point, when the policy space is one-dimensional. However, this alternative is a Condorcet winner, and so predicting it as a policy outcome is not particularly surprising. Indeed, if the model predicted something else, the space of alternatives would most likely have been restricted by the model in some way or another. Kalandrakis (2002) looks at continuing programs in the setting of a divide-the-dollar game, and finds that the ideal points of the players emerge as policy outcomes, with probability one. This result is disturbing because rarely in political environments does a legislative dictator emerge, with probability one, in every round.

While continuing programs have been largely ignored in the formal literature, we would expect these types of programs to be the most interesting from the standpoint of political science. Legislators are keenly aware of the fact that policy sets precedent, and that today's status quo greatly effects the types of alternatives that are feasible tomorrow. An example is President Bush's 2001 tax cut package, which mandated the gradual phase-out of the estate tax by the year 2010, only to return to its 2001 levels in 2011. A lobbyist in favor of the complete abolishment of the estate tax was quoted as saying "In Washington terms, it's the finality we needed. It's very difficult for Congress to reinstate a tax once it's been repealed." Thus, a bill eliminating a tax for one year and then reinstating it the next is effectively similar to a bill eliminating the tax forever. Once the status quo of "no tax" has been set, it is virtually impossible to defeat the status quo with a policy mandating "tax."

In these chapters I model policy alternatives as not only yielding utility today, but also leading to streams of future policy that are dependent upon the status quo. I find that, even in the absence of a game form, players are not indifferent between different policies which provide them with the same level of utility. This is because the space of alternatives which defeat each policy, and which each policy defeats, matters. I show that in dynamic environments, the space of alternatives which can and cannot defeat a policy, or the future agenda conditioned upon that policy, may have as much impact on individual decision making as the substance of the policy itself. These models provide one answer to the question of "why so much stability?" Here, cooperative outcomes emerge and are sustained as a consequence of looking at the probabilistic path of legislation a policy can lead to over time. Thus, even though these models are ultimately sophisticated preference aggregation techniques, they yield well defined sets of best alternatives.

Possibly most interesting is link between these chapters and cooperative game theory. Cooperative game theory examines the types of allocations that coalitions of agents can procure for themselves, while remaining agnostic as to how these allocations arise, and how they are enforced. In all of the chapters presented here, outcomes often emerge which are in keeping with those predicted by cooperative solution concepts such as the von Neumann-Morgenstern stable set. These outcomes are a consequence of players "thinking ahead," or conditioning how they bargain upon the idea that policies selected today should stand up to tomorrow's agenda. A consequence of all of these chapters is that players are induced into taking the payoffs of others into account when voting over and proposing policies, not because of a behavioral assumption such as altruism or inequality aversion, but because they know that when collective choices are being deliberated upon, the behavior of others in large part determines how policies are chosen. In this sense, fairness is induced by the foresight of the players involved. Perhaps modeling foresight in such a way can provide a first step toward a behavioral rationalization of cooperative game theory.

# Chapter 2 A Distributive N-Amendment Game with Endogenous Agenda Formation

## 2.1 Introduction

Much work has been done on the fact that under the assumption of a finite agenda, amendment procedures can be solved by backward induction, yielding well defined results. McKelvey (1979) shows that when players vote sincerely, an agenda can be constructed such that any point in $\mathbb{R}^N$ can be supported as the unique outcome of an amendment game, be it Pareto efficient or not. However, when players vote sophisticatedly, the possible set of outcomes that can be supported by an amendment game becomes significantly smaller (Banks, 1985), and is in fact a subset of the uncovered set. And when we allow sophisticated players to set the agenda themselves, this outcome shrinks to a single point, M*, in a three-player, two-amendment game with a maximin equilibrium concept (Banks and Gasmi, 1987).

This point, M*, possesses certain characteristics which are attractive from a normative point of view. If the ideal points of the three players are equidistant from each other, then M* is the barycenter of the Pareto set, giving all players equal utility. As the ideal points become less symmetrical, M* tends toward the closer pair of ideal points, leaving the player whose preferences are less similar in a disadvantageous position. The authors specifically examine a two-dimensional spatial setting, however they demonstrate that their analysis can be easily applied to a distributive, divide-the-dollar game, in which case M* = $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.[1] Yet Banks and Gasmi do not establish whether M* is an equilibrium outcome if we allow for more amendments—intuitively, it seems as though any two players could conspire to pick an alternative on their contract curve such that both could be better off. Thus, a natural question to ask is whether the M* result is completely driven by limiting

[1]Duggan (2003) provides a formal proof of this in the case where the order of proposers is known. He also establishes the existence of a subgame-perfect equilibrium in an endogenous agenda formation game similar to that of Banks and Gasmi, where the order of the proposers is known and each player proposes once.

the number of possible amendments to two.

This chapter answers this question in a distributive setting by characterizing those finite agendas (and resulting outcomes) which could and would arise by allowing players to propose as many amendments as they wish, until no new alternative exists that defeats every other previously proposed alternative via the majority preference relation. Using a maximin solution concept similar to the one used by Banks and Gasmi, I find that the M* result can *not* be supported as a maximin-Nash equilibrium outcome when then the two-amendment restriction is removed. However, a unique set of equilibrium outcomes to this game does exist. I will show that this set equals the three-element von Neumann-Morgenstern stable set.

Allowing for an unspecified number of amendment proposals by players is appealing for several reasons. First, the setting is quite informal. It is a bargaining process between three players in which individuals are randomly chosen to make proposals until no one wishes to make another proposal. This setting is much more natural than a setting in which the order of the players is known, or in which players are only allowed to make a specified number of proposals. And second, because the end of the proposal process is endogenously determined, the final outcome will be stable in that no other alternative can defeat it, if players vote sophisticatedly. It is well known that in a three player divide-the-dollar game, there is no stable bargaining equilibrium, or core. The existence of an equilibrium in this game relies on sophisticated voting and the nature of the amendment procedure—since it is solved recursively, any new proposal must be weakly majority preferred to all of the amendments that preceded it.

The chapter proceeds as follows: Section 2.2 describes the structure of voter preferences, the structure of the amendment game itself, the notion of the Banks set over an infinite alternative space, and the equilibrium concept used in the game. Section 2.3 presents some results about the Banks set needed for solving the game, and then proves that the Banks set equals the uncovered set in a distributive three-player game with linear preferences. Section 2.4 solves for the set of outcomes of the N-amendment game, and Section 2.5 concludes.

## 2.2 The Model

### 2.2.1 Setup and Preliminary Definitions

Consider a voting game consisting of a set $I$ of players, where $I = \{1, 2, 3\}$, and let the set of alternatives be $\Delta = \{(x_1, x_2, x_3) \in [0, 1]^3 : \sum_{i \in I} x_i = 1\}$. Let $x_i$ denote the $i$-th component of a vector $x$, and let $x^i$ be an element of $\Delta$. For each $i$ define $u_i : \Delta \cup \emptyset \to \mathbb{R}$ by

$u_i(x) = x_i$ for $x \in \Delta$ and

$u_i(\emptyset) = -1.$

In words, we are looking at a divide-the-dollar game; preferences are assumed to be linear and the alternative space is the unit simplex. If no policy is chosen, all players receive a payoff of negative one. Many of the following definitions and lemmas can be made in the context of a more general set of alternatives and preferences. Assume this set of alternatives is convex, and that preferences are strictly quasi-concave and continuous over this set. I will call this set of alternatives $X \subset \mathbb{R}^N$.

The point $x$ is *strictly majority preferred* to $y$, written $xPy$, if

$$|\{i \in I : u_i(x) > u_i(y)\}| \geq 2.$$

Define $P(y)$ as the set of all points in $X$ that are strictly majority preferred to $y$, so that $P(y) = \{x \in X : xPy\}$. Similarly, $P^{-1}(y) = \{x \in X : yPx\}$. Let $\overline{P}(\cdot)$ and $\overline{P}^{-1}(\cdot)$ be the closures of $P(\cdot)$ and $P^{-1}(\cdot)$, respectively.[2]

Given a finite set of alternatives $B = \{x^1, ..., x^t\} \subseteq X$ with $T = \{1, 2, ..., t\}$, define an *agenda*, $A$, to be a permutation of $B$, and let $\mathbf{A}$ denote the set of all agendas composed of

[2]The notation $\overline{P}$ is also used by Banks and Gasmi, and simply represents the weak majority preference relation. In much of the literature it is termed $R$.

elements of $B$. Thus,

$$A \in \mathbf{A} = \{(x^{\phi(1)}, x^{\phi(2)}, ..., x^{\phi(t)}) \in B^t : \phi : T \to T \text{ and } \phi \text{ is } 1-1\}.$$

Voting over the elements in an agenda follows an *amendment procedure*. Letting $y^i = x^{\phi(i)}$, for a given agenda $A$, a decision over the alternatives in an agenda is arrived at by: $(i)$ comparing $y^t$ to $y^{t-1}$ via the weak majority preference relation; $(ii)$ comparing the winner to $y^{t-2}$, and so on until a single remaining alternative is reached. This alternative is the voting outcome. If a player is indifferent between two alternatives on an agenda, I assume that he votes for the alternative proposed later in the agenda formation process. Although this assumption has been used frequently, as in Banks and Gasmi (1987) and Austen-Smith (1987), Duggan (2003) shows that it poses problems in proving the existence of subgame-perfect equilibria. This is because while the assumption guarantees the existence of a best response for the last proposer, it does not guarantee the existence of a best response for the next-to-last proposer in all subgames. In the setting I examine, the assumption poses no problem because I am not considering subgame perfection. However, this assumption greatly effects the equilibrium that I construct, and this will be discussed in further detail in Section 2.4.

### 2.2.2 Game Form and Solution Concepts

#### N-Amendment Games and Sophisticated Agendas

The N-amendment game considered here begins as one player is randomly selected to propose an alternative to be considered, called the *bill*. After the bill is proposed, another player is randomly selected to propose an amendment to the bill. This player may be the same person who proposed the initial bill. Once this amendment is proposed, another player is randomly selected to propose an amendment to the amendment. And so on. The process continues until no players remain that wish to amend the last proposed amendment; i.e., until there is no remaining point on the simplex that is weakly majority preferred to every other previously proposed alternative. The bill and subsequent amendments constitute an agenda. Once the agenda is set, the players then vote on the agenda via an amendment procedure.

Suppose a bill, $x^1$, and $t-1$ amendments, $(x^2, ..., x^t)$, are proposed. The resulting agenda is $A = (x^1, ..., x^t)$. In the subsequent voting game, $x^t$ is paired against $x^{t-1}$, with the winner paired against $x^{t-2}$ and so on. Consequently, the assumption of sophisticated voting greatly restricts the types of amendments players choose to propose. If an amendment is chosen that is not weakly majority preferred to *every* amendment proposed before it, then that amendment cannot change the outcome of the game in any way and is therefore irrelevant. The following definition of a sophisticated equivalent agenda formalizes this idea.

*Definition:* Given an agenda $A = (x^1, ..., x^t)$, the *sophisticated equivalent agenda* $A^* = (x^{1^*}, ..., x^{t^*})$ is defined as,

i) $x^{1^*} = x^1$

ii) for $1 < i \leq t$,

$$x^{i^*} = \begin{cases} x^i & \text{if } x^i \neq x^j \text{ and } x^i \in \overline{P}(x^{j^*}), \forall j < i \\ x^{i-1^*} & \text{otherwise} \end{cases}$$

and the *reduced form*, $A'$, of the sophisticated equivalent agenda $A^*$ is defined as the truncated version of $A^*$; if $A^* = (x^1, x^2, x^3, x^3)$ then $A' = (x^1, x^2, x^3)$. Consequently, the effective strategy space for player $i$ after alternatives $(x^1, ..., x^t)$ have been proposed is $\{\overline{P}(x^t) \cap \overline{P}(x^{t-1}) \cap \cdots \cap \overline{P}(x^1)\} \setminus \{x^1, ..., x^t\}$, and I will restrict my attention to this space. [3]

**Externally Stable Chains**

Given a set $X$ and a binary relation, $B$, on $X$, a *chain* $Y$ is defined as a subset of $X$ such that $B$ restricted to $Y$ is a *linear order*. A binary relation $B$ is a linear order if it is complete, transitive, and anti-symmetric (for all $x, y \in X$, $xBy$ and $yBx$ implies $y = x$). Because

[3] Austen-Smith (1987) shows that when agendas are formed endogenously, sophisticated voting is observationally equivalent to sincere voting. This is precisely the reason that the effective strategy space is so restricted after a sequence of proposals have been made.

we are considering sophisticated equivalent agendas, the related binary relation is the weak majority preference relation, $\overline{P}$, which is clearly not anti-symmetric. Define $H \subseteq X$ to be a *weak chain* if a binary order $\overline{P}$ restricted to $H$ is complete and transitive, but not necessarily anti-symmetric. Thus, the alternatives in the reduced form of a sophisticated equivalent agenda constitute a weak chain. Since the remainder of the chapter focuses solely on weak chains, I will from now on refer to *weak chains* as *chains*.

Let $\mathbf{H}$ be the set of all finite chains in $X$. Then a chain $H \in \mathbf{H}$ is *externally stable* if and only if there exists no $x \in X \setminus H$ such that for all $y \in H$, $x\overline{P}y$. Thus, there is no alternative outside of $H$ that is weakly preferred to every alternative in $H$. Let $\mathbf{H}^* \subseteq \mathbf{H}$ be the set of externally stable chains in $\mathbf{H}$, and for all $H^* \in \mathbf{H}^*$, define $X^{H^*}$ to be the set of maximal elements of $H^*$, so that for all $x \in X^{H^*}$ and all $y \in H^*$, $x\overline{P}y$. Note that every chain has at least one maximal element.

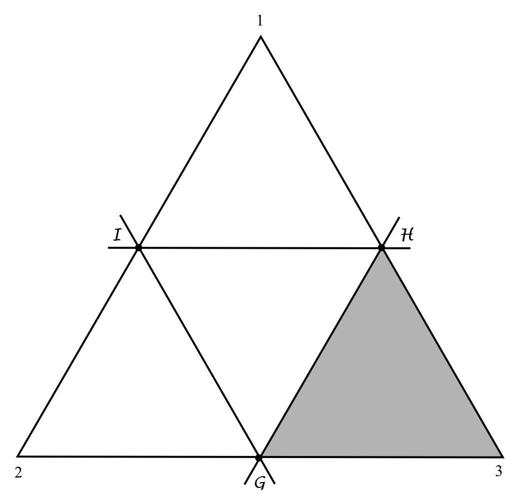



Figure 2.1: A minimal externally stable chain.

An externally stable chain $V \subseteq X$ is a *von Neumann-Morgenstern stable set* if for all $x, y \in V$, it is not the case that $xPy$. This property is termed *internal stability*. In the game considered here, only one finite externally stable chain also satisfies the property of internal stability. This chain is called the *simple von Neumann-Morgenstern stable set*. The simple von Neumann-Morgenstern stable set refers to the unique finite stable set, which consists of the points $\{(\frac{1}{2}, \frac{1}{2}, 0), (\frac{1}{2}, 0, \frac{1}{2}), (0, \frac{1}{2}, \frac{1}{2})\}$.

A further restriction can be made on the set of externally stable chains. Consider Figure 2.1 which depicts the ideal points of the three players, three linear indifference curves, and the simple von Neumann-Morgenstern stable set $\{\mathcal{G}, \mathcal{H}, \mathcal{I}\}$. The line connecting points $\mathcal{H}$ and $\mathcal{I}$ represents all outcomes where Player 1 receives a payoff of $\frac{1}{2}$. The shaded region consists of all alternatives that are weakly majority preferred to both points $\mathcal{H}$ and $\mathcal{G}$. It is apparent that the points $(\mathcal{G}, \mathcal{H}, \mathcal{I})$ form an externally stable chain. However, there are an infinite number of externally stable chains that can be constructed with starting points $(\mathcal{G}, \mathcal{H})$ and final element $\mathcal{I}$. For example, any point $p$ could be inserted into the chain between $\mathcal{H}$ and $\mathcal{I}$ to form another externally stable chain $(\mathcal{G}, \mathcal{H}, p, \mathcal{I})$. Moreover, an arbitrarily large number of points could be inserted into the chain, yielding an arbitrarily long externally stable chain. Figure 2.2 illustrates this; pick a random point, $p^1 \in P(\mathcal{G}) \cap P(\mathcal{H})$, and add it to the chain. Then add point $p^2 \in P(\mathcal{G}) \cap P(\mathcal{H}) \cap P(p^1)$ to the chain. Then add point $p^3 \in P(\mathcal{G}) \cap P(\mathcal{H}) \cap P(p^1) \cap P(p^2)$ to the chain. And so on. By picking alternatives in such a fashion, an infinite number of arbitrarily long externally stable chains can be constructed with beginning elements $(\mathcal{G}, \mathcal{H})$. However, all such chains must have the same final element $\mathcal{I}$, since $\mathcal{I}$ is strictly majority preferred to every point in the set $\overline{P}(\mathcal{G}) \cap \overline{P}(\mathcal{H})$. Thus, the shortest, or *minimal*, externally stable chain with beginning elements $(\mathcal{G}, \mathcal{H})$ is $(\mathcal{G}, \mathcal{H}, \mathcal{I})$.

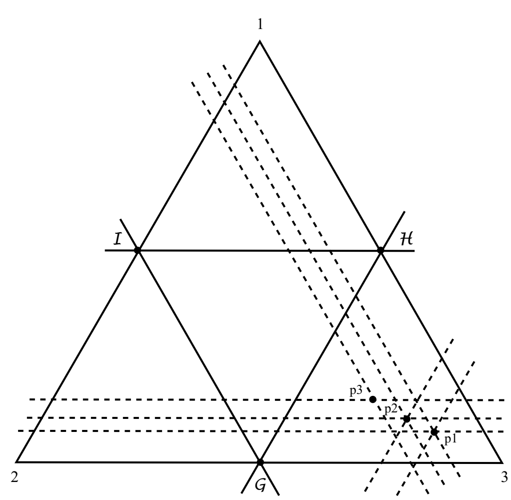



Figure 2.2: Constructing an arbitrarily long externally stable chain.

*Definition:* An *minimal externally stable chain* is an externally stable chain $(x^1, ..., x^t)$, where for all $i \leq t$, $(x^1, ..., x^{i-1}, x^{i+1}, ..., x^t)$ is not externally stable.

In other words, you cannot remove an element of a minimal externally stable chain and still have an externally stable chain. This implies that for all $i \leq t$, there exists a $y \in \overline{P}(x^1) \cap \cdots \cap \overline{P}(x^{i-1}) \cap \overline{P}(x^{i+1}) \cap \cdots \cap \overline{P}(x^t)$ such that either $x^i P y$ or $x^i = y$. Let $\mathbf{H^M} \subset \mathbf{H}^*$ denote the set of all minimal externally stable chains.

**The Uncovered Set**

For any set $X$, and any $x, y \in X$, the *covering relation* $\mathcal{C} \subseteq X \times X$ is defined as,

$x\mathcal{C}y$ if and only if
i) $y \in P^{-1}(x)$, and
ii) $\{P^{-1}(y) \cap X\} \subseteq \{P^{-1}(x) \cap X\}$.[4]

In words, $x$ "covers" $y$ if and only if $x$ strictly beats $y$ and if $y$ strictly beats $z$, then $x$ also strictly beats $z$.

**The Banks set, with an infinite policy space**

In his 1985 paper, Banks defines the set $S(X)$, later termed the *Banks set*, as the set of all outcomes achievable as sophisticated voting outcomes under some agenda. In his definition it is assumed that the policy space is finite and that an agenda is an ordering of every element in the policy space. Banks then proves that this set is equivalent to the set of maximal elements of maximal chains, and that this set is always a subset of the uncovered set. However, his definition cannot be extended to the case of an infinite policy space

[4]McKelvey's (1986) definition also requires that iii) $P(x) \subseteq P(y)$. Under his definition of covering, the entire simplex is uncovered in a divide-the-dollar game. Under my definition, the vertices of the simplex are covered. For a discussion of different definitions of the covering relation and their implications in a divide-the-dollar setting, see Penn (2001).

because there is no well defined way to characterize every possible *countable* ordering of every element in the space. Furthermore, it is unclear how an amendment procedure would progress over this infinitely long agenda. To deal with this problem I have extended the definition of the Banks set over an infinite policy space:

$$S(X) = \{x^j \in X : \exists H^* \in \mathbf{H}^* \text{ with } x^j \in X^{H^*}\}.$$

This definition is in keeping with that proposed by Banks for the case of a finite policy space, and implicitly assumes that the voting process takes place in discrete time and terminates in finite time. Restricting our attention to finite externally stable chains also enables us to maintain the result that the Banks set is a subset of the uncovered set (proved in Lemma 4). It is possible to construct an infinitely long externally stable chain whose limit point is covered. Consider the chain $H = \{x^1, x^2, ...\}$, where

$$x^i = \begin{cases} (1 - (\frac{3}{4})^i, (\frac{3}{4})^i, 0) & \text{if } i \text{ is odd} \\ (1 - (\frac{3}{4})^i, 0, (\frac{3}{4})^i) & \text{if } i \text{ is even.} \end{cases}$$

The beginning of $H$ is pictured in Figure 2.3. This is an externally stable chain with limit point $(1, 0, 0)$, where $(1, 0, 0)$ is not an element of the uncovered set (proved in Proposition 3).

Moreover, because sophisticated equivalent agendas are chains, it is natural to relate the concept of an externally stable chain to that of a endogenously formed sophisticated equivalent agenda. If an agenda has been proposed that is not externally stable, then there exists a point that is weakly majority preferred to every element of the agenda.

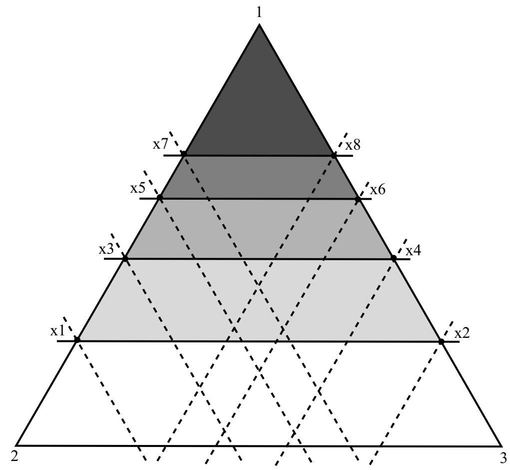



Figure 2.3: An infinitely long externally stable chain whose limit point is covered.

### Solution Concept and Extensive Form

We are now ready to specify the extensive form of the game. Using the definition provided by Osborne and Rubinstein (1994), we are considering an infinite game of perfect information with a possibly infinite horizon. The game is one of perfect information because players are perfectly informed of all past events when taking actions. Denote by $\mathbf{A}^t$ the set of all possible agendas of length $t$, with generic element $A^t$. Define $A^0 = \emptyset$ and $\mathbf{A}^\infty$ to be the set of infinitely long sequences of distinct elements. Let $\mathbf{A} = \bigcup_{n\in\mathbb{N}} \mathbf{A}^n \cup \mathbf{A}^\infty$, with generic element $A$. Let $\boldsymbol{\pi} = \bigcup_{n\in\mathbb{N}}\{1,2,3\}^n$, with generic element $\pi$. At a $\pi \in \boldsymbol{\pi}$, the extensive form of the game can be represented by the triple $< I, \mathbf{A}, \pi >$, where

- $I$ equals the set of players defined in Section 2.2.1.
- $\mathbf{A}$ equals the set of (possibly infinite) histories.
  - A history, $A$, is *terminal* if $A \in \mathbf{A}^\infty$ or $A \in \mathbf{H}^*$, the set of externally stable chains.
- $\pi$ assigns to each nonterminal history a player in $I$.

It is assumed that players vote sophisticatedly once a terminal node has been reached, and this assumption implies that for the game defined above and given a terminal history, $A$, players receive payoffs equal to zero if $A$ is infinitely long, and equal to the sophisticated voting outcome of agenda $A$ if $A \in \mathbf{H}^*$.

The solution concept employed here is maximin-Nash.[5] A maximin-Nash equilibrium is a strategy profile such that no player can guarantee himself a higher minimum possible utility by utilizing a different strategy. Throughout, I will abuse notation by referring to an agenda $A^t$ as a set, when it is actually a sequence. Thus, $x^k \in A^t$ is the $k^{th}$ element of agenda $A^t$, and $X \setminus A^t$ is the set of all alternatives minus the alternatives that are part of the sequence $A^t$. A *strategy* is a sequence of functions, $x_i^t(\cdot)_{t=1}^{\infty}$, such that $x_i^t : A^{t-1} \to X \setminus A^{t-1}$. Since players are chosen randomly to propose items to the agenda, let $FH(A^t)$ be the set of finite reachable maximal chains in periods $t+1$, $t+2$, ... under strategies $x_i^t$. Then, for some $\pi \in \boldsymbol{\pi}$, a given $H^* \in FH(A^t)$ would look like

$$H^* = (A^t, x_i^{t+1}(A^t), x_{\pi(t+2)}^{t+2}(A^t, x_i^{t+1}(A^t)), ...).$$

Let $FM(A^t)$ denote the corresponding set of sophisticated voting outcomes. Finally, let $Y(A) = \{x \in X : (A, x, A^q) \in \mathbf{H}^* \Rightarrow (A, A^q) \in \mathbf{H}^*, \text{ all } q, A^q \in \mathbf{A}^q\}$.

A *maximin-Nash equilibrium* at a given $\pi \in \boldsymbol{\pi}$ is defined as a profile of strategies, $\hat{x}_i^t(\cdot)$, such that for each $i \in N$ and $t \in \mathbb{N}$ and for all $\hat{A}^{t-1} = (x^1, ..., x^{t-1}) \in \mathbf{A^{t-1}}$ such that

$$x^k \in \hat{A}^{t-1} \Rightarrow x^k = \hat{x}_{\pi(k)}^k(x^1, ..., x^{k-1}),$$

- $\hat{x}_i^t(\hat{A}^{t-1}) = \underset{x \in F_{\hat{A}^{t-1}}}{\operatorname{argmax}} \left\{ \min_{y \in FM(\hat{A}^{t-1}, x)} u_i(y) \right\},$

  where $F_{\hat{A}^{t-1}} = \left( \bigcap_{x \in \hat{A}^{t-1}} \overline{P}(x) \right) \setminus Y(\hat{A}^{t-1})$ .

Thus, $\hat{x}$ is a maximin-Nash equilibrium if $\hat{x}_i^t$ maximizes Player $i$'s minimum possible

[5] *Stackelberg* equilibria are a subgame-perfect refinement of Nash in which players move sequentially. Although players also move sequentially in this game, I call this equilibrium maximin-Nash because I am not considering subgame perfection.

utility when agenda $\hat{A}^{t-1}$ has occurred along the equilibrium path of play. The second condition implies that the effective strategy space after a sequence of proposals has been made equals the set of policies which weakly defeat every previously proposed alternative via majority rule, and which, when appended to agenda $\hat{A}^{t-1}$, form the beginnings of some minimal-externally stable chain. Thus, there exists an $H^M \in \mathbf{H^M}$ such that $H^M = (\hat{A}^{t-1}, \hat{x}_i^t(\hat{A}^{t-1}), ...)$. This implies that players will not propose superfluous alternatives, or alternatives which are not the outcome of the game, and could not change the outcome of the game under any circumstance.

Given this definition, the maximin operation is defined only along the path of play. Because of this, the equilibrium concept used here is not subgame-perfect, and thus not equivalent to the *minimax-Stackelberg* equilibrium concept used by Banks and Gasmi.[6] Although the equilibrium concept I use is quite weak, I will show that it yields a unique set of equilibrium predictions. Furthermore, any minimax-Stackelberg equilibrium must also be a maximin-Nash equilibrium, and so the results of Section 2.4 demonstrate that if a subgame-perfect equilibrium exists, then it must also yield the same predictions.[7]

It is not clear that the equilibrium concept defined above is the only, or even the best, concept to use in the context of this game. I use it for three reasons. First, because one of the main motivations of this chapter is to test whether the Banks-Gasmi M* result remains robust when the assumption of a three-item agenda is relaxed, an equilibrium concept is needed that is consistent with the concept used by Banks and Gasmi. Maximin-Nash is a weaker equilibrium concept than the minimax-Stackelberg concept that they use, and I use it here solely for purposes of tractability. However, since the game yields unique Nash predictions, we can infer that if a minimax-Stackelberg equilibrium exists, then it must yield the same predictions as those generated here.

[6] Although Banks and Gasmi call their equilibrium concept *minimax*-Stackelberg, players are actually *maximin*-utility maximizers. To be in keeping with their terminology however, I will also refer to the concept as minimax-Stackelberg.

[7] As a helpful referee pointed out, proving the existence of a subgame-perfect equilibrium in this context poses a real challenge, because the action spaces of the players are infinite and non-compact, and a player's actions in one period restrict the actions of the next player in a deterministic way.

Second, when the order of the players is unknown Banks and Gasmi use a maximin framework, as opposed to an expected utility framework, for tractability purposes. The problem becomes extremely difficult when a maximin assumption is not made. Furthermore, in the environment considered here it is unclear whether a subgame-perfect equilibrium even exists.

