CONSTRUCTION OF SOLUTIONS TO PARTIAL DIFFERENTIAL EQUATIONS BY THE USE OF TRANSFORMATION GROUPS

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ABSTRACT

A systematic approach is given for finding similarity solutions to partial differential equations by the use of transformation groups.

If a one-parameter group of transformations leaves invariant a partial differential equation and its accompanying boundary conditions, then the number of variables can be reduced by one. In order to find the group of a given partial differential equation, the "classical" and "non-classical" methods are discussed. Initially no special boundary conditions are imposed since the invariances of the equation are used to find the general class of invariant boundary conditions.

New exact solutions to the heat equation are derived. In addition new exact solutions are found for the transition probability density function corresponding to a particular class of first order non-linear stochastic differential equations. The equation of nonlinear heat conduction is considered from the classical point of view.

The conformal group in n "space-like" and m "time-like" dimensions, $C(n,m)$, which is the group leaving invariant

$$\left(\sum_{i=1}^{n} - \sum_{i=n+1}^{n+m} \right) \frac{\partial^2 u}{\partial x_i^2} = 0,$$ is shown to be locally isomorphic to

$SO(n+1, m+1)$ for $n + m \geqslant 3$. Thus locally compact operators, besides pure rotations, leave invariant Laplace's equation in $n \geqslant 3$ dimensions. These are used to find closed bounded geometries for which the number of variables in Laplace's equation can be reduced.

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Introduction

In this thesis using group theory we give a systematic approach for finding similarity solutions to partial differential equations.

Originally ([1], [2], [3]) the group-theoretic method was applied to ordinary differential equations. If we are able to find a one-parameter group of transformations which leaves invariant a given ordinary differential equation, then by using a simple recipe we can reduce the order of the equation by one.

In the case of a system of partial differential equations, invariance of the given system and its accompanying boundary conditions under a one-parameter group of transformations leads to a reduction of one in the number of variables. The invariants of the group become the new variables. A familiar example of such a reduction is the class of self-similar solutions which corresponds to invariance under the similitudinous (stretching) group. These are solutions of the form $u(x,y) = x^{\alpha} f\left(xy^{\beta}\right)$ where α and β are to be determined such that the original partial differential equation with dependent variable u and independent variables x and y, is reduced to an ordinary differential equation with variables f and xy^{β}.

Since a Lie group is generated by its infinitesimals, it would seem natural to try to find the largest possible set of infinitesimals leaving invariant a given system of equations. The set of infinitesimals correspondingly determines boundary conditions for which a reduction in the number of variables is possible. Sometimes (e.g. equation of nonlinear heat conduction, Fokker-Planck equation) the given system of partial differential equations contains some arbitrary function. Applying the group-theoretic method, we would try to find the most general form of this function which leads to a reduction in the number of variables.

It turns out that for most problems we are able to find the "classical" set of infinitesimals (which form a Lie algebra) since the classical set of "determining equations" corresponding to the original system is usually solvable by elementary means. However the "non-classical" set of infinitesimals is much harder to find since here the determining equations (fewer in number with the classical case included as a subcase) are often more difficult to solve than the original system. But it must be emphasized that, in principle, any solution to the determining equations reduces the number of variables.

In applying group theory to differential equations a knowledge of representation theory is unnecessary although sometimes helpful as shown in the final part of the thesis. Moreover in using the similarity method it is immaterial whether or not the equations are linear. However, we show that for linear equations one can set up eigenvalue problems by the superposition principle.

We consider examples of elliptic, hyperbolic, and parabolic equations. The one-dimensional heat, Fokker-Planck, and nonlinear diffusion equations, and also a special hyperbolic equation, are studied in detail for similarity solutions.

For the linear heat equation we obtain new "classical" solutions corresponding to particular moving boundary conditions. Some examples of "non-classical" solutions to the heat equation are also given.

Looking at the Fokker-Planck equation from the classical point of view, we are able to extend the class of forcing functions which lead to "exact" solutions.

A systematic classical analysis of the equation of nonlinear heat conduction is given in Chapter V.

In the final part of this thesis we investigate the conformal group $C(n,m)$, i.e., a particular subgroup of the group leaving invariant $\left[\sum_{i=1}^{n} - \sum_{i=n+1}^{n+m}\right] \frac{\partial^2 u}{\partial x_i^2} = 0$. We prove that for $n+m \geqslant 3$ this group is locally isomorphic to $SO(n+1, m+1)$, the group of "rotations" which leaves invariant the quadratic form $\left[\sum_{i=1}^{n+1} - \sum_{i=n+2}^{n+m+2}\right] x_i^2$. From the proof we are able to find compact transformation in addition to elliptic rotations and using these we show that for Laplace's equation in n dimensions there exist closed bounded geometries other than concentric spheres for which the number of variables may be reduced.

After reading Chapters I and II, one could read independently as a unit any one of Chapters III to VI.

Throughout this thesis a repeated index implies summation over the index.

The most complete recent works on group properties of partial differential equations are found in [4], [5].

Chapter I

The Reduction of the Number of Variables of a System of Partial Differential Equations due to Invariance under a One-parameter Transformation Group

1.1 Definition of a One-parameter Lie Group of Transformations.

Let

$$x'^i = f^i(x^1,\ x^2,\ \dots,\ x^n;\ \epsilon) \qquad (1.1.1)$$

for $i = 1, \dots, n$ be a set of transformations of the n variables $x^1, \dots, x^n$ depending on the continuous parameter ϵ. Then in order that these transformations form a Lie group of transformations it is sufficient that:

(1) For each i, $x'^i = x^i$ when $\epsilon = 0$, i.e., the identity element exists.

(2) If $x''^i = f^i(x'^1,\ x'^2,\ \dots,\ x'^n;\ \delta)$, then there exists a function φ defining the law of combination between parameters such that $x''^i = f^i(x^1, x^2, \dots, x^n; \varphi(\epsilon, \delta))$, i.e., the closure property is satisfied.

(3) $\varphi(a, \varphi(b,c)) = \varphi(\varphi(a,b),c)$, i.e. the associative property is satisfied.

(4) For each $\epsilon \ \exists$ a $\delta = \epsilon^{-1}$ such that $\varphi(\epsilon, \epsilon^{-1}) = 0$.

