A METHODOLOGY FOR THE SYNTHESIS OF ROBUST CONTROL SYSTEMS FOR MULTIVARIABLE SAMPLED-DATA PROCESSES

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Η εργασια αυτη αφιερωνεται με αγαπη
στους γονεις μου Αγη και Δημητρα
και τον αδελφο μου Νικο

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ABSTRACT

The problem of the synthesis of multivariable controllers which are robust with respect to model-plant mismatch is addressed. A two-step design procedure based on the Internal Model Control (IMC) structure is used. In the first step the IMC controller is designed assuming no modeling error, and in the second step the IMC filter is designed to preserve the closed-loop characteristics in spite of model-plant mismatch.

Two alternatives are provided for the first step. One of them allows the designer to satisfy structural performance specifications, in terms of the structure of the closed-loop interactions, their magnitude and duration. The closed-loop transfer function matrix is directly designed. The method requires only standard linear algebra operations and includes the construction of the IMC or the feedback controller in state-space. The second approach involves the minimization of the appropriately weighted H_2-norm of the sensitivity transfer function matrix, that relates the errors to the external inputs (setpoints or disturbances). A method is given for the meaningful selection of a full matrix weight so that the H_2-error is minimized for a set of external input directions and their linear combinations. The procedure is extended to open-loop unstable systems. In both approaches, special care is taken to avoid intersample rippling.

The design of the filter in the second step is formulated as an optimization problem over the filter parameters. The objective function is constructed by using

the Structured Singular Value theory so that the maximum singular value of the sensitivity transfer function remains bounded in spite of modeling error. The selection of the frequency bound is based on the properties of the design that was obtained in the first step. Analytic gradient expressions have been developed for the objective function. The optimization problem is an unconstrained one, solved with standard gradient search techniques. An iterative method for the selection of the appropriate sampling time is proposed, which explicitly takes into account model uncertainty information and performance specifications.

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CHAPTER I

INTRODUCTION

The first section of this chapter discusses very briefly the past and current trends in process control. The discussion is far from being complete and it is merely intended to point out the need for research in the topics presented in this thesis. The second section presents an overview of the contents of the thesis.

1. Past and Current Trends in Process Control Theory and Applications.

The vast amount of work that has been published in the area of control theory in the last 30 years gives rise to some questions:

- Haven't most of the practical control problems been solved by now?
- If not, then what problems did this work address?

The answer to these questions lies in the fact that the problems that the practicing engineer required control theory to provide answers to, have changed with time. For the first half of this time period, the reality in the chemical industry was that gains from control were very small compared to those from other tasks, like for example, steady-state optimization of operating conditions. The mere stability of the process was often satisfactory and since single-loop PID controllers were usually able to provide that, the use of more advanced techniques did not seem necessary. On the other hand, designs obtained with the Optimal Control Theory tended to behave very differently when applied to the real systems than to their mathematical models. The reason for that is that the models do not exactly describe the actual process and the inaccuracies associated with them were not taken into account by the theory (Foss, 1973; Kestenbaum et al., 1976). As a result, practitioners sooner or later lost their faith in the theoretical developments, but were satisfied with the fact that their semi-empirical tools were sufficient. At the same time theoreticians from a number of disciplines proceeded to describe almost every conceivable control problem as an optimization problem based on the known process model and then solve it. Although very few successfull applications

came out of this work because of the model uncertainties, this work made control theory a hard science by creating the mathematical foundation, which most of today's developments build on.

In the late 1970's a much greater need for improved control became present in the chemical industry. Tighter product specifications and increased competition contributed to this. The steep energy cost increase made it more important to remain close to efficient operating conditions in spite of disturbances, and it also led to integrated process designs. Such processes have many uncertain variables and suffer from increased interactions between the units, facts that make them inherently difficult to control, to the point that control considerations at the stage of process design are often necessary. As the control requirements increased, it became clear that the semi-empirical tools available to the practitioner were not sufficient any more. The pressing need to control their processes led the industry to the development of new approaches, like the Model Algorithmic Control (Richalet et al., 1978) and the Dynamic Matrix Control (Cutler and Ramaker, 1979) The initial success of these algorithms led to the study of the underlying principles and the development of the Internal Model Control (IMC) structure (Garcia and Morari, 1982). At the same time in other disciplines, similar considerations led to the quantification of conditions for robustness with respect to model-plant mismatch (Doyle and Stein, 1981) and the H_∞-type of approach (Zames, 1981). The introduction of the Structured Singular Value (SSV) theory (Doyle, 1982) opened the way for the development of methods for the direct synthesis of robust controllers by quantifying the concept of robust performance in a non-conservative way.

Today the need for good control of the industrial processes is greater than ever because of the continuously changing economic environment in which the industry operates. Such changes force the plant to operate at conditions different

from the ones for which it was originally designed, with inefficient operation as the result. The modification of the process design would require the market to remain unchanged for a substantial period of time in order to produce a profit. Since this period is usually longer than what can be expected of the market, the alternative is the use of optimization systems to maintain optimal operation for the existing equipment, an approach that drives the processes over a wider range of operating conditions than ever before (Garcia and Prett, 1986). This fact adds significantly to the already pressing need for control systems that perform reasonably well in the presence of model-plant mismatch.

2. Thesis Overview.

The objective of this thesis is the development of a synthesis methodology for control systems that are robust with respect to modeling error. A major consideration in this work was the fact that in process control applications it is highly desirable or required that the operators are provided with tuning parameters with clear physical meaning and effect. This is not necessarily the case for applications of control theory in other disciplines, as, for example, in the aerospace industry where online tuning of the control system is clearly out of the question. As a result, the synthesis procedure had to be streamlined in ways appropriate for applications to process control, a fact that posed a number of restrictions.

Since an advanced control algorithm will be implemented on a digital computer, it was decided to formulate the procedure with sampled-data processes in mind. Chapter II reviews a number of well-known digital control algorithms for single-input single-output (SISO) systems. The IMC structure is used to illustrate and study their respective problems as a function of the sampling time. A new algorithm is then proposed that combines their advantages and is free of their problems. Chapter III builds on the results of Chapter II and it provides a method for the synthesis of a control system that is robust with respect to

modeling error. The limitation that modeling error poses on the achievable performance is quantified and compared with the one that the sampling poses. The result is a procedure for the selection of the appropriate sampling time, which takes explicitly into account model uncertainty.

Chapter IV examines multi-input multi-output (MIMO) processes. The algorithm that was developed in Chapter II is extended to MIMO systems. The effect of inherently limiting characteristics like time delays and undesirable zeros and their directions is quantified for the multivariable case and a procedure is proposed for the direct synthesis of the desired closed-loop transfer function matrix. The designer can select among different structures of the closed-loop interactions by using a quantitative method that requires only linear algebra operations. Chapter V is set in the continuous domain. It presents a two-step synthesis method for MIMO continuous sytems, which uses an Integral Squared Error type of objective in the first step and incorporates robustness properties by minimizing an appropriate SSV objective function over the parameters of a low pass filter in the second step. The method also extends the IMC procedure to open-loop unstable plants. Chapter VI extends the procedures of Chapters III and V to MIMO sampled-data systems and presents an alternative to the method of Chapter IV, which can be used for open-loop unstable systems. The SSV is used to achieve robustness to modeling error. Chapter VII includes the conclusions and suggestions for future research.

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CHAPTER II

DIGITAL CONTROLLERS FOR SISO SYSTEMS: A REVIEW AND A NEW ALGORITHM

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Digital controllers for SISO systems: a review and a new algorithm

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Several digital control algorithms for linear single-input single-output systems are examined and the effect of the sampling period on their performance is analysed in terms of rippling, overshoot and settling time. The problem is addressed in the frequency domain (z-transform) and it is shown that each controller works for some classes of systems but that none works for all. The similarities and differences of these controllers are established and an explanation of their deficiencies is given based on the location of the zeros of the discrete system. The insight gained leads to a simple new rule for the design of a controller which combines the advantages of the different algorithms but at the same time is free of their problems. A single tuning parameter is included which directly affects the closed-loop speed of response and bandwidth. The parameter can be used to detune the controller in the event that the real system differs from the model on which the controller design is based. No tuning is necessary when the available model is exact, unless smaller values for the manipulated variable, at the cost of a slower response, are preferred.

1. Introduction

In the literature one can find a very large number of digital control algorithms for single-loop systems. Numerous design criteria are used and often they are formulated so as to satisfy the special requirements of a specific process. The synthesis of many of those controllers is quite complicated and it is usually very difficult to know when and why problems may occur.

A common property of the control algorithms examined in this paper is the computational simplicity of their synthesis; it so happens that all of them can in fact be described as pole–zero placement controllers, though that may not be the way they were initially designed. Each of them tries to satisfy some commonly accepted criteria, such as:

(i) A performance criterion for the discrete output of the system (for example, minimization of the sum of squared errors).

(ii) Behaviour between the sampling points: hidden oscillations in the system output can be caused by ringing of the manipulated variable or by unobservable oscillatory open-loop modes.

(iii) Settling time (for example, the requirement to reach the set-point in a finite number of time steps).

(iv) Overshoot and/or undershoot.

(v) Manipulated variable: large values must be avoided because they usually cannot be implemented due to saturation of the manipulated variable.

In this paper we compare the set-point step responses obtained with various controllers. In the first part, an analysis of the problems is given for the case of no

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modelling error and a new algorithm which avoids the disadvantages of the examined controllers is proposed. The second part deals with stability issues of the new algorithm when a mismatch between the model and the plant exists.

2. Evaluation of the control algorithms

In the case of no modelling error, the classical feedback (Fig. 1 (*a*)) is equivalent, for command following, to the open-loop structure shown in Fig. 1 (*b*), where $G_c(z)$ may be considered as the controller to be designed.

There is a simple relation between $G_c(z)$ and the controller of the classical feedback structure $C(z)$:

$$G_c(z) = \frac{C(z)}{1 + C(z)G(z)} \tag{1}$$

where $G(z)$ is the transfer function of the plant. We can similarly obtain $C(z)$ from $G_c(z)$ by

$$C(z) = \frac{G_c(z)}{1 - G_c(z)G(z)} \tag{2}$$

Designing $G_c(z)$ and then obtaining $C(z)$ by eqn. (2) is an established technique (see, for example, Jury and Schroeder (1956)). This technique will be used throughout the paper because it allows presentation of the results in a transparent manner.

For the pulse transfer function of the plant we shall use the expression

$$G(z) = K\frac{(z - a_1^-)\dots(z - a_k^-)(z - a_{k+1}^+)\dots(z - a_{n-1}^+)}{(z - p_1)\dots(z - p_n)}z^{-N} \tag{3}$$

where N is the largest integer such that NT is less than or equal to the dead-time (T being the sampling time). The superscripts '$-$' and '$+$' denote the zeros inside and outside the unit circle respectively and k is the number of zeros inside the unit circle. $G(z)$ is assumed to be open-loop stable.

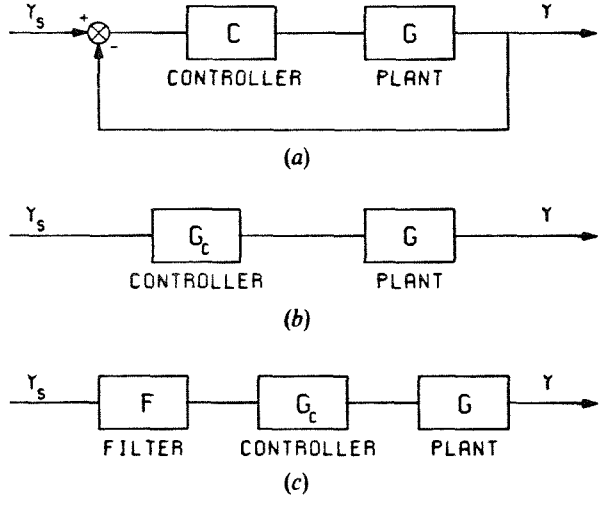

Figure 1. (*a*) Classical feedback structure. (*b*) Open-loop structure equivalent to the feedback for the exact model. (*c*) The open-loop structure with the filter included.

Digital controllers for SISO systems

2.1. *Digital control algorithms*

(i) *Dahlin's controller*

This is a well known controller included in current textbooks like Smith (1972) and Kuo (1977). It was introduced by Dahlin (1968), who proposed a method for designing the controller by specifying the closed-loop transfer function $V(z)$ to be first order with a dead-time equal to that of the plant and a steady-state gain of 1·0, thus:

$$V(z) = \frac{[1 - \exp(-T/\lambda)]z^{-1-N}}{1 - \exp(-T/\lambda)z^{-1}} \tag{4}$$

where λ is the time constant of the closed-loop response and T is the sampling interval. From $V(z)$ and $G(z)$, the classical feedback controller is found to be

$$C_{DC}(z) = \frac{[1 - \exp(-T/\lambda)]z^{-1-N}}{1 - \exp(-T/\lambda)z^{-1} - [1 - \exp(-T/\lambda)]z^{-1-N}} \frac{1}{G(z)} \tag{5}$$

The time constant of the closed-loop response serves as a tuning parameter for adjusting the speed of the response.

If we take the equivalent structure of Fig. 1 (*b*) the controller is found from eqn. (1) to be

$$G_{DC}(z) = \frac{1 - \exp(-T/\lambda)}{1 - \exp(-T/\lambda)z^{-1}} \frac{1}{K} \frac{(z-p_1)\ldots(z-p_n)}{z(z-a_1)\ldots(z-a_{n-1})} \tag{6}$$

where the expression for $G(z)$ given by (3) is used. Hence we can rearrange to get the structure of Fig. 1 (*c*) with controller

$$G_C(z) = \frac{1}{K} \frac{(z-p_1)\ldots(z-p_n)}{z(z-a_1)\ldots(z-a_{n-1})} \tag{7}$$

and a filter

$$F(z) = \frac{1 - \exp(-T/\lambda)}{1 - \exp(-T/\lambda)z^{-1}} \tag{8}$$

The reason for separating out $F(z)$ and calling it a 'filter' will become apparent in § 4 of the paper dealing with modelling errors. Note that for $\lambda = 0$ (no filter), Dahlin's controller becomes a deadbeat controller which brings the output to the set-point value after one sampling period. The stability problems are discussed in § 2.2.

Dahlin also proposed a modification to be used when there are problems with the initial algorithm; we will examine this modified controller in detail in § 2.2, after the reasons for the problems are explained.

(ii) *The controller that minimizes the sum of the squared errors of the output*

This controller is designed so as to minimize the objective function

$$\Phi = \sum_{j=0}^{\infty} (y_j - r)^2$$

where y_j is the value of the system output at $t = jT$ and r the desired set-point. The

problem is solved in the z-domain by Chang (1961) and in the state space of Kucera (1972).

The controller for the structure of Fig. 1 (*b*) is (Kucera 1972):

$$G_{SE}(z) = \frac{1}{K}\frac{\left(1-\frac{1}{a_{k+1}^{+}}\right)\cdots\left(1-\frac{1}{a_{n-1}^{+}}\right)}{(1-a_{k+1}^{+})\cdots(1-a_{n-1}^{+})}\frac{(z-p_1)\cdots(z-p_n)}{z(z-a_1^{-})\cdots(z-a_k^{-})\left(z-\frac{1}{a_{k+1}^{+}}\right)\cdots\left(z-\frac{1}{a_{n-1}^{+}}\right)} \tag{9}$$

(iii) *Output and state deadbeat controllers*

The *output deadbeat controller* is defined as the controller that drives the discrete output of the system to the set-point value and keeps it there, after a minimum number of time steps. The earliest discussion of this kind of controller was given by Bergen and Ragazzini (1954). Kalman (1954) raised the question of the behaviour of the continuous output between the sampling points. Jury and Schroeder (1956) and Jury (1958) studied the problem by using the modified z-transform and Kalman and Bertram (1959) in the state-space.

The design of the *state deadbeat controller* is based on the idea of having the states of the system at equilibrium when the discrete output reaches the set-point so that the continuous output will remain there as well. The minimum number of time steps required for all the states to be at equilibrium is at most equal to the number of the states, which is equal to the order of the system. For SISO systems it is easier to design this controller using the z-transform. As a design criterion, the manipulated variables are required to remain constant after the output of the system has reached the set-point (minimal prototype controllers; see, Ragazzini and Franklin (1958) and Luyben (1973)).

Increasing the specified settling time by one or more sampling intervals introduces additional degrees of freedom which allow one to prescribe a value for the manipulated variable (Isermann 1981) or to optimize an objective function or to prescribe the system output (Janiszowski 1983).

Output deadbeat

The controller for the open-loop structure of Fig. 1 (*b*) is (Kucera 1972)

$$G_{OD}(z) = \frac{1}{K}\frac{(z-p_1)\cdots(z-p_n)}{(z-a_1^{-})\cdots(z-a_k^{-})z^{n-k}(1-a_{k+1}^{+})\cdots(1-a_{n-1}^{+})} \tag{10}$$

State deadbeat

The controller for the open-loop structure is

$$G_{SD}(z) = \frac{1}{K}\frac{(z-p_1)\cdots(z-p_n)}{(1-a_1)\cdots(1-a_{n-1})z^{n}} \tag{11}$$

Note that for state deadbeat control the closed-loop characteristic equation has $(n+N)$ roots at the origin (Åström and Wittenmark 1984).

(iv) *Vogel–Edgar controller*

Vogel and Edgar (Vogel 1982) derived a control algorithm by using the general pole–zero placement controller design procedure presented by Åström and Witten-

mark (1980). For the conventional feedback structure, the Vogel–Edgar controller is

$$C_{VE}(z) = \frac{1}{K} \frac{[1-\exp(-T/\lambda)](z-p_1)\dots(z-p_n)}{z^n[1-\exp(-T/\lambda)z^{-1}](1-a_1)\dots(1-a_{n-1}) - [1-\exp(-T/\lambda)](z-a_1)\dots(z-a_{n-1})z^{-N}} \tag{12}$$

For the equivalent structure of Fig. 1 (*b*), this controller becomes

$$G_{VE}(z) = \frac{1-\exp(-T/\lambda)}{1-\exp(-T/\lambda)z^{-1}} \frac{1}{K} \frac{(z-p_1)\dots(z-p_n)}{(1-a_1)\dots(1-a_{n-1})z^n} \tag{13}$$

Comparing (13) with (8) and (11) we note that the Vogel–Edgar controller is equivalent to the state deadbeat controller (11) with an added filter (8). The filter parameter λ is used for tuning, as in the case of Dahlin's controller. By comparing this controller with Dahlin's, Vogel and Edgar (Vogel 1982) found their's to be superior for second-order systems.

The expressions for the controller $G_c(z)$ of the open-loop structure (Fig. 1 (*b*)) are summarized for the discussed control algorithms in Table 1.

Dahlin	$G_{DC}(z) = F(z)\frac{1}{K}\frac{(z-p_1)\dots(z-p_n)}{z(z-a_1)\dots(z-a_{n-1})}$
Output deadbeat	$G_{OD}(z) = \frac{1}{K}\frac{(z-p_1)\dots(z-p_n)}{(1-a_{k+1}^+)\dots(1-a_{n-1}^+)(z-a_1^-)\dots(z-a_k^-)z^{n-k}}$
Minimum sum of squared errors	$G_{SE}(z) = \frac{1}{K}\frac{\left(1-\frac{1}{a_{k+1}^+}\right)\dots\left(1-\frac{1}{a_{n-1}^+}\right)}{(1-a_{k+1}^+)\dots(1-a_{n-1}^+)} \times \frac{(z-p_1)\dots(z-p_n)}{z(z-a_1^-)\dots(z-a_k^-)\left(z-\frac{1}{a_{k+1}^+}\right)\dots\left(z-\frac{1}{a_{n-1}^+}\right)}$
State deadbeat	$G_{SD}(z) = \frac{1}{K}\frac{(z-p_1)\dots(z-p_n)}{(1-a_1)\dots(1-a_{n-1})z^n}$
Vogel–Edgar	$G_{VE}(z) = F(z)\frac{1}{K}\frac{(z-p_1)\dots(z-p_n)}{(1-a_1)\dots(1-a_{n-1})z^n}$

Where

$$F(z) = \frac{1-\exp(-T/\lambda)}{1-\exp(-T/\lambda)z^{-1}}$$

the plant is

$$G(z) = K\frac{(z-a_1^-)\dots(z-a_k^-)(z-a_{k+1}^+)\dots(z-a_{n-1}^+)}{(z-p_1)\dots(z-p_n)}z^{-N}$$

and the superscripts '−' and '+' denote 'inside' and 'outside' the unit circle, respectively.

Table 1. Controller $G_c(z)$ of the open-loop structure of Fig. 1 (*b*).

2.2. *Analysis of the deficiencies of the algorithms*

From Table 1, it is apparent that all these control algorithms are in fact pole–zero placement controllers and that the poles of the controller and the closed-loop transfer function are explicitly related to the zeros of the plant. Hence, the performance of these controllers is strongly related to the location of the zeros of $G(z)$ on the complex plane.

Åström *et al.* (1984) have proved the following theorems regarding the effect of the sampling period on the location of the zeros of sampled systems.

For the system shown in Fig. 2 the pulse transfer function $G(z)$ between y and u is

$$G(z) = (1 - z^{-1})\mathscr{Z}\mathscr{L}^{-1}\left\{\frac{A(s)}{s}\right\} \tag{14}$$

Figure 2. Pulse transfer function: $G(z) = (1 - z^{-1}) \cdot \mathscr{Z}\mathscr{L}^{-1}\{A(s)/s\}$.

Theorem 1 (Åström *et al.* 1984)

Let $A(s)$ be a strictly proper ($m < n$) rational function:

$$A(s) = K\frac{(s - v_1)\ldots(s - v_m)}{(s - w_1)\ldots(s - w_n)} \tag{15}$$

Then as $T \to 0$, m zeros of $G(z)$ go to 1 as $\exp(v_i T)$ and the remaining $(n - m - 1)$ zeros of $G(z)$ go to the zeros of $B_{n-m}(z)$, where

$$B_k(z) = b_1^k z^{k-1} + b_2^k z^{k-2} + \cdots + b_k^k \tag{16}$$

and

$$b_i^k = \sum_{l=1}^{i} (-1)^{i-l} l^k \binom{k+1}{i-l}, \quad i = 1, 2, \ldots, k \tag{17}$$

Some of the zeros of the polynomials B_k are outside or on the unit circle. The 'unstable' zeros for a few values of k are listed in Table 2.

k	Unstable zero of B_k
2	−1
3	−3·732
4	−1, −9·899
5	−2·322, −23·20
6	−1, −4·542, −51·22
7	−1·868, −8·160, −109·3
8	−1, −3·138, −13·96, −228·5
9	−1·645, −4·957, −23·14, −471·4

Table 2. Unstable zeros of the polynomial $B_k(z)$ of eqn. (16).

Theorem 2 (Åström *et al.* 1984)

Let $A(s)$ be a strictly proper rational transfer function with $A(0) \neq 0$ and $\mathrm{Re}(w_i) < 0$. Then all zeros of the pulse transfer function $G(z)$ given by eqn. (14) go to zero as the sampling period T goes to infinity.

(i) *Dahlin's controller*

There are two problems associated with this controller:

(*a*) When there exist zeros of $G(z)$ outside the unit circle then it follows from (5) or (6) that the transfer function between the set-point and the input to the plant is unstable. As for the transfer function between the output and the set-point, it will be unstable unless exact cancellation occurs between the unstable poles of $G_{DC}(z)$ and the unstable zeros of $G(z)$.

It is important to notice that $G(z)$ may have unstable zeros not only when there are RHP zeros in the Laplace transfer function of the plant, but even for systems with only LHP zeros or no zeros at all. As we can see from Theorem 1 and Table 2, a system with three poles and no zeros in the Laplace transfer function, for example, will have a zero outside the unit circle for 'small' sampling periods.

(*b*) Even when all the zeros are inside the unit circle, there will still be problems if there are zeros close to -1. That would cause ringing of the controller output (oscillations with period twice the sampling period) and its effect on the system output would be rippling *between* the sampling instants. In addition, the ringing of the manipulated variable may cause unnecessary equipment wear. This problem is illustrated in Example 1.

Example 1

For $T = 3$,

$$G(s) = \frac{1}{(10s+1)(25s+1)} \Rightarrow G(z) = 0.0157\frac{(z+0{\cdot}869)}{(z-0{\cdot}887)(z-0{\cdot}741)}$$

For $\lambda = 0$ the response is shown in Fig. 3. For $\lambda = 3$ the response is shown in Fig. 4. As λ increases, the response becomes slower but the oscillations remain, though their amplitude is reduced.

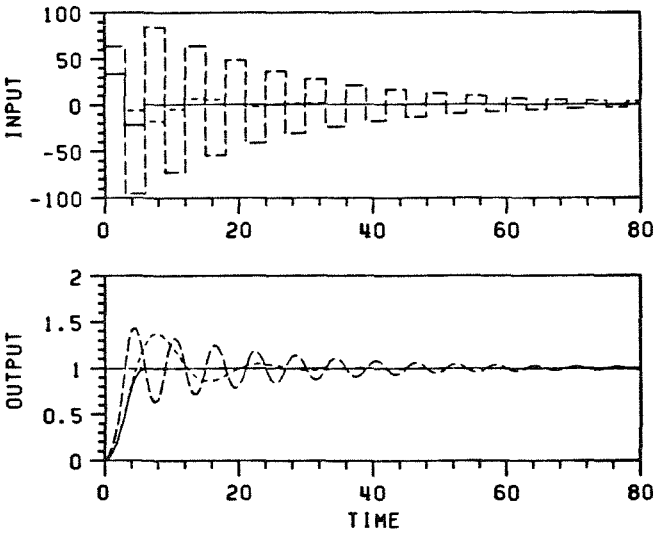

Figure 3. $G(s) = 1/(10s+1)(25s+1)$, $T = 3$, $\lambda = 0$: —— Dahlin's controller (also the one that minimizes the sum of squared errors); ··· modified Dahlin; —— the new algorithm; –·– set-point.

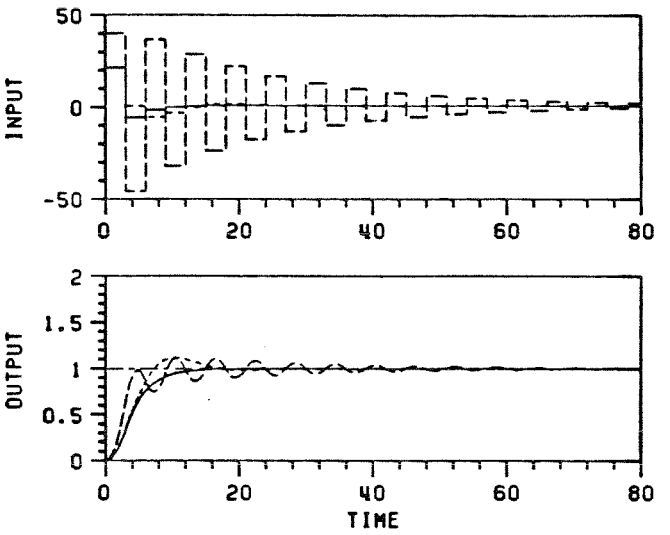

Figure 4. $G(s) = 1/(10s + 1)(25s + 1)$, $T = 3$, $\lambda = 3$: —— Dahlin's controller; ··· modified Dahlin; —— the new algorithm; –·– set-point.

Note that zeros near -1 may occur, depending on $G(s)$, not only for small Ts but also for intermediate Ts. This will be the case, for example, for a third-order system with no zeros in the Laplace transfer function (see Table 2 and Theorems 1 and 2).

Modified Dahlin controller

The modification introduced by Dahlin (1968) is to substitute zeros at the origin for the zeros of $G(z)$ (in (5)) which are unstable or cause ringing, while keeping the same steady-state gain for $C_{DC}(z)$.

Assume that there are m zeros of $G(z)$ that have to be discarded (unstable or those that cause ringing). The expression for the modified Dahlin controller can be derived from eqn. (5).

$$C'_{DC}(z) = \frac{[1 - \exp(-T/\lambda)]z^{-1}}{1 - \exp(-T/\lambda)z^{-1} - [1 - \exp(-T/\lambda)]z^{-1-N}} \frac{1}{K}$$
$$\times \frac{(z - p_1)\ldots(z - p_n)}{z^m(1 - a_1)\ldots(1 - a_m)(z - a_{m+1})\ldots(z - a_{n-1})} \tag{18}$$

For the system of Example 1, the modified controller will give the response shown in Fig. 3, which is clearly better than that obtained with the unmodified algorithm. By increasing λ, the response can be improved significantly (Fig. 4).

As we will see in § 3 of this paper, the modification would be much more effective if the substitution of the 'ringing' zeros of $G(z)$ with zeros at the origin were applied to the controller $G_{DC}(z)$ (eqn. (6)) of the equivalent structure of Fig. 1 (*b*). When the modification is made on $C_{DC}(z)$, the problem may remain, as Example 2 illustrates. An additional disadvantage is that we do not know beforehand for which systems problems will appear.

Example 2

For $T = 1$,

$$G(s) = \frac{\exp(-s)}{(2s+1)(5s+1)} \Rightarrow G(z) = 0{\cdot}0398\frac{(z+0{\cdot}792)}{(z-0{\cdot}819)(z-0{\cdot}607)}z^{-1}$$

The responses with both the unmodified and the modified Dahlin controllers are shown in Fig. 5. Although the modification is supposed to eliminate ringing, the time during which the oscillation of the manipulated variable persists, increases significantly when the modification is made.

Let us now examine the performance of the modified Dahlin controller when unstable zeros are present.

It follows from eqn. (18) that as $\exp(-T/\lambda) \to 1$ ($\lambda \to \infty$), $(m+N)$ of the roots of the closed-loop characteristic polynomial go to the origin and one goes to $+1$ (from inside the unit circle). Hence, there is no doubt that a sufficiently large λ can be found to stabilize the system. The problem is that this tuning is necessary even when the model is exact and we do not know beforehand where to look for $\exp(-T/\lambda)$.

A system with an unstable zero is examined in Example 3.

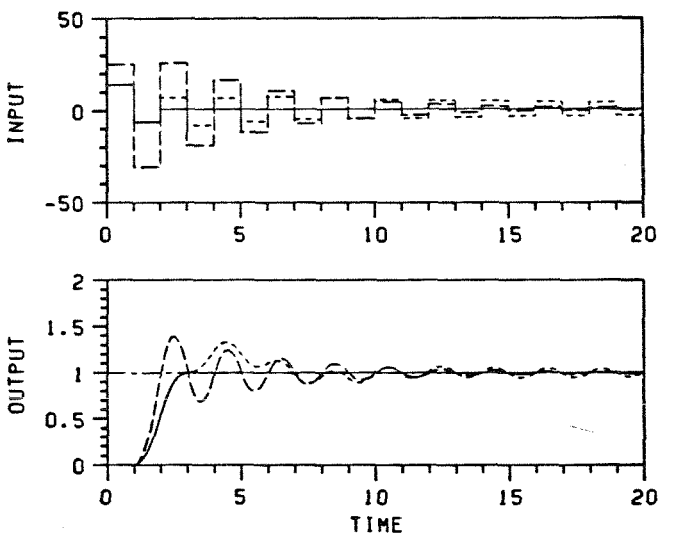

Figure 5. $G(s) = \exp(-s)/(2s+1)(5s+1)$, $T = 1$, $\lambda = 0$: ——— Dahlin's controller; ··· modified Dahlin; —— the new algorithm; –·– set-point.

Example 3

For $T = 0{\cdot}1$,

$$G(s) = 3{\cdot}333\frac{(-s+1{\cdot}5)}{(s+1)(s+2)(s+2{\cdot}5)} \Rightarrow G(z) = -0{\cdot}01316\frac{(z-1{\cdot}162)(z+0{\cdot}792)}{(z-0{\cdot}905)(z-0{\cdot}819)(z-0{\cdot}779)}$$

Both zeros (one is unstable, and one causes ringing) have to be substituted by zeros at the origin when applying eqn. (18). Then

$\lambda = 0$	unstable.
$\lambda = 0{\cdot}718$ ($\exp(-T/\lambda) = 0{\cdot}87$)	stable (at the edge of instability, i.e. undamped oscillations of the discrete, as well as of the continuous output).
$\lambda = 1{\cdot}95$ ($\exp(-T/\lambda) = 0{\cdot}95$)	response shown in Fig. 6.

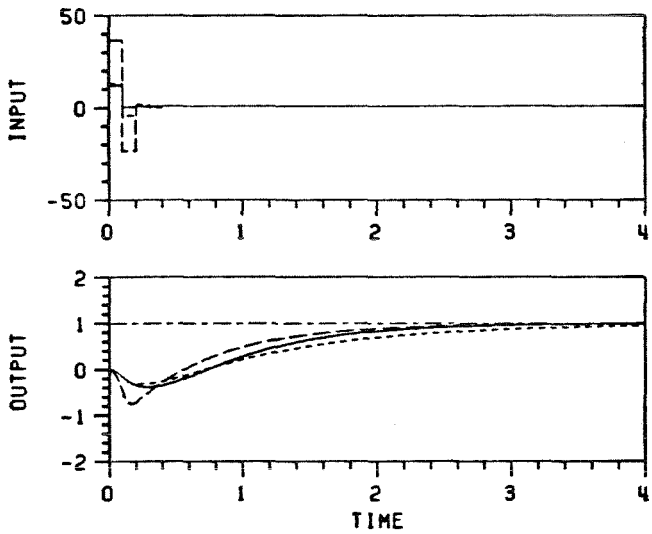

Figure 6. $G(s) = 3{\cdot}333(-s + 1{\cdot}5)/(s + 1)(s + 2)(s + 2{\cdot}5)$, $T = 0{\cdot}1$: ··· modified Dahlin, $\lambda = 1{\cdot}95$; – – – the new algorithm, $\lambda = 0$; —— the new algorithm, $\lambda = 0{\cdot}25$; –·– set-point.

For $\lambda = 1{\cdot}95$, the roots of the closed-loop characteristic polynomial are at 0·30, 0·56 and 0·91. Hence, one of them is close to 1/1·162 (i.e. the inverse of the unstable zero). The other two are near enough to the origin for their effect on the response to be close to the effect that two roots at the origin would have. These observations form part of the basis of the new algorithm introduced in § 3.

(ii) *Output deadbeat*

There are two problems associated with this controller.

(*a*) Potential ringing of the controller output. This will happen when there are zeros with negative real part *inside* the unit circle; the closer they are to -1 the worse the rippling.

(*b*) The second problem is associated with the way the control system handles the unstable zeros. Since those zeros cannot be used as poles of $G_c(z)$, in order to have steady-state gain 1 for the closed-loop transfer function, i.e. $V(1) = 1$, the terms $(1 - a_{k+1}^+), \ldots, (1 - a_{n-1}^+)$ have to be included in the denominator of the controller and they remain in the closed-loop transfer function $V(z)$, too. We have

$$V(z) = \frac{(z - a_{k+1}^+) \ldots (z - a_{n-1}^+)}{z^{n-k}(1 - a_{k+1}^+) \ldots (1 - a_{n-1}^+)} z^{-N} \tag{19}$$

From eqn. (19) we find that for a set-point step change the first non-zero value of the system output will occur at time $t = (N + 1)T$ and will be of magnitude

$$y_1 = \frac{1}{(1 - a_{k+1}^+) \ldots (1 - a_{n-1}^+)} \tag{20}$$

Equation (20) implies that if some of those zeros are near $+1$, y_1 will be large and significant undershoot or overshoot will occur. From Theorem 1, we see that this

happens for small sampling times when the zeros outside the unit circle are introduced by RHP zeros of the Laplace transfer function.

The response for the system of Example 3 is shown in Fig. 7, where oscillations, as well as large undershoot and overshoot, occur.

(iii) *The controller that minimizes the sum of the squared errors of the output*

There is only one problem with this controller, but it is nevertheless important, i.e. ringing of the manipulated variable and, as a result, rippling of the system output between the samples when there are zeros with negative real part near the unit circle.

On the other hand, this controller does not suffer the second problem of the output deadbeat controller, illustrated in Fig. 7. The first non-zero value of the system output is

$$y_1 = \frac{\left(1-\frac{1}{a^+_{k+1}}\right)\ldots\left(1-\frac{1}{a^+_{n-1}}\right)}{(1-a^+_{k+1})\ldots(1-a^+_{n-1})} \tag{21}$$

Hence when one of those a^+s is near $+1$, then its inverse, $1/a^+$, is also near $+1$, in fact it is closer to $+1$ so that $|y_1| < 1$.

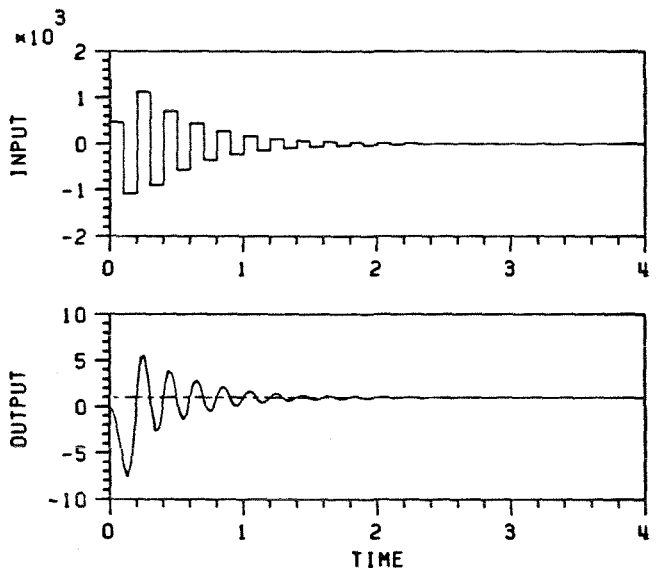

Figure 7. $G(s) = 3{\cdot}333(-s+1{\cdot}5)/(s+1)(s+2)(s+2{\cdot}5)$, $T = 0{\cdot}1$: —— output deadbeat controller; –·– set-point.

For the case of Example 1, this controller is identical to Dahlin's (its response can be seen in Fig. 3). Another case is examined in Example 4.

Example 4

For $T = 1{\cdot}65$,

$$G(s) = \frac{2}{(s^2+1{\cdot}2s+1)(s+2)} \Rightarrow G(z) = 0{\cdot}4168\frac{(z+0{\cdot}0708)(z+1{\cdot}058)}{(z^2-0{\cdot}184z+0{\cdot}138)(z-0{\cdot}0369)}$$

By using the inverse of $-1{\cdot}058$ as a pole for the controller $G_c(z)$ (according to eqn. (9)), we get the oscillatory response shown in Fig. 8.

In both Figs. 3 and 8 we see that this controller clearly minimizes the sum of the squared errors of the *discrete* output, but at the same time no attention is paid to the system output between the samples.

(iv) *State deadbeat*

Equation (11) shows that the ringing problem of the controller output will not occur. Nevertheless, the state deadbeat controller frequently suffers from large overshoot or undershoot of the discrete output itself before equilibrium is reached. The reason is similar to that discussed for the output deadbeat controller, but now the problem arises not only for zeros of $G(z)$ outside the unit circle, but also for zeros inside the unit circle.

The closed-loop transfer function is

$$V(z) = \frac{(z-a_1)\dots(z-a_{n-1})}{(1-a_1)\dots(1-a_{n-1})z^n} z^{-N} \tag{22}$$

The first non-zero value of the discrete output for a set-point step change is

$$y_1 = \frac{1}{(1-a_1)\dots(1-a_{n-1})} \tag{23}$$

Equation (23) implies that when there are zeros near $+1$, y_1 will be large, which results in significant overshoot or undershoot or both.

From Theorem 1 we know that this will happen for small Ts whenever the Laplace transfer function of the system has zeros either in the LHP or the RHP. Also note that for systems with more than one zero in the Laplace transform, all the zeros contribute to the large value of y_1.

The problem is illustrated in Example 5.

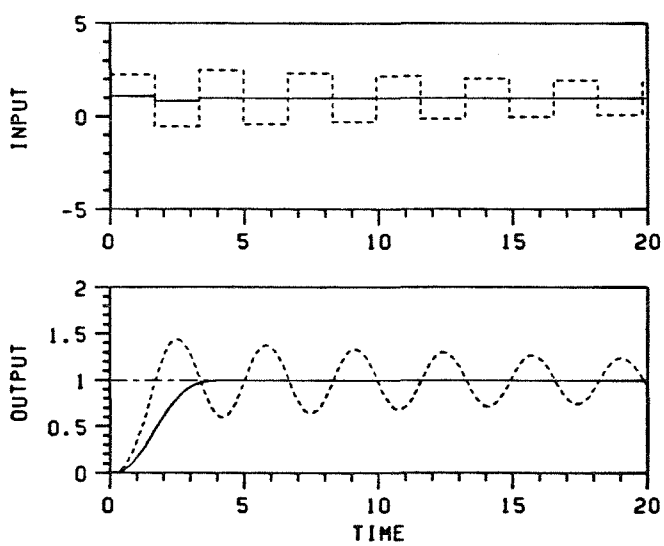

Figure 8. $G(s) = 2/(s^2 + 1{\cdot}2s + 1)(s+2)$, $T = 1{\cdot}65$, $\lambda = 0$: ··· the controller that minimizes the sum of squared errors; —— the new algorithm; –·– set-point.

Example 5

For $T = 0{\cdot}1$,

$$G(s) = 1{\cdot}5\frac{(s+2)}{(s+1)(s+3)} \Rightarrow G(z) = 0{\cdot}136\frac{(z-0{\cdot}819)}{(z-0{\cdot}905)(z-0{\cdot}741)}$$

The response is given in Fig. 9; the overshoot is about 450%.

A case with a RHP zero is the following.

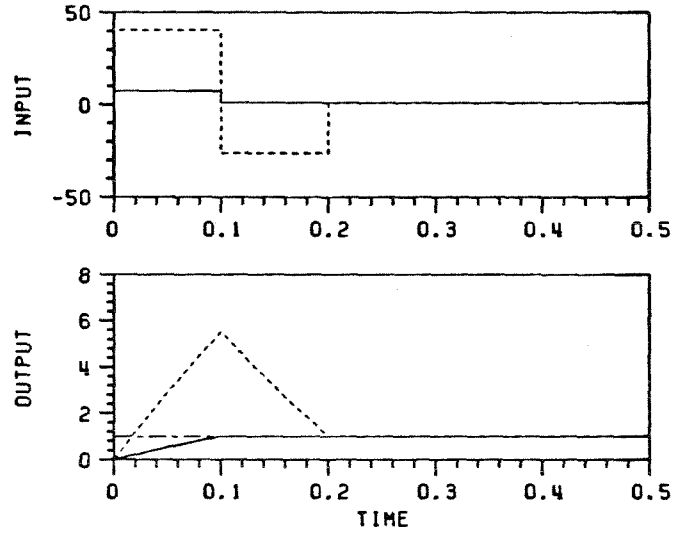

Figure 9. $G(s) = 1{\cdot}5(s+2)/(s+1)(s+3)$, $T = 0{\cdot}1$; ··· state deadbeat (also Vogel–Edgar, $\lambda = 0$); —— the new algorithm, $\lambda = 0$ (also Vogel–Edgar, $\lambda = 0{\cdot}5$); –·– set-point.

Example 6

For $T = 0{\cdot}1$,

$$G(s) = 0{\cdot}5\frac{(-s+2)}{(s^2+1{\cdot}5s+1)} \Rightarrow G(z) = -0{\cdot}0416\frac{(z-1{\cdot}223)}{(z^2-1{\cdot}851z+0{\cdot}861)}$$

The response with an extremely large undershoot is shown in Fig. 10.

(v) *Vogel–Edgar controller*

This controller is identical to the state deadbeat controller when the tuning parameter λ is equal to zero. The tuning parameter can only help in certain cases to eliminate the overshoot/undershoot problem of the state deadbeat controller.

The closed-loop transfer function is

$$V(z) = \frac{1-\exp(-T/\lambda)}{1-\exp(-T/\lambda)z^{-1}} \frac{(z-a_1)\ldots(z-a_{n-1})}{(1-a_1)\ldots(1-a_{n-1})z^n} z^{-N} \tag{24}$$

and the first non-zero value of the discrete output is

$$y_1 = \frac{1-\exp(-T/\lambda)}{(1-a_1)\ldots(1-a_{n-1})} \tag{25}$$

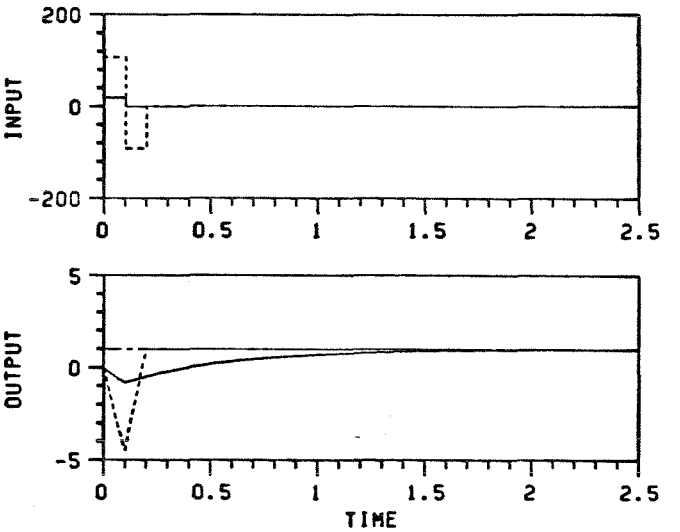

Figure 10. $G(s) = 0{\cdot}5(-s+2)/(s^2+1{\cdot}5s+1)$, $T = 0{\cdot}1$: ··· state deadbeat (also Vogel–Edgar, $\lambda = 0$); —— the new algorithm, $\lambda = 0$ (also Vogel–Edgar, $\lambda = 0{\cdot}497$); –·– set-point.

The tuning parameter can always be made sufficiently large so that y_1 is as small as desired. However for large λ the term $1 - \exp(-T/\lambda)z^{-1}$ in the denominator of the closed-loop transfer function will slow down the response significantly, unless it is cancelled by one of the zeros of the system. Hence, if there is only one zero near $+1$ and inside the unit circle, by selecting λ such that $\exp(-T/\lambda)$ is equal to that zero, not only the overshoot is eliminated, but, at the same time, the response becomes faster. This is illustrated in Fig. 9 for the system of Example 5, where by selecting $\lambda = 0{\cdot}5$, $\exp(-T/\lambda)$ becomes equal to the zero at 0·819.

In the case where there is only one zero near $+1$ but it is outside the unit circle, the best selection for λ would be such that $\exp(-T/\lambda)$ is equal to the inverse of that zero. We can see the result for the system of Example 6, in Fig. 10 ($\lambda = 0{\cdot}497 \Rightarrow \exp(-T/\lambda) = 1/1{\cdot}223$).

Vogel and Edgar did not provide any guidelines for the selection of λ. They suggest gradually increasing λ until the response becomes satisfactory.

In the case where there are two or more zeros near $+1$, λ has to be made very large in order to bring the overshoot or undershoot down to acceptable values (cancellation with only one of the zeros is not sufficient). The result is that the response becomes too slow to be acceptable. This will happen for systems with two or more zeros in the Laplace transfer function, either in the LHP or the RHP. The problem is illustrated in Example 7.

Example 7

For $T = 0{\cdot}1$,

$$G(s) = 2{\cdot}25\frac{(s+1)(s+2)}{(s+0{\cdot}5)(s+1{\cdot}5)(s^2+2{\cdot}5s+6)}$$

$$\Rightarrow G(z) = 0{\cdot}01067\frac{(z-0{\cdot}905)(z-0{\cdot}819)(z+0{\cdot}951)}{(z-0{\cdot}951)(z-0{\cdot}861)(z^2-1{\cdot}726z+0{\cdot}779)}$$

Two of the zeros are close enough to +1 to produce the extremely large overshoot shown in Fig. 11 (for $\lambda = 0$). By selecting λ such that $\exp(-T/\lambda)$ is equal to one of the zeros ($\lambda = 0{\cdot}5$, $\exp(-T/\lambda) = 0{\cdot}819$), the response shown in Fig. 11 is obtained. For $\exp(-T/\lambda)$ equal to the other zero ($\lambda = 1$, $\exp(-T/\lambda) = 0{\cdot}905$), the response is given in Fig. 12. The overshoot is still clearly unacceptable. In order to reduce it to 75%, λ has to be increased to 2·5 and extremely sluggish behaviour results (Fig. 12).

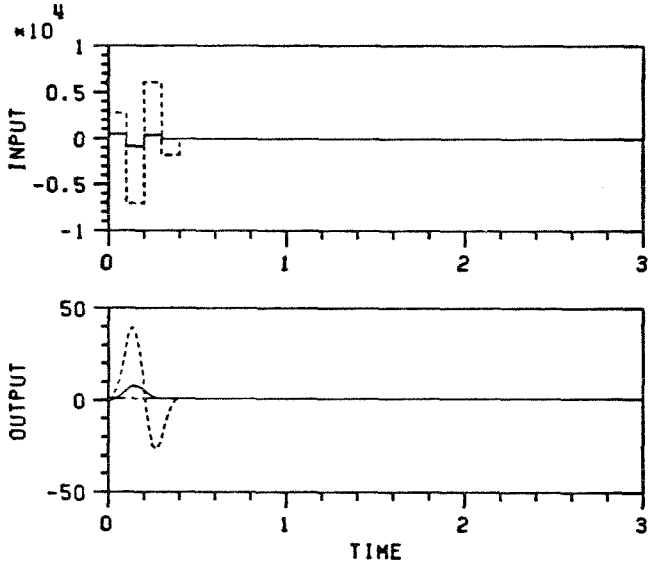

Figure 11. $G(s) = 2{\cdot}25(s+1)(s+2)/(s+0{\cdot}5)(s+1{\cdot}5)(s^2+2{\cdot}5s+6)$, $T = 0{\cdot}1$: ··· state deadbeat (also Vogel–Edgar, $\lambda = 0$); —— Vogel–Edgar, $\lambda = 0{\cdot}5$; –·– set-point.