Third, a maximin framework simply implies a different behavioral assumption than does a traditional expected utility framework. Whether this behavioral assumption is correct or not is an empirical question. However, considering that the game is potentially infinite-horizon, I argue that a simplifying assumption such as maximin is reasonable. More work remains to be done on finding other solution concepts for such games.

## 2.3 Results

The following lemmas and propositions are used in the proofs of the two main results of this chapter. The first result, Theorem 1, is that the Banks set equals the uncovered set in a divide-the-dollar game with linear preferences. The second result, Theorem 2, is that the unique set of maximin-Nash equilibrium outcomes that arise from the game described in Section 2.2.2 is the simple von Neumann-Morgenstern stable set.

First, two definitions.

*Definition:* A *convex component*, $C$ of a set $\overline{P}(x)$ is a maximal convex subset of $\overline{P}(x)$ with respect to inclusion.

*Definition:* A convex component $C$ *dominates* a convex component $C'$ if there exists an $x \in C$ such that $x \in \overline{P}(y)$ for all $y \in C'$.[8]

Lemma 1 proves that, given a point $x \in \Delta$, one convex component of the set $\overline{P}(x)$ dominates the other convex components of $\overline{P}(x)$. This fact is needed in the proof of Theo-

[8]Note that the dominance relation is weak; two components can dominate each other.

rem 1—it is used to show that, given any point $x$ in the uncovered set, an externally stable chain can be constructed with $x$ in its set of maximal elements.

**Lemma 1** *For any $x \in \Delta$, $\overline{P}(x)$ is the union of one, two, or three convex components, and one of these components, $C_i$, dominates all of the others.*

*Proof:* First, if $x = (1, 0, 0)$ then, because there does not exist a $y = (y_1, y_2, y_3) \in \Delta$ such that $y_i < 0$ for some $i$, $\overline{P}(x) = \Delta$, which is the unique convex component of $\overline{P}(x)$. The remaining cases will be handled by the same argument. For example, if $x = (x_1, x_2, x_3)$ with $x_1, x_2 > 0$ and $x_3 = 0$, then $\overline{P}(x)$ consists of the union of two convex components, $C_1$ and $C_2$, where

$$\begin{aligned} C_1 &= \{w \in \Delta : w_1 \geq x_1 \text{ and } w_3 \geq x_3\} \\ C_2 &= \{w \in \Delta : w_2 \geq x_2 \text{ and } w_3 \geq x_3\}. \end{aligned}$$

$C_1$ and $C_2$ are both clearly convex, and intersect only at the point $x$. Similarly, if $x = (x_1, x_2, x_3)$ with $x_i > 0$ for all $i$, then $\overline{P}(x)$ consists of three convex components, where $C_1$ and $C_2$ are as above, and

$$C_3 = \{w \in \Delta : w_1 \geq x_1 \text{ and } w_2 \geq x_2\}.$$

Consider Figure 2.4. First suppose that $x = (x_1, x_2, 0)$, so that $\overline{P}(x)$ consists of two convex components. Also suppose, without loss of generality, that $x_1 \leq x_2$. Let $C_1$ and $C_2$ be defined as above, and consider the point $x' = (x_1, 0, 1 - x_1) \in C_1$. Then $u_1(x') \geq u_1(x'')$, for all $x'' \in C_2$. However, since the *lowest* payoff Player 2 can receive in region $C_2$ is $x_2$, it follows that the *highest* payoff Player 3 can receive in region $C_2$ is $1 - x_2$. However, since $x_1 \leq x_2$, it follows that $1 - x_1 \geq x_3''$, for all $x'' \in C_2$, which implies that $u_3(x') > u_3(x'')$. Thus, the point $x' \in \overline{P}(x'')$, for every $x'' \in C_2$.

Now consider Figure 2.5. Suppose that $x = (x_1, x_2, x_3)$ so that $\overline{P}(x)$ consists of the union of three convex components, as defined above. Suppose that $x_1 \leq x_2 \leq x_3$. Then by

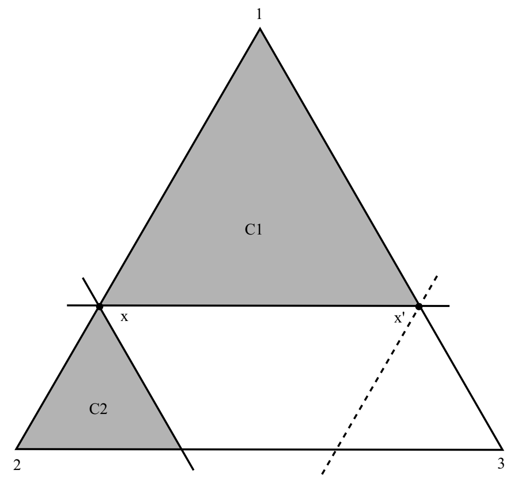



Figure 2.4: Convex component $C_1$ dominates $C_2$ with the point $x'$.

precisely the same argument as above, the region $C_3$ dominates region $C_2$ with the point $x' = (x_1, 1 - x_1, 0)$ and $C_3$ dominates $C_1$ with the point $x'' = (1 - x_2, x_2, 0)$. $\square$

Together, Lemma 2 and Proposition 1 prove that, given any chain $H$, a finite externally stable chain can be constructed by adding points to the end of $H$. This result is needed in the proof of Theorem 2, to show that the only restriction imposed by sophisticated voting is that each amendment must be weakly majority preferred to all prior amendments. In other words, a player can never propose a point that belongs only to chains that are infinitely long; any chain can be ended.

**Lemma 2** *Given any chain* $(x^1, ..., x^t)$ *such that* $\overline{P}(x^1) \cap \cdots \cap \overline{P}(x^t)$ *is convex, a finite externally stable chain can be constructed with beginning elements* $(x^1, ..., x^t)$.

*Proof:* Let $\overline{\mathcal{P}} = \overline{P}(x^1) \cap ... \cap \overline{P}(x^t)$, and assume that $\overline{\mathcal{P}}$ is convex. If $\overline{\mathcal{P}} \setminus \{x^1, ..., x^t\} = \emptyset$ or a singleton (call it $x^{t+1}$), this proof is trivial because either $(x^1, ..., x^t)$ or $(x^1, ..., x^t, x^{t+1})$ is itself an externally stable chain. Suppose then, that $\overline{\mathcal{P}} \setminus \{x^1, ..., x^t\}$ contains multiple elements. Then $\overline{\mathcal{P}}$ is a triangle with sides parallel to the edges of $\Delta$. To see this, note that for any $x$, $\overline{P}(x)$ is a collection of equilateral triangles, with sides represented by the indifference curves of the players. The indifference curves of any given player are parallel

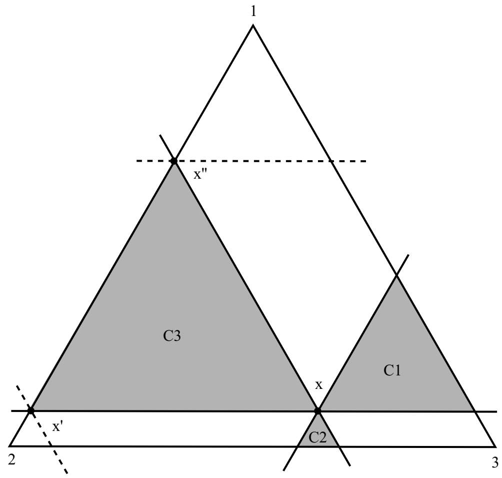



Figure 2.5: $C_3$ dominates $C_2$ with $x'$ and $C_1$ with $x''$.

to each other. Clearly the intersection of two equilateral triangles with parallel sides is an equilateral triangle or an isolated point. Thus, $\overline{\mathcal{P}}$ is either empty, or a collection of equilateral triangles (with edges parallel to the simplex) and (or) isolated points. Since I have assumed that $\overline{\mathcal{P}}$ is convex, and is neither empty nor an isolated point, it follows that $\overline{\mathcal{P}}$ is a triangle with edges parallel to the edges of the simplex.

The boundary of $\overline{\mathcal{P}}$ is defined as the set of points $y^i \in \overline{\mathcal{P}}$ that are not at the center of any $\epsilon$-ball $B_{y^i} \subset \overline{\mathcal{P}}$. Thus, $\partial\overline{\mathcal{P}} = \{y \in \overline{\mathcal{P}} : u_k(y) \leq u_k(x)$, for all $x \in \overline{\mathcal{P}}$, and for some $k \in \{1, 2, 3\}\}$. In words, the boundary of $\overline{\mathcal{P}}$ consists of the set of points that make some player worse off than any other point in $\overline{\mathcal{P}}$.

For any point $y$ in $\partial\overline{\mathcal{P}}$, and for some $k \in \{1, 2, 3\}$, then for all $x \in \overline{\mathcal{P}}$, $u_k(y) \leq u_k(x)$. Define three distinct points $(y^1, y^2, y^3) \subset \partial\overline{\mathcal{P}}$ as follows: let $y^i \in \overline{P}(y^j)$, $\forall i, j$, where $y^1_1 = y^2_1$, $y^3_1 = \min\limits_{x \in \overline{\mathcal{P}}} u_1(x)$, $y^2_2 = y^3_2$, $y^1_2 = \min\limits_{x \in \overline{\mathcal{P}}} u_2(x)$, $y^1_3 = y^3_3$ and $y^2_3 = \min\limits_{x \in \overline{\mathcal{P}}} u_3(x)$. We also know that $\sum\limits_{k=1}^{3} y^i_k = 1$ for $i = 1, 2, 3$. Thus we have a perfectly identified system of nine equations, so the points $y^1, y^2, y^3$ exist and are unique. More specifically, letting $m_i = \min\limits_{x \in \overline{P}} u_i(x)$, then $y^1 = (\frac{1-m_2-m_3+m_1}{2}, m_2, \frac{1-m_1-m_2+m_3}{2})$, $y^2 = (\frac{1-m_2-m_3+m_1}{2}, \frac{1-m_1-m_3+m_2}{2}, m_3)$, and $y^3 = (m_1, \frac{1-m_1-m_3+m_2}{2}, \frac{1-m_1-m_2+m_3}{2})$.

I will now show that $H = (x^1, ..., x^t, y^1, y^2, y^3)$ is an externally stable chain, i.e., $\overline{\mathcal{P}} \subset$

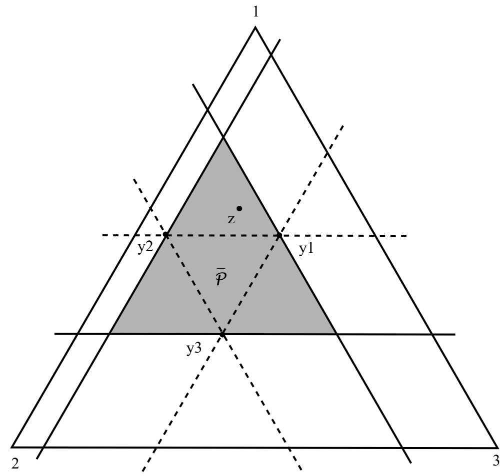



Figure 2.6: Ending an externally stable chain with a "mini stable set."

$P^{-1}(y^1) \cup P^{-1}(y^2) \cup P^{-1}(y^3) \setminus \{y^1, y^2, y^3\}$. To see this, consider Figure 2.6. Without loss of generality, choose a point $z \neq y^1, y^2$ or $y^3$ such that $z \in \overline{\mathcal{P}}$ and $z \in \overline{P}(y^1) \cap \overline{P}(y^2)$. By the definition of $\overline{\mathcal{P}}$ and the points $y^1$ and $y^2$, it follows that $m_2 \leq z_2 < \frac{1-m_1-m_3+m_2}{2}$ and $m_3 \leq z_3 < \frac{1-m_1-m_2+m_3}{2}$. Since $y^3 = (m_1, \frac{1-m_1-m_3+m_2}{2}, \frac{1-m_1-m_2+m_3}{2})$, it is apparent that Players 2 and 3 strictly prefer $y^3$ to $z$, and so $z \in P^{-1}(y^3)$. Consequently, $\overline{\mathcal{P}} \subset P^{-1}(y^1) \cup P^{-1}(y^2) \cup P^{-1}(y^3) \setminus \{y^1, y^2, y^3\}$.

It follows that the chain $H = (x^1, ..., x^t, y^1, y^2, y^3)$ is externally stable. $\square$

**Proposition 1** *Given any chain $(x^1, ..., x^t)$, a finite externally stable chain can be constructed with starting elements $(x^1, ..., x^t)$.*

*Proof:* Let $\overline{\mathcal{P}} = \overline{P}(x^1) \cap \cdots \cap \overline{P}(x^t)$. For any $x \in \Delta \setminus \{(1,0,0), (0,1,0), (0,0,1)\}$, $\overline{P}(x)$ is composed of either two or three convex components, each separated by the indifference curves of two out of three players, and connected only at the point $x$. More formally, the phrase "separated by the indifference curves of two players" implies that for *any* two convex components $C', C'' \subset \overline{P}(x)$ and any points $y \in C'$ and $z \in C''$, there exist two players, $i$ and $j$, such that for all $y$ and $z$, $u_i(y) \geq u_i(z)$ and $u_j(z) \geq u_j(y)$, with equality holding only at $y = z = x$. Thus, $\overline{\mathcal{P}} = C_1 \cup \cdots \cup C_n$, where $C_i$ is a convex component.[9] For

[9] By the discussion in the first paragraph of the proof of Lemma 2, we also know that every $C_i$ is either an isolated point or a triangle with sides parallel to the edges of the simplex.

any two $C', C'' \in \overline{\mathcal{P}}$, $C'$ and $C''$ are separated by indifference curves of at least two players, and are either disjoint or connected at a single point $x^i \in \{x^1, ..., x^t\}$. Thus, $|C' \cap C''| \leq 1$.

I will prove this proposition by constructing an algorithm for finding a sequence of points $y^1, ..., y^n$ such that $(x^1, ..., x^t, y^1, ..., y^n)$ is a chain and $\overline{\mathcal{P}} \cap \overline{P}(y^1) \cap \cdots \cap \overline{P}(y^n)$ is convex. Then, by Lemma 2, there exists a "mini stable set" that ends the chain. Two intermediary steps are needed.

Step 1: First I will show that for any two convex components $C', C'' \subset \overline{\mathcal{P}}$, if the point $v$ is a vertex of $C'$, then $\overline{P}(v) \cap C''$ is convex.

*Proof of Step 1:* By Lemma 1 we know that $\overline{P}(v)$ is the union of one, two, or three convex components. First suppose that $C'' \neq C'$, and suppose that $C''$ has a nonempty intersection with two of the convex components of $\overline{P}(v)$. This implies that *every* player prefers some point in $C''$ to the point $v \in C'$. This contradicts the fact that $C'$ and $C''$ are separated by the indifference curves of at least two players. Thus only one convex component of $\overline{P}(v)$ positively intersects $C''$. Since $C''$ is itself convex, it follows that $\overline{P}(v) \cap C''$ is convex.

Now suppose that $C' = C''$. Every convex component $C$ can be defined by the minimum utility each player can attain in that component, since for all $x \in C$, and for some $k \in \{1, 2, 3\}$, $\partial\overline{C} = \{y \in C : u_k(y) \leq u_k(x)\}$. Let the minimum payoff that Player $i$ can receive in component $C$ be denoted $m_i^C$. It follows that a vertex of $C$ can be defined as any point in $C$ where two players, $i$ and $j$ receive payoffs of $m_i^C$ and $m_j^C$, respectively. This implies that two players weakly prefer every point in $C'$ to the point $v$. Thus, $\overline{P}(v) \cap C' = C'$, which is convex. This proves Step 1.

Step 2: Next, I will show that for any two convex components, $C', C'' \subset \overline{\mathcal{P}}$, either $C'$ dominates $C''$ with one of its vertices, or $C''$ dominates $C'$ with one of its vertices, or both.

*Proof of Step 2:* $C'$ and $C''$ separated by the indifference curves of at least two players implies that one player (call him Player 1) prefers every point in $C'$ to every point in $C''$ and another player (call him Player 2) prefers every point in $C''$ to every point in $C'$. Let

$v' = (m_1^{C'}, m_2^{C'}, 1 - m_1^{C'} - m_2^{C'})$ be the point in $C'$ that leaves Player 3 best off. Define $v''$ similarly and note that these points are distinct and are also vertices of $C'$ and $C''$, respectively. Then $\underset{x \in \{v', v''\}}{\operatorname{argmax}}\, u_3(x)$ dominates the component that it is not an element of. This proves Step 2.

Let $n$ be the number of convex components in $\overline{\mathcal{P}}$. Given these two results, we can now construct a sequence of points $y^1, ..., y^n$ such that $(x^1, ..., x^t, y^1, ..., y^n)$ is a chain and $\overline{\mathcal{P}} \cap \overline{P}(y^1) \cap \cdots \cap \overline{P}(y^n)$ is convex. Let $C_1, ..., C_n$ be any ordering of the convex components of $\overline{\mathcal{P}}$, and to simplify the notation I will first suppose that no two components in $\overline{\mathcal{P}}$ simultaneously dominate each other.

At each stage in the construction, we will compare two components via the dominance relation. Let $j(C_m, C_n)$ and $k(C_m, C_n)$ denote the two players whose indifference curves separate the components being compared ($C_m$ and $C_n$ in this case), and let $l(C_m, C_n)$ denote the third player. Assume $j < k$. Then the sequence can be constructed as

$$y^1 = \underset{x \in C_1 \cup C_2}{\operatorname{argmax}} \{x_{l(C_1,C_2)} \in \{1 - m^{C_1}_{j(C_1,C_2)} - m^{C_1}_{k(C_1,C_2)}, 1 - m^{C_2}_{j(C_1,C_2)} - m^{C_2}_{k(C_1,C_2)}\}\}$$

and for $2 \leq i \leq n$,

$$\begin{aligned} y^i = \; & \underset{x \in S}{\operatorname{argmax}} \{x_{l_o} \in \{1 - m_{j_o}^{C_{y^{i-1}} \cap \overline{\mathcal{P}_i}} - m_{k_o}^{C_{y^{i-1}} \cap \overline{\mathcal{P}_i}}, 1 - m_{j_o}^{C_{i+1} \cap \overline{\mathcal{P}_i}} - m_{k_o}^{C_{i+1} \cap \overline{\mathcal{P}_i}}\}\} \\ & \text{where } \overline{\mathcal{P}_i} = \bigcap_{q=1}^{i-1} \overline{P}(y^q), \\ & j_o = j(C_{y^{i-1}} \cap \overline{\mathcal{P}_i}, C_{i+1} \cap \overline{\mathcal{P}_i}) \text{ (with } k_o \text{ and } l_o \text{ defined similarly)}, \\ & C_{y^{i-1}} = \{C \subset \overline{\mathcal{P}} : y^{i-1} \in C\}, \\ & \text{and } S = \{C_{y^{i-1}} \cup C_{i+1}\} \cap \overline{\mathcal{P}_i}. \end{aligned}$$

In words, first $C_1$ and $C_2$ are compared via the dominance relation, and by Step 2 we know that one of these components will dominate the other with one of its vertices, $y^1$.

Then the dominant component intersected with $\overline{P}(y^1)$ is compared to $C_3$ intersected with $\overline{P}(y^1)$. By Step 1 we know that both of these sets are convex, and hence are themselves convex components. By Step 2, one will dominate the other with vertex $y^2$. This process is repeated $n$ times, where $n$ is finite because there are a finite number of convex components. Then the constructed sequence is transitive with respect to the weak majority preference relation because each $y^i$ is chosen from the set $S$, which is restricted to $\overline{\mathcal{P}_i} = \bigcap_{t=1}^{i-1} \overline{P}(y^t)$. Because convex components are sequentially dominated (i.e., eliminated) as the sequence is constructed, a single component will remain once the process is complete. By Lemma 2, the chain can be ended by constructing a "mini stable set" on this remaining component.

In the case that two components dominate each other, then for some $i$ two distinct points will satisfy $\underset{x \in S}{\operatorname{argmax}}\{x_l \in \{1 - m_j^{C_{y^{i-1}} \cap \overline{\mathcal{P}_i}} - m_k^{C_{y^{i-1}} \cap \overline{\mathcal{P}_i}}, 1 - m_j^{C_{i+1} \cap \overline{\mathcal{P}_i}} - m_k^{C_{i+1} \cap \overline{\mathcal{P}_i}}\}\}$. To extend the proof to this case, simply propose both points as $y^i$ and $y^{i+1}$ and continue as before. □

Proposition 2 shows that in a divide-the-dollar game with linear preferences, the uncovered set equals the unit simplex minus its vertices. This result is used in the proof of Theorem 1 to precisely define the uncovered set, and is used indirectly in the proof of Theorem 2, to precisely define the Banks set.

**Proposition 2** *The uncovered set $U(\Delta)$ equals $\Delta \setminus \{(1,0,0),(0,1,0),(0,0,1)\}$*

*Proof: $\Delta \setminus \{(1,0,0),(0,1,0),(0,0,1)\} \subset U(\Delta)$:* proof by contradiction. Let $x = (x_1, x_2, x_3) \in \Delta \setminus \{(1,0,0), (0,1,0), (0,0,1)\}$, and assume that there exists $y = (y_1, y_2, y_3) \in \Delta$ such that $y \mathcal{C}_\Delta x$. Then, without loss of generality, we can assume that $y_1 > x_1$, $y_2 > x_2$, $y_3 < x_3$, $x_1 > 0$, and $x_3 > 0$. Choose some $\epsilon > 0$ such that $x_3 > y_3 + \epsilon$ and $\epsilon < y_1$. Consider the point $y' = (0, 1 - y_3 - \epsilon, y_3 + \epsilon)$. It is clear that $y' \in \Delta$; $\sum_{i=1,2,3} y'_i = 1$ and $1 - y_3 - \epsilon > 1 - x_3 > 0$. Then $x_1 > 0$ implies that $u_1(x) > u_1(y')$ and $x_3 > y_3 + \epsilon$ implies that $u_3(x) > u_3(y')$, which together imply that $y' \in P^{-1}(x)$. Similarly, $\epsilon < y_1$ implies that $1 - y_3 - \epsilon > 1 - y_3 - y_1 = y_2$, which in turn implies that $u_2(y') > u_2(y)$, and $y_3 + \epsilon > y_3$ implies that $u_3(y') > u_3(y)$. It now follows that $y \in P^{-1}(y')$, which contradicts the fact that $y \mathcal{C}_\Delta x$.

$U(\Delta) \subset \Delta \setminus \{(1,0,0),(0,1,0),(0,0,1)\}$ *:* Let $x = (1,0,0)$. It is clear that there does not exist a $y \in \Delta$ such that $y \in P^{-1}(x)$ because there does not exist a $y = (y_1, y_2, y_3) \in \Delta$ such that $y_i < 0$ for some $i$. Thus, $P^{-1}(x) = \emptyset$. Consider the point $y' = (0, \frac{1}{2}, \frac{1}{2})$. Then $u_2(y') > u_2(x)$ and $u_3(y') > u_3(x)$ imply $x \in P^{-1}(y')$. Also, $\emptyset = P^{-1}(x) \subset P^{-1}(y')$. Thus, $y' \mathcal{C}_\Delta x$. By symmetry it follows that $U(\Delta) \subset \Delta \setminus \{(1,0,0),(0,1,0),(0,0,1)\}$. $\square$

Lemma 3 extends Banks' 1985 proof that the Banks set is a subset of the uncovered set to the setting of an infinite alternative space. Using this fact, the proof of Theorem 1 is simplified to showing that the uncovered set is also a subset of the Banks set.

**Lemma 3** *The Banks set is a subset of the uncovered set (*$S(\Delta) \subseteq U(\Delta)$*).*

*Proof:* Since we know that $U(\Delta) = \Delta \setminus \{(1,0,0),(0,1,0),(0,0,1)\}$, this statement can be proved by showing that $(1,0,0),(0,1,0)$, and $(0,0,1) \notin S(\Delta)$. Suppose the contrary, that there exists a finite externally stable chain, $H^*$ with $x^{H^*}$ being a maximal element, and without loss of generality let $x^{H^*} = (1,0,0)$. Since $P^{-1}(x^{H^*}) = \emptyset$, it must be the case that for all $x^i \in H^*$, either $x_2^i$ or $x_3^i = 0$. This implies that the entire chain must lie on two edges of the simplex. The following paragraphs demonstrate that if this is the case, then there must exist an $\epsilon$-neighborhood around $(1,0,0)$ that no point on the chain can defeat.

Let $x^m = \underset{x^i \in H^* \setminus \{x^{H^*}\}}{\operatorname{argmax}} \{x_1^i\}$. In words, $x^m$ is the point on the chain not equal to $(1,0,0)$ that maximizes Player 1's payoff. Without loss of generality, let $x^m = (x_1^m, x_2^m, 0)$. Let $z = (\frac{x_1^m+1}{2}, 1 - \frac{x_1^m+1}{2}, 0)$. It is clear that $z \notin H$, since $z \neq x^m$ and $z$ gives Player 1 a higher payoff than any other point in the set $\{H^* \setminus \{x^{H^*}\}\}$.

I will now show that $z \in \overline{P}(x^i)$ for all $x^i \in H^*$. Let $x^i \in H^* \setminus \{x^{H^*}\}$. Either $x_2^i$ or $x_3^i = 0$. Suppose $x_2^i = 0$. Then $z_2 > x_2^i$ and $z_1 > x_1^i$ imply that $z \in P(x^i)$. Now suppose $x_3^i = 0$. Then $z_3 = x_3^i$ and $z_1 > x_1^i$ imply that $z \in \overline{P}(x^i)$. Finally, $z_2 > x_2^{H^*}$ and $z_3 = x_3^{H^*}$ imply that $z \in \overline{P}(x^{H^*})$. Thus the chain $H^*$ is not externally stable, a contradiction. $\square$

**Theorem 1** *The Banks set equals the uncovered set (*$S(\Delta) = U(\Delta)$*).*

*Proof:* By Lemma 3 we know $S(\Delta) \subseteq U(\Delta)$. I will prove that $U(\Delta) \subseteq S(\Delta)$ by showing that for any $x^* = (x_1^*, x_2^*, x_3^*) \in U(\Delta)$, a finite externally stable chain can be constructed with $x^*$ in its set of maximal elements. Consider Figure 2.7.

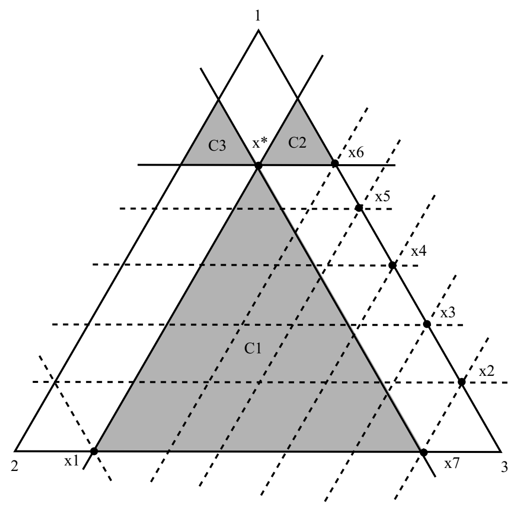



Figure 2.7: Constructing an externally stable chain with maximal element $x^* \in U(\Delta)$.

By Proposition 2 we can assume, without loss of generality, that $x_1^* > 0$ and $x_2^* > 0$. By Lemma 2 we know that $\overline{P}(x^*)$ is the union of either two or three convex components, and that one of these components dominates the others. Call this dominant component $C_1$, and assume that $C_1 = \{x \in \Delta : u_2(x) \geq u_2(x^*) \text{ and } u_3(x) \geq u_3(x^*)\}$ (which implies that $x_1^* \geq x_2^*$ and $x_1^* \geq x_3^*$). Let $C_2 = \{x \in \Delta : u_1(x) \geq u_1(x^*) \text{ and } u_3(x) \geq u_3(x^*)\}$ and let $C_3 = \{x \in \Delta : u_1(x) \geq u_1(x^*) \text{ and } u_2(x) \geq u_2(x^*)\}$.

Choose an arbitrary $\alpha \in (0, 1)$, and fix $\epsilon$ such that $\epsilon = \alpha \cdot x_2^*$.

*Step 1:* Let the first point in our chain be $x^1 = (0, 1 - x_3^*, x_3^*)$.

*Step 2:* Let the second point in our chain be $x^2 = (x_2^*, 0, 1 - x_2^*)$.

*Step 3$(i)$, $i \geq 3$:* If $i \cdot x_2^* < x_1^*$, let $x^i = (i \cdot x_2^* - \epsilon, 0, 1 - i \cdot x_2^* + \epsilon)$ and continue to Step $3(i+1)$. Otherwise, if $i \cdot x_2^* \geq x_1^*$, then let $x^i = (x_1^*, 0, 1 - x_1^*)$, and continue to Step 4.

*Step 4:* Let $j$ be the first integer such that $i \cdot x_2^* \geq x_1^*$. If $x_1^* > x_3^*$, the chain can then be ended with the points $x^{j+1} = (0, x_2^*, 1 - x_2^*)$ and $x^{j+2} = x^*$. If $x_1^* = x_3^*$, the chain can be ended with the points $x^{j+1} = (0, x_2^*, 1 - x_2^*)$, $x^{j+2} = (x_1^*, 1 - x_1^*, 0)$, and $x^{j+3} = x^*$.