(5) The functions x'^i are differentiable as many times as is necessary with respect to $\underline{x}$ and the parameter ϵ. $(\underline{x} = (x^1, x^2, \dots, x^n))$

1.2 Infinitesimal Transformations

We now expand x'^i about the identity, i.e. we consider

$$x'^i = f^i(x^1,\ x^2,\ \dots,\ x^n; \epsilon) = f^i(x^1, x^2, \dots, x^n; 0) + \epsilon\left(\frac{\partial f^i}{\partial \epsilon}\right)_{\epsilon=0} + \frac{\epsilon^2}{2!}\left(\frac{\partial^2 f^i}{\partial \epsilon^2}\right)_{\epsilon=0} + O(\epsilon^3).$$

Let

$$\left(\frac{\partial f^i}{\partial \epsilon}\right)_{\epsilon=0} = \xi^i(\underline{x})$$

$\{\xi^i\}$ for $i = 1, \ldots, n$ is known as the set of infinitesimals of the group (1.1.1). The operator $L = \xi^j \frac{\partial}{\partial x^j}$ is defined to be the <u>infinitesimal operator</u> of the one-parameter group of transformations (1.1.1)

Under the action of transformations (1.1.1) a function $u(\underline{x})$ is transformed to a new function $u'(\underline{x})$ by the formula $u' = u(\underline{x}')$.

Expanding u' about $\epsilon = 0$ we see that

$$u' = u(\underline{x}) + \epsilon\left(\frac{\partial u'}{\partial \epsilon}\right)_{\epsilon=0} + \frac{\epsilon^2}{2!}\left(\frac{\partial^2 u'}{\partial \epsilon^2}\right)_{\epsilon=0} + \ldots + \frac{\epsilon^n}{n!}\left(\frac{\partial^n u'}{\partial \epsilon^n}\right)_{\epsilon=0} + O\left(\epsilon^{n+1}\right)$$

But

$$\left(\frac{\partial u'}{\partial \epsilon}\right)_{\epsilon=0} = \left(\frac{\partial x'^i}{\partial \epsilon}\right)_{\epsilon=0}\left(\frac{\partial u'}{\partial x'^i}\right)_{\epsilon=0} = \xi^i \frac{\partial u}{\partial x^i} = Lu$$

$$\Longrightarrow \quad \left(\frac{\partial^2 u'}{\partial \epsilon^2}\right)_{\epsilon=0} = \left(\frac{\partial}{\partial \epsilon}\left(\frac{\partial u'}{\partial \epsilon}\right)\right)_{\epsilon=0} = L\left(\frac{\partial u'}{\partial \epsilon}\right)_{\epsilon=0} = L^2 u$$

In general,

$$\left(\frac{\partial^n u'}{\partial \epsilon^n}\right)_{\epsilon=0} = L^n u$$

We see that a Lie transformation is completely specified by its infinitesimal operator. Symbolically, we write $u' = e^{\epsilon L} u$. From now on our study of a continuous group will concentrate on the study of the infinitesimals generated by the group.

1.3 <u>Extended Groups.</u>

In studying group properties of differential equations one must know how derivatives are transformed under group transformations.

Let

$$\left\{\begin{matrix} u_i\,, & i = 1, \ldots, m \\ x^j\,, & j = 1, \ldots, n \end{matrix}\right\}$$

represent dependent and independent variables respectively. Consider a Lie group of transformations:

$$\left\{\begin{aligned} u'_i &= u'_i(x^1, x^2, \ldots, x^n, u_1, u_2, \ldots, u_m; \epsilon) \\ x'^j &= x'^j(x^1, x^2, \ldots, x^n, u_1, u_2, \ldots, u_m; \epsilon) \end{aligned}\right\} \tag{1.3.1}$$

with corresponding infinitesimals $U_i(x^1, x^2, \ldots, x^n, u_1, u_2, \ldots, u_m)$ and $X^j(x^1, x^2, \ldots, x^n, u_1, u_2, \ldots, u_m)$.

We define the total derivative operator

$$\frac{D}{Dx^j} = \frac{\partial}{\partial x^j} + \frac{\partial u_m}{\partial x^j}\frac{\partial}{\partial u_m} \tag{1.3.2}$$

so that

$$\frac{Du_i}{Dx_j} = \frac{\partial u_i}{\partial x^j} = u_{i,j}$$

Consider

$$\frac{\partial u'_i}{\partial x'^j} = u'_{i,j} = \frac{Du'_i}{Dx'^j} = \frac{Du'_i}{Dx^m}\frac{Dx^m}{Dx'^j}$$

$$= \frac{D\left[u_i + \epsilon\, U_i + O(\epsilon^2)\right]}{Dx^m} \cdot \frac{D\left[x'^m - \epsilon X^m + O(\epsilon^2)\right]}{Dx'^j}.$$

$$= u_{i,j} + \epsilon\left[\frac{DU_i}{Dx^j} - \frac{DX^m}{Dx^j}u_{i,m}\right] + O(\epsilon^2) = u_{i,j} + \epsilon V_i^j + O(\epsilon^2) \tag{1.3.3}$$

Expanding the right side by using (1.3.2) we find that

$$u'_{i,j} = u_{i,j} + \epsilon\left[\frac{\partial U_i}{\partial x^j} + \frac{\partial U_i}{\partial u_m}u_{m,j} - \frac{\partial X^m}{\partial x^j}u_{i,m} - \frac{\partial X^m}{\partial u_\ell}u_{\ell,j}\,u_{i,m}\right] + O(\epsilon^2) \tag{1.3.4}$$

This gives us the first extension of group (1.1.1).

We get the second extension by finding how $u_{i,jk}$ transforms. Formula (1.3.3) tells us that

$$u'_{i,jk} = u_{i,jk} + \epsilon\left[\frac{DV_i^j}{Dx^k} - \frac{DX^m}{Dx^k}\, u_{i,jm}\right] + O(\epsilon^2) = u_{i,jk} + \epsilon V_i^{jk} + O(\epsilon^2) \qquad (1.3.5)$$

Expanding the right side of (1.3.5) we find that

$$V_i^{jk} = \frac{\partial^2 U_i}{\partial x^j \partial x^k} + \frac{\partial^2 U_i}{\partial x^j \partial u_m}\, u_{m,k} + \frac{\partial^2 U_i}{\partial x^k \partial u_m}\, u_{m,j}$$

$$+ \frac{\partial^2 U_i}{\partial u_m \partial u_\ell}\, u_{m,j}\, u_{\ell,k} + \frac{\partial U_i}{\partial u_m}\, u_{m,jk}$$

$$- \frac{\partial^2 X^m}{\partial x^j \partial x^k}\, u_{i,m} - \frac{\partial^2 X^m}{\partial x^j \partial u^\ell}\, u_{\ell,k}\, u_{i,m}$$

$$- \frac{\partial X^m}{\partial x^j}\, u_{i,km} - \frac{\partial^2 X^m}{\partial x^k \partial u_\ell}\, u_{\ell,j}\, u_{i,m}$$

$$- \frac{\partial^2 X^m}{\partial u_p \partial u_\ell}\, u_{p,k}\, u_{\ell,j}\, u_{i,m} - \frac{\partial X^m}{\partial u_\ell}\Big[u_{i,m}\, u_{\ell,jk}$$

$$+ u_{\ell,j}\, u_{i,mk} + u_{\ell,k}\, u_{i,jm}\Big] - \frac{\partial X^m}{\partial x^k}\, u_{i,jm}\,.$$