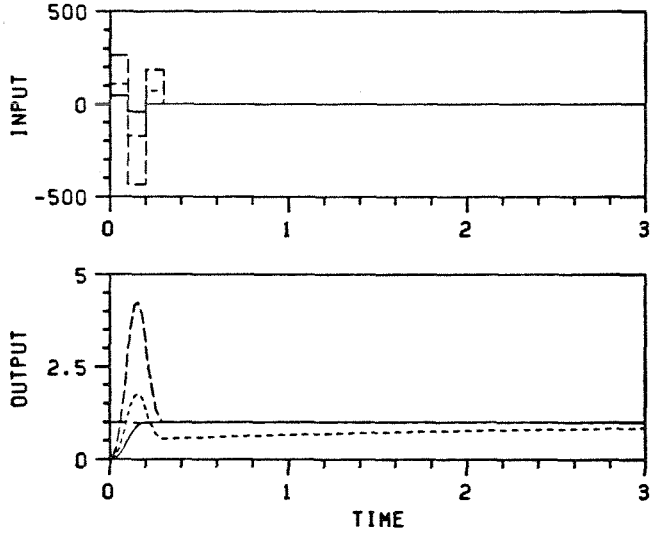

Figure 12. $G(s) = 2.25(s+1)(s+2)/(s+0{\cdot}5)(s+1{\cdot}5)(s^2+2{\cdot}5s+6)$, $T = 0{\cdot}1$: —— Vogel–Edgar, $\lambda = 1$; ··· Vogel–Edgar, $\lambda = 2{\cdot}5$; —— the new algorithm, $\lambda = 0$; –·– set-point.

2.3. *Discussion*

There are two basic problems associated with the performance of the controllers examined. One is rippling of the system output between the samples due to ringing of the manipulated variable and the other is large overshoot and/or undershoot of the discrete output itself. Dahlin's controller will also be unstable when the system has zeros outside the unit circle and the modified Dahlin controller has to be stabilized in this case by appropriate selection of the tuning parameter.

The transfer functions of the controller $G_c(z)$ of the open-loop structure (Fig. 1 (*b*)) are given in Table 1. In § 2.2 we explained how the problems are related to the location of the zeros of the model and how the location depends on the continuous system Laplace transfer function and on the sampling period T (Theorems 1 and 2).

The relations between the continuous system, the discrete controller, the sampling period and the associated problems are given in Table 3. The entries in the first

Continuous system	Control algorithm	Small T	$T_{cr}-\Delta T$	$T_{cr}+\Delta T$
LHP zero	DC	—	—†	Unstable†
	OD	—	—†	Overshoot†
	SE	—	—†	—†
	SD	Overshoot	Overshoot†	Overshoot†
	VE	One zero: — More: overshoot	Overshoot†	Overshoot†
RHP zero	DC	Unstable	Unstable	Rippling
	OD	Undershoot	—	Rippling
	SE	—	Rippling	Rippling
	SD	Undershoot	—	—
	VE	One zero: — More: undershoot	—	—
Excess zeros $n-m$ even‡	DC	$n-m=2$: rippling $n-m\geqslant 4$: unstable	Unstable	Rippling
	OD	Rippling	—	Rippling
	SE	Rippling	Rippling	Rippling
	SD	—	—	—
	VE	—	—	—
Excess zeros $n-m$ odd ($\geqslant 3$)	DC	Unstable	Unstable	Rippling
	OD	—	—	Rippling
	SE	—	Rippling	Rippling
	SD	—	—	—
	VE	—	—	—

† Two or more zeros (see § 2.3).
‡ For $n-m=2$ only the column for small T applies.
DC Dahlin's controller
OD Output deadbeat controller
SE Minimizes sum of squared errors of the output
SD State deadbeat controller
VE Vogel–Edgar controller
T_{cr} Sampling period at which the zero of the discrete system crosses the unit circle
ΔT Range around T_{cr} for which the corresponding problem occurs
m, n Degrees of numerator and denominator of $G(s)$

Table 3. Summary, of the problems of the examined control algorithms.

column refer to the Laplace transfer function of the system; hence the table shows what effect a LHP or a RHP zero or a pole excess $(n-m)$ of 2 or more, can have on the performance of each controller for various ranges of the sampling period T.

How small T has to be in order for the problems in Table 3 to arise, depends on the continuous system and the type of problem. An overshoot of 100% caused by a LHP zero at small Ts can occur for $T \leqslant 0{\cdot}75\tau$ for a real zero and for $T \leqslant 0{\cdot}9\tau$ for a pair of complex-conjugate zeros, where τ is the inverse of the distance of the zero from the origin. Similarly for a RHP zero, we may get a significant undershoot for $T \leqslant 0{\cdot}75\tau$ for a real zero and for $T \leqslant 0{\cdot}9\tau$ for complex zeros. The rippling caused by excess zeros at small Ts (when $(n-m)$ is even) can occur for T up to $0{\cdot}75\tau_D$, where τ_D is the dominant time constant of the system, but usually the problem appears for smaller Ts which can start as low as $0{\cdot}2\tau_D$.

T_{cr} denotes the critical sampling period at which a zero crosses the unit circle. Rippling may occur around T_{cr} for a range from $T_{cr}-\Delta T$ to $T_{cr}+\Delta T$. The value of T_{cr} for each zero of $G(z)$ has to be calculated from the Laplace transfer function $G(s)$. ΔT appears to be a fraction of T_{cr}: it is about $0{\cdot}3T_{cr}$ when we have a zero of $G(z)$ corresponding to a RHP zero of $G(s)$ and about $0{\cdot}7T_{cr}$ when we have an excess zero crossing the unit circle. The reason for the difference is that a zero corresponding to a RHP zero covers a greater distance on the complex plane than an excess zero, as T goes from zero to infinity, and therefore it moves faster with T.

When $G(s)$ has more than one zero in the LHP and when these zeros are much smaller than the poles (by an order of magnitude or more), then the corresponding discretized system $G(z)$ can have a zero either inside or outside the unit circle, depending on the sampling time. From Theorems 1 and 2, it follows that if such a zero crosses the unit circle, then there are two T_{cr}s for which a crossing occurs. For the smaller T_{cr} at which the zero leaves the unit circle, the problems of the examined algorithms are listed in Table 3. For the larger T_{cr} (at which the zero returns to the unit circle), all algorithms will generate significant overshoot before $t = T$ and, in addition, have all the other problems particular to each algorithm. The reason is that the open-loop step response of systems of this type shows large overshoot and the second T_{cr} is larger than the time t at which this overshoot occurs. This problem is not of practical importance since such a large T would result in a folding frequency π/T smaller than the bandwidth of $G(s)$ and would therefore not be selected.

From an inspection of the open-loop step response, it can easily be determined if any of the zeros of $G(z)$ arising from LHP zeros of $G(s)$ cross the unit circle as follows. Assume that $G(s)$ has been scaled so that $G(0) > 0$. If the unit step response is negative for some t' and if $T = t'$ then $G(z)$ has an odd number of zeros on the real axis between $+1$ and $+\infty$. The converse is also true. The reason is that K in (3) is the value of the open-loop step response at $t = T$. For all the systems tested with such crossings there was always a range of T for which the number of zeros between $+1$ and $+\infty$ was odd. Hence, the above condition on the step response seems to be both necessary and sufficient for the existence of such crossings. However no proof is available.

3. A new algorithm

In § 2 it became clear that the controller $G_c(z)$ of the structure of Fig. 1 (*b*) should not have poles with negative real part close to the unit circle. These poles cause oscillation of the controller output (with period equal to twice the sampling period if

the poles are negative reals) and as a result the process output exhibits *intersample* rippling.

Hence, when there are zeros of $G(z)$ with negative real part, one should not use those zeros, or their inverses, as poles of $G_c(z)$, but use poles at the origin instead.

When there are unstable zeros of $G(z)$ with positive real part, one should use their inverses as poles of $G_c(z)$. If zeros close to $+1$ were substituted with poles at the origin, then, in order to avoid steady-state offset (i.e. in order to have $G_c(1) = G(1)^{-1}$), large undershoot or overshoot would have to be accepted.

In addition, for all the control algorithms examined in § 2, the zeros of $G_c(z)$ are chosen to be equal to the poles of $G(z)$.

Hence a simple rule to design the controller $G_c(z)$ is the following.

(1) Use as zeros of G_c, the poles of $G(z)$ (they are assumed to be stable.

(2) Use as poles of G_c, the zeros of $G(z)$ with positive real part which are inside the unit circle, the inverses of those with positive real part which are outside the unit circle and as many at the origin as there are zeros with negative real part.

(3) An additional pole of G_c at the origin must be present because of the inherent time delay of a discrete system.

(4) The steady-state gain of G_c should be:

$$G_c(1) = \frac{1}{G(1)}$$

Let the superscripts '$-$' and '$+$' denote 'inside' and 'outside' the unit circle, respectively, and the subscripts '$+$' and '$-$' denote 'positive' and 'negative' real part. We have

$$G_c(z) = \frac{1}{K} \frac{\left(1 - \frac{1}{a^+_{+,k+1}}\right) \dots \left(1 - \frac{1}{a^+_{+,l}}\right)}{(1 - a^+_{+,k+1}) \dots (1 - a^+_{+,l})(1 - a_{-,l+1}) \dots (1 - a_{-,n-1})} \times \frac{(z - p_1) \dots (z - p_n)}{z^{n-l}(z - a^-_{+,1}) \dots (z - a^-_{+,k}) \left(z - \frac{1}{a^+_{+,k+1}}\right) \dots \left(z - \frac{1}{a^+_{+,l}}\right)} \tag{26}$$

where k zeros $(a^-_{+,1}, \dots, a^-_{+,k})$ have positive real part and are inside the unit circle, $l-k$ $(a^+_{+,k+1}, \dots, a^+_{+,l})$ have positive real part and are outside the unit circle and $n-1-l$ $(a_{-,l+1}, \dots, a_{-,n-1})$ have negative real part.

A filter of the form $F(z) = (1-\alpha)/(1-\alpha z^{-1})$ (where $0 \leqslant \alpha < 1$) should also be included. Theoretical justification for the inclusion of the filter will be given in § 4. At this point one can, nevertheless, note that the tuning parameter can be used to lower the values of the manipulated variable. This of course will be accomplished at the cost of a slower response.

The classical feedback controller $C(z)$ for the structure of Fig. 1 (*a*) can be obtained from

$$C(z) = \frac{F(z)G_c(z)}{1 - F(z)G_c(z)G(z)} \tag{27}$$

The behaviour of the new algorithm was examined for the examples of § 2 and, as the corresponding plots show, it performs very well (Figs. 3–12).

The algorithm is expected to work well all the time, unless of course there are unobservable open-loop oscillatory modes (Åström and Wittenmark 1984). This will happen if the sampling period matches the frequency of any oscillatory mode of the continuous open-loop system. The problem can be met by appropriate selection of the sampling period (for example, less than half the period of oscillations of the continuous system impulse response (Jury 1957)). The controller $G_c(z)$ can then be designed from $G(z)$ as above.

It should also be noted that this design procedure can be extended to external system inputs other than steps. One might already have observed that one can obtain the new algorithm of eqn. (26) by writing the controller transfer function $G_{SE}(z)$ (structure of Fig. 1 (*b*)) that minimizes the sum of the squared errors and substituting the poles with negative real part with poles at the origin, while keeping the same steady-state gain. The same procedure can be used for inputs other than steps to yield a problem-free controller. One should first obtain the transfer function of the controller (structure of Fig. 1 (*b*)) that minimizes the sum of the squared errors for the specified input and then make the correction mentioned above.

4. Stability in the presence of modelling error

For no modelling error, the classical feedback structure of Fig. 1 (*a*) is equivalent to the open-loop structure of Fig. 1 (*b*), where eqns. (1) and (2) relate the controllers $C(z)$ and $G_c(z)$. In the case where the model is not exact, let us denote the model by $\tilde{G}(z)$ and the actual plant by $G(z)$. Then the feedback structure is equivalent to that of Fig. 13 (*a*) where the two added $\tilde{G}(z)$ blocks cancel each other. This structure can be simplified to that in Fig. 13 (*b*).

The relationships between the controller $G_c(z)$ in Fig. 13 (*b*) and the classical feedback controller $C(z)$ are

$$G_c(z) = \frac{C(z)}{1 + C(z)\tilde{G}(z)} \tag{28}$$

$$C(z) = \frac{G_c(z)}{1 - G_c(z)\tilde{G}(z)} \tag{29}$$

For an exact model, the structure of Fig. 13 (*b*) simplifies for command-following to that shown in Fig. 1 (*b*), i.e. it becomes open-loop, which makes the design of G_c transparent. For example, if Dahlin had substituted the zeros which cause ringing in the corresponding $G_c(z)$ instead of $C(z)$ he would have got the desired result regarding ringing, without the complications and problems of the modified Dahlin controller.

On the other hand, when the model $\tilde{G}(z)$ is not exact, the structure of Fig. 13 (*b*) is not effectively open-loop and stability problems can arise.

A very useful property of this structure is that any controller $G_c(z)$ which satisfies $G_c(1) = \tilde{G}(1)^{-1}$ yields zero offset (Garcia and Morari 1982). From Fig. 13 (*b*), we find

$$u(z) = \frac{G_c(z)}{1 + G_c(z)[G(z) - \tilde{G}(z)]} y_s(z) \tag{30}$$

$$y(z) = \frac{G(z)G_c(z)}{1 + G_c(z)[G(z) - \tilde{G}(z)]} y_s(z) \tag{31}$$

where u is the manipulated variable.

For stability, it is necessary and sufficient that both of the following characteristic equations have their roots strictly inside the unit circle:

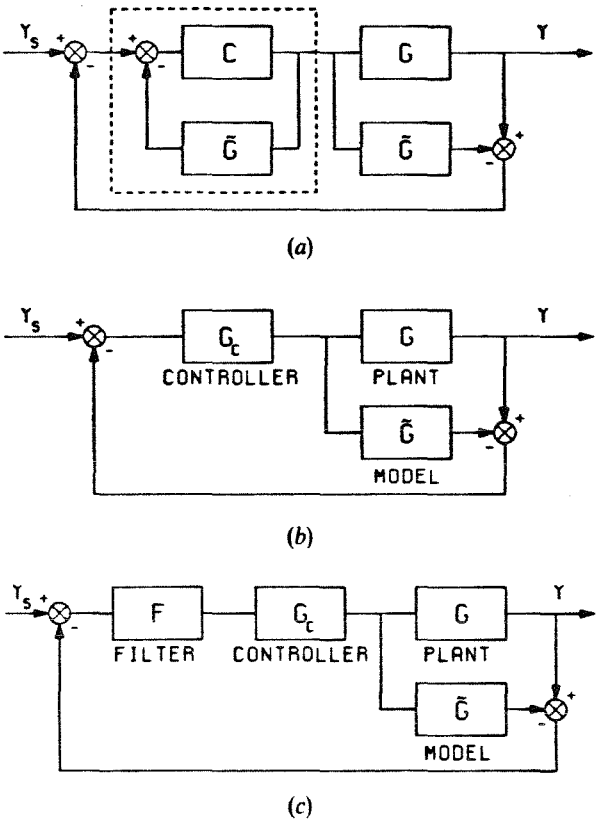

Figure 13. (*a*) Structure equivalent to the classical feedback. (*b*) Simplification of the structure shown in (*a*). (*c*) The equivalent structure with the filter included.

$$\frac{1}{G_c(z)} + [G(z) - \tilde{G}(z)] = 0 \tag{32}$$

$$\frac{1}{G(z)G_c(z)} + \frac{1}{G(z)}[G(z) - \tilde{G}(z)] = 0 \tag{33}$$

By adding a filter $F(z)$ to the controller $G_c(z)$ we get the structure in Fig. 13 (*c*). The importance of the filter is shown by the following theorem (Garcia and Morari 1985).

Theorem 3

Let $G(z)$ and $\tilde{G}(z)$ be stable, $G_c(1) = 1/\tilde{G}(1)$ and

$$F(z) = \frac{1-\alpha}{1-\alpha z^{-1}}, \quad 0 \leqslant \alpha < 1 \tag{34}$$

Then there exists an α^* $(0 \leqslant \alpha^* < 1)$ such that the system is closed-loop stable for all α in the range $\alpha^* \leqslant \alpha < 1$ if and only if $G(z)$ and $\tilde{G}(z)$ satisfy

$$G(1)\tilde{G}(1) > 0 \tag{35}$$

Hence, by using the structure of Fig. 13 (*c*) and a $G_c(z)$ such that $G_c(1) = 1/\tilde{G}(1)$,

stability can always be achieved for any mismatch of model and plant, as long as the steady-state gain of the model has the correct sign.

This structure does not have to be actually implemented, since it is entirely equivalent to the classical feedback. The conventional feedback controller $C(z)$ can be obtained from

$$C(z) = \frac{F(z)G_c(z)}{1 - F(z)G_c(z)\tilde{G}(z)} \tag{36}$$

An advantage of using the structure of Fig. 13 (*c*) for design purposes is that it provides a clear physical meaning to the tuning parameter α.

5. Conclusions

The purpose of this paper was to present and explain, in a transparent manner, the problems of some well known digital controllers for SISO systems. Based on the results of this study, a simple rule was derived for the design of a controller which will always perform well. The reason for trying to establish a simple rule is to make sure that this design procedure is of use to the practicing engineer.

The algorithm developed is free of the basic problems of the controllers examined, i.e. intersample rippling and overshoot or undershoot. In the case where all the unstable zeros of the pulse transfer function of the system $G(z)$ have negative real part, it yields a deadbeat controller which drives the discrete output of the system to the set-point in a finite number of time steps. When $G(z)$ has unstable zeros with positive real part, the controller drives the output to the set-point asymptotically in order to avoid large overshoot or undershoot. When all the zeros, stable or unstable, have positive real part, it minimizes the sum of the squared errors of the output.

An advantage of the proposed structure is that a tuning parameter is included (filter of Fig. 13 (*c*)) whose physical meaning and effect is clear to the designer. If the model is exact, no tuning is necessary in contrast with the Dahlin and Vogel–Edgar controllers. In the case of a mismatch between the system and the model, stability can always be guaranteed when some simple conditions hold. It is also important to note that the same tuning parameter can be used to reduce the values of the manipulated variable at the cost of a slower response.

ACKNOWLEDGEMENT

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CHAPTER III

DESIGN OF ROBUST DIGITAL CONTROLLERS AND SAMPLING TIME SELECTION FOR SISO SYSTEMS

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Design of robust digital controllers and sampling-time selection for SISO systems

EVANGHELOS ZAFIRIOU† and MANFRED MORARI†

The stability of a digital control system and its performance in terms of the continuous plant output are studied. A two-step controller design is proposed. In the first step, the assumption of no modelling error is made and a controller that combines properties of the algorithm that minimizes the sum of squared errors and a deadbeat-type algorithm is designed so that no intersample rippling appears. In the second step, a filter is designed so that appropriate conditions which guarantee robust stability and performance in the presence of model–plant mismatch are satisfied. The effect of the sampling time on the achievable performance and the robustness properties of the system is examined and the results are incorporated in a complete procedure for sampling-time selection and robust controller design. Finally, the procedure and some theoretical implications are illustrated with examples.

1. Introduction

The importance of obtaining control designs which are robust with respect to model–plant mismatch has been well emphasized in the literature in the last few years. For sampled-data systems, although information on the plant output is available only at the sample points and the manipulated variable is discrete, it is important that robust performance is guaranteed in terms of the continuous plant output. The internal model control (IMC) structure will be used to make some qualitative aspects of the problem clear and to derive quantitative robustness conditions. A synthesis method that makes use of these conditions will also be developed.

The selection of the sampling time is an integral and very important part of any control system design. The sampling time directly affects the achievable performance and the robustness properties of the control system. A clear qualitative understanding and a quantification of these relations will be attempted, which will then lead to a criterion for sampling-time selection built into the controller synthesis method.

Finally, the theoretical results will be incorporated into a complete step by step procedure for robust controller design and sampling-time selection.

2. System description and design goals

2.1. *Structure*

The classical feedback structure is shown in Fig. 1 (*a*). Wave lines are used to represent paths along which the signals are digital. The transfer function of the zero-order hold is

$$H(s) = \frac{1 - \exp(-sT)}{s} \tag{1}$$

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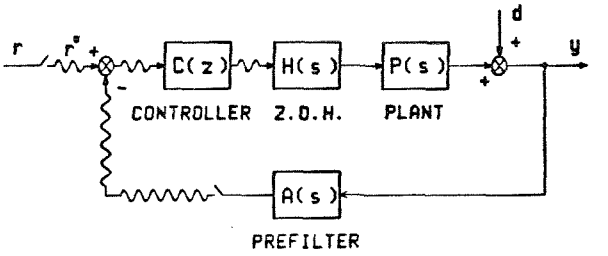
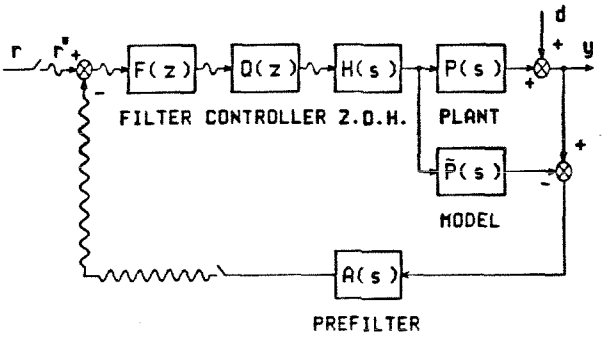


(*a*)



(*b*)

Figure 1. (*a*) Classical feedback structure. (*b*) Internal model control structure.

where T is the sampling time. $A(s)$ is an analogue anti-aliasing prefilter. A detailed explanation of the problem of aliasing can be found in digital control books (see, for example, Åström and Wittenmark 1984, and Franklin and Powell 1980). Briefly one can see the problem by looking at (2) which relates a continuous signal $a(s)$ to its z-transform $a^*(z)$:

$$a^*(\exp i\omega T) = \frac{1}{T}\sum_{k=-\infty}^{\infty} a(i\omega + ik\,2\pi/T) \tag{2}$$

Equation (2) shows that the value of a^* at a frequency ω is the sum of the values of the continuous signal a at the frequencies $\omega + k\,2\pi/T$ divided by T. The result is that after sampling, a high-frequency disturbance or measurement noise cannot be distinguished from an equivalent low-frequency one. The prefilter serves the function of cutting off high-frequency components from the analogue signals before sampling, when that is necessary. It is clear from (2) that $a^*(\exp i\omega T)$ is periodic in ω with period $2\pi/T$. It is also important to note that for a rational function $a^*(z)$ we have $\overline{a^*(z)} = a^*(\bar{z})$, where the overbar indicates the complex conjugate, and therefore for $\pi/T < \omega < 2\pi/T$ we have

$$a^*(\exp i\omega T) = \overline{a^*(\exp -i\omega T)} = \overline{a^*(\exp i(2\pi/T - \omega)T)} \tag{3}$$

Hence in addition to the periodicity, a rational z-transform $a^*(z)$ has the property that its values for frequencies greater than π/T are uniquely determined by those for $0 \leqslant \omega \leqslant \pi/T$.

The IMC structure was introduced by Garcia and Morari (1982). This structure is

a theoretical tool that simplifies the design of a controller which is robust to plant–model mismatch. This will become clear in the following sections of this paper. In Fig. 1 (*b*) an IMC structure which includes both digital and analogue signals is given. $\tilde{P}(s)$ is the process model and $P(s)$ the actual process. Note that though the block $\tilde{P}(s)$ appears in the structure, one will not have to implement an analogue block for the model. The structure that will be implemented is that of Fig. 1 (*a*) and the feedback controller $C(z)$ can be obtained from the IMC controller $Q(z)$ and filter $F(z)$ by

$$C(z) = \frac{F(z)Q(z)}{1 - F(z)Q(z)\tilde{P}_A^*(z)} \tag{4}$$

where

$$\tilde{P}_A^*(z) = \mathscr{Z}\mathscr{L}^{-1}\{H(s)\tilde{P}(s)A(s)\} \tag{5}$$

When (4) holds the mappings between the inputs r, d and the output y are the same for the two structures.

Let

$$\tilde{P}^*(z) = \mathscr{Z}\mathscr{L}^{-1}\{H(s)\tilde{P}(s)\} \tag{6}$$

$$P^*(z) = \mathscr{Z}\mathscr{L}^{-1}\{H(s)P(s)\} \tag{7}$$

$$P_A^*(z) = \mathscr{Z}\mathscr{L}^{-1}\{H(s)P(s)A(s)\} \tag{8}$$

$$d^*(z) = \mathscr{Z}\mathscr{L}^{-1}\{d(s)\} \tag{9}$$

$$d_A^*(z) = \mathscr{Z}\mathscr{L}^{-1}\{A(s)d(s)\} \tag{10}$$

$$y^*(z) = \mathscr{Z}\mathscr{L}^{-1}\{y(s)\} \tag{11}$$

Then the continuous plant output $y(s)$ is given by

$$y(s) = \frac{F(\exp sT)Q(\exp sT)H(s)P(s)}{1 + F(\exp sT)Q(\exp sT)[P_A^*(\exp sT) - \tilde{P}_A^*(\exp sT)]} \times [r^*(\exp sT) - d_A^*(\exp sT)] + d(s) \tag{12}$$

In the above the transformation $z = \exp sT$ is used. Sampling of (12) yields

$$y^*(z) = \frac{F(z)Q(z)P^*(z)}{1 + F(z)Q(z)[P_A^*(z) - \tilde{P}_A^*(z)]}[r^*(z) - d_A^*(z)] + d^*(z) \tag{13}$$

The digital controller output $u(z)$ is given by

$$u(z) = \frac{F(z)Q(z)}{1 + F(z)Q(z)[P_A^*(z) - \tilde{P}_A^*(z)]}[r^*(z) - d_A^*(z)] \tag{14}$$

2.2. *Plant uncertainty description*

In order to be able to design a control system which is robust with respect to model–plant mismatch one should have some bounds on this mismatch, in other words one should know how 'far' the actual process can be from the model.

The most commonly used descriptions of plant uncertainty for control purposes is the additive and multiplicative uncertainty (Doyle and Stein 1981). This kind of description can be obtained from bounds on the values of the estimated parameters of

the process model, either analytically, as illustrated in the examples of § 6, or numerically.

We can write

$$P(s) = \tilde{P}(s) + E_a(s) \tag{15}$$

$$P(s) = \tilde{P}(s)[1 + E_m(s)] \tag{16}$$

where for the additive and the multiplicative uncertainty, E_a and E_m, we have

$$|E_a(i\omega)| \leqslant l_a(\omega) \quad \forall\omega \tag{17}$$

$$|E_m(i\omega)| \leqslant l_m(\omega) \quad \forall\omega \tag{18}$$

and the bounds $l_a(\omega)$ and $l_m(\omega)$ are known. Note that

$$l_a(\omega) = |\tilde{P}(i\omega)| \cdot l_m(\omega) \tag{19}$$

Typically, $l_m(\omega)$ becomes equal to 1 or greater for high frequencies where nothing is known about the phase characteristics.

Equation (12) indicates that we need to obtain a bound for $P_A^*(\exp sT) - \tilde{P}_A^*(\exp sT)$. We have from (5), (8), and (15)

$$P_A^*(z) - \tilde{P}_A^*(z) = \mathscr{Z}\mathscr{L}^{-1}\{H(s)[P(s) - \tilde{P}(s)]A(s)\} = \mathscr{Z}\mathscr{L}^{-1}\{H(s)E_a(s)A(s)\}$$

and from (2) it follows

$$P_A^*(\exp i\omega T) - \tilde{P}_A^*(\exp i\omega T) = \frac{1}{T}\sum_{k=-\infty}^{\infty} HE_aA(i\omega + ik\,2\pi/T) \tag{20}$$

Equations (17) and (20) can now be used to obtain the following bound

$$|P_A^*(\exp i\omega T) - \tilde{P}_A^*(\exp i\omega T)| \leqslant \frac{1}{T}\sum_{k=-\infty}^{\infty} |HA(i\omega + ik\,2\pi/T)|\, l_a(\omega + k\,2\pi/T) \triangleq l_a^*(\omega) \tag{21}$$

Since the plant is a physical system, $P(s)$ and $\tilde{P}(s)$ are strictly proper and so $|E_a(i\omega)| \to 0$ at least as fast as $1/\omega$, as $\omega \to \infty$. Hence we can always obtain a bound $l_a(\omega)$ in (17) such that $l_a(\omega) \to 0$, as $\omega \to \infty$. Also $|HA(i\omega)| \to 0$ at least as fast as $1/\omega$ as $\omega \to \infty$ even if $A(s) = 1$ and therefore $|HA(i\omega)|\,l_a(\omega) \to 0$ faster than $1/\omega$ as $\omega \to \infty$, which implies that the sum in (21) converges. Still, note that if a prefilter $A(s)$ is used, the property $l_a(\omega) \to 0$ as $\omega \to \infty$ is not needed for convergence. Finally, note that for computational purposes only a few terms in (21) need be considered. The reason for this is that $A(s)$ is small for ω larger than π/T in order to cut off the high-frequency components. Also from (1) it follows that $H(i\omega)/T$ is small for $\omega \geqslant \pi/T$. Hence for computational purposes one need only consider two or three terms in (21). Actually there is one dominant term in (20) and (21), which is the one for which $-\pi/T \leqslant \omega + k\,2\pi/T \leqslant \pi/T$. Hence for $0 \leqslant \omega \leqslant \pi/T$, the dominant term corresponds to $k = 0$.

2.3. *Design goals and procedure*

2.3.1. *Zero offset*

The property of zero steady-state offset for some class of external inputs is an essential property of the control system. The conditions that have to be satisfied in order for this to happen impose certain requirements on the controller $Q(z)$, the filter $F(z)$ and the anti-aliasing prefilter $A(s)$, described by the following theorem.

Theorem 1

For an open-loop stable plant and provided that the closed-loop system is stable, the necessary and sufficient conditions for no offset for the class of external inputs $r(s)$ and $d(s)$ with all poles in the open left-half plane except l poles at $s = 0$ where $l \leqslant m$ and m is specified, are the following:

$$F(1)Q(1)\tilde{P}^*(1) = 1 \tag{22}$$

$$\frac{d^k}{dz^k}(F(z)Q(z)\tilde{P}^*(z))\bigg|_{z=1} = 0, \quad k = 1, \ldots, m-1 \tag{23}$$

$$A(0) = 1 \tag{24}$$

$$\frac{d^k}{ds^k}A(s)\bigg|_{s=0} = 0, \quad k = 1, \ldots, m-1 \tag{25}$$

For $m = 1$, only (22) and (24) apply.

Proof

See Appendix A.

The implications of the above relations on the design of Q and F will be considered in subsequent sections. Let us discuss only briefly the design of the prefilter $A(s)$, whose performance specification is quite simple, namely to cut off high-frequency components. Most digital control books (Åström and Wittenmark 1984, Franklin and Powell 1980) discuss different types of anti-aliasing prefilters, which satisfy (24). In the case of $m > 1$, a simple modification can be used; let us write

$$A(s) = A_1(s)A_m(s) \tag{26}$$

where

$$A_m(s) = \frac{c_{m-1}s^{m-1} + \ldots + c_1 s + 1}{(\tau s + 1)^{m-1}}, \quad m \geqslant 2 \tag{27}$$

and $A_1(s)$ is an appropriate prefilter for $m = 1$. Then for a specified τ, (25) can be used to compute the coefficients $c_1, \ldots, c_{m-1}$. Qualitatively it is clear that the use of $A_m(s)$ to satisfy (25) should not significantly change the behaviour of $A_1(s)$. The reason is that (25) simply adds some properties at $\omega = 0$ and this can be done without affecting the high-frequency properties of $A_1(s)$. A large τ should be used to push the effect of $A_m(s)$ towards $\omega = 0$. Indeed for a usual second-order $A_1(s) = \omega_0^2/(s^2 + 2\omega_0\zeta s + \omega_0^2)$ and for $m = 2$ (ramp inputs), (25) yields $c_1 = \tau + 2\zeta/\omega_0$ and therefore for a sufficiently large τ, $A_m(s)$ does not significantly affect the high-frequency performance of $A(s)$.

2.3.2. *IMC design procedure*

The purpose of the control system is to guarantee stability and good performance not only when the model is exact but also in the presence of model–plant mismatch. The IMC structure gives rise naturally to a two-step design procedure. From Fig. 1 (*b*) it is clear that when no modelling error is present, the design of the IMC controller $Q(z)$ reduces to the design of an open-loop controller. Indeed for $P_A^* = \tilde{P}_A^*$ and $P = \tilde{P}$, (12) becomes

$$y(s) = Q(\exp sT)H(s)\tilde{P}(s)(r^*(\exp sT) - d_A^*(\exp sT)) + d(s) \tag{28}$$

where the filter F is assumed to be the identity. Hence in the first step, $Q(z)$ can be designed so that some desired response is achieved. Inherent performance limitations exist, imposed by non-minimum-phase elements and potential intersample rippling, but the simple form of (28) simplifies the design considerably. For example, stability is not an issue if $P = \tilde{P}$, since then for an open-loop stable plant, a stable Q is all that is required for overall stability. Section 3 deals with the design of Q and the effect of sampling time on the achievable performance.

A mismatch between the model and the plant will generate a feedback signal which may cause performance deterioration or instability. The IMC filter $F(z)$ is used to take care of this problem by acting on this signal before it is fed to $Q(z)$. The filter should be designed so that stability and acceptable performance are guaranteed for a given set of possible plants. Section 4 of the paper deals with the derivation of the robustness conditions that have to be satisfied, and the filter design.

The fact that in the first step of the procedure Q is designed so that no offset is produced for a given class of inputs when $F(z) = 1$, means that according to Theorem 1, $Q(z)$ has to satisfy

$$Q(1)\tilde{P}^*(1) = 1 \tag{29}$$

$$\frac{d^k}{dz^{-k}}(Q(z)\tilde{P}^*(z))\bigg|_{z=1} = 0, \quad k = 1, \ldots, m-1 \tag{30}$$

Then clearly a filter $F(z)$ will satisfy (22) and (23) for a $Q(z)$ that satisfies (29) and (30) if and only if

$$F(1) = 1 \tag{31}$$

$$\frac{d^k}{dz^{-k}}F(z)\bigg|_{z=1} = 0, \quad k = 1, \ldots, m-1 \tag{32}$$

3. Controller design for no modelling error

3.1. *Effect of sampling on performance*

In § 1, it was mentioned that sampling puts a limitation on the achievable performance. We shall now demonstrate this fact quantitatively. Consider (28)

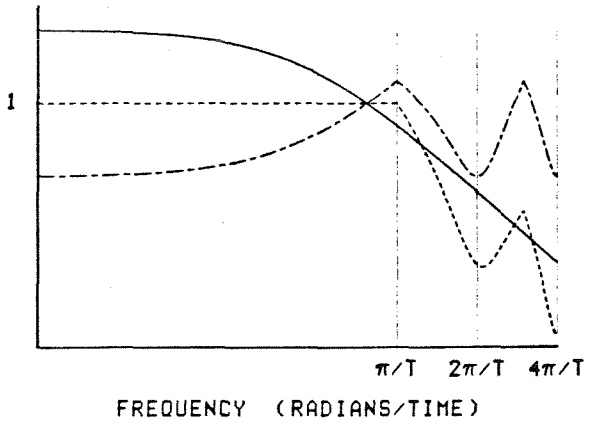


Figure 2. Effect of sampling on performance (logarithmic plot): ——— $|\tilde{P}(i\omega)|$;—·— $|Q(\exp i\omega T)|$; ---- $|Q(\exp i\omega T)\tilde{P}(i\omega)|$.

obtained from (12) for no modelling error. $H(s)r^*(\exp sT)$ and $H(s)d^*_A(\exp sT)$ are zero-order-hold reconstructions of the set-point $r(s)$ and output disturbance $d(s)$. Though these are not exact, we shall assume that they are, in order to demonstrate the limitation that comes from the periodicity of Q. Then (28) implies that for good performance $\tilde{P}(s)Q(\exp sT)$ should be close to one.

In Fig. 2 a typical Bode plot of $\tilde{P}(s)$ is shown. For perfect performance we need a Q equal to the inverse of $\tilde{P}(s)$. However, as described by (3), Q is periodic in ω with period $2\pi/T$ and its values for frequencies greater than π/T are uniquely determined by those for $\omega \leqslant \pi/T$. In Fig. 2, an ideal Q is plotted which inverts $\tilde{P}(s)$ for ω up to π/T. In order for this to be accomplished, Q has actually to be of infinite order. However even for this Q, it is clear in Fig. 2 that the closed-loop transfer function $\tilde{P}(s)Q(\exp sT)$ cannot have a bandwidth larger than π/T.

3.2. *The controller* Q

We have seen that sampling limits the achievable performance. The question that arises is how to design Q so that for a given sampling time T we obtain the 'best' possible performance. In addition, the design method should be simple enough so that designing Q for more than one sampling time is not time-consuming. The necessity for repeating the design for more than one T will become apparent in § 5.

A detailed study of the advantages and disadvantages and the theoretical reasons behind them, for a number of well-known digital control algorithms led Zafiriou and Morari (1985) to a simple method for designing Q from the model $\tilde{P}^*(z)$. To do so one should first obtain the controller Q_{SE} which minimizes the sum of squared errors between the sampled system output y^* and a specified external input. Then to obtain $Q(z)$ one should substitute the poles of $Q_{SE}(z)$ which have a negative real part with poles at the origin while preserving the property of zero steady-state offset. The reason for the substitution is that poles with a negative real part produce undesirable intersample rippling in the continuous plant output, which does not reveal itself in the sum of squared errors, computed only at discrete points in time. The introduction of poles at the origin, aims at incorporating in the design some of the advantages of a deadbeat-type response, while at the same time avoiding known problems of deadbeat controllers like overshoot and undershoot (Zafiriou and Morari 1985).

We can always write $\tilde{P}^*(z)$ as

$$\tilde{P}^*(z) = K\frac{(z-a_1)\ldots(z-a_{n-1})}{(z-p_1)\ldots(z-p_n)}z^{-N} \tag{33}$$

where N is the largest integer such that NT is less than or equal to the dead time. All the poles are assumed to be stable.

Let $v(s)$ be the external input which we want our system to follow (for v a setpoint) or reject (for v an output disturbance). All poles of $v(s)$ are assumed to be in the open left-half plane, except possibly some at $s = 0$. It is important however to note that in order for the control system to yield zero steady-state offset for all inputs with m or less poles at $s = 0$, the controller $Q(z)$ must have been designed for an input $v(s)$ which has m such poles. Let

$$v^*(z) = \mathscr{Z}\mathscr{L}^{-1}\{v(s)\} \tag{34}$$

Write

$$v^*(z) = z^{-N_v}(v_0 + v_1z^{-1} + v_2z^{-1} + \ldots) \tag{35}$$

where v_l, $l = 0, 1, \ldots$, are the values of $\mathscr{L}^{-1}\{v(s)\}$ at time $t = (N_v + l)T$, $v_0 \neq 0$. Define

$$v_N^*(z) = [z^{N_v} v^*(z) - (v_0 + v_1 z^{-1} + \ldots + v_N z^{-N})] z^{N+1}$$
$$= v_{N+1} + v_{N+2} z^{-1} + v_{N+3} z^{-2} + \ldots \tag{36}$$

This represents the part of $\mathscr{Z}^{-1}\{v^*(z)\}$ for time points greater than the dead-time of the plant, moved by $(N+1)T$ to the left on the time axis. Another way to compute $v_N^*(z)$ is to find from $\mathscr{L}^{-1}\{v(s)\}$ the Laplace transform of the continuous time function that corresponds to these points and then take the z-transform from tables.

Without loss of generality, assume that $a_1, \ldots, a_f$ and $a_{v1}, \ldots, a_{vh}$ are the zeros of $\tilde{P}^*(z)$ and $v^*(z)$ respectively, which are outside the unit circle. Define

$$\tilde{P}_+^*(z) = \prod_{j=1}^{f} \frac{(1 - a_j^{-1})(z - a_j)}{(1 - a_j)(z - a_j^{-1})} \tag{37}$$

$$v_+^*(z) = \prod_{j=1}^{h} \frac{(1 - a_{vj}^{-1})(z - a_{vj})}{(1 - a_{vj})(z - a_{vj}^{-1})} \tag{38}$$

and

$$\tilde{P}_-^*(z) = (\tilde{P}_+^*(z))^{-1} \tilde{P}^*(z) z^N \tag{39}$$

$$v_-^*(z) = (v_+^*(z))^{-1} v^*(z) z^{N_v} \tag{40}$$

Then $Q_{SE}(z)$ is given by

$$Q_{SE}(z) = (\tilde{P}_-^*(z) v_-^*(z))^{-1} \{(\tilde{P}_+^*(z) v_+^*(z))^{-1} v_N^*(z) z^{-1}\}_- \tag{41}$$

where $\{\cdot\}_-$ is obtained by taking a partial fraction expansion of $\{\cdot\}$ and discarding the terms with poles *outside* the unit circle. Terms with poles at $z = 1$ are retained. The constant term is zero because $\{\cdot\}$ is strictly proper. The steps used to arrive at (41) are given in Appendix B.

In the case of set-point following, one often has available and supplies to the controller future values of the set-point, which one wants the system output to follow after some time steps. By doing so, better servo behaviour is accomplished. In this case, $Q_{SE}(z)$ can be obtained by using $v_{N_0}^*(z)$ instead of $v_N^*(z)$ in (41), where $v_{N_0}^*$ is defined as follows:

$$v_{N_0}^*(z) = [z^{N_v} v^*(z) - (v_0 + v_1 z^{-1} + \ldots + v_{N_0} z^{-N_0})] z^{N+1} \tag{42}$$

with

$$N_0 = \max\{N - N_p, 0\} \tag{43}$$

and N_p the number of time steps ahead for which the set-point is supplied. How this conclusion is reached, is discussed in Appendix B.

$Q(z)$ can now be obtained from $Q_{SE}(z)$ as

$$Q(z) = q(z) Q_{SE}(z) B(z) \tag{44}$$

where

$$q(z) = z^{-\kappa} \prod_{j=1}^{\kappa} \frac{(z - \pi_j)}{(1 - \pi_j)} \tag{45}$$

$$B(z) = \sum_{j=0}^{m-1} b_j z^{-j} \tag{46}$$

and π_j, $j = 1, \ldots, \kappa$, are the poles of $Q_{SE}(z)$ with a negative real part. The coefficients b_j, $j = 0, \ldots, m-1$, will be determined so that the controller produces zero steady-state

offset for all inputs $v(s)$ with m or less poles at $s=0$. By its construction, $Q_{SE}(z)$ produces no offset and therefore it satisfies (29) and (30). Then clearly $Q(z)$ will satisfy (29) and (30) if and only if

$$q(1)B(1)=1 \tag{47}$$

$$\frac{d^k}{dz^k}(q(z)B(z))\bigg|_{z=1}=0, \quad k=1,\ldots,m-1 \tag{48}$$

Equation (47) yields

$$b_0=1-(b_1+\ldots+b_{m-1}) \tag{49}$$

Note that for $m=1$, only (47) need be considered and then (46) and (49) yield $B(z)=1$. Equation (48) is equivalent to

$$\frac{d^k}{d\lambda^k}(q(\lambda^{-1})B(\lambda^{-1}))\bigg|_{\lambda=1}=0, \quad k=1,\ldots,m-1 \tag{50}$$

Note that $q(\lambda^{-1})$ is a polynomial in λ and therefore its derivatives with respect to λ can be computed easily. Hence simple successive substitution will reduce (50) to

$$\frac{d^k}{d\lambda^k}B(\lambda^{-1})\bigg|_{\lambda=1}=\gamma_k, \quad k=1,\ldots,m-1 \tag{51}$$

where the γs are known. Equation (51) can be written in terms of the bs as

$$N_{m-1}\begin{bmatrix} b_1 \\ \vdots \\ b_{m-1}\end{bmatrix}=\begin{bmatrix}\gamma_1 \\ \vdots \\ \gamma_{m-1}\end{bmatrix} \tag{52}$$

where the matrix N_k is defined for some k as a matrix of dimension $(m-1)\times k$ whose elements v_{ij} are

$$v_{ij}=\begin{cases} 0 & \text{for } i>j \\ \dfrac{j!}{(j-i)!} & \text{for } i\leqslant j\end{cases} \tag{53}$$

Equation (52) can be solved by successive substitution. Note that for the simple case of $m=2$ we get

$$b_1=\gamma_1=\sum_{j=1}^{\kappa}\frac{\pi_j}{(1-\pi_j)}, \quad \text{for } m=2 \tag{54}$$

The case of $v(s)=1/s$ is of special interest because this is the most commonly considered input. In this case $v^*(z)=v_N^*(z)=v_-^*(z)=z/(z-1)$ and $v_+^*(z)=1$. Then (41) yields $Q_{SE}(z)=(z\tilde{P}_-^*(z))^{-1}$ and therefore the $Q(z)$ obtained from (44) for $B(z)=1$ can actually be constructed by using the following rules (Zafiriou and Morari 1985).

(i) Use as zeros of $Q(z)$ the poles of $\tilde{P}^*(z)$.

(ii) Use as poles of $Q(z)$ the zeros of $\tilde{P}^*(z)$ with a positive real part that are inside the unit circle, also the inverses of those with a positive real part which are outside the unit circle, and as many at the origin as there are zeros with a negative real part.

(iii) An additional pole of $Q(z)$ at the origin must be present because of the inherent time delay of a sampled-data system.

(iv) The steady-state gain of $Q(z)$ should be $Q(1) = 1/\tilde{P}^*(1)$.

This controller designed for step inputs combines the advantages of the algorithm that minimizes the sum of squared errors and of deadbeat-type algorithms. In the case where all the unstable zeros of $\tilde{P}^*(z)$ have a negative real part, it yields a deadbeat controller which drives the discrete output of the system to the set-point in a finite number of time steps. When $\tilde{P}^*(z)$ has unstable zeros with a positive real part, the controller drives the output to the set-point asymptotically in order to avoid large overshoot or undershoot. When all the zeros, stable or unstable, have a positive real part, it minimizes the sum of the squared errors of the output. The same properties are maintained for a controller designed for inputs other than steps according to (41), (44), (45) and (46), when the minimum number of coefficients b_i necessary to satisfy (47) and (48) is used.

4. Filter design for model-plant mismatch

In this section conditions for stability and good performance in the presence of a modelling error are derived and a method for designing a filter so that these conditions are satisfied is proposed. Also the effect of sampling time on robustness is discussed.

4.1. *Robust stability*

As mentioned earlier, the plant is assumed to be open-loop stable and therefore, when $P = \tilde{P}$, all that is required for overall stability is that Q and F are also stable. Application of the 'small gain' theorem to (13) and (14) and use of (21) will then yield the following stability condition for $P \neq \tilde{P}$ (see Doyle and Stein 1981).

Theorem 2

Let $P(s)$, $Q(z)$ and $F(z)$ be stable. Then the system in Fig. 1 (*b*) is stable

$$\forall\, P(s) \text{ s.t. } |P_A^*(\exp i\omega T) - \tilde{P}_A^*(\exp i\omega T)| \leqslant l_a^*(\omega) \quad \text{for } 0 \leqslant \omega \leqslant \pi/T$$

if and only if

$$|Q(\exp i\omega T)F(\exp i\omega T)| \cdot l_a^*(\omega) < 1 \quad \text{for } 0 \leqslant \omega \leqslant \pi/T \tag{55}$$

Note that the periodicity and (3) imply that if (55) holds for $0 \leqslant \omega \leqslant \pi/T$, then it holds for all ω. Note also that stability of the system in Fig. 1 (*b*) is equivalent to stability of the classical feedback structure in Fig. 1 (*a*), provided that $C(z)$ is related to Q and F through (4).

The above condition is both necessary and sufficient if stability for all the plants in the set described by (21) is required. Any conservativeness comes only from the fact that this set is, in general, larger than that of the plants which one actually needs to consider. For a specified sampling time, $l_a^*(\omega)$ is obtained from (21) and $Q(z)$ constructed according to § 3.2. If one then selects $F(z) = 1$, the condition (55) may or may not be satisfied. If not, one should use an $F(z) \neq 1$ to achieve it. The simplest filter $F(z)$ is a first-order one:

$$F_1(z) = \frac{(1-\alpha)z}{z-\alpha} \tag{56}$$

This filter satisfies (31) but not (32) and therefore it can be used only for external inputs $r(s)$ with one or less poles at $s = 0$ ($m = 1$) when no offset is required. We shall assume here that this is the case. The structure of the filter for $m \geqslant 2$ will be discussed in § 4.3.1.

It is clear from (56) that by changing α we can affect the value of $|F(\exp i\omega T)|$ at every frequency except $\omega = 0$. Hence it is important to examine (55) at $\omega = 0$. From (21) we get $l_a^*(0) = l_a(0)$ since $H(i(2\pi k)/T) = 0$ for $k = \pm 1, \pm 2, \ldots$, and $H(0)/T = A(0) = 1$. Then (29), (19) and the fact that $\tilde{P}^*(1) = \tilde{P}(0)$ imply that (55) will be satisfied for $\omega = 0$ and an $F(z)$ given by (56) if and only if $l_m(0) < 1$. All that this means is that the error between the steady-state gain of the actual plant and that of the model should not be more than 100% of the gain of the model, which is a rather easily satisfied condition. For example, if all the possible plants have steady-state gains with the same sign, then one can always choose an appropriate gain for the model so that $l_m(0)$ is less than one.

Hence if (55) is not satisfied we can always increase the time constant of the filter until it does. Clearly such an α will always exist provided that $|Q(\exp i\omega T)|l_a^*(\omega)$ is finite for all $0 \leqslant \omega \leqslant \pi/T$ and of course that $l_m(0) < 1$. However increasing the filter time-constant means that we are simply reducing the closed-loop bandwidth of the nominal system (i.e. no modelling error) and in § 3.1 we saw that this is equivalent to using a larger sampling time T. This becomes clearer if we write (55) as

$$|\tilde{P}(i\omega)Q(\exp i\omega T)F(\exp i\omega T)| \leqslant |\tilde{P}(i\omega)|/l_a^*(\omega) \tag{57}$$

One can see that the bandwidth of the left-hand-side term can be reduced by either increasing α in $F(z)$ or leaving $F(z) = 1$ and increasing T. A graphical depiction of the above is given in Fig. 3. Note that in Fig. 3 the right-hand-side term of (56) is assumed to be independent of T by using the approximation $l_a^*(\omega) \approx l_a(\omega)$. For illustrative purposes, this is a reasonable approximation for $0 \leqslant \omega \leqslant \pi/T$ but it should not be used to check (55); $l_a^*(\omega)$ should be computed from (21).