To see that this sequence of points forms an externally stable chain, first we must check that there does not exist a $y \in \Delta$ such that $y \in \overline{P}(x^1) \cap ... \cap \overline{P}(x^*) \setminus \{x^1, ..., x^*\}$. Suppose that such a $y$ did exist. Then $y \in \overline{P}(x^*)$ implies that either $[x_1^* \leq y_1$ and $x_2^* \leq y_2]$ or that $[x_1^* \leq y_1$ and $x_3^* \leq y_3]$ or that $[x_2^* \leq y_2$ and $x_3^* \leq y_3]$. The reader may find it helpful to refer to Figure 2.7.

Suppose the first case (i.e., $y \in C_3$). Since $y_2 \leq 1 - y_1 \leq 1 - x_1^* \leq 1 - x_3^*$, it follows that $u_2(x^1) \geq u_2(y)$, with equality possibly holding only if $y = (x_1^*, 1 - x_1^*, 0)$ and $x_3^* = x_1^*$. However, in this case, $y = x^{j+2}$. Similarly, $u_3(x^1) \geq u_3(y)$ with equality holding only at $x^*$. Thus, Players 2 and 3 strictly prefer $x^1$ to $y$, so that $x^1 \in P(y)$.

Suppose the second case ($y \in C_2$). Then $u_2(x^{j+1}) \geq u_2(y)$, with equality holding only at $x^*$ and $u_3(x^{j+1}) \geq u_3(y)$ with equality holding only if $y = (x_1^*, 0, 1 - x_1^*)$ and $x_2^* = x_1^*$. However, $x^j = (x_1^*, 0, 1 - x_1^*)$.

Suppose the third case ($y \in C_1$). Then $y$ is such that $y_2 \geq x_2^*$, $y_3 \geq x_3^*$, $y \neq x^*$, $y \neq (0, x_2^*, 1 - x_2^*)$, and $y \neq (0, 1 - x_3^*, x_3^*)$. Then there exists a positive integer $i$ such that $i \cdot x_2^* \leq y_1 < min\{x_1^*, (i+1) \cdot x_2^* - \epsilon\}$. Find the smallest such $i$ that satisfies these inequalities, and note that when these inequalities are satisfied, Player 1 prefers $x^{i+1}$ to $y$. If $x_1^* < (i+1) \cdot x_2^* - \epsilon$, then the smallest such $i$ satisfying these inequalities is actually $j$ from the last iteration of Step 3. In Figure 2.7, this is when $j = 6$. Then $y_3 < 1 - (i+1) \cdot x_2^* + \epsilon < 1 - x_1^*$, which implies that Players 1 and 3 strictly prefer $x^j$ to $y$. Thus $x^j \in P(y)$. If $(i+1) \cdot x_2^* - \epsilon \leq x_1^*$, then it follows that $y_3 \leq 1 - y_1 - y_2 \leq 1 - y_1 - x_2^* \leq 1 - x_2^* - i \cdot x_2^* < 1 - (i+1) \cdot x_2^* + \epsilon$. $\epsilon > 0$ implies that Player 3 prefers $x^{i+1}$ to $y$. Thus Players 1 and 3 prefer $x^{i+1}$ to $y$, and so $x^{i+1} \in P(y)$.

Last, we must check that $\overline{P}$ restricted to our sequence is transitive. First note that for all $i = 2, ..., j$, $x^i$ weakly beats $x^{i-1}, ..., x^2$, because Player 2 is indifferent between all of these alternatives. It is clear that $u_1(x^2) > u_1(x^1) = 0$. To see that $u_3(x^2) > u_3(x^1)$, note that since $x_2^*$ is the worst payoff Player 2 can receive from any point in region $C_1$, it follows that $1 - x_2^*$ is the payoff from a point in region $C_1$ that leaves Player 3 best off. Thus, $x^2 \in P(x^1)$. Similarly, $\{x^3, ..., x^j\} \in P(x^1)$. It is clear that Player 1 prefers these allocations to $x^1$ because $u_1(x^1) = 0$. To see why Player 3 prefers these allocations to $x^1$, note that $x_3^1 = x_3^*$ and that $x_3^j = 1 - x_1^*$. Since $u_3(x^j) < u_3(x^i)$ for all $i \in \{2, ..., j-1\}$ and $1 - x_1^* = x_3^* + x_2^* > x_3^*$, it follows that $\{x^3, ..., x^j\} \in P(x^1)$. Next, $x^{j+1} \in \overline{P}(x^1)$ since $u_1(x^{j+1}) = u_1(x^1) = 0$ and $u_3(x^{j+1}) = 1 - x_2^* = x_1^* + x_3^* > x_3^* = u_3(x^1)$. Also, $x^{j+1} \in \overline{P}(x^2) \cap ... \cap \overline{P}(x^j)$, because $u_2(x^{j+1}) > 0 = u_2(x^i)$ for $i \in \{2, ..., j\}$ and $u_3(x^{j+1}) = 1 - x_2^* \geq u_3(x^i)$ for $i \in \{2, ..., j\}$. If $x_1^* = x_3^* > x_2^*$, then $x^{j+2} = (x_1^*, 1 - x_1^*, 0)$. Since no point in our sequence so far gives Player 1 a better payoff than $x_1^*$, $u_1(x^{j+2}) \geq u_1(x^i)$, for all $i < j+2$. Similarly, the only points in our sequence so far that give player 2 a strictly positive payoff are $x^1$ and $x^{j+1}$. However, $u_2(x^1) = 1 - x_3^* = 1 - x_1^* = u_2(x^{j+2})$ and $u_2(x^{j+1}) = x_2^* < 1 - x_1^*$. Thus, $x^{j+2} \in \overline{P}(x^i)$, for all $i < j+2$. Finally, $x^*$ is weakly majority preferred to every previously proposed point because no point yet proposed gives either Player 1 or Player 2 higher utility than $x^*$. Thus, $\overline{P}$ restricted to our sequence is transitive. □

Banks and Bordes (1988) show that the Banks set (defined analogously, but for a finite policy space) is a subset of a different definition of the uncovered set than used here; namely when $xCy \Leftrightarrow xPy$ and $\forall z \in X \setminus \{x, y\}, y\overline{P}z \Rightarrow x\overline{P}z$. The authors also demonstrate that the two different definitions of the uncovered set may not be the same in finite spaces. In this setting the uncovered set equals the entire policy space under the definition used by Banks and Bordes. Interestingly, in this continuous space, the Banks set is a subset of the uncovered set under both definitions. [10]

## 2.4 Equilibrium

The previous section demonstrated that in a distributive setting the Banks set, or the set of finite agendas that admit no further possible amendments, coincides with the uncovered set, and has full measure in the policy space $\Delta$. Thus, in this setting the concept of the Banks set does not reduce the set of outcomes that are feasible in an amendment game. This section demonstrates that imposing a further restriction that individuals play maximin-Nash

[10] I thank an anonymous referee for pointing this out.

strategies reduces the set of feasible outcomes to a finite subset of the Banks set.

The following theorem provides a set of solutions to the three-player, N-amendment game described in Section 2.2.2 in which players are randomly chosen to amend an agenda until no player wishes to make another proposal. Banks and Gasmi (1987) examine a similar game in which proposers are selected similarly but are only allowed a single proposal. In contrast to their universalistic result, M*, I find that the outcome to the game considered here is a point in the simple von Neumann-Morgenstern stable set, where one player is guaranteed a payoff of zero. This outcome of a minimal winning coalition is in keeping with Riker's size principle and much of the literature on endogenous agenda formation in divide-the-dollar games, such as Baron and Ferejohn (1989).

The proof of this theorem relies on the symmetry of the divide-the-dollar game and the definition of maximin-Nash equilibria, which implies that the only information known about a player is his ideal point and the effective strategy space after a sequence of proposals has been made. Thus, if the effective strategy space after a sequence of proposals has been made is perfectly symmetric about the ideal points of two players, then those players are identical up to a relabeling of their names, and are guaranteed the same interim maximum minimum payoff. The following lemma proves this fact.

If $x \in \Delta$ is such that $x = (x_1, x_2, x_3)$, let the function $\phi_{12} : \Delta \to \Delta$ be defined so that $\phi_{12}(x) = (x_2, x_1, x_3)$. Thus $\phi_{12}(x)$ permutes the first two elements of $x$. $\phi_{13}$ and $\phi_{23}$ are defined analogously.

**Lemma 4** *For any two players $i, j \in I$, if $x \in \hat{A}^{t-1}$ implies $\phi_{ij}(x) \in \hat{A}^{t-1}$, then $y \in \hat{x}_i^t(\hat{A}^{t-1})$ implies $\phi_{ij}(y) \in \hat{x}_j^t(\hat{A}^{t-1})$.*

*Proof:* Suppose not. Then there exists a $y \in \operatorname*{argmax}_{F_{\hat{A}^{t-1}}} \left\{ \min_{z \in FM(\hat{A}^{t-1}, y)} u_i(z) \right\}$ such that $\phi_{ij}(y) \notin \operatorname*{argmax}_{F_{\hat{A}^{t-1}}} \left\{ \min_{z \in FM(\hat{A}^{t-1}, \phi_{ij}(y))} u_j(z) \right\}$. Without loss of generality, let $i = 1$ and $j = 2$. We know that for all $x \in \hat{A}^{t-1}$, it follows that $\phi_{12}(x) \in \hat{A}^{t-1}$. Thus,

$$y \in F_{\hat{A}^{t-1}} \Leftrightarrow \phi_{12}(y) \in F_{\hat{A}^{t-1}},$$

because the majority preference relation is anonymous. Similarly,

$$z \in FM(\hat{A}^{t-1}, y) \Leftrightarrow \phi_{12}(z) \in FM(\hat{A}^{t-1}, \phi_{12}(y)).$$

Since $u_1(z) = u_2(\phi_{12}(z))$ for all $z \in \Delta$, we have a contradiction. Thus, $y \in \hat{x}_i^t(\hat{A}^{t-1})$ implies $\phi_{ij}(y) \in \hat{x}_j^t(\hat{A}^{t-1})$. $\Box$

The following corollary follows immediately.

**Corollary 1** *If $x \in \hat{A}^{t-1}$ implies $\phi_{ij}(x) \in \hat{A}^{t-1}$, then at time $t-1$ Players $i$ and $j$ are guaranteed the same maximin level of utility.*

**Theorem 2** *The unique set of maximin-Nash equilibrium outcomes of the game described in Section 2.2.2 is the three-element simple von Neumann-Morgenstern stable set,* $\{(0, \frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, 0, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}, 0)\}$.

*Proof:* I will prove this theorem by proving three claims. First, I will show that if $\mathcal{G}$ has been proposed first and Player 1 is selected to propose next, the outcome of the game will be either $\mathcal{H}$ or $\mathcal{I}$. Second, I will show that $\hat{x}_2^2(\mathcal{G}) = \mathcal{H}$, so that if $\mathcal{G}$ has been proposed and Player 2 is selected to propose next, he will propose $\mathcal{H}$. By Lemma 4, this claim is equivalent to the claim that $\hat{x}_3^2(\mathcal{G}) = \mathcal{I}$, so that if Player 3 is selected to propose second, he will propose $\mathcal{I}$. And third, I will show that $\hat{x}_1^1(\emptyset) = \mathcal{G}$, so that Player 1 will always propose $\mathcal{G}$ if selected to propose first.

To see that these three claims prove the theorem, first note that any externally stable chain beginning with two points in the von Neumann-Morgenstern stable set must end with the third point.[11] By this, it follows that for all $i$, $\hat{x}_i^3(\mathcal{G}, \mathcal{H}) = \mathcal{I}$, and $\hat{x}_i^3(\mathcal{G}, \mathcal{I}) = \mathcal{H}$.

**Claim 1:** If $A = (\mathcal{G}, \hat{x}_1^2(\mathcal{G}), ...)$, then $FM(A) = \{\mathcal{H}, \mathcal{I}\}$.

*Proof of Claim 1:* It is clear that $\text{argmax}_{x \in \overline{P}(\mathcal{G})}\{u_1(x)\} = \{\mathcal{H}, \mathcal{I}\}$. For all $i \in I$, $\hat{x}_i^3(\mathcal{G}, \mathcal{H}) =$

[11] For further clarification, see Figure 2.1 and the discussion of minimal externally stable chains that accompanies it.

$\mathcal{I}$ and $\hat{x}_i^3(\mathcal{G}, \mathcal{I}) = \mathcal{H}$, and so it follows that Player 1 can *guarantee* himself a payoff of $\frac{1}{2}$ by proposing either $\mathcal{H}$ or $\mathcal{I}$. Since Player 1 can guarantee himself a payoff of $\frac{1}{2}$ with certainty and is playing a maximin-Nash strategy, it follows that he will choose $x_1^2(\mathcal{G})$ so that the outcome of the game will be $\mathcal{H}$ or $\mathcal{I}$ with certainty. Thus, if $A = (\mathcal{G}, \hat{x}_1^2(\mathcal{G}), ...)$, $FM(A) = \{\mathcal{H}, \mathcal{I}\}$.

**Claim 2:** $\hat{x}_2^2(\mathcal{G}) = \mathcal{H}$ (or $\hat{x}_3^2(\mathcal{G}) = \mathcal{I}$ ).

*Proof of Claim 2:* Suppose that $\hat{x}_2^2(\mathcal{G}) \neq \mathcal{H}$ with $\hat{x}_2^2(\mathcal{G}) = (x_1^2, x_2^2, x_3^2) \in \overline{P}(\mathcal{G})$. Consider Figure 2.8.[12] Since Player 2 could have guaranteed himself a payoff of $\frac{1}{2}$ by proposing $\mathcal{H}$ by the logic above, it follows that $\min_{x \in FM(\mathcal{G}, \hat{x}_2^2)}\{u_2(x)\}$ is weakly greater than $\frac{1}{2}$.

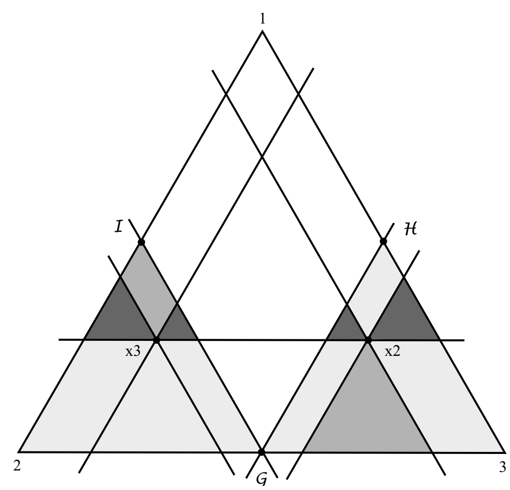



Figure 2.8: If Player 2 proposes $\hat{x}_2^2(\mathcal{G}) \neq \mathcal{H}$.

Now suppose that Player 3 is selected to propose next. Since Player 2 is already guaranteed a payoff of at least $\frac{1}{2}$, the highest payoff Player 3 could guarantee himself is also $\frac{1}{2}$. Suppose he chooses the point $x^3 = (x_1^2, x_3^2, x_2^2) \in \overline{P}(\mathcal{G}) \cap \overline{P}(\hat{x}_2^2(\mathcal{G})) \setminus \{\mathcal{G}, \hat{x}_2^2(\mathcal{G})\}$. By Proposition 1, an externally stable chain exists that begins with these three points.

After $x^3$ is proposed, the effective strategy space becomes $\overline{\mathcal{P}} = \overline{P}(\mathcal{G}) \cap \overline{P}(\hat{x}_2^2(\mathcal{G})) \cap$

[12]Throughout the figures in this proof, the union of all shaded regions represents the effective policy space after the first policy is proposed, and the more darkly shaded regions represent the effective policy space after the second, and possibly third, policies are proposed.

$\overline{P}(x^3) \setminus \{\mathcal{G}, \hat{x}_2^2(\mathcal{G}), x^3\}$. Note that for each $x = (x_1, x_2, x_3) \in \{\mathcal{G}, \hat{x}_2^2, x^3\}$, there is an $x' \in \{\mathcal{G}, \hat{x}_2^2, x^3\}$ such that $x' = (x_1, x_3, x_2)$. Thus it is apparent that $\overline{\mathcal{P}}$ is nonempty if $\hat{x}_2^2(\mathcal{G}) \neq \mathcal{I}$, and for all $y \in \overline{\mathcal{P}}$ such that $y = (y_1, y_2, y_3)$, there exists a $y' \in \overline{\mathcal{P}}$ such that $y' = (y_1, y_3, y_2)$. Furthermore, $\hat{x}_2^2(\mathcal{G})$ will not equal $\mathcal{I}$ because then Player 2 would be guaranteeing himself a payoff of zero when he could have guaranteed himself a payoff of $\frac{1}{2}$.

By Corollary 1, Player 3 is now guaranteed the same maximum minimum payoff of $\frac{1}{2}$ as Player 2. However, since the point $(0, \frac{1}{2}, \frac{1}{2}) = \mathcal{G}$ has already been proposed, there is no point left in $X$ that gives both players a payoff of $\frac{1}{2}$, and so we have a contradiction. Thus, $\hat{x}_2^2(\mathcal{G}) = \mathcal{H}$. By exactly the same logic, $\hat{x}_3^2(\mathcal{G}) = \mathcal{I}$

**Claim 3:** $\hat{x}_1^1(\emptyset) = \mathcal{G}$.

*Proof of Claim 3:* Suppose that Player 1 proposes a point $\hat{x}_1^1(\emptyset) = (x_1^1, x_2^1, x_3^1) \neq \mathcal{G}$. Since Player 1 could have guaranteed himself a payoff of $\frac{1}{2}$ by proposing $\mathcal{G}$, it follows that $\min_{x \in FM(\hat{x}_1^1(\emptyset))}\{u_1(x)\} \geq \frac{1}{2}$. To prove this claim, I have divided the simplex into six regions, or cases. For each case, two things must be proved: first, that if $\hat{x}_1^1(\emptyset)$ is in the region considered then Player 1's maximin payoff is not strictly greater than $\frac{1}{2}$, which is what $\hat{x}_1^1(\emptyset) = \mathcal{G}$ guaranteed her. And second, there exists no $x \neq \mathcal{G}$ in the region considered such that $x \in \hat{x}_1^1(\emptyset)$.

The six cases (or regions of the simplex) considered for this proof are as follows. First, the case where $x_1^1 = x_2^1$, second, the case where $x_1^1 > \frac{1}{2}$ *or* $x_2^1 > \frac{1}{2}$, third, the case where $x_2^1 > x_3^1 > 0$, fourth, the case where $x_2^1 = x_3^1$, fifth, the case where $x_3^1 \geq \frac{1}{2}$, and sixth, the case where $x_3^1 > x_2^1 > 0$.

*Case 1*: $x_1^1 = x_2^1$. First assume $\hat{x}_1^1(\emptyset)$ is such that $x_1^1 = x_2^1 \neq x_3^1$. By Corollary 1, Player 2 is also guaranteed a maximum minimum payoff of $\frac{1}{2}$ and Player 3 is guaranteed a payoff of zero. Thus, Player 1's maximin payoff is not larger than $\frac{1}{2}$.

Suppose that Player 3 is selected to propose next, and chooses $x^2 = (x_3^1, x_2^1, x_1^1)$. By Corollary 1 this strategy guarantees him a maximum minimum payoff of at least $\frac{1}{2}$, a contradiction since both 1 and 2 were also guaranteed at least $\frac{1}{2}$.

If $\hat{x}_1^1(\emptyset)$ is such that $x_1^1 = x_2^1 = x_3^1$, then we immediately have a contradiction, because by Corollary 1 all players are guaranteed the same payoff, but could not all be guaranteed a payoff weakly greater than $\frac{1}{2}$. Thus, Player 1 cannot guarantee himself a payoff weakly greater than $\frac{1}{2}$ when $x_1^1 = x_2^1$.

For Cases 2-4, suppose Player 2 is selected to propose next and chooses the point $x^2 = (x_2^1, x_1^1, x_3^1)$. By the same argument as above, Players 1 and 2 are identical up to a relabeling of their names, and so Player 2 is now also guaranteed a payoff of at least $\frac{1}{2}$. As this is the highest payoff Player 2 can guarantee himself, $x^2$ is indeed an equilibrium proposal for him to make. It follows that Player 1 cannot guarantee himself a maximin payoff greater than $\frac{1}{2}$, and that Player 3 will receive a payoff of zero with certainty. Let $\overline{\mathcal{P}}$ be the effective strategy space after $\hat{x}_1^1(\emptyset)$ and $x^2$ have been proposed, so that $\overline{\mathcal{P}} = \overline{P}(\hat{x}_1^1(\emptyset)) \cap \overline{P}(x^2) \setminus \{\hat{x}_1^1(\emptyset), x^2\}$.

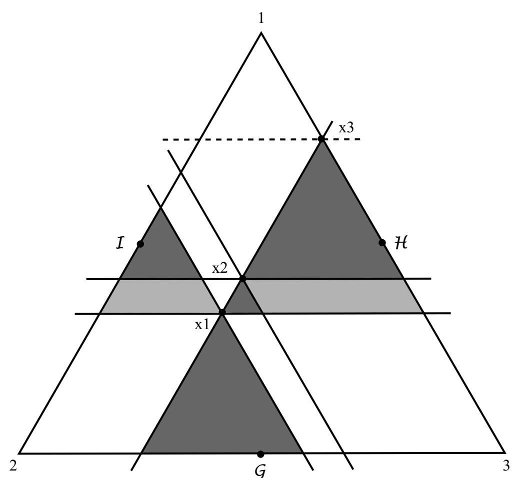



Figure 2.9: If Player 1 proposes $\hat{x}_1^1(\emptyset) \neq \mathcal{G}$ with $x_2^1 > x_3^1 > 0$.

*Case 2*: $x_1^1 > \frac{1}{2}$ *or* $x_2^1 > \frac{1}{2}$. By the strategy of Player 2 defined above, both Players 1 and 2 are guaranteed a maximin payoff of $\frac{1}{2}$. However, if either $x_1^1$ or $x_2^1$ is greater than $\frac{1}{2}$, then $\hat{x}_1^1(\emptyset) \in P(\mathcal{I})$, where $\mathcal{I} = (\frac{1}{2}, \frac{1}{2}, 0)$, and so $\mathcal{I} \notin FM(\hat{x}_1^1(\emptyset))$. Thus, we have a contradiction. It follows that Player 1 cannot guarantee himself a maximin payoff weakly greater than $\frac{1}{2}$ when either $x_1^1 > \frac{1}{2}$ or $x_2^1 > \frac{1}{2}$.

*Case 3*: $x_2^1 > x_3^1 > 0$. Consider Figure 2.9. Suppose that Player 3 is now recognized to make the third proposal. The point $x^3 = (1 - x_3^1, 0, x_3^1)$ is clearly in $\overline{\mathcal{P}}$ because $1 - x_3^1 = x_1^1 + x_2^1 > x_1^1$ and $1 - x_3^1 > x_2^1$, which implies that Player 1 strictly prefers $x^3$ to both $x^1$ and $x^2$, and Player 3 is indifferent between all three points. Also, this point $x^3$ is strictly preferred by Player 1 and Player 3 to every point in $\overline{\mathcal{P}}$ that gives Player 3 a payoff of zero.

By Proposition 1 we know that a finite externally stable chain exists that begins with the points $(\hat{x}_1^1(\emptyset), x^2, x^3)$, and so it must follow that there exists an $H^* \in \mathbf{H}^*$ such that $H^* \setminus \{x^3\} \notin \mathbf{H}^*$, since the addition of $x^3$ to the chain eliminated all externally stable chains with maximal elements which allocated zero to Player 3. Thus, there is an equilibrium proposal Player 3 can choose that guarantees him a payoff strictly greater than zero. This contradicts the fact that both Players 1 and 2 each get a payoff greater than or equal to $\frac{1}{2}$ with certainty. It follows that Player 1 cannot guarantee himself a maximin payoff weakly greater than $\frac{1}{2}$ when $x_2^1 > x_3^1 > 0$.

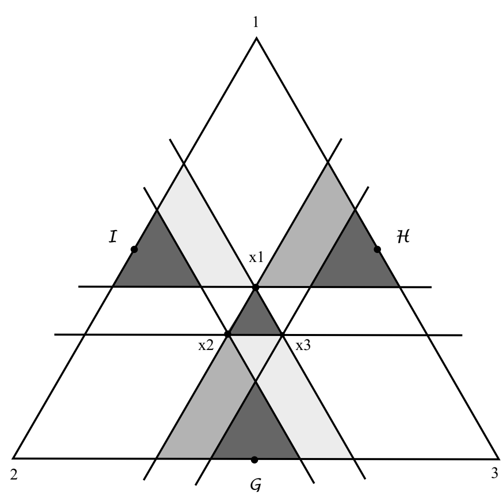



Figure 2.10: If Player 1 proposes $\hat{x}^1(\emptyset) \neq \mathcal{G}$ with $x_2^1 = x_3^1$.

*Case 4*: $x_2^1 = x_3^1$. Consider Figure 2.10. It is clear that $x_1^1 \leq \frac{1}{2}$, because if not, then the point $\mathcal{I} \notin \overline{\mathcal{P}}$, and consequently Players 1 and 2 could not both receive payoffs of $\frac{1}{2}$. Suppose that

Player 3 gets selected to make the third proposal, and chooses point $x^3 = (x^1_3, x^1_2, x^1_1)$. The effective strategy space is now symmetric about the ideal points of all three players, and so by Corollary 1 every player is guaranteed the same maximin payoff. However, since all players cannot receive at least $\frac{1}{2}$, we have a contradiction. It follows that Player 1 cannot guarantee himself a maximin payoff weakly greater than $\frac{1}{2}$ when $x^1_2 = x^1_3$.

For Cases 5 and 6, suppose Player 3 is selected to propose next and chooses the point $x^2 = (x^1_3, x^1_2, x^1_1)$. By Corollary 1, Players 1 and 3 are identical up to a relabeling of their names, and so Player 3 is now also guaranteed a payoff of at least $\frac{1}{2}$. As this is the highest payoff Player 3 could guarantee himself, $x^2$ is indeed an equilibrium proposal for him to make. It follows that Player 1 cannot guarantee himself a maximin payoff greater than $\frac{1}{2}$, and that Player 2 will receive a payoff of zero with certainty.

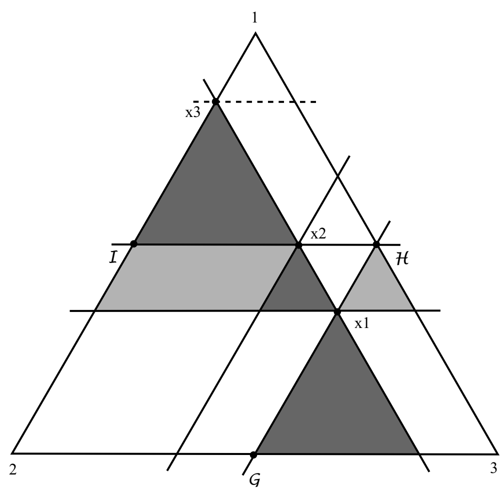



Figure 2.11: If Player 1 proposes $\hat{x}^1(\emptyset) \neq \mathcal{G}$ with $x^1_3 = \frac{1}{2}$.

*Case 5*: $x^1_3 \geq \frac{1}{2}$. Clearly it is not the case that $x^1_3 > \frac{1}{2}$, since then $\hat{x}^1_1(\emptyset) \in P(\mathcal{H})$, and so $\mathcal{H} \notin FM(\hat{x}^1_1(\emptyset))$. Thus, $x^1_3 = \frac{1}{2}$. Consider Figure 2.11. We also know that $x^1_2 \neq \frac{1}{2}$ by Case 4, and that $x^1_1 \neq \frac{1}{2}$, since $\mathcal{H} \notin FM(\mathcal{H})$. Thus, $x^1_1 > 0$ and $x^1_2 > 0$. However, $x^3 = (1 - x^1_2, x^1_2, 0) \in \overline{\mathcal{P}}$, since it is weakly preferred to both $\hat{x}^1_1 = (x^1_1, x^1_2, \frac{1}{2})$ and $x^2 = (\frac{1}{2}, x^1_2, x^1_1)$. Since $x^3_1 = 1 - x^1_2 > \frac{1}{2}$, $x^3$ strictly defeats $\mathcal{H}$, which is the only alternative in $\overline{\mathcal{P}}$ that gives Player 2 a payoff of zero. Thus, if Player 2 is chosen to propose third, he can propose $x^3$ and guarantee himself a payoff strictly greater than zero, a contradiction. It follows that Player 1 cannot guarantee himself a maximin payoff weakly greater than $\frac{1}{2}$ when $x_3^1 \geq \frac{1}{2}$.

*Case 6*: $x_3^1 > x_2^1 > 0$. The proof of this case is equivalent to the proof of Case 3, with the labels of Players 2 and 3 reversed.