Similarly to find the q-th extension one "merely" applies the operator $\frac{D}{Dx^{j_q}}$ to the (q-1)th extension and adds the term

$$-\frac{DX^m}{Dx^{j_q}}\, u_{i,j_1 j_2 \cdots j_{q-1} m}$$

where

$$j_i = 1, 2, \ldots, n$$

$$\text{for} \qquad i = 1, 2, \ldots, q.$$

1.4 Translation Groups

We now prove that any one-parameter Lie group of transformations is equivalent to a one-parameter group of translations.

Proof.

Let a group (1.1.1) be given. Consider n functions F_i such that

$$\left\{ \begin{array}{l} F_i(\underline{x}') = F_i(\underline{x}), \; i = 2, \ldots, n \\ F_1(\underline{x}') = F_1(\underline{x}) + \epsilon \end{array} \right\}$$

Since $x'^j = x^j + \epsilon X^j + O(\epsilon^2)$, $j = 1, 2, \ldots, n$ we have:

$$\frac{dF_i}{0} = \frac{dx^1}{X^1} = \frac{dx^2}{X^2} = \cdots = \frac{dx^n}{X^n} \qquad (1.4.1)$$

$$i = 2, \ldots, n$$

The solution of (1.4.1) leads to $n-1$ integrals which are independent invariants of the given group. These become $n-1$ new variables $F_2, F_3 \ldots, F_n$. The canonical variable F_1 is found by solving

$$\frac{dF_1}{1} = \frac{dx^1}{X^1} = \frac{dx^2}{X^2} \cdots = \frac{dx^n}{X^n}.$$

Thus from a given one-parameter group we have generated a one-parameter translation group. Of course, in practice it may be too difficult to solve (1.4.1).

1.5 Reduction of the Number of Variables of a System of Partial Differential Equations.

Let

$$G_q\left(x^i,\ u_j,\ \frac{\partial u_i}{\partial x^j},\ \frac{\partial^2 u_i}{\partial x^i \partial x^k}\right) = 0$$

$q = 1, \ldots, r$, represent a system S of second order partial differential equations. The Lie group of transformations (1.3.1) is said to leave system S invariant iff

$$G_q\left(x'^i,\ u'_j,\ \frac{\partial u'_i}{\partial x'^j},\ \frac{\partial^2 u'_i}{\partial x'^j \partial x'^k}\right) = 0$$

for

$$q = 1, 2, \ldots, r$$

which, from (1.3) is true iff

$$X^i \frac{\partial G_q}{\partial x^i} + U_i \frac{\partial G_q}{\partial u_i} + V_i^j \frac{\partial G_q}{\partial u_{i,j}} + V_i^{jk} \frac{\partial G_q}{\partial u_{i,jk}} = 0 \quad \text{for} \quad q = 1, 2, \ldots, r .$$

We now introduce canonical variables. Let the first n such variables be the new independent variables labeled $y^1, y^2, \ldots, y^n$ and the remainder will be our new dependent variables labeled $v_1, v_2, \ldots, v_m$. Then, without loss of generality, in terms of these variable's the group of transformations (1.3.1) becomes

$$\begin{aligned} y'^1 &= y^1 + \epsilon \\ y'^i &= y^i \quad , \quad i = 2, \ldots, n \qquad (1.5.1) \\ v'_j &= v_j \quad , \quad j = 1, \ldots, m \end{aligned}$$

with system S being transformed to a new system S' :

$$H_q\left(y^i,\ v_j,\ \frac{\partial v_i}{\partial y^j},\ \frac{\partial^2 v_i}{\partial y^j \partial y^k}\right) = 0 \quad \text{for} \quad q = 1, \ldots, r. \tag{1.5.2}$$

Before proceeding any further in this discussion we show how one passes from S to S'.

Consider

$$\frac{Du_\ell}{Dy^j} = \frac{\partial u_\ell}{\partial y^j} + \frac{\partial u_\ell}{\partial v_k}\frac{\partial v_k}{\partial y^j} = \frac{\partial u_\ell}{\partial x^\alpha}\left[\frac{\partial x^\alpha}{\partial y^j} + \frac{\partial x^\alpha}{\partial v_k}\frac{\partial v_k}{\partial y^j}\right] = \frac{\partial u_\ell}{\partial x^\alpha}\frac{Dx^\alpha}{Dy^j}$$

$$\Longrightarrow \text{symbolically} \quad \frac{\partial u_\ell}{\partial x^\alpha} = \frac{\dfrac{D\left(x_1, \ldots, x^{\alpha-1}, u_\ell, x^{\alpha+1}, \ldots, x^n\right)}{D(y^1, y^2, \ldots, y^n)}}{\dfrac{D(x^1, x^2, \ldots, x^n)}{D(y^1, y^2, \ldots, y^n)}}$$

Thus we demand that

$$\frac{D(x^1, x^2, \ldots, x^n)}{D(y^1, y^2, \ldots, y^n)} = \begin{vmatrix} \dfrac{Dx^1}{Dy^1} & \cdots & \dfrac{Dx^1}{Dy^n} \\ \vdots & & \vdots \\ \dfrac{Dx^n}{Dy^1} & \cdots & \dfrac{Dx^n}{Dy^n} \end{vmatrix}$$

$$= \det \left\| \frac{Dx^i}{Dy^j} \right\| \neq 0$$

i.e. the transformation from $\underline{x}$ to $\underline{y}$ is locally one-to-one.

$$\frac{D^2 u_\ell}{Dy^j\, Dy^k} = \frac{\partial u_\ell}{\partial x^\alpha}\,\frac{D^2 x^\alpha}{Dy^j\, Dy^k} + \frac{\partial^2 u_\ell}{\partial x^\alpha\, \partial x^\beta}\,\frac{Dx^\alpha}{Dy^j}\,\frac{Dx^\beta}{Dy^k}$$

$$= \frac{\partial^2 u_\ell}{\partial y^j\, \partial y^k} + \frac{\partial^2 u_\ell}{\partial y^j\, \partial v_\alpha}\,\frac{\partial v_\alpha}{\partial y^k},$$

$$+ \frac{\partial^2 u_\ell}{\partial y^k\, \partial v_\alpha}\,\frac{\partial v_\alpha}{\partial y^j} + \frac{\partial^2 u_\ell}{\partial v_\alpha\, \partial v_\beta}\,\frac{\partial v_\alpha}{\partial y^j}\,\frac{\partial v_\beta}{\partial y^k}$$

$$+ \frac{\partial u_\ell}{\partial v_\alpha}\,\frac{\partial^2 v_\alpha}{\partial y^j\, \partial y^k}$$