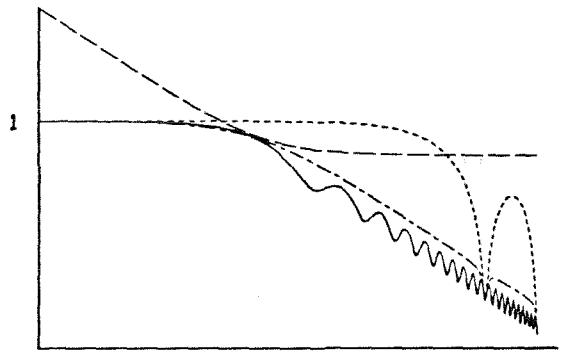


Figure 3. Effect of sampling on robust stability (logarithmic plot): ———— $1/l_m(\omega)$; ——— $|Q(\exp i\omega T)\tilde{P}(i\omega)|$, $T = T_1$; ------ $|Q(\exp i\omega T)\tilde{P}(i\omega)|$, $T = T_2 < T_1$; – – – $|F_1(\exp i\omega T)Q(\exp i\omega T)\tilde{P}(i\omega)|$, $T = T_2$.

4.2. *Robust performance*

In § 3.1 it was demonstrated that since the values of $Q(z)$ for frequencies larger than π/T are uniquely determined from its values for $0 \leqslant \omega \leqslant \pi/T$, by using a digital controller one can only try to guarantee good performance for frequencies less than

π/T. The same holds for $F(z)$, of course, since it is a rational function in z. Therefore, we shall now proceed to obtain a condition for acceptable performance in the presence of model–plant mismatch by considering all frequencies such that $0 \leqslant \omega \leqslant \pi/T$. According to § 2.2, there is one dominant term in the infinite sum in (20) which for $0 \leqslant \omega \leqslant \pi/T$ is the term corresponding to $k = 0$. Hence (20), yields the following approximation:

$$P_A^*(\exp i\omega T) - \tilde{P}_A^*(\exp i\omega T) \approx \frac{1}{T} H(i\omega)E_a(i\omega)A(i\omega) \quad \text{for } 0 \leqslant \omega \leqslant \pi/T \tag{58}$$

Similarly, from (10) we obtain

$$d_A^*(\exp i\omega T) \approx \frac{1}{T} A(i\omega)d(i\omega) \quad \text{for } 0 \leqslant \omega \leqslant \pi/T \tag{59}$$

Note that if no prefilter is used, i.e. if $A(s) = 1$, then in order to obtain (59), T must be such that $d(s)$ is small for $\omega > \pi/T$.

Substitution of (15), (58) and (59) into (12) yields for the response of the system $y(s)$ to a disturbance $d(s)$:

$$y(i\omega) \approx \frac{1 - \tilde{P}(i\omega)Q(\exp i\omega T)F(\exp i\omega T)H(i\omega)A(i\omega)/T}{1 + E_a(i\omega)Q(\exp i\omega T)F(\exp i\omega T)H(i\omega)A(i\omega)/T} d(i\omega), \quad \text{for } 0 \leqslant \omega \leqslant \frac{\pi}{T} \tag{60}$$

Note that the transfer function between the error $(y(s) - r(s))$ and the setpoint $r(s)$ is the same as in (60) with A in the numerator substituted by the identity.

A limitation on the performance deterioration caused by model–plant mismatch can now be set by requiring that the magnitude of the function connecting y and d in (60) is bounded by a designer-specified function of ω. This transfer function is similar to the sensitivity function defined between the error $y(s) - r(s)$ and $r(s)$ or $d(s)$ for continuous control systems. For sampled-data systems however one cannot obtain an equation in the form of (60) without making the approximations in (58) and (59), because (12) describes a time-varying relation between $y(s)$ and $d(s)$. Let us use the notation

$$K(s) = Q(\exp sT)F(\exp sT)H(s)A(s)/T \tag{61}$$

For robust performance we require

$$\left| \frac{1 - \tilde{P}(i\omega)K(i\omega)}{1 + E_a(i\omega)K(i\omega)} \right| \leqslant S(\omega) \quad \forall E_a \text{ s.t. } |E_a(i\omega)| \leqslant l_a(\omega) \quad \text{for } 0 \leqslant \omega \leqslant \pi/T \tag{62}$$

where $S(\omega)$ is designer-specified. Note however that $S(\omega)$ cannot be chosen arbitrarily small because even for $E_a = 0$, the left-hand side of (62) may be non-zero. The selection of $S(\omega)$ will be discussed in § 4.3.2.

We shall now proceed to write (62) in a different form without making any conservative steps. The idea behind the following steps is based on the concept of the structured singular value, introduced by Doyle (1982).

We can write (62) $\Leftrightarrow$

$$1 + \frac{1 - \tilde{P}(i\omega)K(i\omega)}{1 + E_a(i\omega)K(i\omega)} \frac{1}{S(\omega)} \Delta(\omega) \neq 0,$$

$$\forall E_a \text{ s.t. } |E_a(i\omega)| \leqslant l_a(\omega) \quad \text{and} \quad \forall \Delta \text{ s.t. } |\Delta(\omega)| < 1 \quad \text{for } 0 \leqslant \omega \leqslant \pi/T$$

$$\Leftrightarrow 1 + K(i\omega)l_a(\omega)\frac{E_a(i\omega)}{l_a(\omega)} + \frac{1 - \tilde{P}(i\omega)K(i\omega)}{S(\omega)} \Delta(\omega) \neq 0$$

$$\forall E_a \text{ s.t. } |E_a(i\omega)/l_a(\omega)| \leqslant 1 \quad \text{and} \quad \forall \Delta \text{ s.t. } |\Delta(\omega)| < 1 \quad \text{for } 0 \leqslant \omega \leqslant \pi/T$$

$$\Leftrightarrow L(\omega) \triangleq |K(i\omega)| l_a(\omega) + \frac{|1 - \tilde{P}(i\omega)K(i\omega)|}{S(\omega)} \leqslant 1 \quad \text{for } 0 \leqslant \omega \leqslant \pi/T \tag{63}$$

Hence the above condition is non-conservative. It may however be somewhat optimistic because of the approximations in (58) and (59). Any such optimism though is related only to performance. Robust stability is guaranteed from (55). Finally, note that (63) is different from (55) in the sense that increasing the sampling time does not lead to satisfaction of (63). An optimization over the filter time-constant in an effort to satisfy (63) is necessary at each T and the result is improved as T is reduced.

4.3. *Filter design*

This section deals with the design of a filter so that the robustness conditions derived in §§ 4.1 and 4.2 are satisfied.

4.3.1. *Filter structure*

In the case where $m \geqslant 2$ a filter described by (56) is not sufficient. However we can write

$$F(z) = (\beta_0 + \beta_1 z^{-1} + \ldots + \beta_w z^{-w}) \frac{(1-\alpha)z}{z-\alpha} \tag{64}$$

where the coefficients $\beta_0, \ldots, \beta_w$ are such that $F(z)$ satisfies (31) and (32), for some specified α. Equation (31) implies that we must have

$$\beta_0 = 1 - (\beta_1 + \ldots + \beta_w) \tag{65}$$

The following theorem allows the computation of $\beta_1, \ldots, \beta_w$.

Theorem 3

Equation (32) is satisfied for an $F(z)$ given by (64) and (65) if and only if the coefficients $\beta_1, \ldots, \beta_w$ ($w \geqslant m-1$) satisfy

$$N_w \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_w \end{bmatrix} = \left.\begin{bmatrix} -\alpha/(1-\alpha) \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right\} (m-1) \tag{66}$$

where the elements of the matrix N_w of dimension $(m-1) \times w$ are defined in (53).

Proof

See Appendix C.

For a choice of $w > m-1$, there are more than one solution to (66) and then one can obtain $\beta_1, \ldots, \beta_w$ as the minimum-norm solution (see Stewart 1973). Then, as $w \to \infty$ the norm of this solution goes to zero and from (64) and (65) it follows that the properties of $F(z)$ are not significantly different from those of $F_1(z)$. Finally, note that for $m = 2$, one should choose a $w \geqslant 2$ in order to avoid the trivial solution $F(z) = 1$.

Then the minimum-norm solution for $m = 2$ and $w \geq 2$, is

$$\beta_k = -\frac{6k\alpha}{(1-\alpha)w(w+1)(2w+1)}, \quad k = 1, \ldots, w \tag{67}$$

4.3.2. *Selection of the weight* $S(\omega)$

The choice of $S(\omega)$ depends on the performance requirements set by the designer. However, one can use as a guide a function $S_0(\omega)$ determined by the model $\tilde{P}(s)$. Let $\zeta_1, \ldots, \zeta_\eta$ be the right half-plane zeros of $\tilde{P}(s)$ and τ_D its time delay. Define

$$\tilde{P}_+(s) = \prod_{j=1}^{\eta} \frac{(-s+\zeta_j)}{(s+\zeta_j)} \exp(-\tau_D s) \tag{68}$$

Then the optimal sensitivity function in terms of minimizing the integral squared error for a step input is $1 - \tilde{P}_+(s)$ (Frank 1974; see also Kwakernaak and Sivan 1972). Hence a reasonable choice is

$$S(\omega) \geq S_0(\omega) = |1 - \tilde{P}_+(i\omega)| \tag{69}$$

The above sensitivity function however is achieved only by a non-proper controller. The properness requirement simply adds to (69) the condition that $S(\infty) \geq 1$, even if there are no right half-plane zeros and time delays. Also note that though $S_0(0) = 0$, there is no need to choose $S(0) = 0$, since $Q(z)$ and $F(z)$ have been designed so that the conditions (22)–(25) are satisfied, which guarantee no steady-state offset under modelling error, provided that stability is maintained.

4.3.3. *Computation of* α

The filter parameter α has to be adjusted in an effort to satisfy (55) and (63). Equation (55) is equivalent to placing a lower bound α^* on α. This can be obtained from a Bode plot of $(|Q(\exp i\omega T)| l_a^*(\omega))^{-1}$. If this quantity is never less than one, then $\alpha^* = 0$. If it attains values less than one, then α^* can be found from the Bode plot of a first-order filter so that (55) is satisfied. For example, if ω_l is the frequency at which the above quantity becomes equal to 0·7, then α should be larger than approximately $\exp(-T\omega_l)$, i.e.

$$\alpha^* = \exp(-T\omega_l) \tag{70}$$

Note that as explained in § 4.3.1, the properties of $F(z)$ in (64) are practically the same as those of $F_1(z)$ in (56) for a sufficiently large w. Subsequently one should obtain

$$\psi(T) = \min_{\alpha^* \leq \alpha < 1} \left(\max_{0 \leq \omega \leq \pi/T} L(\omega) \right) \tag{71}$$

where $L(\omega)$ is defined in (63).

The above minimization can be done by simply computing $L(\omega)$ for a number of values for α. The computational effort is very small. It is advisable however that one write $\alpha = \exp(-T/\tau)$ where τ is in $[\tau^*, \infty)$ with $\alpha^* = \exp(-T/\tau^*)$ and minimize over τ. Then it is, in general, sufficient to only examine τs such that $1/\tau$ is in the frequency range where any significant changes in the value of $S(\omega)$ occur. The optimal τ will be denoted by τ_{opt} and the corresponding α by α_{opt}. Both τ_{opt} and α_{opt} are functions of T.

5. Sampling-time selection

5.1. *Sampling-time bounds*

These are imposed from the following:

(*a*) *Open-loop bandwidth*: Let ω_B be the frequency at which $|\tilde{P}(i\omega_B)/\tilde{P}(0)| = 0{\cdot}7$. Then letting π/T be less than ω_B, clearly makes no engineering sense. For example, it may not be possible to take care of undesirable open-loop response characteristics like overshoot described by a peak at a frequency less than ω_B, since the controller can only guarantee good performance up to π/T. Therefore one should choose

$$\pi/T \geqslant \omega_B \tag{72}$$

(*b*) *Expected disturbances*: Let the frequency content of any expected disturbances be negligible for frequencies larger than ω_d. Then Shannon's sampling theorem (Åström and Wittenmark 1984) implies that if one wants to reconstruct those disturbances then one has to use a π/T at least as large as ω_d. If not then the aliasing problem will appear unless an anti-aliasing analogue prefilter is used.

(*c*) *Prefilter*: Since this is an analogue device, hardware and cost considerations put a limit ω_A on how small the prefilter cut-off frequency can be. Hence one should choose

$$\pi/T \geqslant \min\{\omega_d, \omega_A\} \tag{73}$$

(*d*) *Digital computer*: It is clear that, depending on the particular machine to be used and the total load that it should accommodate, there exists a lower bound T_{comp} on the possible sampling times. Combination with (72) and (73) yields:

$$T_{comp} \leqslant T \leqslant \pi/\max\{\omega_B, \min\{\omega_d, \omega_A\}\} \tag{74}$$

5.2. *Initial choice for* T

The discussion in § 4.1 on (57) which is illustrated in Fig. 3, indicates that a reasonable starting point would be a T_{init} such that

$$T_{init} = \pi/\omega_c \tag{75}$$

where ω_c is the smallest frequency at which $|\tilde{P}(i\omega)|/l_a(\omega) = 0{\cdot}7 (\Leftrightarrow 1/l_m(\omega) = 0{\cdot}7)$, since such a T would tend to satisfy the robust stability condition (57) for $F(z) = 1$. If there is no ω at which this happens, then one can choose as ω_c the corner frequency at which $1/l_m(\omega)$ settles to its value for $\omega \to \infty$. T_{init} of course should be kept within the limits specified by (74).

5.3. *Iteration on the sampling time*

The first step in each iteration is to design $Q(z)$ and $F(z)$ for a given T. Then depending on the value of the quantity $\psi(T)$ defined in (71), either a new value for T is determined or the decision is taken to terminate the iteration. Two cases are possible:

(*a*) $\psi(T) > 1$: Then the performance condition (63) is not satisfied; stability however is guaranteed through satisfaction of (55). A smaller T should be used next. If there has been no trial at a T where $\psi \leqslant 1$, then a $T_{next} = T/10$ (moving π/T to the right by one decade) is a reasonable choice. If the bounds of (74) are violated, T should be set equal to the corresponding bound.

(*b*) $\psi(T) \leqslant 1$: Both the robust stability and performance requirements are satisfied and therefore the design is an acceptable one. However, it may be that the same requirements are also satisfied for a larger sampling time. To find the largest T where the conditions are satisfied, defined as

$$T_0 = \max \{T : \psi(T) \leqslant 1\} \tag{76}$$

one should increase T. If no larger T has been tried, then a reasonable choice is to move π/T by one or one-half decade to the left. If a larger T where $\psi > 1$ is known, then T_{next} can be chosen as the geometric mean of the two values.

Finally if no T can be found within the limits of (74), for which $\psi(T) \leqslant 1$, that means that the performance requirements set by the designer through the choice of $S(\omega)$ are too strict to be satisfied. The only course of action is to choose a different $S(\omega)$ and repeat the procedure. A plot of $L(\omega)$ as a function of ω for $\alpha = \alpha_{opt}$ at the smallest T that was used can help locate the frequency range where $S(\omega)$ was too strict.

6. Illustrations

The controller design for two systems will be presented. The first example will serve as an illustration of the design procedure. In the second, the procedure will be applied on a system that is difficult to control and a high-frequency external input will be considered in order to demonstrate that fast sampling does not necessarily help to achieve good performance.

6.1. *Example* 1

Let

$$\tilde{P}(s) = \frac{3}{(s+1)(s+3)} \tag{77}$$

A delay-type uncertainty is assumed, i.e.

$$P(s) = \tilde{P}(s) \exp(-\tau_D s) \tag{78}$$

where

$$0 \leqslant \tau_D \leqslant 0{\cdot}05 \tag{79}$$

then (16) and (78) $\Rightarrow |E_m(i\omega)| = \sqrt{2}\,(1 - \cos(\omega\tau_D))^{1/2} \Rightarrow$ (from (79))

$$l_m(\omega) = \begin{cases} 2 & \text{for } \omega \geqslant 20\pi \\ \sqrt{2}\,(1 - \cos(0{\cdot}05\omega))^{1/2} & \text{for } 0 \leqslant \omega \leqslant 20\pi \end{cases} \tag{80}$$

and $l_a(\omega)$ can be obtained through (19).

Bounds on T can now be obtained from (74). For the system of (77), $\omega_B = 0{\cdot}92$. The assumption will be made that no high-frequency disturbances are expected ($\omega_d = \omega_B$) and that $T_{comp} \to 0$. Finally, no prefilter will be used ($\omega_A = \infty$). Then (74) yields

$$0 < T \leqslant \pi/\omega_B = 3{\cdot}4 \tag{81}$$

Since $\tilde{P}_+(s) = 1$, the only restriction on $S(\omega)$ is that it is larger than 1 at $\omega = \infty$. Its shape depends on how strict a performance requirement one wishes to set. A Bode

plot of $|\tilde{P}(i\omega)|$ is helpful in this respect. For this design, a choice of

$$S(\omega) = 0{\cdot}4\left[\frac{(\omega^2/2^2+1)}{(\omega^2/10^2+1)}\right]^{1/2} \tag{82}$$

is made based on the observation that at $\omega = 2$, $|\tilde{P}(i\omega)|$ is small enough ($\approx 0{\cdot}35$) to justify a relaxation of the performance requirement. Also $S(\infty) = 2 > 1$. It should be noted that the above choice is a rather strict performance requirement, but it is justified because the system is not inherently difficult to control and the uncertainty is small.

Equation (80) yields a value $\omega_c = 31$ and then from (75) we get $T_{init} = 0{\cdot}101$. An iteration on T according to the outline in § 5.3 yields the values shown in Table 1.

T	$\psi(T)$	$1/\tau_{opt}$	ω_l
0·1013	1·22	7·713	($\alpha^* = 0$)
0·0101	0·90	6·581	31·9
0·0320	0·98	7·124	41·0

Table 1.

At the final choice of $T = 0{\cdot}032$, the controller $Q(z)$ designed according to the rules in § 3.2 is given by

$$Q(z) = \frac{345{\cdot}9(z^2 - 1{\cdot}877z + 0{\cdot}8797)}{z^2} \tag{83}$$

and the filter for $\alpha = \alpha_{opt} = \exp(-T/\tau_{opt})$ is

$$F(z) = \frac{0{\cdot}2041z}{z - 0{\cdot}7959} \tag{84}$$

The controller for the classical feedback structure of Fig. 1 (*a*) can be obtained through (4).

In Fig. 4 (*a*) the response to a step set-point input is given for $P(s) = \tilde{P}(s)$. In Fig. 4 (*b*) a model–plant mismatch is assumed and $P(s) = \tilde{P}(s)\exp(-0{\cdot}05s)$ is used. For comparison, simulations are also given for a continuous controller

$$QF_{cont}(s) = \frac{1}{3}\frac{(s+1)(s+3)}{(0{\cdot}0737s+1)^2} \tag{85}$$

designed with the same performance condition and bound $S(\omega)$. Also the controller obtained by a Tustin approximation at $T = 0{\cdot}032$ of the classical feedback controller corresponding to $QF_{cont}(s)$ is simulated. It is clear that the design obtained with the proposed design procedure performs robustly under modelling error. On the other hand, the discrete approximation of the continuous controller tends not to be robust and a smaller sampling time would be required to improve it. It should, however, be repeated that this is a system which is rather easy to control and which was chosen solely to illustrate the design procedure.

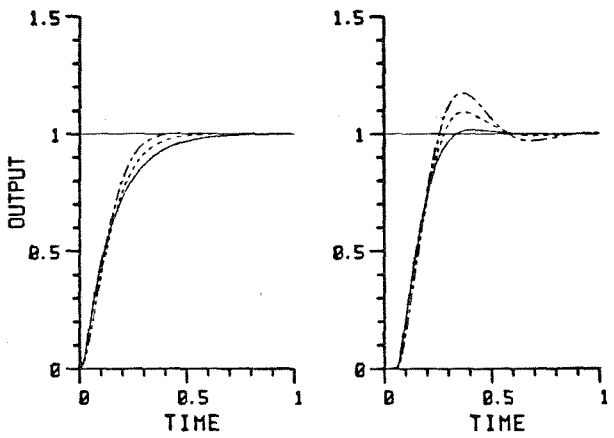


Figure 4. Example 1, step set-point response: (*a*) $P = \tilde{P}$, (*b*) $P \neq \tilde{P}$, $\tau_D = 0{\cdot}05$; ——— $Q(z)F(z)$, $T = 0{\cdot}032$; – – – – $QF_{cont}(s)$; – · – Tustin approximation of $QF_{cont}(s)$ for $T = 0{\cdot}032$.

6.2. *Example* 2

Let

$$P(s) = K\frac{-0{\cdot}5s + 1}{(s+1)(0{\cdot}25s+1)}\exp(-\tau_D s) \tag{86}$$

where

$$0{\cdot}95 \leqslant K \leqslant 1{\cdot}05 \tag{87}$$

$$0{\cdot}35 \leqslant \tau_D \leqslant 0{\cdot}45 \tag{88}$$

and the nominal values $K = 1$, $\tau_D = 0{\cdot}4$ are used for the model $\tilde{P}(s)$.

The $|E_m(i\omega)| = |K\exp[i(0{\cdot}4 - \tau_D)\omega] - 1|$ and after some algebra

$$l_m(s) = \begin{cases} 2{\cdot}05 & \text{for } \omega > 20\pi \\ [2{\cdot}1025 - 2{\cdot}1\cos(0{\cdot}05\omega)]^{1/2} & \text{for } 10\pi \leqslant \omega \leqslant 20\pi \\ [2{\cdot}1025 - 1{\cdot}9\cos(0{\cdot}05\omega)]^{1/2} & \text{for } 0 \leqslant \omega \leqslant 10\pi \end{cases} \tag{89}$$

For this system $\omega_B = 1{\cdot}2$. High-frequency disturbances requiring an $\omega_d = 63$ ($\pi/\omega_d = 0{\cdot}05$) are considered possible and T_{comp} is assumed to be practically zero. Two cases will be distinguished with respect to the use of a prefilter:

Case I. No limitations on the use of a prefilter, i.e. $\omega_A = 0$. Then (74) yields

$$0 < T \leqslant \pi/\omega_B = 2{\cdot}6 \tag{90}$$

Case II. No prefilter can be used. Hence $\omega_A = \infty$ and from (74)

$$0 < T \leqslant \pi/\omega_d = 0{\cdot}05 \tag{91}$$

In Fig. 5 a plot of $S_0(\omega)$ is shown. $S(\omega)$ is chosen as

$$S(\omega) = 0{\cdot}4\left[\frac{(\omega^2/0{\cdot}1^2 + 1)}{(\omega^2+1)}\right]^{1/2} \tag{92}$$

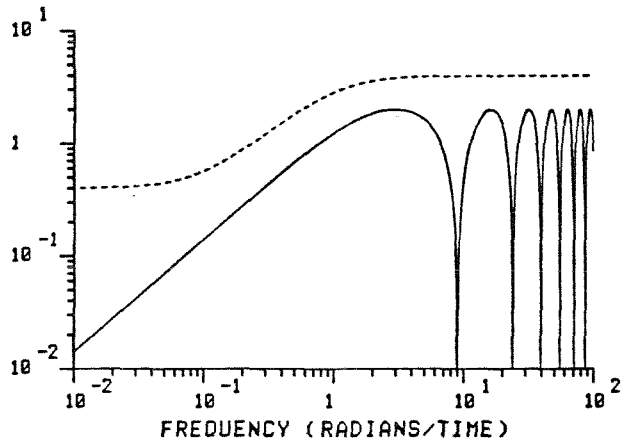


Figure 5. Example 2: ——— $S_0(\omega)$; ---- $S(\omega)$.

so that (69) is satisfied. It is clear that the non-minimum-phase elements in $\tilde{P}(s)$ limit the achievable performance even for no modelling error (Holt and Morari 1985 a, b).

Equation (89) yields $\omega_c = 30$ and then from (75) we obtain $T_{init} = 0{\cdot}105$. This T_{init} is outside the bound in (91) and therefore in Case II a $T_{init} = 0{\cdot}05$ will be used.

Case I: A second-order Butterworth prefilter with a cutoff frequency of $\pi/2T$ is selected for each sampling time. The iteration on T yields the values shown in Table 2.

T	$\psi(T)$	$1/\tau_{opt}$	α^*
0·105	0·963	2·157	0
0·331	1·027	2·603	0
0·186	0·986	2·317	0
0·248	1·000	3·070	0

Table 2.

For the final choice of $T = 0{\cdot}248$, the controller $Q(z)$ and the filter $F(z)$ are given by

$$QF_{\mathrm{I}}(z) = 1{\cdot}458\frac{z^2 - 1{\cdot}150z + 0{\cdot}2889}{z^2 - 1{\cdot}088z + 0{\cdot}2900} \tag{93}$$

The anti-aliasing prefilter is

$$A(s) = \frac{1}{0{\cdot}02499s^2 + 0{\cdot}2236s + 1} \tag{94}$$

Case II: For $T = T_{init} = 0{\cdot}05$, $\psi(T) = 0{\cdot}936 < 1$ and therefore this is the final choice since (91) allows no larger T. We have $1/\tau_{opt} = 2{\cdot}062 < \omega_l = 27$ which yields

$$QF_{\mathrm{II}}(z) = 1{\cdot}0585\frac{z^2 - 1{\cdot}770z + 0{\cdot}7788}{z^2 - 1{\cdot}8065z + 0{\cdot}8159} \tag{95}$$

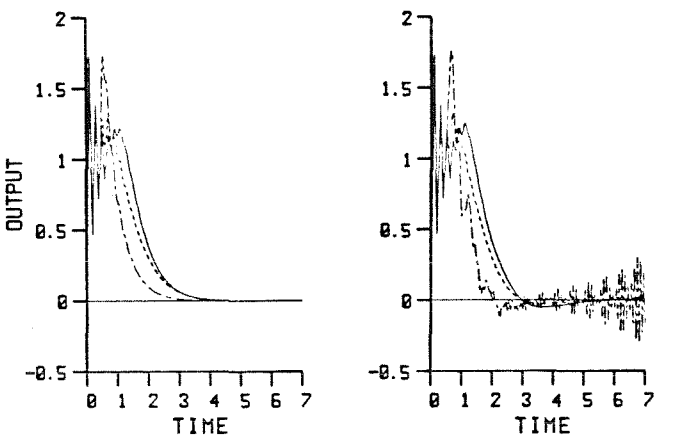
The response of the two control systems to a high-frequency disturbance

$$d(s) = \frac{1}{s} \cdot \frac{1}{(0{\cdot}001003s^2 + 0{\cdot}006334s + 1)} \tag{96}$$

justifying an $\omega_d = 63$, is shown in Fig. 6 (*a*) for $P(s) = \tilde{P}(s)$ ($K = 1$, $\tau_D = 0{\cdot}4$) and in Fig. 6 (*b*) for $K = 1{\cdot}05$ and $\tau_D = 0{\cdot}45$. The two designs are robust to model–plant mismatch and behave quite similarly. This was to be expected because the faster sampling in Case II does not aim at a faster response, but at avoiding the aliasing problem that would appear if no prefilter were used. The speed of response is determined by the robustness requirements. To demonstrate this, we shall proceed to design the controller $Q(z)$ for the particular input $v(s) = d(s)$ for $T = 0{\cdot}05$ and ignore any robustness requirements. Equations (33)–(41) are used to obtain $Q_{SE}(z)$ and then (44) yields

$$Q_d(z) = 6{\cdot}247 \frac{0{\cdot}8000z^4 - 1{\cdot}250z^3 + 1{\cdot}093z^2 - 1{\cdot}222z + 0{\cdot}5947}{z^3(z - 0{\cdot}9045)} \tag{97}$$

One can see in Fig. 6 (*a*) that the response is faster, though not much, since as mentioned earlier the non-minimum-phase elements limit the achievable performance. However, even this small improvement for the nominal case ($P = \tilde{P}$) is paid for by instability in the presence of modelling error as Fig. 6 (*b*) shows. Note that for $Q = Q_d$ and $\alpha = 0$, (55) is not satisfied, therefore indicating potential instability.



(*a*) (*b*)

Figure 6. Example 2, response to $d(s)$: (*a*) $P = \tilde{P}$, (*b*) $P \neq \tilde{P}$, $K = 1{\cdot}05$, $\tau_D = 0{\cdot}45$; ——— $QF_I(z)$; $T = 0{\cdot}248$; – – – – $QF_{II}(z)$, $T = 0{\cdot}05$; – · – $Q_d(z)$, $T = 0{\cdot}05$.

7. Conclusions

Two main goals were accomplished in this paper. The first was the derivation of conditions that guarantee robust stability and performance for sampled-data systems and the development of a controller synthesis method. The conditions that were obtained can easily be checked and the computation effort required for the design is small. Any particular external input (set-point or disturbance) can be considered and the performance requirements are defined through a designer-specified frequency weight for the selection of which guidelines are given.

The second goal was the illustration of the effect of the sampling time on the achievable performance and the robustness properties of the control system. These relations were quantified in an iterative procedure for robust controller design and sampling-time selection. This design procedure has the advantage that it can easily be programmed on the computer in an interactive form.

Appendix A. Proof of Theorem 1

The disturbance $d(s)$ is fed through $A(s)$ before it is sampled and therefore for no offset it is clear that we need

$$\lim_{\text{time}\to\infty} \mathscr{L}^{-1}\{d(s) - A(s)d(s)\} = 0 \tag{A 1}$$

Application of the final-value theorem on (A 1) yields

$$\lim_{s\to 0} (s[1 - A(s)]d(s)) = 0 \tag{A 2}$$

Since (A 2) must be satisfied for all $d(s)$ with m or less poles at $s = 0$, we need $[1 - A(s)]$ to have m zeros at $s = 0$, which will be the case if and only if (24) and (25) hold.

Equation (A 1) implies that $\lim_{\text{time}\to\infty} \mathscr{Z}^{-1}\{d^*(z) - d_A^*(z)\} = 0$ or $\lim_{z\to 1}((1 - z^{-1}) \times [d^*(z) - d_A^*(z)]) = 0$. Hence for offset considerations, $d_A^*(z)$ can be substituted for $d^*(z)$ in (13). Consider an external input $v(s)$ and let

$$v^*(z) = \mathscr{Z}\mathscr{L}^{-1}\{v(s)\} \tag{A 3}$$

Then for both cases: (i) $v^*(z) = -r^*(z)$, $d_A^*(z) = 0$; (ii) $v^*(z) = d_A^*(z)$, $r^*(z) = 0$, (13) yields after substitution of d_A^* for d^*:

$$y^*(z) - r^*(z) = \frac{1 - F(z)Q(z)\tilde{P}_A^*(z) + F(z)Q(z)[P_A^*(z) - P^*(z)]}{1 + F(z)Q(z)[P_A^*(z) - \tilde{P}_A^*(z)]}\, v^*(z) \tag{A 4}$$

Assume $P = \tilde{P}$; then (A 4) becomes

$$y^*(z) - r^*(z) = [1 - F(z)Q(z)\tilde{P}^*(z)]v^*(z) \tag{A 5}$$

The final-value theorem implies that for no offset we need

$$\lim_{z\to 1}((1 - z^{-1})[1 - F(z)Q(z)\tilde{P}^*(z)]v^*(z)) = 0 \tag{A 6}$$

If $v(s)$ has l poles at $s = 0$, then from (A 3) it follows that $v^*(z)$ has l poles at $z = 1$. Hence (A 6) will be satisfied for all $l \leqslant m$ if and only if $(1 - F(z)Q(z)\tilde{P}^*(z))$ has m zeros at $z = 1$, i.e. if and only if (22) and (23) hold. Note that (A 1) means that the steady-state value of a signal in the class of inputs considered, going through $[1 - A(s)]$ is zero. This will remain zero even after passing through some other stable systems, say P or $\tilde{P}$. Hence

$$\lim_{z\to 1}((1 - z^{-1})[P_A^*(z) - P^*(z)]v^*(z)) = \lim_{z\to 1}((1 - z^{-1})[\tilde{P}_A^*(z) - \tilde{P}^*(z)]v^*(z)) = 0$$

and therefore

$$\lim_{z\to 1}((1 - z^{-1})[1 - F(z)Q(z)\tilde{P}_A^*(z) + F(z)Q(z)[P_A^*(z) - P^*(z)]]v^*(z))$$

$$= \lim_{z\to 1}((1 - z^{-1})[1 - F(z)Q(z)\tilde{P}^*(z)]v^*(z)) = 0 \tag{A 7}$$

Then from (A 4) and (A 7) it follows that the offset is zero even when $P \neq \tilde{P}$. Hence conditions (22), (23), (24) and (25) are all that is needed.

Appendix B. Proof of (41)

Sampling of (28) yields

$$y^*(z) = \tilde{P}^*(z)Q(z)r^*(z) + [1 - \tilde{P}^*(z)Q(z)]d^*(z) \tag{B 1}$$

Note that to obtain (B 1) from (28) we have to assume $A(s) = 1$. If this is not the case, then we should use $d_A^*(z)$ instead of $d^*(z)$ in (B 1), which yields

$$y^*(z) = \tilde{P}^*(z)Q(z)r^*(z) + [1 - \tilde{P}^*(z)Q(z)]d_A^*(z) \tag{B 2}$$

This equation cannot be obtained from (28) by sampling. However it can be used for control purposes because using an $A(s) \neq 1$ means that we opted for the rejection of $A(s)d(s)$ instead of $d(s)$.

$Q_{SE}(z)$ is the stable proper rational function which minimizes Φ:

$$\Phi = \sum_{k=0}^{\infty} (\mathscr{Z}^{-1}\{r^*(z) - y^*(z)\})^2 \tag{B 3}$$

Then by substituting (B 1) or (B 2) into (B 3) we obtain

$$\Phi = \sum_{k=0}^{\infty} (\mathscr{Z}^{-1}\{[1 - \tilde{P}^*(z)Q(z)]v^*(z)\})^2 \tag{B 4}$$

whether v is a set-point ($v = r$, $d = 0$) or a disturbance ($r = 0$, $v = d$ *or* $v = Ad$).

We now have to perform the following to assure that we shall obtain a proper Q. From (33) we see that because of the time delay, Q does not affect the first $N + 1$ terms in (B 4). If $v^*(z)$ also contains a delay N_v, i.e. $z^{N_v}v^*(z)$ is semi-proper, then this introduces an additional number of N_v terms in (B 4) which are not affected by Q. By using (35) we obtain

$$\begin{aligned}[1 - \tilde{P}^*(z)Q(z)]v^*(z) &= z^{-N_v}[z^{N_v}v^*(z) - \tilde{P}^*(z)Q(z)z^{N_v}v^*(z)] \\ &= z^{-N_v}(v_0 + v_1z^{-1} + \ldots + v_Nz^{-N}) + z^{-N_v}z^{-N-1} \\ &\quad \times [v_{N+1} + v_{N+2}z^{-1} + \ldots - z^{N+1}\tilde{P}^*(z)Q(z)z^{N_v}v^*(z)]\end{aligned} \tag{B 5}$$

It is clear that in (B 5) the first term of the right-hand side involves only the first $N_v + N + 1$ terms in (B 4) while the second, which contains Q, involves only the remainder of the terms. Hence minimizing Φ is equivalent to minimizing $\hat{\Phi}$:

$$\begin{aligned}\hat{\Phi} &= \sum_{k=N_v+N+1}^{\infty} (\mathscr{Z}^{-1}\{z^{-N_v-N-1}[v_N^*(z) - z^{N+1}\tilde{P}^*(z)Q(z)z^{N_v}v^*(z)]\})^2 \\ &= \sum_{k=0}^{\infty} (\mathscr{Z}^{-1}\{v_N^*(z) - z^{N_v+N+1}\tilde{P}^*(z)Q(z)v^*(z)\})^2\end{aligned} \tag{B 6}$$

where $v_N^*(z)$ is defined in (36).

By applying Parseval's theorem on the right-hand side of (B 6) we obtain

$$\hat{\Phi} = \frac{1}{2\pi}\int_{-\pi}^{\pi} |v_N^*(\exp i\theta) - \exp[i(N_v + N + 1)\theta]\tilde{P}^*(\exp i\theta)Q(\exp i\theta)v^*(\exp i\theta)|^2 \, d\theta \tag{B 7}$$

For the $\tilde{P}_+^*(z)$ and $v_+^*(z)$ defined by (37) and (38) we can easily check that

$$|\tilde{P}_+^*(\exp i\theta)| = 1 \quad \text{for } -\pi \leqslant \theta \leqslant \pi \tag{B 8}$$

$$|v_+^*(\exp i\theta)| = 1 \quad \text{for } -\pi \leqslant \theta \leqslant \pi \tag{B 9}$$

since any complex zeros of $\tilde{P}^*(z)$ and $v^*(z)$ come in complex conjugate pairs.

Use of (39), (40), (B 8) and (B 9) in (B 7) yields

$$\hat{\Phi} = \frac{1}{2\pi}\int_{-\pi}^{\pi} |[\tilde{P}^*_+(\exp i\theta)v^*_+(\exp i\theta)]^{-1} v^*_N(\exp i\theta)\exp(-i\theta) - \tilde{P}^*_-(\exp i\theta)v^*_-(\exp i\theta)Q(\exp i\theta)|^2\, d\theta \tag{B 10}$$

Define

$$f_1(z) = (\tilde{P}^*_+(z)v^*_+(z))^{-1} v^*_N(z)z^{-1} \tag{B 11}$$

$$f_2(z) = \tilde{P}^*_-(z)v^*_-(z) \tag{B 12}$$

Then $f_2(z)Q(z)$ is strictly proper and stable (poles strictly inside the unit circle) except possible for some poles of $f_2(z)$ at $z = 1$. $f_1(z)$ is strictly proper but not stable. Write

$$f_1(z) = \{f_1(z)\}_+ + \{f_1(z)\}_- \tag{B 13}$$

where $\{f_1\}_+$ contains only the unstable poles (strictly outside the unit circle), and $\{f_1\}_-$ only the stable poles. Any poles at $z = 1$ are included in $\{f_1\}_-$. The reason is that we shall assume at this point that the optimal $Q(z)$ is such that these poles are cancelled in both $f_1 - f_2Q$ and $\{f_1\}_- - f_2Q$. It should, however, be verified that the optimal Q has this property. We can obtain $\{f_1\}_+$ and $\{f_1\}_-$ from f_1, by partial-fraction expansion. The constant term is zero since $f_1(z)$ is strictly proper.

Let $L_2(-\pi, \pi)$ be the space of functions $f(\exp i\theta)$ which are square-integrable with respect to θ, i.e. for which

$$\int_{-\pi}^{\pi} |f(\exp i\theta)|^2\, d\theta < \infty \tag{B 14}$$

The inner product in this space is defined by

$$\langle f_a, f_b\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{f_a(\exp i\theta)} f_b(\exp i\theta)\, d\theta \tag{B 15}$$

where the overbar indicates complex conjugate. Then by using (B 10) to (B 15) we obtain

$$\begin{aligned}\hat{\Phi} &= \langle \{f_1\}_+ + \{f_1\}_- - f_2Q, \{f_1\}_+ + \{f_1\}_- - f_2Q\rangle \\ &= \langle \{f_1\}_+, \{f_1\}_+\rangle + \langle \{f_1\}_- - f_2Q, \{f_1\}_- - f_2Q\rangle \\ &\quad + \langle \{f_1\}_+, \{f_1\}_- - f_2Q\rangle + \langle \{f_1\}_- - f_2Q, \{f_1\}_+\rangle\end{aligned} \tag{B 16}$$

Note that the first of the four terms in the right-hand side of (B 16) is independent of Q. As for the last two, they are both zero because they represent the inner products between a strictly proper stable and a strictly proper but totally unstable function and these two subspaces of $L_2(-\pi, \pi)$ are orthogonal (Francis and Zames 1983).

Hence our problem reduces to minimizing $\langle \{f_1\}_- - f_2Q, \{f_1\}_- - f_2Q\rangle$. The obvious solution to this is

$$Q_{SE}(z) = f_2^{-1}(z)\,\{f_1(z)\}_- \tag{B 17}$$

Note that the above $Q_{SE}(z)$ is stable and proper. Also we have $\{f_1\}_- - f_2Q_{SE} = 0$ and $f_1 - f_2Q_{SE} = \{f_1\}_+$, which has no poles at $z = 1$ and therefore the assumption that Q is such that the poles at $z = 1$ cancel out in both $f_1 - f_2Q$ and $\{f_1\}_- - f_2Q$ holds.

Hence the above $Q_{SE}(z)$ is acceptable and therefore it is the solution we were seeking.

In the case of set-point-following where future values of the set-point are supplied to the controller, which we want our system output to follow after N_p time steps, the objective function Φ in (B 3) should be written

$$\Phi = \sum_{k=0}^{\infty} (\mathscr{Z}^{-1}\{z^{-N_p} r^*(z) - y^*(z)\})^2 \tag{B 18}$$

By following the same steps used to find $Q_{SE}(z)$, we can easily see that (B 17) is obtained, but with $v_N^*(z)$ substituted with $v_{N_0}^*(z)$ in (B 11), where $v_{N_0}^*(z)$ is defined by (42).

Appendix C. Proof of Theorem 3

The following lemma will be used.

Lemma

Let $h(\lambda) = (1-\alpha)/(1-\alpha\lambda)$. Then

$$h^{(k)}(\lambda) = (1-\alpha)k!\alpha^k(1-\alpha\lambda)^{-(k+1)} \tag{C 1}$$

where the superscript (k) denotes the kth derivative.

Proof (by induction)

$k = 1$ $\quad \dfrac{d}{d\lambda} h(\lambda) = (1-\alpha)\alpha(1-\alpha\lambda)^{-2}$

$k = n$ $\quad$ Let $h^{(n)}(\lambda) = (1-\alpha)n!\alpha^n(1-\alpha\lambda)^{-(n+1)}$ (C 2)

$k = n+1$ From (C 2) we get

$$h^{(n+1)}(\lambda) = (1-\alpha)n!\alpha^n \frac{d}{d\lambda}(1-\alpha\lambda)^{-(n+1)} = (1-\alpha)(n+1)!\alpha^{n+1}(1-\alpha\lambda)^{-(n+2)} \qquad \square$$

Proof of Theorem 3

Equation (32) is equivalent to

$$\frac{d^k}{d\lambda^k} F(\lambda^{-1})\bigg|_{\lambda=1} = 0, \quad k = 1, \ldots, m-1 \tag{C 3}$$

From (57) we get

$$F(\lambda^{-1}) = \Gamma(\lambda)h(\lambda) \tag{C 4}$$

where

$$\Gamma(\lambda) = \beta_0 + \beta_1\lambda + \ldots + \beta_w\lambda^w \tag{C 5}$$

From the Lemma we get

$$h^{(k)}(1) = k!\alpha^k(1-\alpha)^{-k} \tag{C 6}$$

then (C 3) for $k = 1$ yields

$$\Gamma^{(1)}(1)h(1) + \Gamma(1)h^{(1)}(1) = 0$$

or

$$\Gamma^{(1)}(1) = -h^{(1)}(1) = -\alpha/(1-\alpha) \tag{C 7}$$

We shall now show that (C 3) for $k = 2, \ldots, m-1$ yields $\Gamma^{(k)}(1) = 0$ for $k = 2, \ldots, m-1$. The proof will be by induction.

$k = 2$ Equation (C 3) for $k = 2$ yields

$$\Gamma^{(2)}(1)h(1) + 2\Gamma^{(1)}(1)h^{(1)}(1) + \Gamma(1)h^{(2)}(1) = 0$$

or by using (C 6) and (C 7)

$$\Gamma^{(2)}(1) = 0$$

$k \leqslant n$ for $2 \leqslant n < m-1$

Let $\Gamma^{(k)}(1) = 0$ (C 8)

$k = n+1$ Equation (C 3) for $k = n+1$ yields, because of (C 8)

$$\Gamma^{(n+1)}(1)h(1) + (n+1)\Gamma^{(1)}(1)h^{(n)}(1) + \Gamma(1)h^{(n+1)}(1) = 0$$

or by using (C 6) and (C 7)

$$\Gamma^{(n+1)}(1) = 0$$

Hence by induction

$$\Gamma^{(k)}(1) = 0, \quad k = 2, \ldots, m-1 \tag{C 9}$$

But one can easily see that

$$\begin{bmatrix} \Gamma^{(1)}(1) \\ \vdots \\ \Gamma^{(m-1)}(1) \end{bmatrix} = N_w \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_w \end{bmatrix} \tag{C 10}$$

and therefore Theorem 3 follows from (C 7), (C 9) and (C 10). □

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Addendum to Chapter III.

The following equation is entirely equivalent to (41), but rather simpler to use. It is obtained as a special case of the general MIMO controller, described by Theorem 2.2.2 of Chapter VI for open-loop stable or unstable systems.

$$Q_{SE} = (\tilde{P}^*_-(z)v^*_-(z))^{-1}\{(\tilde{P}^*_+(z))^{-1}z^N v^*_-(z)\}_* \qquad (41')$$

where the notation $\{.\}_*$ indicates that after a partial fraction expansion is taken, only the strictly proper terms corresponding to the poles of $v^*_-(z)$ are retained.

CHAPTER IV

DIGITAL CONTROLLER DESIGN FOR MULTIVARIABLE SYSTEMS WITH STRUCTURAL CLOSED–LOOP PERFORMANCE SPECIFICATIONS

DIGITAL CONTROLLER DESIGN FOR MULTIVARIABLE SYSTEMS WITH STRUCTURAL CLOSED-LOOP PERFORMANCE SPECIFICATIONS

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Abstract

The problem of the direct design of the closed-loop transfer function matrix is addressed for multivariable discrete systems. The limitations imposed by unstable zeros, time delays and the structure associated with these are quantified. A design procedure is formulated that provides the designer with quantitative measures for evaluating the tradeoffs between different closed-loop interaction structures and durations. The problem of intersample rippling is also considered. The procedure requires only linear algebra operations, includes the eventual construction of the feedback controller in state space and is presented in a way that allows its straightforward computer implementation.

1. Introduction

One can find in the control literature numerous different types of criteria for synthesizing or evaluating a control system. In most cases a number of performance considerations is lumped together into some objective function, which is then optimized with respect to the control system. Such approaches have been proven satisfactory in many cases. However there are situations in which one cannot simply optimize a single scalar objective function. In process control, such a case is that of setpoint tracking for multivariable systems. Quite often it is necessary to look at the closed-loop transfer function matrix relating the setpoints to the process outputs and require that certain elements of the matrix are equal to zero, so that setpoint changes in some outputs do not upset other important ones. Also, one may sometimes wish to allow such closed-loop interactions in order to improve setpoint tracking for the important outputs at the expense of upsetting less valuable ones. The same arguments carry over to certain cases of disturbance rejection. The paper treats setpoint tracking and disturbance rejection in a uniform way.

2. Achievable Input/Output Mappings

The discretized plant is described by the transfer matrix P(z), which is obtained by adding a zero order hold in front of the continuous plant and then taking the z–transform $P(z)$ is assumed to be square.

Let H_{oi}(z) denote the transfer matrix between output o and input i. We can define the following relations with respect to Fig. 1.

$$H_{ur} = C(I + PC)^{-1} \tag{2.0.1}$$

$$H_{ud} = -H_{ur} \tag{2.0.2}$$

$$H_{yr} = PC(I + PC)^{-1} = PH_{ur} \tag{2.0.3}$$

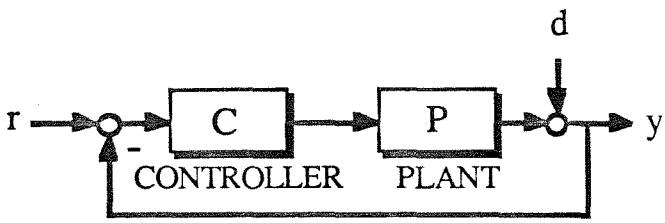


Figure 1 . Feedback control structure.

$$H_{yd} = (I + PC)^{-1} = I - H_{yr} = I - PH_{ur} \tag{2.0.4}$$

From (2.0.4) it follows that if the control system provides good setpoint tracking ($H_{yr}r \approx r$) then one has also good disturbance rejection ($H_{yd}d \approx 0$), provided that the disturbance d is of a type similar to the setpoint r. If this is not the case, then one has to design a Two– Degree–of–Freedom controller (Vidyasagar, 1985), whose design can actually be separated into designing two different controllers C, one for setpoint tracking and one for disturbance rejection and then appropriately combine them in one unified block structure (see, e.g., Morari et al., 1987). Hence, it is sufficient to cover here only the design of C (Fig. 1) for good setpoint tracking or disturbance rejection.

From (2.0.1) we can obtain

$$C = H_{ur}(I - PH_{ur})^{-1} \tag{2.0.5}$$

and so designing C is equivalent to designing H_{ur}, which is the IMC controller (Garcia and Morari, 1982) or the parameter of the Q -Parametrization (Zames, 1981). It can be shown (e.g., Callier and Desoer, 1982) that necessary and sufficient conditions for the internal stability of the system in Fig. 1 are

Condition C1:

i) H_{ur} stable

ii) PH_{ur} stable

iii) $H_{ur}P$ stable

iv) $(I - PH_{ur})P$ stable

C1.ii,iii,iv are implied by C1.i if P is stable. Hence the following assumption, which will be made throughout this paper allows to consider only C1.i:

Assumption A1: P is stable.

It should be pointed out however, that for setpoint tracking, the above assumption need not be made. In that case, the use of the Two–Degree–of–Freedom structure

makes it sufficient to consider C1.i only, even when P is unstable. The problem is then reduced to the one discussed in this paper in which A1 holds (Vidyasagar, 1985; Morari et al. 1987).

The controller $C(z)$ has to be causal since future measurements of the plant output are not known. It follows from (2.0.5) that an equivalent condition is

Condition C2: H_{ur} causal

One can see from the above discussion that the control objective can be reduced to finding an $H_{yr}(z)$ with the desired structure and properties, which can be produced through (2.0.3) by an $H_{ur}(z)$ that satisfies C1.i and C2. However looking only at $H_{yr}(z)$ for checking the performance of the control system may be insufficient because of the phenomenon of intersample rippling. This phenomenon is present when $H_{ur}(z)$ has poles near (-1,0) which are cancelled by zeros of $P(z)$ in (2.0.3). Hence, in order to make it sufficient to judge performance by looking at $H_{yr}(z)$ only, $H_{ur}(z)$ must also satisfy the following condition.

Condition C3: H_{ur} cancels no zeros of P that are "near" (-1,0).

One can use a number of different regions on the z–plane to define "near" (-1,0) (Astrom and Wittenmark, 1984). A simple and satisfactory in practice way to do that, is to include all zeros with negative real part (Zafiriou and Morari, 1985).

3. Characterization of All Permissible $H_{yr}(z)$

From (2.0.3) it follows

$$H_{ur} = P^{-1}H_{yr} \tag{3.0.1}$$

Hence the conditions of section 2 on H_{ur} can be translated into the following condition on H_{yr}:

Condition C4: H_{yr} is a stable, causal transfer matrix that makes $P^{-1}H_{yr}$ causal and cancels the poles of P^{-1} (zeros of P) that are outside the unit circle or near (-1,0).