It follows that Player 1 will always propose $\hat{x}_1^1(\emptyset) = \mathcal{G}$. $\square$

It is clear that the solution provided by Theorem 2 depends greatly upon the tie-breaking assumption used. If ties are broken in favor of the status quo, or broken randomly, it is not clear that a solution to this game would exist. The fact that the agenda constructed in equilibrium is not only externally stable but also *internally* stable reaffirms the importance of the tie-breaking assumption, because no element of the constructed agenda strictly defeats any other element via majority rule. A good example of the effect of tie-breaking rules on outcomes is the two-player analogue to this game, in which players are randomly chosen to propose divisions of the dollar until no one wishes to make another proposal, and the constructed agenda is then voted upon via unanimity rule. In this setting, the only externally stable (weak) chain consists of every possible allocation of the dollar, and is thus uncountably infinite. This is because every policy weakly majority defeats every other. In this example both players get a payoff of zero.

There is a large literature on two-player bargaining games, and much of it has stemmed from the work of Nash (1950, 1953). Nash approached the problem axiomatically, by developing a list of five properties that any bargaining outcome should satisfy. Outcomes should be Pareto efficient, individually rational (players should not get less from bargaining than disagreement), scale covariant (changing the size of the pie should similarly change payoffs), independent of irrelevant alternatives (eliminating alternatives that would not have been chosen should not effect outcomes), and symmetric (if the game treats the players symmetrically). The Nash bargaining solution to this problem is the allocation $(\frac{1}{2}, \frac{1}{2})$.

Rubinstein (1982) also solves a two-player bargaining game in which players sequentially propose divisions of a pie that can be either accepted or rejected by the other player. One difference between the game presented in this paper and Rubinstein's game is that Rubinstein's players discount the future; the longer players take to reach an agreement, the smaller the size of the pie gets. However, in Rubinstein's game (with a pie equal to one) the equilibrium allocation converges to $(\frac{1}{2}, \frac{1}{2})$, the Nash bargaining solution, as discount rates converge to one (i.e., as players become increasingly patient). Thus, in Rubinstein's game, payoffs converge to an efficient and equitable outcome. In this game, payoffs are not efficient.

The opposite tie-breaking rule, biased in favor of the status quo, implies that we are now considering strict external stability instead of weak external stability. In this case, every policy is itself a strict maximal chain, because no policy strictly defeats any other. The first proposer will propose his ideal point, and this will be the outcome of the game. Thus, these two different tie-breaking rules each violate an axiom of Nash's bargaining solution; the first violates Pareto efficiency, the second violates symmetry. Both examples demonstrate that the assumption of external stability in an equilibrium agenda may not be appropriate in every bargaining game—in this case, a bargaining game under unanimity rule.

In a constant-sum game under unanimity rule, no alternative strictly defeats any other, and so the only externally stable chain of alternatives is uncountably infinite. The solution presented in Theorem 2.4 also hints at the possibility that external stability may not be an appropriate assumption in situations with many players. In a three-player constant-sum game, the von Neumann-Morgenstern stable set forms the shortest externally stable chain. If we hypothesize that this is also the case in constant-sum games with more than three players, then the length of any possible externally stable chain will increase exponentially with the number of players. In the three-player case, the size of the stable set is three. With five players it grows to ten, with seven payers thirty-five, and so on. In a large legislature, an agenda could conceivably require thousands of amendments in order to form an externally stable chain. Thus, the assumption of external stability may be more appropriate in situations where a small number of players bargain informally, by tossing out ideas.

## 2.5 Conclusions

Games of endogenous agenda selection pose many difficult modeling problems. One such issue is the determination of the length of an agenda, and whether this length should be endogenously or exogenously set. If the number of items on an agenda is not fixed, then there is no well-specified end to the proposal process and we cannot backward induct to solve for an equilibrium outcome. However, restricting the number of items on an agenda can also lead to problems. As Banks and Gasmi's results demonstrate, restricting players to a specific number of proposals each can lead to minimax-Stackelberg equilibrium agendas which are not externally stable. This chapter takes up where Banks and Gasmi left off to ask the question of how agendas should naturally end when players are allowed to make policy proposals for as long as they wish. While the equilibrium concept considered here is somewhat unusual, it generates precise predictions in a very informal institutional setting, and these predictions can be directly compared to those of Banks and Gasmi.

This chapter presents two results; first, it shows that the set of possible outcomes that can be supported by a distributive amendment game with sophisticated voting (i.e., the Banks set) equals the uncovered set, which has full measure over the space of alternatives. Second, this chapter characterizes the unique minimax-Nash solution to an endogenous agenda formation game with three players, and shows that this solution can be any of the three points in the von Neumann-Morgenstern stable set. Thus, while virtually any division of the dollar can be supported as a sophisticated voting outcome, allowing players to determine the agenda themselves narrows the set of potential outcomes to three unique points. These two results provide a link between the concept of the Banks set, which has previously been defined only in terms of a finite policy space, and endogenous agenda formation when an arbitrarily large number of proposals are allowed for. They are also in keeping with much of the divide-the-dollar literature, such as Baron and Ferejohn (1989), which predicts the emergence of minimal winning coalitions when agendas are endogenously determined.

This chapter also provides an interesting example of how finite and infinite-stage amendment games are played out differently. I find that the solution to the N-amendment game differs substantially from the universalistic M* solution to the two-amendment game found by Banks and Gasmi, even though in both games only three proposals are made. Allowing for an arbitrarily large number of amendments does not change the number of amendments actually made, but dramatically changes *which* amendments are made. In particular, it motivates the first two proposers to work as quickly as possible to disenfranchise the third player. Thus, this noncooperative game generates the same outcome as that generated by a cooperative game in which two players collude—namely, a point in the simple von Neumann-Morgenstern stable set.

# Chapter 3 A Model of Farsighted Voting

## 3.1 Introduction

Formal modeling has perhaps had its most important impact on the study of legislative politics. Beginning with the field of social choice theory a half-century ago, scholars sought a formal means of directly aggregating individual preferences into collective outcomes. However, the social choice theoretic technique proved to have limited predictive power; the most well-known social choice theoretic results tell us that there is no normatively appealing means of aggregating individual preferences, and that, generically, any given policy can defeat any other via an amendment agenda. These "impossibility" and "chaos" theorems led many to believe that the direct aggregation of preferences into outcomes was not a promising approach to the study of collective choice. As a solution to this dilemma, Shepsle (1979) presented the idea of a "structure-induced equilibrium," in which institutional detail is combined with social choice theory to yield core-based predictions. Shepsle's argument is that, for a model to have predictive power, some specific institutional form must generally be assumed. While the notion of SIE has today been discarded in favor of other equilibrium concepts such as Nash, Shepsle's argument in favor of institutions foreshadowed the course of formal modeling.

Institutional models today are generally noncooperative and game-theoretic, where the game form (or institutional venue) is assumed exogenous. One of the advantages of these models is that they have strong predictive power, as Nash equilibria virtually always exist. However the game theoretic approach can also prove problematic, particularly when institutions are regarded as solutions to the so-called chaos problem. First, the predictions of many models are not robust to slight institutional changes; Nash equilibria in particular are highly sensitive to specific institutional detail. And second, when strict institutional assumptions are made, they can effect outcomes in extreme, and sometimes quite unrealistic, ways. Diermeier and Krehbiel (2002) argue that it is through the *comparison* of institutional models that the link between institutions and outcomes should be drawn. Institutional models in isolation should serve a more methodological, and less predictive, role.

In this chapter I argue that the inability of social choice theory to yield predictions arises not solely because the majority preference relation is unstable, but in large part because social choice theory implicitly assumes a static environment. It fails to take into account the fact that individuals may not only have immediate tastes over policies, but also preferences over future turns of events. Thus, the behavior of a voter may depend on both his short and long-term interests. This is captured in this model by adding a new dimension to the standard social choice-theoretic framework. Here, individuals rank policies not only on the basis of the utility they yield, but also with respect to the types of alternatives they can and cannot defeat.

This model differs from much of the bargaining literature in several respects. First, unlike Baron and Ferejohn (1989), I model bargaining as a dynamic process, in which choices made today effect those feasible tomorrow. As in Baron and Ferejohn's model, proposals are made sequentially, however in this chapter proposals are not made by the players themselves. Rather, policies to be pitted against the status quo arise exogenously. The next chapter endogenizes the proposal process and in this way is more similar to Baron and Ferejohn's model. Another difference between this model and much of the bargaining literature is that the bargaining process does not end once a policy has been agreed upon. Instead, the chosen policy becomes the reversion point of the next round of bargaining. A last difference is that this model often makes statistical, rather than point, predictions. In this way, it is similar to McKelvey and Palfrey's notion of *quantal response equilibrium* although, again, the focus here is on the types of policies likely to emerge as a consequence of bargaining over time.

Specifically, the chapter examines how individuals evaluate policies in a setting of repeated interaction, when they are aware that any policy enacted today will become tomorrow's status quo, and will lead to a future stream of legislation which is to some extent dependent upon it. The focus is on continuing programs, in which policies remain in effect until new legislation is enacted. Examples of such programs include entitlements, regulation, and both distributive and redistributive programs.[1] Formal models to date have not been able to make compelling predictions in the setting of a continuing program. I will demonstrate that modeling farsightedness in this way is not only a more realistic approach to the study of repeated bargaining, but that it also yields compelling predictions in a variety of legislative environments. Furthermore, it does so in an "institution-free" way, by keeping the level of institutional detail to a minimum. The chapter focuses on two main questions. First, is there a way of evaluating policies in terms of what they are likely to produce over time? And second, what do these individual-level evaluations imply about the types of outcomes likely to emerge when programs are continuing?

The formal setup of the model is that of an infinite-horizon continuing program which legislators vote on in discrete time. For every potential status quo, there exists a density from which alternatives to replace that status quo are drawn, and this density is known by all individuals. For each potential policy, individuals "look ahead", and iteratively calculate a long-term value based not only on the utility the policy yields, but also on the utility yielded by policies which will likely defeat it in future rounds. Individuals then vote upon the choices presented to them using these valuations. In equilibrium, the value every player assigns to a policy equals the true expected value of that policy, given the valuations of the other players. Using this information, we can then calculate a probability distribution over the types of outcomes likely to emerge.

I show in this setting that, in the absence of a game form, players are not indifferent between different policies which provide them with the same level of utility. This is because the space of alternatives which defeat each policy, and which each policy defeats, are substantively different. In the setting of a continuing program in which status quos are endogenously determined, we can expect a certain path dependence to be observed in policy outcomes, and this path dependence cannot be captured in a simple one-shot model of voting. In this model, it is precisely the probabilistic path which a policy leads to that defines that policy. One consequence is that players are induced into taking the payoffs of others into account when voting, not because of a behavioral assumption such as altruism,

[1] See Baron (1996) for a more detailed discussion of continuing programs.

inequality aversion, or sophisticated voting, but because they know that the behavior of others in large part determines which policies are enacted in the future.

A closely related paper is that of Kalandrakis (2002), in which the author analyzes an infinitely repeated divide-the-dollar game with an endogenous reversion point, where the status quo in any round is determined by the bargaining outcome of the previous round. He finds that a Markov Perfect Nash equilibrium (in stage-undominated vote strategies) of the game is characterized by a situation in which the proposer in each round allocates himself the entire dollar, and this allocation is approved by a majority of players. This result is interesting because it negates two common theories about the outcomes of repeated bargaining games—namely, chaos and centrality. Under the characterized equilibrium, only a finite number of outcomes are ever achieved with positive probability and once the steady-state distribution is reached, every subsequent proposal allocates everything to the proposer. Furthermore, the methodology used (MPNESUV) is appealing from a game-theoretic point of view.

However, a distressing aspect of this result is that it seems entirely unrealistic. It is hard to imagine any kind of democratic process by which a legislative dictator emerges, with certainty, in every round. The author addresses this point in his paper by stating that the strange result may generate intuition for the argument that, in actuality, budgets are deliberated under an exogenous reversion point. While this may be the case, it may also be the case that different primitives of the model are incorrect; in reality, budget deliberations may be history-dependent, policies to replace the status quo may arise probabilistically rather than deterministically, and deliberators may tremble when casting votes. Thus, while Kalandrakis' model tells us that only the most extreme policy allocations are ever generated with positive probability, and McKelvey (1979) tells us that collective preference can be manipulated to generate virtually any policy outcome, this model demonstrates that there exists a methodological middle ground. By taking the agenda-setting process to be exogenous and probabilistic, and by allowing players to ex ante evaluate policies on the basis of what they are likely to produce over time, a unique distribution over observed outcomes can be found.

The chapter proceeds as follows: Section 3.2 describes the notation used and presents the Markov model. Section 3.4 proves some analytic results. Under very general conditions I prove that there exists a unique self-generating value function when the number of players is large, or when players vote according to a stationary rule. Then I show that, regardless of the number of players, there exists a self-generating value function in certain settings. Section 3.5 provides some simple examples of the model in the setting of a unidimensional, finite policy space. Section 3.6 discusses the specific applications of the model in greater detail and presents numerical results concerning these applications in more complex policy spaces. Section 3.7 concludes.

## 3.2 A Model of Farsighted Valuations

### 3.2.1 Notation

I assume a set $N = \{1, 2, ..., n\}$ of *voters* (where $n$ is odd), a compact set $X \subset \mathbb{R}^m$ of *alternatives*, or *policies*, and, for each $i \in N$, voter preferences are represented by a real-valued *utility function*, $u_i : X \to \mathbb{R}_+$. When the set $X$ is infinite, also assume that these utility functions are differentiable, and that their derivatives are uniformly bounded by some constant $U$. A nonempty subset $C \subset N$ is called a *coalition*. I will restrict attention to *simple* and *anonymous* games, so that given a collection of coalitions $W$ with $C \in W$, then $C \subseteq C'$ implies $C' \in W$. Anonymity implies that the voting rules considered here are $q$-rules, such that for some fixed integer $q > n/2$, $W = \{C \subseteq N : |C| \geq q\}$. The collection $W$ can be considered the set of *winning* or *decisive* coalitions.

### 3.2.2 The Markov Model, in Finite and Continuous Policy Spaces

Policy selection is modeled as a Markov process, and individual valuations over policies are conditioned upon this process. First, assume $X$ is finite. Players' valuations at time $t$ are represented by a vector of continuous value functions $v_t : X \to \mathbb{R}^n$. Player $i$'s value function at time $t$, $v_{it}$ is the $i^{th}$ element of vector $v_t$. We can define $\mathbb{R}^X$ to be the set of all measurable functions from $X$ into the real line. Then $v_{it} \in \mathbb{R}^X$ and $v_t \in \prod_{i \in N} \mathbb{R}^X$. The function $v_{it}$ is represented by

$$v_{i0}(x) = u_i(x) \tag{3.1}$$

and

$$v_{it+1}(x) = u_i(x) + \delta \sum_{y \in X} \left[ v_{it}(y) p(v_t(x), v_t(y)) + v_{it}(x)(1 - p(v_t(x), v_t(y))) \right] Q(y). \tag{3.2}$$

The function $v_{i0}$ equals the utility player $i$ receives from alternative $x$. The probability of transitioning from state $x$ to state $y$ at time $t+1$, given the two states are paired against each other, is represented by $p(v_t(x), v_t(y)) \in [0, 1]$. $Q(y)$ is the probability mass from which alternatives $y$ to replace the status quo are drawn. $\delta \in [0, 1)$ is a discount factor.

In the infinite case, Equation 3.2 is written

$$v_{it+1}(x) = u_i(x) + \delta \int_{y \in X} v_{it}(y) p(v_t(x), v_t(y)) + v_{it}(x)(1 - p(v_t(x), v_t(y)))\, dQ(y) \tag{3.3}$$

and $Q(y)$ is instead a density. In this case, $Q$ is assumed to have full support, and to be continuous and differentiable in $y$. Let $\mathcal{V}$ be the space of continuous, real-valued functions taking $X$ to $\mathbb{R}_+$. Then $v_{it} \in \mathcal{V}$ and $v_t \in \mathcal{V}^n$.

Note that there are two types of transitions playing into the above equation, $p$ and $Q$. I will refer to these as "transition probabilities" and "transition densities" (or "masses", when $X$ is finite), respectively. Intuitively, alternatives to replace the status quo arise probabilistically, picked from a stationary transition density $Q$. Since $Q$ is exogenous, the model assumes that legislators do not explicitly set the agenda themselves, but have fixed beliefs over the types of alternatives which will be added to the agenda. These beliefs could be uniform over all alternatives (uninformative), or could be generated by fixed external pressures from political parties, special interests, constituencies, or some function of the ideal points of the legislators themselves. However, once such an alternative is picked, it must then be pitted against the status quo, and will defeat the status quo with some transition probability $p$.

Transition probabilities are possibly nonstationary because legislators retrospectively update the values that they assign to policies, and vote according to these updated values. A legislator could have initially assigned a very high value to policy $x$. However if $x$ is replaced by a stream of future policies that the legislator dislikes, then the value he assigns to $x$ will be brought down in subsequent rounds, and this will be reflected in how he votes. The following two assumptions are made throughout the formal analysis of the model.

**Assumption 1** *Transition probability assumption*

For all $x, y \in X$, $p(v_t(x), v_t(y))$, or the probability of transitioning from policy $x$ to policy $y$ at time $t+1$, given $x$ and $y$ are put to a vote and given value function $v_t$, can be written as the probability of victory of $y$ over $x$:

$$p(v_t(x), v_t(y)) = \sum_{C \in W} \prod_{i \in C} p_i(v_i(x), v_i(y)) \prod_{i \notin C} (1 - p_i(v_i(x), v_i(y))) \tag{3.4}$$

where $p_i(v_i(x), v_i(y)) \in [0,1]$ represents Player i's probability of voting for $y$ over $x$ given value function $v$. It is assumed that $p_i$ is independent of $p_j$ for all $i, j \in N$, that $p_i(v_i(x), v_i(y)) + p_i(v_i(y), v_i(x)) = 1$, and that $p_i$ is increasing in $v_i(y) - v_i(x)$. Since $p$ is a function of $v$, which is indexed by time in the model, the transition functions are possibly nonstationary. An example of a nonstationary transition function to be discussed later is one in which players vote probabilistically, according to a logistic function. In this case, $p_i(v_i(x), v_i(y)) = \frac{e^{\lambda v_i(y)}}{e^{\lambda v_i(x)} + e^{\lambda v_i(y)}}$ for some $\lambda \in \mathbb{R}_+$.

**Assumption 2** *Differentiability and non-determinism of individual transition probabilities*

For the remainder of the formal analysis, it is assumed that for all $i$, $p_i(v_i(x), v_i(y))$ is continuous and differentiable in both of its arguments, and that these derivatives are uniformly bounded by some constant. Is is also assumed that for all $x, y \in X$, $p_i(v_i(x), v_i(y)) \in (0,1)$, so that individuals can never vote with probability one for one alternative over another. While the assumption is made solely to simplify the analysis, the reader should note that it is always possible to approximate a discontinuous function with such a continuous and differentiable one. For example, the logistic vote function converges to the deterministic case as $\lambda$ is driven to infinity.

Underlying the model of individual valuations over alternatives is a Markov process by which policies are realized. A Markov process has the property that, given a current realized state of the world (or status quo policy), future states of the world are independent of the past. Thus, the Markov process defines a probability measure over $X$, conditioned upon a status quo $x \in X$ which tells us how likely any given policy is to replace status quo $x$. The following definition states this process explicitly.

**Definition:** The *conditional transition measure* $f^{v_{t+1}}(Z|x)$ represents the relative likelihood of transitioning to a policy in the set $Z \subseteq X$ at time $t+1$, given a status quo $x$:

$$f^{v_{t+1}}(Z|x) = \int_{y \in Z} p(v_t(x), v_t(y))dQ(y) + \mathbf{1}_{\{x \in Z\}} \int_{y \in X} (1 - p(v_t(x), v_t(y)))dQ(y). \tag{3.5}$$

Since the vector $v_t$ plays into this function, the process is dependent upon time $t$. Thus the Markov process is said to be *nonstationary*. The following measures are also useful in understanding the relationship between individual valuations and realized policy outcomes.

**Definition:** The *marginal transition measure* $f_X^{v_{t+1}}(x)$ represents the relative likelihood of $x$ being the status quo at time $t+1$, and the marginal transition measure $f_Y^{v_{t+1}}(y)$ represents the relative likelihood of transitioning to policy $y$ at time $t+1$. These measures are defined recursively, with

$$f_X^{v_1}(x) = \frac{1}{\int_{y \in X} dy} \tag{3.6}$$

$$f_Y^{v_t}(y) = \int_{x \in X} f_X^{v_t}(x) \cdot f^{v_t}(y|x)dx \tag{3.7}$$

and

$$f_X^{v_{t+1}}(x) = f_Y^{v_t}(x). \tag{3.8}$$

**Definition:** The *transition measure* $f^{v_{t+1}}(x, y)$ represents the relative likelihood of transitioning from policy $x$ to policy $y$ at time $t+1$, and is simply the product of the conditional and partial transition measures:

$$\begin{aligned} f^{v_{t+1}}(x, y) &= f^{v_{t+1}}(y|x) \cdot f_X^{v_{t+1}}(x) \\ &= f^{v_{t+1}}(y|x) \cdot f_Y^{v_t}(x) \end{aligned} \tag{3.9}$$

Of most interest to us is the marginal transition measure, $f_X^{v_{t+1}}(x)$, or the likelihood policy $x$ is the observed status quo at time $t+1$. In the numerical simulations of Section 3.6, I calculate both the equilibrium value functions of players and the equilibrium distribution over outcomes that these value functions generate, as represented by the marginal transition measure.

## 3.3 Dynamically Stable Voting Equilibria

The main focus of the following analysis is to prove the existence of, and numerically compute, value functions which are self-generating. These functions are of interest because they represent equilibria in beliefs. When a player behaves according to such a function, the value he assigns to a policy equals the true future expected value of that policy. When this holds for *all* players, then the vote strategies of players generate value functions which generate the same vote strategies. Thus, beliefs and behavior are entirely consistent with each other. The following equilibrium concept captures this notion.

Let $\mathcal{M}_X$ be the set of probability measures over $X$. Let $\mathcal{P}$ be the set of functions taking $\mathbb{R}^n \times \mathbb{R}^n$ to $[0, 1]$. When the set $X$ is infinite, then at a given $u \in \mathcal{V}^n$, $p \in \mathcal{P}$, and $Q \in \mathcal{M}_X$, a *dynamically stable voting equilibrium* is a collection of value functions, $v = \{v_i\}_{i \in N}$, such that for all $i \in N$ and $x \in X$,

$$v_i(x) = u_i(x) + \delta \int_{y \in X} v_i(y)p(v_i(x), v_i(y)) + v_i(x)(p(v_i(y), v_i(x)))dQ(y).$$

The case of a finite $X$ is defined analogously. Thus, given the Markov process defined in Section 3.2.2 a dynamically stable voting equilibrium is reached at a fixed point, when $v_{t+1} = v_t$.

If we define the functions $p_i$ to be deterministic, so that $p_i(v_i(x), v_i(y)) = 1$ if $v_i(y) \geq v_i(x)$ and zero otherwise, then at a dynamically stable voting equilibrium, $v^*$, the collection of functions $p_i$ would constitute a Nash equilibrium. In this case, the $v^*$ vector represents the expected utility functions of the players, and strategies as specified by the functions $p_i$ are consistent with the maximization of these expected utility functions. The proof of this is straightforward.

**Lemma 5** *If $p_i(v_i(x), v_i(y)) = 1$ if $v_i(y) \geq v_i(x)$ and zero otherwise, then at a dynamically stable voting equilibrium, $v^*$, the collection of functions $p_i$ constitute a Nash equilibrium.*

*Proof:* Let $p_i(v_i^*(x), v_i^*(y)) : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \{0, 1\}$ denote Player $i$'s *strategy*, the probability with which he votes for $y$ over $x$, given valuations $v_i^*$. Let $p(\{p_i(v_i^*(x), v_i^*(y))\}_{i=1}^n)$ denote the probability that $y$ defeats $x$, given that players vote according to strategies $p_i$. Since we are considering a simple and anonymous game, $p(1, \{p_j(v_j^*(x), v_j^*(y))\}_{j \neq i}) \geq p(0, \{p_j(v_j^*(x), v_j^*(y))\}_{j \neq i})$. Thus, if Player $i$ votes for $y$ over $x$, then the likelihood that $y$ defeats $x$ is weakly greater than it would have been had Player $i$ voted for $x$ over $y$.

The functions $v_i(x)$ and $v_i(y)$ denote Player $i$'s respective payoffs from policies $x$ and $y$ being selected. Let $U_i(p_i, p_{-i})$ denote Player $i$'s payoff from playing strategy $p_i$, given that the other players are playing strategies $p_{-i}$. Assume, without loss of generality, that $v_i^*(x) \geq v_i^*(y)$. Then if $p_i(v_i^*(x), v_i^*(y)) = 1$ if $v_i^*(y) \geq v_i^*(x)$ and zero otherwise, $U_i(0, p_{-i}) = v_i^*(x)p(0, \{p_j(v_j^*(x), v_j^*(y))\}_{j \neq i}) + v_i^*(y)p(0, \{p_j(v_j^*(x), v_j^*(y))\}_{j \neq i}) \geq v_i^*(x)p(1, \{p_j(v_j^*(x), v_j^*(y))\}_{j \neq i}) + v_i^*(y)p(1, \{p_j(v_j^*(x), v_j^*(y))\}_{j \neq i})$. Thus, the proposed strategies $p_i$ constitute a Nash equilibrium. □

Another possibility is that $p_i$ has a logistic form, so that for all $i \in N$, $p_i(v_i(x), v_i(y)) = \frac{e^{v_i(y)}}{e^{v_i(x)} + e^{v_i(y)}}$. Although our equilibrium concept bears a close resemblance to the notion of a quantal response equilibrium in this example, there are some subtle differences. The main difference is that in the model presented here, players vote sincerely and do not condition upon the consequences of their actions when voting. Thus, pivot probabilities are not taken into account. In a quantal response equilibrium, players condition their votes on the expected consequences of those votes. For example, if the chance that any player is pivotal is low enough, then players will be observed as voting with near fifty-fifty probability over any two alternatives, because the effect of each vote is essentially zero. Because of this, it is not clear that the two equilibrium concepts will yield the same fixed points. However, both models are similar in that they assume a fixed functional form over behavior, and both yield statistical predictions. The similarities between this equilibrium concept and quantal response equilibrium will be discussed in further detail in the next chapter.

## 3.4 Analytic Results

At any given time $t$, $v_t$ is a function of $v_{t-1}$. Let this function be called $g$, so that $v_{t+1}(\cdot) = g(v_t(\cdot))$. The following proposition proves that if transition probabilities are stationary (i.e., people do not alter how they vote over time), then there exists a unique dynamically stable voting equilibrium. Moreover, the Markov process will limit to this vector of functions.

**Proposition 3** *If transition probabilities, $p$, are stationary, then $g$ is a contraction mapping.*

*Proof:* Endow $\mathcal{V}$ with the following metric and the topology induced by it: $\rho(v_i, w_i) = \sup_{x\in X} |v_i(x) - w_i(x)|$ and for $v, w \in \mathcal{V}^n$, $\rho(v, w) = \max\limits_{i\in N} \rho(v_i, w_i)$. We must show that for any two vectors of functions $v = (v_1, ..., v_n), w = (w_1, ..., w_n) \in \mathcal{V}^n$, $\rho(g(v), g(w)) < \gamma\rho(v, w)$, for a $\gamma \in [0, 1)$. Redefine the domain of $p$ so that $p : \{\prod_{i\in N} \mathbb{R}^X \times \prod_{i\in N} \mathbb{R}^X \times X \times X\} \to [0, 1]$. Then stationarity in $p$ implies that for all $v, w \in \mathcal{V}^n$ and all $x, y \in X$, $p(v(x), v(y), x, y) = p(w(x), w(y), x, y) = p^*(x, y)$. Choose any $v, w \in \mathcal{V}^n$ and let $\xi = \rho(v, w)$. Then for every $i \in N$,

$$\rho(g(v_i), g(w_i)) = \delta * \sup_{x\in X} |\int_{y\in X} (v_i(y) - w_i(y))p^*(x, y) + (v_i(x) - w_i(x))p^*(y, x)dQ(y)|.$$

However, we know that $p^*(x, y) \in [0, 1]$, so the maximum value the argument of the integral could take for any given $y$ is $\max\{|v_i(y) - w_i(y)|, |v_i(x) - w_i(x)|\}$. Our worst case scenario is that $Q$ assigns all of its weight to the policy which maximizes this argument.

This implies that the maximum value $\rho(g(v), g(w))$ could take is $\delta \sup_{y \in X} |v_i(y) - w_i(y)|$ which is equal to $\delta\xi$ which is strictly less than $\gamma\xi$ for $\gamma \in (\delta, 1)$. Since $\delta \in [0, 1)$, we know such a $\gamma$ exists. Thus $g$ is a contraction mapping. □

The next proposition proves that the Markov process will limit to a unique dynamically stable voting equilibrium even when transitions are nonstationary, provided that the total number of players is sufficiently large. This implies that with enough players, the process described in Section 3.2.2 is a *tâtonnement*, or equilibrium-seeking, process. Note that although the proof assumes that $X$ is infinite, the same logic can be used to prove the result when $X$ is finite. The integrals are simply replaced by sums.