One can show that

$$\det \left\| \frac{Dx^\alpha}{Dy^j}\,\frac{Dx^\beta}{Dy^k} \right\|$$

$$= \det \left\| \frac{Dx^i}{Dy^j} \otimes \frac{Dx^i}{Dy^j} \right\|$$

$$= \left[\det \left\| \frac{Dx^i}{Dy^j} \right\| \right]^{2n}$$

Thus, for example

$$
\frac{\partial^2 u_\ell}{\partial x^1 \partial x^1} = \frac{\begin{vmatrix} \dfrac{D^2 u_\ell}{Dy^{1\,2}} - \dfrac{\partial u_\ell}{\partial x^\alpha}\dfrac{D^2 x^\alpha}{Dy^1 Dy^1} & \dfrac{Dx^1}{Dy^1} & \dfrac{Dx^2}{Dy^1} & \cdots & \dfrac{Dx^n}{Dy^1} & \dfrac{Dx^n}{Dy^1} \\ \dfrac{D^2 u_\ell}{Dy^1 Dy^2} - \dfrac{\partial u_\ell}{\partial x^\alpha}\dfrac{D^2 x^\alpha}{Dy^1 Dy^2} & \dfrac{Dx^1}{Dy^1} & \dfrac{Dx^2}{Dy^2} & \cdots & \dfrac{Dx^n}{Dy^1} & \dfrac{Dx^n}{Dy^2} \\ & \vdots & & & \vdots & \\ \dfrac{D^2 u_\ell}{Dy^{n^2}} - \dfrac{\partial u_\ell}{\partial x^\alpha}\dfrac{D^2 x^\alpha}{Dy^{n^2}} & \dfrac{Dx^1}{Dy^n} & \dfrac{Dx^2}{Dy^n} & \cdots & \dfrac{Dx^n}{Dy^n} & \dfrac{Dx^n}{Dy^n} \end{vmatrix}}{\left[\operatorname{Det} \left\| \dfrac{Dx^i}{Dy^j} \right\| \right]^{2n}}
$$

Since system S' is invariant under (1.5.1), there exist solutions v_j which are independent of y^1 since both $v'_j = v_j$ and

$$
\begin{aligned}
v'_j &= v_j(y'^1, y'^2, \ldots, y'^n) \\
&= v_j(y^1 + \epsilon, y^2, \ldots, y^n)
\end{aligned}
$$

are solutions to (1.5.2). Assuming that (1.5.1) also leaves the boundary conditions invariant and that the solution is unique, we must have $\dfrac{\partial v_j}{\partial y^1} = 0$ for $j = 1, 2, \ldots, m$. Thus the number of variables has been reduced by one.

Chapter II

The Linear Heat Equation

2.1 Derivation of the group for $u_{xx} - u_t = 0$

We now find the group of the one-dimensional heat equation

$$u_{xx} - u_t = 0 \tag{2.1.1}$$

Let the Lie group of transformations

$$\left\{ \begin{aligned} u' &= u'(x,t,u;\epsilon) \\ x' &= x'(x,t,u;\epsilon) \\ t' &= t'(x,t,u;\epsilon) \end{aligned} \right\} \tag{2.1.2}$$

leave (2.1.1) and its accompanying boundary conditions invariant. Invariance implies that $v = u'$ satisfies

$$v_{x'x'} - v_{t'} = 0 \tag{2.1.3}$$

iff $u_{xx} - u_t = 0$, and in terms of the new variables the original boundary conditions must be satisfied.

If $u = \theta(x,t)$ is a solution to (2.1.1), then, of course, $v = \theta(x', t')$ is a solution to (2.1.3). But having found the group (2.1.2) leaving invariant (2.1.1), we obtain another solution to (2.1.3), namely, $v = u'(x,t,\theta(x,t);\epsilon)$. Assuming a unique solution to (2.1.3), we must demand that

$$u'(x,t,\theta(x,t);\epsilon) = \theta(x',t') \tag{2.1.4}$$

We consider the $O(\epsilon)$ terms in the expansions of u', x', t':

$$\begin{cases} u' = u + \epsilon U(x,t,u) + O(\epsilon^2) \\ x' = x + \epsilon X(x,t,u) + O(\epsilon^2) \\ t' = t + \epsilon T(x,t,u) + O(\epsilon^2) \end{cases}$$

where

$$U = \left(\frac{\partial u'}{\partial \epsilon}\right)_{\epsilon=0}, \quad X = \left(\frac{\partial x'}{\partial \epsilon}\right)_{\epsilon=0}, \quad T = \left(\frac{\partial t'}{\partial \epsilon}\right)_{\epsilon=0}$$

Expanding (2.1.4) and equating $O(\epsilon)$ terms, we get (after replacing $\theta(x,t)$ by u):

$$U(x,t,u) = X(x,t,u)\,\frac{\partial u}{\partial x} + T(x,t,u)\,\frac{\partial u}{\partial t} \tag{2.1.5}$$

(2.1.5) is the general partial differential equation of an invariant surface.

The characteristic equations corresponding to (2.1.5) are:

$$\frac{du}{U(x,t,u)} = \frac{dx}{X(x,t,u)} = \frac{dt}{T(x,t,u)} \tag{2.1.6}$$

In principle (2.1.6) is solvable, and thus we obtain

$$u = u(x,t,\eta\,;\,F(\eta)\,) \tag{2.1.7}$$

where the dependence of u on x and t is known explicitly, η is some similarity variable found from solving (2.1.6) (which will be independent of u if $\frac{X}{T} = fn(x,t)$) and the dependence of u on η involves some arbitrary function $F(\eta)$. Substituting (2.1.7) into (2.1.1), we reduce it to an ordinary differential equation of at most second order with independent variable η and dependent variable $F(\eta)$.

Obviously our first problem is to find the largest possible class of infinitesimals U, X, T for (2.1.1).