The time delays in $P(z)$, which make P^{-1} non–causal, appear as zeros at infinity. We shall now exploit this fact to make the treatment of time delays and undesirable zeros of P uniform. The transformation $\lambda = z^{-1}$ will be used. Define

$$\hat{P}(\lambda) \stackrel{\text{def}}{=} P(\lambda^{-1}) \leftrightarrow P(z) \tag{3.0.2}$$

$$\hat{H}_{yr}(\lambda) \stackrel{\text{def}}{=} H_{yr}(\lambda^{-1}) \leftrightarrow H_{yr}(z) \tag{3.0.3}$$

Let $a_1, \ldots, a_f$ be the zeros of P(z), which according to C4 we do not wish to appear as poles of $P(z)^{-1}H_{yr}(z)$. These will appear in $\hat{P}(\lambda)^{-1}$ as poles at $b_1, \ldots, b_f$ where

$$b_i = 1/a_i, \qquad i = 1, \ldots, f \tag{3.0.4}$$

The time delays in $P(z)$ will give rise to zeros at 0 in $\hat{P}(\lambda)$ and consequently the non–causal terms in $P(z)^{-1}$ will produce poles at 0 in $\hat{P}(\lambda)^{-1}$. Hence C4 is equivalent to:

Condition C5:

i) $H_{yr}(z)$ is a stable, causal transfer matrix

ii) $\hat{P}(\lambda)^{-1}\hat{H}_{yr}(\lambda)$ has no poles at $b_0, b_1, \ldots, b_f$.

In the above the following notation was used:

$$b_0 = 0 \tag{3.0.5}$$

Some additional notation and definitions are now needed. $P(z)$ (and $\hat{P}(\lambda)$) is assumed to have dimension $r \times r$ and to be of normal rank r. In the following it will be assumed that $\hat{P}(\lambda)$ has no poles at $b_0, \ldots, b_f$. This is certainly the case for b_0 since all elements of $P(z)$ are proper, but in general P(z) may have poles at $\alpha_1, \ldots, \alpha_f$ resulting in poles at $b_1, \ldots, b_f$ in $\hat{P}(\lambda)$. The existence of poles and zeros at the same location is a clearly multivariable characteristic (Kailath, 1980). The assumption that this is not the case for $P(z)$ serves in considerably simplifying

the notation and it is not restrictive since such a phenomenon is caused by exact cancellations in $det[P(z)]$ which will not happen if a slight perturbation in the terms of $P(z)$ is introduced. Let $\{n_0, n_1, \ldots, n_f\}$ be a set of integers greater or equal to zero, such that

$$\hat{P}^{(k)}(b_i) = 0, \quad ((k = 0, \ldots, n_i - 1), i = 0, \ldots, f) \tag{3.0.6}$$

$$rank[\hat{P}^{(n_i)}(b_i)] \neq 0, \qquad i = 0, \ldots, f \tag{3.0.7}$$

where $\hat{P}^{(k)}(\lambda)$ is the kth derivative of $\hat{P}(\lambda)$. Also let $m_i, i = 0, \ldots, f$, be the order of the zero b_i of $\hat{P}(\lambda)$ as this order is defined from the Smith–McMillan form of $\hat{P}(\lambda)$ (Desoer and Schulman, 1974). The computation of m_i without going through the Smith–McMillan form is briefly discussed in Section 4.2. From (3.0.6), (3.0.7) and the definition of the order of a zero, it follows that

$$m_i \geq n_i, \qquad i = 0, \ldots, f \tag{3.0.8}$$

The following theorem quantifies C5.ii:

Theorem 1.

Condition C5.ii holds if and only if both (a) and (b) hold:

a) $\hat{H}_{yr}(\lambda) = (\lambda - b_i)^{n_i} \hat{H}_i(\lambda), \qquad i = 0, \ldots, f$

where $\hat{H}_i(\lambda)$ is a rational $r \times r$ matrix in λ, with no poles at b_i.

b) for any $i = 0, \ldots, f$ such that $m_i > n_i$, the columns of

$$[\hat{H}_i^{(0)}(b_i)^T \ldots \frac{1}{(m_i - n_i - 1)!} \hat{H}_i^{(m_i - n_i - 1)}(b_i)^T]^T$$

are in the column space of

$$M_i \stackrel{\text{def}}{=} \begin{bmatrix} \frac{1}{n_i!}\hat{P}^{(n_i)}(b_i) & 0 & \ldots & 0 \\ \frac{1}{(n_i+1)!}\hat{P}^{(n_i+1)}(b_i) & \frac{1}{n_i!}\hat{P}^{(n_i)}(b_i) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{(m_i-1)!}\hat{P}^{(m_i-1)}(b_i) & \frac{1}{(m_i-2)!}\hat{P}^{(m_i-2)}(b_i) & \ldots & \frac{1}{n_i!}\hat{P}^{(n_i)}(b_i) \end{bmatrix} \tag{3.0.9}$$

where the superscript (k) indicates k^{th} derivative and T the transpose of a matrix.

Proof: See Appendix A.

The value of Theorem 1 lies in the fact that it provides a characterization of all acceptable $\hat{H}_{yr}(\lambda)$ without requiring the inversion of $\hat{P}(\lambda)$. The theorem applies to the general case. However in practice one is usually faced with a situation where the order of the zeros $a_1,\ldots,a_f$ of the model $P(z)$ is equal to 1. Hence of the zeros $b_0,b_1,\ldots,b_f$ of $\hat{P}(\lambda)$ only b_0 has an order larger than 1. The fact that b_0 is equal to zero (Eq. (3.0.5)) can then be used to obtain a simpler form of Theorem 1. The following two Corollaries describe these situations:

Corollary 1.

Let the order of the zero a_i of $P(z)$ be equal to one. Then $\hat{P}(\lambda)^{-1}\hat{H}_{yr}(\lambda)$ has no poles at b_i if and only if the columns of $\hat{H}_{yr}(b_i)$ are in the column space of $\hat{P}(b_i)(=P(a_i))$.

Proof: It follows directly from Theorem 1 for $m_\ell = 1$.

Corollary 2.

Let $P(z)$ have the impulse response coefficient description

$$P(z) = z^{-N}(A_0 + A_1 z^{-1} + A_2 z^{-2} + \ldots) \tag{3.0.10}$$

where

$$rank[A_0] \neq 0 \tag{3.0.11}$$

$$N \geq 0 \tag{3.0.12}$$

Then

$$n_0 = N \tag{3.0.13}$$

$$M_0 = \begin{bmatrix} A_0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ A_{m_0-1-N} & \ldots & A_0 \end{bmatrix} \tag{3.0.14}$$

and $\hat{P}(\lambda)^{-1}\hat{H}_{yr}(\lambda)$ has no poles at $b_0 = 0$ if and only if both (a) and (b) hold:

a) $\hat{H}_{yr}(\lambda) = \lambda^N \hat{H}_0(\lambda)$ where $\hat{H}_0(\lambda)$ is a rational matrix in λ with no poles at $b_0 = 0$.

b) if $m_0 > N$, the columns of $[\hat{H}_0^{(0)}(0)^T \ldots \frac{1}{(m_0-N-1)!}\hat{H}_0^{(m_0-N-1)}(0)^T]^T$ are in the column space of M_0.

Proof: From (3.0.2) we get $\hat{P}(\lambda) = P(\lambda^{-1}) = \lambda^N(A_0 + A_1\lambda + A_2\lambda^2 + \ldots)$. Equations (3.0.13) and (3.0.14) can now be obtained by repeated differentiation and evaluation at $\lambda = 0$. The rest follows as a restatement of Theorem 1 for this special case.

4. Construction of $H_{yr}(z)$.

4.1 The Form of $H_{yr}(z)$.

Theorem 1 and its Corollaries quantify the restrictions that are imposed on H_{yr} from the zeros and time delays. The designer can select any overall transfer function H_{yr} he considers appropriate for the particular system, provided that it satisfies those restrictions. The choice can be made between decoupled and non–decoupled response and the location of the non–zero elements of $H_{yr}(z)$ can be directly specified. A detailed procedure for doing so will be developed in this section and quantitative criteria for the evaluation of different designs will be obtained. Before proceeding, the form of the non–zero elements of $H_{yr}(z)$ should be discussed. The possibilities are of course infinite but three simple rules will be stated and the reasoning behind them briefly explained.

Rule 1. For a given set of locations for the non–diagonal elements of H_{yr} which are allowed to be non–zero, the design should be such that in each diagonal element of $\hat{H}_{yr}(\lambda)$, every term $(\lambda - b_i)^{\kappa_i}, i = 0, \ldots, f$, has the smallest possible power κ_i.

Rule 2. If in a diagonal element of $\hat{H}_{yr}(\lambda)$, a factor $(\lambda - b_i)^{\kappa_i}$ has to appear, then one should use the factor $\frac{(1-b_i^{-1})^{\kappa_i}(\lambda-b_i)^{\kappa_i}}{(1-b_i)^{\kappa_i}(\lambda-b_i^{-1})^{\kappa_i}}$ if b_i has positive real part and

the term $\frac{(\lambda-b_i)^{\kappa_i}}{(1-b_i)^{\kappa_i}}$ otherwise.

Rule 3. The non–zero, non–diagonal elements of $\hat{H}_{yr}(\lambda)$ should have the form $\lambda^{\delta}(\beta_0+\beta_1\lambda+\ldots+\beta_\nu\lambda^{\nu})(1-\lambda)$.

The reasoning behind Rule 1 is that one wishes the effect of the undesirable zeros and time delays on the response of an element of the output vector to the corresponding setpoint or disturbance, to be as small as possible.

The problem that Rule 2 addresses is exactly the same as the one for the SISO case. Rule 2 is a rule obtained by Zafiriou and Morari (1985). Briefly, it introduces the pole at the inverse of the zero in order to minimize the sum of the squared errors to an external step input. In the case where the zero has negative real part this action would introduce intersample rippling, which is avoided by making a deadbeat type selection. At the same time no significant overshoot or undershoot appears. It is also possible to do the design for external inputs other than steps. This would result in a different expression for the factor in Rule 2 (Zafiriou and Morari, 1986a), which however can be used without any changes in the procedure that will be developed in the following sections.

Rule 3 makes sure that the steady–state gain of the non–diagonal elements of $H_{yr}(z)$ is zero, by including the term $(1-\lambda) \leftrightarrow (1-z^{-1})$. Also the parameters $\beta_0,\ldots,\beta\nu$ have physical meaning because $z^{-\delta}(\beta_0+\beta_1 z^{-1}+\ldots+\beta_\nu^{-\nu})$ is the step response for the corresponding pair of system output and external input. Hence one wishes to have ν small and at the same time the magnitudes of $\beta_0,\ldots,\beta_\nu$ to be small. The trade–off between these two goals will be discussed in section 4.3.iii.

The above three rules are not really restrictive and they will simplify the design procedure. It should be noted that as a result of those rules, the closed–loop steady–state gain will be

$$H_{yr}(1) = I \qquad (4.1.1)$$

Hence the control system will be such that no steady–state offset is produced for external inputs r (or d) with one pole at z = 1 (step–like inputs). For inputs with more poles at z = 1 (ramp–like, etc.), the no–offset property holds when the appropriate factor is used in Rule 2 (Zafiriou and Morari, 1986).

4.2. Zeros of P(z)

The first step towards the construction of H_{yr} is clearly the computation of the zeros of $P(z)$ and of their respective orders as well as the computation of the order m_0 of the zero $b_0 = 0$ of $\hat{P}(\lambda)$. For a square system $P(z)$ the zeros can be computed as the roots of $det[P(z)] = 0$, provided that there are no cancellations with any poles. Numerically better techniques for the computation of the zeros can be found in the literature (Laub and Moore, 1978) and a number of software packages for this computation exist.

The following theorem provides a method for computing the order of a zero without having to find the Smith–McMillan form.

Theorem 2. (Van Dooren et al., 1979; rephrased)

Let

$$M_{i,k} \stackrel{\text{def}}{=} \begin{bmatrix} \frac{1}{n_i!}\hat{P}^{(n_i)}(b_i) & \dots & 0 \\ \vdots & \ddots & \vdots \\ \frac{1}{k!}\hat{P}^{(k)}(b_i) & \dots & \frac{1}{n_i!}\hat{P}^{(n_i)}(b_i) \end{bmatrix} \tag{4.2.1}$$

then

$$m_i = min\{k | rank[M_{i,k}] - rank[M_{i,k-1}] = r\}, \qquad i = 0, \dots, f \tag{4.2.2}$$

It was mentioned in Section 3 that usually in practice the order of the zeros $b_1, \dots, b_f$ is one. Theorem 2 is then useful in computing the order m_0 of b_0. In this case $M_{0,k}$ can be written in terms of the impulse response matrices defined in (3.0.10):

$$M_{0,k} = \begin{bmatrix} A_0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ A_{k-N} & \dots & A_0 \end{bmatrix} \tag{4.2.3}$$

The discussion of some computational aspects is necessary at this point. Theorem 2 explicitly requires the computation of the rank of $M_{i,k}$ for all $k = n_i, \ldots, m_i$. Also in order to use Theorem 1 effectively in a design procedure it is necessary to reduce $M_i (= M_{i,m_i-1})$ in (3.0.9) to a form with linearly independent columns so that a basis for its column space is available. The Singular Value Decomposition (SVD) is a very reliable method for both purposes. However, its application on the matrices $M_{i,k}$ whose dimension can grow very large might be difficult and time–consuming. Van Dooren et al. (1979) have exploited the Toeplitz matrix form of $M_{i,k}$ to develop a fast recursive algorithm that performs the rank search in a numerically stable way. In each step the rank of $M_{i,k}$ is computed for some k by obtaining the SVD of an $r \times r$ matrix. At the same time $M_{i,k}$ is reduced to a form with linearly independent columns. Hence to obtain m_i and an orthonormal basis for the column space of M_i one has to obtain the SVD of only $(m_i - n_i + 1)$ matrices of dimension $r \times r$.

4.3 Design of a Column of $H_{yr}(z)$

The requirements of Theorem 1 apply to each column of H_{yr} separately and so each column can be designed independently. Let us write

$$H_{yr}(z) = [\, h_1(z) \quad \ldots \quad h_r(z) \,] \tag{4.3.1}$$

where $h_j(z)$ has dimension $r \times 1, j = 1, \ldots, r$. Also let

$$\hat{h}_j(\lambda) = h_j(\lambda^{-1}) \leftrightarrow h_j(z), \qquad j = 1, \ldots, r \tag{4.3.2}$$

We shall now proceed with the design of $\hat{h}_j(\lambda)$ for some j. Let U_i be a matrix whose columns form an orthonormal basis for the column space of M_i given in (3.0.9). U_i can be obtained from M_i with the procedure of Van Dooren et al. (1979) briefly discussed in Section 4.2. Also let

$$\rho_i = rank[M_i] = rank[U_i] \tag{4.3.3}$$

According to Theorem 1 we must have

$$\hat{h}_j(\lambda) = (\lambda - b_i)^{n_i} \hat{h}_{j,i}(\lambda), \qquad i = 0, \ldots, f \tag{4.3.4}$$

where

$$\eta_{j,i} \stackrel{\text{def}}{=} [\hat{h}_{j,i}^{(0)}(b_i)^T \cdots \frac{1}{(m_i - n_i - 1)!} \hat{h}_{j,i}^{(m_i - n_i - 1)}(b_i)^T]^T \tag{4.3.5}$$

is a linear combination of the columns of U_i, i.e.,

$$\eta_{j,i} = U_i \chi_i^1 \tag{4.3.6}$$

where χ_i^1 is any vector of dimension ρ_i. The freedom allowed in the choice of χ_i^1 will now be gradually reduced by requiring certain properties for $\hat{h}_j(\lambda)$, according to the designer's specifications and decisions. First, the limitations imposed by the desired structure of h_j will be quantified. Then, the undesirable zeros and time delays that have to be present in the diagonal element will be determined and the design of this element will be reduced to that of a SISO system. Finally, the non-diagonal elements will be designed so that the closed-loop interactions are minimized. It should be pointed that if for some i we have $m_i = n_i$, then part (b) of Theorem 1 and therefore (4.3.6) do not apply for that i and so all equations in this section corresponding to that particular i should be ignored.

i) Structure of h_j

Let the design specification be that the $\ell_1, \ell_2, \ldots, \ell_g$ elements of $h_j(z)$ be identically equal to zeros, where

$$g \leq r - 1 \tag{4.3.7}$$

$$\ell_k \neq j, \qquad k = 1, \ldots, g \tag{4.3.8}$$

We shall use ℓ to denote the set

$$\ell \stackrel{\text{def}}{=} \{\ell_1, \ldots, \ell_g\} \tag{4.3.9}$$

Define

$$\bar{\ell} \stackrel{\text{def}}{=} \{1,2,\ldots,r\} - \{j\} - \ell \tag{4.3.10}$$

Let

$$e_k \stackrel{\text{def}}{=} [0\ldots0 \quad 1 \quad 0\ldots0]^T \tag{4.3.11}$$

where the 1 is the k^{th} element and e_k has dimension $r \times 1$.

Define

$$e^{\ell} \stackrel{\text{def}}{=} \begin{bmatrix} e_{\ell_1}^T \\ \vdots \\ e_{\ell_g}^T \end{bmatrix} \tag{4.3.12}$$

$$\Lambda_i^{\ell} \stackrel{\text{def}}{=} diag[\underbrace{e^{\ell},\ldots,e^{\ell}}_{(m_i-n_i)}], \qquad i = 1,\ldots,f \tag{4.3.13}$$

In order for the specified elements of h_j to be zero, the vector χ_i^1 must solve:

$$\Lambda_i^{\ell}\eta_{j,i} = 0$$

or

$$\Lambda_i^{\ell}U_i\chi_i^1 = 0 \tag{4.3.14}$$

Let

$$\rho_i^{\ell} = rank[\Lambda_i^{\ell}M_i] = rank[\Lambda_i^{\ell}U_i] \tag{4.3.15}$$

Then $\rho_i \geq \rho_i^{\ell}$. Hence the null space of $\Lambda_i^{\ell}U_i$ has dimension

$$\xi_i^{\ell} = \rho_i - \rho_i^{\ell} \tag{4.3.16}$$

Let V_i^{ℓ} be a matrix whose columns form an orthonormal basis for the null space of $\Lambda_i^{\ell}U_i$. Both V_i^{ℓ} and ρ_i^{ℓ} can be obtained from an SVD of $\Lambda_i^{\ell}U_i$. Then the solutions to (4.3.14) are:

$$\chi_i^1 = V_i^{\ell}\chi_i^2 \tag{4.3.17}$$

where χ_i^2 can be any vector of dimension ξ_i^{ℓ} when $\xi_i^{\ell} \neq 0$. If $\xi_i^{\ell} = 0$ then of course $V_i^{\ell} = 0$ and $\chi_i^1 = 0$.

Hence we must have

$$\eta_{j,i} = U_i V_i^{\ell} \chi_i^2 \tag{4.3.18}$$

where $\eta_{j,i}$ was defined in (4.3.5). Note that (4.3.18) includes the case $\xi_i^{\ell} = 0$, where $V_i^{\ell} = 0$ yields $\eta_{j,i} = 0$.

ii) Diagonal element of h_j

We shall now proceed with the determination of the j^{th} element of h_j. Up to this point no assumption has been made on the order of the zeros $b_0, b_1, \ldots, b_f$. However if more than one zero has order larger than 1, then the number of possible choices to be examined at this point could grow enormously. On the other hand, in practice one is usually faced with a situation where the order and degree (as defined from the Smith–McMillan form (Desoer and Schulman, 1974); also see Lemma A.1 in Appendix A) of the zeros $a_1, \ldots, a_f$ of $P(z)$ and therefore of the zeros $b_1, \ldots, b_f$ of $\hat{P}(\lambda)$, is equal to 1. The following assumption will be made here to simplify the procedure.

Assumption A2: The degree of the zeros $a_1, \ldots, a_f$ of $P(z)$ is equal to 1.

No assumption is made however about the zero $b_0 = 0$ of $\hat{P}(\lambda)$ corresponding to time delays in P(z). We shall examine the two cases separately.

a) $b_i, i = 1, \ldots, f$

It follows from A2 that for the order of the zeros m_i we have $m_i = 1$. Also since $r \geq 2$, A2 implies that $n_i = 0$. Then from (4.3.4), (4.3.5), (4.3.6) it follows that since 0 is a linear combination of the columns of U_i, the highest power of $(\lambda - b_i)$ that is sufficient to include in the elements of $\hat{h}_j(\lambda)$ is $(\lambda - b_i)^1$. However according to Rule 1 we wish to have the smallest possible power in the j^{th} element, and that is $(\lambda - b_i)^0 = 1$. In order for this to be possible, the following equation must have a solution

$$e_j^T \eta_{j,i} = 1$$

or

$$e_j^T U_i V_i^{\ell} \chi_i^2 = 1 \tag{4.3.19}$$

where e_j is defined in (4.3.11). Eq. (4.3.19) will have no solution only if the matrix $e_j^T U_i V_i^{\ell}$ is identically zero. If this happens for some i, then the factor $(\lambda - b_i)$ must be included in the j^{th} element of $\hat{h}_j(\lambda)$. Let us assume that this matrix is zero only for $i = 1, \ldots, \phi_1$. Also let the zeros $b_1, \ldots, b_{\phi_2} (\phi_2 \leq \phi_1)$ have positive real part and the zeros $b_{\phi_2+1}, \ldots, b_{\phi_1}$ have negative real part. Then according to Rule 2, the factor

$$\varsigma_0(\lambda) = \prod_{i=1}^{\phi_2} \frac{(1 - b_i^{-1})(\lambda - b_i)}{(1 - b_i)(\lambda - b_i^{-1})} \prod_{i=\phi_2+1}^{\phi_1} \frac{(\lambda - b_i)}{(1 - b_i)} \tag{4.3.20}$$

should be included in the j^{th} element.

Note that one does not always have to follow Rule 1. One may wish to include the factor $(\lambda - b_i)$ for some i in the j^{th} element even when one does not have to do it, if that will result in significantly smaller interactions (non–diagonal elements) and if the j^{th} output is not so important. The procedure for determining the magnitude of the interactions will then be exactly the same (see Section 4.3.iii) and at the end the designer can decide whether inclusion of $(\lambda - b_i)$ pays off. A simple qualitative way to figure out a priori whether it may pay off, without going through the whole design procedure, is the following. For $m_i = 1$ and $n_i = 0$ we have $rank[U_i] = r - 1$. U_i consists of the first $(r - 1)$ columns of the left singular vector matrix in an SVD of M_i. The r^{th} column u_i is orthogonal to all the columns of U_i. If the j^{th} element of u_i is large compared to the k^{th} elements where k belongs to the set $\bar{\ell}$ defined in (4.3.10), then it is likely that inclusion of $(\lambda - b_i)$ in the j^{th} element will result in significantly smaller interactions in the non–zero non–diagonal elements of h_j.

b) $b_0 (= 0)$ (Time delays)

Define

$$\varsigma_\tau(\lambda) = \lambda^\tau \varsigma_0(\lambda) \tag{4.3.21}$$

where τ is an integer. Then according to Rule 1 we need to find the smallest τ such that $\varsigma_\tau(\lambda)$ is possible as the j^{th} element of $\hat{h}_{j,0}(\lambda)$. From (4.3.5), (4.3.6) it follows that in order for a τ to be possible, the following equation must have a solution

$$\epsilon_j \eta_{j,0} = Z_\tau$$

or

$$\epsilon_j U_0 V_0^{\ell} \chi_0^2 = Z_\tau \tag{4.3.22}$$

where

$$\epsilon_j = diag[\underbrace{e_j^T, \ldots, e_j^T}_{(m_0 - n_0)}] \tag{4.3.23}$$

and

$$Z_\tau = [\varsigma_\tau^{(0)}(0) \cdots \frac{1}{(m_0 - n_0 - 1)!} \varsigma_\tau^{(m_0 - n_0 - 1)}(0)]^T \tag{4.3.24}$$

Hence one can obtain the smallest possible τ as

$$\tau_0 = min\{\tau \in N_0 | rank[\epsilon_j U_0 V_0^{\ell} | Z_\tau] = rank[\epsilon_j U_0 V_0^{\ell}]\} \tag{4.3.25}$$

where N_0 is the set of positive integers, including zero.

Still, contrary to Rule 1, one may wish to choose a τ larger than τ_0 if that results in smaller interactions for a given set ℓ. Eq. (4.3.22) should, of course, have a solution for this τ, i.e., the rank condition in (4.3.25) should hold. In the following paragraph τ_0 is used, but any other possible τ can be used instead, without affecting the procedure for determining the magnitude of the interactions in Section 4.3.iii.

The j^{th} element of $\hat{h}_j(\lambda)$ has been completely determined at this point as

$$e_j^T \hat{h}_j(\lambda) = \lambda^N \varsigma_{\tau_0}(\lambda) \tag{4.3.26}$$

Let us now quantify the limitations that the selection of this diagonal element imposes on $\chi_i^2, i = 0, 1, \ldots, f$. The following equations have to be satisfied.

$$e_j^T U_i V_i^{\ell} \chi_i^2 = b_i^{n_o} \varsigma_{\tau_0}(b_i), \qquad i = 1, \ldots, f \tag{4.3.27}$$

$$\epsilon_j U_0 V_0^{\ell} \chi_0^2 = Z_{\tau_0} \tag{4.3.28}$$

Let $\chi_i^0, i = 0, \ldots, f$, be a particular solution for each corresponding equation, obtained with some method for solving systems of linear equations. Also let $W_i^{\ell}, i = 1, \ldots, f$ be a matrix whose columns form an orthonormal basis for the null space of $e_j^T U_i V_i^{\ell}$, and W_0^{ℓ} the corresponding matrix for $\epsilon_j U_0 V_0^{\ell}$. These matrices and their ranks w_i^{ℓ}, can be obtained from an SVD. Then the χ_i^2's that solve the set of equations (4.3.27), (4.3.28) are:

$$\chi_i^2 = \chi_i^0 + W_i^{\ell} \chi_i^3, \qquad i = 0, 1, \ldots, f \tag{4.3.29}$$

where χ_i^3 is any vector of dimension w_i^{ℓ}, when $w_i^{\ell} \neq 0$. If $w_i^{\ell} = 0$ then $W_i^{\ell} = 0$ and $\chi_i^2 = \chi_i^0$.

From (4.3.18) and (4.3.29) we obtain

$$\eta_{j,i} = U_i V_i^{\ell} \chi_i^0 + U_i V_i^{\ell} W_i^{\ell} \chi_i^3, \qquad i = 0, 1, \ldots, f \tag{4.3.30}$$

iii). Non-diagonal elements of h_j

The part of the procedure that was developed in Section 4.3.i makes sure that the elements of h_j corresponding to the set ℓ defined in (4.3.9), are identically equal to zero. We shall now proceed to compute the terms in the non–zero non–diagonal elements of h_j, i.e., the elements corresponding to the set $\bar{\ell}$, defined in (4.3.10). To do so the freedom allowed in the choice of χ_i^3 will be used.

Let $\bar{\ell}_1, \bar{\ell}_2, \ldots, \bar{\ell}_q$ be the elements of the set $\bar{\ell}$. According to Rule 3 the $\bar{\ell}_k^{th}$ element of $\hat{h}_j(\lambda)$ should be of the form

$$e_{\bar{\ell}_k}^T \hat{h}_j(\lambda) = B_k(\lambda) \stackrel{\text{def}}{=} \lambda^{\delta_k}(\beta_{k,0} + \beta_{k,1}\lambda + \ldots + \beta_{k,\nu}\lambda^{\nu})(1 - \lambda), \qquad k = 1, \ldots, q \tag{4.3.31}$$

From Corollary 2.a it follows:

$$\delta_k = n_0 = N, \qquad k = 1, \ldots, q \tag{4.3.32}$$

The values of $\beta_{k,0}, \ldots, \beta_{k,\nu}, k = 1, \ldots, q$ will be computed from (4.3.5) and (4.3.30). Note that any of the $\beta's$ can be zero, including the first ones, β_0, β_1, etc.

Define

$$e^{\bar{\ell}} \stackrel{\text{def}}{=} \begin{bmatrix} e^T_{\bar{\ell}_1} \\ \vdots \\ e^T_{\bar{\ell}_q} \end{bmatrix} \tag{4.3.33}$$

$$\epsilon^{\bar{\ell}} \stackrel{\text{def}}{=} \begin{bmatrix} \epsilon_{\bar{\ell}_1} \\ \vdots \\ \epsilon_{\bar{\ell}_q} \end{bmatrix} \tag{4.3.34}$$

where $\epsilon_{\bar{\ell}_k}$ is defined as in (4.3.23) for $\bar{\ell}_k$ instead of j. As explained in Section 4.3.ii.a, we have $n_i = 0$ for $i = 1, \ldots, f$. From (4.3.4), (4.3.5), (4.3.31), (4.3.32) it follows:

$$e^{\bar{\ell}}\eta_{j,i} = \begin{bmatrix} B_1(b_i) \\ \vdots \\ B_q(b_i) \end{bmatrix} = diag[\underbrace{\gamma_{i,\nu}, \ldots, \gamma_{i,\nu}}_{q}] \begin{bmatrix} \theta_{1,\nu} \\ \vdots \\ \theta_{q,\nu} \end{bmatrix}, \qquad i = 1, \ldots, f \tag{4.3.35}$$

where

$$\gamma_{i,\nu} \stackrel{\text{def}}{=} [\,(1-b_i)b_i^N \quad (1-b_i)b_i^{(N+1)} \quad \ldots \quad (1-b_i)b_i^{(N+\nu)}\,], \qquad i = 1, \ldots, f \tag{4.3.36}$$

$$\theta_{k,\nu} \stackrel{\text{def}}{=} [\,\beta_{k,0} \quad \beta_{k,1} \quad \ldots \quad \beta_{k,\nu}\,]^T, \qquad k = 1, \ldots, q \tag{4.3.37}$$

It also follows that

$$\epsilon^{\bar{\ell}}\eta_{j,0} = \begin{bmatrix} B_1^0 \\ \vdots \\ B_q^0 \end{bmatrix} = diag\,\underbrace{[\Gamma_\nu, \ldots, \Gamma_\nu]}_{q} \begin{bmatrix} \theta_{1,\nu} \\ \vdots \\ \theta_{q,\nu} \end{bmatrix} \tag{4.3.38}$$

where

$$B_k^0 = [\,\beta_{k,0} \quad (\beta_{k,1} - \beta_{k,0}) \quad \ldots \quad (\beta_{k,m_0-N-1} - \beta_{k,m_0-N-2})\,]^T, \qquad k = 1, \ldots, q \tag{4.3.39}$$

$$\beta_{k,\mu} = 0 \qquad for \qquad \mu > \nu, \qquad k = 1, \ldots, q \tag{4.3.40}$$

and Γ_ν is a matrix containing the first $\nu + 1$ columns of $[\Gamma 000 \ldots]$ with

$$\Gamma = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ -1 & 1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & -1 & 1 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & -1 & 1 \end{bmatrix}_{(m_0 - N) \times (m_o - N)} \tag{4.3.41}$$

Then the use of (4.3.30) in (4.3.35), (4.3.38) and combination of the resulting equations yields:

$$K_\nu \theta_\nu = T_1 + T_2 X \tag{4.3.42}$$

where

$$\theta_\nu \stackrel{\text{def}}{=} \begin{bmatrix} \theta_{1,\nu} \\ \vdots \\ \theta_{q,\nu} \end{bmatrix} \tag{4.3.43}$$

$$K_\nu \stackrel{\text{def}}{=} \begin{bmatrix} diag[\Gamma_\nu, \ldots, \Gamma_\nu] \\ diag[\gamma_{1,\nu}, \ldots, \gamma_{1,\nu}] \\ \vdots \\ diag[\underbrace{\gamma_{f,\nu}, \ldots, \gamma_{f,\nu}}_{q}] \end{bmatrix} \tag{4.3.44}$$

$$X \stackrel{\text{def}}{=} [(\chi_0^3)^T \quad (\chi_1^3)^T \ldots (\chi_f^3)^T]^T \tag{4.3.45}$$

$$T_1 \stackrel{\text{def}}{=} \begin{bmatrix} \epsilon^{\bar{\ell}} U_0 V_0^\ell \chi_0^0 \\ e^{\bar{\ell}} U_1 V_1^\ell \chi_1^0 \\ \vdots \\ e^{\bar{\ell}} U_f V_f^\ell \chi_f^0 \end{bmatrix} \tag{4.3.46}$$

$$T_2 \stackrel{\text{def}}{=} diag[\epsilon^{\bar{\ell}} U_0 V_0^\ell W_0^\ell, e^{\bar{\ell}} U_1 V_1^\ell W_1^\ell, \ldots, e^{\bar{\ell}} U_f V_f^\ell W_f^\ell] \tag{4.3.47}$$

Equation (4.3.42) can also be written as

$$[K_\nu | -T_2] \begin{bmatrix} \theta_\nu \\ X \end{bmatrix} = T_1 \tag{4.3.48}$$

Hence the smallest possible ν for h_j and for the particular choice of set ℓ, can be obtained as the smallest ν for which (4.3.48) has a solution, i.e.,

$$\nu_{min} = min\{\nu \in N_0 | rank[K_\nu | -T_2] = rank[K_\nu | -T_2 | T_1]\} \tag{4.3.49}$$

However instead of trying to minimize ν, a better alternative to use a larger ν and use the extra degrees of freedom to minimize the sum of the squared errors for the step response of the $\bar{\ell}_1, \ldots, \bar{\ell}_q$ system outputs to the j^{th} external input (j^{th} element of r or d). This means minimizing

$$J_\nu \stackrel{\text{def}}{=} \sum_{k=1}^{q} (\phi_k \sum_{\mu=0}^{\nu} \beta_{k,\mu}^2) = \theta_\nu^T \Phi^T \Phi \theta_\nu \tag{4.3.50}$$

where $\phi_k, k = 1, \ldots, q$ are optional weights (positive real numbers) and

$$\Phi = diag[\phi_1^{1/2} I_{\nu+1}, \ldots, \phi_q^{1/2} I_{\nu+1}] \tag{4.3.51}$$

where $I_{\nu+1}$ is the $(\nu+1) \times (\nu+1)$ identity matrix.

Equation (4.3.42) can be written as

$$K_\nu \Phi^{-1} (\Phi \theta_\nu) = T_1 + T_2 X \tag{4.3.52}$$

Hence the $\Phi\theta_\nu$ that minimizes J_ν can be obtained as the minimum norm solution to (4.3.52). For ν large enough, K_ν is full rank, i.e., $rank[K_\nu] = q(f + m_0 - N)$, and for a given X, the solution is

$$\theta_\nu(X) = \Phi^{-1} F_\nu^* (F_\nu F_\nu^*)^{-1} (T_1 + T_2 X) \tag{4.3.53}$$

where

$$F_\nu \stackrel{\text{def}}{=} K_\nu \Phi^{-1} \tag{4.3.54}$$

and the superscript * indicates complex conjugate transpose. Note that although the matrices involved may be complex, the solution θ_ν will be real because any complex zeros of P(z) come in complex conjugate pairs. However the form in which the solution is given in (4.3.53) may cause numerical problems in some cases. One can avoid them by computing the pseudo–inverse $F_\nu^\dagger = F_\nu^* (F_\nu F_\nu^*)^{-1}$ from an SVD of F_ν (Stewart (1973), p. 324).

One can now compute X by minimizing $J_\nu(X)$ for the solution $\theta_\nu(X)$ of (4.3.53). From (4.3.53) we get

$$J_\nu(X) = (T_1 + T_2X)^*(F_\nu F_\nu^*)^{-1}(T_1 + T_2X)$$
$$= (T_1 + T_2X)^*(F_\nu^\dagger)^* F_\nu^\dagger (T_1 + T_2X) \qquad (4.3.55)$$

By setting the gradient of $J_\nu(X)$ equal to zero we get

$$T_2^*(F_\nu^\dagger)^* F_\nu^\dagger T_2 X = -T_2^*(F_\nu^\dagger)^* F_\nu^\dagger T_1 \qquad (4.3.56)$$

from which a solution X which minimizes $J_\nu(X)$ can be obtained. The optimum θ_ν can then be computed from (4.3.53).

It is clear that by increasing ν, the value of the obtained minimum of J_ν will either be reduced or it will remain the same. Hence the designer has the option to choose interactions with smaller magnitude in exchange for a longer duration of the interactions. The knowledge of the value of this minimum at the limit as $\nu \to \infty$ would be quite helpful in making this decision. From (4.3.55) we see that we need to compute $F_\nu F_\nu^*$ as $\nu \to \infty$. The fact that the elements of $\gamma_{i,\nu}, i = 1, \ldots, f$, are terms in a geometric progression, allows us to do so easily when $b_1, \ldots, b_f$ are inside the unit circle. We cannot do so however if some of them are outside the unit circle, i.e., when some of the undesirable zeros of $P(z)$ are inside the unit circle. This is actually a situation, where for numerical reasons it would be strongly recommended to compute $F_\nu^\dagger$ from an SVD of F_ν as mentioned above.

5. Construction of $H_{ur}(z)$ and $C(z)$

After the desired $H_{yr}(z)$ has been designed, $H_{ur}(z)$ can be obtained from (3.0.1):

$$H_{ur}(z) = P(z)^{-1} H_{yr}(z) \qquad (3.0.1)$$

Substitution of (3.0.1) into (2.0.5) yields:

$$C(z) = H_{ur}(z)[I - H_{yr}(z)]^{-1} \qquad (5.0.1)$$

If one attempted to construct $H_{ur}(z)$ and $C(z)$ from (3.0.1), (5.0.1) by doing the computations in terms of transfer function matrices, the procedure would be extremely tedious. Instead, the computations can be made quite simply by working in the state space. One can obtain realizations of $P(z)$, $H_{yr}(z)$ to get the state space descriptions:

$$P(z) = C(zI - A)^{-1}B + D \tag{5.0.2}$$

$$H_{yr}(z) = C_0(zI - A_0)^{-1}B_0 + D_0 \tag{5.0.3}$$

$P(z)$ represents a physical system and so it can be assumed to be strictly proper, i.e., D = 0. Then from Corollary 2 it follows $D_0 = 0$. Construction of $H_{yr}(z)$, $C(z)$ involves the following steps.

Step 1. Inversion of $P(z)$.

Silverman (1969) developed a computationally simple algorithm for the inversion of a linear multivariable system, whose state space description is known. The result of the inversion will be

$$P(z)^{-1} = (C_1(zI - A_1)^{-1}B_1 + D_1)(K_0 + K_1 z + \ldots + K_{m_0} z^{m_0})z^N \tag{5.0.4}$$

where

$$A_1 = A - B\bar{D}^{-1}\bar{C} \tag{5.0.5}$$

$$B_1 = B\bar{D}^{-1} \tag{5.0.6}$$

$$C_1 = -\bar{D}^{-1}\bar{C} \tag{5.0.7}$$

$$D_1 = \bar{D}^{-1} \tag{5.0.8}$$

m_0 is the order of the zero b_0 of $\hat{P}(\lambda)$ obtained from Corollary 2 and N is defined in (3.0.10). The matrices $\bar{C}, \bar{D}, K_0, \ldots, K_{m_o}$ are determined with Silverman's (1969) procedure.

Step 2. Computation of $H_{ur}(z)$.

The following Theorem will be used.

Theorem 3

Let

$$G(z) = C(zI - A)^{-1}B + D \tag{5.0.9}$$

then

$$G(z)z^k = C(zI - A)^{-1}A^kB + \sum_{\ell=1}^{k} CA^{\ell-1}Bz^{k-\ell} + Dz^k, \quad \forall \quad k \geq 1 \tag{5.0.10}$$

Proof. See Appendix B.

We can now apply Theorem 3 to $P(z)^{-1}$, to obtain

$$\begin{aligned} P(z)^{-1} =& C_1(zI - A_1)^{-1}(\sum_{i=0}^{m_0} A_1^{i+N}B_1K_i) \\ &+ \sum_{i=0}^{m_0}\sum_{\ell=1}^{i+N} C_1A_1^{\ell-1}B_1K_iz^{i+N-\ell} + \sum_{i=0}^{m_0} D_1K_iz^{i+N} \\ =& C_2(zI - A_2)^{-1}B_2 + D_{2,0} + D_{2,1}z + \ldots + D_{2,m_0+N}z^{m_0+N} \end{aligned} \tag{5.0.11}$$

where

$$A_2 = A_1 \tag{5.0.12}$$

$$B_2 = \sum_{i=0}^{m_0} A_1^{i+N}B_1K_i \tag{5.0.13}$$

$$C_2 = C_1 \tag{5.0.14}$$

$$D_{2,k} = \sum_{i=0}^{m_0} C_1A^{N-1-k+i}B_1K_i, \qquad k = 0, \ldots, N-1 \tag{5.0.15}$$

$$D_{2,k} = D_1K_{k-N} + \sum_{i=k-N+1}^{m_0} C_1A^{N-1-k+i}B_1K_i, \quad k = N, \ldots, N+m_0 \tag{5.0.16}$$

Then from (3.0.1), (5.0.3), (5.0.11) we get

$$\begin{aligned} H_{ur}(z) =& (C_2(zI - A_2)^{-1}B_2C_0 + D_{2,0}C_0)(zI - A_0)^{-1}B_0 \\ &+ (\sum_{i=1}^{m_0+N} D_{2,i}z^i)C_0(zI - A_0)^{-1}B_0 \end{aligned} \tag{5.0.17}$$

Application of Theorem 3 on the second term of the right–hand side yields the term

$$(\sum_{i=1}^{m_0+N} D_{2,i}C_0A_0^i)(zI-A_0)^{-1}B_0 + \sum_{i=1}^{m_0+N} D_{2,i}C_0A_0^{i-1}B_0 + \sum_{i=1}^{m_{0+N-1}} \Psi_i z^i$$

where the fact $(zI-A)^{-1}A^k = A^k(zI-A)^{-1}$ was used. However, by construction, $P(z)^{-1}H_{yr}(z)$ is proper. Therefore $\Psi_i = 0$ for all $i = 1,\ldots,m_0+N-1$. Hence

$$\begin{aligned} H_{ur}(z) = (&C_2(zI-A_2)^{-1}B_2C_0 + D_{2,0}C_0 \\ &+ \sum_{i=1}^{m_0+N} D_{2,i}C_0A_0^i)(zI-A_0)^{-1}B_0 \\ &+ \sum_{i=1}^{m_0+N} D_{2,i}C_0A_0^{i-1}B_0 \end{aligned} \tag{5.0.18}$$

All that is necessary now is to compute the product of two proper transfer function matrices, whose state space descriptions are known. The following Theorem takes care of that:

Theorem 4. (Doyle, 1984)

Let

$$G_1(z) = C_1(zI-A_1)^{-1}B_1 + D_1 \tag{5.0.19}$$

$$G_2(z) = C_2(zI-A_2)^{-1}B_2 + D_2 \tag{5.0.20}$$

then

$$G_1(z)G_2(z) = C(zI-A)^{-1}B + D \tag{5.0.21}$$

where

$$A = \begin{bmatrix} A_1 & B_1C_2 \\ 0 & A_2 \end{bmatrix}, B = \begin{bmatrix} B_1D_2 \\ B_2 \end{bmatrix}, C = [C_1|D_1C_2], D = [D_1D_2]$$

Application of Theorem 4 on (5.0.18) yields a state space description for H_{ur}.

Step 3. Computation of $C(z)$

All that is needed is to compute a state space description of $(I - H_{yr}(z))^{-1}$. After that, application of Theorem 4 on (5.0.1) will give a state–space description for $C(z)$. From (5.0.3) we get

$$I - H_{yr}(z) = -C_0(zI - A_0)^{-1}B_0 + I \tag{5.0.22}$$

and a state space description of $(I - H_{yr}(z))^{-1}$ can be easily computed as

$$(I - H_{yr}(z))^{-1} = C_0(zI - (A_0 + B_0C_0))^{-1}B_0 + I \tag{5.0.23}$$

The result of the described procedure is state space descriptions of $H_{ur}(z)$ and $C(z)$. One can always obtain a matrix transfer function form, but since the control law can be easily implemented with a state space description it would be advisable to avoid further computations by implementing it as such. It is important to point out however that the realizations obtained for $H_{ur}(z)$ and $C(z)$ are not minimal. It is essential to obtain minimal realizations of them before the implementation so that the undesirable zeros $a_1, \ldots, a_f$ of $P(z)$ do not appear as poles of $H_{ur}(z)$.

6. Illustrations

The first example in this section is used to illustrate the tradeoff between the time duration of the closed loop interactions and the magnitude of the sum of squared errors that they cause. This simple example is also used to demonstrate the procedure step by step. The second example examines different structures for H_{yr} and illustrates how the structure associated with a zero outside the UC can produce large or small closed-loop interactions, depending on the structure chosen for H_{yr}.

6.1 Example 1

Consider the system

$$P(z) = \begin{bmatrix} \frac{0.6}{z-0.4} & \frac{0.5}{z-0.5} \\ 1.2\frac{0.5}{z-0.5} & \frac{0.6}{z-0.4} \end{bmatrix} \tag{6.0.1}$$

Computation of the roots of $det[P(z)]$ shows that the system has one zero outside the UC, at $z = 1.547$. Garcia and Morari (1985) pointed out that an acceptable lower triangular H_{yr} is

$$H_{yr,1}(z) = \begin{bmatrix} z^{-1} & 0 \\ 3.095(1-z^{-1})z^{-1} & \frac{-z+1.547}{1.547z-1}z^{-1} \end{bmatrix} \tag{6.0.2}$$

Clearly the interactions in output 2 for a setpoint change in output 1, are very large in magnitude (over 300% the setpoint change) although of short duration.

We shall now use the procedure of Section 4.3, to design a lower triangular $H_{yr}(z)$. For the time delays ($b_0 = 0$) we have $n_0 = m_0 = 1$. Hence part (b) of Theorem 1 (or Corollary 2) does not apply for $i = 0$ and therefore (4.3.6) does not apply for $i = 0$. Thus none of the equations or subsections of Section 4.3 that correspond to $i = 0$ should be considered. For the zero $a_1 = 1.547$ ($b_1 = a_1^{-1}$) we have $n_1 = 0, m_1 = 1$. Also

$$U_1 = \begin{bmatrix} 0.675 \\ 0.739 \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{6.0.3}$$

and $\rho_1 = 1$.

i) Design of $h_1(z)$.

In this case, ℓ is the empty set and $\bar{\ell} = \{2\}$. Hence $\rho_1^{\ell} = 0$ and therefore $\xi_1^{\ell} = 1, V_1^{\ell} = 1$. Thus (4.3.19) has a solution for $i = 1$ and as a result b_1 should not be included in $\varsigma_0(\lambda)$. Thus from (4.3.20) it follows that $\varsigma_0(\lambda) = 1$. The case of b_0 should not be considered as mentioned above. Hence $\varsigma_{\tau_0}(\lambda) = \varsigma_0(\lambda) = 1$ and from (4.3.26) it follows that the first element of the column (diagonal element) is equal to $\lambda(= z^{-1})$. Then (4.3.27) need be satisfied for $i = 1$. Since the null space of $e_1^T U_1 V_1^{\ell}$ is the empty set, it follows that $W_1^{\ell} = 0$ and

$$\chi_1^2 = \chi_1^0 = b_1/u_1 \tag{6.0.4}$$

We shall now proceed with the design of the second element of the column. In (4.3.31), (4.3.32) we have $q = 1, \bar{\ell}_1 = 2, \delta_1 = 1$. In (4.3.44),(4.3.46) the part

corresponding to b_0 is omitted and so from (4.3.36), (4.3.44), (4.3.46), (6.0.4), it follows:

$$K_\nu = [\,(1-b_1)b_1 \quad (1-b_1)b_1^2 \quad \dots \quad (1-b_1)b_1^{\nu+1}\,] \tag{6.0.5}$$

$$T_1 = b_1 u_2/u_1 \tag{6.0.6}$$

From $W_1^\ell = 0$ and (4.3.47) it follows that $T_2 = 0$. In this case $\Phi = I$ and from (4.3.53) we get

$$\theta_\nu = K_\nu^*(K_\nu K_\nu^*)^{-1}T_1 \tag{6.0.7}$$

(4.3.37), (4.3.50), (6.0.5), (6.0.6), (6.0.7) yield

$$\beta_{i,j} = \frac{(1+b_1)b_1^j u_2}{(1-b_1^{2\nu+2})u_1}, \qquad j = 0,\dots,\nu \tag{6.0.8}$$

$$J_\nu = \frac{(1+b_1)u_2^2}{(1-b_1)(1-b_1^{2\nu+2})u_1^2} \tag{6.0.9}$$

For $\nu = 0$ we get $\beta_{1,0} = 3.095$, i.e., the design in (6.0.2). However the error caused by the interactions in this case is $J_0 = 9.58$, which is quite large. Increasing the duration ν of the interactions reduces J_ν as (6.0.9) indicates. Since $b_1 < 1$ we can compute the limit:

$$\lim_{\nu\to\infty} J_\nu = \frac{(1+b_1)u_2^2}{(1-b_1)u_1^2} = 5.58 \tag{6.0.10}$$

The designer can of course select a relatively small ν, for which J_ν is sufficiently close to the limit given by (6.0.10). A plot of J_ν as a function of ν is given in Fig. 2. One can see that a selection of $\nu = 4$, is satisfactory. It yields $J_\nu = 5.65$. For $\nu = 4$, the second element of $h_1(z)$ becomes equal to $(1.82 + 1.18z^{-1} + 0.76z^{-2} + 0.49z^{-3} + 0.32z^{-4})(1-z^{-1})z^{-1}$.

ii) Design of $h_2(z)$.