**Proposition 4** *There exists an $M \in \mathbb{N}$ such that whenever $n = |N| > M$, the function $g(v_t) = v_{t+1}$ is a contraction mapping*

*Proof:* For $w, z \in \mathcal{V}^n$, let $\rho(w_i, z_i) = \max_{x \in X} |w_i(x) - z_i(x)|$, and let $\rho(w, z) = \max_{i \in N} \rho(w_i, z_i)$. We must show that for any $w, z \in \mathcal{V}^n$, $\rho(g(w), g(z)) < \rho(w, z)$.

Let $g_i : \mathcal{V}^n \to \mathcal{V}$ be such that for any $v_t \in \mathcal{V}^n$, $g_i(v_t) = v_{it+1}$. Thus, $g = (g_1, ..., g_n)$. First consider the gradient vector $\nabla g_i$. For all $x \in X$,

$$g_i(v(x)) = u_i(x) + \delta \int_{y \in X} v_i(y)p(v(x), v(y)) + v_i(x)(1 - p(v(x), v(y)))dQ(y).$$

Thus, the components of $\nabla g_i(v(x))$ can be defined using the partial derivatives

$$\frac{\partial g_i(v(x))}{\partial v_i(x)} = \delta[1 - \int_{y \in X} p(v(x), v(y))dQ] + \delta \int_{y \in X} (v_i(y) - v_i(x)) \frac{\partial p(v(x), v(y))}{\partial v_i(x)} dQ \quad (3.10)$$

and for all $j \in N \setminus \{i\}$,

$$\frac{\partial g_i(v(x))}{\partial v_j(x)} = \delta \int\limits_{y \in X} (v_i(y) - v_i(x)) \frac{\partial p(v(x), v(y))}{\partial v_j(x)} dQ. \tag{3.11}$$

Using Assumption 1 we get that for all $i \in N$,

$$\frac{\partial p(v(x), v(y))}{\partial v_i(x)} = \frac{\partial p_i(v_i(x), v_i(y))}{\partial v_i(x)} Z_i(\{p_j(v_j(x), v_j(y))\}_{j \in N \setminus \{i\}}) \tag{3.12}$$

where, letting $C_i^M$ equal the set of minimal winning coalitions that $i$ is in,

$$Z_i(\{p_j(v_j(x), v_j(y))\}_{j \in N \setminus \{i\}}) = \sum_{C \in C_i^M} \prod_{j \in C \setminus \{i\}} p_j(v_j(x), v_j(y)) \prod_{j \notin C} (1 - p_j(v_j(x), v_j(y))).$$

$Z_i(\{p_j(v_j(x), v_j(y))\}_{j \in N \setminus \{i\}})$ represents the probability that Player $i$'s vote is pivotal given that all other players $j$ vote according to the functions $p_j(v_j(x), v_j(y))$. McKelvey and Patty (2002, Lemma 1) prove that when people vote probabilistically (i.e when for all $j \in N$, and all $x, y \in X$, $p_j(v_j(x), v_j(y)) \in (0, 1)$), all pivot probabilities $Z_i(\cdot) \to 0$ as $|N|$ gets large.

Combining Equations 3.11 and 3.12, we get for all $j \in N \setminus \{i\}$

$$\frac{\partial g_i(v(x))}{\partial v_j(x)} = \delta \int\limits_{y \in X} (v_i(y) - v_i(x)) \frac{\partial p_j(v_j(x), v_j(y))}{\partial v_j(x)} Z_j(\{p_k(v_k(x), v_k(y))\}_{k \in N \setminus \{j\}}) dQ.$$

By Assumption 2 we know that for all $j \in N$ and $x, y \in X$, $\frac{\partial p_j(v_j(x), v_j(y))}{\partial v_j(x)}$ is bounded by some constant. We also know that the difference $|v_j(y) - v_j(x)|$ is bounded by a constant, since $\delta < 1$ and utility is bounded. Since $Z_j(\cdot) \to 0$ as $|N| \to \infty$, it follows that for any $\epsilon > 0$ there exists an $M \in \mathbb{N}$ such that for all $n = |N| > M$,

$$\frac{\partial g_i(v(x))}{\partial v_j(x)} < \epsilon.$$

Using Equation 3.10, by the same logic it follows that for any $\epsilon > 0$ there exists an $M \in \mathbb{N}$ such that for all $n = |N| > M$,

$$\frac{\partial g_i(v(x))}{\partial v_i(x)} < \delta[1 - \int_{y \in X} p(v(x), v(y)dQ] + \epsilon.$$

Define $|\nabla g(v)|$ such that

$$|\nabla g(v)| = \max_{\{i,j\} \in N} \left( \max_{x \in X} \left| \frac{\partial g_i(v(x))}{\partial v_j(x)} \right| \right).$$

Since $\delta[1 - \int_{y \in X} p(v(x), v(y))dQ] \in (0, 1)$ for all $\delta < 1$, it follows that for $|N|$ sufficiently large (i.e., $\epsilon$ sufficiently small), $|\nabla g(v)| < 1$.

By the Mean Value Theorem we know that

$$\rho(g(w), g(z)) \leq \rho(w, z)|\nabla g(v)|$$

for some $v$ on the line segment between $w$ and $z$. Since, for any $v \in \mathcal{V}^n$, $|\nabla g(v)| < 1$ for $|N|$ sufficiently large, it follows that

$$\rho(g(w), g(z)) < \rho(w, z).$$

Thus, there exists an $M \in \mathbb{N}$ such that for all $n = |N| > M$, the function $g$ is a contraction mapping. $\square$

The final two propositions prove that when transitions are nonstationary there exists a dynamically stable voting equilibrium regardless of the number of players. When $X$ is finite, the proof relies only upon Assumptions 1 and 2. When $X$ is infinite an additional assumption is needed.

**Proposition 5** *If $X$ is finite, then there exists a dynamically stable voting equilibrium.*

*Proof:* Since $\delta < 1$ and $u_i$ is real-valued for all $i \in N$, the upper bound any individual's value function could take is $\frac{1}{1-\delta} \max_{x \in X} u_i(x)$, and the lower bound is zero. Thus, for every $v_t \in \prod_{i \in N} \mathbb{R}^X$, $v_t \in \prod_{i \in N} [0, \frac{1}{1-\delta} \max_{x \in X} u_i(x)]^X$, and so the set of value functions is bounded. Furthermore, the set of value functions is convex, since the convex combination of two bounded functions taking $X$ to $\mathbb{R}$ is itself bounded. Last, the set of value functions is closed, trivially. It follows that the set of value functions taking $X$ into the real numbers $\mathbb{R}$ is a nonempty, closed, bounded and convex subset of a finite-dimensional vector space, $\mathbb{R}^X$.

The mapping $g : \prod_{i \in N} \mathbb{R}^X \to \prod_{i \in N} \mathbb{R}^X$, such that $g(v_t) = v_{t+1}$ (see Equation 3.2) is single-valued by definition, and is continuous by the continuity of every $p_i(v_{it}(x), v_{it}(y))$. By Brouwer's Fixed Point Theorem, there exists a $v_t \in \prod_{i \in N} \mathbb{R}^X$ such that $g(v_t) = v_t$. Thus, there exists a dynamically stable voting equilibrium. $\square$

For the infinite case, the following assumption is needed, along with a definition and a Lemma.

**Assumption 3** *Multiplicative separability and boundedness of derivative of* $p$

Let $\rho(v(r), v(s))$ represent the following metric:

$$\rho(v(r), v(s)) = \max_{i \in N} |v_i(r) - v_i(s)|.$$

Assume that for all $x, y \in X$ and all $v \in \mathcal{V}^n$,

$$|\frac{\partial}{\partial x} p(v(x), v(y))| \leq \max_{i \in N} |v_i'(x) * B|,$$

where $B \in \mathbb{R}$ is a constant and

$$|B| * \max_{r,s \in X} \rho(v(r), v(s)) < \frac{1-\delta}{\delta}.$$

First, note that this condition is merely a sufficient, and not necessary, condition for the existence of a fixed point. It may be the case that existence can be obtained in far less restrictive environments. And second, while this assumption may seem strange, many commonly used vote functions satisfy it. Consider the example where individuals vote probabilistically, according to a logistic function, and the voting rule is unanimity.

**Example 1** *The implications of Assumption 3 under logistic voting and unanimity.*

Recall that in this case, $p_i(v_i(x), v_i(y)) = \frac{e^{\lambda v_i(y)}}{e^{\lambda v_i(x)}+e^{\lambda v_i(y)}}$. Since $\lambda$ is not assumed to be fixed in this example, redefine the domain of $p_i$ to be $\mathcal{V} \times \mathcal{V} \times \mathbb{R}_+$, so that $p_i$ is now also a function of $\lambda \in \mathbb{R}_+$. Thus,

$$\frac{\partial}{\partial x} p_i(v_i(x), v_i(y), \lambda) = v_i'(x) h(v_i(x), v_i(y), \lambda),$$

where

$$h(v_i(x), v_i(y), \lambda) = -\lambda e^{\lambda(v_i(x)+v_i(y))} / (e^{\lambda v_i(x)} + e^{\lambda v_i(y)})^2.$$

Also assume that for all $i \in N$ and $x \in X$, $u_i(x) \in [0, 1]$.

By Assumption 1 we get

$$\begin{aligned}
&\frac{\partial}{\partial x} p(v(x), v(y)) = \\
&\sum_{C \in W} \Big[ \big[ \sum_{i \in C} v_i'(x) h(v_i(x), v_i(y), \lambda) \prod_{j \in C \setminus \{i\}} p_j(v_j(x), v_j(y), \lambda) \prod_{j \notin C} (1 - p_j(v_j(x), v_j(y), \lambda)) \big] \\
&\quad - \big[ \sum_{i \notin C} v_t'(x) h(v_i(x), v_i(y), \lambda) \prod_{j \in C} p_j(v_j(x), v_j(y), \lambda) \prod_{j \notin C \cup i} (1 - p_j(v_j(x), v_j(y), \lambda)) \big] \Big] \\
&\quad = \sum_{i \in N} v_i'(x) h(v_i(x), v_i(y), \lambda) Z_i(\{p_j(v_j(x), v_j(y), \lambda)\}_{j \in N \setminus \{i\}})
\end{aligned} \tag{3.13}$$

where, letting **1** be an indicator function,

$$\begin{aligned}
&Z_i(\{p_j(v_j(x), v_j(y), \lambda)\}_{j \in N \setminus \{i\}}) = \\
&\qquad \sum_{C \in W} \Big[ \mathbf{1}_{\{i \in C\}} \prod_{j \in C \setminus \{i\}} p_j(v_j(x), v_j(y), \lambda) \prod_{j \notin C} (1 - p_j(v_j(x), v_j(y), \lambda)) \\
&\quad - \mathbf{1}_{\{i \notin C\}} \prod_{j \in C} p_j(v_j(x), v_j(y), \lambda) \prod_{j \notin C \cup \{i\}} (1 - p_j(v_j(x), v_j(y), \lambda)) \Big].
\end{aligned} \tag{3.14}$$

If we let $C_i^M$ be the set of minimal winning coalitions that $i$ is a member of, then we can rewrite

$$Z_i(\{p_j(v_j(x), v_j(y), \lambda)\}_{j\in N\setminus\{i\}})$$
$$= \sum_{C\in C_i^M} \prod_{j\in C\setminus\{i\}} p_j(v_j(x), v_j(y), \lambda) \prod_{j\notin C} (1 - p_j(v_j(x), v_j(y), \lambda)).$$

The function $Z_i$ can be thought of as the probability that Player $i$'s vote is pivotal in determining the winning outcome. Since we are considering unanimity rule, $Z_i$ is maximized when the $n-1$ other players vote with their maximum probability for $y$ over $x$. Since utility is restricted to the $[0,1]$ interval, values are bounded below by zero and above by $\frac{1}{1-\delta}$. Thus, the maximum probability with which a player can vote for one alternative over another is $e^{\frac{\lambda}{1-\delta}}/(1+e^{\frac{\lambda}{1-\delta}})$.
Using Equation 3.13 we get

$$|\frac{\partial}{\partial x} p(v(x), v(y), \lambda)| \leq n \max_{i\in N} |v_i'(x)| \max_{j\in N} |h(v_j(x), v_j(y), \lambda)| \left(\frac{e^{\frac{\lambda}{1-\delta}}}{1+e^{\frac{\lambda}{1-\delta}}}\right)^{n-1}.$$

By the assumption of logistic voting and the assumption that utility is bounded below by zero, we get

$$\max_{j\in N} |h(v_j(x), v_j(y), \lambda)| \leq \frac{\lambda}{4}.$$

Thus,

$$B = \frac{n\lambda}{4(1+e^{\frac{\lambda}{\delta-1}})^{n-1}}.$$

Since utility is restricted to the interval $[0,1]$ we get that for all $v$,

$$\max_{r,s\in X} \rho(v(r), v(s)) \leq \frac{1}{1-\delta}.$$

Thus, $|B| * \max_{r,s\in X} \rho(v(r), v(s)) < \frac{1-\delta}{\delta}$ for $\lambda \geq 0$ such that

$$\lambda < \frac{4(\delta-1)^2(1+e^{\frac{\lambda}{\delta-1}})^{n-1}}{\delta n}.$$

The right side of this equation is always strictly greater than zero, and so for any given $n$, there exists a $\lambda > 0$ such that Assumption 3 is met. Furthermore, for any $\lambda \in \mathbb{R}_+$, there exists an $n \in \mathbb{N}$ such that Assumption 3 is met, as the right side of the above equation approaches infinity as $n$ gets large. $\square$

**Definition:** A set of real-valued functions $\mathcal{V}^* \subset \mathcal{V}$ is *equicontinuous* if for all $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\rho(s,t) < \delta \text{ and } v_i \in \mathcal{V}^* \Rightarrow |v_i(s) - v_i(t)| < \epsilon.$$

For the purposes of the following proofs, we are concerned in particular with a set $\mathcal{B}_M^n \subset \mathcal{V}^n$ of vectors of differentiable functions taking $X$ to $\mathbb{R}$ whose derivatives are uniformly bounded by the constant $M$. This set is equicontinuous; let $M$ be a bound for the derivatives of the functions in $\mathcal{B}_M$, and recall that for $v \in \mathcal{V}^n$, $\rho(v(s), v(t)) = \max_{i\in N} |v_i(s) - v_i(t)|$. For a $\theta \in X$ and $J$ equal to the dimensionality of the policy space, let $|\nabla v_i(\theta)| = \max_{j\in J} |\frac{\partial v_i}{\partial \theta_j}|$. Then, by an extension of the Mean Value Theorem, $\rho(s,t) < \delta$ implies that $\rho(v(s), v(t)) = \max_i |\bigtriangledown v_i(\theta)|\rho(s,t) \leq M\delta$, for some $\theta$ on the line segment between $s$ and $t$. Thus, given $\epsilon > 0$, the choice $\delta = \epsilon/(M+1)$ demonstrates that $\mathcal{B}_M$, and thus $\mathcal{B}_M^n$, is equicontinuous.

**Lemma 6** *If Assumption 3 holds, then the function $g(v_t) = v_{t+1}$ maps a closed, bounded, and equicontinuous subset of $\mathcal{V}^n$ into itself.*

*Proof:* Boundedness is attained because $\delta < 1$. Let $\mathcal{B}_M^n$ be the set of vectors of differentiable functions whose derivatives are uniformly bounded by the constant $M$. The set $\mathcal{B}_M^n$

is closed. I will show that there exists an $M \in \mathbb{R}_+$ such that for any $v \in \mathcal{V}^n$, if $v \in \mathcal{B}_M^n$, then $g(v) \in \mathcal{B}_M^n$. By Equation 3.3 we know that for all $i$,

$$g(v_i(x)) = u_i(x) + \delta \int_{y \in X} v_i(y)p(v_i(x), v_i(y)) + v_i(x)(1 - p(v_i(x), v_i(y)))dQ(y)$$

and thus,

$$\begin{aligned}\frac{\partial}{\partial x} g(v_i(x)) &= u_i'(x) + \delta v_i'(x)(1 - \int_{y \in X} p(v(x), v(y))dQ(y)) \\ &+ \delta \int_{y \in X} (v_i(y) - v_i(x)) \frac{\partial}{\partial x} p(v(x), v(y))dQ(y).\end{aligned}$$

Using Assumption 3 we get

$$\begin{aligned}&\frac{\partial}{\partial x} g(v_i(x)) \leq u_i'(x) \\ &+ \delta \max_{j \in N} |v_j'(x)| \big(1 - \int_{y \in X} p(v(x), v(y))dQ(y) + |B| \int_{y \in X} (v_i(y) - v_i(x))dQ(y)\big) \\ &< \max_{j \in N} u_j'(x) + \gamma \max_{j \in N} |v_j'(x)|.\end{aligned}$$

for some $\gamma \in [0, 1)$.

Let $U = \max_{j \in N} u_j'(x)$. $U$ is assumed to be bounded. Now let $M = \frac{U}{1-\gamma}$. Then if $v \in \mathcal{B}_M^n$ we get

$$\begin{aligned}\frac{\partial}{\partial x} g(v_i(x)) &< U + \gamma M \\ &= U + \gamma \frac{U}{1 - \gamma} \\ &= \frac{U}{1 - \gamma} \\ &= M.\end{aligned}$$

Thus, for all $i \in N$ and $x \in X$, $g(v_i(x)) \in \mathcal{B}_M$, and so $g(v(x)) \in \mathcal{B}_M^n$. It follows that $g$ maps a closed, bounded and equicontinuous subset of $\mathcal{V}^n$ into itself. $\square$

Using this lemma, we can now establish the existence of a dynamically stable voting equilibrium.

**Proposition 6** *If Assumption 3 holds, then there exists a dynamically stable voting equilibrium when $X$ is infinite.*

*Proof:* The Heine-Borel Theorem in a function space tells us that a subset $\mathcal{V}^* \subset \mathcal{V}$ is compact if and only if it is closed, bounded, and equicontinuous.[2] Lemma 6 proves that the set of value functions can be restricted to the compact set $\mathcal{B}^n_M$. Since the function $g : \mathcal{B}^n_M \to \mathcal{B}^n_M$ such that $g(v_t) = v_{t+1}$ is continuous, we need only convexity of the set of value functions to prove that there exists a self-generating value function.

Take the convex combination of any two value functions, $v, w \in \mathcal{B}^n_M$, so that for any $\gamma \in [0,1]$, $\gamma v(x) + (1-\gamma)w(x) = z(x)$. Clearly $z$ is continuous, since $v$ and $w$ are continuous. Furthermore, $z'(x) = \gamma v'(x) + (1-\gamma)w'(x) \leq M$. Thus, $z$ is differentiable, and the derivative of $z$ is bounded by the constant $M$. It follows that $z \in \mathcal{B}^n_M$, and that $\mathcal{B}^n_M$ is convex. By Brouwer's Fixed Point Theorem, there exists a $v$ such that $g(v) = v$. $\square$

## 3.5 One-Dimensional Examples

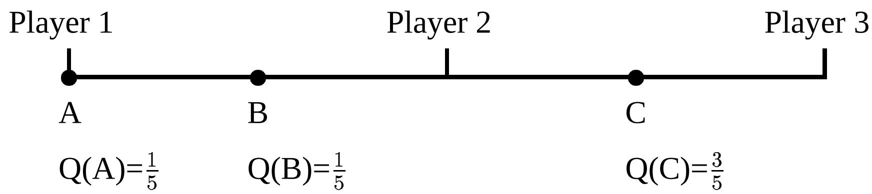
In this section I will present some simple examples of dynamically stable equilibria in a finite one-dimensional spatial setting under majority rule, when voters have single-peaked preferences. In all of the examples I will assume that $\delta = .9$ and that voting is deterministic, with

$$p_i(v_{it}(x), v_{it}(y)) = \begin{cases} 1 & \text{if } v_{it}(y) > v_{it}(x) \\ 0 & \text{if } v_{it}(y) < v_{it}(x) \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

**Example 2** *When value functions are not monotone in utility.*

[2] This theorem is a direct consequence of the Arzela-Ascoli Propagation Theorem.

Let $n = 3$, and $X = \{A, B, C\}$. The following figure depicts the spatial location of the ideal points of the three players, the locations of the three policies, and the frequency ($Q$) by which each policy is chosen to replace the status quo.




The above figure generates the following two tables, which show the utility functions of the three players and the set of valuations yielded in the long term, at a dynamically stable voting equilibrium.

**Myopic Utility**

| $i$ | $u_i(A)$ | $u_i(B)$ | $u_i(C)$ |
|---|---|---|---|
| 1 | 1 | $\frac{3}{4}$ | $\frac{1}{4}$ |
| 2 | $\frac{1}{2}$ | $\frac{3}{4}$ | $\frac{3}{4}$ |
| 3 | 0 | $\frac{1}{4}$ | $\frac{3}{4}$ |

**Equilibrium Valuations**

| $i$ | $v_i(A)$ | $v_i(B)$ | $v_i(C)$ |
|---|---|---|---|
| 1 | $\frac{185}{41}$ | $\frac{105}{23}$ | $\frac{80}{23}$ |
| 2 | $\frac{295}{41}$ | $\frac{15}{2}$ | $\frac{15}{2}$ |
| 3 | $\frac{225}{41}$ | $\frac{125}{23}$ | $\frac{150}{23}$ |

The following table summarizes the above information by depicting individuals' rankings over the alternatives, in the short and long term. If Player $i$ strictly prefers policy $x$ to policy $y$, it is notated $x \succ y$. If Player $i$ is indifferent between the two, it is written $x \sim y$.

**Individuals' Rankings of Alternatives**

| | Short Term | Long Term |
|---|---|---|
| Player 1 | A $\succ$ B $\succ$ C | B $\succ$ A $\succ$ C |
| Player 2 | B $\sim$ C $\succ$ A | B $\sim$ C $\succ$ A |
| Player 3 | C $\succ$ B $\succ$ A | C $\succ$ A $\succ$ B |

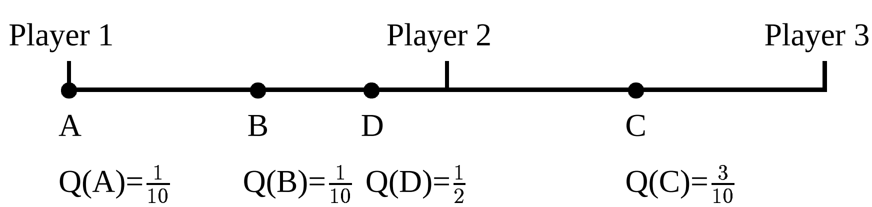
As is consistent with a traditional spatial model, the median voter is indifferent between $B$ and $C$, and strictly prefers both of these policies to $A$ in both the short and long run.

However, Player 1 (with ideal point A) strictly prefers $B$ to his own ideal point in the long run. This is because when $C$ is given an advantage over the other two policies by being chosen more often from density $Q$, the policy which makes Player 1 best off over time is not his ideal point, but the policy closest to his ideal point which defeats $C$, his least favorite policy. Thus, it is in Player 1's best interest to concede some utility in the current round to reap higher rewards in future rounds. Finally, Player 3 strictly prefers $A$ to $B$ in the long run, even though $B$ is closer to his ideal point. Loosely speaking, this is because at $A$ there is a 60 percent chance of transitioning to $C$, Player 3's favorite policy, while at $B$, this chance drops to 30 percent.

While this example is not surprising, it provides a clear picture of how this model works, and and demonstrates that the predictions that this model yields are often quite intuitive. In the next example I will add a fourth policy to the same three-player setting considered above, and show that the independence of irrelevant alternatives property fails to hold in the long term.

**Example 3** *Failure of IIA.*

Consider the same setting and players as above, but now add a new policy $D$ to policy space $X$. The following figure depicts the spatial location of the ideal points of the three players, the locations of the four policies, and the frequency ($Q$) by which each policy is chosen to replace the status quo. Note that the new frequencies of $A$, $B$, and $C$ are half of what they were in the previous example (and so the relative frequencies of these policies are the same as in the previous example).




As in the previous example, the above spatial setting generates the following two tables, which show the utility functions of the three players and the long-term valuations yielded at a dynamically stable voting equilibrium.

**Myopic Utility**

| $i$ | $u_i(A)$ | $u_i(B)$ | $u_i(C)$ | $u_i(D)$ |
|---|---|---|---|---|
| 1 | 1 | $\frac{3}{4}$ | $\frac{1}{4}$ | $\frac{3}{5}$ |
| 2 | $\frac{1}{2}$ | $\frac{3}{4}$ | $\frac{3}{4}$ | $\frac{9}{10}$ |
| 3 | 0 | $\frac{1}{4}$ | $\frac{3}{4}$ | $\frac{2}{5}$ |

**Equilibrium Valuations**

| $i$ | $v_i(A)$ | $v_i(B)$ | $v_i(C)$ | $v_i(D)$ |
|---|---|---|---|---|
| 1 | $\frac{20242}{3367}$ | $\frac{14043}{2368}$ | $\frac{12193}{2368}$ | 6 |
| 2 | $\frac{28013}{3367}$ | $\frac{318}{37}$ | $\frac{318}{37}$ | 9 |
| 3 | $\frac{13428}{3367}$ | $\frac{9637}{2368}$ | $\frac{11487}{2368}$ | 4 |

Again, the above information can be summarized in the following table, which depicts individuals' rankings over alternatives in both the short and long term.

**Individuals' Rankings of Alternatives**

| | Short Term | Long Term |
|---|---|---|
| Player 1 | A $\succ$ B$\succ$ D $\succ$ C | A $\succ$ D $\succ$ B $\succ$ C |
| Player 2 | D $\succ$ B $\sim$ C $\succ$ A | D $\succ$ B $\sim$ C $\succ$ A |
| Player 3 | C $\succ$ D $\succ$ B $\succ$ A | C $\succ$ B $\succ$ D $\succ$ A |

Recall that the relative frequencies of our initial three policies (as defined by $Q$) are the same as in Example 1, only halved. However, the presence of a Condorcet winner, $D$, makes the long-run behavior of the three initial policies very similar, as they are all likely to be defeated by the Condorcet winner. Thus, starting utility differentiates these policies more than their long-run behavior does. It follows that the players' rankings over the initial three alternatives changes with the addition of this fourth policy, and becomes monotone with respect to starting utility. For example, before $D$ was added to the policy space, Player 1 (in the long term) ranked the alternatives $B \succ A \succ C$. After the addition of $D$, he ranks the initial three policies $A \succ B \succ C$. Also note that we can directly verify that $D$ is a Condorcet winner; at a fixed point, for any Condorcet winner $c$, $v_{it}(c) = v_{io}(c) + \delta v_{it}(c)$, which implies that $v_{it}(c) = v_{io}(c)/(1 - \delta)$.

## 3.6 Numerical Results

What follows is a look at several numerical simulations of this model in continuous policy spaces. The first setting is that of a three-player constant sum game and the second setting is that of a three-player, two-dimensional spatial model where players have convex preferences. The simulations were run by discretizing the policy space into approximately nine hundred policies and then iterating the Markov process until it converged numerically to an approximate dynamically stable voting equilibrium. The graphs that follow depict both the equilibrium value functions of the players and the equilibrium marginal transition measure $f_X(x)$ over alternatives. $f_X(x)$ represents the likelihood that any given alternative is a future observed policy. In all of the simulations it is assumed that the voting rule is majority rule, that $\delta = 0.9$, and that players vote deterministically, as in the previous examples.

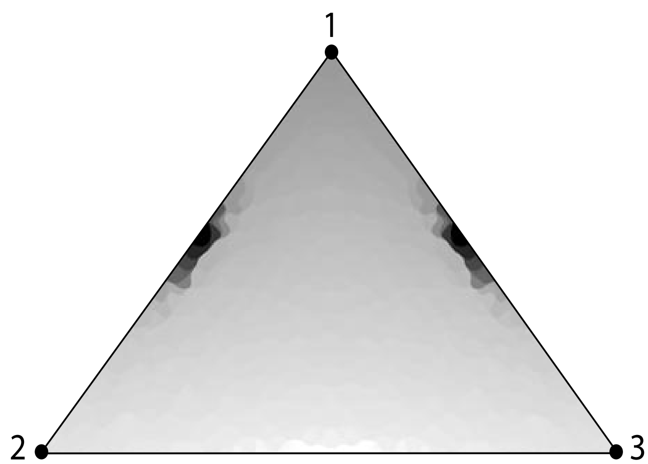



Figure 3.1: Player 1's value function with uniform $Q$.

**Example 4** *Three players divide a dollar when $Q(y)$ is uniform.*

Figure 3.1 is a graph of Player 1's value function. The setting is a divide-the-dollar game in which players have linear preferences. The two-dimensional unit simplex is pictured, and Player 1's ideal point (the policy $x = (1, 0, 0)$) is at the top of the simplex. The bottom of the simplex denotes those policies which give Player 1 no portion of the dollar. The darkest areas correspond to the policies which yield the highest values, and the lightest areas denote the policies which yield the lowest values. It is apparent that the policies which Player 1 values most are not Player 1's ideal point, but rather those which divide the dollar about equally between himself and one other player, or $(\frac{1}{2}, \frac{1}{2}, 0)$ and $(\frac{1}{2}, 0, \frac{1}{2})$. In social choice theory the set of policies which divide the dollar evenly between all members of a minimal winning coalition, in this case $\{(\frac{1}{2}, \frac{1}{2}, 0), (0, \frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, 0, \frac{1}{2})\}$, is referred to as the von Neumann-Morgenstern stable set.