In general, for linear equations

$$\left\{ \begin{aligned} \frac{X(x,t,u)}{T(x,t,u)} &= u \quad fn(x,t) \\ \frac{U(x,t,u)}{T(x,t,u)} &= u \quad fn(x,t) + fn(x,t) \end{aligned} \right\} . \tag{2.1.8}$$

For illustrative purposes, in considering the heat equation initially we shall assume arbitrary U, X, T and show that finally we arrive at the form (2.1.8).

We now find the $O(\epsilon)$ terms in the expansion of $u'_{x'x'} - u'_{t'}$:

$$\begin{aligned} u'_{t'} &= u'_t + u'_x x_{t'} \\ &= \left[u_t + \epsilon(U_t + U_u u_t)\right]\left[1-\epsilon T_t - \epsilon T_u u_t\right] - \epsilon u_x\left[X_t + X_u u_t\right] + O(\epsilon^2) \\ &= u_t + \epsilon\left[-X_u u_t u_x - T_u u_t u_t + (U_u - T_t) u_t - X_t u_x + U_t\right] + O(\epsilon^2) \end{aligned}$$

In the same way

$$u'_{x'} = u_x + \epsilon\left[-T_u u_t u_x - X_u u_x u_x + (U_u - X_x) u_x - T_x u_t + U_x\right] + O(\epsilon^2)$$

Thus for invariance

$$\begin{aligned} u'_{x'x'} = (u'_{x'})_x x_{x'} + (u'_{x'})_t t_{x'} = u_{xx} + \epsilon\Big[&-T_{uu} u_x u_x u_t - X_{uu} u_x u_x u_x \\ &- 2T_u u_x u_{tx} - (3X_u + 2T_{xu}) u_x u_t - T_u u_t u_t + (U_{uu} - 2X_{ux}) u_x u_x - 2T_x u_{tx} \\ &+ (U_u - 2X_x - T_{xx}) u_t + (2U_{xu} - X_{xx}) u_x + U_{xx}\Big] + O(\epsilon^2) = u'_{t'} \end{aligned} \tag{2.1.9}$$

with $u'_{t'}$ as given above. In deriving (2.1.9) we have made use of the given equation (2.1.1).

2.2 The Classical Group of the Heat Equation

S. Lie [6] found the classical group of the heat equation although he did not proceed any further to derive the resulting ordinary differential equations for the different cases. The classical group corresponds to equating to zero the coefficients of terms with the same derivatives of u, i.e., the coefficients of $u_x u_{tx}, u_t u_x, u_x u_x, u_{tx}, u_t u_t$, u_t, u_x, and the remaining terms in the expansion of $u'_{x'x'} - u'_{t'}$. This is clearly a sufficient, but, as will be shown in 2.4, not a necessary condition for finding similarity solutions. However, for any problem, linear or nonlinear, the classical group results from solving a set of linear equations, whereas this is not the case non-classically, even if the original equation is linear.

Successively equating to zero the coefficients of $u_x u_{tx}, u_t u_x$, $u_x u_x$ in (2.1.9), we find that:

$$\left\{\begin{array}{l} T_u = 0 \\ X_u = 0 \\ U_{uu} = 0 \end{array}\right\} \qquad (2.2.1)$$

$$\Longrightarrow \qquad \left\{\begin{array}{lcl} U(x,t,u) & = & f(x,t)u + g(x,t) \\ X(x,t,u) & = & X(x,t) \\ T(x,t,u) & = & T(x,t) \end{array}\right\} \qquad (2.2.2)$$

Then successively equating to zero the coefficients of u_{tx}, u_t, u_x, u and the remaining terms, we find that:

$$T_x = 0 \implies T = T(t) \tag{2.2.3}$$

$$2X_x - T'(t) = 0 \tag{2.2.4}$$

$$X_t - X_{xx} + 2f_x = 0 \tag{2.2.5}$$

$$f_{xx} - f_t = 0 \tag{2.2.6}$$

$$g_{xx} - g_t = 0 \tag{2.2.7}$$

Thus $g(x,t)$ is any solution to (2.1.1). At first we shall only consider the subgroup for which $g(x,t) = 0$ and deal with the contrary case in 2.3.

Solving (2.2.4) for X we are led to:

$$S = \frac{xT'(t)}{2} + A(t) \tag{2.2.4'}$$

with arbitrary $A(t)$.

Substituting (2.2.4') into (2.2.5) and solving for f we obtain:

$$f = -\frac{x^2}{8} T''(t) - \frac{xA'(t)}{2} + B(t) \tag{2.2.5'}$$

where $B(t)$ is arbitrary.

Substitution of (2.2.5') in (2.2.6) yields:

$$\frac{-T''(t)}{4} + \frac{x^2 T'''(t)}{8} + \frac{xA''(t)}{2} - B'(t) = 0 \tag{2.2.6'}$$

Solving (2.2.6') we finally obtain the classical group of the heat equation:

$$\left(\begin{array}{l} X = \kappa + \delta t + \beta x + \gamma x t \\ \\ T = \alpha + 2\beta t + \gamma t^2 \\ \\ f = -\gamma\left[\frac{x^2}{4} + \frac{t}{2}\right] - \frac{\delta x}{2} + \lambda \end{array} \right) \tag{2.2.8}$$

where $\alpha, \beta, \gamma, \delta, \kappa, \lambda$ are 6 arbitrary parameters.

In x-t space, the group (2.2.8) is a subgroup of the projective group. All of the parameters, except for γ, individually represent "trivial" transformations. κ represents translation invariance in x, α translation in t, δ represents invariance under a Galilean transformation, and β represents similitudinous invariance which is the invariance used to find the source solution to the heat equation.

Since we are concerned with a linear homogeneous equation, we notice that one of the parameters, λ, does not enter in the invariance of a boundary curve. If $x = a(t)$ is an invariant boundary, then infinitesimally $X = a'(t)T$, i.e.,

$$\frac{dx}{X} = \frac{dt}{T} \tag{2.2.9}$$

Next we solve (2.2.9) to find the most general invariant "classical" boundary:

$$\frac{dx}{dt} = \frac{x}{2}\frac{T'(t)}{T(t)} + \frac{\delta t + \kappa}{T(t)}$$

Setting $x = y\sqrt{T(t)}$, we get:

$$\frac{dy}{dt} = \frac{\delta t + \kappa}{(\alpha + 2\beta t + \gamma t^2)^{3/2}}$$

In general, four cases are distinguished:

<u>Case I</u> $\beta^2 \neq \alpha\gamma$. Then

$$\eta = \frac{x - (At + B)}{\sqrt{\alpha + 2\beta t + \gamma t^2}} \tag{2.2.10}$$

where

$$A = \frac{\kappa\gamma - \delta\beta}{\alpha\gamma - \beta^2}\,, \qquad B = \frac{\kappa\beta - \delta\alpha}{\alpha\gamma - \beta^2}$$

and, as the constant of integration, η becomes the independent variable. Solving for u, we find that

$$u = F(\eta)\left[\frac{1}{(\alpha + 2\beta t + \gamma t^2)^{1/4}}\right]\left(\frac{\gamma t + \beta - \sqrt{\beta^2 - \alpha\gamma}}{\gamma t + \beta + \sqrt{\beta^2 - \alpha\gamma}}\right)^{\rho} e^{-\left\{\frac{t}{4}\left[A^2 - \gamma\eta^2\right] + \frac{A\eta}{2}\sqrt{\alpha + 2\beta + \gamma t^2}\right\}} \tag{2.2.11}$$

where

$$\rho = \frac{1}{2\sqrt{\beta^2 - \alpha\gamma}}\left\{\frac{\beta}{2} + \lambda + \frac{1}{4\gamma}\left[\delta^2 + A^2(\alpha\gamma - \beta^2)\right]\right\}$$

Of course (2.2.11) only makes sense if $\beta^2 > \alpha\gamma$ and here t is restricted to <u>one</u> of the ranges

$$|\gamma t + \beta| > \sqrt{\beta^2 - \alpha\gamma}\,,$$

$$|\gamma t + \beta| < \sqrt{\beta^2 - \alpha\gamma}\,.$$

If $\beta^2 < \alpha\gamma$ then there is no restriction on t and $\left(\frac{\gamma t + \beta - \sqrt{\beta^2 - \alpha\gamma}}{\gamma t + \beta + \sqrt{\beta^2 - \alpha\gamma}}\right)^{\rho}$ becomes

$$\exp\left[\frac{1}{\sqrt{\alpha\gamma - \beta^2}}\left\{\frac{\beta}{2} + \lambda + \frac{1}{4\gamma}\left[\delta^2 + A^2(\alpha\gamma - \beta^2)\right]\right\} \cdot \arctan\frac{\sqrt{\alpha\gamma - \beta^2}}{\gamma t + \beta}\right]$$

Substituting (2.2.11) into (2.1.1) we obtain the following ordinary differential equation for $F(\eta)$:

$$F'' + \beta\eta F' + \left[D\eta^2 + E\right] F = 0 \tag{2.2.12}$$

where primes denote differentiation with respect to η and $D = \frac{\alpha\gamma}{4}$,

$$E = \frac{A^2(\beta^2 - \alpha\gamma)}{4\gamma} - \left[\lambda + \frac{\delta^2}{4\gamma}\right]$$

Now let $z = \eta\sqrt[4]{\beta^2 - \alpha\gamma}$ and $F(\eta) = G(z)\, e^{-\beta\eta^2/4}$. Then (2.2.12) becomes

$$G'' + \left[\tfrac{1}{2} + \nu - \tfrac{1}{4} z^2\right] G = 0 \qquad (2.2.12')$$

where

$$\nu = \frac{E - \beta/2}{\sqrt{\beta^2 - \alpha\gamma}} - \tfrac{1}{2}$$

The parabolic cylinder functions $D_\nu(z)$ are particular solutions to this equation, which, for the values $\nu = n =$ integer, are easily related to the Hermite functions.

<u>Case II</u> $\beta^2 = \alpha\gamma$, $\gamma \neq 0$. Thus we set $\gamma = 1$.

Here the invariant boundary is:

$$\eta = \left\{ x + \delta + \left(\frac{\kappa - \delta\beta}{2}\right)\frac{1}{t+\beta} \right\} \frac{1}{t+\beta} \qquad (2.2.13)$$

where η, as the constant of integration, becomes our similarity variable. Solving for u, we get:

$$u(x,t) = \frac{F(\eta)}{\sqrt{\beta + t}}\; e^{\left[\frac{A^2}{3(t+\beta)^3} + \frac{B}{t+\beta} + \frac{A\eta}{t+\beta} - \frac{\eta^2 t}{4}\right]} \qquad (2.2.14)$$

where

$$A = \frac{\delta\beta - \kappa}{4}$$

$$B = -\left[\frac{\beta}{2} + \lambda + \frac{\delta^2}{4}\right]$$

Substitution of (2.2.14) into (2.1.1) results in the following ordinary differential equation for $F(\eta)$:

$$F'' + \beta\eta F' + \left\{ \frac{\beta^2}{4}\eta^2 + 2A\eta - (\lambda + \frac{\delta^2}{4}) \right\} F = 0 \qquad (2.2.15)$$

Letting $z = (\kappa - \delta\beta)^{1/3}\eta$ and $F(\eta) = G(z)\, e^{-\beta\eta^2/4}$, (2.2.15) becomes

$$G'' + \left[-\frac{z}{2} + \nu \right] G = 0 \qquad (2.2.15')$$

where

$$\nu = \frac{-\left[\lambda + \frac{\delta^2}{4} + \frac{\beta}{2}\right]}{\left[\delta\beta - \kappa\right]^{2/3}}$$

Airy's integral is a particular solution of (2.2.15').

<u>Case III</u> $\beta^2 = \alpha\gamma$, $\beta = \gamma = 0$, $\alpha \neq 0$.

Here the invariant boundary is of the form

$$\eta = x - \frac{\delta t^2}{2} - \kappa t \qquad (2.2.16)$$

As before, η plays the role of a similarity variable, and the resulting solution is

$$u(x,t) = F(\eta)\, e^{\left[-\frac{\delta^2}{12}t^3 - \frac{\delta\kappa t^2}{4} + \lambda t - \frac{\delta}{2}\eta t\right]} \qquad (2.2.17)$$

with $F(\eta)$ satisfying

$$F'' + \kappa F' + \left[\frac{\delta}{2}\eta - \lambda\right] F = 0 \qquad (2.2.18)$$

If we let $z = -\delta^{\frac{1}{3}}\eta$ and $F(\eta) = G(z)\, e^{-\frac{\kappa}{2}\eta}$, then (2.2.18) becomes

$$G'' + \left[-\frac{z}{2} + \nu\right] G = 0 \qquad (2.2.18')$$

with

$$\nu = -\frac{\left[\frac{\kappa^2}{4} + \lambda\right]}{\delta^{2/3}} \quad \text{(same eqn. as in Case II)}$$

Burgers [7] recently consider this case, deriving this class of solutions by other special means.