Since we require the first element of the column to be zero, we have $\ell = \{1\}$ and therefore $\bar{\ell}$ is the empty set. Hence $\rho_1^\ell = 1, \xi_1^\ell = 0, V_1^\ell = 0$. Then (4.3.19) does

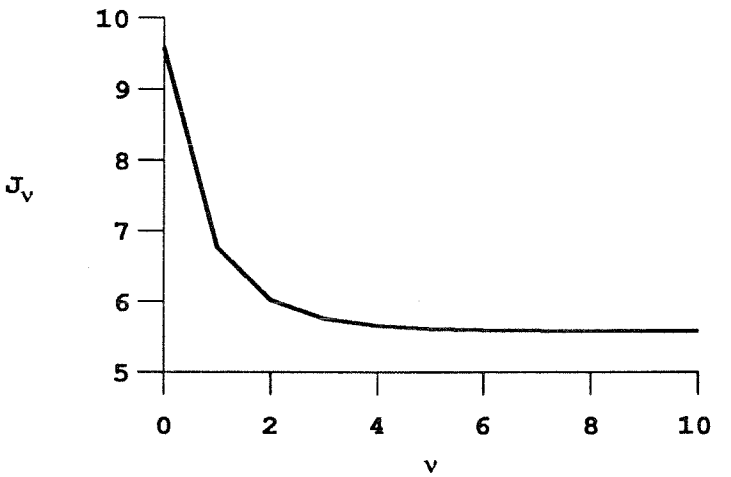


Figure 2 . Example 1; J_ν for column 1.

not have a solution for $i = 1$. As a result the second element (diagonal element) has to have a zero at $\lambda = b_1$. From (4.3.20) we get

$$\varsigma_0(\lambda) = \frac{(1 - b_1^{-1})(\lambda - b_1)}{(1 - b_1)(\lambda - b_1^{-1})} \tag{6.0.11}$$

Then, since $\varsigma_{r_0}(\lambda) = \varsigma_0(\lambda)$, (4.3.26) implies that the diagonal element is $\lambda\varsigma_0(\lambda)$. Substitution of z^{-1} for λ and $a_1(= 1.547)$ for b_1^{-1}, produces the expression in (6.0.2).

6.2 Example 2.

Consider the system:

$$P(z) = \begin{bmatrix} \frac{0.90}{z-0.35} & \frac{0.50}{z-0.35}z^{-1} & \frac{1.00}{z-0.35} \\ \frac{2.70}{z-0.60}z^{-1} & \frac{5.80}{z-0.60}z^{-1} & \frac{0.60}{z-0.60}z^{-1} \\ \frac{0.40}{z-0.50} & \frac{-0.45}{z-0.50} & \frac{1.00}{z-0.50}z^{-1} \end{bmatrix} \tag{6.0.12}$$

The computation of the roots of $det[P(z)]$ yields one zero outside the UC at $z = a_1 = 1.3088$. We shall limit ourselves to the design of the first column of H_{yr}. Two different structures will be examined:

$$h_1 = \begin{bmatrix} x \\ x \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} x \\ 0 \\ x \end{bmatrix} \tag{6.0.13}$$

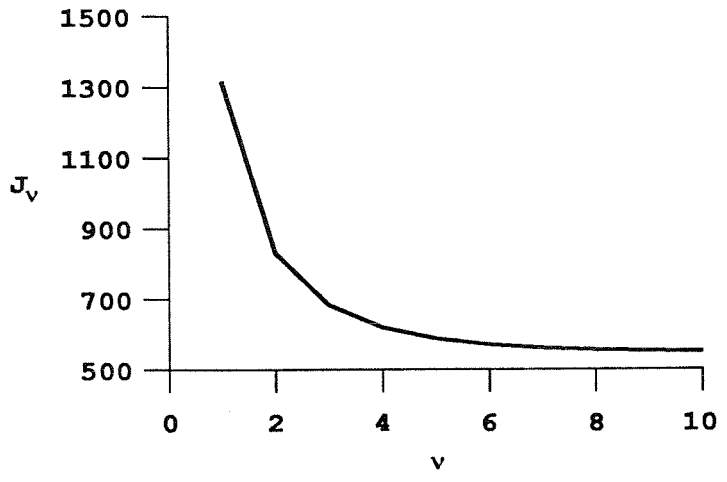
The SVD of $P(a_1)$ yields the following left singular vector matrix:

$$U = \begin{bmatrix} 0.125 & -0.700 & -0.703 \\ 0.992 & 0.0689 & 0.107 \\ -0.0267 & -0.711 & 0.703 \end{bmatrix} \tag{6.0.14}$$

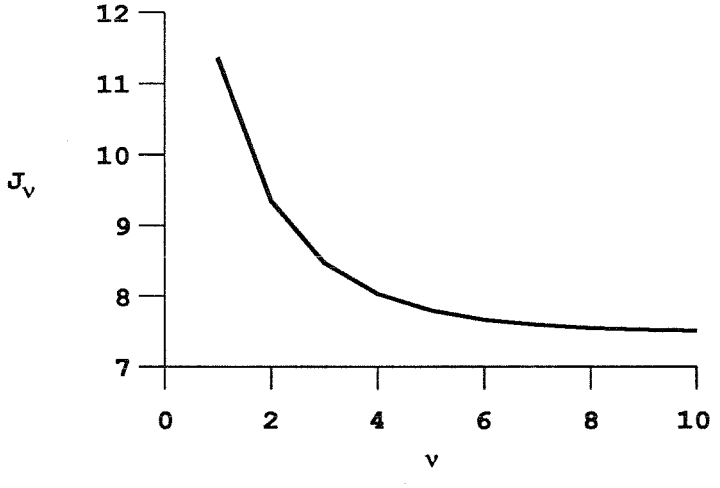
The two first columns of U form U_1. The third, u, is orthogonal to U_1. Then from Corollary 1 it follows that $u^* H_{yr}(a_1) = 0$ for all acceptable H_{yr}'s. (6.0.14) suggests that if the first structure of (6.0.13) is selected, the value of the non-diagonal element at $z = a_1$ will have to be larger than the one for the second structure, because of the smaller corresponding element in u.

The consideration of the time delays (b_0) makes the situation even more favorable for the second structure. We have $n_0 = 1, m_0 = 2$ and

$$U_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{6.0.15}$$



(a)



(b)

Figure 3 . Example 2; J_ν for column 1.

(a) $h_1 = [\,x \quad x \quad 0\,]^T$

(b) $h_1 = [\,x \quad 0 \quad x\,]^T$

The fact that the second row of U_0 is zero, allows in the case of the second structure satisfaction of (4.3.14) for $i = 0$ without using any of the available degrees of freedom. This results in a nonzero w_0^ℓ and T_2 and the additional freedom in choosing χ_0^3 through (4.3.56) reduces J_ν even more.

The above qualitative observations are confirmed from the quantitative results of the design procedure. The corresponding plots of J_ν vs. ν shown in Fig. 3 for both structures of (6.0.13) show a huge difference in the closed-loop interactions for the two structures.

7. Conclusions

The results in this paper quantify the effects of the undesirable zeros and time delays of a multivariable discrete system on its closed-loop performance, in a way that can be used for the direct design of the closed loop transfer function matrix. The designer is provided with quantitative criteria for comparing different designs and evaluating the tradeoffs. The entire procedure is based on linear algebra operations and its implementation on the computer is straightforward.

The design is based on the knowledge of a system model. Hence it may not be robust to model-plant mismatch. However it can be used in the first step of the controller design for the standard two-step Internal Model Control design procedure, in which robustness properties are incorporated in the second step with the design of a low pass filter. Details on the filter design can be found in the literature (Zafiriou and Morari, 1986b,c).

APPENDIX

A) Proof of Theorem 1

The following Lemma will be used in the proof:

Lemma A.1. (Van Dooren et al., 1979; Vandewalle et al., 1974). Let a rational matrix $A(\lambda)$ of normal rank r have the following Laurent expansion at α:

$$A(\lambda) = \sum_{k=-\ell}^{\infty} (\lambda - \alpha)^k A_k(\alpha) \tag{A.1}$$

Define

$$T_k(\alpha) = \begin{bmatrix} A_{-\ell}(\alpha) & A_{-\ell+1}(\alpha) & \dots & A_k(\alpha) \\ 0 & A_{-\ell}(\alpha) & \dots & A_{k-1}(\alpha) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_{-\ell}(\alpha) \end{bmatrix} \tag{A.2}$$

$$\rho_k(\alpha) \stackrel{\text{def}}{=} rank[T_k(\alpha)] - rank[T_{k-1}(\alpha)] \tag{A.3}$$

Let p and z be a pole and a zero respectively of $A(\lambda)$ of orders ω_p, ω_z and degrees δ_p, δ_z, as these are defined from the Smith–McMillan form of $A(\lambda)$ (Van Dooren et al., 1979; Desoer and Schulman, 1974).

The following hold:

i) $\omega_p = -min\{k|\rho_k \neq 0\}$

ii) $\omega_z = min\{k|\rho_k = r\}$

iii) $\delta_p = \sum_{k=-\omega_p}^{-1} \rho_k$

iv) $\delta_z = \sum_{k=0}^{\omega_z} (r - \rho_k)$

Proof of Theorem 2. $\hat{P}(\lambda)^{-1}$ has as its poles exactly the zeros of $\hat{P}(\lambda)$ with the same order and degree (Desoer and Schulman, 1974). Hence since b_i is a zero of $\hat{P}(\lambda)$ of order m_i, it is also a pole of $\hat{P}(\lambda)^{-1}$ and we can write

$$\hat{P}(\lambda)^{-1} = \sum_{k=1}^{m_i} (\lambda - b_i)^{-k} R_{i,k} + G_i(\lambda), \qquad i = 0, 1, \dots, f \tag{A.4}$$

where

$$rank[R_{i,m_i}] \neq 0, \qquad i = 0, 1, \dots, f \tag{A.5}$$

and $G_i(\lambda)$ has no poles at $b_i, i = 0, 1, \ldots, f$.

Postmultiplication of (A.4) with $\hat{H}_{yr}(\lambda)$ yields

$$\hat{P}(\lambda)^{-1}\hat{H}_{yr}(\lambda) = \sum_{k=1}^{m_i}(\lambda - b_i)^{-k}R_{i,k}\hat{H}_{yr}(\lambda) + G_i(\lambda)\hat{H}_{yr}(\lambda) \qquad (A.6)$$

Now take a partial fraction expansion for each term in the sum of the right–hand side of (A.6) to obtain:

$$\begin{aligned}\hat{P}(\lambda)^{-1}\hat{H}_{yr}(\lambda) &= \sum_{k=1}^{m_i}[\sum_{h=0}^{k-1}(\lambda - b_i)^{-k+h}R_{i,k}\frac{1}{h!}\hat{H}_{yr}^{(h)}(b_i) + R_{i,k}G_k^H(\lambda)] + G_i(\lambda)\hat{H}_{yr}(\lambda) \\ &= \sum_{k=1}^{m_i}((\lambda - b_i)^{-k}\sum_{h=k}^{m_i}R_{i,h}\frac{1}{(h-k)!}\hat{H}_{yr}^{(h-k)}(b_i)) \\ &+ \sum_{k=1}^{m_i}R_{i,k}G_k^H(\lambda) + G_i(\lambda)\hat{H}_{yr}(\lambda) \qquad (A.7)\end{aligned}$$

where $G_k^H(\lambda)$ has no poles at b_i. Also recall that $G_i(\lambda)$ has no poles at b_i either. Hence in order for Condition C5.ii to hold we must have for all $i = 0, \ldots, f$:

$$\sum_{h=k}^{m_i}R_{i,h}\frac{1}{(h-k)!}\hat{H}_{yr}^{(h-k)}(b_i) = 0, \qquad k = 1, \ldots, m_i \qquad (A.8)$$

Satisfaction of (A.8) is equivalent to requiring that the columns of $[\hat{H}_{yr}^{(0)}(b_i)^T \ldots \frac{1}{(m_i-1)!}\hat{H}_{yr}^{(m_i-1)}(b_i)^T]^T$ are in the null space of N_i, where:

$$N_i \stackrel{\text{def}}{=} \begin{bmatrix} R_{i,m_i} & 0 & \ldots & 0 \\ R_{i,m_{i-1}} & R_{i,m_i} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ R_{i,1} & R_{i,2} & \ldots & R_{i,m_i} \end{bmatrix} \qquad (A.9)$$

We shall now proceed to determine the null space of N_i.

Postmultiply both sides of (A.4) with $\hat{P}(\lambda)$ to obtain:

$$I = \sum_{k=1}^{m_i}(\lambda - b_i)^{-k}R_{i,k}\hat{P}(\lambda) + G_i(\lambda)\hat{P}(\lambda) \qquad (A.10)$$

Since I has no poles at $b_0,\ldots,b_f$, taking a partial fraction expansion leads to a condition similar to (A.8), in exactly the same manner. Hence (A.10) yields

$$\sum_{h=k}^{m_i} R_{i,h}\frac{1}{(h-k)!}\hat{P}^{(h-k)}(b_i)=0, \qquad k=1,\ldots,m_i \tag{A.11}$$

The equations implied by (A.11) for $k=\ell,\ldots,m_i$ can be put together in the matrix form:

$$N_i[\underbrace{0\ldots0}_{\ell-1} \quad \hat{P}^{(0)}(b_i)^T \quad \ldots \quad \frac{1}{(m_i-\ell)!}\hat{P}^{(m_i-\ell)}(b_i)^T]^T=0, \qquad \ell=1,\ldots,m_i \tag{A.12}$$

The equations obtained from (A.12) for $\ell=1,\ldots,m_i$ can be written together as:

$$N_iL_i=0, \qquad i=0,\ldots,f \tag{A.13}$$

where

$$L_i \stackrel{\text{def}}{=} \begin{bmatrix} \hat{P}^{(0)}(b_i) & 0 & \ldots & 0 \\ \hat{P}^{(1)}(b_i) & \hat{P}^{(0)}(b_i) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{(m_i-1)!}\hat{P}^{(m_i-1)}(b_i) & \frac{1}{(m_i-2)!}\hat{P}^{(m_i-2)}(b_i) & \ldots & \hat{P}^{(0)}(b_i) \end{bmatrix} \tag{A.14}$$

Hence the column space of L_i is a subspace of the null space of N_i. It will now be shown that it is exactly the null space of N_i.

As explained earlier, the order ω_p of the pole b_i of $\hat{P}(\lambda)^{-1}$ is equal to the order m_i of the zero b_i of $\hat{P}(\lambda)$, i.e., equal to m_i:

$$\omega_p(b_i)=m_i \tag{A.15}$$

Lemma A.1 will now be applied on $A(\lambda)=\hat{P}(\lambda)^{-1}$, for $\alpha=b_i$. From (A.1), (A.4), (A.5) it follows that $\ell=m_i$ and $A_{-k}=R_{i,k}$ for $k=1,\ldots,m_i$. By using (A.15) and Lemma A.1.iii we get

$$\delta_p(b_i)=\sum_{k=-m_i}^{-1}\rho_k(b_i)=rank[T_{-1}(b_i)]=rank[N_i] \tag{A.16}$$

since $T_{m_i-1}(b_i) = 0$ and $T_{-1}(b_i)$ can be obtained from N_i by simply permuting its rows and columns.

By definition the order ω_z of the zero b_i of $\hat{P}(\lambda)$ is equal to m_i:

$$\omega_z(b_i) = m_i \tag{A.17}$$

Lemma A.1 will now be applied on $A(\lambda) = \hat{P}(\lambda)$, for $\alpha = b_i$. In this case, since $\hat{P}(\lambda)$ is assumed to have no poles at b_i, we have $\ell \leq 0$ and $A_k = \frac{1}{k!}\hat{P}^{(k)}(b_i)$ for $k = 1, \ldots, m-1$. By using (A.17) and Lemma A.1.iv we obtain

$$\begin{aligned}\delta_z(b_i) &= \sum_{k=0}^{m_i}(r - \rho_k(b_i)) = \sum_{k=0}^{m_i-1}(r - \rho_k(b_i)) \\ &= m_i r - rank[T_{m_i-1}(b_i)] = m_i r - rank[L_i]\end{aligned} \tag{A.18}$$

since from Lemma A.1.ii we have $\rho_{m_i}(b_i) = r$, and we also have $T_{-1}(b_i) = 0$ and $T_{m_i-1}(b_i)$ can be obtained from L_i by permutting its rows and columns.

The degree δ_z of the zero b_i of $\hat{P}(\lambda)$ is the same as the degree δ_p of the pole b_i of $\hat{P}(\lambda)^{-1}$ and so from (A.16), (A.18) we get

$$rank[N_i] + rank[L_i] = m_i r \tag{A.19}$$

But N_i and L_i are matrices of dimension $m_i r \times m_i r$. Therefore (A.13) and (A.19) imply that the column space of L_i is exactly the null space of N_i.

Hence from (A.8) it follows that Condition C5.ii is satisfied if and only if the columns of $[\hat{H}_{yr}^{(0)}(b_i)^T \ldots \frac{1}{(m_i-1)!}\hat{H}_{yr}^{(m_i-1)}(b_i)^T]^T$ are in the column space of L_i. From (3.0.6) it follows that the first $n_i r$ rows and the last $n_i r$ columns of L_i are identically zero. Hence

$$\hat{H}_{yr}^{(k)}(b_i) = 0, \qquad k = 0, \ldots, n_i - 1$$

which implies that

$$\hat{H}_{yr}(\lambda) = (\lambda - b_i)^{n_i}\hat{H}_i(\lambda) \tag{A.20}$$

where $\hat{H}_i(\lambda)$ has no poles at b_i. (A.20) completes the proof of part (a) of Theorem 1. If $m_i = n_i$, then this is the only requirement since then $L_i = 0$. If however $m_i > n_i$ then $rank[L_i] \neq 0$ and additional requirements on $\hat{H}_i(\lambda)$ are necessary. We have for $\ell = n_i, \ldots, m_i - 1$:

$$\frac{1}{\ell!}\hat{H}_{yr}^{(\ell)}(b_i) = \frac{1}{\ell!}\begin{pmatrix} \ell \\ n_i \end{pmatrix} n_i!\hat{H}_i^{(\ell-n_i)}(b_i) = \frac{1}{(\ell-n_i)!}\hat{H}_i^{(\ell-n_i)}(b_i)$$

and so the requirement on $\hat{H}_i(\lambda)$ is that the columns of $[\hat{H}_i^{(0)}(b_i)^T \ldots \frac{1}{(m_i-n_i-1)!}\hat{H}_i^{(m_i-n_i-1)}(b_i)^T]^T$ are in the column space of M_i, where M_i is defined in (3.0.9). QED

B) Proof of Theorem 3

The proof is by induction

$k = 1$:

$$\begin{aligned} G(z)z &= C(zI - A)^{-1}zB + Dz \\ &= C(zI - A)^{-1}(A + zI - A)B + Dz \\ &= C(zI - A)^{-1}AB + CB + Dz \end{aligned}$$

$k = n$: Let

$$G(z)z^n = C(zI - A)^{-1}A^nB + \sum_{\ell=1}^{n} CA^{\ell-1}Bz^{n-\ell} + Dz^n \qquad (B.1)$$

hold.

$k = n + 1$: From (B.1) it follows that

$$G(z)z^{n+1} = C(zI - A)^{-1}A^nzB + \sum_{\ell=1}^{n} CA^{\ell-1}Bz^{n+1-\ell} + Dz^{n+1}$$

and by using the result for $k = 1$ we obtain

$$G(z)z^{n+1} = C(zI - A)^{-1}A^{n+1}B + \sum_{\ell-1}^{n+1} CA^{\ell=1}Bz^{n+1-\ell} + Dz^{n+1} \qquad QED$$

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CHAPTER V

ROBUST H_2-TYPE IMC CONTROLLER DESIGN VIA THE STRUCTURED SINGULAR VALUE

ROBUST H_2-TYPE IMC CONTROLLER DESIGN VIA THE STRUCTURED SINGULAR VALUE

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Abstract.

The two-step Internal Model Control procedure is used for the synthesis of robust controllers for multivariable open-loop stable or unstable plants. In the first step the controller is designed so that the Integral Squared Error (ISE) is minimized for every external input (setpoint or output disturbance) direction in a set and their linear combinations. Open-loop stable and unstable plants are treated in a uniform way. In the second step a low-pass filter is designed so that stability and good performance characteristics are maintained in the presence of model-plant mismatch. The problem is formulated as a minimization of the Structured Singular Value (SSV) for robust performance over the filter parameters. The optimization problem is unconstrained and analytic computation of the gradients is in general possible. Special filter structures are used for open-loop unstable or ill-conditioned plants. A method is also given for computing the worst over all possible plants ISE, for a particular external input.

1. Preliminaries

1.1. Internal Model Control

The Internal Model Control (IMC) structure, introduced by Garcia and Morari (1982), has been widely recognized as very useful in clarifying the issues related to the mismatch between the model used for controller design and the actual process. The IMC structure (Fig. 1a), is mathematically equivalent to the classical feedback structure (Fig. 1b). The IMC controller Q and the feedback C are related through

$$Q = C(I + \tilde{P}C)^{-1} \tag{1.1.1}$$

$$C = Q(I - \tilde{P}Q)^{-1} \tag{1.1.2}$$

where $\tilde{P}$ is the process model.

$\underline{P = \tilde{P}.}$

In this case the overall transfer function connecting the set-points r and disturbances d to the errors $e = y - r$, where y are the process outputs, is

$$e = y - r = (I - PQ)(d - r) \stackrel{\text{def}}{=} \tilde{E} \quad (d - r) \tag{1.1.3}$$

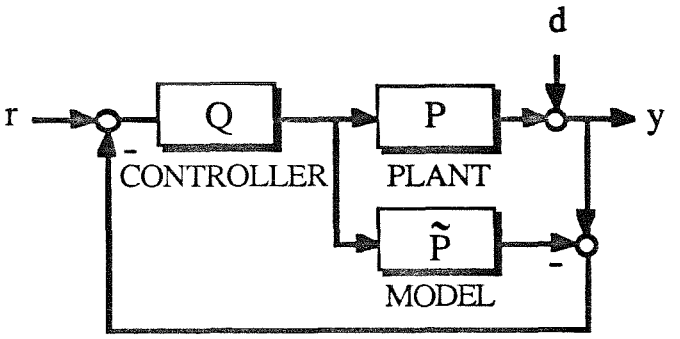
Hence the IMC stucture becomes effectively open-loop (Fig. 2a) and the design of Q is simplified. Note that the IMC controller is identical to the parameter of the Q-parametrization (Zames, 1981). Also the addition of a diagonal filter F by writing

$$Q = \tilde{Q}F \tag{1.1.4}$$

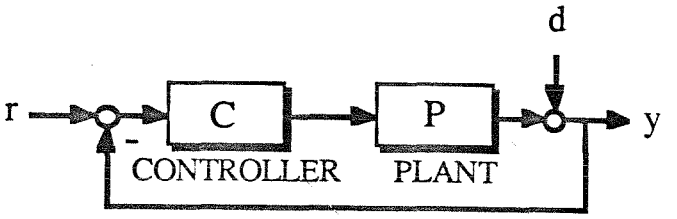
introduces parameters (the filter time constants) which can be used for adjusting on-line the speed of response for each process output.

$\underline{P \neq \tilde{P}.}$

The model-plant mismatch generates a feedback signal in the IMC stucture which can cause performance deterioration or even instability. Since the relative



(a)

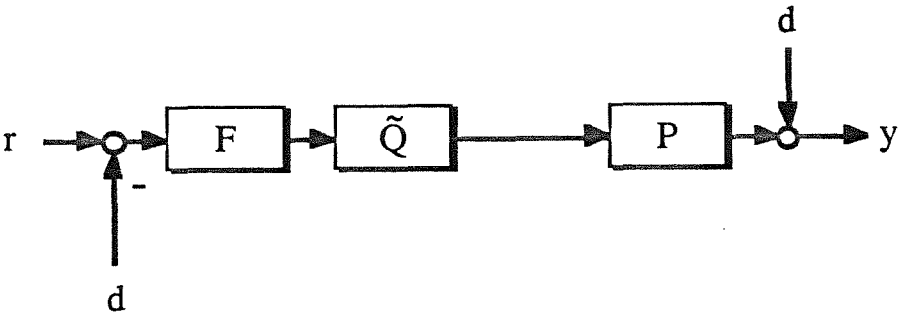


(b)

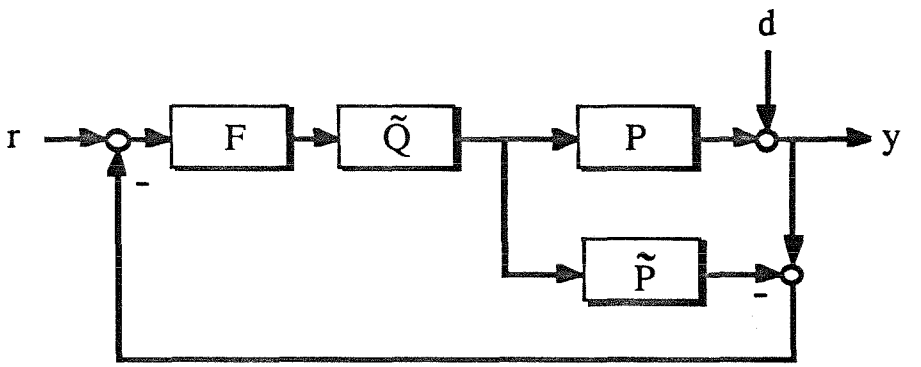
Figure 1.

(a) Internal Model Control structure.

(b) Feedback Control structure.



(a)



(b)

Figure 2. IMC structure with the filter F.

(a) $P = \tilde{P}$.

(b) $P \neq \tilde{P}$.

modeling error is larger at higher frequencies, the addition of the low-pass filter F (Fig. 2b) adds robustness characteristics into the control system. In this case the closed-loop transfer function is

$$e = y - r = (I - P\tilde{Q}F)(I - (P - \tilde{P})\tilde{Q}F)^{-1}(d - r) \stackrel{\text{def}}{=} E \quad (d - r) \qquad (1.1.5)$$

Hence the IMC structure gives rise rather naturally to a two–step design procedure:

Step 1: Design $\tilde{Q}$, assuming $P = \tilde{P}$.

Step 2: Design F so that the closed-loop characteristics that $\tilde{Q}$ produces in Step 1, are preserved in the presence of model-plant mismatch ($P \neq \tilde{P}$).

Note that the feedback controller C, given from (1.1.2), includes integral action if and only if Q inverts at steady-state the model $\tilde{P}$, i.e.,

$$\tilde{Q}(0) = \tilde{P}(0)^{-1} \qquad (1.1.6)$$

$$F(0) = I \qquad (1.1.7)$$

1.2. Internal Stability

A linear time invariant control system is internally stable if the transfer functions between any two points of the control system are stable. A more detailed discussion of the concept of internal stability can be found in the literature (e.g., Morari et al., 1987).

Examination of the feedback structure of Fig. 1b results in the requirement that all elements in the matrix $IS1$ in (1.2.1) are stable.

$$IS1 = \begin{pmatrix} C(I+PC)^{-1} & PC(I+PC)^{-1} & C(I+PC)^{-1}P & (I+PC)^{-1}P \end{pmatrix} \qquad (1.2.1)$$

For the remainder of this section we shall assume that $P = \tilde{P}$. The additional requirements to take care of modeling error are discussed in Section 3.3. Use of

(1.1.1) or (1.1.2) in (1.2.1) yields

$$IS1 = (Q \quad PQ \quad QP \quad (I - PQ)P) \tag{1.2.2}$$

Note that stability of each element in (1.2.2) implies internal stability when the control system is implemented as the feedback structure in Fig. 1b, where C is obtained from the Q used in (1.2.2) through (1.1.2).

In order for the control system to be stable when implemented in the IMC stucture of Fig. 1a, internal stability arguments (Morari et al.,1987) lead to the requirement that all elements of IS2 are stable.

$$IS2 = (Q \quad PQ \quad QP \quad (I - PQ)P \quad PQP \quad P) \tag{1.2.3}$$

Hence if the process P is open-loop unstable, $IS2$ will also be unstable and the control system has to be imlemented in the feedback stucture of Fig. 1b. Still, the two step IMC design procedure can be used for the design of Q, as described in the following sections. C can then be obtained from (1.1.2) and the structure in Fig. 1b implemented. However special care has to be taken in the construction of C so that all the common RHP zeros of Q and $(I - PQ)$ are cancelled in (1.1.2). Minimal or balanced realization software can be used to accomplish that.

Note that when the process is open-loop stable, it follows from (1.2.2) and (1.2.3) that the only requirement for internal stability is that Q is stable.

2. Step 1: Design of $\tilde{Q}$.

Throughout this section the assumption is made that $P = \tilde{P}$.

2.1. Performance Objectives.

The performance objective adopted in this paper is to minimize the Integral Squared Error (ISE) for the error signal e given by (1.1.3). This is an H_2-type objective. Other objectives like an $H_\infty-$type can be used (Zafiriou and Morari, 1986) but they will not be discussed here.

For a specified external system input v ($v = d$ for $r = 0$; $v = -r$ for $d = 0$), the ISE is given by the square of the L_2-norm of e:

$$\Phi(v) \stackrel{\text{def}}{=} ||e||_2^2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^*(i\omega)e(i\omega) \quad d\omega \tag{2.1.1}$$

where the superscript * denotes complex conjugate transpose.

From (1.1.3) we get

$$\Phi(v) = ||e||_2^2 = ||\tilde{E}v||_2^2 = ||(I - P\tilde{Q})v||_2^2 \tag{2.1.2}$$

Hence one objective could be

$$\min_{\tilde{Q}} \Phi(v) \tag{O1}$$

for a particular input $v = (v_1 \quad v_2 \quad \dots \quad v_n)^T$, where $\tilde{Q}$ satisfies the internal stability requirements of section 1.2 and the superscript T denotes transpose.

Minimizing the ISE just for one vector v however is not very meaningful, because of the different directions in which the disturbances enter the process or the setpoints are changed. What is desirable is to find a $\tilde{Q}$, that minimizes $\Phi(v)$ for every single v in a set of external inputs v of interest for the particular process. For an $n \times n$ P this set can be defined as

$$\mathcal{V} = \{v^i(s) | i = 1, ..., n\} \tag{2.1.3}$$

where $v^1(s), ..., v^n(s)$ are vectors that describe the expected directions and frequency content of the external system inputs, e.g., steps, ramps or other types of inputs.

The objective can then be written as

$$\min_{\tilde{Q}} \Phi(v) \qquad \forall v \in \mathcal{V} \tag{O2}$$

under the constraint that $\tilde{Q}$ satisfies the internal stability requirements. It should be noted however that a linear time invariant $\tilde{Q}$ that solves (O2) does not necessarily exist. In Section 2.3, it will be shown that this is the case for some $\mathcal{V}$'s. An

alternative objective in such a case would be:

$$\min_{\tilde{Q}} \left[\Phi(v^1) + \Phi(v^2) + ... + \Phi(v^n)\right] \qquad (O3)$$

In this case the objective is to minimize the sum of the ISE's that each of the inputs v^i would cause when applied to the system separately. One should note that this is a deterministic interpretation of a similar stochastic optimal control problem (Youla et al., 1976).

2.2. Parametrization of All Stabilizing $\tilde{Q}$'s.

The process P can in general be open-loop unstable. The following assumption simplifies the solution of the optimization problem:

Assumption A.1. If π is a pole of the model $\tilde{P}$ in the open RHP, then:

a. The order of π is equal to 1.

b. $\tilde{P}$ has no zeros at $s = \pi$.

c. The residual matrix that corresponds to π is full rank.

Assumption A.1.a is made to simplify the notation and it is the usual case. The results in this paper can be extended to higher order poles. A.1.b is true for SISO systems but not necessarily for MIMO. However, the assumption is not restrictive because the presence of a zero at $s = \pi$ implies an exact cancellation in $det[\tilde{P}(s)]$, which usually does not happen after a slight perturbation in the coefficients of $\tilde{P}$ is introduced. A.1.c is always true for SISO systems, but it can be quite restrictive for MIMO systems. Instead of A.1.c however, an additional assumption can be made on the external input for which the optimal controller is designed. This is discussed in Section 2.3.

Assumption A.1 is not made for poles at $s = 0$ because more than one such poles may appear in an element of $\tilde{P}$, introduced by capacitances that are present in the process. The following assumption true for all practical process control problems is made:

Assumption A.2. Any poles of $\tilde{P}$ or P on the imaginary axis are at $s = 0$. Also $\tilde{P}$ has no finite zeros on the imaginary axis.

Let $\pi_1, ..., \pi_k$ be the poles of each element of $\tilde{P}$ in the open RHP. Define the allpass

$$b_p(s) = \prod_{i=1}^{k} \frac{-s + \pi_i}{s + \pi_i^*} \tag{2.2.1}$$

If A.1.c does not hold, then define

$$b_p(s) = 1 \tag{2.2.1'}$$

The following Theorem holds:

Theorem 2.2.1.

Assume that $Q_0(s)$ satisfies the internal stability requirements of section 1.2, i.e., it produces a matrix $IS1$ with stable elements. Then all Q's that make $IS1$ stable are given by

$$Q(s) = Q_0(s) + b_p(s)^2 Q_1(s) \tag{2.2.2}$$

where Q_1 is any stable transfer matrix such that:

i) If A.1.c holds, PQ_1P has no poles at $s = 0$.

ii) If A.1.c does not hold, PQ_1P has no poles in the closed RHP.

Proof: See Appendix A.1.

2.3. Objective (O1)

This is the first step towards obtaining a solution to (O2), if such a solution exists. In this section we only consider one particular input v. For every external input v that will be considered in this paper, the following assumption can be made without loss of generality:

Assumption A.3.

a. The poles of each nonzero element of v in the open RHP (if any) are the first k' poles π_i of the plant in the open RHP, each with degree 1.

b. If A.1.c does not hold, then every nonzero element of v (or $\hat{v}$) includes all the open RHP poles of $\tilde{P}$, each of them with degree 1.

To simplify the arguments in the paper, we shall assume that if A.3.b is satisfied, then A.1.c is not. In this way the proper choices in the definitions and the proofs will be made on the basis of A.1.c. If both A.1.c and A.3.b hold, then the results that apply to the case where A.1.c does not hold but A.3.b does, are still correct.

Define

$$b_v(s) = \prod_{i=1}^{k'} \frac{-s + \pi_i}{s + \pi_i^*} \tag{2.3.1}$$

If A.1.c does not hold, then define

$$b_v(s) = 1 \tag{2.3.1'}$$

A different assumption is made for the poles of v at $s = 0$:

Assumption A.4. Let l_i be the maximum number of poles at $s = 0$ that an element of the i^{th} row of P has. Then $v_i(s)$ has at least l_i poles at $s = 0$. Also v has no other poles on the imaginary axis and its elements have no finite zeros on the imaginary axis.

The above assumptions are not restrictive in the case where v is an output disturbance d, because in a practical situation we want to design for an output disturbance produced by a disturbance that has passed through the process. Hence, d usually includes the unstable process poles (e.g., an output disturbance d produced by a disturbance in the manipulated variables). Note that the control system will still reject other disturbances with fewer unstable poles, without producing steady-state offset. The assumption is different for poles at $s = 0$ because their number in each row of $\tilde{P}$ can be different, since capacitances may be associated with only certain process outputs. Also the output disturbance may

have more poles at $s = 0$ than the process (e.g., a persistent disturbance in the manipulated variables).

The assumptions may be restrictive in the case of setpoints though. However for setpoint tracking the use of the Two-Degree-of-Freedom structure, which will be discussed briefly in Section 2.5, allows us to disregard the existence of any unstable poles of P and therefore this assumption need not be made for setpoints.

Let $v_0(s)$ be the scalar allpass that includes the common RHP zeros of the elements of v. Factor v as follows:

$$v(s) = v_0(s)\,(\,\hat{v}_1(s) \quad \ldots \quad \hat{v}_n(s)\,)^T \stackrel{\text{def}}{=} v_0(s)\hat{v}(s) \tag{2.3.2}$$

The plant P can be factored into a stable allpass portion P_A and a minimum phase (MP) portion P_M such that

$$P = P_A P_M \tag{2.3.3}$$

The term MP used for a system in the context of this paper basically means that the inverse of the system is stable. Hence P_A is stable and such that $P_A^*(i\omega)P_A(i\omega) = I$. Also P_M^{-1} is stable. This "inner-outer" factorization can be accomplished through the spectral factorization of $P(-s)^T P(s)$. Details on these problems can be found in the literature (Anderson, 1967; Chu, 1985; Doyle, 1984). When P is strictly proper, then one has to add a small constant term in P in order to apply these methods. This approach works satisfactorily in practice.

The following theorem holds:

Theorem 2.3.1.

The set of controllers $\tilde{Q}$ that solve (O1) satisfy

$$\tilde{Q}\hat{v} = b_p b_v^{-1} P_M^{-1} \{ b_p^{-1} b_v P_A^{-1} \hat{v} \}_* \tag{2.3.4}$$

where the operator $\{.\}_*$ denotes that after a partial fraction expancion of the operand all terms involving the poles of P_A^{-1} are omitted. Furthermore, for $n \geq 2$

the number of stabilizing controllers that satisfy (2.3.4) is infinite. Guidelines for the construction of such a controller are given in the proof.

Proof: See Appendix A.2

Note that not every controller that satisfies (2.3.4) is a stabilizing one. Improper stabilizing controllers that satisfy (2.3.4) are accepted because the use of a filter with an appropriate zero-pole roll-off in the second step of the design procedure will produce a proper $Q(s)$.

2.4. Objectives (O2) and (O3)

We shall consider objective (O3) first. Define

$$V(s) \stackrel{\text{def}}{=} (v^1(s) \quad v^2(s) \quad \dots \quad v^n(s)) \tag{2.4.1}$$

where $v^1, ..., v^n$ satisfy assumption A.3. An additional assumption on V is needed:

Assumption A.5.

a. V has no zeros at the location of its unstable poles or on the imaginary axis and V^{-1} cancels the poles of $\tilde{P}$ at $s = 0$ in $V^{-1}\tilde{P}$.

b. If A.1.c holds, the residual matrices for the open RHP poles of V are full rank; if A.1.c does not hold, then V^{-1} cancels the open RHP poles of $\tilde{P}$ in $V^{-1}\tilde{P}$.

Note that satisfaction of assumptions A.3.b and A.4 for each column of V does not necessarily imply satisfaction of A.5. However such a V can be easily constructed. One way is to obtain V as $\tilde{P}$ times a matrix with no open RHP poles and no zeros on the imaginary axis. This case corresponds to the physically meaningful situation, where the output disturbances are produced by disturbances in the manipulated variables. Another simple way is to use a diagonal V, in which case satisfaction of A.3 and A.4 by every column of V implies satisfaction of A.5 by V. This situation is discussed further in Corollary 2.4.1.

Factor V similarly to P (use $V(s)V^T(-s)$ if spectral factorization theory is used):

$$V = V_M V_A \tag{2.4.2}$$

The following Theorem holds:

Theorem 2.4.1.

The controller

$$\tilde{Q} = b_p b_v^{-1} P_M^{-1} \{ b_p^{-1} b_v P_A^{-1} V_M \}_* V_M^{-1} \tag{2.4.3}$$

is the unique solution to (O3).

Proof: See Appendix A.3.

Let us now consider the more meaningful objective (O2). Clearly, a $\tilde{Q}$ that solves (O2) will also solve (O3). Hence from Theorem 2.4.1, it follows that if a solution to (O2) exists, it is given by (2.4.3). Factor each of the v^i in the way used in (2.3.2):

$$v^i(s) = v_0^i(s)\hat{v}^i(s) \tag{2.4.4}$$

Define

$$\hat{V} \stackrel{\text{def}}{=} (\hat{v}^1 \quad \hat{v}^2 \quad \dots \quad \hat{v}^n) \tag{2.4.5}$$

Theorem 2.4.2.

i) If $\hat{V}(z)$ is non-minimum phase, then there exists no solution to (P3).

ii) If $\hat{V}(z)$ is minimum phase, then use of $\hat{V}$ instead of V_M in (2.4.3) yields exactly the same $\tilde{Q}$, which also solves (O2) and it minimizes $\Phi(v)$ for any v that is a linear combination of v^i's that have the same v_0^i's.

Proof: See Appendix A.4.

The following corollary to Theorem 2.4.2 holds:

Corollary 2.4.1.

Let

$$V = diag(v_1, v_2, \dots, v_n) \tag{2.4.6}$$

where $v_1(s), \dots, v_n(s)$ are scalars. Then use of $\hat{V}$ instead of V_M in (2.4.3) yields

exactly the same $\tilde{Q}$, which minimizes $\Phi(v)$ for the following n vectors:

$$v = \begin{pmatrix} v_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ v_2 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ v_n \end{pmatrix} \tag{2.4.7}$$

and their multiples, as well as for the linear combinations of those directions that correspond to v_i's with the same open RHP zeros in the same degree and the same time delays.

2.5. Two-Degree-of-Freedom Structure

From the discussion of the Internal Stability requirements in Section 1.2, it follows that RHP poles in the plant limit the possible choices of Q and thus the achievable performance. This however need not be so for setpoint tracking. Consider the general feedback structure of Fig. 3. For the disturbance behavior it is irrelevant if the controller is implemented as one block C as in Fig. 1b, or as two blocks as in Fig. 3. Hence the achievable disturbance rejection is restricted both by the RHP zeros and poles of P as the quantitative results of the previous sections indicate.

Let us now proceed from the point where a stabilizing $\tilde{Q}$ and the corresponding C have been found through the results of the previous sections, which produce a satisfactory disturbance response. We can then split C into two blocks C_1 and C_2 such that C_1 is minimum phase and C_2 is stable. Then one can easily see that the only RHP zeros of the stabilized system $PC_1(I + PC_1C_2)^{-1}$ are those of the process P. Thus C_3 can be designed without regard for the RHP poles of P and the achievable setpoint tracking is restricted by the RHP zeros of P only.

In summary, the achievable disturbance response of a system is restricted by the presence of the plant RHP zeros and poles regardless of how complicated a controller is used. If the Two-Degree-of-Freedom controller shown in Fig. 3 is employed, the achievable setpoint response is restricted by the RHP zeros only.

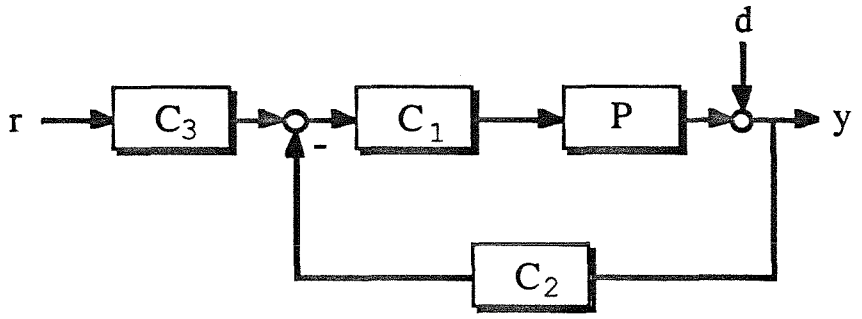


Figure 3. Two-degree-of-freedom feedback structure.

A more rigorous discussion can be found in Vidyasagar (1985).

3. Model Uncertainty.

3.1. Structured Singular Value.

Potential modeling errors, described as uncertainty associated with the process model, can appear in different forms and places in a multivariable model. This fact makes the derivation of non-conservative conditions that guarantee robustness with respect to model-plant mismatch difficult. The Structured Singular Value (SSV), introduced by Doyle (1982), takes into account the structure of the model uncertainty and it allows the non-conservative quantification of the concept of robust performance.

For a constant complex matrix M the definition of the SSV $\mu_{\mathbf{\Delta}}(M)$ depends also on a certain set $\mathbf{\Delta}$. Each element Δ of $\mathbf{\Delta}$ is a block diagonal complex matrix with a specified dimension for each block, i.e.,

$$\mathbf{\Delta} = \left\{ diag(\Delta_1, \Delta_2, ..., \Delta_n) | \Delta_j \in \mathbf{C}^{m_j \times m_j} \right\} \tag{3.1.1}$$

Then

$$\frac{1}{\mu_{\mathbf{\Delta}}(M)} = \min_{\Delta \in \mathbf{\Delta}} \{ \bar{\sigma}(\Delta) | det(I - M\Delta) = 0 \} \tag{3.1.2}$$

and $\mu_{\mathbf{\Delta}}(M) = 0$ if $det(I - M\Delta) \neq 0 \qquad \forall \Delta \in \mathbf{\Delta}$. Note that $\bar{\sigma}$ is the maximum singular value of the corresponding matrix.

Details on how the SSV can be used for studying the robustness of a control system can be found in Doyle (1985), where a discussion of the computational problems is also given. For three or fewer blocks in each element of $\mathbf{\Delta}$, the SSV can be computed from

$$\mu_{\mathbf{\Delta}}(M) = \inf_{D \in \mathbf{D}} \bar{\sigma}(DMD^{-1}) \tag{3.1.3}$$

where

$$\mathbf{D} = \{ diag(d_1 I_{m_1}, d_2 I_{m_2}, ..., d_n I_{m_n}) | d_j \in \mathbf{R}_+ \} \tag{3.1.4}$$

and I_{m_j} is the identity matrix of dimension $m_j \times m_j$. For more than three blocks, (3.1.3) still gives an upper bound for the SSV.

3.2. Block Structure.

In order to effectively use the SSV for designing F, some rearrangement of the block structure is necessary. The IMC structure of Fig. 1a can be written as that of Fig. 4a, where $v = d - r$, $e = y - r$ and

$$G = \begin{pmatrix} 0 & 0 & \tilde{Q} \\ I & I & \tilde{P}\tilde{Q} \\ -I & -I & 0 \end{pmatrix} \tag{3.2.1}$$

where the blocks 0 and I have appropriate dimensions.

The structure in Fig. 4a can always be transformed into that in Fig. 4b, where Δ is a block diagonal matrix with the additional property that

$$\bar{\sigma}(\Delta) \leq 1 \qquad \forall\omega \tag{3.2.2}$$

The superscript u in G^u denotes the dependence of G^u not only on G but also on the specific uncertainty description available for the model $\tilde{P}$. Only some of the more common types will be covered here to demonstrate how this is done, but it is straightforward to apply the same concepts to other types of uncertainty descriptions, like parametric uncertainty.

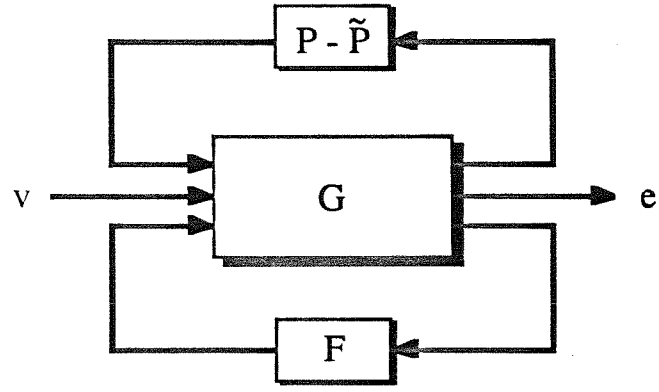
i) Multivariable Additive Uncertainty.

The information on the model uncertainty is of the form

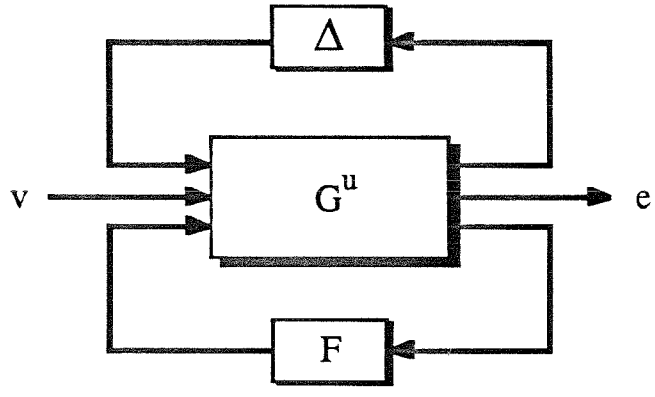
$$\bar{\sigma}(P - \tilde{P}) \leq l_a(\omega) \tag{3.2.3}$$

where l_a is a known function of frequency. In this case we can easily write $P - \tilde{P} = l_a\Delta$ where $\bar{\sigma}(\Delta) \leq 1$ and so obtain

$$G^u = G^a = \begin{pmatrix} l_aI & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} G \tag{3.2.4}$$



(a)



(b)

Figure 4. Model uncertainty block diagrams.

ii) Multivariable Input Multiplicative Uncertainty.

$$\bar{\sigma}(\tilde{P}^{-1}(P-\tilde{P})) \leq l_i(\omega) \tag{3.2.5}$$

where l_i is known. Then

$$G^u = G^i = G \begin{pmatrix} l_i\tilde{P} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \tag{3.2.6}$$

iii) Multivariable Output Multiplicative Uncertainty.

$$\bar{\sigma}((P-\tilde{P})\tilde{P}^{-1}) \leq l_o(\omega) \tag{3.2.7}$$

$$G^u = G^o = \begin{pmatrix} l_o\tilde{P} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} G \tag{3.2.8}$$

iv) Element by Element Additive Uncertainty.

For each element p_{ij} of P we have

$$|p_{ij} - \tilde{p}_{ij}| \leq l_{ij}(\omega), \qquad i = 1,...,n; \quad j = 1,...,n \tag{3.2.9}$$

Then

$$P - \tilde{P} = J_1 \Delta L J_2 \tag{3.2.10}$$

where

$$L = diag(l_{11}, l_{12}, ..., l_{1n}, l_{21}, ..., l_{nn}) \tag{3.2.11}$$

$$J_1 = \begin{pmatrix} 1 & \dots & 1 & 0 & \dots & 0 & \dots & \dots & 0 & \dots & 0 \\ 0 & \dots & 0 & 1 & \dots & 1 & \dots & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 & \dots & \dots & 1 & \dots & 1 \end{pmatrix} \tag{3.2.12}$$

$$J_2 = \begin{pmatrix} I_n \\ I_n \\ \vdots \\ I_n \end{pmatrix} \tag{3.2.13}$$

From (3.2.10) it follows that

$$G^u = G^{ebe} = \begin{pmatrix} LJ_2 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} G \begin{pmatrix} J_1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \tag{3.2.14}$$

Note that all the above relations yield a G^u already partitioned as

$$G^u = \begin{pmatrix} G^u_{11} & G^u_{12} & G^u_{13} \\ G^u_{21} & G^u_{22} & G^u_{23} \\ G^u_{31} & G^u_{32} & G^u_{33} \end{pmatrix} \tag{3.2.15}$$

Then Fig. 4b can be written as Fig. 5 with

$$\begin{aligned} G^F &= \begin{pmatrix} G^u_{11} & G^u_{12} \\ G^u_{21} & G^u_{22} \end{pmatrix} + \begin{pmatrix} G^u_{13} \\ G^u_{23} \end{pmatrix} (I - FG^u_{33})^{-1} F \begin{pmatrix} G^u_{31} & G^u_{32} \end{pmatrix} \\ &\stackrel{\text{def}}{=} \begin{pmatrix} G^F_{11} & G^F_{12} \\ G^F_{21} & G^F_{22} \end{pmatrix} \end{aligned} \tag{3.2.16}$$

3.3. Robust Stability.

We now require that the matrix $IS1$ as given by (1.2.1) is stable for all possible plants P. The design of $\tilde{Q}$ according to Section 2 resulted in a stable $IS1$ for $P = \tilde{P}$. In order for $IS1$ to remain stable we need to satisfy the requirements that as we move in a "continuous" way from the model $\tilde{P}$ to the plant P, no closed-loop poles cross the imaginary axis and no such poles suddenly appear in the RHP. The latter requirement is satisfied if we assume that the model and the plant have the same number of RHP poles. If this is not true, a different sufficient condition is that G^F is a stable matrix and only stable Δ's are possible. The SSV can be used to determine if any crossings of the imaginary axis occur. Then we can say that the system is stable for any of the plants in the set defined from the bounds on the model uncertainty and which have the same number of RHP poles as the model, if and only if (Doyle, 1985)

$$\mu_\Delta(G^F_{11}) < 1 \qquad \forall\omega \tag{3.3.1}$$

3.4. Robust Performance.

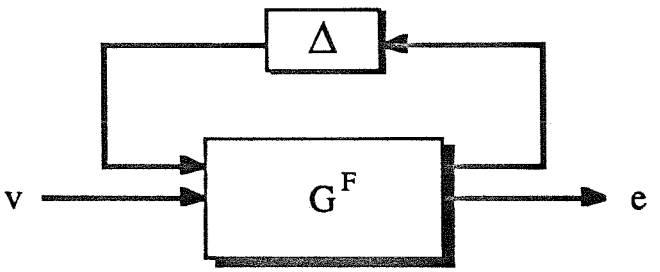


Figure 5. SSV block diagram.