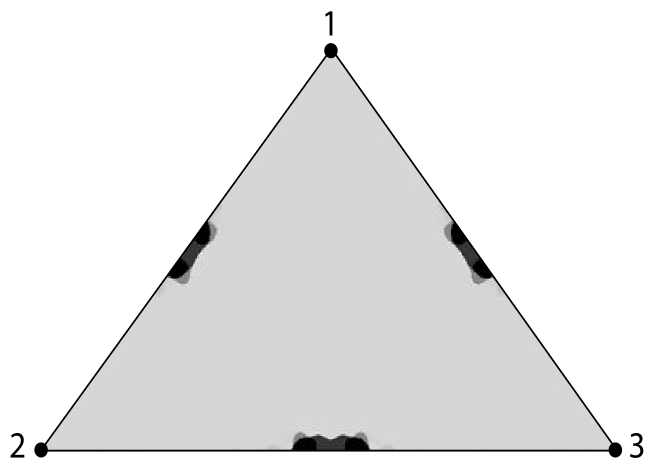



Figure 3.2: Density over outcomes when $Q$ is uniform.

Figure 3.2 depicts $f_X(x)$, the density over observed policy outcomes. This and all subsequent pictures of $f_X(x)$ were generated by drawing approximately 200,000 policies from the density $f_X(x)$ and then plotting their frequencies. The darkest areas correspond to the most frequently observed policies. In this example, only a small subset of the total policy space is ever observed with positive probability. In particular, the points in the stable set appear to constitute a majority rule top cycle set with respect to players' value functions. Figure 3.1 demonstrates this—since the setting is symmetric, it is clear that each of Player 1's most-preferred policies is also the most-preferred policy of another player.

**Example 5** *Three players divide a dollar when $Q$ draws heavily from the "corners" of the simplex.*

In this series of pictures the same divide-the-dollar setting is considered, however $Q$ is no longer uniform. In these examples, $Q$ draws heavily from the "corners" of the simplex, or from those policies which give most of the dollar to a single player. This particular $Q$ was chosen so as to be observationally similar to the equilibrium agenda-setting process that

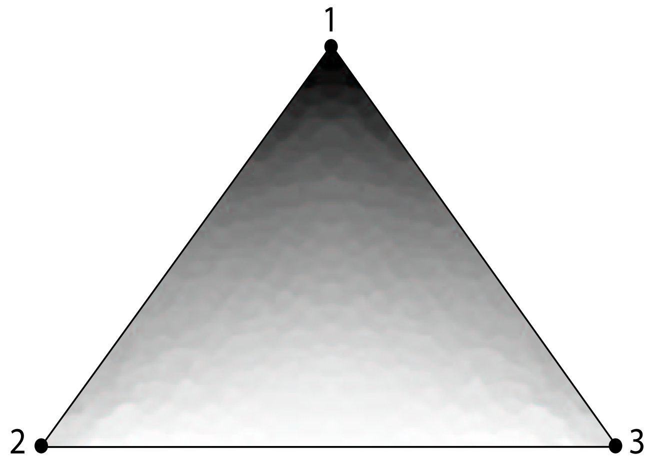



Figure 3.3: Player 1's value function when $Q$ draws heavily from the "corners".

Kalandrakis considers, in which players repeatedly propose their ideal points.[3]

Figure 3.3 shows that when alternatives to replace the status quo are picked from this different density, Player 1's ranking over alternatives changes. In this case, holding the portion of the dollar given to Player 2 or 3 fixed, Player 1's value is now monotone in his utility. This is because $Q$ weights most heavily those policies which are the least likely to defeat any other policy, given our voting rule. Thus, every policy is likely to remain in effect for a relatively long time once enacted, and so the utility a policy yields is a close proxy for what it is likely to yield over time. Interestingly, Player 1's preferences have become concavified; holding his utility constant, the policies which he prefers most are those which give the remainder of the dollar to only one other player.

Figure 3.4 shows that when $Q$ draws policies which lie near the ideal points of the players, the ideal points of the players tend to emerge most often as outcomes. It is not particularly surprising that the most frequently observed policies are those which are most frequently proposed. However, it is interesting to note that, as in Kalandrakis' model, if a player is going to get no portion of the dollar, he prefers the allocation which gives the entire dollar to another player. This is because once such an allocation passed, there is a

[3]The proposed agendas in Kalandrakis' model and this example are not observationally equivalent because agendas cannot be deterministic in this model. Every policy must have some $\epsilon$-probability of being proposed.

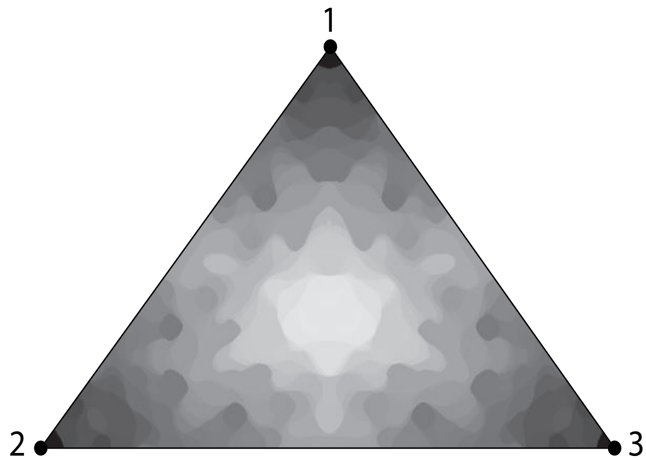



Figure 3.4: Density over outcomes when $Q$ draws heavily from the "corners".

high probability that he will get the entire dollar in a subsequent round.

This example is important because it hints at the possibility that it is really the endogeneity of the agenda-setting process in Kalandrakis' model that is driving the result. In the next chapter I show that the model presented here can also be extended to the case where players are probabilistic agenda-setters. Depending upon the parameters of the model (such as how much randomness there is in players' vote and proposal functions and how highly players discount the future), results are obtained which are both in keeping with, and antithetical to, Kalandrakis'.

**Example 6** *Three players divide a dollar with different rates of discounting.*

In this example, $Q$ is again uniform, but now players place different weights on future events; $\delta$ is not the same for each player. Let $\delta_i$ denote Player $i$'s discount rate. For Player 1, a payoff at time $t+1$ is worth $0.9$ of what a payoff at time $t$ is worth (i.e., $\delta_1 = .9$), but for Players 2 and 3, it is worth only $0.3$ of a time $t$ payoff (i.e., $\delta_2 = \delta_3 = 0.3$).

Player 1's long-term value function looks much the same as in Figure 3.3. However, the density over observed outcomes differs, and in particular, predicts the stable set alternatives $(\frac{1}{2}, \frac{1}{2}, 0)$ and $(\frac{1}{2}, 0, \frac{1}{2})$ as emerging with highest probability. This is pictured in Figure 3.5. Thus, by being more "patient" than the other two players, Player 1 is more likely to be included in a winning coalition. Similarly, if $\delta_1 = \delta_2 = 0.9$ and $\delta_3 = 0.3$, Figure 3.6

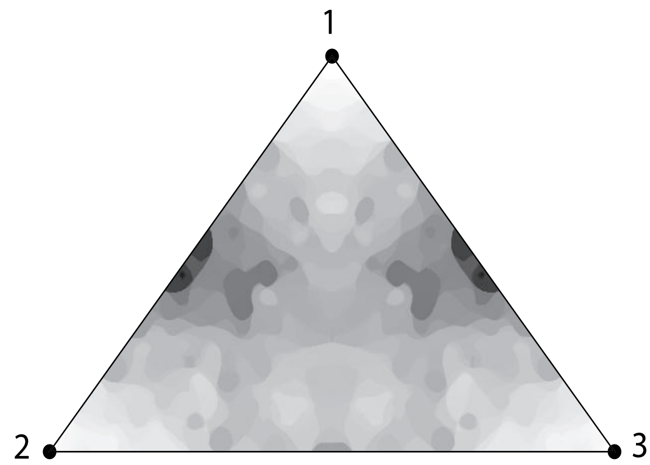



Figure 3.5: Density over outcomes with uniform Q, $\delta_1 = 0.9$, and $\delta_2 = \delta_3 = 0.3$.

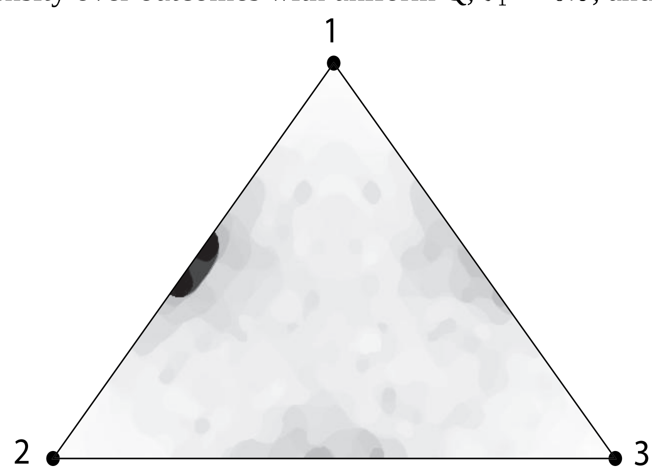



Figure 3.6: Density over outcomes with uniform Q, $\delta_1 = \delta_2 = 0.9$, and $\delta_3 = 0.3$.

shows us that the most frequently observed outcome, $(\frac{1}{2}, \frac{1}{2}, 0)$, corresponds to a coalition consisting of Players 1 and 2.

**Example 7** *A three-player, two-dimensional spatial model with circular preferences and uniform $Q$.*

The last series of pictures depict a two-dimensional spatial model, where the ideal points of the three players are no longer symmetric, but are located at $(0, \frac{1}{2})$, $(0, 0)$, and $(1, 0)$. The policy space is bounded by the lines connecting the ideal points of the three players, and

$Q(y)$ is assumed to be uniform. In this example preferences are assumed to be circular, so that players are indifferent between all policies equidistant from their ideal points. Figure 3.7 depicts the spatial location of the ideal points of the three players and their indifference curves. In this example, the policy space is equal to the Pareto set, or the set of alternatives $p \in X$ such that there is no other alternative $x \in X$ with the property that $u_i(x) \geq u_i(p)$ for all $i \in N$ and $u_i(x) > u_i(p)$ for some $i$.

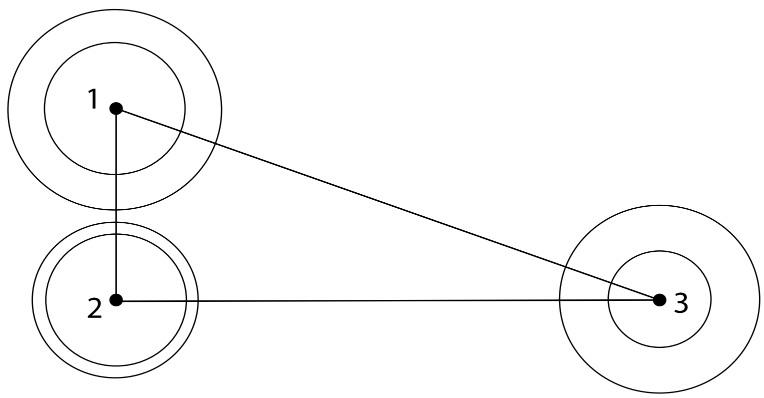



Figure 3.7: Two-dimensional spatial model with circular preferences.

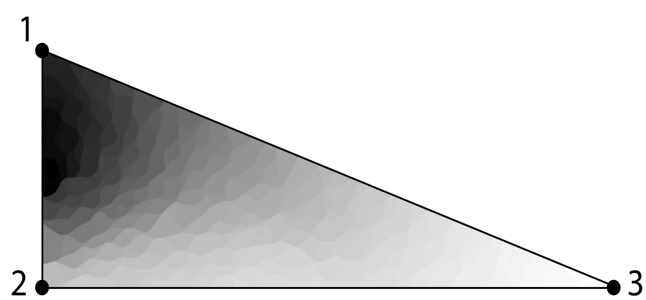



Figure 3.8: Player 1's value function with uniform $Q$ and circular preferences.

In this setting, the von Neumann-Morgenstern stable set approximately equals the points $\{(0, .19), (.28, .36), (.19, 0)\}$. Figure 3.8 depicts the value function of Player 1, whose ideal point is located at $(0, \frac{1}{2})$. In Figure 3.8, we can see that Player 1's most valued-alternative is approximately $(0, .25)$, closer to the alternative in the stable set corresponding to a coalition between himself and the player whose ideal point is $(0, 0)$ than to his own ideal point.

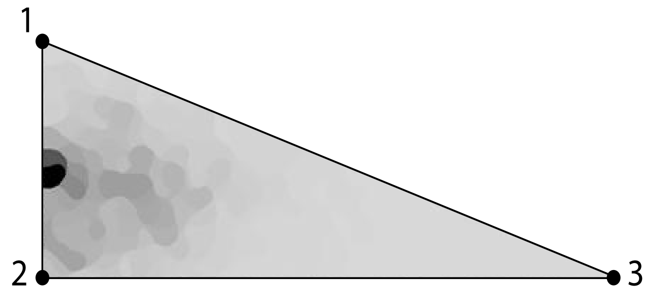



Figure 3.9: Density over outcomes with uniform $Q$ and circular preferences.

Figure 3.9 shows us that the most observed outcome is approximately $(0, .22)$, close to the alternative in the stable set corresponding to a coalition between Players 1 and 2, the two players whose ideal points are closest to each other. This alternative is essentially a core.

**Example 8** *Three-player, two-dimensional spatial model with elliptical preferences and uniform $Q$.*

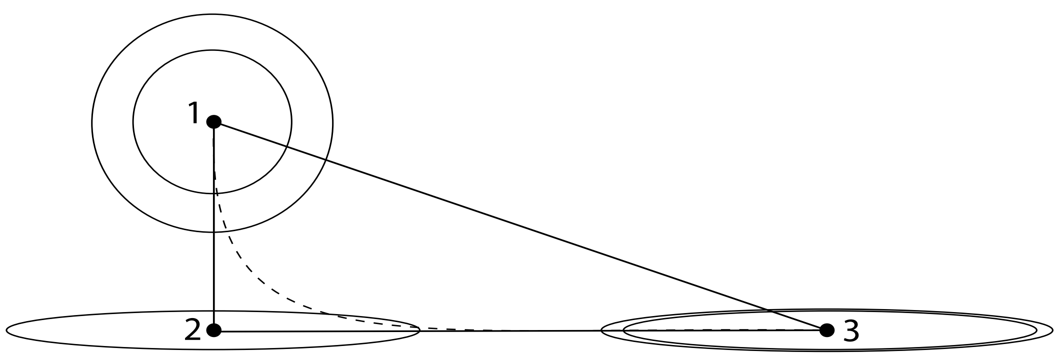



Figure 3.10: Two-dimensional spatial model with elliptical preferences.

In this last example, the preferences of Players 2 and 3 are now elliptical rather than circular, and are defined by the equation

$$u_i(x_1, x_2) = \sqrt{(p_{i1} - x_1)^2 + 100(p_{i2} - x_2)^2},$$

where $p_i = (p_{i1}, p_{i2})$ is the ideal point of player $i$. Thus, Players 2 and 3 value the second (or $y$) dimension of the policy space ten times more than the first. The preferences of Player 1 have remained unchanged. Pictured in Figure 3.10 are the ideal points of the three players and their indifference curves. The dotted curve represents the contract curve of Players 1 and 3, and is the upper bound of the Pareto set.

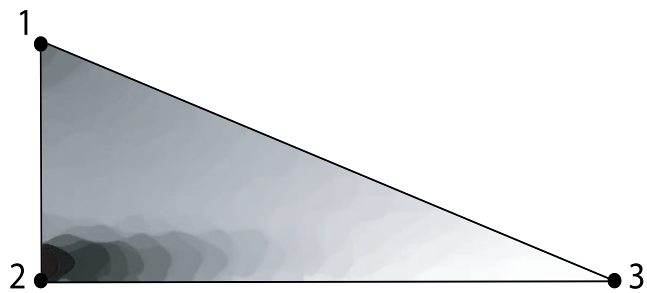



Figure 3.11: Player 1's value function with uniform $Q$ and elliptical preferences.

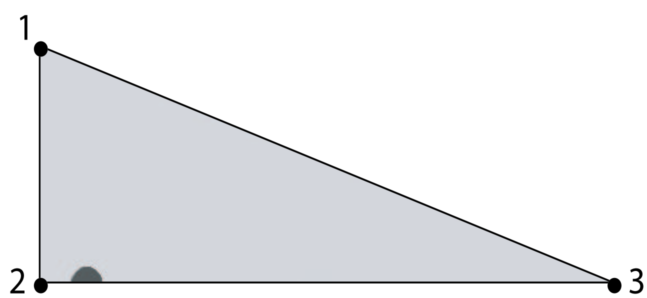



Figure 3.12: Density over outcomes with uniform $Q$ and elliptical preferences.

Interestingly, even though Player 1's utility function is the same as in the previous example, his value function is quite different than both his utility function and his value function in the previous example (when the preferences of the other two players were circular). Figure 3.11 shows that Player 1's most-preferred alternatives now lie close to the origin, the ideal point of Player 2. The reason for this is similar to the intuition behind Example 2. Since $Q$ is uniform and the indifference curves of Players 2 and 3 both favor policies which lie close to the $y$-axis, Player 1 knows that policies close to his own ideal point will surely be defeated. This is because the point $(0, \frac{1}{2})$, Player 1's ideal point, is the alternative in $X$ which is *farthest* from the $y$-axis. Thus, he is willing to concede utility along the second dimension of the policy space in order to collude with Player 2 along the first dimension.

Given the preferences of the players, the von Neumann-Morgenstern stable set approximately equals $\{(.14, .03), (0, .01), (.09, 0)\}$. Figure 3.12 shows that there exists a single alternative, $(.09, .02)$, which arises with near certainty. This alternative is close to the alternative in the stable set corresponding to a coalition between Players 1 and 3, whose "flattened" indifference curves now lie closer together than in the previous example.

## 3.7 Conclusions

This chapter presents a model of how individuals value policies in the environment of a continuing program, in which status quos are endogenously determined. In this setting, any policy enacted today will lead to a future stream of policies which are dependent upon it, to the extent that every policy must have defeated its predecessor. The chapter first presents a formal means of evaluating policies in terms of the types of outcomes they are likely to produce over time. It then uses these equilibrium evaluations to make predictions about the types of outcomes likely to emerge when programs are continuing and thus, when policies are continually being selected and reevaluated.

Policy evaluation is modeled as a dynamic process in which individuals vote on policies based upon the utility they believe the policies will yield in the long run. The focus of the analysis is to examine the types of outcomes that emerge when players' beliefs and voting strategies are consistent with one another. When this consistency is achieved, the value every player assigns to a policy equals the true future expected value of that policy, given the valuations that the *other* players assign to every policy. This is defined as a dynamically stable voting equilibrium.

I show in this setting that, in the absence of a game form, players are not indifferent between different policies which provide them with the same level of utility. This is because the space of alternatives which defeat each policy, and which each policy defeats, are substantively different. This model demonstrates that in dynamic environments, the space of alternatives which can and cannot defeat a policy may have as much impact on individual decision making as the substance of the policy itself.

Last, and possibly most interesting, is link between this model and cooperative game theory. Cooperative game theory examines the types of allocations that coalitions can procure for themselves, while remaining agnostic as to how these allocations arise, and how they are enforced. In the examples and numerical simulations presented here, outcomes often emerge which are in keeping with those predicted by cooperative solution concepts like the von Neumann-Morgenstern stable set and the core. Perhaps modeling foresight can provide a first step toward a behavioral rationalization of cooperative game theory.

# Chapter 4 A Model of Farsighted Voting, with Endogenous Agenda Formation

## 4.1 Introduction

A maintained assumption of the previous chapter was that policies to replace the status quo arose probabilistically, drawn from a fixed distribution. Thus, the model assumed that legislators do not explicitly set the agenda themselves, but have fixed beliefs over the types of alternatives which will be added to the agenda. This assumption was made for several reasons. The first was to construct a model of bargaining over a continuing program that could yield compelling predictions. Kalandrakis (2002) presents a model of dynamic agenda-setting and bargaining with an endogenous reversion point, and finds that the ideal points of the players emerge as policy outcomes with probability one, in a Markov-perfect Nash equilibrium. This result is, in large part, an artifact of the assumption that the players themselves are deterministic agenda-setters, and capable of proposing any policy. In reality, should a legislator deliberating over a collective choice be capable of proposing his ideal point in every round, with probability one? Because all players involved know the exact strategy of the next proposer, there is no incentive for players to propose policies that can last. In equilibrium, the policy outcome jumps from the ideal point of one player to another. In the previous chapter, the assumption of an *exogenous* agenda simply reflected the notion that players vote with the knowledge that the future is uncertain. Yet over time, some alternatives will be more likely to be brought to the floor than others, and legislators are aware of, and condition their future behavior upon, this. An exogenous distribution over future proposals could reflect the idea that the hands of an agenda-setter are often tied by the interests of his constituents, or the state of nature.

Second, a fixed agenda provided us with a simple parameter to vary. For instance, in Examples 4 and 5 of the previous chapter, the static agenda was varied from being uniform over the policy space, to drawing heavily from the "corners" of the two-dimensional unit simplex, or the ideal points of the three players. In the first case the model predicted policy outcomes in keeping with those predicted by cooperative game theory. In the second, the model predicted the same result found by Kalandrakis. Thus, we could hypothesize that it was the deterministic nature of the agenda-setting process that was driving Kalandrakis' result, which allowed players to repeatedly propose their ideal points.

However, the assumption of an exogenously drawn agenda leaves us unsatisfied because there are clearly so many instances in which it is inappropriate. If the definition of the alternatives is truly the supreme instrument of power, we would expect the process of agenda-setting to be an essential part of legislative debate. Furthermore, McKelvey has shown that for virtually any policy outcome, there exists an agenda that can induce it. Thus, the agenda may be the *most* essential aspect of legislative bargaining and debate. The definition of the alternatives on a legislative agenda dictates the issues that are defined as important to society and worthy of attention, and directly governs whose problems get attention and whose do not. In this sense the choice of agenda can, and often does, produce distinct winners and losers.

In this chapter I extend the model of the previous chapter to allow for endogenous agenda-setting on the part of legislators. Again, policy evaluation is modeled as a dynamic process in which individuals vote on policies based upon the utility they believe the policies will yield in the long run. However, now individuals also *propose* policies using these same valuations. I find that the specification of how proposers behave greatly effects both the types of policy outcomes likely to emerge, and the induced preferences of voters in the long term. For example, do individuals propose policies "sincerely", so that a given proposer is more likely to suggest an alternative that gives him a higher long-run valuation? Or do players propose "strategically", and condition upon the status quo at hand to propose a policy likely to defeat it? In Section 4.5 different specifications of behavior are discussed, and examples are provided.

Surprisingly, I find that some of the conclusions from the previous chapter still hold, and some do not. When players propose sincerely, we get an outcome similar to Kalandrakis; players propose policies near their ideal points and outcomes jump from one such policy to another. However, when players make proposals strategically, cooperative outcomes can emerge. An interesting finding is that there exists an "intermediate" level of randomness in players' strategies that generates cooperation. When Players are complete randomizers, or when they are entirely deterministic, cooperation rarely emerges. However, when players tremble only slightly in their strategies they become more capable of thinking ahead, and condition how they bargain on the idea that policies selected today should stand up to tomorrow's agenda.

The chapter proceeds as follows: Section 4.2 describes the notation used and presents the Markov model. Section 4.4 proves two analytic results. I first show that there always exists a self-generating value function. Then I prove that there exists a unique self-generating value function when the number of players is large, and the Markov process limits to this unique function. Section 4.5 provides analytic and numerical examples of the model in different legislative settings. Section 4.6 concludes.

## 4.2 The Model

### 4.2.1 Notation and Assumptions

I assume a set $N = \{1, 2, ..., n\}$ of *voters* (where $n$ is odd), a finite set $X \subset \mathbb{R}^m$ of *alternatives*, or *policies*, and, for each $i \in N$, voter preferences are represented by a real-valued *utility function*, $u_i : X \to \mathbb{R}_+$. A nonempty subset $C \subset N$ is called a *coalition*. I will restrict attention to *simple* and *anonymous* games, so that given a collection of coalitions $W$ with $C \in W$, then $C \subseteq C'$ implies $C' \in W$. Anonymity implies that the voting rules considered here are $q$-rules, such that for some fixed integer $q > n/2$, $W = \{C \subseteq N : |C| \geq q\}$. The collection $W$ can be considered the set of *winning* or *decisive* coalitions.

### 4.2.2 The Markov Model, with Endogenous Agenda Formation

Policy selection is modeled as a Markov process, and individual valuations over policies are conditioned upon this process. Players' valuations at time $t$ are represented by a vector of continuous value functions $v_t : X \to \mathbb{R}^n$. Player $i$'s value function at time $t$, $v_{it}$ is the $i^{th}$ element of vector $v_t$. We can interpret $\mathbb{R}^X$ as the set of all functions from $X$ into the real line. Then $v_{it} \in \mathbb{R}^X$ and $v_t \in \prod_{i\in N} \mathbb{R}^X$. The function $v_{it}$ is represented by

$$v_{i0}(x) = u_i(x) \tag{4.1}$$

and

$$v_{it+1}(x) = u_i(x) \tag{4.2}$$

$$+ \quad \delta \sum_{y\in X} \left[ v_{it}(y)p(v_t(x), v_t(y)) + v_{it}(x)(1 - p(v_t(x), v_t(y))) \right] Q(v_t(y)|v_t(x), v_t). \tag{4.3}$$

The function $v_{i0}$ equals the utility Player $i$ receives from alternative $x$, and the function $v_{it}(x)$ represents the expected utility Player $i$ receives from having policy $x$ enacted at time zero, *given that $t$ more policies will be enacted after it.* The probability of transitioning from state $x$ to state $y$ at time $t + 1$, given the two states are paired against each other, is represented by $p(v_t(x), v_t(y)) \in [0, 1]$. $Q(v_t(y)|v_t(x), v_t)$ is the probability mass from which alternatives $y$ to replace status quo $x$ are drawn. $\delta \in [0, 1)$ is a discount factor.

Individuals have an effect on policy selection through both the agenda setting process and the subsequent vote that is taken. These two procedures are represented by the $p$ and $Q$ functions in the above equation. Alternatives to replace the status quo arise probabilistically, picked from a transition measure, $Q(v(y)|v(x), v)$, which is dependent upon the valuations of the voters and the status quo policy. Thus, the likelihood that a policy is selected to be put to a vote against the status quo depends both upon how highly the voters value that policy, and upon the likelihood that it defeats the status quo. If policies $x$ and $y$ both defeat the status quo, but many voters assign a high value to policy $x$ and a low value to policy $y$, $x$ will be more likely than $y$ to be brought to a vote against the existing status quo. Similarly, the likelihood that $x$ will defeat the existing status quo $z$, or $p(v(z), v(x))$, will also depend upon the voters' valuations of $x$ and $z$. Let $s_i$ be the likelihood that Player $i$ is the agenda-setter at any given time. Thus, $\sum_{i\in N} s_i = 1$.

The following assumptions are made about the functional forms of $Q$ and $p$.

**Assumption 4** *Transition measure assumption*

For all $y \in X$, $Q(v(y), v|v(x))$ equals the probability that alternative $y$ is chosen to be put to a vote against the existing status quo $x$, given value vector $v$. It is assumed that

$$Q(v(y)|v(x), v) = \sum_{i \in N} s_i Q_i(v(y)|v(x), v),$$

where $Q_i(v(y)|v(x), v)$ represents the likelihood that, if chosen to make a policy proposal, Player $i$ will propose $y$ given value vector $v$ and status quo $x$. It is assumed that $Q_i(v(y)|v(x), v) \in (0, 1)$ for all $i$, and $\sum_{y \in X} Q_i(v(y)|v(x), v) = 1$, so that every policy has some probability of being proposed by Player $i$. It is also assumed that $Q_i$ is continuous and differentiable in all of its arguments.