Case IV. $\alpha = \beta = \gamma = 0, \quad \delta \neq 0$

The invariant boundary here is $\eta = t$. (2.2.19)

The resulting solution is

$$u(x,t = F(t)\, e^{\frac{x}{4(t+\kappa)}[\lambda - x]} \tag{2.2.20}$$

with $F(t)$ satisfying

$$F' = \frac{[\lambda^2 - 8(t+\kappa)]}{16(t+\kappa)^2} F(t) \tag{2.2.21}$$

$$\Longrightarrow \qquad u(x,t) = \phi \frac{e^{-\frac{1}{4(t+\kappa)}(x - \frac{\lambda}{2})^2}}{\sqrt{t}} \tag{2.2.22}$$

with ϕ an arbitrary constant.

Next we consider a boundary condition in more detail. If

$$x = a(t) \tag{2.2.23}$$

is an invariant boundary along which $u(x,t) = W(t)$, then in each case (2.2.23) corresponds to some value of $\eta = \sigma$, say, along which $U(\sigma) = \tau$. Referring to Case III, an invariant boundary would be $x = \sigma + \delta t^2/2 + \kappa t$ along which

$$W(t) = \tau\, e^{\left[-\frac{\delta^2}{12}t^3 - \frac{\delta\kappa}{4}t^2 + \lambda t - \frac{\delta\sigma}{2}t\right]}$$

Often we are interested in the situation where along some boundary $U = 0$ $(\tau = 0)$ and a source is present [7]. Under such circumstances, in each of the four cases the parameter λ does not enter in the boundary condition. Thus we can pick those values of λ (ν in the first three cases) which allow the source to be represented and lead to the proper behaviour for $x \to \infty$. Here λ (or ν), in fact, plays the role of an eigenvalue and the superposition principle is used to find the required solution.

2.3 An Example Where $g(x,t) \neq 0$.

Up to now we have assumed that $g(x,t) = 0$. A trivial solution to (2.2.7) is $g(x,t) = C$ = constant. We find the resulting solution u, assuming that $f = 0$, $X = x$, $T = 2t$. Then

$$\frac{du}{C} = \frac{dx}{x} = \frac{dt}{2t}$$

$\Longrightarrow$ $u = C \ln x + F(\eta)$. Letting $F(\eta) = -C \ln \eta + G(\eta)$ we find that $V(\eta)$ satisfies

$$G'' + \eta \frac{G'}{2} = \frac{C}{2} \tag{2.3.2}$$

$$\Longrightarrow \qquad G' = Ce^{-\frac{\eta^2}{4}} \int_{\alpha}^{\eta} e^{\eta^2/4} \, d\eta .$$

2.4 The Non-classical Group of the Heat Equation.

In deriving the classical group of the heat equation we did not make use of relationship (2.1.5) which connects u_x and u_t. In addition from (2.1.6) we see that it is not the functions X, T, U which are needed, but the ratios $\frac{X}{T}$, $\frac{U}{T}$, provided that $T \neq 0$.

Let

$$\left\{ \begin{array}{l} Y = \frac{X}{T} \\ \\ V = \frac{U}{T} \end{array} \right\} \tag{2.3.1}$$

Then

$$u_t = V - Y u_x \tag{2.3.2}$$

$$\Longrightarrow \qquad u_{tx} = (V_x - YV) + (V_u - Y_x + Y^2)\, u_x - Y_u\, u_x u_x \, .$$

To derive the non-classical group of the heat equation we substitute in these expressions for u_t and u_{tx} in the expansion of $u'_{x'x'} - u'_t$ and then successively equate to zero the coefficients of $u_x u_x u_x$, $u_x u_x$, u_x and the remaining terms.

The resulting non-classical determining equations are:

$$Y_{uu} = 0 \tag{2.3.3}$$

$$V_{uu} - 2Y_{xu} + 2YY_u = 0 \tag{2.3.4}$$

$$Y_t - 2VY_u + 2V_{xu} - Y_{xx} + 2YY_x = 0 \tag{2.3.5}$$

$$V_{xx} - 2\, VY_x - V_t = 0 \tag{2.3.6}$$

Differentiating (2.3.4) with respect to u

$$\Longrightarrow \qquad V_{uuu} + 2Y_u\, Y_u = 0$$

Then differentiating (2.3.5) thrice with respect to u leads to

$$V_{uuu}\, Y_u = 0$$

Then using (2.3.4) again we find that

$$\begin{Bmatrix} V_{uu} = 0 \\ Y_u = 0 \end{Bmatrix} \qquad (2.3.7)$$

Therefore the determining equations now become (we let $V = uF(x,t) + G(x,t)$)

$$Y_t + 2F_x - Y_{xx} + 2YY_x = 0 \qquad (2.3.8)$$

$$F_{xx} - 2FY_x - F_t = 0 \qquad (2.3.9)$$

$$G_{xx} - 2GY_x - G_t = 0 \qquad (2.3.10)$$

Letting $Y = 2\nu_x$, (2.3.8) $\Longrightarrow$ $F = \nu_{xx} - \nu_t - 2\nu_x^2 + M(t)$.
If $M(t) = 0$ then (2.3.9) $\Longrightarrow$

$$\nu_{xxxx} - 2\nu_{txx} - 4\nu_x\nu_{xxx} - 8\nu_{xx}\nu_{xx} + 4\nu_{xx}\nu_t + 8\nu_{xx}\nu_x^2 + 4\nu_x\nu_{xt} = 0 \qquad (2.3.11)$$

Clearly nonlinear (2.3.11) is a more complicated equation than the original linear heat equation. However <u>any</u> solution to (2.3.11) reduces the heat equation to an ordinary differential equation.

The classical case results when we set $\nu_{xxx} = 0$ ($Y_{xx} = 0$).

It is interesting to note that if $F = 0$, then (2.3.8) becomes the Burgers equation. Setting $Y = -H_x/H$ where H solves the heat equation, we can show that the resulting solution u satisfies the relation $u_x = H$.

We now give particular examples of non-classical solutions. $\nu(x,t) = \gamma \ell n x$ is a solution to (2.3.11) if $\gamma = -\frac{1}{2}, -\frac{3}{2}$.

<u>Case I</u> $\gamma = -\frac{1}{2}$

Here $$Y = -\frac{1}{x}$$

$$V = 0$$

$$\Longrightarrow \qquad \eta = \frac{x^2}{2} + t$$

Substituting $u = F(\eta)$ into (2.1.1) we find that $F'' = 0 \Longrightarrow$ $u = A(\frac{x^2}{2} + t) + b$ with a, b arbitrary constants.

Case II $\gamma = -\frac{3}{2}$

Here $$V = \frac{-3u}{x^2}$$

$$Y = -\frac{3}{x}$$

Thus $\eta = \frac{x^2}{2} + 3t$ and $u = xF(\eta)$. Again $F''(\eta) = 0 \Longrightarrow$ $u = ax(\frac{x^2}{2} + 3t) + b$ with a, b arbitrary constants.