In the first step of the IMC design procedure a controller $\tilde{Q}$ is obtained, which produces satisfactory disturbance rejection and/or setpoint tracking. This response is described by the "sensitivity" function $\tilde{E}$ given by (1.1.3). Since $\tilde{E}$ connects the external inputs to the error e, a well-designed control system produces a relatively "small" $\tilde{E}$. A measure of the magnitude of the known $\tilde{E}$ is its maximum singular value. Let $b(\omega)$ be a frequency function such that

$$\bar{\sigma}(\tilde{E}(i\omega)) < b(\omega) \qquad \forall\omega \tag{3.4.1}$$

When $P \neq \tilde{P}$, the sensitivity function E is described by (1.1.5). Note that $E = \tilde{E}$ when $P = \tilde{P}$. In order for the performance of the control system to remain robust with respect to model-plant mismatch we have to keep E small in spite of the modeling error. Hence we require that

$$\sup_{\omega} \bar{\sigma}(b(\omega)^{-1}E(i\omega)) < 1 \qquad \forall\Delta \in \mathbf{\Delta} \tag{3.4.2}$$

We can now use the properties of the SSV (Doyle,1985) to obtain

$$\sup_{\omega} \bar{\sigma}(b(\omega)^{-1}E(i\omega)) < 1 \qquad \forall\Delta \in \mathbf{\Delta} \iff \sup_{\omega} \mu_{\mathbf{\Delta}^0}(G^b) < 1 \tag{3.4.3}$$

where

$$G^b = \begin{pmatrix} I & 0 \\ 0 & b^{-1} \end{pmatrix} G^F \tag{3.4.4}$$

$$\mathbf{\Delta}^0 = \{diag(\Delta, \Delta^0) | \Delta \in \mathbf{\Delta}, \Delta^0 \in \mathbf{C}^{n\times n}\} \tag{3.4.5}$$

The worst possible ISE that any plant within the uncertainty bounds can produce for a particular input v is given by the following theorem.

Theorem 3.4.1.

For a specified v define

$$G^x \stackrel{\text{def}}{=} \begin{pmatrix} I & 0 \\ 0 & x \end{pmatrix} G^F \begin{pmatrix} I & 0 \\ 0 & v \end{pmatrix} \tag{3.4.6}$$

where x is a scalar function of ω and the blocks 0 have the appropriate dimensions (in general non-square). Augment G^x, which is in general a "tall" matrix, to obtain a square matrix:

$$G^x_{full} = \begin{pmatrix} G^x & 0 \end{pmatrix} \tag{3.4.7}$$

Then

$$\mu_{\Delta^o}(G^x_{full}(i\omega)) = 1 \iff x(\omega) = x_0(\omega) \qquad \forall\omega \tag{3.4.8}$$

defines a function x_0 of frequency and

$$\sup_{\Delta\in\mathbf{\Delta}} ||Ev||_2 = ||x_0^{-1}||_2 \tag{3.4.9}$$

Proof: See Appendix B.1.

Note that as it turned out $x_0^{-1} = \sup_{\Delta\in\mathbf{\Delta}} \bar{\sigma}(Ev)$, but the only way to compute it is through (3.4.8). Also without loss of generality x can be assumed to be positive since the value of $\mu_{\Delta^o}(G^x_{full})$ depends only on $|x|$. The following theorem simplifies the problem of computing x_0.

Theorem 3.4.2.

Let

$$M^x = \begin{pmatrix} M_{11} & M_{12} \\ xM_{21} & xM_{22} \end{pmatrix} \tag{3.4.10}$$

where x a positive scalar.

Then $\inf_{D\in\mathbf{D}} \bar{\sigma}(DM^xD^{-1})$ is a non-decreasing function of x, where $\mathbf{D} = \{diag(D_1, D_2)\}$.

Proof: See Appendix B.2.

Note that G^x_{full} is a special case of the above M and so Theorem 3.4.2 applies to (3.4.8).

4. Step 2: Design of F.

4.1. Filter Structure.

The filter parameters can now be computed so that the robustness conditions that were discussed in Section 3 are satisfied. To do so, some structure will

have to be assumed for F, which can be of any general type that the designer wishes. However in order to keep the number of variables in the optimization problem small, a rather simple structure like a diagonal F with first or second order terms would be recommended. In most cases this is not restrictive because the potentially higher orders of the model $\tilde{P}$ have been included in the controller $\tilde{Q}$ that was designed in the first step of the IMC procedure and which is in general a full matrix. Some additional restrictions on the filter exist in the case of an open-loop unstable plant. Also the use of more complex filter structure may be necessary in cases of highly ill-conditioned systems ($\bar{\sigma}(\tilde{P})/\underline{\sigma}(\tilde{P})$ very large).

i) Open-loop unstable plants.

The IMC filter $F(s)$ is chosen to be a diagonal rational function that satisfies the following requirements.

a. Pole-zero excess. The controller $Q = \tilde{Q}F$ must be proper. Assume that the designer has specified a pole-zero excess of κ for the filter $F(s)$.

b. Internal stability. $IS1$ in (1.2.2) must be stable.

c. Asymptotic setpoint tracking and/or disturbance rejection. $(I-\tilde{P}\tilde{Q}F)v$ must be stable.

Write

$$F(s) = diag(f_1(s), ..., f_n(s)) \tag{4.1.1}$$

Under assumptions A.1,2,3,4,5, (b),(c) are equivalent to the following conditions. Let π_i $(i = 1, k)$ be an open RHP pole of $\tilde{P}$ and $\pi_0 = 0$ and m_{0l} the largest multiplicity of such a pole in any element of the l^{th} row of V. From assumptions A.1, 2, 3, 4, 5 and the fact that $\tilde{Q}$ makes IS1, $(I - P\tilde{Q})V$ stable, it follows that the l^{th} element , f_l of the filter F must satisfy:

$$f_l(\pi_i) = 1, \qquad i = 0, 1, ..., k \tag{4.1.2}$$

$$\frac{d^j}{ds^j} f_l(s)|_{s=\pi_0} = 0, \quad j = 1, ..., m_{0l} - 1 \tag{4.1.3}$$

(4.1.2) clearly shows the limitation that RHP poles place on the robustness properties of a control system designed for an open-loop unstable plant. Since because of (4.1.2) one cannot reduce the nominal ($P = \tilde{P}$) closed-loop bandwidth of the system at frequencies corresponding to the RHP poles of the plant, one can only tolerate a relatively small model error at those frequencies.

Experience has shown that the following structure for a filter element $f_l(s)$ is reasonable:

$$f_l(s) = \frac{a_{\nu_l-1,l}s^{\nu_l-1} + ... + a_{1,l}s + a_{0,l}}{(\lambda s+1)^{\kappa+\nu_l-1}} \tag{4.1.4}$$

where

$$\nu_l = m_{0l} + k \tag{4.1.5}$$

and then compute the numerator coefficients for a specific tuning parameter λ from (4.1.2), (4.1.3). This involves solving a system of ν_l linear equations with ν_l unknowns.

Example 4.1.1. Assume that we have a pole-zero excess of κ and there is only one pole π. Then

$$f(s) = \frac{(\lambda\pi+1)^{\kappa}}{(\lambda s+1)^{\kappa}} \tag{4.1.6}$$

If $\pi = 0$, (4.1.9) reduces to the standard filter for stable systems $f(s) = (\lambda s+1)^{-\kappa}$.

Example 4.1.2. Assume that $\kappa = 2$ and the only pole is a double pole at $s = 0$. Then

$$f(s) = \frac{3\lambda s+1}{(\lambda s+1)^3} \tag{4.1.7}$$

ii) Ill-conditioned plants.

The problems arise because the optimal controller $\tilde{Q}$ designed for $\tilde{P}$ tends to be an approximate inverse of $\tilde{P}$ and as a result $\tilde{Q}$ is ill-conditioned as well. The result is that although robust stability is achieved through significant detuning of the diagonal filter, the robust performance condition is usually not satisfied. For example, Skogestad et al. (1986) have shown that an ill-conditioned P or C

(and therefore an inverting ill-conditioned $\tilde{Q}$ as well, because of (1.1.2)) can cause problems when input uncertainty is present. A way to address this problem is to try to use a filter that acts directly on the singular values of $\tilde{Q}$, at the frequency where the condition number of $\tilde{Q}$ is highest, say ω^*. Let

$$\tilde{Q}(i\omega^*) = U_Q \Sigma_Q V_Q^* \tag{4.1.8}$$

be the SVD of $\tilde{Q}$ at ω^* and let R_u, R_v, be real matrices that solve the pseudo-diagonalization problems:

$$U_Q^* R_u \approx I \tag{4.1.9}$$

$$V_Q^* R_v \approx I \tag{4.1.10}$$

Then for the IMC controller Q that includes the filter, use the expression

$$Q(s) = R_u F_1(s) R_u^{-1} \tilde{Q}(s) F_2(s) \tag{4.1.11}$$

or

$$Q(s) = \tilde{Q}(s) R_v F_1(s) R_v^{-1} F_2(s) \tag{4.1.12}$$

where $F_1(s)$, $F_2(s)$ are diagonal filters, such that $F_1(0) = F_2(0) = I$ if integral action is desired. Note for F_1, m_0 should be used in (4.1.3), (4.1.5), for all l, instead of m_{0l}, where $m_0 = \max_l m_{0l}$.

It should be pointed out that the success of this approach depends on how good any of the pseudo-diagonalizations (4.1.9) or (4.1.10) is. The diagonalization will be perfect if U_Q or V_Q is real. This will happen if $\omega^* = 0$, which is the case when the problems arise because the plant is ill-conditioned at steady-state, as for example high purity distillation columns are.

One can put this control structure in the form of Fig. 4, as follows. Define

$$F(s) = diag(F_1(s), F_2(s)) \tag{4.1.13}$$

$$\tilde{Q}_A(s) = R_u \qquad or \qquad \tilde{Q}(s)R_v \tag{4.1.14}$$

$$A(s) = R_u^{-1}\tilde{Q}(s) \qquad or \qquad R_v^{-1} \tag{4.1.15}$$

depending on whether (4.1.11) or (4.1.12) is used. Obtain G^u by substituting $\tilde{Q}$ with $\tilde{Q}_A$ in (3.2.1). Then in Fig. 4 use instead of G^u, $G^{u,ill}$, where

$$G^{u,ill} = \begin{pmatrix} G^u_{11} & G^u_{12} & G^u_{13} & 0 \\ G^u_{21} & G^u_{22} & G^u_{23} & 0 \\ 0 & 0 & 0 & A \\ G^u_{31} & G^u_{32} & G^u_{33} & 0 \end{pmatrix} \tag{4.1.16}$$

4.2. Objective

We can write

$$F \stackrel{\text{def}}{=} F(s;\Lambda) \tag{4.2.1}$$

where Λ is an array with the filter parameters.

The problem can now be formulated as a minimization problem over the elements of the array Λ. A constraint is that the part of Λ corresponding to denominator time constants should be such that F is a stable transfer function. However the problem can be turned into an unconstrained one by writing the denominator of each element of F as a product of polynomials of degree 2 and one of degree 1 if the order is odd, with the constant terms of the polynomials equal to 1. Then the stability requirement translates into the requirement that the coefficients (elements of Λ) are positive, which is a constraint that can be eliminated by writing λ_k^2 or $|\lambda_k|$ instead of λ_k for the corresponding filter parameters.

Our goal is to satisfy (3.4.3). The filter parameters can be obtained by solving

$$\min_{\Lambda} \sup_{\omega} \mu_{\Delta^0}(G^b) \tag{O4}$$

It may be however that the optimum values for (O4), still do not manage to satisfy (3.4.3). The reason may be that an F with more parameters is required, but more often that the performance requirements set by the selection of $b(\omega)$ in (3.4.1)

are too tight to satisfy in the presence of model-plant mismatch. In this case one should choose a less tight bound b and resolve (O4). Note that satisfaction of the Robust Performance condition (3.4.3) implies satisfaction of the Robust Stability condition (3.3.1) as well.

A different objective can be set in the case where the ISE for a particular external input direction v is of special interest to the designer. The objective is then to minimize (3.4.9) for a specified v (set-point or disturbance). Hence the filter parameters are obtained by solving

$$\min_{\Lambda} ||x_0^{-1}||_2 \qquad (O5)$$

It should be pointed that contrary to the problems addressed in Section 2, where a minimization for a set of v's could be carried out, (O5) cannot be solved for a set of v's. The reason is the presence of modeling error in the problem definition.

Finally one should note that in both (O4) and (O5), the objective functions are not convex. Hence a local minimum could be the result of solving (O4) or (O5). Use of a number of different initial values for Λ can help circumvert this problem. Also, good initial guesses can usually be obtained for the filter parameters (elements of Λ) by mathcing them with the frequencies where the peaks of $\mu_{\Delta^0}(G^b)$ appear for $F = I$.

4.3. Computational Issues.

i) Objective (O4).

The computation of μ in (O4) is made through (3.1.3); details can be found in Doyle (1982). As it was pointed out in Doyle (1985), the minimization of the Frobenius norm instead of the maximum singular value yields D's which are very close to the optimal ones for (3.1.3). Note that the minimization of the Frobenius norm is a very simple task. In the computation of the supremum in (O4) only a

finite number of frequencies is considered. Hence (O4) is transformed into

$$\min_{\Lambda} \max_{\omega \in \Omega} \inf_{D \in \mathbf{D}^0} \bar{\sigma}(DG^bD^{-1}) \qquad (O4')$$

where Ω is a set containing a finite number of frequencies and $\mathbf{D}^0$ is the set corresponding to Δ^0 according to (3.1.1) and (3.1.4). Define

$$\Phi_\infty(\Lambda) \stackrel{\text{def}}{=} \max_{\omega \in \Omega} \inf_{D \in \mathbf{D}^0} \bar{\sigma}(DG^bD^{-1}) \qquad (4.3.1)$$

The analytic computation of the gradient of Φ_∞ with respect to Λ is in general possible. This is not the case when the two or more largest singular values of DG^bD^{-1} are equal. However this is quite uncommon and although the computation of a generalized gradient is possible, experience has shown the use of a mean direction to be satisfactory. A similar problem appears when the $\max_{\omega\in\Omega}$ is attained at more than one frequencies, but again the use of a mean direction seems to be sufficient. We shall now proceed to obtain the expression for the gradient of $\Phi_\infty(\Lambda)$ in the general case.

Assume that for the value of Λ where the gradient of $\Phi_\infty(\Lambda)$ is computed, the $\max_{\omega\in\Omega}$ is attained at $\omega = \omega_0$ and that the $\inf_{D\in\mathbf{D}^0} \bar{\sigma}(DG^b(i\omega_0)D^{-1})$ is obtained at $D = D_0$, where only one singular value σ_1 is equal to $\bar{\sigma}$. Let the singular value decomposition (SVD) be

$$D_0G^b(i\omega_0)D_0^{-1} = \begin{pmatrix} u_1 & U \end{pmatrix} \begin{pmatrix} \sigma_1 & 0 \\ 0 & \Sigma \end{pmatrix} \begin{pmatrix} v_1^* \\ V^* \end{pmatrix} \qquad (4.3.2)$$

Then for the element of the gradient vector corresponding to the filter parameter λ_k we have under the above assumptions:

$$\frac{\partial}{\partial \lambda_k}\Phi_\infty = \frac{\partial}{\partial \lambda_k}\sigma_1(D_0G^b(i\omega_0)D_0^{-1}) \qquad (4.3.3)$$

because $\nabla_{D_0}(\sigma_1) = 0$ since we are at an optimum with respect to the D's. To simplify the notation use

$$A = D_0G^b(i\omega_0)D_0^{-1} = U_A\Sigma_AV_A^* \qquad (4.3.4)$$

By using the properties of the SVD we obtain from (4.3.2)

$$AA^* = U_A\Sigma_A^2U_A^* \Rightarrow u_1^*\frac{\partial}{\partial\lambda_k}(AA^*)u_1 = u_1^*U_A\frac{\partial}{\partial\lambda_k}(\Sigma_A^2)U_A^*u_1$$

$$\Rightarrow u_1^*(\frac{\partial}{\partial\lambda_k}(A)A^* + A\frac{\partial}{\partial\lambda_k}(A^*))u_1 = u_1^*U_A(2\Sigma_A\frac{\partial}{\partial\lambda_k}(\Sigma_A))U_A^*u_1$$

$$\Rightarrow u_1^*\frac{\partial}{\partial\lambda_k}(A)v_1\sigma_1 + \sigma_1v_1^*\frac{\partial}{\partial\lambda_k}(A^*)u_1 = 2\sigma_1\frac{\partial}{\partial\lambda_k}(\sigma_1)$$

$$\Rightarrow \frac{\partial}{\partial\lambda_k}(\sigma_1) = \mathrm{Re}\left[u_1^*\frac{\partial}{\partial\lambda_k}(D_0G^b(i\omega_0)D_0^{-1})v_1\right] \tag{4.3.5}$$

Use of (4.3.3), (3.2.16), (3.4.4), (4.3.5), and of the property

$$\frac{d}{dz}(M(z)^{-1}) = -M(z)^{-1}\frac{d}{dz}(M(z))M(z)^{-1} \tag{4.3.6}$$

where M(z) is a matrix, yields after some algebra

$$\frac{\partial}{\partial\lambda_k}\Phi_\infty = \mathrm{Re}\Bigg[u_1^*D_0\begin{pmatrix} G_{13}^u \\ b^{-1}wG_{23}^u \end{pmatrix}(I - FG_{33}^u)^{-1}\frac{\partial}{\partial\lambda_k}(F(i\omega_0))$$
$$(I - FG_{33}^u)^{-1}\begin{pmatrix} G_{31}^u & G_{32}^u \end{pmatrix}D_0^{-1}v_1\Bigg] \tag{4.3.7}$$

where F, G_{ij}^u, b, w are computed at $\omega = \omega_0$. The derivatives of F with respect to its parameters (elements of A) depend on the particular form that the designer selected and they can be easily computed.

ii) Solution of (O5).

The first issue in this case is the computation of x_0. Note that this computation has to be made at every frequency ω. In practice only a set Ω with a finite number of frequencies is used, from which $||x_0^{-1}||_2$ can be computed approximately. Theorem 3.4.2 indicates that any basic descent method should be sufficient. The fact that it is possible to obtain an analytic expression for the gradient of $\mu_{\Delta^0}(G_{full}^x(i\omega))$ with respect to x, simplifies the problem even further. This is possible when (3.1.3) is used for the computation of μ and the two largest singular values of $DG_{full}^xD^{-1}$ for the optimal D's at the value of x where the

gradient is computed, are not equal to each other. If this not the case a mean direction can be used as mentioned in the H_∞ case above.

Let the $\inf_{D\in\mathbf{D}^o} \bar{\sigma}(DG^x_{full}(i\omega)D^{-1})$ be attained for $D_0 = D_0(\omega;x)$ and let σ_1 be the maximum singular value and u_1, v_1 the corresponding singular vectors. Then the same steps for obtaining (4.3.5) are valid. Hence by using (3.4.6) and (3.4.7) we get after some algebra

$$\frac{\partial}{\partial x}\left(\mu_{\Delta^o}(G^x_{full}(i\omega))\right) = \mathrm{Re}\left[u_1^* D_0 \begin{pmatrix} 0 & 0 & 0 \\ WG^F_{21} & WG^F_{22}v & 0 \end{pmatrix} D_0^{-1} v_1\right] \tag{4.3.8}$$

The second computational issue is the solution of (O5). To obtain the gradient of $||x_0^{-1}||_2$ with respect to the filter parameters, we need to compute the gradient of $x_0(\omega)$ with respect to these parameters for every frequency $\omega \in \Omega$. From the definition of x_0 in (3.4.8) we see that as some filter parameter λ_k changes, $x_0(\omega)$ will also change so that $\mu_{\Delta^o}(G^x_{full}(i\omega))$ remains constantly equal to 1. Hence we can write

$$\frac{\partial \mu}{\partial x_0}\ \frac{\partial x_0}{\partial \lambda_k} + \frac{\partial \mu}{\partial \lambda_k} = 0 \Longrightarrow \quad \frac{\partial x_0}{\partial \lambda_k} = -\frac{\partial \mu}{\partial \lambda_k} \Big/ \frac{\partial \mu}{x_0} \tag{4.3.9}$$

where μ is computed through (3.1.3). The denominator in the right hand side of (4.3.9) is given from (4.3.8). As for the numerator, it can be computed in the same way as (4.3.5) and (4.3.7) but with G^x_{full} instead of G^b:

$$\frac{\partial}{\partial \lambda_k}\left(\mu_{\Delta^o}(G^x_{full}(i\omega))\right) = \mathrm{Re}\left[u_1^* D_0 \begin{pmatrix} G^u_{13} \\ xWG^u_{23} \end{pmatrix} (I - FG^u_{33})^{-1} \frac{\partial}{\partial \lambda_k}(F(i\omega))(I - FG^u_{33})^{-1} \begin{pmatrix} G^u_{31} & G^u_{32}v & 0 \end{pmatrix} D_0^{-1} v_1\right] \tag{4.3.10}$$

Hence $\partial x_0/\partial \lambda_k$ can be computed from (4.3.8), (4.3.9), (4.3.10).

5. Illustration.

The design of a robust IMC controller will be demonstrated for a 2×2 high purity distillation column. Skogestad and Morari (1986) suggested that a very

simple model can be used to underline the control problems inherent in the LV-configuration. For the particular example they examined, the following model was used:

$$\tilde{P}(s) = \frac{1}{75s+1}\begin{pmatrix} 0.878 & -0.864 \\ 1.082 & -1.096 \end{pmatrix} \tag{5.0.1}$$

The problems arise from the fact that high purity distillation columns are ill-conditioned at steady-state. For the case in (5.0.1) the condition number is equal to 142. Hence any controller Q based on the inverse of $\tilde{P}$ will also be ill-conditioned and this might result in an unrobust control system.

The $\tilde{P}$ above is MP and so

$$\tilde{Q}(s) = \tilde{P}(s)^{-1} \tag{5.0.2}$$

The IMC filter has now to be designed for robustness. To do so we need to have some uncertainty description as explained in Section 3.2. Input uncertainty will assumed in this case. This type of uncertainty is defined in Section 3.2.ii. Skogestad and Morari (1986) proposed the following uncertainty bound l_i and performance bound b (used in (3.4.2)):

$$l_i(s) = 0.2\frac{5s+1}{0.5s+1} \tag{5.0.3}$$

$$b(s) = \frac{20s}{10s+1} \tag{5.0.4}$$

First contrary to Section 4.1.ii, a simple diagonal filter will be used to demonstrate the problem. A gradient search procedure based on the analytic gradient expressions in Section 4.3, is used to solve (O4') for a one parameter Λ, i.e., a scalar times identity filter is employed. The result is

$$F(s) = \frac{1}{7.28s+1}I \tag{5.0.5}$$

Plots of μ for robust stability (given by (3.3.1)) and robust performance (given by (3.4.3)) are shown in Fig. 6a. Clearly, although the system is guaranteed to

remain stable in the presence of modeling error (within the bound in (5.0.3)), the performance is expected to deteriorate. This is confirmed by the simulations shown in Fig. 7. A step setpoint change in output 1 is made:

$$v(s) = r(s) = \begin{pmatrix} 1/s \\ 0 \end{pmatrix} \tag{5.0.6}$$

Note that the actual magnitude of the change in not important since the system is linear. The value 1 is used to facilitate the comparisons. For the nominal case ($P = \tilde{P}$), the outputs are decoupled and the performance is acceptable (Fig. 7a). However in the case where

$$P(s) = \tilde{P}(s) \begin{pmatrix} 1.2 & 0 \\ 0 & 0.8 \end{pmatrix} \tag{5.0.7}$$

the performance deteriorates to the point where it is totally unacceptable (Fig. 7b). Note that the plant in (5.0.7) includes a 20% error in each plant input and it is within the bound in (5.0.3). The same plant is used in all other simulations in this section when model-plant mismatch is assumed. Filters with different elements and higher orders and zeros were also used in the optimization but the results showed that these filters produce no worthwhile difference for the one-filter controller. The reason is that one diagonal filter cannot in general significantly affect the condition number of $\tilde{Q}$.

We shall now proceed and use the filter structure suggested for ill-conditioned systems in Section 4.1.ii. In this case $\omega^* = 0$ and therefore the diagonilizations (4.1.9) and (4.1.10) are exact. Hence (4.1.11) and (4.1.12) produce the same Q. Objective (O4′) was solved with a gradient search method using the analytic gradient expressions of Section 4.3. Different filter orders were used and a few different initial guesses were tried to avoid local minima. The final result for filters with two parameters in each element, was:

$$F_1(s) = \begin{pmatrix} \frac{0.244s+1}{(0.184s+1)^2} & 0 \\ 0 & \frac{0.00284s+1}{(8.72s+1)^2} \end{pmatrix} \tag{5.0.8}$$

$$F_2(s) = \begin{pmatrix} \frac{0.164s+1}{(0.446s+1)^2} & 0 \\ 0 & \frac{0.213s+1}{(0.476s+1)^2} \end{pmatrix} \qquad (5.0.9)$$

The values of μ for robust stability and performance are shown in Fig. 6b. The clear improvement over the diagonal filter is verified by the simulations in Fig. 8. It is interesting to note that the responses in Figs. 7 and 8 are similar for the nominal case. Hence, as expected, the increase in robustness was not the result of additional detuning, which is something that if it was sufficient it would have been accomplished by the minimization problem solved for the diagonal filter. The reason for the improvement is that the two-filter structure acted directly on the singular values of $\tilde{Q}$ and, in addition to appropriate detuning, it also reduced its condition number at the critical frequency range.

Finally, a last comparison will be made between the performance obtained by the two-filter IMC controller and the "true" $\mu-$optimum controller, defined as the result of minimizing $\sup_\omega \mu_{\Delta^o}(G^b)$ over not a specified filter structure but over any stabilizing controller Q or C. However, the iterative approach suggested by Doyle (1985) for solving this problem is not guaranteed to converge and indeed, it has often failed to converge. For this particular example though, Skogestad and Morari (1986) obtained this $\mu-$optimal controller. The values of μ for robust performance and stability are shown if Fig. 6c for the same bounds as in (5.0.3) and (5.0.4). Clearly the difference is not significant and this is verified by the simulations shown in Fig. 9.

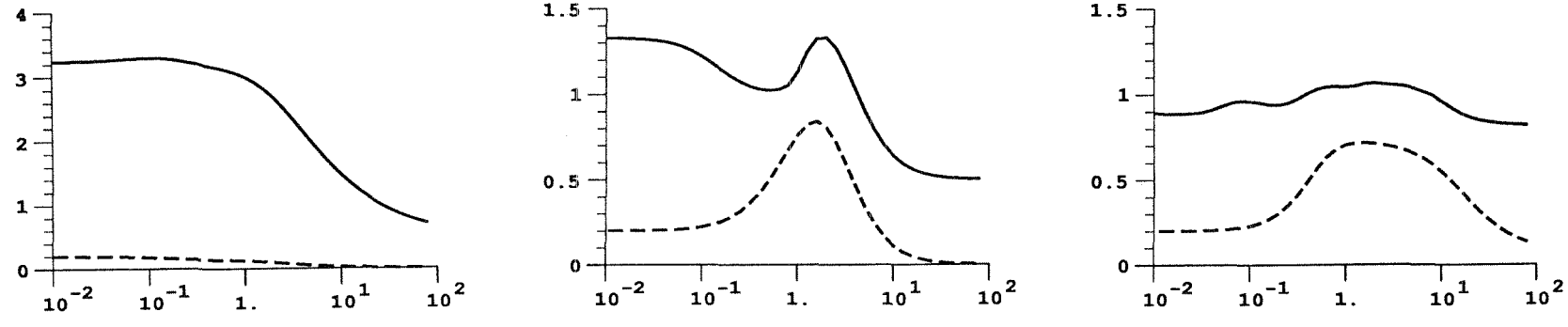
6. Concluding Remarks.

The work presented in this paper extends the two-step IMC design procedure to open-loop unstable systems. It proposes a meaningfull way for the selection of the "weight" in the H_2-minimization problem solved in the first step of the synthesis procedure, through Theorem 2.4.2 and Corollary 2.4.1. The method quantifies the problem of the design of the IMC filter via the use of the Structured

Singular Value and provides analytic expressions for the gradients of the objective functions. The special filter structures needed for open-loop unstable plants and ill-conditioned plants are given. The promise of the approach as a practical way for the design of robust multivariable controllers is demonstrated by the high purity distillation column example examined in the paper.

Figure 6. Plots of $\mu(\omega)$.

Solid line: Robust Performance. Dashed line: Robust stability.



(a) (b) (c)

(a) One-filter IMC controller.

(b) Two-filter IMC controller.

(c) μ-optimal controller.

- Dashed lines: Setpoints; Solid lines: Outputs.

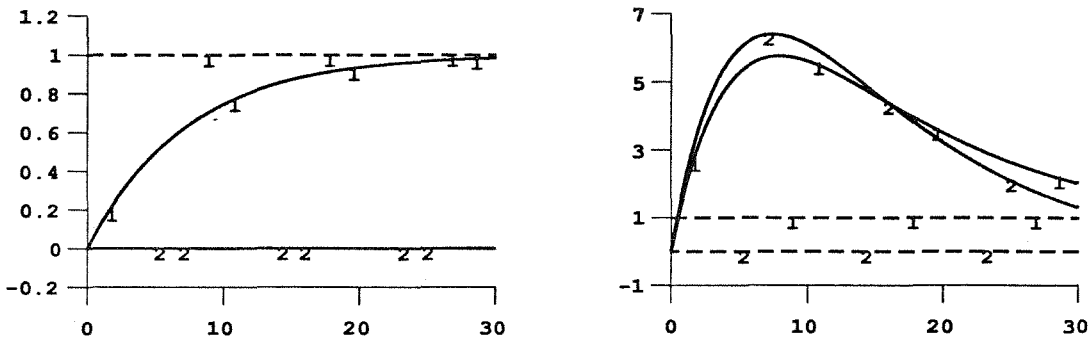


Figure 7. Time response for the one-filter IMC controller.

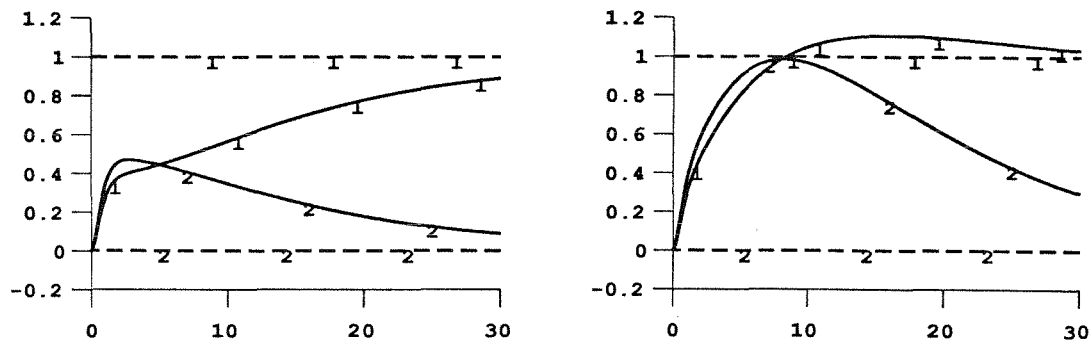


Figure 8. Time response for the two-filter IMC controller.

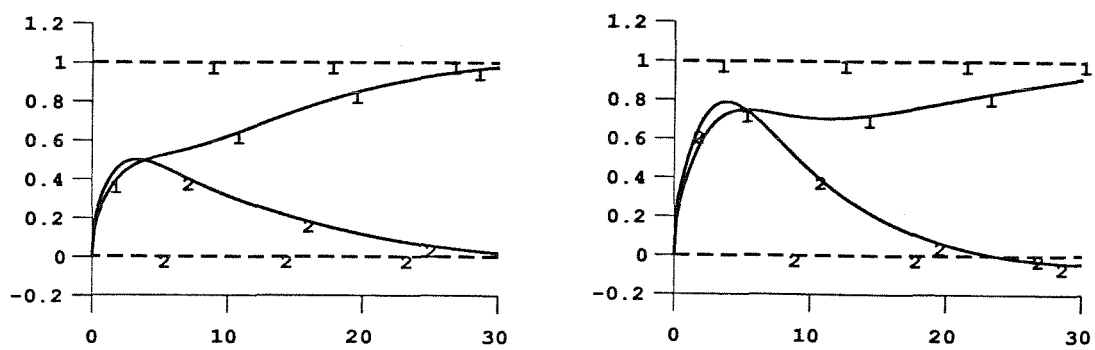


Figure 9. Time response for the μ-optimal controller.

(a) (b)

(a) $P = \tilde{P}$.

(b) $P \neq \tilde{P}$.

Appendix A

A.1. Proof of Theorem 2.2.1.

i) We shall show that any Q given by (2.2.2) makes $IS1$ stable. From substitution of (2.2.2) into (1.2.2) it follows that all that is required is that $(Pb_p^2Q_1 \quad b_p^2Q_1P \quad Pb_p^2Q_1P)$ be stable. From the properties of Q_1, it follows that the third element in the above matrix is stable. Stability of the other two follows by pre- and post-multiplication of that element by P^{-1}, since according to assumptions A.1, A.2, P has no zeros at the location of its unstable poles and these are the only possible unstable poles in the above matrix.

ii) Assume that Q makes $IS1$ stable. Then the difference matrix

$$IS1(Q) - IS1(Q_0) = ((Q-Q_0) \quad P(Q-Q_0) \quad (Q-Q_0)P \quad P(Q-Q_0)P) \tag{A.1.1}$$

is stable. The fact that P has no zeros at the location of the unstable poles makes the stability of the matrix in (A.1.1) equivalent to the stability of $(Q - Q_0)$, $P(Q - Q_0)P$. Then, when assumption A.1.c holds, we can write $P = b_p\hat{P}$, where $\hat{P}$ has no zeros at the unstable poles of P and its only unstable poles are at $s = 0$. So, it follows that $(Q - Q_0) = b_p^2Q_1$ with Q_1 stable and such that PQ_1P have no poles at $s = 0$. If A.1.c does not hold, Q_1 should also have the property that it makes PQ_1P stable.

A.2. Proof of Theorem 2.3.1.

We shall assume that a Q_0 exists, which in addition to the properties mentioned in Theorem 2.2.1, it also produces a matrix $(I - PQ_0)V^0$ with no poles at $s = 0$, where V^0 is a diagonal matrix with l_v poles at $s = 0$ in every element, with l_v the maximum number of such poles in any element of v. If assumption A.1.c does not hold, then each column of V^0 also satisfies assumption A.3.b and Q_0 makes $(I - PQ_0)V^0$ stable. Its existence will be proven by construction. Substitution of (2.2.2) into (2.1.2) and use of the fact that pre- or post-multiplication

of a function with an allpass does not change its L_2-norm, yields:

$$\Phi(v) = ||b_p^{-1} b_v P_A^{-1}(I - PQ_0)\hat{v} - b_p b_v P_M Q_1 \hat{v}||_2^2$$
$$\stackrel{\text{def}}{=} ||f_1 - f_2 Q_1 \hat{v}||_2^2 \qquad (A.2.1)$$

L_2, the space of functions square integrable on the imaginary axis, can be decomposed into two subspaces, H_2 the subspace of functions with analytic continuations in the LHP (strictly unstable functions) and its orthogonal complement $H_2^\perp$ that includes all stable functions. Then, if f_1 is an L_2 function, it can be uniquely decomposed into two orthogonal functions $\{f_1\}_+ \in H_2$ and $\{f_1\}_- \in H_2^\perp$:

$$f_1 = \{f_1\}_+ + \{f_1\}_- \qquad (A.2.2)$$

From (A.2.1) one can see that if improper Q's are allowed, then f_1 may not be an L_2 function. However, in order for $\Phi(v)$ to be finite, the optimal Q_1 has to make $f_1 - f_2 Q_1 \hat{v}$ strictly proper. The assumption will be made that is the case and it will be verified at the solution has this property. Hence to proceed we shall use the convention that when a decomposition as in (A.2.2) of a function is obtained through a partial fraction expansion, all improper and the constant terms are included in $\{.\}_-$.

When A.1.c holds, inspection of (A.2.1) shows that $f_2 Q_1 \hat{v}$ can have no poles in the closed RHP except possibly for some poles at $s = 0$ introduced by $\hat{v}$. f_1 however has no poles at $s = 0$ because $(I - PQ_0)V^0$ has no such poles. Hence for $\Phi(v)$ to be finite, $f_2 Q_1 \hat{v}$ should have no poles at $s = 0$. Hence the optimal Q_1 has to be such that these poles are cancelled. When A.1.c does not hold, then the fact that $(I - PQ_0)V^0$ is stable and A.1.b imply that an acceptable Q_1 and therefore the optimal Q_1 is such that $f_2 Q_1 v$ is stable. We shall assume that Q_1 has this property. It should be verified at the end however that the solution indeed has the property. We can then write

$$\Phi(v) = ||\{f_1\}_+||_2^2 + ||\{f_1\}_- - f_2 Q_1 \hat{v}||_2^2 \qquad (A.2.3)$$

The first term in the right hand side of (A.2.3) does not depend on Q_1. Hence for solving (O1) we only have to look at the second term. The obvious solution is

$$Q_1\hat{v} = f_2^{-1}\{f_1\}_- \qquad (A.2.4)$$

Clearly such a Q_1 produces a stable $f_2Q_1\hat{v}$ as it was assumed. Also $f_1 - f_2Q_1\hat{v} = \{f_1\}_+$, which has no improper or constant terms. It should now be proved that Q_1's that satisfy the internal stability requirements exist among those described by (A.2.4) so that the obvious solution is a true solution. For $n = 1$, (A.2.4) yields a unique Q_1, which can be shown to satisfy the requirements by following the arguments in the Proof of Theorem 2.4.1 in Appendix A.3. For $n \geq 2$ write

$$Q_1 \stackrel{\text{def}}{=} (\, q_1 \quad q_2 \,) \qquad (A.2.5)$$

$$\hat{V}_2 \stackrel{\text{def}}{=} (\, \hat{v}_2 \quad \dots \quad \hat{v}_n \,)^T \qquad (A.2.6)$$

where without loss of generality the first element of v is assumed to be nonzero. Also q_1 is $n \times 1$ and q_2 is $n \times (n-1)$. Then from (A.2.4) it follows that

$$Q_1 = (\, \hat{v}_1^{-1}(f_2^{-1}\{f_1\}_- - q_2\hat{V}_2) \quad q_2 \,) \qquad (A.2.7)$$

We now need to show that a stable q_2 exists such that Q_1 is stable and produces a PQ_1P with no poles at $s = 0$ (or in the closed RHP, when A.1.c does not hold). Select a q_2 of the form:

$$q_2(s) = \hat{q}_2(s)s^{3l_v}\prod_{i=1}^{k}(s - \pi_i)^3 \qquad (A.2.8)$$

where $\hat{q}_2$ is stable. Then from (A.2.7) it follows that in order for PQ_1P not to have any poles at $s = 0$ it is sufficient that $P\hat{v}_1^{-1}f_2^{-1}\{f_1\}_-\{P\}_{1^{st}row}$ have no such poles. This holds because the poles in the P on the left cancel with the P_M^{-1} in f_2^{-1} and v_1 (and so $\hat{v}_1$ as well) has by assumption A.4 at least as many poles

at $s = 0$ as the 1st row of P. When A.1.c does not hold, then the same type of argument and the fact that A.3.b holds imply that PQ_1P has no poles in the open RHP either. Let us now examine the stability of Q_1. The only poles in the open RHP may come from $\hat{v}_1^{-1}$. Let α be such a pole (zero of v_1). Then for stability we need to find $\hat{q}_2$ such that

$$\hat{q}_2(\alpha)\hat{V}_2(\alpha) = \alpha^{-3l_v}\prod_{i=1}^{k}(\alpha - \pi_i)^{-3} f_2^{-1}(\alpha)\{f_1\}_-(\alpha) \qquad (A.2.9)$$

The above equation always has a solution because the vector $\hat{V}_2(\alpha)$ is not identically zero since any common RHP zeros in v were factored out in v_0.

We shall now proceed to obtain an expression for $Q\hat{v}$. (2.2.2) and (A.2.7) yield

$$\begin{aligned} Q\hat{v} &= b_p b_v^{-1} P_M^{-1}\left[b_p^{-1} b_v P_A^{-1} P Q_0 \hat{v} - \{b_p^{-1} b_v P_A^{-1} P Q_0 \hat{v}\}_- + \{b_p^{-1} b_v P_A^{-1}\hat{v}\}_-\right] \\ &= b_p b_v^{-1} P_M^{-1}\left[\{b_p^{-1} b_v P_A^{-1} P Q_0 \hat{v}\}_{0+} + \{b_p^{-1} b_v P_A^{-1}\hat{v}\}_-\right] \qquad (A.2.10) \end{aligned}$$

where $\{.\}_{0+}$ indicates that in the partial fraction expansion all poles in the closed RHP are retained. For (A.2.10), these poles are the poles of $b_p^{-1} b_v \hat{v}$ in the closed RHP; $P_A^{-1} P Q_0 = P_M Q_0$ is strictly stable because Q_0 is a stabilizing controller. When A.1.c holds, the stability of $(I - PQ_0)P$ and the fact that the residues of P at the open RHP poles are full rank imply that at these poles $I - PQ_0 = 0$. Also the fact that $(I - PQ_0)V^0$ has no poles at $s = 0$ implies that $(I - PQ_0)$ and its derivatives up to and including the $(l_v - 1)^{th}$ are also equal to zero at $s = 0$. When A.1.c does not hold, the fact that $(I - PQ_0)V^0$ is stable and that the columns of this diagonal V^0 satisfy A.3.b, imply that $I - PQ_0 = 0$ at $0, \pi_1, ..., \pi_k$. Hence (A.2.10) simplifies to (2.3.4).

We now need to show that a stabilizing controller Q_0 with the property that $(I - PQ_0)V^0$ is stable exists. The selection of a V^0 with the properties mentioned in the beginning of this section and its use instead of V in (2.4.3) yields such a controller.

A.3. Proof of Theorem 2.4.1.

The L_2-norm for a matrix $G(s)$ analytic on the imaginary axis is given by

$$||G||_2 = \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} trace[G^*(i\omega)G(i\omega)] \quad d\omega\right)^{1/2} \qquad (A.3.1)$$

Then from (2.1.2), (A.3.1) it follows that

$$\Phi(v^1) + \Phi(v^2) + \ldots + \Phi(v^n) = ||(I - P\tilde{Q})V||_2^2 \stackrel{\text{def}}{=} \Phi(V) \qquad (A.3.2)$$

The minimization of $\Phi(V)$ follows the steps in the proof of Theorem 2.3.1 in Appendix A.2 up to (A.2.4), with V_M used instead of $\hat{v}$. In this case l_v is the maximum number of poles at $s = 0$ in any element of V. From the equivalent to (A.2.4) equation we obtain

$$Q_1 = f_2^{-1}\{f_1\}_- V_M^{-1} \qquad (A.3.3)$$

where V_M is used instead of $\hat{v}$ in f_1. We now have to establish that Q_1 is stable and produces a PQ_1P with no poles at $s = 0$ (or in the closed RHP, when A.1.c does not hold). In the case where b_p, b_v are not equal to identity, the stability of Q_1 follows from the full rank conditions in A.1.c and A.5.b. In PQ_1P the poles at $s = 0$ of the P on the left cancel with the P_M^{-1} in f_2^{-1}. As for the P on the right, the same follows from assumption A.5.a. When A.1.c does not hold, the same arguments are true for the open RHP poles as well. Then in the same way that (2.3.4) follows from (A.2.7), (2.4.3) follows from (A.3.3).

A.4. Proof of Theorem 2.4.2.

A stabilizing controller that solves (O2) has to solve (O1) for all v^i, $i = 1,\ldots,n$. Satisfying (2.3.4) for every v^i is equivalent to

$$\tilde{Q} = b_p b_v^{-1} P_M^{-1}\{b_p^{-1} b_v P_A^{-1}\hat{V}\}_* \hat{V}^{-1} \qquad (A.4.1)$$

Hence the above $\tilde{Q}$ is the only potential solution to (O2). However it is not necessary a stabilizing controller since not only stabilizing $\tilde{Q}$'s satisfy (2.3.4) for

some v. Indeed if $\hat{V}$ is non-minimum phase, $\hat{V}^{-1}$ is unstable and this results in an unstable $\tilde{Q}$, which is therefore unacceptable. Hence in such a case, there exists no solution to (O2), which completes the proof of part (i) of the theorem.

In the case where $\hat{V}^{-1}$ is stable ($\hat{V}$ minimum phase), the controller given by (A.4.1) is stable and therefore it is the same as the one given by (2.4.3). This fact can be explained as follows. We have

$$V = \hat{V} V_0 \tag{A.4.2}$$

where

$$V_0 = diag(v_0^1, v_0^2, \ldots, v_0^n) \tag{A.4.3}$$

Since $\hat{V}^{-1}$ is stable, (A.4.2) represents a factorization of V similar to that in (2.4.2). From spectral factorization theory it follows that

$$\hat{V}(s) = V_M(s) A \tag{A.4.4}$$

where A is a constant matrix, such that $AA^* = I$. Then from (2.4.3) it follows that the use of $\hat{V}$ instead of V_M does not alter $\tilde{Q}$ because A cancels.

Let us now assume without loss of generality that the first j v^i's have the same v_0^i's. Consider a v that is a linear combination of $v^1, \ldots, v^j$:

$$v(s) = \alpha_1 v^1(s) + \ldots + \alpha_j v^j(s) \tag{A.4.5}$$

Then it follows that

$$v_0(s) = v_0^1(s) = \ldots = v_0^j(s) \tag{A.4.6}$$

$$\hat{v}(s) = \alpha_1 \hat{v}^1(s) + \ldots + \alpha_j \hat{v}^j(s) \tag{A.4.7}$$

One can easily check that a $\tilde{Q}$ that satisfies (2.3.4) for $\hat{v}^1, \ldots, \hat{v}^j$, will also satisfy (2.3.4) for the $\hat{v}$ given by (A.4.7) because of the property

$$\{\alpha_1 f_1(s) + \ldots + \alpha_j f_j(s)\}_* = \alpha_1 \{f_1(s)\}_* + \ldots + \alpha_j \{f_j(s)\}_* \tag{A.4.8}$$

But then from Theorem 2.3.1 it follows that if a stabilizing controller $\tilde{Q}$ satisfies (2.3.4) for $\hat{v}$, then it minimizes the ISE $\Phi(v)$.

Appendix B

B.1. Proof of Theorem 3.4.1.

For a matrix K partitioned as

$$K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \qquad (B.1.1)$$

define

$$R(K,\Delta) \stackrel{\text{def}}{=} K_{22} + K_{21}\Delta(I - K_{11}\Delta)^{-1}K_{12} \qquad (B.1.2)$$

Then the transfer function relating v to e in Fig. 5 is $R(G^F,\Delta)$ and since Fig. 1a and Fig. 5 are equivalent, we get by using (1.1.5)

$$E = R(G^F,\Delta) \qquad (B.1.3)$$

The properties of the SSV and (3.4.8) imply (Doyle,1985) that

$$\sup_{\Delta\in\mathbf{\Delta}} \bar{\sigma}(R(G^{x_0}_{full},\Delta)) = 1 \qquad (B.1.4)$$

From (3.4.6), (3.4.7), (B.1.2), (B.1.3), it follows after some algebra that

$$R(G^{x_0}_{full},\Delta) = \begin{pmatrix} x_0 Ev & 0 \end{pmatrix} \qquad (B.1.5)$$

Then from (B.1.4),(B.1.5) and the definition of the singular values, it follows, since x_0Ev is a vector:

$$\sup_{\Delta\in\mathbf{\Delta}} (x_0^2 v^* E^* E v) = 1 \qquad \forall\omega$$

$$\Longrightarrow \sup_{\Delta\in\mathbf{\Delta}} \int_{-\infty}^{+\infty} v^* E^* E v \quad d\omega \quad = \int_{-\infty}^{+\infty} x_0^{-2} \quad d\omega$$

$$\Longleftrightarrow (3.4.9) \qquad QED$$

B.2. Proof of Theorem 3.4.2.