**Assumption 5** *Partial derivatives of transition measure assumption*

It is assumed that the function $Q_i(v(y)|v(x), v)$ is dependent upon $v_j$, ($j \neq i$) only through the transition probability function $p(v(x), v(y))$. Thus, players care about the valuations of others only insofar as the votes of others determine future outcomes. Formally, this assumption implies that

$$\frac{\partial Q_i(v(y)|v(x), v)}{\partial v_j(x)} = \frac{\partial p(v(x), v(y))}{\partial v_j(x)} \cdot \frac{\partial Q_i(v(y)|v(x), v)}{\partial p(v(x), v(y))}.$$

**Assumption 6** *Transition probability assumption*

For all $x, y \in X$, $p(v(x), v(y))$, or the probability of transitioning from policy $x$ to policy $y$, given $x$ and $y$ are put to a vote and given value vector $v$, can be written as the probability of victory of $y$ over $x$:

$$p(v(x), v(y)) = \sum_{C \in W} \prod_{i \in C} p_i(v_i(x), v_i(y)) \prod_{i \notin C} (1 - p_i(v_i(x), v_i(y))) \tag{4.4}$$

where $p_i(v_i(x), v_i(y))$ represents Player i's probability of voting for $y$ over $x$ given value function $v_i$. It is assumed that $p_i(v_i(x), v_i(y)) \in (0, 1)$, and that $p_i(v_i(x), v_i(y)) + p_i(v_i(y),$

$v_i(x)) = 1$. It is also assumed that $p_i$ is continuous and differentiable in both of its arguments, and that $p_i$ is increasing in $v_i(y) - v_i(x)$.

## 4.3 Dynamically Stable Voting Equilibria, With Endogenous Proposals

As in the previous chapter, the focus of the following analysis is to prove the existence of, and numerically compute, value functions which are self-generating. When a player behaves according to such a function, the value he assigns to a policy equals the true future expected value of that policy. When this holds for *all* players, then the strategies of players generate value functions which generate the same strategies. Thus, beliefs and behavior are entirely consistent with each other. The difference between this model and the previous chapter is that now proposals are made endogenously.

Let $\mathcal{P}$ be the set of all functions taking taking $\mathbb{R}^n \times \mathbb{R}^n$ to the interval $[0, 1]$. Let $\mathcal{M}^X$ be the set of probability measures over the set $X$. Last, let $\mathcal{Q}$ be the set of functions taking the set $\mathbb{R}^n \times \prod_{x \in X} \mathbb{R}^n$ to $\mathcal{M}^X$. Then, at a given $u \in \prod_{i \in N} \mathbb{R}^X$, $p \in \mathcal{P}$, and $Q \in \mathcal{Q}$, a *dynamically stable voting equilibrium with endogenous proposals* is a collection of value functions, $v = \{v_i\}_{i \in N}$, such that for all $i \in N$ and $x \in X$,

$$v_i(x) = u_i(x) + \delta \sum_{y \in X} v_i(y) p(v_i(x), v_i(y)) + v_i(x)(p(v_i(y), v_i(x))) Q(v(y)|v(x), v).$$

Given the Markov process defined in Section 4.2.2, a dynamically stable voting equilibrium with endogenous proposals occurs at a fixed point, when $v_{t+1} = v_t$.

As in the previous chapter, this equilibrium concept also bears a close resemblance to quantal response equilibrium in some settings. Assuming that individual transition probabilities take a logistic form, the problem we ran into in the previous chapter was that in this model players vote sincerely, and do not condition upon the consequences of their votes. In the examples presented in Section 4.5, both sincere *and* strategic functional forms are assumed over individual proposal strategies $Q_i$, and in the strategic case this model bears a closer resemblance to QRE than the model presented in the previous chapter. For example, consider a logistic form over proposal strategies, so that for each player $i$,

$$Q_i(v(y)|v(x), v) = \frac{e^{\lambda(v_i(y)p(v(x),v(y))+v_i(x)(1-p(v(x),v(y))))}}{\sum_{z\in X} e^{\lambda(v_i(z)p(v(x),v(z))+v_i(x)(1-p(v(x),v(z))))}} \tag{4.5}$$

for a $\lambda \in \Re_+$. In this case, players make proposals based not only on how much they value policies, but also on the likelihood that a policy will defeat the status quo at hand. However, this concept is not the same as quantal response equilibrium because, given the specification of the individual transition probabilities, $p_i$, players can only condition on their own valuations and not on the likelihood that a policy will defeat the status quo. This is because $p_i$ is defined as a function of $v_i$. An alternate specification could be used, so that $p_i$ is instead a function of $v$, the vector of all players' valuations. For example, consider the case where players condition their votes on the likelihood that their votes are pivotal in determining the winning outcome, so that $i$'s expected value from casting a vote for $y$ is

$$Z_i(\{p_j(v^*(x), v^*(y))\}_{j\neq i})v_i^*(y),$$

or the likelihood that $i$'s vote is pivotal in the vote between $y$ and $x$ times his valuation of $y$. Letting $C_i^M$ equal the set of minimal winning coalitions that $i$ is in,

$$Z_i(\{p_j(v^*(x), v^*(y))\}_{j\in N\setminus\{i\}}) = \sum_{C\in C_i^M} \prod_{j\in C\setminus\{i\}} p_j(v^*(x), v^*(y)) \prod_{j\notin C}(1 - p_j(v^*(x), v^*(y))).$$

When players vote according to these expected values, we can define $p_i$ such that

$$p_i(v(x), v(y)) = \frac{e^{\lambda(Z_i(\{p_j(v^*(x),v^*(y))\}_{j\neq i})v_i^*(y))}}{e^{\lambda(Z_i(\{p_j(v^*(x),v^*(y))\}_{j\neq i})v_i^*(y))} + e^{\lambda(Z_i(\{p_j(v^*(y),v^*(x))\}_{j\neq i})v_i^*(x))}}. \tag{4.6}$$

If these two specifications of $p_i$ and $Q_i$ are used, then it is not difficult to demonstrate that the behavior generated by a dynamically stable voting equilibrium with endogenous proposals is equivalent to a quantal response equilibrium.

## 4.4 Analytic Results

It is assumed throughout that as $|N| \to \infty$, $s_i \to 0$ for all $i$, so that as the number of players gets large, the likelihood that any particular player is chosen to be the proposer gets small.

**Proposition 7** *There exists a self-generating vector of value functions.*

*Proof:* Since $\delta < 1$ and $u_i$ is real-valued for all $i \in N$, the upper bound any individual's value function could take is $\frac{1}{1-\delta} \max\limits_{x \in X} u_i(x)$, and the lower bound is zero. Thus, for every $v_t \in \prod_{i \in N} \mathbb{R}^X$, $v_t \in \prod_{i \in N} [0, \frac{1}{1-\delta} \max\limits_{x \in X} u_i(x)]^X$, and so the set of value functions is bounded. Furthermore, the set of value functions is convex, since the convex combination of two bounded functions taking $X$ to $\mathbb{R}$ is itself bounded. Last, the set of value functions is closed, as the set $\prod_{i \in N} [0, \frac{1}{1-\delta} \max\limits_{x \in X} u_i(x)]^X$ is closed. It follows that the set of value functions taking $X$ into the real numbers $\mathbb{R}$ is a nonempty, closed, bounded and convex subset of a finite-dimensional vector space, $\mathbb{R}^X$.

The mapping $g : \prod_{i \in N} \mathbb{R}^X \to \prod_{i \in N} \mathbb{R}^X$, such that $g(v_t) = v_{t+1}$ (see Equation 4.2) is single-valued by definition, and is continuous by the continuity of every $p_i(v_{it}(x), v_{it}(y))$ and $Q_i(v_t(y)|v_t(x), v_t)$. By Brouwer's Fixed Point Theorem, there exists a $v_t \in \prod_{i \in N} \mathbb{R}^X$ such that $g(v_t) = v_t$. Thus, there exists a self-generating vector of value functions. $\square$

The next proposition proves that when the number of players is large, the sequence $\{v_t\}$ defined by Equations 4.1 and 4.2 is such that $v_t \to v^*$ as $t \to \infty$. Thus, the Markov process will limit to a unique self-generating vector of value functions, or dynamically stable voting equilibrium with endogenous proposals.

**Proposition 8** *There exists an $M \in \mathbb{N}$ such that whenever $n = |N| > M$, the function $g(v_t) = v_{t+1}$ is a contraction mapping.*

*Proof:* For $w, z \in \prod_{i \in N} \mathbb{R}^X$, let $\rho(w_i, z_i) = \max\limits_{x \in X} |w_i(x) - z_i(x)|$, and let $\rho(w, z) = \max\limits_{i \in N} \rho(w_i, z_i)$. We must show that for any $w, z \in \prod_{i \in N} \mathbb{R}^X$, $\rho(g(w), g(z)) < \rho(w, z)$.

Let $g_i : \prod_{i \in N} \mathbb{R}^X \to \mathbb{R}^X$ be such that for all $v_t \in \mathbb{R}^X$, $g_i(v_t) = v_{it+1}$. Thus, $g = (g_1, ..., g_n)$. First consider the gradient vector $\nabla g_i$ with respect to $v$. For all $x \in X$,

$$g_i(v(x)) = u_i(x) + \delta \sum_{y \in X} v_i(y)p(v(x), v(y)) + v_i(x)(1 - p(v(x), v(y)))Q(v(y)|v(x), v).$$

Thus, the components of $\nabla g_i(v(x))$ can be defined using the partial derivatives

$$\begin{aligned} \frac{\partial g_i(v(x))}{\partial v_i(x)} = & \\ & \delta[1 - \sum_{y \in X} p(v(x), v(y))Q(v(y)|v(x), v)] \\ + & \delta \sum_{y \in X} (v_i(y) - v_i(x)) \frac{\partial p(v(x), v(y))}{\partial v_i(x)} Q(v(y)|v(x), v) \\ + & \delta \sum_{y \in X} \left[ v_i(y)p(v(x), v(y)) + v_i(x)(1 - p(v(x), v(y))) \right] \frac{\partial Q(v(y)|v(x), v)}{\partial v_i(x)} \end{aligned} \tag{4.7}$$

and for all $j \in N \setminus \{i\}$,

$$\begin{aligned} \frac{\partial g_i(v(x))}{\partial v_j(x)} = & \\ & \delta \sum_{y \in X} (v_i(y) - v_i(x)) \frac{\partial p(v(x), v(y))}{\partial v_j(x)} Q(v(y)|v(x), v) \\ + & \delta \sum_{y \in X} \left[ v_i(y)p(v(x), v(y)) + v_i(x)(1 - p(v(x), v(y))) \right] \frac{\partial Q(v(y)|v(x), v)}{\partial v_j(x)}. \end{aligned} \tag{4.8}$$

Using Assumption 4 we get that for all $i \in N$,

$$\frac{\partial Q(v(y)|v(x), v)}{\partial v_i(x)} = \sum_{j \in N} s_j \frac{\partial Q_j(v(y)|v(x), v)}{\partial v_i(x)}. \tag{4.9}$$

It follows that

$$\frac{\partial Q(v(y)|v(x),v)}{\partial v_i(x)} \leq s_i \frac{\partial Q_i(v(y)|v(x),v)}{\partial v_i(x)} + (1-s_i)\max_{j\neq i} \frac{\partial Q_j(v(y)|v(x),v)}{v_i(x)}.$$

Since $Q_i$ is a continuously differentiable function over a compact set, the derivatives of $Q_i$ must be bounded by a constant. Let $\overline{\partial Q} = \max_{i\in N}\big(\max_{x,y\in X} \frac{\partial Q_i(v(y)|v(x),v)}{\partial v_i(x)}\big)$. Similarly, let $\overline{\partial Q_{-i}/\partial p} = \max_{j\neq i} \frac{\partial Q_j(v(y)|v(x),v)}{\partial p(v(x),v(y))}$, where $\overline{\partial Q_{-i}/\partial p}$ is also bounded by a constant.

Using Assumption 6 we get that for all $i \in N$,

$$\frac{\partial p(v(x),v(y))}{\partial v_i(x)} = \frac{\partial p_i(v_i(x),v_i(y))}{\partial v_i(x)} Z_i(\{p_j(v_j(x),v_j(y))\}_{j\in N\setminus\{i\}}) \tag{4.10}$$

where, letting $C_i^M$ equal the set of minimal winning coalitions that $i$ is in,

$$Z_i(\{p_j(v_j(x),v_j(y))\}_{j\in N\setminus\{i\}}) = \sum_{C\in C_i^M} \prod_{j\in C\setminus\{i\}} p_j(v_j(x),v_j(y)) \prod_{j\notin C}(1-p_j(v_j(x),v_j(y))).$$

For ease of notation, let $Z_i(\{p_j(v_j(x),v_j(y))\}_{j\in N\setminus\{i\}}) = Z_i(\cdot)$. $Z_i(\cdot)$ represents the probability that Player $i$'s vote is pivotal given that all other players $j$ vote according to the functions $p_j(v_j(x),v_j(y))$. McKelvey and Patty (2002, Lemma 1) prove that when people vote probabilistically (i.e when for all $j \in N$, and all $x, y \in X$, $p_j(v_j(x),v_j(y)) \in (0,1)$), all pivot probabilities $Z_i(\cdot) \to 0$ as $|N|$ gets large. We also know that since $p_i$ is a continuously differentiable function over a compact set, the derivatives of $p_i$ are bounded by a constant. Let Let $\overline{\partial p} = \max_{i\in N}\big(\max_{x,y\in X} \frac{\partial p_i(v_i(x),v_i(y))}{\partial v_i(x)}\big)$.

Let $\overline{v} = \max_{i\in N}\big(\max_{x,y\in X} \rho(v_i(x),v_i(y))\big)$. We know that $\overline{v}$ is bounded by a constant because utility is bounded and $\delta < 1$. Then, combining Equations 4.8, 4.9 and 4.10, and using Assumption 5, we get for all $j \in N \setminus \{i\}$

$$\frac{\partial g_i(v(x))}{\partial v_j(x)} = \tag{4.11}$$
$$\begin{aligned}
& \delta \sum_{y \in X} (v_i(y) - v_i(x)) \frac{\partial p_j(v_j(x), v_j(y))}{\partial v_j(x)} Z_j(\cdot) Q(v(y)|v(x), v) \\
+ \quad & \delta \sum_{y \in X} \left[ v_i(y) p(v(x), v(y)) + v_i(x)(1 - p(v(x), v(y))) \right] \sum_{k \in N} \frac{\partial Q_k(v(y)|v(x), v)}{\partial v_j(x)} \\
< \quad & \delta\big( \overline{v}\, \overline{\partial p}\, Z_j(\cdot) + \ |X|\, \overline{v}\, \left[ s_j\, \overline{\partial Q} + (1 - s_j) \overline{\partial p}\, \overline{\partial Q_{-j}/\partial p}\, Z_j(\cdot) \right] \big).
\end{aligned}$$

Since $Z_j(\cdot) \to 0$ as $|N| \to \infty$, and since $s_i \to 0$ as $|N| \to \infty$ by assumption, it follows that for any $\epsilon > 0$ there exists an $M \in \mathbb{N}$ such that for all $n = |N| > M$,

$$\frac{\partial g_i(v(x))}{\partial v_j(x)} < \epsilon.$$

Using Equation 4.7, by the same logic it follows that for any $\epsilon > 0$ there exists an $M \in \mathbb{N}$ such that for all $n = |N| > M$,

$$\frac{\partial g_i(v(x))}{\partial v_i(x)} < \delta(1 + \epsilon).$$

Define $|\nabla g(v)|$ such that

$$|\nabla g(v)| = \max_{\{i,j\} \in N} \left( \max_{x \in X} \left| \frac{\partial g_i(v(x))}{\partial v_j(x)} \right| \right).$$

Since $\delta < 1$, it follows that for any $v \in \mathbb{R}^X$, and for $|N|$ sufficiently large (i.e., $\epsilon$ sufficiently small), $|\nabla g(v)| < 1$.

By the Mean Value Theorem we know that

$$\rho(g(w), g(z)) \leq \rho(w, z) |\nabla g(v)|$$

for some $v$ on the line segment between $w$ and $z$. Since $|\nabla g(v)| < 1$ for $|N|$ sufficiently large, it follows that

$$\rho(g(w), g(z)) < \rho(w, z).$$

Thus, there exists an $M \in \mathbb{N}$ such that for all $n = |N| > M$, the function $g$ is a contraction mapping. □

## 4.5 Examples

Although the results of Section 4.4 assume no specific functional form for $p_i$ and $Q_i$, for the purposes of Examples 9 and 10 it is assumed that both possess a logistic form, and that players make proposals based solely on their own valuations of policies and propose policies irrespective of the status quo at hand. Thus,

$$p_i(v_i(x), v_i(y)) = \frac{e^{\lambda v_i(y)}}{e^{\lambda v_i(x)} + e^{\lambda v_i(y)}}$$

and

$$Q_i(v(y)|v(x), v) = \frac{e^{\lambda v_i(y)}}{\sum_{x \in X} e^{\lambda v_i(x)}},$$

for some $\lambda \in \mathbb{R}_+$. I will refer to this as *sincere* proposing, because players make proposals based solely on their valuations of policies, and not on the likelihood that their proposal will defeat the status quo at hand. In all of the examples, it is assumed that $s_i = \frac{1}{n}$ for all $i \in N$, and that $\delta = 0.9$.

Example 9 examines this specification numerically, in a divide-the-dollar setting. The policy space, $\Delta$, equal to the two-dimensional unit simplex, has been uniformly discretized into approximately nine hundred alternatives, and it is assumed that $\lambda = 5$ and $n = 3$. In this setting it is assumed that for each $x = (x_1, x_2, x_3) \in \Delta$, $u_i(x) = x_i$.

**Example 9** *A divide-the-dollar game, with sincere proposals and* $\lambda = 5$*.*

In this example, players propose policies sincerely and probabilistically, and at time $t$, are more likely to propose policies which yielded them a high value at time $t - 1$. Below, the value function of Player 1 is pictured, where Player 1's ideal point (the policy $x = (1, 0, 0)$) is located at the top of the simplex. Dark areas correspond to the policies which yield Player 1 a high long-term value.

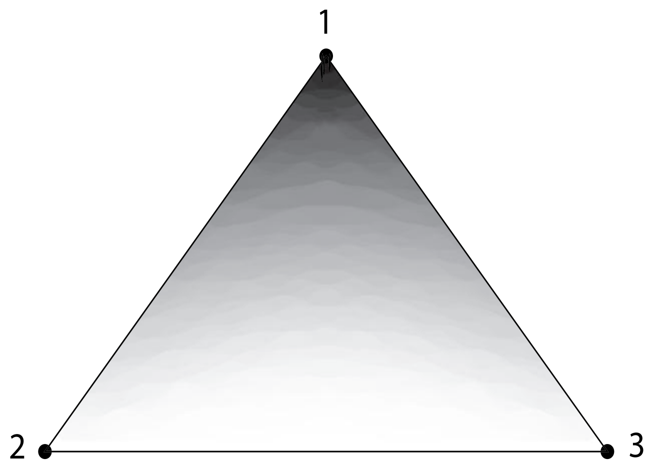



Figure 4.1: Player 1's value function under probabilistic "sincere" proposing.

Figure 4.2 depicts the equilibrium density over observed outcomes, given players propose sincerely and probabilistically. Dark areas correspond to the more frequently observed policies. In this figure we can see that the policies most likely to emerge as outcomes are the ideal points of the players. This result is similar to the Markov-perfect equilibrium found by Kalandrakis (2002), for a game in which players vote in discrete time over two alternatives, and the status quo in a given round is the previous round's winner. In each round, the alternative to be pitted against the status quo is chosen strategically, by a randomly chosen player. Interestingly, even though players propose sincerely in this example, the outcomes are similar to those observed in Kalandrakis' game; he finds that after a finite number of rounds, every implemented policy is the ideal point of some player. This result is also generated when we allow proposals to occur exogenously, but picked from a distribution which places more weight on the ideal points of the three players, as seen in Example 5 of the previous chapter. Thus, the ideal points of the three players may emerge as outcomes in three very different games; one in which proposals are endogenous and sincere, one in which proposals are endogenous and strategic, and one in which proposals are exogenously chosen. However, the similarity between all of these settings is that the ideal points of the three players are the alternatives most likely to be added to the agenda. This leads us to question whether such a $Q$ is reasonable.

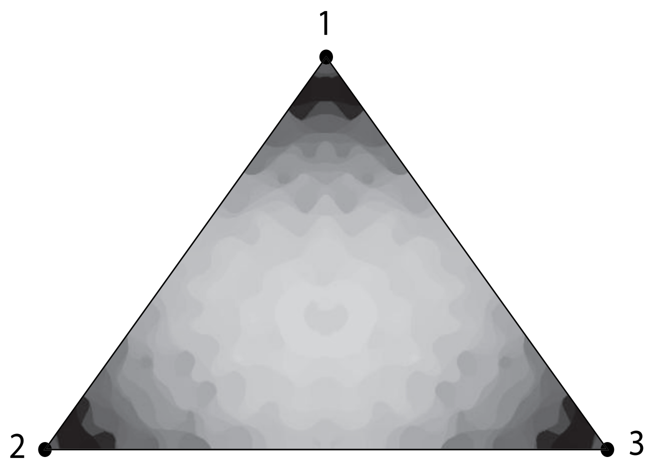



Figure 4.2: Density over outcomes under probabilistic “sincere” proposing.

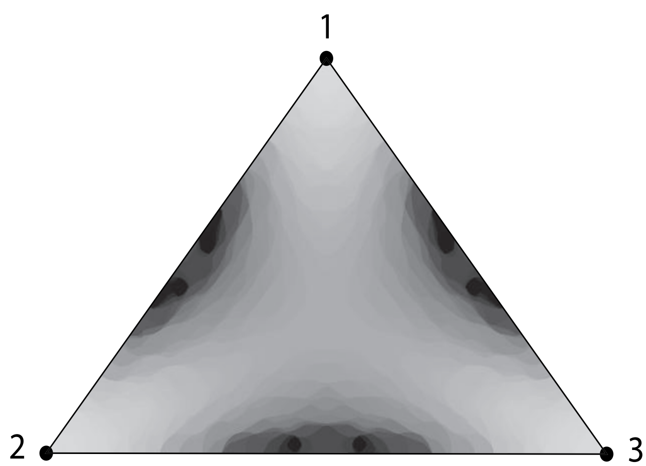



Figure 4.3: A graph depicting the relative area of the simplex defeated by each policy.

Figure 4.3 graphs policies as a function of the area of the policy space that they defeat, via the majority preference relation. The darker policies defeat a larger area of the policy space, when players vote according to their long-term valuations. Interestingly, even though the ideal points of the players are the most frequently observed outcomes, this figure shows us that the policies which are the most "fit", or defeat the most other policies when players vote according to their long-term valuations, are actually centered around the points $\{(\frac{1}{2},\frac{1}{2},0),(0,\frac{1}{2},\frac{1}{2}),(\frac{1}{2},0,\frac{1}{2})\}$, or the simple von Neumann-Morgenstern stable set. This is not the result we get when players vote according to their short-term utility. The policy that defeats the most others when players vote according to their utility is the centroid, $(\frac{1}{3},\frac{1}{3},\frac{1}{3})$.

A possible explanation for the fact that the ideal points of the three players are the most frequently observed outcomes in this example is that $\lambda = 5$; if $\lambda$ is too low, players may find it in their best interest to propose their ideal points because there is a moderate likelihood that the other players will "mess up" and vote in favor of it. However, it turns out that this is not the case. The following figures depict the equilibrium value function of Player 1 and the density over outcomes in this same setting, but when $\lambda = 20$. This, players are far less likely to tremble in their vote and proposal strategies.

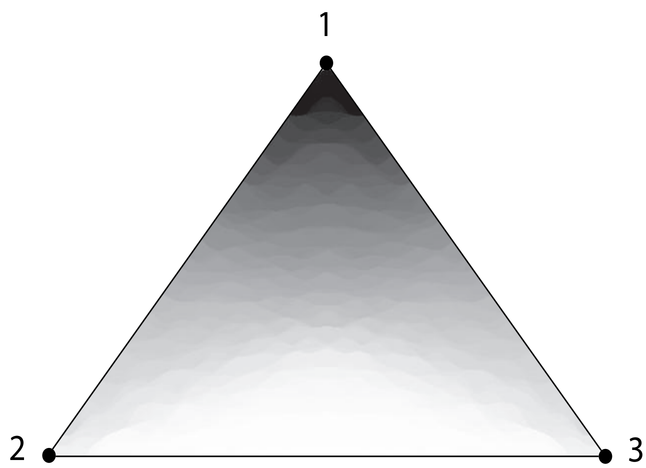



Figure 4.4: Player 1's value function under sincere proposing, with $\lambda = 20$.

We can see in comparing Figures 4.2 and 4.5 that when players propose alternatives sincerely, increasing the size of $\lambda$, or decreasing the amount of randomness in players' vote and proposal functions, actually leads to outcomes that are even more centered about the ideal points of the three players.

Examples 10 and 11 examine this sincere specification analytically, under the most extreme

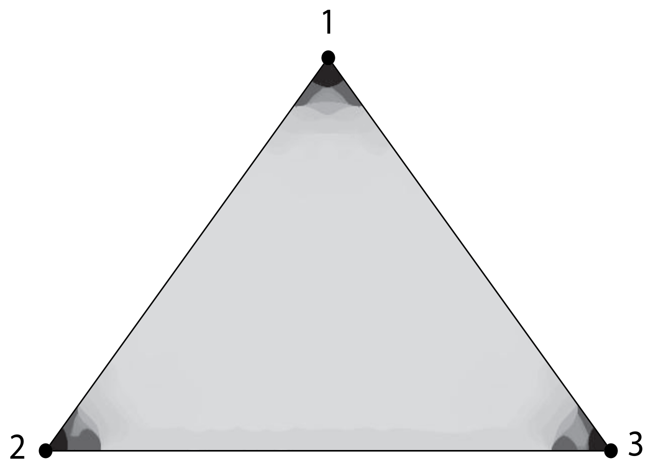



Figure 4.5: Density over outcomes under sincere proposing, with $\lambda = 20$.

cases of logistic voting; In Example 10, $\lambda = 0$ and in Example 11, voting is deterministic (corresponding to $\lambda = \infty$). Since the policy space in Example 11 is continuous, it is assumed instead that

$$Q_i(v(y)|v(x), v) = \frac{e^{\lambda v_i(y)}}{\int_{x \in X} e^{\lambda v_i(x)} dx}.$$

**Example 10** *When players randomize completely ($\lambda = 0$).*

In this example it is assumed that players randomize completely, both in their proposing and vote strategies. Thus,

$$p_i(v_{it}(x), v_{it}(y)) = \frac{1}{2} \text{ for all } x, y \in X$$

and

$$Q_i(v_t(y)|v_t(x), v_t) = \frac{1}{|X|} \text{ for all } y \in X.$$

Let Player $i$'s average value at time $t$ be denoted $\overline{v}_{it}$, so that

$$\overline{v}_{it} = \sum_{y \in X} v_{it}(y) \cdot \frac{1}{|X|}.$$

Given $p_i$ and $Q_i$ as defined above, at a fixed point

$$v_i(x) = u_i(x) + \delta(\frac{1}{2}v_i(x) + \frac{1}{2}\overline{v}_i)$$

and thus,

$$v_i(x) = (\frac{2}{2-\delta})(u_i(x) + \frac{\delta}{2}\overline{v}_i).$$

Using this formula we get

$$\begin{aligned}
\overline{v}_i &= \frac{1}{|X|}\sum_{y\in X}(\frac{2}{2-\delta})(u_i(y) + \frac{\delta}{2}\overline{v}_i) \\
&= \frac{\delta}{2-\delta}\overline{v}_i + \frac{2}{2-\delta}\frac{1}{|X|}\sum_{y\in X} u_i(y) \\
&= \frac{2}{2-2\delta}\frac{1}{|X|}\sum_{y\in X} u_i(y) \\
&= \frac{1}{1-\delta}\overline{u}_i,
\end{aligned}$$

and solving for a fixed point we get

$$v_i^*(x) = \frac{2}{2-\delta}u_i(x) + \frac{\delta}{(2-\delta)(1-\delta)}\overline{u}_i.$$

It follows that in this example, long-term valuations are simply a linear transformation of starting utility. Furthermore, the equilibrium distribution over observed outcomes will be uniform over the entire policy space. □

**Example 11** *A divide-the-dollar game with deterministic voting and proposals.*

Although Section 4.2 assumes that the policy space is finite, assume for the purposes of this example, that $|X| = \Delta$, the two-dimensional unit simplex. Let $|N| = 3$. Thus, for every $x = (x_1, x_2, x_3) \in X$, $x_i \geq 0$, $\sum_i x_i = 1$, and $u_i(x) = x_i$. Last, assume that players vote for and propose policies deterministically, so that

$$p_i(v_{it}(x), v_{it}(y)) = \begin{cases} 1 & \text{if } v_{it}(y) > v_{it}(x) \\ 0 & \text{otherwise} \end{cases}$$

and, letting $Y_i = \{y : y \in \underset{z \in X}{\operatorname{argmax}}\, v_{it}(z)\}$,

$$Q_i(v_t(y)|v_t(x), v_t) = \begin{cases} \frac{1}{\int_{z \in Y_i} dz} & \text{if } y \in Y_i \\ 0 & \text{otherwise.} \end{cases}$$

In case of a tie, players vote for the status quo.[1] $Q_i(v_t(y)|\ v_t(x), v_t)$ has support equal to the set of policies which maximize Player $i$'s time $t$ value function. Since value functions are not necessarily continuous in this example, such maxima need not exist. However, I will demonstrate that this setup yields a fixed point vector of value functions.