Let $0 < x_2 \leq x_1$. Then we can write $x_2 = x_1\beta$, where $0 < \beta \leq 1$. From (3.4.10) we have

$$\begin{aligned} DM^{x_2}D^{-1} &= \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \beta I \end{pmatrix} M^{x_1}D^{-1} \\ &= \begin{pmatrix} I & 0 \\ 0 & \beta I \end{pmatrix} DM^{x_1}D^{-1} \end{aligned} \qquad (B.2.1)$$

Then the properties of the singular values yield

$$\begin{aligned} (B.2.1) &\Longrightarrow \bar{\sigma}(DM^{x_2}D^{-1}) \leq \bar{\sigma}\begin{pmatrix} I & 0 \\ 0 & \beta I \end{pmatrix} \quad \bar{\sigma}(DM^{x_1}D^{-1}) \\ &\Longrightarrow \bar{\sigma}(DM^{x_2}D^{-1}) \leq \bar{\sigma}(DM^{x_1}D^{-1}) \qquad \forall D \in \mathbf{D} \\ &\Longrightarrow \inf_{D\in\mathbf{D}} \bar{\sigma}(DM^{x_2}D^{-1}) \leq \inf_{D\in\mathbf{D}} \bar{\sigma}(DM^{x_1}D^{-1}) \qquad QED \end{aligned}$$

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CHAPTER VI

INTERNAL MODEL CONTROL: ROBUST DIGITAL CONTROLLER SYNTHESIS FOR MULTIVARIABLE OPEN-LOOP STABLE OR UNSTABLE SYSTEMS

INTERNAL MODEL CONTROL: ROBUST DIGITAL CONTROLLER SYNTHESIS FOR MULTIVARIABLE OPEN–LOOP STABLE OR UNSTABLE SYSTEMS

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Abstract

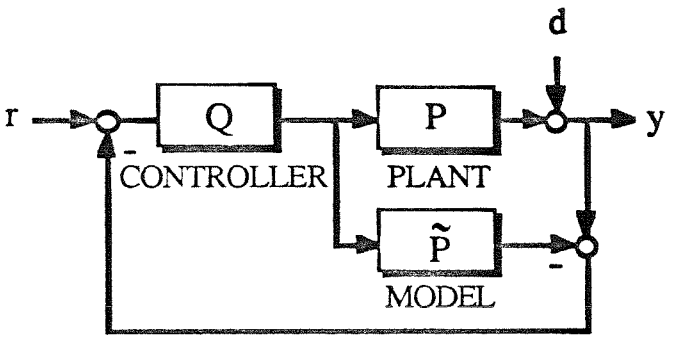
The two–step Internal Model Control procedure is used for the synthesis of multivariable discrete controllers for open–loop stable or unstable plants. The plant models used in the proposed method are transfer function matrices. In the first step the controller is designed so that the L_2–error (sum of squared errors) is minimized for every setpoint or disturbance vector in a set and their linear combinations. A modification is then introduced to avoid the potential problem of intersample rippling. In the second step a low–pass filter is designed so that stability and good performance characteristics are maintained in the presence of model–plant mismatch. The continuous plant output is considered in order to avoid bad intersample behavior. The filter parameters are obtained as the result of a minimization of a non–conservative robustness measure, the Structured Singular Value. Special filter structures have to be used for open–loop unstable or ill–conditioned plants.

The design of a control system requires the use of a process model, either explicitly or implicitly. However, modeling error is unavoidable and it results in a mismatch between the model and the actual plant. Other reasons for such a mismatch are nonlinearities that manifest themselves as modeling error when a linear model is used for the controller design. Whatever the reasons for the modeling error, the result is that a controller designed for a particular model may perform quite differently when it is implemented on the actual process. Modern control theory addresses this problem with the design of controllers that are robust with respect to model-plant mismatch, i.e., of control systems that will perform within certain design specifications, provided that the modeling error does not exceed certain bounds.

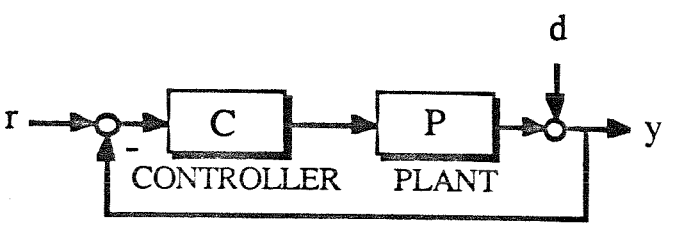
The Internal Model Control (IMC) structure, introduced by Garcia and Morari (1982), has been widely recognized as very useful in clarifying the issues related to the mismatch between the model used for controller design and the actual process. An important characteristic of the structure is that it gives rise to a two step controller synthesis procedure. In the first step the assumption is made that the model and the plant are the same, and as a result the design of an IMC controller with desired performance characteristics in this first step is significantly simplified. The second step deals with the design of a low–pass filter such that robustness with respect to model-plant mismatch is guaranteed, in the sense that the system performance will remain close to the one for the nominal case (no modeling error), even when such a mismatch exists.

1. Internal Model Control.

The IMC structure (Fig. 1a), is mathematically equivalent to the classical feedback structure (Fig. 1b). $\tilde{P}(z)$ represents the model and $P(z)$ the actual discretized plant. $P(z)$ is obtained by adding a zero order hold in front of the continuous plant and then taking the z-transform. $P(z)$ is assumed to be a square



(a)



(b)

Figure 1.

(a) Internal Model Control structure.

(b) Feedback Control structure.

$n \times n$ matrix of full normal rank, i.e., it may be singular only at a finite number of z's. The IMC controller $Q(z)$ and the feedback $C(z)$ are related through

$$Q = C(I + \tilde{P}C)^{-1} \tag{1.0.1}$$

$$C = Q(I - \tilde{P}Q)^{-1} \tag{1.0.2}$$

where $\tilde{P}$ is the process model. Note that throughout this paper, unless otherwise pointed, all the transfer function matrices are z transforms.

Some advantages of using the IMC structure can be seen by examining the structure for $P = \tilde{P}$ and $P \neq \tilde{P}$:

$\underline{P = \tilde{P}}$.

In this case the overall transfer function connecting the setpoints $r(z)$ and disturbances $d(z)$ to the errors $e = y - r$, where $y(z)$ are the process outputs is

$$e = y - r = (I - PQ)(d - r) \stackrel{\text{def}}{=} \tilde{E}(d - r) \tag{1.0.3}$$

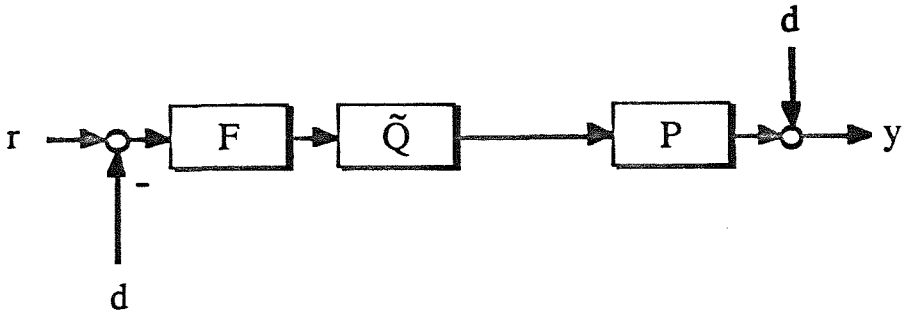
Hence the IMC structure becomes effectively open-loop (Fig. 2a) and the design of Q is simplified. Note that the IMC controller is identical to the parameter of the Q-parametrization (Zames, 1981). Also the addition of a diagonal filter F by writing

$$Q = \tilde{Q}F \tag{1.0.4}$$

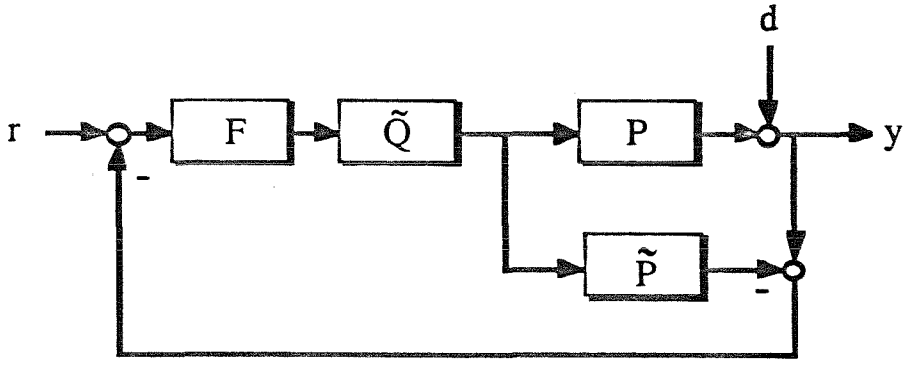
introduces parameters (the filter time constants) which can be used for adjusting on-line the speed of response for each process output.

$\underline{P \neq \tilde{P}}$.

The model-plant mismatch generates a feedback signal in the IMC structure which can cause performance deterioration or even instability. Since the relative modeling error is larger at higher frequencies, the addition of the low-pass filter F (Fig. 2b) adds robustness characteristics into the control system. In this case



(a)



(b)

Figure 2. IMC structure with the filter F.

(a) $P = \tilde{P}$.

(b) $P \neq \tilde{P}$.

the closed-loop transfer function is

$$e = y - r = (I - P\tilde{Q}F)(I - (P - \tilde{P})\tilde{Q}F)^{-1}(d - r) \stackrel{\text{def}}{=} E(d - r) \qquad (1.0.5)$$

Hence the IMC structure gives rise rather naturally to a two–step design procedure:

Step 1: Design $\tilde{Q}$, assuming $P = \tilde{P}$.

Step 2: Design F so that the closed-loop characteristics that $\tilde{Q}$ produces in Step 1, are preserved in the presence of model-plant mismatch ($P \neq \tilde{P}$).

In previous IMC work, Garcia and Morari (1985a) proposed for open-loop stable plants the direct design of the closed-loop transfer function. In Step 1 this approach was rigorously quantified in its general form by Zafiriou and Morari (1986b) by using the concept of zero directions, who also took into account inter-sample rippling and modified the approach so that time delays and outside the unit circle (UC) zeros were considered in one single step. However, the extension of the above approach or of the impulse response formulation of the problem (Garcia and Morari, 1985b) to open-loop unstable systems is very awkward. The method proposed in this paper takes care of both open-loop stable and unstable plants in a general way by minimizing the Sum of Squared Errors (SSE) for a set of setpoints or disturbance vectors and their linear combinations, if possible.

With respect to the second step, this paper proposes the design of the IMC filter by minimizing an appropriate robustness condition. Potential modeling errors, described as uncertainty associated with the process model, can appear in different forms and places in a multivariable model. This fact makes the derivation of non-conservative conditions that guarantee robustness with respect to model-plant mismatch quite difficult. The Structured Singular Value (SSV), introduced by Doyle (1982), has gained a lot of popularity recently, because it takes into account the structure of the model uncertainty and it allows the non-conservative

quantification of the concept of robust performance. The objective function for the minimization problem is formulated in such a way so that the continuous plant output is considered and the problem of intersample rippling does not occur.

2. Step 1: Design of $\tilde{Q}$.

Throughout this section the assumption is made that $P = \tilde{P}$. Details on the definition of multivariable zeros and poles and their degrees and orders can be found in the literature (Desoer and Schulman, 1974). In general a pole of an element of $P(z)$ is also a pole of $P(z)$ and the roots of $det[P(z)] = 0$ are the zeros of the matrix $P(z)$.

2.1. Stabilizing Q's.

It will be assumed that there are no open-loop unstable poles of the continuous plant, which after sampling do not appear in the discretized plant, i.e., no open-loop poles become unobservable because of sampling. If this is not the case, then a small change of sampling time will make those poles observable.

2.1.1 Internal Stability.

The concept of Internal Stability can be motivated by pointing out that the signals between blocks constituting a control system are subject to (possibly very small) errors. In practice it cannot be tolerated that these small errors lead to unbounded signals at some other location in the control system.

Definition 1: A linear time invariant control system is internally stable, if the transfer functions between any two points of the control system are stable, i.e., have all poles inside the UC (for discrete systems).

In a control system many different points can be selected for signal injection and observation but most of the choices are equivalent. For example, for the system in Fig. 3a, y and e differ only by a bounded signal r and their observation reveals the same information about internal stability. Also, from the point of view of internal stability the effect of d and r on u is equivalent. Simple arguments

of this type reveal that there are only two "independent" outputs, which can be chosen as y and u and two "independent" inputs which can be chosen as r and u'. Thus the classic feedback system is stable if and only if all elements in the transfer matrix in (2.1.1) have all their poles inside the UC.

$$\begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} PC(I+PC)^{-1} & (I+PC)^{-1}P \\ C(I+PC)^{-1} & -C(I+PC)^{-1}P \end{pmatrix} \begin{pmatrix} r \\ u' \end{pmatrix} \tag{2.1.1}$$

Equivalently the internal stability requirements for the classical feedback structure is that all elements in the matrix IS1 in (2.1.2) are stable:

$$IS1 = \begin{pmatrix} C(I+PC)^{-1} & PC(I+PC)^{-1} & C(I+PC)^{-1}P & (I+PC)^{-1}P \end{pmatrix} \tag{2.1.2}$$

The internal stability condition clarifies the fact that mere cancellation of unstable poles by zeros is not enough to guarantee the stability of the system. It becomes evident that instability arising from unstable pole-zero cancellations is not due to inexact cancellation (as it has been argued in the past) but is solely due to the fact that the cancellation does not satisfy the internal stability requirements.

Use of (1.0.1) or (1.0.2) in (2.1.2) yields

$$IS1 = \begin{pmatrix} Q & PQ & QP & (I-PQ)P \end{pmatrix} \tag{2.1.3}$$

Note that stability of each element in (2.1.3) implies internal stability when the control system is implemented as the feedback structure in Fig. 1b, where C is obtained from the Q used in (2.1.3) through (1.0.2).

In order for the control system to be stable when implemented in the IMC structure of Fig. 1a, we have to examine the transfer functions between all possible system inputs and outputs. From the block diagram in Fig. 3b we note that there are three independent system inputs r, u_1 and u_2 and three independent outputs y, u and $\tilde{y}$. For no model error ($P = \tilde{P}$) the inputs and outputs are related through

the following transfer matrix.

$$\begin{pmatrix} y \\ u \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} PQ & (1-PQ)P & P \\ Q & -QP & 0 \\ PQ & -PQP & P \end{pmatrix} \begin{pmatrix} r \\ u_1 \\ u_2 \end{pmatrix} \tag{2.1.4}$$

In order for the matrix in (2.1.4) to be stable we need that the matrix IS2 be stable:

$$IS2 = (Q \quad PQ \quad QP \quad (I-PQ)P \quad PQP \quad P) \tag{2.1.5}$$

Hence if the process P is open-loop unstable, $IS2$ will also be unstable and the control system has to be implemented in the feedback structure of Fig. 1b. Still, the-two step IMC design procedure can be used for the design of Q, as described in the following sections. C can then be obtained from (1.0.2) and the structure in Fig. 1b implemented. In this case, special attention should be paid to the construction of C, so that all the common on or outside the UC zeros of Q and $(I-PQ)$ are cancelled in (1.0.2). Minimal or balanced realization software can be used to accomplish that.

Note that when the process is open-loop stable, it follows from (2.1.3) or (2.1.5) that the only requirement for internal stability is that Q is stable.

2.1.2. Parametrization of All Stabilizing Q's.

The process P can in general be open-loop unstable. The following assumption simplifies the solution of the optimization problem:

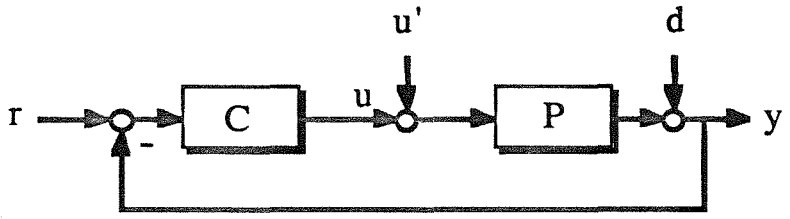
Assumption A.1. If π is a pole of the model $\tilde{P}$ outside the UC, then:

a. The order of π is equal to 1.

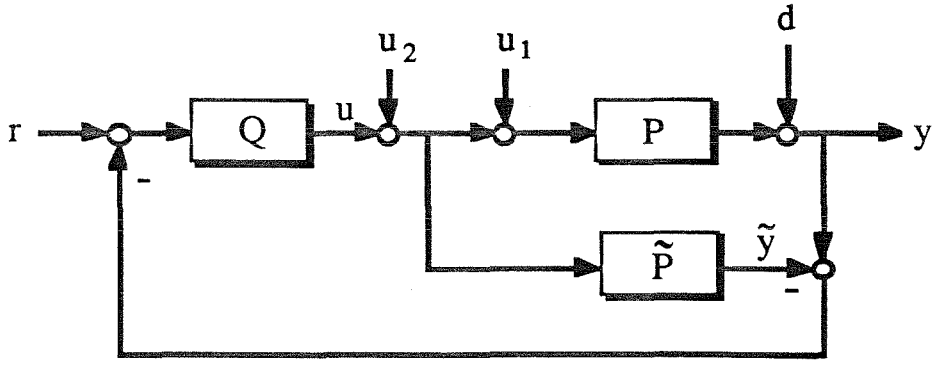
b. $\tilde{P}$ has no zeros at $z = \pi$.

c. The residual matrix that corresponds to π is full rank.

Assumption A.1.a is made to simplify the notation and it is the usual case. The results in this paper can be extended to higher order poles. A.1.b is true for SISO systems but not necessarily for MIMO (Kailath, 1980). However, the



(a)



(b)

Figure 3. Internal Stability.

(a) Feedback structure.

(b) IMC structure.

assumption is not restrictive because the presence of a zero at $z = \pi$ implies an exact cancellation in $det[\tilde{P}(z)]$, which usually does not happen after a slight perturbation in the coefficients of $\tilde{P}$ is introduced. A.1.c is always true for SISO systems, but it can be quite restrictive for MIMO systems. Instead of A.1.c however, an additional assumption can be made on the external input for which the optimal controller is designed. This is discussed in Section 2.2.1.

Assumption A.1 is not made for poles at $z = 1$ because more than one such poles may appear in an element of $\tilde{P}$, introduced by capacitances that are present in the process. The following assumption true for all practical process control problems is made:

Assumption A.2. Any poles of $\tilde{P}$ or P on the UC are at $z = 1$. Also $\tilde{P}$ has no zeros on the UC.

Let $\pi_1, ..., \pi_k$ be the poles of each element of $\tilde{P}$ outside the UC. Define the allpass

$$b_p(z) = \prod_{i=1}^{k} \frac{(1-(\pi_i^*)^{-1})(z-\pi_i)}{(1-\pi_i)(z-(\pi_i^*)^{-1})} \tag{2.1.6}$$

where the superscript (*) denotes complex conjugate (and transpose when applied to a matrix). If A.1.c does not hold, define

$$b_p(z) = 1 \tag{2.1.6'}$$

The following Theorem holds, where "proper" means that the degree of the numerator in any element of $P(z)$ is less than or equal to that of its denominator.

Theorem 2.1.1.

Assume that $Q_0(z)$ is a proper transfer matrix that satisfies the internal stability requirements of Section 1.2, i.e., it produces a matrix IS1 with stable elements. Then all proper Q's that make IS1 stable are given by

$$Q(z) = Q_0(z) + b_p(z)^2 Q_1(z) \tag{2.1.7}$$

where Q_1 is any proper and stable transfer matrix such that:

i) If A.1.c holds, PQ_1P has no poles at $z = 1$.

ii) If A.1.c does not hold, PQ_1P has no poles on or outside the UC.

Proof: See Appendix A.1.

Note that if $P(z)$ is stable, then Theorem 2.1.1 implies that any proper and stable $Q(z)$ is acceptable, as it was remarked in Section 2.1.1.

2.2. H_2-Optimal $\tilde{Q}(z)$.

2.2.1. Definitions.

We define L_2 as the Hilbert space of complex valued functions $f(z)$ defined on the unit circle (UC $= \{e^{i\theta} | -\pi \leq \theta < \pi\}$) and square-integrable with respect to θ. For a vector function f, the norm on L_2 is given by:

$$||f||_2 = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f^*(e^{i\theta})f(e^{i\theta}) \quad d\theta\right)^{1/2} \tag{2.2.1}$$

L_2 can be decomposed into two subspaces, H_2 the closed subspace of functions having analytic continuations inside the UC and its orthogonal complement $H_2^{\perp}$. Note that with these definitions a constant function is in H_2. H_2 also includes all rational z-transfer functions that are strictly unstable, i.e., which have all their poles strictly outside the UC (including poles at $z = \infty$ (improper transfer functions)). All strictly proper (numerator degree less than denominator degree), stable rational z-transfer functions, are in $H_2^{\perp}$.

If $f(z)$ is proper, stable and $\{f_k\}$ is the time domain sequence of vectors corresponding to it, i.e.,

$$\{f_k\} = Z^{-1}\{f(z)\} \tag{2.2.2}$$

then we also have

$$||f||_2 = \left(\sum_{k=0}^{\infty} f_k^T f_k\right)^{1/2} \tag{2.2.3}$$

where the superscript (T) denotes transpose. Hence if the error signal $e(z)$ is stable, then the square of its L_2-norm is equal to the SSE.

For a specified external system input v ($v = d$ for $r = 0; v = -r$ for $d = 0$), define by using (1.0.3)

$$\phi(v) \stackrel{\text{def}}{=} ||e||_2^2 = ||\tilde{E}v||_2^2 = ||(I - P\tilde{Q})v||_2^2 \tag{2.2.4}$$

Then $\phi(v)$ is the SSE for the particular input v. For every external input v that will be considered in this paper the following assumption can be made without loss of generality:

Assumption A.3.

a. The poles of each nonzero element of v outside the UC (if any) are the first k' poles π_i of the plant outside the UC, each with degree 1.

b. If A.1.c does not hold, then every nonzero element of v (or $\hat{v}$) includes all the outside the UC poles of $\tilde{P}$, each with degree 1.

To simplify the arguments in the paper, we shall assume that if A.3.b is satisfied, then A.1.c is not. In this way the proper choices in the definitions and the proofs will be made on the basis of A.1.c. If both A.1.c and A.3.b hold, then the results that apply to the case where A.1.c does not hold but A.3.b does, are still correct.

Define

$$b_v(z) = \prod_{i=1}^{k'} \frac{(1 - (\pi_i^*)^{-1})(z - \pi_i)}{(1 - \pi_i)(z - (\pi_i^*)^{-1})} \tag{2.2.5}$$

If A.1.c does not hold, define

$$b_v(z) = 1 \tag{2.2.5'}$$

A different assumption is made for the poles of v at $z = 1$:

Assumption A.4. Let l_i be the maximum number of poles at $z = 1$ that an element of the i^{th} row of P has. Then the i^{th} element of v has at least l_i poles at $z = 1$. Also v has no other poles on the UC and its elements have no zeros on the UC.

The above assumptions are not restrictive in the case where v is an output disturbance d, because in a practical situation we want to design for an out-

put disturbance produced by a disturbance that has passed through the process. Hence, d usually includes the unstable process poles (e.g., an output disturbance d produced by a disturbance in the manipulated variables). Note that the control system will still reject other disturbances with fewer unstable poles, without producing steady-state offset. The assumption is different for poles at $z = 1$ because their number in each row of $\tilde{P}$ can be different, since capacitances may be associated with only certain process outputs. Also the output disturbance may have more poles at $z = 1$ than the process (e.g., a persistent disturbance in the manipulated variables).

The assumptions may be restrictive in the case of setpoints though. However for setpoint tracking the use of the Two-Degree-of-Freedom structure, which will be discussed briefly in Section 2.5, allows us to disregard the existence of any unstable poles of P and therefore this assumption need not be made for setpoints.

The plant P can be factored into an allpass portion P_A and a minimum phase (MP) portion P_M:

$$P = P_A P_M \tag{2.2.6}$$

P_A is stable and such that $P_A^*(e^{i\theta})P_A(e^{i\theta}) = I$. Also P_M^{-1} is stable. P_M has the additional property that both P_M and P_M^{-1} are proper. In the case where P is scalar, this factorization can be easily accomplished by writing P_A as a scalar allpass (similarly to b_p or b_v) containing as zeros the outside the UC zeros of P, times the time delays of the plant so that P_M is semi-proper (numerator degree = denominator degree). In the general multivariable case, this "inner-outer" factorization can be accomplished through the spectral factorization of $P(z^{-1})^T P(z)$, where (T) denotes transpose. Details on these problems can be found in the literature (Motyka and Cadzow, 1967; Anderson et al., 1974; Chu, 1985; Doyle et al., 1984).

2.2.2. Minimization of $\phi(\mathbf{v})$ for One Specific $\mathbf{v}$.

The objective in this section is to consider only one specific input $v(z)$ and solve the problem:

$$\min_{\tilde{Q}\in\mathcal{Q}} \phi(v) \qquad (P1)$$

where $\mathcal{Q}$ denotes the set of all stabilizing Q's described by Theorem 2.1.1.

Let $v_0(z)$ be the scalar allpass with the property $v_0(1) = 1$, which includes the common outside the UC zeros and time delays of the elements of $v(z)$. Write

$$v(z) = v_0(z)\hat{v}(z) \qquad (2.2.7)$$

where $\hat{v}(z)$ is a vector. Hence $\hat{v}$ is proper with at least one element semi-proper and there is no point z on or outside the UC where $\hat{v}$ becomes identically zero.

The following theorem holds:

Theorem 2.2.1.

Any stabilizing $\tilde{Q}$ that solves (P1) satisfies

$$\tilde{Q}\hat{v} = zb_p b_v^{-1} P_M^{-1}\{z^{-1}b_p^{-1}b_v P_A^{-1}\hat{v}\}_* \qquad (2.2.8)$$

where the operator $\{\cdot\}_*$ denotes that after a partial fraction expansion of the operand, only the strictly proper terms are retained except for those corresponding to poles of P_A^{-1}. Furthermore, for $n \geq 2$ the number of stabilizing controllers that satisfy (2.2.8) is infinite. Guidelines for the construction of such a controller are given in the proof.

Proof: See Appendix A.2.

Note that not every $\tilde{Q}$ satisfying (2.2.8) is necessarily a stabilizing controller.

2.2.3. Minimization of ϕ(v) for a Set of v's.

Minimizing the SSE just for one vector v is not very meaningful, because of the different directions in which the disturbances enter the process or the setpoints are changed. What is desirable is to find a $\tilde{Q}$, that minimizes $\phi(v)$ for every single

v in a set of external inputs v of interest for the particular process. For an $n \times n$ P, let us consider the n vectors $v^i(z), i = 1, \ldots, n$. Define

$$V \stackrel{\text{def}}{=} (v^1 \quad v^2 \quad \ldots \quad v^n) \tag{2.2.9}$$

where $v^1, \ldots, v^n$ satisfy assumption A.3. An additional assumption on V is needed:

Assumption A.5.

a. V has no zeros at the location of its unstable poles or on the UC and V^{-1} cancels the poles of $\tilde{P}$ at $z = 1$ in $V^{-1}\tilde{P}$.

b. If A.1.c holds, the residual matrices for the outside the UC poles of V are full rank; if A.1.c does not hold, then V^{-1} cancels the outside the UC poles of $\tilde{P}$ in $V^{-1}\tilde{P}$.

Note that satisfaction of assumptions A.3.b and A.4 for each column of V does not necessarily imply satisfaction of A.5. However such a V can be easily constructed. One way is to obtain V as $\tilde{P}$ times a matrix with no outside the UC poles and no zeros on the UC. This case corresponds to the physically meaningful situation, where the output disturbances are produced by disturbances in the manipulated variables. Another simple way is to use a diagonal V, in which case satisfaction of A.3 and A.4 by every column of V implies satisfaction of A.5 by V. This situation is discussed further in Corollary 2.2.1.

Factor V similarly to P (use $V(z)V(z^{-1})^T$ if spectral factorization theory is used):

$$V = V_M V_A \tag{2.2.10}$$

Let us now consider the problem:

$$\min_{\tilde{Q} \in \mathcal{Q}} [\phi(v^1) + \phi(v^2) + \ldots \phi(v^n)] \tag{P2}$$

Hence the $\tilde{Q}$ that solves (P2) minimizes the sum of the squares of the L_2-errors (SSE) that each of the inputs v^i would cause when applied to the system separately.

Theorem 2.2.2.

The controller

$$\tilde{Q} = zb_p b_v^{-1} P_M^{-1} \{z^{-1} b_p^{-1} b_v P_A^{-1} V_M\}_* V_M^{-1} \tag{2.2.11}$$

is the unique solution to (P2).

Proof: See Appendix A.3.

A more meaningful objective would be to solve:

$$\min_{\tilde{Q} \in \mathcal{Q}} \phi(v^i) \qquad \forall i = 1, \ldots, n \tag{P3}$$

However a $\tilde{Q}$ that solves (P3) will also solve (P2). Then from Theorem 2.2.2 it follows that if a solution to (P3) exists, it is given by (2.2.11). Factor each of the v^i in the way used in (2.2.7):

$$v^i(z) = v_0^i(z)\hat{v}^i(z) \tag{2.2.14}$$

Define

$$\hat{V} \stackrel{\text{def}}{=} (\hat{v}^1 \quad \hat{v}^2 \quad \ldots \quad \hat{v}^n) \tag{2.2.15}$$

Theorem 2.2.3.

i) If $\hat{V}(z)$ is non-minimum phase (i.e., $\hat{V}^{-1}$ is unstable or improper), then there exists no solution to (P3).

ii) If $\hat{V}(z)$ is minimum phase, then use of $\hat{V}$ instead of V_M in (2.2.11) yields exactly the same $\tilde{Q}$, which also solves (P3) and it minimizes $\phi(v)$ for any v that is a linear combination of v^i's that have the same v_0^i's.

Proof: See Appendix A.4.

The following corollary to Theorem 2.2.3 holds:

Corollary 2.2.1:

Let

$$V = diag(v_1, v_2, \ldots, v_n) \tag{2.2.16}$$

where $v_1(z), \ldots, v_n(z)$ are scalars. Then use of $\hat{V}$ instead of V_M in (2.2.11) yields exactly the same $\tilde{Q}$, which minimizes $\phi(v)$ for the following n vectors:

$$v = \begin{pmatrix} v_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ v_2 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ v_n \end{pmatrix} \tag{2.2.17}$$

and their multiples, as well as for the linear combinations of those directions that correspond to v_i's with the same outside the UC zeros in the same degree and the same time delays.

2.2.4. Setpoint Prediction

In the case of setpoint tracking, future values of r are often known and supplied to the computer ahead of time. Then if at time t the setpoint value that is fed to the control algorithm as $Z^{-1}\{r(z)\}$ is the one we wish the plant output to reach at time $t + NT$, when T is the sampling time, the objective function has to be modified to:

$$\phi_N(v) = ||(z^{-N}I - P\tilde{Q})r||_2^2 \tag{2.2.18}$$

When the above objective function is used in the minimization problems (P1), (P2), (P3), the resulting expressions for the H_2-optimal controller are the same as in Theorems 2.2.1, 2.2.2, 2.2.3, but with the term z^{-N-1} used instead of z^{-1} inside $\{\cdot\}_*$. All the steps in the proofs remain the same when (2.2.18) is used rather than (2.2.4).

2.3. Intersample Rippling.

The H_2-optimal controller minimizes the SSE. Therefore it completely disregards the plant output's behavior between the sample points. The result is that

the H_2-optimal controller may produce an excellent performance at the sample points but suffer from severe intersample rippling. Zafiriou and Morari (1985) examined this type of controller for SISO systems and showed that the problem is caused by zeros of $P(z)$ that are close to the point (-1,0) on the z-plane, which give rise to poles of the H_2-optimal $\tilde{Q}$ that are close to (-1,0). A modification was introduced to substitute such poles in $\tilde{Q}$ with poles at $z = 0$. The new $\tilde{Q}$ was shown to be free of the problem of intersample rippling and to combine desirable deadbeat type characteristics to those of the H_2-optimal. This section extends the modification to MIMO systems and general open-loop stable or unstable plants. It should be pointed out that this modification is sufficient only if there are no open-loop oscillatory poles in the continuous plant transfer function, which have become unobservable after sampling.

Let $\tilde{Q}_H(z)$ be an H_2-optimal $\tilde{Q}$ obtained according to the previous sections. Also let $\delta(z)$ be the least common denominator of $P(z)$, and κ_i, $i = 1,\ldots,\rho$ be the roots of $\delta(z)$ close to (-1,0) (or in general with negative real part). Define

$$q_-(z) = z^{-\rho} \prod_{j=1}^{\rho} \frac{(z-\kappa_j)}{(1-\kappa_j)} \tag{2.3.1}$$

Then $\tilde{Q}_H$ is modified as follows:

$$\tilde{Q}(z) = \tilde{Q}_H(z) q_-(z) b(z) \tag{2.3.2}$$

where the scalar $b(z)$ is selected so that IS1 in (2.1.3) and $(I - P\tilde{Q})V$ remain stable. $\pi_i, i = 0,1,\ldots,k$ are the unstable roots (including $\pi_0 = 1$) of the least common denominator of $P(z), V(z)$. Let the multiplicity of each of them be m_i. Note that the multiplicity of the ones that are outside the UC is equal to 1, according to A.1, A.3. Remember that according to assumptions A.3, A.4, V has at least as many poles at $z = 1$ as P and each pole of V outside the UC is also a pole of P. Then since $\tilde{Q}_H$ makes IS1 and $(I - P\tilde{Q}_H)V$ stable, it follows that the requirements on

$b(z)$ are:

$$\frac{d^j}{dz^j}(1-\tilde{q}_-(z)b(z))|_{z=\pi_i}=0, \qquad (j=0,\ldots,m_i-1), i=0,1,\ldots,k \tag{2.3.3}$$

We can write

$$b(z)=\sum_{j=0}^{m-1} b_j z^{-j} \tag{2.3.4}$$

where

$$m=\sum_{i=0}^{k} m_i \tag{2.3.5}$$

and then compute the coefficients $b_j, j=0,\ldots,m-1$ from (2.3.3). Note that since none of the π_i's is 0 or ∞, (2.3.3) is equivalent to

$$\frac{d^j}{d\lambda^j}(1-q_-(\lambda^{-1})b(\lambda^{-1}))|_{\lambda=\pi_i^{-1}}=0, \qquad (j=0,\ldots,m_i-1), i=0,1,\ldots,k \tag{2.3.6}$$

Both $\tilde{q}_-(\lambda^{-1})$ and $b(\lambda^{-1})$ are polynomials in λ and therefore their derivatives with respect to λ can be computed easily. Then (2.3.6) yields a system of m linear equations with m unknowns $(b_0, b_1, \ldots, b_{m-1})$. The resulting controller $\tilde{Q}$ combines the desirable properties of the H_2-optimal controller and deadbeat type controllers.

2.4. $\tilde{Q}$ for Specific Cases.

This section looks at simplified forms of the general expressions for the H_2-optimal controller, for specific systems and external inputs.

2.4.1. Stable P.

We have $b_p = b_v = 1$. Then (2.2.11) simplifies to

$$\tilde{Q}_H = zP_M^{-1}\{z^{-1}P_A^{-1}V_M\}_* V_M^{-1} \tag{2.4.1}$$

2.4.2. Minimum phase P.

$P(z)$ cannot truly be MP for a physical system. Even if the Laplace transfer matrix representing the continuous plant is MP but strictly proper (as a physical

system), the discretized plant $P(z)$ will still have a delay of one unit because of the sampling. Hence $P_A = z^{-1}I$, $P_M = zP$ and (2.2.11) yields

$$\tilde{Q}_H = P^{-1}(I - b_p b_v^{-1} K V_M^{-1}) \tag{2.4.2}$$

where K is the constant term in a partial fraction expansion of $b_p^{-1}b_v V_M$ or since b_p^{-1}, b_v, V_M are semi-proper, K is the product of the constant terms of the PFE's of b_p^{-1}, b_v, V_M. After some algebra we get

$$K = V_{M,0} \prod_{j=k'+1}^{k} (-\pi_j)^{m_i} \tag{2.4.3}$$

where k, k', π_j are defined in assumptions A.2, A.3, and $V_{M,0}$ is the first non-zero matrix in the impulse response description of $V(z)$, which can be obtained by long division and is equal to the constant term in the PFE of $V_M(z)$.

2.4.3. Example.

Consider the continuous MP system

$$P_c(s) = \frac{b}{-s+b}, \qquad b > 0 \tag{2.4.4}$$

and assume that a step disturbance acts at the process input, i.e., the continuous external input $V_c(s)$ is

$$V_c(s) = d_c(s) = \frac{b}{s(-s+b)} \tag{2.4.5}$$

Then for a sampling time T we have

$$P(z) = \frac{1 - e^{bT}}{z - e^{bT}} \tag{2.4.6}$$

$$V(z) = \frac{(1 - e^{bT})z}{(z-1)(z - e^{bT})} \tag{2.4.7}$$

$$V_M(z) = zV(z) \tag{2.4.8}$$

Note that $e^{bT} > 1$ since $b > 0$. The H_2-optimal controller can be obtained from (2.4.2). We have $b_p = b_v$ and so from (2.4.8)

$$K = V_{M,0} = 1 - e^{bT} \tag{2.4.9}$$

Substitution of (2.4.6), (2.4.7), (2.4.8), (2.4.9) into (2.4.3) yields

$$\tilde{Q}_H(z) = \frac{(z - e^{bT})((1 + e^{bT})z - e^{bT})}{(1 - e^{bT})z^2} \tag{2.4.10}$$

From Section 2.3 it follows that in this case we have $\tilde{Q}(z) = \tilde{Q}_H(z)$

2.5. Two-Degree-of-Freedom Structure.

From the discussion of the Internal Stability requirements in Section 2.1, it follows that unstable plant poles limit the possible choices of Q and thus the achievable performance. This however need not be so for setpoint tracking. Consider the general feedback structure of Fig. 4. For the disturbance behavior it is irrelevant if the controller is implemented as one block C as in Fig. 1b, or as two blocks as in Fig. 4. Hence the achievable disturbance rejection is restricted both by the outside the UC zeros and poles of P as the quantitative results of the previous sections indicate.

Let us now proceed from the point where a stabilizing $\tilde{Q}$ and the corresponding C have been found through the results of the previous sections, which produce a satisfactory disturbance response. We can then split C into two blocks C_1 and C_2 such that C_1 is minimum phase and C_2 is stable. Then one can see that the only outside the UC zeros of the stabilized system $PC_1(I + PC_1C_2)^{-1}$ are those of the process P. Thus C_3 can be designed without regard for the unstable poles of P and the achievable setpoint tracking is restricted by the outside the UC zeros of P only.

In summary, the achievable disturbance response of a system is restricted by the presence of the plant zeros and poles that lie outside the UC regardless of how complicated a controller is used. If the Two-Degree-of Freedom controller shown in Fig. 4 is employed, the achievable setpoint response is restricted only by the zeros. A more rigorous discussion can be found in Vidyasagar (1985).

3. Step 2: Design of F.

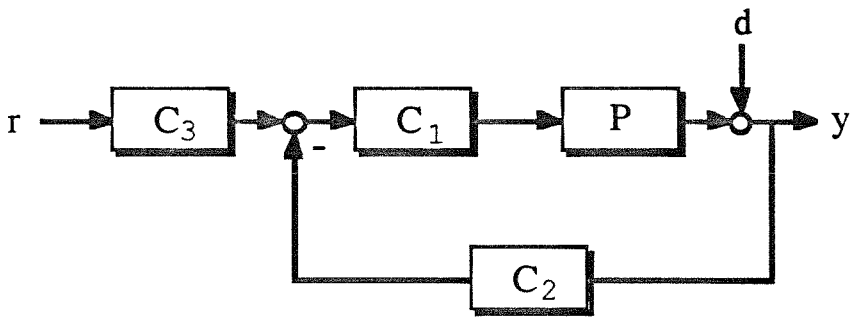


Figure 4. Two-degree-of-freedom feedback structure.

This section deals with the design of the IMC filter $F(z)$ so that the performance characteristics obtained in Step 1 are preserved in the presence of model-plant mismatch.

3.1. Robustness Conditions.

3.1.1. Structured Singular Value.

Potential modeling errors, described as uncertainty associated with the process model, can appear in different forms and places in a multivariable model. This fact makes the derivation of non-conservative conditions that guarantee robustness with respect to model-plant mismatch difficult. The Structured Singular Value (SSV), introduced by Doyle (1982), takes into account the structure of the model uncertainty and it allows the non-conservative quantification of the concept of robust performance.

For a constant complex matrix M the definition of the SSV $\mu_{\mathbf{\Delta}}(M)$ depends also on a certain set $\mathbf{\Delta}$. Each element Δ of $\mathbf{\Delta}$ is a block diagonal complex matrix with a specified dimension for each block, i.e.

$$\mathbf{\Delta} = \left\{ diag(\Delta_1, \Delta_2, ..., \Delta_n) | \Delta_j \in \mathbf{C}^{m_j \times m_j} \right\} \tag{3.1.1}$$

Then

$$\frac{1}{\mu_{\mathbf{\Delta}}(M)} = \min_{\Delta \in \mathbf{\Delta}} \{ \bar{\sigma}(\Delta) | det(I - M\Delta) = 0 \} \tag{3.1.2}$$

and $\mu_{\mathbf{\Delta}}(M) = 0$ if $det(I - M\Delta) \neq 0 \qquad \forall \Delta \in \mathbf{\Delta}$. Note that $\bar{\sigma}$ is the maximum singular value of the corresponding matrix.

Details on how the SSV can be used for studying the robustness of a control system can be found in Doyle (1985), where a discussion of the computational problems is also given. For three or fewer blocks in each element of $\mathbf{\Delta}$, the SSV can be computed from

$$\mu_{\mathbf{\Delta}}(M) = \inf_{D \in \mathbf{D}} \bar{\sigma}(DMD^{-1}) \tag{3.1.3}$$

where

$$\mathbf{D} = \{diag(d_1 I_{m_1}, d_2 I_{m_2}, ..., d_n I_{m_n}) | d_j \in \mathbf{R}_+\} \tag{3.1.4}$$

and I_{m_j} is the identity matrix of dimension $m_j \times m_j$. For more than three blocks, (3.1.3) still gives an upper bound for the SSV.

3.1.2 Model Uncertainty

In order to effectively use the SSV for designing F, some rearrangement of the block structure is necessary. The IMC structure of Fig. 1a can be written as that of Fig. 5a, where $v = d - r, e = y - r$ and

$$G = \begin{pmatrix} 0 & 0 & \tilde{Q} \\ I & I & \tilde{P}\tilde{Q} \\ -I & -I & 0 \end{pmatrix} \tag{3.1.5}$$

where the blocks 0 and I have appropriate dimensions. The block $(P - \tilde{P})$ represents the model-plant mismatch. In order to design a control system that takes into account this modeling error, we need to have some information on how large this mismatch can be. For example we might know a bound $l_a(\omega)$, where ω is the frequency $(z = e^{i\omega T})$, on the additive error:

$$\bar{\sigma}(P - \tilde{P}) \leq l_a \tag{3.1.6}$$

where $\bar{\sigma}(\cdot)$ is the maximum singular value of $(\cdot)$. However (3.1.6) represents only a very simple way to describe model uncertainty. For multivariable systems, such uncertainty may appear in many different places in the matrix, like specific parameters, elements of P, the inputs or outputs of P, etc. It may then be very conservative to lump this information into (3.1.6). However, provided that we can write P as a linear fractional transformation of its uncertain points, the structure in Fig. 5a can always be transformed into that in Fig. 5b, where Δ is a block diagonal matrix with the additional property that

$$\bar{\sigma}(\Delta) \leq 1 \qquad \forall\omega \tag{3.1.7}$$

The superscript u in G^u denotes the dependence on G^u not only on G but also on the specific uncertainty description available for the model $\tilde{P}$. We shall not demonstrate in detail here, how G^u can be obtained from G. For some common cases of model uncertainty, the expressions can be found in the literature (Zafiriou and Morari, 1986c). For the simple case described by (3.1.6), this can be accomplished by simply multiplying the first row of G with l_a.

Let G^u be partitioned as

$$G^u = \begin{pmatrix} G^u_{11} & G^u_{12} & G^u_{13} \\ G^u_{21} & G^u_{22} & G^u_{23} \\ G^u_{31} & G^u_{32} & G^u_{33} \end{pmatrix} \tag{3.1.8}$$

Then Fig. 5b can be written as Fig. 6 with

$$G^F = \begin{pmatrix} G^u_{11} & G^u_{12} \\ G^u_{21} & G^u_{22} \end{pmatrix} + \begin{pmatrix} G^u_{13} \\ G^u_{23} \end{pmatrix} (I - FG^u_{33})^{-1} F (G^u_{31} \qquad G^u_{32})$$

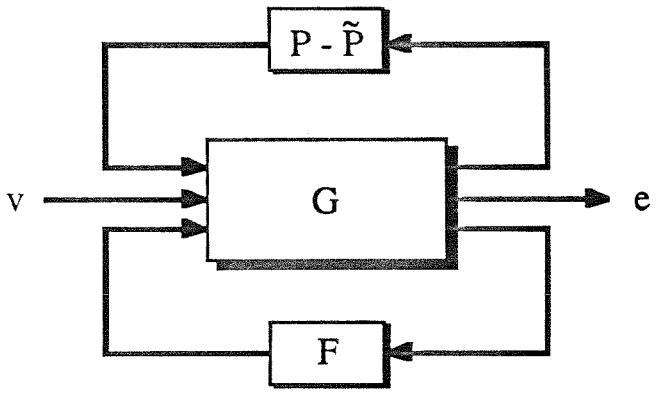
$$\stackrel{\text{def}}{=} \begin{pmatrix} G^F_{11} & G^F_{12} \\ G^F_{21} & G^F_{22} \end{pmatrix} \tag{3.1.9}$$

Note that for $P, \tilde{P}$, z-transforms and therefore periodic in ω, the block Δ will also be periodic. Hence in this case only the frequencies from 0 to π/T need be considered in (3.1.7). However in Section 3.1.4 it will become apparent that in order to avoid bad intersample behavior, we also have to consider the continuous plant, described by some Laplace transfer function $P_c(s)$ (and $\tilde{P}_c(s)$ for the model). Clearly the modeling error in the description of the discretized plant is related to that in the continuous plant description. For example, let us assume that we have a bound on the additive uncertainty for the continuous plant:

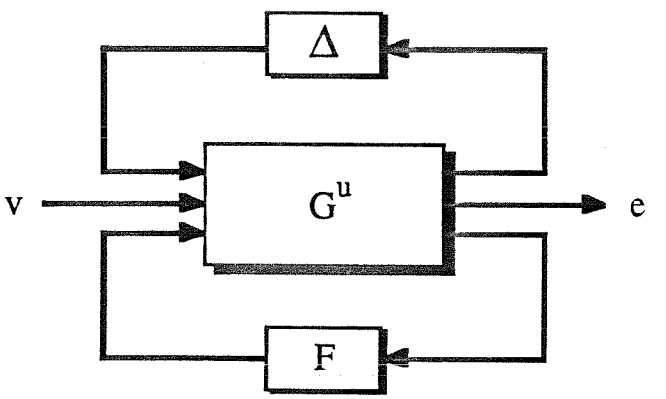
$$\bar{\sigma}(P_c(i\omega) - \tilde{P}_c(i\omega)) \leq l_c(\omega) \tag{3.1.10}$$

Then for the discretized plant we have

$$P(z) - \tilde{P}(z) = \mathcal{Z}\mathcal{L}^{-1}\{H(s)(P_c(s) - \tilde{P}_c(s))\} \tag{3.1.11}$$



(a)



(b)

Figure 5. Model uncertainty block diagrams.

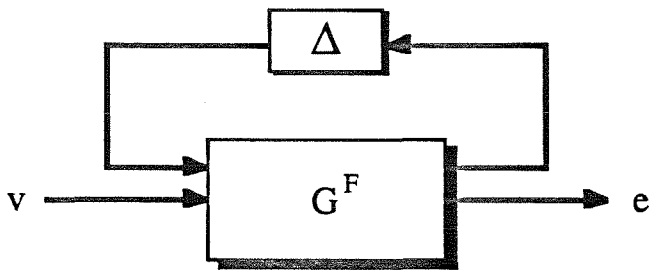


Figure 6. SSV block diagram.

where $H(s)$ is the zero order hold. Then from the property of any z transform $a(z) = Z\mathcal{L}^{-1}\{a_c(s)\}$:

$$a(e^{i\omega T}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} a(i\omega + ik2\pi/T) \qquad (3.1.12)$$

the singular value property $\bar{\sigma}(A+B) \leq \bar{\sigma}(A) + \bar{\sigma}(B)$, and (3.1.10), (3.1.11), it follows that

$$\bar{\sigma}(P(e^{i\omega T}) - \tilde{P}(e^{i\omega T})) \leq \frac{1}{T}\sum_{k=-\infty}^{\infty} |H(i\omega + ik2\pi/T)| l_c(i\omega + ik2\pi/T) \qquad (3.1.13)$$

$H(s)$ is small at frequencies higher than π/T and goes to 0 as fast as $1/\omega$ as $\omega \to \infty$. Therefore only a few terms around $k = 0$ are important in the infinite sum. Also note that for a physical system, $l_c(\omega) \to 0$ at least as fast as $1/\omega$, as $\omega \to \infty$, and hence the sum converges. Then the bound l_a in (3.1.6) can be set equal to the right hand side of (3.1.13).

However, it is not always possible to obtain in a non-conservative way a mathematical description for the uncertainty in the z-domain, starting from the uncertainty in the s-domain. If first principles models are not available, these descriptions may be the result of experiments conducted with different sampling times, of which one is small enough to approximate the continuous system. A discussion of identification techniques is beyond the scope of this paper. Details on such methods and the resulting modelling error can be found in the literature (e.g., Jenkins and Watts (1969), Astrom and Wittenmark (1984)).

3.1.3. Robust Stability

We now require that the matrix IS1 as given by (2.1.2) is stable for all possible plants P. The design of $\tilde{Q}$ according to Section 2 resulted in a stable $IS1$ for $P = \tilde{P}$. In order for $IS1$ to remain stable we need to satisfy the requirements that as we move in a "continuous" way from the model $\tilde{P}$ to the plant P, no closed-loop poles cross the UC and no such poles suddenly appear outside the

UC. The latter requirement is satisfied if we assume that the model and the plant have the same number of poles outside the UC. If this is not the case, another sufficient condition is that G^F is a stable matrix and only stable Δ's are possible. The SSV can be used to determine if any crossings of the UC occur. Then we can say that the system is stable for any of the plants in the set defined from the bounds on the model uncertainty and which have the same number of outside the UC poles as the model, if and only if (Doyle, 1985)

$$\mu_{\Delta}(G_{11}^F) < 1, \qquad 0 \leq \omega \leq \pi/T \tag{3.1.14}$$

3.1.4 Robust Performance

The problem of intersample rippling was addressed for the first step of the design procedure in Section 2.3. There, a simple modification was sufficient because the model $\tilde{P}(z)$ was known exactly. In this section however we have to consider the situation where $P \neq \tilde{P}$ and as a result we have to examine the continuous plant output $y_c(s)$ in order to avoid bad intersample behavior. The obstacle in doing so is the fact that the relation between $y_c(s)$ and $r_c(s)$ or $d_c(s)$ (continuous setpoint and disturbance descriptions) is linear but time varying because of the sampling, and so no transfer function exists that describes this relation. The approach that will be followed in this paper is to obtain a transfer function approximation for the frequencies of interest.