Note that at time $t = 0$, $Q_i(v_0(y)|v_0(x), v_0) = 1$ for $y$ such that $y_i = 1$ and, for $j \neq i$, $y_j = 0$. Thus, at time $t = 1$, Player $i$'s value function is

$$\begin{aligned} v_{i1}(x) &= x_i + \delta \frac{1}{3}\Big( \\ & \quad v_{i0}(1,0,0)p(v_0(x), v_0((1,0,0))) + v_{i0}(x)(1 - p(v_0(x), v_0((1,0,0)))) \\ &+ v_{i0}(0,1,0)p(v_0(x), v_0((0,1,0))) + v_{i0}(x)(1 - p(v_0(x), v_0((0,1,0)))) \\ &+ v_{i0}(0,0,1)p(v_0(x), v_0((0,0,1))) + v_{i0}(x)(1 - p(v_0(x), v_0((0,0,1))))\Big) \\ &= x_i + \delta v_{i0}(x) \\ &= x_i + \delta x_i. \end{aligned}$$

Iterating this process, we can see that for all $i$ and at time $t+1$, $v_{it+1}(x) = x_i + \delta v_{it}(x)$. This is because any status quo policy, $x \in \Delta$ will defeat any policy $y \in \cup_{j \in N} Y_j$ with probability one, and so $i$'s payoff from having $x$ enacted today equals his payoff from having $x$ enacted for every subsequent round, or $\frac{1}{1-\delta} x_i$. Thus, a fixed point is generated at $v_i^*(x) = \frac{1}{1-\delta} x_i$, and so the long-term valuations of players simply equal their short-term utility, multiplied by a constant. The equilibrium distribution over observed outcomes predicts the first status quo alternative as the outcome in every subsequent round, with probability one. $\square$

[1]Under the definition of logistic voting given at the beginning of this section, ties should be broken probabilistically, so that players vote with equal probability for policies they are indifferent between. The deterministic tie-breaking assumption used in this example is made so that a maximum value function exists when calculating $Q_i$.

As in Example 10, here long-term valuations are also a linear transformation of initial utility, and as in Example 10, the predictions yielded by this example are not particularly compelling. Thus, the idea that players make proposals based solely on their own long-term valuations does not appear to be a promising approach. In the next series of examples, Players utilize information about other players' valuations when making policy proposals, and propose policies which leave them better off *conditional upon defeating the status quo at hand*. In particular, I will assume the following functional form of $Q_i$:

$$Q_i(v(y)|v(x), v) = \frac{e^{\lambda(v_i(y)p(v(x),v(y))+v_i(x)(1-p(v(x),v(y))))}}{\sum_{z\in X} e^{\lambda(v_i(z)p(v(x),v(z))+v_i(x)(1-p(v(x),v(z))))}}, \tag{4.12}$$

for a $\lambda \in \Re_+$. As discussed earlier, this type of proposal strategy, where players condition upon both the value they receive from a policy *and* upon the probability that it defeats the status quo, will be referred to as *strategic* proposing.

**Example 12** *A divide-the-dollar game, with strategic proposals and* $\lambda = 5$.

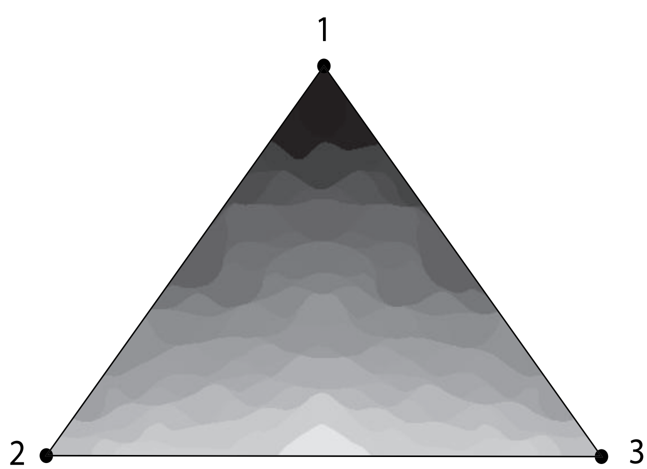



Figure 4.6: Player 1's value function under probabilistic "strategic" proposing.

In this example I assume the functional form of $Q$ seen in Equation 4.12, in which players propose policies based in part on the likelihood a policy will defeat the status quo

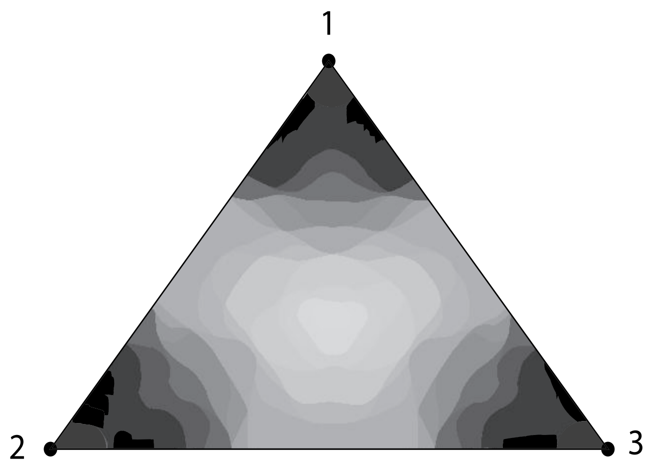



Figure 4.7: Density over outcomes under probabilistic "strategic" proposing.

at hand. The policy space has been discretized to include about 120 alternatives.[2] However, as Figures 4.6 and 4.7 show, there appears to be little difference between the predictions generated by this *strategic* functional form of $Q$ and the *sincere* functional form assumed in Example 9. In both instances, the ideal points of the players are predicted to be the most frequently observed outcomes. In this example, this may be because the value of $\lambda = 5$ is relatively low; players may find it in their best interest to propose their ideal points because there is a moderate likelihood that the other players will "mess up", and vote in favor of it. In the next example we will examine what happens when we increase the size of $\lambda$.

**Example 13** *A divide-the-dollar game, with strategic proposals and* $\lambda = 20$*.*

In this example I assume the same functional form of $Q$ seen in Equation 4.12 and Example 12, in which players propose policies based in part on the likelihood a policy will defeat the status quo at hand. However, here it is assumed that $\lambda = 20$, which implies that although players propose and vote for policies probabilistically, the size of the random shock that would induce a player to vote "incorrectly" (i.e., against his observable value) is much smaller than in the previous example.

[2]The mesh by which the policy space is discretized is substantially coarser in this example than in Example 9. This is because the functional form of $Q$ used in this example requires considerably more processing power, as $Q$ is now dependent upon the status quo.

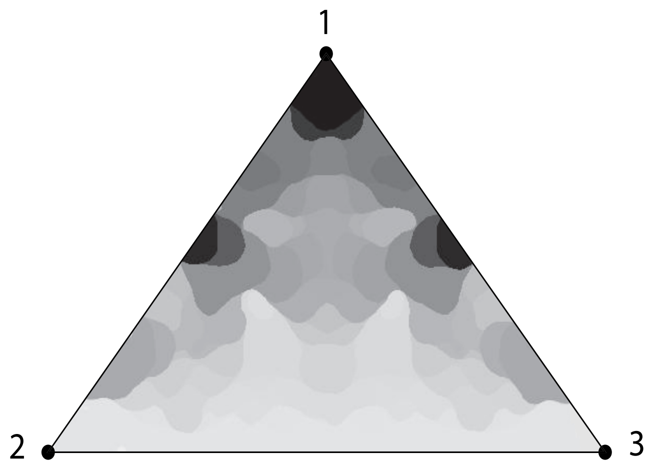



Figure 4.8: Player 1's value function under strategic proposing, with $\lambda = 20$.

Increasing the size of $\lambda$ appears to have a significant effect on both the equilibrium value functions of the players, and on the equilibrium density over observed outcomes. Figure 4.6 showed us that when there was a moderate likelihood of players "messing up" in their voting and proposing strategies, the favorite policy of Player 1 was his ideal point. Figure 4.8 shows that when vote and proposal strategies become closer to being deterministic, Player 1 now has three favorite policies: he still favors his ideal point, but he also favors the stable set alternatives which yield him half of the dollar just as much. Figure 4.9 shows that when players' long-term preferences are such, cooperative outcomes tend to emerge most often, over time.

This result is exciting because it hints at the possibility that there is an intermediate level of strategy in players' proposal and vote functions that generates cooperation. The results generated in Kalandrakis' paper and in Example 10 also appear to support this. In Kalandrakis' model, there is no randomness in players' strategies and players are purely strategic. In every round the proposer is aware of exactly the policy (or policies) that defeat the status quo and leave him best off. Since an absorbing set of outcomes is reached once any player receives the entire dollar, the path of play is predetermined and leads to this set. In Example 10 players are complete randomizers. The result is that long-term valuations are simply a linear transformation of initial utility; players' favorite policies in the long run

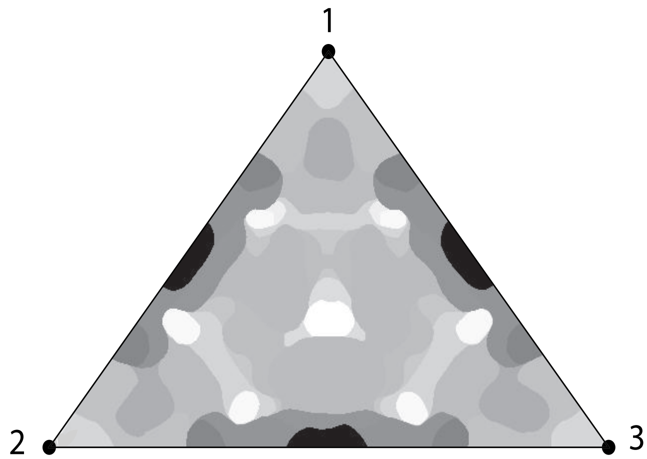



Figure 4.9: Density over outcomes under strategic proposing, with $\lambda = 20$.

continue to be their ideal points and every policy is equally likely to be an outcome.

In this example, players are more strategic than in the last example, because here the random element in players' strategies is small. However, a randomness still exists because players are behaving probabilistically. Thus, they cannot condition entirely upon the future proposal strategies of the other players as they do in Kalandrakis' model. When players possess this intermediate level of strategy, they become capable of thinking ahead, and condition how they bargain on the idea that policies selected today should stand up to tomorrow's, possibly uncertain, agenda. Interestingly, cooperative outcomes emerge.

**Example 14** *The dynamics of endogenous agenda formation.*

In this last example we will look at the dynamics of how agendas are formed in this setting. We consider a constant-sum game with policy space $X$ where $|X| = 6$ and $X = \{(1,0,0), (0,1,0), (0,0,1), (\frac{1}{2},\frac{1}{2},0), (\frac{1}{2},0,\frac{1}{2}), (0,\frac{1}{2},\frac{1}{2})\} = \{1,2,3,A,B,C\}$. Thus, $X$ consists of the three ideal points of the players (denoted by the Player's number), and the three elements of the simple von Neumann-Morgenstern stable set (denoted $A$,$B$,$C$). Players propose strategically, so that $Q$ has the functional form seen in Equation 4.12. In this example, it is also assumed that $\lambda = 20$.

Figure 4.10 depicts the ideal points of the three players and the locations of the six policies in both the short and long term. The long-term locations were calculated by renor-

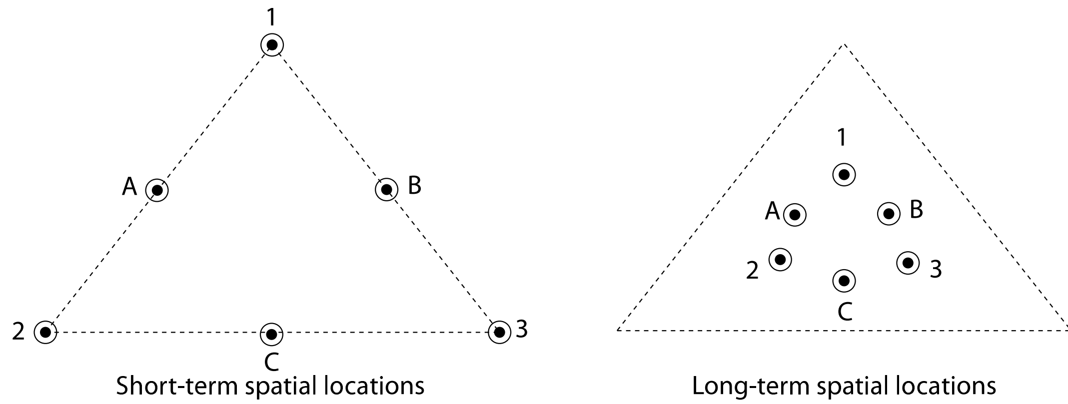



Figure 4.10: Spatial locations of policies in the short and long term.

malizing the value function vectors from $\frac{1}{1-\delta}$, or ten, to one. Since we are considering a constant-sum game, value functions remain constant-sum, and so this is possible. As this figure and the table of long-term valuations demonstrate, in the long run the policies all yield similar values of about $\frac{1}{3}$ of the total pie to each player, because of the dynamic of how the agenda-setting process is played out. For a given status quo policy in $X$, the table of transition measures gives us the likelihood that any other policy is proposed to replace it, and so allows us to examine the dynamics of how agendas are set in equilibrium.

**Myopic Utility**

| $i$ | $u_i(1)$ | $u_i(2)$ | $u_i(3)$ | $u_i(A)$ | $u_i(B)$ | $u_i(C)$ |
|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 |
| 2 | 0 | 1 | 0 | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ |
| 3 | 0 | 0 | 1 | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ |

**"Farsighted" Valuations**

| $i$ | $v_i(1)$ | $v_i(2)$ | $v_i(3)$ | $v_i(A)$ | $v_i(B)$ | $v_i(C)$ |
|---|---|---|---|---|---|---|
| 1 | 3.859 | 3.070 | 3.070 | 3.602 | 3.602 | 2.796 |
| 2 | 3.070 | 3.859 | 3.070 | 3.602 | 2.796 | 3.602 |
| 3 | 3.070 | 3.070 | 3.859 | 2.796 | 3.602 | 3.602 |

**Equilibrium transition measures**

| Status Quo | 1 | 2 | 3 | A | B | C |
|---|---|---|---|---|---|---|
| 1 | 0.115 | 0.020 | 0.020 | 0.109 | 0.109 | 0.627 |
| 2 | 0.020 | 0.115 | 0.020 | 0.109 | 0.627 | 0.109 |
| 3 | 0.020 | 0.020 | 0.115 | 0.627 | 0.109 | 0.109 |
| A | 0.339 | 0.339 | 0.004 | 0.004 | 0.157 | 0.157 |
| B | 0.339 | 0.004 | 0.339 | 0.157 | 0.004 | 0.157 |
| C | 0.004 | 0.339 | 0.339 | 0.157 | 0.157 | 0.004 |

In Figure 4.11 the six policies are depicted, with arrows pointing to the alternatives that are most likely to be proposed to be pitted against them. This example gives us some insight as to why cooperative outcomes are more difficult to attain when agenda formation is endogenous. In this figure we see that, at the ideal point of a particular player, the policy most likely to be pitted against the status quo is the outcome corresponding to a coalition consisting of the two players that were disenfranchised. However, we can see that these partnerships are fickle, because the alternatives most likely to be pitted against a *cooperative* status quo policy are the least cooperative policies of all—the ideal points of the players in the winning coalition.

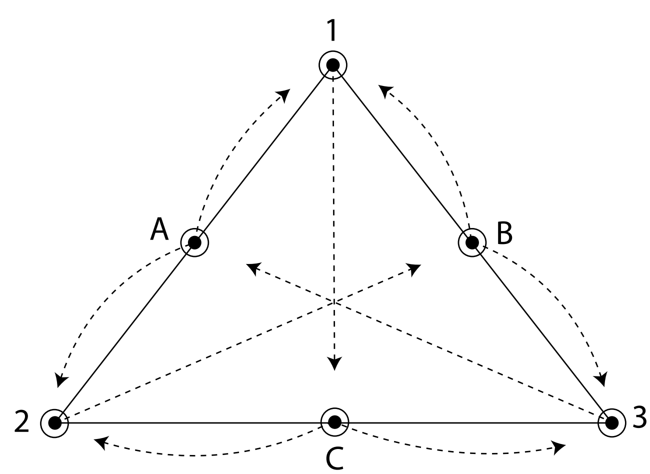



Figure 4.11: The dynamics of endogenous agenda formation.

In other words, at any given time behavior depends greatly on whether a player is presently being disenfranchised or not. If a player is not a member of a winning coalition, he will attempt to collude with someone. However, if he *is* a member of a winning coalition in one round, he will attempt to cheat his coalition partner in the next. Yet even given these ungentlemanly proposal strategies, we still witness cooperative outcomes as occurring roughly 95 percent of the time in this setting.

## 4.6 Conclusions

Schattschneider (1960) writes, “The definition of the alternatives is the supreme instrument of power.” If this is true, we should expect the process of agenda-setting to be an essential part of how individuals arrive at a collective choice. Alternatives on an agenda reveal the issues that are important to society and worthy of attention, and directly govern whose problems get attention and whose do not. In this way, the choice of agenda can, and often does, produce distinct winners and losers, and is necessarily the subject of debate and negotiation.

In this chapter I extend the model of the previous chapter to allow for endogenous agenda-setting on the part of legislators. Again, policy evaluation is modeled as a dynamic process in which individuals vote on policies based upon the utility they believe the policies will yield in the long run. However, here individuals are also agenda-setters, and propose alternatives to be voted upon using these same valuations. I find that the specification of how proposals are made greatly effects both the types of policy outcomes likely to emerge, and the induced preferences of voters in the long term. Specifically, I find that given a sincere specification of proposal strategies, a result is generated in which the ideal points of the players emerge most frequently as policy outcomes. However, when players propose alternatives strategically by conditioning their proposal upon the status quo at hand, cooperation can emerge if players’ strategies are neither too random nor too precise. Thus, when a small amount of uncertainty is incorporated into players’ vote and proposal strategies, players become able to “think ahead”, and propose policies capable of standing up to a future agenda.

The idea that moderate uncertainty drives cooperation is not far-fetched. If the future agenda is selected through a deterministic process, then players may be incapable of manipulating the future agenda through their choice of current policy. Yet if all strategies are purely random, players again are incapable of effecting future outcomes, because anything can happen. In the presence of moderate uncertainty about the future, players can condition their choice of policy upon the likely agenda it will generate, and choose alternatives that are the most resilient to future foes.

One difference between this chapter and the previous chapter is that previously players could condition against a random agenda because their vote strategies were not purely random. In the examples presented in this chapter it is assumed that players are equally rational in all aspects of their behavior, and thus are as equally likely to tremble when voting as when proposing. Clearly, if players are deterministic voters and random proposers, some of the examples of the previous chapter would be generated. Whether legislators truly have as much say in the choice of agenda as is assumed here is an empirical question, but as I argued earlier, it is unlikely that legislators deliberating over a collective outcome would (or could) repeatedly propose their ideal points. Rather, we would expect items on an agenda to also reflect the state of the world, or the issues of importance to constituents.

Another difference between the results of this chapter and the previous chapter stems from the observation that adding endogeneity to a dynamic model can quickly produce knife-edged results. For example, if we assume that players are more likely to propose policies that yield them higher value, we quickly generate Kalandrakis' result, because ideal points are proposed most frequently. By adding this additional layer of endogeneity to the model of the previous chapter, we need to exert far more care in specifying the functional forms that proposing and vote strategies take. This leaves us with the thought that, in certain circumstances, perhaps modeling a process such as agenda formation in continuing programs as *exogenous* can actually produce stronger predictions. Particularly in a dynamic setting such as this, in which status quos are endogenously determined, a misstep in identifying one of the parameters of the model can rapidly snowball as the process is iterated hundreds of times.

How agendas are actually set in dynamic environments remains a rich topic for future research. In the House of Representatives, for example, the Speaker of the House possesses considerable control over the legislative agenda in that he refers all bills to committee. The Democratic Caucus and Republican Conference also greatly influence the legislative agenda, both directly and indirectly. Party organizations elect the Speaker of the House, make committee assignments, and assist party members in formulating and defending their programs. Very rarely is an agenda-setting process direct; agendas are generally formed through delegation. A question for future research is how the delegation of agenda-setting powers to groups with no voting power, such as political parties, can alter the effective issue space. In particular, if the agenda defines the issues that are of importance to us, we could expect partisan specialization and issue salience to emerge endogenously as a consequence of foresight on the part of agenda-setters.

# Chapter 5 Conclusions

*Most things people want they cannot get by themselves.*
—William H. Riker, 1988.

The formation of stable coalitions is indisputably central to political life, and yet game theorists do not understand it. Without the threat of punishment, how can cooperation between members of a coalition be maintained when it is always in the best interest of someone to defect? In these chapters I argue that the answer to this question lies in the fact that interaction between individuals is virtually always dynamic, and that politics rarely revolves around a single issue or outcome. If a group of individuals is simply dividing up a single dollar between its members, it is clearly in the best interest of everyone involved to try to procure as much of the dollar as possible. However, in reality politics involves collective choices that must be made over the course of many years, and choices made today will greatly effect the types of choices that are feasible tomorrow. Furthermore, once a selfish equilibrium has been reached, it is often very difficult to get out of. An empirical example is the difficulty seen in establishing representative democracy in countries that once had dictatorships. A formal example discussed throughout the previous chapters is the game of repeated policy selection solved by Anastassios Kalandrakis. The author elegantly discovers that in continuing programs, purely selfish outcomes form an "absorbing state." Once such an outcome is reached, every subsequent proposal and outcome must be of the same degenerate form. Traditional game theory also hints at the possibility of cooperation in dynamic environments. It is only in the infinitely repeated prisoner's dilemma that "cooperate, cooperate" can be supported as an equilibrium.

In Chapters 3 and 4 I look at a setting of political debate that has been largely ignored by formal theorists—the setting of a continuing program. In continuing programs, policies enacted today remain in effect until new legislation is enacted that overturns them. This type of environment is politically important because most policies that we care about *are* continuing programs. Examples include entitlements, social policies, and both distributive and redistributive policies. In fact, it is difficult to find examples of policies that are not continuing. However, this setting has been largely ignored by game theorists because it is difficult to model formally. If we think of players as strategically selecting policies today to effect policy tomorrow, which in turn effects policy in the next round and so forth, we can, in very few rounds, quickly become left with an intractable mess.

While continuing programs have been largely ignored in the formal literature, we would expect these types of programs to be the most interesting from the standpoint of political science. Legislators are keenly aware of the fact that policy sets precedent, and that today's status quo greatly effects the types of alternatives that are feasible tomorrow. An example provided earlier was President Bush's 2001 tax cut package, which mandated the gradual phase-out of the estate tax by the year 2010, only to return to its 2001 levels in 2011. A lobbyist in favor of completely eliminating the tax was quoted as saying "In Washington terms, it's the finality we needed. It's very difficult for Congress to reinstate a tax once it's been repealed."[1] Thus, a bill eliminating a tax for one year and then reinstating it the next is politically equivalent to a bill eliminating the tax forever. Once the status quo of "no tax" has been set, it is effectively impossible to defeat the status quo with a policy mandating "tax." Examples such as this are not difficult to come by. Another closely related example is the provision of any kind of right or benefit to a group, such as granting women the vote, or reading offenders their Miranda rights.

Chapters 3 and 4 stem from the idea that in a setting in which current policies have an effect on future legislation, legislators may often forgo some satisfaction with respect to a policy in the short term to get a more important policy passed in the future. I argue that social choice theory fails to account for this type of dynamic because the theory assumes that we are bargaining over a single outcome. It fails to recognize the fact that oftentimes programs are continuing, and the behavior of a voter may depend on both his short- and long-term interests. The model of continuing programs I present is non game-theoretic, and

[1] Dan Blackenberg, lobbyist for the National Federation of Independent Business, quoted in the Washington Business Journal on June 4, 2001.

instead captures the notion of short- versus long-term gain by adding a new dimension to the standard social choice-theoretic framework. Here, individuals rank policies not only on the basis of the utility they yield today, but also with respect to the types of alternatives they will and will not be capable of defeating in the future. The types of alternatives a policy can defeat are conditioned upon a probabilistic future agenda. In Chapter 3 this agenda is assumed to be exogenous and static; thus some alternatives are simply more likely than others to be brought to the floor in the future, regardless of the status quo at hand, and players know this. In Chapter 4 I allow the agenda setting process to be endogenous, so that players propose alternatives themselves. Thus, players can condition upon the current status quo policy to propose an alternative that defeats it and leaves them better-off.

The most important result of these two chapters is that in dynamic environments, the space of alternatives which can and cannot defeat a policy, or the future agenda conditioned upon that policy, may have as much impact on individual decision making as the substance of the policy itself. In this way, these models provide one answer to the question of "why so much stability?" Cooperative outcomes emerge and are sustained as a consequence of looking at the probabilistic path of legislation a policy can lead to over time. This is because foresight often results in players voting for cooperative alternatives over alternatives which yield them higher utility in the short run. Even though these models are ultimately sophisticated preference aggregation techniques, they can yield well defined sets of best alternatives, and can do so in environments in which social choice theory fails to provide any predictions.

In Chapter 2 I look at a game of endogenous agenda formation in which three players bargain over a single dollar by proposing allocations and then voting upon the proposals. The paper models a bargaining process with no predetermined end, and so players are allowed to make proposals for as long as they wish. In a setting similar to one examined by Banks and Gasmi, I show that if the end of a negotiation process is not predetermined, players propose alternatives by utilizing information about the types of policies that can be combined into externally stable agendas, or agendas that are immune to amendment. I find that uncertainty about the future length of the agenda drives the first two proposers to collude, and that, although there is no restriction to the number of items allowed on an agenda per se, in equilibrium only three proposals are actually made. Interestingly, the outcome generated here differs from the universalistic outcome generated by Banks and Gasmi in the same setting, only with the number of items on an agenda restricted to three. Thus, while in both games the same number of proposals are made, removing the restriction on the number of proposals a player *can* make dramatically changes the policy outcome. It forces the first two proposers to act preemptively to collude to disenfranchise the third.

Removing the restriction on agenda length is appealing from a modeling standpoint, because in reality it is hard to imagine a process of negotiation in which individuals are allowed a single proposal. However, as in the previous two chapters, we are left with a result that is difficult to interpret, because by endogenizing the length of the agenda we are left with an outcome that is not as normatively appealing as the universalistic outcome generated by Banks and Gasmi. The point to be drawn from this chapter is not that universalistic outcomes are a consequence of players only being allowed to make a single proposal, or that allowing for an endogenous end to a proposal process truly drives first-movers to collude so drastically. Rather, this chapter and Banks and Gasmi's article demonstrate a similar phenomenon. In both, players propose alternatives that restrict the future effective policy space in a favorable way. When players are allowed a single proposal each, the final agenda need not be immune to future amendment. In this setting, the last mover is given an advantage, in that he can choose a policy that defeats the previous two proposals and maximizes his utility. Thus, the first two proposers backward induct to realize that the best they can get is a third each. In Chapter 2, the requirement that agendas be immune to amendment gives the first two proposers an advantage, because in this setting there happens to exist an externally stable chain of length three. Whether either of these results would hold in more general settings, or in settings with more than three players, is unclear. However, they both demonstrate the emergence of collusive behavior in environments in which proposals alter the future policy space in a deterministic way.

As stated in the previous chapter, how agendas are actually set in dynamic environments remains an important topic for future theoretical research. In the House of Representatives, for example, the Speaker of the House possesses considerable control over the legislative agenda in that he refers all bills to committee. The Democratic Caucus and Republican Conference also greatly influence the legislative agenda, both directly and indirectly. For example, party organizations elect the Speaker of the House, make committee assignments, and assist party members in formulating and defending their programs. Very rarely is an agenda-setting process as direct as in Chapters 2 and 4, and as in Banks and Gasmi's article; agendas are generally formed through delegation. A question for future research is how the delegation of agenda-setting powers to groups with no voting power, such as political parties, can alter the effective issue space. In particular, if the agenda defines the issues that are of importance to us, we could expect partisan specialization and issue salience to emerge endogenously as a consequence of foresight on the part of both voters and agenda-setters.

There exists an interesting link between these chapters and cooperative game theory. Cooperative game theory examines the types of allocations that coalitions of agents can procure for themselves, while remaining silent as to how these allocations arise, and how they are enforced. In all of the chapters presented here, outcomes emerge which are in keeping with those predicted by cooperative solution concepts such as the von Neumann-Morgenstern stable set. These outcomes are a consequence of players conditioning how they bargain upon the idea that policies selected today should stand up to tomorrow's agenda. A result of all of these chapters is that players are induced into taking the payoffs of others into account when voting over and proposing policies because they know that when collective choices are deliberated upon, the behavior of others in large part determines which policies are enacted. Thus, fair outcomes are induced through the foresight of players themselves.

Last, I have also tried to demonstrate in all of these models that, given a policy space and individuals with preferences over that space, there is a good deal of information that still remains untapped in the traditional social choice theoretic literature. Examples of such types of information are the variety of policies an alternative is defeated by, or can defeat, and the types of policies that can be combined into agendas that are immune to amendment. By utilizing this kind of information in dynamic settings, we can generate interesting predictions even with a minimum of institutional detail, in environments that concepts such as the Banks set, the uncovered set, and the core cannot.

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