The digital control system is actually implemented as shown in Fig. 7a. The thick lines in the block diagram represent paths along which the signals are described by Laplace transforms, while the thin lines represent digital signals. The block $\gamma(s)$ is an anti-aliasing analog prefilter. Details on the problem of aliasing can be found in the literature (Astrom and Wittenmark, 1980). If the IMC structure is implemented, the block diagram is described in Fig. 7b. The

following notation is used:

$$y(z) = \mathcal{Z}\mathcal{L}^{-1}\{y_c(s)\} \tag{3.1.15}$$

$$d(z) = \mathcal{Z}\mathcal{L}^{-1}\{d_c(s)\} \tag{3.1.16}$$

$$r(z) = \mathcal{Z}\mathcal{L}^{-1}\{r_c(s)\} \tag{3.1.17}$$

$$\tilde{P}_\gamma(z) = \mathcal{Z}\mathcal{L}^{-1}\{\gamma(s)H(s)\tilde{P}_c(s)\} \tag{3.1.18}$$

Note that when one wishes to use a $\gamma(s) \neq 1$ in the case of an open-loop unstable plant, the simplest way to avoid any internal stability problems is to use $\tilde{P}_\gamma(z)$ instead of $\tilde{P}(z)$ in Step 1. For the rest of this section we shall assume that $\gamma(s) = 1$ in order to simplify the notation. From Fig. 7b it follows that

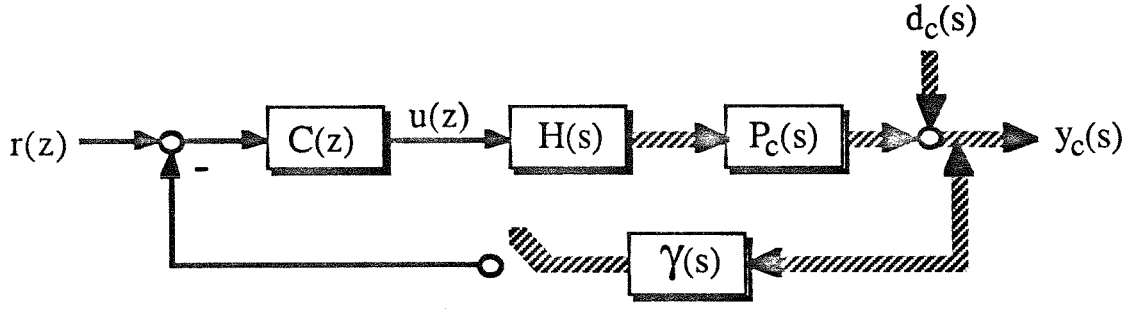
$$\begin{aligned} e_c(s) &= y_c(s) - r_c(s) \\ &= (d_c(s) - r_c(s)) - P_c(s)H(s)\tilde{Q}(e^{sT})F(e^{sT}) \\ &\quad (I - (P(e^{sT}) - \tilde{P}(e^{sT}))\tilde{Q}(e^{sT})F(e^{sT}))^{-1}(d(z) - r(z)) \end{aligned} \tag{3.1.19}$$

We shall now obtain an approximation to (3.1.19) by considering the frequencies $0 \leq \omega \leq \pi/T$. Note that because of the periodicity of $Q(z)$, these are the only frequencies that one can influence independently by using a digital controller. From (3.1.12) it follows that if $a_c(s)$ is small for $\omega > \pi/T$, then

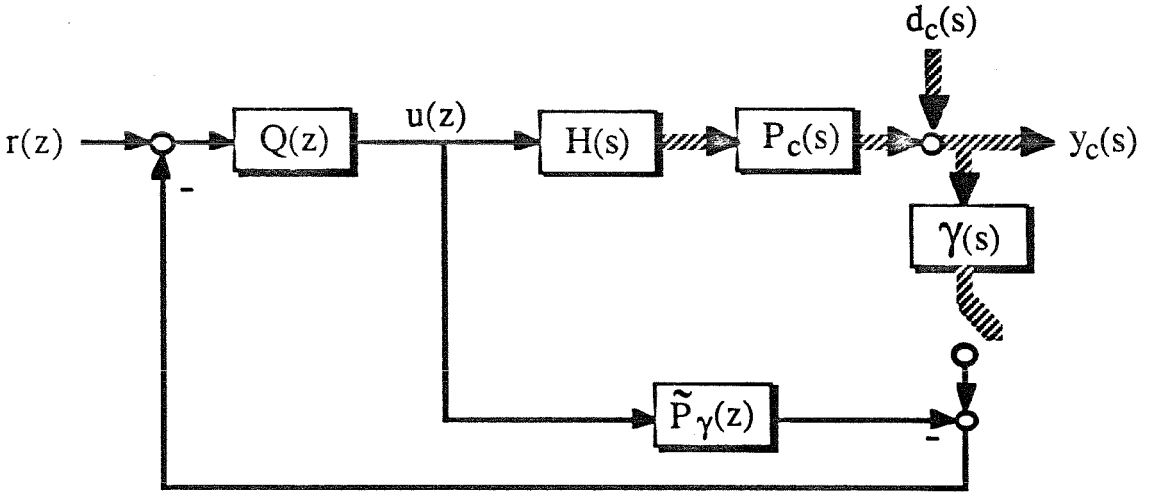
$$a(e^{i\omega T}) \approx \frac{1}{T}a_c(i\omega), \qquad 0 \leq \omega \leq \pi/T \tag{3.1.20}$$

Use of (3.1.20) for all the z-transforms in (3.1.19) yields the approximation

$$\begin{aligned} e_c(i\omega) \approx &(I - P_c(i\omega)Q(e^{i\omega T})F(e^{i\omega T})H(i\omega)/T) \\ &(I - (P_c(i\omega) - \tilde{P}_c(i\omega))\tilde{Q}(e^{i\omega T})F(e^{i\omega T})H(i\omega)/T)^{-1}(d_c(i\omega) - r_c(i\omega)) \\ \overset{\text{def}}{=} &E_c(i\omega)(d_c(i\omega) - r_c(i\omega)) \end{aligned} \tag{3.1.21}$$



(a)



(b)

Figure 7. Control system implemented on the continuous plant.

(a) Feedback structure.

(b) IMC structure.

Note that the above approximation is valid when the input signals r_c, d_c are small for $\omega > \pi/T$. Clearly, for setpoints one should always select a sampling time small enough to allow tracking of r_c. Note that in reality one does not really have an $r_c(s)$. We can always however assume that $r_c(s)$ represents in the time domain a staircase function that corresponds to the points of $Z^{-1}\{r(z)\}$. For disturbances, if one expects high frequency content at $\omega > \pi/T$ and one cannot reduce T any more, then one should use an anti-aliasing prefilter, whose function is to cutoff frequencies higher than π/T.

Let us use the notation $\tilde{E}_c(i\omega) = E_c(i\omega)$ when $P_c = \tilde{P}_c (\Leftrightarrow P = \tilde{P})$. In the first step of the IMC design procedure, $\tilde{Q}$ is obtained so that it produces satisfactory disturbance rejection and/or setpoint tracking. Since $\tilde{E}_c$ connects the external inputs to the error e_c, a well-designed control system produces a relatively "small" $\tilde{E}_c$. A measure of the magnitude of the known $\tilde{E}_c$ is its maximum singular value. Let $b(\omega)$ be a frequency function such that

$$\bar{\sigma}(\tilde{E}_c(i\omega)) < b(\omega), \qquad 0 \leq \omega \leq \pi/T \tag{3.1.22}$$

When $P \neq \tilde{P}$, the "sensitivity" function E_c is described by (3.1.21). In order for the performance of the control system to remain robust with respect to model-plant mismatch we have to keep e_c small in spite of the modeling error. Similarly to the discussion in Section 3.1.2, we can represent the relation between e_c and $v_c (= d_c - r_c)$ in block diagrams of the form of Figures 5 and 6. The only difference is that we now use $(H(s)\tilde{P}_c(s)/T)$ instead of $\tilde{P}(z) (= \tilde{P}(e^{sT}))$ in G and that the block Δ is obtained from the modeling error in $H(s)(P_c(s) - \tilde{P}_c(s))/T$ and so G^u depends on the continuous plant uncertainty as well. We shall use the subscript c to indicate that.

Then we require:

$$\max_{0 \leq \omega \leq \pi/T} \bar{\sigma}(b(\omega)^{-1} E_c(i\omega)) < 1 \qquad \forall \Delta \in \mathbf{\Delta} \tag{3.1.23}$$

We can now use the properties of the SSV (Doyle, 1985) to obtain

$$\max_{0\le\omega\le\pi/T} \bar{\sigma}(b(\omega)^{-1}E_c(i\omega)) < 1 \quad \forall\Delta\in\mathbf{\Delta} \Longleftrightarrow \max_{0\le\omega\le\pi/T} \mu_{\mathbf{\Delta}^0}(G_c^b) < 1 \qquad (3.1.24)$$

where

$$G_c^b = \begin{pmatrix} I & 0 \\ 0 & b^{-1} \end{pmatrix} G_c^F \qquad (3.1.25)$$

$$\mathbf{\Delta}^0 = \{diag(\Delta, \Delta^0) | \Delta\in\mathbf{\Delta}, \Delta^0 \in C^{n\times n}\} \qquad (3.1.26)$$

3.2 Solution to the Filter Synthesis Problem

In this section the filter design problem is formulated and solved as a parameter optimization problem.

3.2.1 Filter Form.

At this point some structure has to be assumed for F, which can be of any general type that the designer wishes. However in order to keep the number of variables in the optimization problem small, a rather simple structure like a diagonal F with first or second–order terms would be recommended. In most cases this is not restrictive because the potentially higher orders of the model $\tilde{P}$ have been included in the controller $\tilde{Q}$ that was designed in the first step of the IMC procedure and which is in general a full matrix. The use of more complex filter structure may be necessary in cases of highly ill-conditioned systems ($\bar{\sigma}(\tilde{P})/\underline{\sigma}(\tilde{P})$ very large). The filter structure for such systems is discussed in detail in Zafiriou and Morari (1986c).

Some additional restrictions on the filter exist in the case of an open-loop unstable plant. The filter $F(z)$ is chosen to be a diagonal rational function that satisfies the following requirements.

a. Internal Stability. IS1 in (2.1.3) must be stable.

b. Asymptotic setpoint tracking and/or disturbance rejection. $(I-\tilde{P}\tilde{Q}F)v$ must be stable.

Write

$$F(z) = diag(f_1(z), \ldots, f_n(z)) \tag{3.2.1}$$

Then, assumptions A.1, 2, 3, 4, 5 and the facts that $\tilde{Q}(z)$ is designed to make IS1 and $(I - P\tilde{Q})V$ stable, imply that the requirements on an element f_l of F are:

$$\frac{d^j}{dz^j}(1 - f_l(z))|_{z=\pi_i} = 0,\ j = 0, \ldots, m_i - 1((m_{ol} - 1) \text{ for } i = 0), i = 0, 1, \ldots, k \tag{3.2.2}$$

where $\pi_0 = 1$ and m_{ol} is the highest multiplicity of π_0 as pole of an element of the l^{th} row of V. Note that for $j = 0$, (3.2.2) yields

$$f_l(\pi_i) = 1, \qquad i = 0, 1, \ldots, k \tag{3.2.3}$$

(3.2.3) clearly shows the limitation that poles outside the UC place on the robustness properties of a control system designed for an open-loop unstable plant. Since because of (3.2.3) one cannot reduce the nominal ($P = \tilde{P}$) closed-loop bandwidth of the system at frequencies corresponding to the unstable poles of the plant, one can only tolerate a relatively small model error at those frequencies.

One can now select for a filter element, the form

$$f(z) = \Phi(z) f_1(z) \tag{3.2.4}$$

where

$$f_1(z) = \frac{(1 - \alpha)z}{z - \alpha} \tag{3.2.5}$$

$$\Phi(z) = \sum_{j=0}^{w} \beta_j z^{-j} \tag{3.2.6}$$

and the coefficients $\beta_0, \ldots, \beta_w$ are computed so that (3.2.2) is satisfied for some specified α. The parameter α can be used as a tuning parameter.

Note that for $k = 0, \pi_0 = 1, m_{ol} = 1$, we only need $\Phi(z) = 1$. For the general case, (3.2.2) can be transformed into a system of ν_l linear equations with $\beta_0, \ldots \beta_w$

as unknowns where ν_l is given by

$$\nu_l = m_{ol} + m_1 + \ldots + m_k \tag{3.2.7}$$

Since none of the π_i is 0 or ∞, (3.2.2) is equivalent to

$$\frac{d^j}{d\lambda^j}(1 - f_l(\lambda^{-1}))|_{\lambda=\pi_i^{-1}} = 0,$$

$$j = 0, 1, \ldots, m_i - 1 \,((m_{ol} - 1) \quad \text{for} \quad i = 0),\ i = 0, 1, \ldots, k \tag{3.2.8}$$

Then the fact that $\Phi(\lambda^{-1})$ is a polynomial in λ, and the following theorem can help simplify the necessary algebra.

Theorem 3.2.1. (Zafiriou and Morari, 1986a).

For the $f_1(z)$ given by (3.2.5) we have

$$\frac{d^j}{d\lambda^j} f_1(\lambda^{-1}) = (1 - \alpha) j! \alpha^j (1 - \alpha\lambda)^{-(j+1)} \tag{3.2.9}$$

One should select $w \geq \nu_l - 1$ so that the system of linear equations has one or more solutions. When $w \geq \nu_l$ we have an underdetermined system and then $\beta_0, \ldots, \beta_w$ can be obtained as the minimum norm solution. Note that for $\nu_l = 2$ one sould select $w \geq 2$ in order to avoid the trivial solution $f(z) = 1$.

Let us now examine the usual situation where $m_i = 1$ for $i = 1, \ldots, k$. Then (3.2.2) is equivalent to:

$$\frac{d^j}{dz^j}(1 - f(z))|_{z=\pi_0=1} = 0, \quad j = 0, \ldots, m_{ol} - 1 \tag{3.2.10}$$

$$f(\pi_i) = 1, \qquad i = 1, \ldots, k \tag{3.2.11}$$

Then for this special case, the following theorem holds:

Theorem 3.2.2

When $m_i = 1$ for $i = 1, \ldots, k$, the coefficients $\beta_0, \ldots, \beta_w$, have to satisfy

$$\begin{pmatrix} \Pi \\ N \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_w \end{pmatrix} = \begin{pmatrix} f_1(\pi_k)^{-1} \\ \vdots \\ f_1(\pi_0)^{-1} \\ -\alpha/(1-\alpha) \\ 0 \\ \vdots \\ 0 \end{pmatrix} \stackrel{\text{def}}{=} \chi \tag{3.2.12}$$

where

$$\Pi = \begin{bmatrix} 1 & \pi_k^{-1} & \ldots & \pi_k^{-w} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \pi_0^{-1} & \ldots & \pi_0^{-w} \end{bmatrix} \tag{3.2.13}$$

and the elements ν_{ij} of the $(m_{ol} - 1) \times (w + 1)$ matrix N are defined by

$$\nu_{ij} = \begin{cases} 0 & \text{for } i \geq j \\ \frac{(j-1)!}{(j-1-i)!} & \text{for } i < j \end{cases} \tag{3.2.14}$$

Proof: It follows directly from Theorem 3 of Zafiriou and Morari (1986a) and (3.2.11).

In general a $w \geq \nu_l - 1$ should be selected and $\beta_0, \beta_1, \ldots, \beta_w$ be obtained as the minimum norm solution to (3.2.12):

$$\begin{pmatrix} \beta_0 \\ \vdots \\ \beta_w \end{pmatrix} = A^T (A A^T)^{-1} \chi \tag{3.2.15}$$

where

$$A \stackrel{\text{def}}{=} \begin{pmatrix} \Pi \\ N \end{pmatrix} \tag{3.2.16}$$

Note that in general it is numerically preferable to compute the pseudo-inverse in (3.2.15) from a singular value decomposition (SVD) of A.

Example 3.2.1: Assume that P and V have one common unstable pole at $\pi \neq 1$, and that $m_{ol} = 0$. Then

$$A = [1 \quad \pi^{-1} \ldots \pi^{-w}] \tag{3.2.17}$$

$$\chi = \frac{\pi - \alpha}{(1-\alpha)\pi} \tag{3.2.18}$$

and (3.2.15) yields

$$\beta_j = \frac{1-\pi^{-2}}{1-\pi^{-2(w+1)}} \frac{(\pi-\alpha)\pi^{-j-1}}{1-\alpha}, \qquad j = 0, \ldots, w \tag{3.2.19}$$

Note that

$$\sum_{j=0}^{w} \beta_j^2 = \left(\frac{\pi-\alpha}{(1-\alpha)\pi}\right)^2 \frac{1-\pi^{-2}}{1-\pi^{-2(w+1)}} \longrightarrow \frac{(\pi-\alpha)^2(\pi^2-1)}{(1-\alpha)^2\pi^4} \tag{3.2.20}$$

since $\pi > 1$. Hence when π is close to (1,0), the properties of $f(z)$ in (3.2.4) are similar to those of $f_1(z)$.

Example 3.2.2. (Zafiriou and Morari, 1986a):

Assume that $k = 0$ and that $m_{ol} = 2$. Then the minimum norm solution is

$$\beta_j = -\frac{6j\alpha}{(1-\alpha)w(w+1)(2w+1)}, \qquad j = 1, \ldots, w \tag{3.2.21}$$

$$\beta_0 = 1 - \sum_{j=1}^{w} \beta_j \tag{3.2.22}$$

The norm of this solution goes to 0 as $w \to \infty$ and so the properties of $f(z)$ in (3.2.4) are similar to those of $f_1(z)$ when w is large enough.

3.2.2 Objective

We can write

$$F \stackrel{\text{def}}{=} F(z;\Lambda) \tag{3.2.23}$$

where Λ is an array with the filter parameters.

The problem can now be formulated as a minimization problem over the elements of the array Λ. A constraint is that the elements of Λ corresponding to denominator poles should be such that F is a stable transfer function. Note that if a diagonal filter with elements given by (3.2.4) is used, then each element of Λ corresponds to same α_j, which has to be inside the UC for F to be stable.

Then the problem can be turned into an unconstrained one by using some λ_j as an element of Λ and writing

$$\alpha_j = e^{-T/\lambda_j^2}$$

or

$$\alpha_j = e^{-T/|\lambda_j|} \tag{3.2.24}$$

Hence any λ_j in $(-\infty, \infty)$ produces an α_j in (0,1]. Note that if one wishes to use in (3.2.4) a higher order $f_1(z)$ with more parameters, one can write the denominator of each element of F as a product of polynomials of degree 2 and one of degree 1 if the order is odd. Then similarly to (3.2.24) one can write the roots of the polynomials of degree 2 as e^{Tp_1}, e^{Tp_2}, where p_1, p_2 are the roots of $\lambda_2^2 x^2 + \lambda_1^2 x + 1 = 0$ or $|\lambda_2| x^2 + |\lambda_1| x + 1 = 0$ and in doing so turn the problem into an unconstrained one in λ_1, λ_2.

Our goal is to satisfy (3.1.24). The filter parameters can be obtained by solving

$$\min_{\Lambda} \max_{0 \le \omega \le \pi/T} \mu_{\Delta^o}(G_c^b) \tag{P4}$$

It may be however that the optimum values for (P4), still do not manage to satisfy (3.1.24). The reason is usually that the performance requirements set by the selection of $b(\omega)$ in (3.1.22) are too tight to satisfy in the presence of model-plant mismatch. In this case one should choose a less tight bound b and resolve (P4). Note that satisfaction of the Robust Performance condition (3.1.24) does not necessarily imply satisfaction of the Robust Stability condition (3.1.14). Hence when a solution to (P4) is found, one should check if (3.1.14) holds. Note that if the uncertainty description for the continuous plant (used in (3.1.24)) and the discretized plant (used in (3.1.14)) are equivalent, then satisfaction of (3.1.24) is usually sufficient for satisfaction of (3.1.14). If this does not happen then one can always substitute the objective function in (P4) with the μ function given in

(3.1.14) and proceed with the minimization only up to the point where (3.1.14) is satisfied. The computational issues remain the same as those discussed in Section 3.2.3 for (P4). Note however that the result of (P4) may often be a local minimum. To circumvert this problem, (P4) should be solved for a number of starting points. Good initial guesses can often be obtained for the filter parameters by matching them with the frequencies where the peaks of $\mu_{\Delta^0}(G_c^b)$ appears for $F = I$.

3.2.3 Computational Issues

The computational of μ in (P4) is made through (3.1.3); details can be found in Doyle (1982). As it was pointed out in Doyle (1985), the minimization of the Frobenius norm instead of the maximum singular value yields D's which are very close to the optimal ones for (3.1.3). Note that the minimization of the Frobenius norm is a very simple task. In the computation of the maximum in (P4) only a finite number of frequencies will be considered. Hence (P4) is transformed into

$$\min_{\Lambda} \max_{\omega \in \Omega} \inf_{D \in \mathbf{D}^0} \bar{\sigma}(DG_c^b D^{-1}) \qquad (P4')$$

where Ω is a set containing a finite number of frequencies in $[0, \pi/T]$ and $\mathbf{D}^0$ is the set corresponding to $\mathbf{Delta}^0$ according to (3.1.1) and (3.1.4). Define

$$\Phi_\infty(\Lambda) \stackrel{\text{def}}{=} \max_{\omega \in \Omega} \inf_{D \in \mathbf{D}^0} \bar{\sigma}(DG_c^b D^{-1}) \qquad (3.2.25)$$

The analytic computation of the gradient of Φ_∞ with respect to Λ is in general possible. This is not the case when the two or more largest singular values of DG^bD^{-1} are equal. However this is quite uncommon and although the computation of a generalized gradient is possible, experience has shown the use of a mean direction to be satisfactory. A similar problem appears when the $\max_{\omega \in \Omega}$ is attained at more than one frequencies, but again the use of a mean direction seems to be sufficient.

Assume that for the value of Λ where the gradient of $\Phi_\infty(\Lambda)$ is computed, the $\max_{\omega \in \Omega}$ is attained at $\omega = \omega_0$ and that the $\inf_{D \in D^0} \bar{\sigma}(DG_c^b(i\omega_0)D^{-1})$ is obtained

at $D = D_0$, where only one singular value σ_1 is equal to $\bar{\sigma}$. Let the singular value decomposition (SVD) be

$$D_0 G_c^b(i\omega_0) D_0^{-1} = \begin{pmatrix} u_1 & U \end{pmatrix} \begin{pmatrix} \sigma_1 & 0 \\ 0 & \Sigma \end{pmatrix} \begin{pmatrix} v_1^* \\ V^* \end{pmatrix} \tag{3.2.26}$$

Then from Zafiriou and Morari (1986 c,d) we have, under the above assumption, that the element of the gradient vector corresponding to the jth element of Λ, λ_j, is given by

$$\frac{\partial}{\partial \lambda_j}\Phi_\infty = Re\Big[u_1^* D_0 \begin{pmatrix} G^u_{c,13} \\ b^{-1} G^u_{c,23} \end{pmatrix} (I - F G^u_{c,33})^{-1} \frac{\partial}{\partial \lambda_j} F|_{\omega=\omega_0}$$
$$(I - F G^u_{c,33})^{-1} (G^u_{c,31} \quad G^u_{c,32}) D_0^{-1} v_1\Big] \tag{3.2.27}$$

where $F, G^u_{c,ij}, b$ are computed at $\omega = \omega_0$. The derivatives of F with respect to its parameters (elements of Λ) depend on the particular form that the designer selected and they can be easily computed.

4. Illustration.

In this section an example is presented to demonstrate the problem of intersample rippling in the H_2-optimal controller and the modification that was discussed in Section 2.3. Consider the continuous system

$$P_c(s) = \begin{pmatrix} \frac{0.50}{s+1} & \frac{1.42}{6s+1} \\ \frac{1.00}{2s+1} & \frac{1.00}{4s+1} \end{pmatrix} \tag{4.0.1}$$

The discretized system (zero order hold included) for a sampling time of $T = 1$, is

$$P(z) = \begin{pmatrix} \frac{0.316}{z-0.368} & \frac{0.218}{z-0.846} \\ \frac{0.393}{z-0.607} & \frac{0.221}{z-0.779} \end{pmatrix} \tag{4.0.2}$$

Computation of the roots of $det[P(z)]$ show that the system in (4.0.2) has two finite zeros, at $a_1 = -0.95$ and $a_2 = 0.75$. The first zero is close to (-1,0) and it is expected to cause intersample rippling when the H_2-optimal controller is used.

From (4.0.2) it follows that $P_A = z^{-1}I$, $P_M = zP$. We shall consider step setpoint changes as external inputs, i.e.,

$$V(z) = \frac{z}{z-1} I \tag{4.0.3}$$

Then (2.4.1) yields

$$\tilde{Q}_H(z) = z^{-1}P^{-1} \tag{4.0.4}$$

Fig. 8a shows the time response of this control system for a unit step change in the setpoint of output 1:

$$v(z) = \begin{pmatrix} z/(z-1) \\ 0 \end{pmatrix} \tag{4.0.5}$$

The prediction of intersample rippling is verified. Note that at the sample points the outputs are indeed exactly at the setpoints producing the minimum SSE.

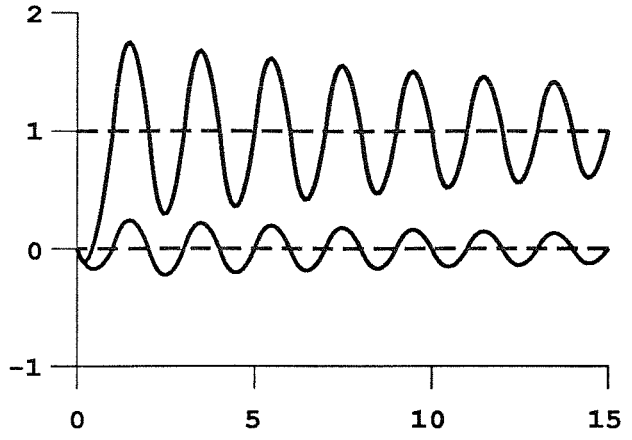
The IMC controller is now obtained from (2.3.2) where $b(z) = 1$ and

$$q_-(z) = \frac{z + 0.95}{1.95z} \tag{4.0.6}$$

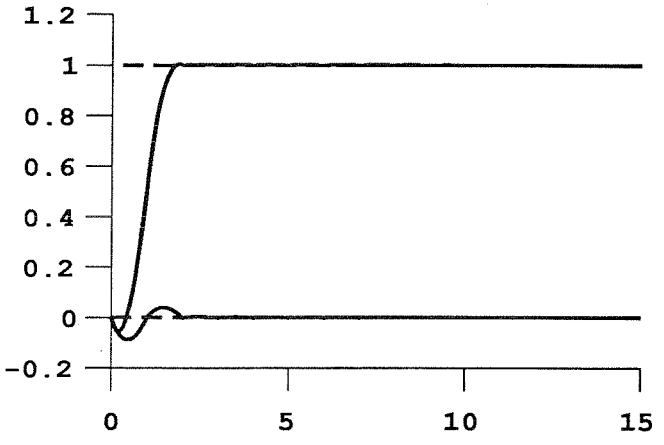
The response for this control system is shown in Fig. 8b. Clearly the problem has disappeared. Finally note that all responses show an inverse response characteristic. This is due to the fact that the continuous system $P_c(s)$ has a right half plane zero.

5. Concluding Remarks.

The results presented in this paper provide a direct synthesis procedure for digital multivariable controllers. The two-step IMC design concept is extended to open-loop unstable systems and the limitations imposed by open-loop unstable poles on achievable performance and robustness are quantified. In the first step the controller is designed for a whole set of external inputs (setpoints or disturbances) and it combines desirable properties of the H_2-optimal and deadbeat type controllers. In the second step the parameters of the low–pass IMC filter are obtained as the result of the optimization of an SSV based objective function, which reflects the performance of the continuous plant outputs, so that bad intersample behavior is avoided. The use of the SSV allows the treatment of general types of model-plant mismatch.



(a)



(b)

Figure 8. Time response.

- Dashed lines: Setpoints; Solid lines: Outputs.

(a) H_2−optimal controller $\tilde{Q}_H$.

(b) IMC controller $\tilde{Q}$.

APPENDIX A

A.1. Proof of Theorem 2.1.1.

The fact that Q_1 has to be proper in order for Q to be proper and vice versa, follows from the properness of Q_0 and b_p.

i) We shall show that any Q given by (2.1.7) makes IS1 stable. From substitution of (2.1.7) into (2.1.3) it follows that all that is required is that $(Pb_p^2Q_1 \quad b_p^2Q_1P \quad Pb_p^2Q_1P)$. be stable. From the properties of Q_1, it follows that the third element in the above matrix is stable. Stability of the other two follows by pre- and post-multiplication of that element by P^{-1}, since according to assumptions A.1, A.2, P has no zeros at the location of its unstable poles and these are the only possible unstable poles in the above matrix.

ii) Assume that Q makes IS1 stable. Then the difference matrix

$$IS1(Q) - IS1(Q_0) = ((Q - Q_0 \quad P(Q - Q_0) \quad (Q - Q_0)P \quad P(Q - Q_0)P) \tag{A.1.1}$$

is stable.

The fact that P has no zeros at the location of the unstable poles makes the stability of the matrix in (A.1.1) equivalent to the stability of $(Q - Q_0)$, $P(Q - Q_0)P$. Then, when assumption A.1.c holds, we can write $P = b_p\hat{P}$, where $\hat{P}$ has no zeros at the unstable poles of P and its only unstable poles are at $z = 1$. So, it follows that $(Q - Q_0) = b_p^2Q_1$ with Q_1 stable and such that PQ_1P have no poles at $z = 1$. If A.1.c does not hold, Q_1 should also have the property that it makes PQ_1P stable.

A.2. Proof of Theorem 2.2.1.

We shall assume that a Q_0 exists, which in addition to the properties mentioned in Theorem 2.1.1, it also produces a matrix $(I - PQ_0)V^0$ with no poles at $z = 1$, where V^0 is a diagonal matrix with l_v poles at $z = 1$ in every element, with l_v the maximum number of such poles in any element of v. If assumption

A.1.c does not hold, then each column of V^0 also satisfies A.3.b and Q_0 makes $(I - PQ_0)V^0$ stable. Its existence will be proven by construction. Substitution of (2.1.7) into (2.2.4) and use of the fact that pre- or post-multiplication of a function with an allpass does not change its L_2-norm, yields:

$$\begin{aligned}\phi(v) &= ||z^{-1}b_p^{-1}b_v P_A^{-1}(I - PQ_0)\hat{v} - z^{-1}b_p b_v P_M Q_1 \hat{v}||_2^2 \\ &\stackrel{\text{def}}{=} ||f_1 - f_2 Q_1 \hat{v}||_2^2 \end{aligned} \qquad (A.2.1)$$

f_1 has no poles at $z = 1$ because $(I - PQ_0)V^0$ has no such poles. Any rational function $f_1(z)$ with no poles on the UC, can be uniquely decomposed into a strictly proper, strictly stable part $\{f_1\}_-$ in $H_2^\perp$ and a strictly unstable part $\{f_1\}_+$ in H_2:

$$f_1 = \{f_1\}_- + \{f_1\}_+ \qquad (A.2.2)$$

When A.1.c holds, inspection of (A.2.1) shows that $f_2 Q_1 \hat{v}$ can have no poles on or outside the UC except possibly for some poles at $z = 1$ introduced by $\hat{v}$. Since f_1 has no poles at $z = 1$, in order for $\phi(v)$ to be finite, $f_2 Q_1 \hat{v}$ should have no poles at $z = 1$. Thus the optimal Q_1 has to cancel any such poles. When A.1.c does not hold, then the fact that $(I - PQ_0)V^0$ is stable and A.1.b imply that an acceptable Q_1 and therefore the optimal Q_1 is such that $f_2 Q_1 v$ is stable. We shall assume that Q_1 has this property. It should be verified at the end however that the solution indeed has the property. Since $f_2 Q_1 \hat{v}$ is strictly proper in addition to being stable, we can write

$$\phi(v) = ||\{f_1\}_+||_2^2 + ||\{f_1\}_- - f_2 Q_1 \hat{v}||_2^2 \qquad (A.2.3)$$

The first term in the right hand side of (A.2.3) does not depend on Q_1. Hence for solving (P1) we only have to look at the second term. The obvious solution is

$$Q_1 \hat{v} = f_2^{-1}\{f_1\}_- \qquad (A.2.4)$$

Clearly such a Q_1 produces a stable $f_2 Q_1 \hat{v}$ as it was assumed. It should now be proved that Q_1's that satisfy the internal stability requirements exist among those described by (A.2.4), so that the obvious solution is a true solution. For $n = 1$, (A.2.4) yields a unique Q_1, which can be shown to satisfy the requirements by following the arguments in the proof of Theorem 2.2.2 in Appendix A.3. For $n \geq 2$ write

$$\hat{v} \stackrel{\text{def}}{=} (\hat{v}_1 \quad \hat{v}_2 \quad \dots \quad \hat{v}_n)^T \tag{A.2.5}$$

$$\hat{V}_2 \stackrel{\text{def}}{=} (\hat{v}_2 \quad \dots \quad \hat{v}_n)^T \tag{A.2.6}$$

$$Q_1 \stackrel{\text{def}}{=} (q_1 \quad q_2) \tag{A.2.7}$$

where without loss of generality the first element of v is assumed to be nonzero. Also q_1 is $n \times 1$ and q_2 is $n \times (n-1)$. Then from (A.2.4) it follows that

$$Q_1 = (\hat{v}_1^{-1}(f_2^{-1}\{f_1\}_- - q_2 \hat{V}_2) \quad q_2) \tag{A.2.8}$$

We now need to show that a proper, stable q_2 exists such that Q_1 is proper, stable and produces a PQ_1P with no poles at $z = 1$ (and no poles outside the UC, when A.1.c does not hold). Select a q_2 of the form:

$$q_2(z) = \hat{q}_2(z)(1 - z^{-1})^{3l_v} \prod_{i=1}^{k} (1 - \pi_i z^{-1})^3 \tag{A.2.9}$$

where $\hat{q}_2$ is proper, stable. Then from (A.2.8) it follows that in order for PQ_1P not to have any poles at $z = 1$ it is sufficient that $P\hat{v}_1^{-1} f_2^{-1}\{f_1\}_-\{P\}_{1^{st}row}$ have no such poles. This holds because the poles at $z = 1$ in the P on the left cancel with the P_M^{-1} in f_2^{-1} and v_1 (and $\hat{v}_1$) has by assumption A.4 at least as many poles at $z = 1$ as the 1st row of P. When A.1.c does not hold, then the same type of argument and the fact that A.3.b holds imply that PQ_1P has no poles outside the UC either. Let us now examine the stability of Q_1. The only poles outside

the UC may come from $\hat{v}_1^{-1}$. Let α be such a pole (zero of v_1). Then for stability we need to find $\hat{q}_2$ such that

$$\hat{q}_2(\alpha)\hat{V}_2(\alpha) = (1-\alpha^{-1})^{-3l_v}\prod_{i=1}^{k}(1-\pi_i\alpha^{-1})^{-3}f_2^{-1}(\alpha)\{f_1\}_-(\alpha) \qquad (A.2.10)$$

The above equation always has a solution because the vector $\hat{V}_2(\alpha)$ is not identically zero since any common outside the UC zeros in v were factored out in v_0. We now need to examine the properness of Q_1. Since P_M^{-1} is proper and $\{f_1\}_-$ is strictly proper, $f_2^{-1}\{f_1\}_-$ is proper. Then if $\hat{v}_1^{-1}$ is improper ($\hat{v}_1$ strictly proper) there exist at least one element in $\hat{V}_2$ that is semi-proper. Hence by solving a system of linear equations we can always select a $\hat{q}_2(z)$ such that of the first impulse response coefficients of $f_2^{-1}\{f_1\}_- - q_2\hat{V}_2$, as many are zero as we need to make the first element of the matrix in (A.2.8) proper.

We shall now proceed to obtain an expression for $Q\hat{v}$. (2.1.7) and (A.2.8) yield

$$\begin{aligned} Q\hat{v} =& zb_pb_v^{-1}P_M^{-1}\left[z^{-1}b_p^{-1}b_vP_A^{-1}PQ_0\hat{v} - \{z^{-1}b_p^{-1}b_vP_A^{-1}PQ_0\hat{v}\}_-\right. \\ &\left. + \{z^{-1}b_p^{-1}b_vP_A^{-1}\hat{v}\}_-\right] \\ =& zb_pb_v^{-1}P_M^{-1}\left[\{z^{-1}b_p^{-1}b_vP_A^{-1}PQ_0\hat{v}\}_{0+} + \{z^{-1}b_p^{-1}b_vP_A^{-1}\hat{v}\}_-\right] \qquad (A.2.11) \end{aligned}$$

where $\{\cdot\}_{0+}$ indicates that in the partial fraction expansion all poles on or outside the UC are retained. For (A.2.11), these poles are the poles of $b_p^{-1}b_v\hat{v}$ on or outside the UC; $P_A^{-1}PQ_0 = P_MQ_0$ is strictly stable and proper because Q_0 is a stabilizing controller. When A.1.c holds, the stability of $(I-PQ_0)P$ and the fact that the residues of P at the outside the UC poles are full rank imply that at these poles $I - PQ_0 = 0$. Also the fact that $(I-PQ_0)V^0$ has no poles at $z=1$ imply that $(I-PQ_0)$ and its derivatives up to the $(l_v-1)^{th}$ are also equal to zero at $z=1$. When A.1.c does not hold, the fact that $(I-PQ_0)V^0$ is stable and

that the columns of this diagonal V^0 satisfy A.3.b, imply that $I - PQ_0 = 0$ at $1, \pi_1, ..., \pi_k$. Thus (A.2.11) simplifies to (2.2.8).

We simply need to establish now that a stabilizing controller Q_0 with the property that $(I - PQ_0)V^0$ has no poles at $z = 1$ exists. The use in (2.2.11) of a V^0 with the properties mentioned at the beginning of this section, instead of V, yields such a controller.

A.3. Proof of Theorem 2.2.2.

The L_2-norm for a matrix $G(z)$ analytic on the UC is given by

$$||G||_2 = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} trace[G^*(e^{i\theta})G(e^{i\theta})] \quad d\theta \right)^{1/2} \qquad (A.3.1)$$

Then from (2.2.1), (A.3.1) it follows that

$$\phi(v^1) + \phi(v^2) + \ldots + \phi(v^n) = ||(I - P\tilde{Q})V||_2^2 \stackrel{\text{def}}{=} \phi(V) \qquad (A.3.2)$$

The minimization of $\phi(V)$ follows the steps in the proof of Theorem 2.2.1 in Appendix A.2 up to (A.2.4), with V_M used instead of $\hat{v}$. In this case l_v is the maximum number of poles at $z = 1$ in any element of V. From the equivalent to (A.2.4) equation we obtain

$$Q_1 = f_2^{-1}\{f_1\}_- V_M^{-1} \qquad (A.3.3)$$

We now have to establish that Q_1 is stable, proper and produces a PQ_1P with no poles at $z = 1$ (nor outside the UC, when A.1.c does not hold).

In the case where b_p, b_v are not equal to identity, the stability of Q_1 follows from the full rank conditions in A.1.c and A.5.b. In PQ_1P the poles at $z = 1$ of the P on the left cancel with the P_M^{-1} in f_2^{-1}. As for the P on the right, the same follows from assumption A.5.a. When A.1.c does not hold, the same arguments are true for the outside the UC poles as well.

Then in the same way that (2.2.8) follows from (A.2.8), (2.2.11) follows from (A.3.3).

A.4. Proof of Theorem 2.2.3.

A stabilizing controller that solves (P3) has to solve (P1) for all v^i, $i = 1,\ldots,n$. Satisfying (2.2.8) for every v^i is equivalent to

$$\tilde{Q} = zb_p b_v^{-1} P_M^{-1}\{z^{-1}b_p^{-1}b_v P_A^{-1}\hat{V}\}_*\hat{V}^{-1} \qquad (A.4.1)$$

Hence the above $\tilde{Q}$ is the only potential solution to (P3). However it is not necessary a stabilizing controller since not only stabilizing $\tilde{Q}$'s satisfy (2.2.8) for some v. Indeed if $\hat{V}$ is non-minimum phase, $\hat{V}^{-1}$ is unstable and/or improper and this results in an unstable and/or improper $\tilde{Q}$, which is therefore unacceptable. Hence in such a case, there exists no solution to (P3), which completes the proof of part (i) of the theorem.

In the case where $\hat{V}^{-1}$ is stable and proper ($\hat{V}$ minimum phase), the controller given by (A.4.1) is stable and proper and therefore it is the same as the one given by (2.2.11). This fact can be explained as follows. We have

$$V = \hat{V}V_0 \qquad (A.4.2)$$

where

$$V_0 = diag(v_0^1, v_0^2, \ldots, v_0^n) \qquad (A.4.3)$$

Since $\hat{V}^{-1}$ is stable and proper, (A.4.2) represents a factorization of V similar to that in (2.2.10). From spectral factorization theory it follows that

$$\hat{V}(z) = V_M(z)A \qquad (A.4.4)$$

where A is a constant matrix such that $AA^* = I$. Then from (2.2.11) it follows that use of $\hat{V}$ does not alter $\tilde{Q}$ because A cancels.

Let us now assume without loss of generality that the first j v^i's have the same v_0^i's. Consider a v that is a linear combination of $v^1,\ldots,v^j$:

$$v(z) = \alpha_1 v^1(z) + \ldots + \alpha_j v^j(z) \qquad (A.4.5)$$

Then it follows that

$$v_0(z) = v_0^1(z) = \ldots = v_0^j(z) \tag{A.4.6}$$

$$\hat{v}(z) = \alpha_1 \hat{v}^1(z) + \ldots + \alpha_j \hat{v}^j(z) \tag{A.4.7}$$

One can easily check that a $\tilde{Q}$ that satisfies (2.2.8) for $\hat{v}^1, \ldots, \hat{v}^j$, will also satisfy (2.2.8) for the $\hat{v}$ given by (A.4.7) because of the property

$$\{\alpha_1 f_1(z) + \ldots + \alpha_j f_j(z)\}_* = \alpha_1 \{f_1(z)\}_* + \ldots + \alpha_j \{f_j(z)\}_* \tag{A.4.8}$$

But then from Theorem 2.2.1 it follows that if a stabilizing controller $\tilde{Q}$ satisfies (2.2.8) for $\hat{v}$, then it minimizes the L_2- error $\phi(v)$.

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CHAPTER VII

CONCLUSIONS

1. Discussion of the Thesis Results.

The objective of this thesis was to provide a rigorous methodology for the direct synthesis of multivariable controllers, which are robust with respect to modeling error. Most of the work was carried out in the sampled-data systems framework since all but the simplest control algorithms are implemented on digital computers. The Internal Model Control Structure (Garcia and Morari, 1982) was selected as the basis on which to build this methodology, because it allows the incorporation of tuning parameters with clear physical meaning in the controller. H_2-type objectives were set because they are more meaningful for process control problems than H_∞-type objectives, which consider an external input (setpoints or disturbances) set that is much larger than it usually need be for a chemical process. The developed methodology however, can be used with such objectives as well (Zafiriou and Morari, 1987).

In summary, the following contributions were made in this thesis:

SISO sampled-data systems control.

The behavior of a number of well-known digital control algorithms was examined and its relation to the sampling time was explained through the effect of the sampling time on the zeros of the discretized process. This study resulted in the development of a new and quite simple algorithm, which combines the advantages of the examined algorithms and which is free of the problems of intersample rippling, overshoot and/or undershoot, independently of the sampling time that is used. One might consider an indication of the practical need for such a simple, yet effective algorithm, the fact that researchers at Kodak (Rochester, NY) were willing to test it on pilot-plant processes and after very good results to program it on software used routinely for controller design (Juba, 1985; Hamer, 1986).

A procedure was developed for the design of a low–pass IMC filter that guarantees robust stability and performance with respect to model-plant mismatch.

The criteria used in the design take explicitly into account the continuous process output, so that the problem of intersample rippling does not appear. The proposed method includes an iterative procedure for the selection of the appropriate sampling time. This was accomplished by quantifying the effect of the sampling time on robustness and by showing that the limitations imposed by sampling on the achievable performance are similar to those imposed by model uncertainty. As a result, the performance specifications and the model uncertainty information (frequency bounds) are explicitly involved in the selection of the sampling time, which thus becomes an integral part of the controller design.

MIMO systems.

In the first step of the IMC design procedure, the achievable performance is limited by certain inherent characteristics of the process model, like time delays and zeros outside the unit circle or close to (-1,0) on the z-plane. For multivariable processes, there is a structure associated with such zeros and delays, which has to be preserved in the closed-loop transfer matrix. A characterization of this structure was obtained, in the form of linear equation constraints. This characterization was used for the development of a procedure, which addresses structural closed-loop performance specifications. The designer can compare different closed-loop achievable transfer matrices, in terms of the structure, magnitude and duration of the interactions and choose on the basis of quantitative measures. This procedure uses only linear algebra operations and is formulated in a way that allows straightforward computer implementation. Also, time delays and undesirable zeros are handled in a unified way in this method, while past approaches (Garcia and Morari, 1985) attempted to treat them separately, resulting in "suboptimal" designs.

An alternative to the above procedure is to minimize some weighted norm of the transfer matrix relating the errors to the setpoints or disturbances. The

problem with such approaches in the control literature has always been the meaningful selection of the weight. The results obtained in this thesis provide a way to choose a full matrix weight for the H_2-norm, by specifying a set of external inputs of interest (setpoints or disturbances). The optimal controller minimizes the H_2-error (Integral Squared Error for continuous systems; Sum of Squared Errors for discrete) for the specified set of external input directions and their linear combinations, when this is possible. In the case of sampled-data systems, a modification is introduced to avoid intersample rippling. This work also extended the IMC design concept to open-loop unstable systems.

Robustness properties are incorporated in the IMC controller by designing a low-pass filter. The Structured Singular Value (SSV) theory (Doyle, 1982) was used to obtain an appropriate objective function. The goal set at this step is to keep the performance of the control system in the presence of model-plant mismatch as close as possible to the one for which it was designed in the first step. The nominal (no model error) performance was used to provide guidelines for the selection of the weight that is used in this step. The filter design problem was then formulated as a minimization over the filter parameters. Special filter structures for open-loop unstable and for ill-conditioned systems were obtained. The complexity of the "min max min" optimization problem made the computation of analytic expressions for the gradient necessary. Such expressions were developed for the general case and the approach was successfully programmed on the computer and tested for a number of cases. The problems caused by the existence of local minima can be overcome in most cases via good initial guesses available because of the clear physical meaning of the filter parameters. For sampled-data systems, the continuous process output and model uncertainty were taken explicitly into account to avoid bad intersample behavior.

2. Research Suggestions.

It is the purpose of this last section of the thesis, to suggest research directions for certain topics directly or indirectly related to the work presented here. Research in some of these topics can improve or extend the applicability of the end product of the thesis, i.e., of the proposed methodology for robust control system design. For some other topics, research in them is either motivated or can be assisted, in the author's opinion, by some of the partial results and experience obtained in the course of the thesis work.

Let us now consider each of these research issues:

- Ill-conditioned systems. The special two-filter structure that was proposed for the case of ill-conditioned systems has been shown to be satisfactory in a number of examples. The advantage of this approach is its simplicity. However, a disadvantage exists in the fact that one is actually correcting in the second step of the design procedure, a "mistake" made in the first step. The problem is that unless model uncertainty is taken into account in the first step, one cannot be certain to avoid this mistake. A reasonable approach that preserves the spirit of the two-step design procedure, would be to consider in the first step the uncertainty structure, but not the actual magnitudes of the uncertainty bounds. Then by taking into account the sensitivity of the control system to modeling error, bad designs in the first step would be avoided.

- Weight modification during the filter design. Meaningful weight selection in the objective functions that are optimized in control problems is one of the most important issues in control theory. In the methodology presented in this thesis, guidelines are given for the selection of physically meamingful weights in both steps of the design procedure. However, one point that is not very clear, is how to best modify the "bound" $b(\omega)$ in the second step, when the solution of the filter optimization problem fails to satisfy the robust performance condition.

The designer has to "relax" the performance specification and it is usually simple to adjust $b(\omega)$ at the appropriate frequency ranges. Still, experience has shown that this is not always so. Work is needed in order to figure out how to automate this adjustment so that the best achievable performance specification is eventually used. Results in this topic will both improve the effectiveness of the method and make its implementation in an "expert system" form easier.

- SSV-optimization problems. The approach proposed for the IMC filter synthesis is the first successful solution of a parameter optimization problem with a Structured Singular Value (Doyle, 1982) objective function. This is very encouraging in the sense that as it turned out, standard gradient search techniques with minor modifications were adequate for solving a seemingly extremely complicated problem, the basic reason being the understanding of the physical meaning of the optimization parameters. Hence it seems reasonable to attempt to solve other control problems by formulating them as such parameter optimizations of an SSV objective function. Candidates for this treatment are problems that require a specific structure for the control system. An obvious case is the problem of synthezing robust decentralized controllers.

- Parametric model uncertainty. In the case where the model uncertainty can be traced to some parameters that are known to be real numbers, the conservativeness, when using the SSV, can be reduced by avoiding to consider the entire disk that encircles them in the complex plane (Doyle, 1985). The author has been involved in work in the computation of the SSV for real parameter uncertainty and it seems that it should be straightforward to extend the filter synthesis procedure of Chapters V and VI to this case.

- Control implications in process design. The need for control considerations at the stage of process design has been recognized in the literature in the last 5-10 years. The results presented in Chapter IV have the potential of being very useful

in this area of research, because they quantify the performance limitations imposed by inherent characteristics of the process design, like time delays, unstable zeros etc. These results have already prompted some work in that direction (Morari et al., 1986), dealing with the analysis aspect of the problem. Although a lot of work still has to be done in the analysis part, much more important is the synthesis part of the problem. The descriptions of the imposed limitations that were obtained in Chapter IV have the simple form of linear equations and they may allow the deduction of properties of the interconnected process structure from the properties of the individual units.

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