Chapter III

Classical Invariances of the Fokker-Planck Equation.

3.1 Derivation of the classical group of the Fokker-Planck equation.

In this chapter we shall find the most general classical set of functions $C(u)$ leading to analytic similarity solutions of the Fokker-Planck equation

$$p_t - p_{uu} - \frac{\partial}{\partial u}\Big(C(u)\,p\Big) = 0 \qquad (3.1.1)$$

where $p(u,t)$ is the transition probability density for having velocity u at time t given that $u = u_0$ at $t = t_0$, and $C(u)$ represents some dissipative force.

The following physically significant conditions are imposed on $C(u)$ and $p(u,t)$:

$$C(-u) = -C(u) \qquad (3.1.1a)$$

$$p(u,t_0) = \delta(u - u_0) \qquad (3.1.1b)$$

$$\int_{-R/2}^{R/2} p(u,t)du = 1 \quad \text{where } R \text{ is the range of interest} \qquad (3.1.1c)$$

and, of course

$$p(u,t) \geq 0 \quad \forall u,t \qquad (3.1.1d)$$

The dynamic equation leading to (3.1.1) is

$$m\frac{du}{dt} + R(u) = F(t)$$

where $R(u)$ is some resistive force depending on the velocity u and $F(t)$ represents white noise. It can be shown that $C(u) = \frac{R(u)}{m}$.

We now find the classical Lie group of transformations leaving invariant (3.1.1, b, c) and show that a Lie group exists for only a restricted class of functions $C(u)$. As for the heat equation, let

$$\left\{\begin{aligned} p' &= p + \epsilon f(u,t)P + O(\epsilon^2) \\ u' &= u + \epsilon U(u,t) + O(\epsilon^2) \\ t' &= t + \epsilon T(u,t) + O(\epsilon^2) \end{aligned}\right\} \tag{3.1.2}$$

correspond to a Lie group of transformations leaving (3.1.1) invariant. Then

$$\begin{aligned} p'_{t'} - p'_{u'u'} - \frac{\partial}{\partial u'}\left(C(u')p'\right) &= p_t - p_{uu} - \frac{\partial}{\partial u}\left(C(u)p\right) \\ &+ \epsilon\left\{ p_u\left(-U_t + U_{uu} - 2f_u + C(u)U_u - C'(u)U - C(u)f\right)\right. \\ &+ p\left(f_t - f_{uu} - C'(u)f - C''(u)U - C(u)f_u\right) \\ &\left.+ p_{uu}\left(2U_u - f\right) + 2p_{ut}T_u\right\} + O(\epsilon^2) \end{aligned} \tag{3.1.3}$$

Substituting $p_t - \frac{\partial}{\partial u}\left(C(u)p\right)$ for p_{uu} in (3.1.3) and then successively equating to zero the coefficients of p_{ut}, p_t, p_u and p we obtain the following determining equations for the classical group of (3.1.1):

$$T_u = 0 \Longrightarrow T = T(t) \tag{3.1.4}$$

$$2U_u - T'(t) = 0 \tag{3.1.5}$$

$$C(u)U_u + U_t + 2f_u + C'(u)U = 0 \tag{3.1.6}$$

$$C'(u)T'(t) - f_t + f_{uu} + C''(u)U + C(u)f_u = 0 \tag{3.1.7}$$

Solving (3.1.5) for U we obtain

$$U = \frac{uT'(t)}{2} + A(t) \qquad (3.1.5')$$

where $A(t)$ is an arbitrary function of t.

Substituting (3.1.5') into (3.1.6) and solving for f we find that

$$f = B(t) - \frac{uC(u)}{4} T'(t) - \frac{C(u)\,A(t)}{2} - \frac{uA'(t)}{2} - \frac{u^2}{8} T''(t) \qquad (3.1.6')$$

where $B(t)$ is arbitrary.

Substitution of (3.1.6') into (3.1.7) yields:

$$\begin{aligned} T'(t) &\left[\frac{C^2(u)}{4} + \frac{uC(u)C'(u)}{4} - \frac{C'(u)}{2} - \frac{uC''(u)}{4}\right] \\ &+ T''(t)\left[\tfrac{1}{4}\right] \\ &+ T'''(t)\left[-\frac{u^2}{8}\right] \\ &+ A(t)\left[\frac{C(u)C'(u)}{2} - \frac{C''(u)}{2}\right] \\ &+ A''(t)\left[-\frac{u}{2}\right] \\ &+ B'(t) \\ &= 0 \end{aligned} \qquad (3.1.8)$$

(3.1.1a) implies that only two cases are allowed.

<u>Case I</u> $\quad T' = B' = 0, \quad A''(t) = \pm\, \alpha^2 A(t)$

<u>Case II</u> $\quad A = 0, \; T''(t) = a\,T'(t), \; B'(t) = bT'(t)$

Both cases lead to the same equation for $C(u)$ and so we shall only consider Case I. Only the case $A''(t) = +\,\alpha^2 A(t)$ is of interest since the contrary case leads to solutions $p(u,t)$ which are negative for some u,t. Thus the equation for $C(u)$ is

$$CC' - C'' - \alpha^2 u = 0$$

$$\Longrightarrow \qquad C' - \frac{C^2}{2} + \frac{\alpha^2 u^2}{2} + \beta = 0 \tag{3.1.9}$$

with α, β arbitrary constants, $\alpha > 0$, say.

Now let $C(u) = \sqrt{2\alpha}\; D(v)$

$$u = \sqrt{\frac{2}{\alpha}}\, v$$

$$\gamma = \frac{\beta}{\alpha}$$

Then

$$D'(v) - D^2 + v^2 + \gamma = 0 \tag{3.1.10}$$

Now let

$$D(v) = -\frac{V'(v)}{V(v)}$$

Then

$$V''(v) - (v^2 + \gamma)\, V(v) = 0 \tag{3.1.11}$$

(3.1.11) is a confluent hypergeometric equation. Because of (3.1.1a), $V(v)$ must be an even function of v. Thus [8]

$V(v) = e^{-\frac{1}{2}v^2}\, M(\frac{1}{4}\gamma + \frac{1}{4}, \frac{1}{2}, v^2)$ where $M(a, b, z)$ is Kummer's function. (The notations $\Phi(a;b;z)$ or ${}_1F_1(a;b;z)$ are also used). Using the properties of Kummer's function [8] we find that

$$C(u) = -\alpha u \left[\left(\frac{\beta}{\alpha} + 1\right) \frac{M\left(\frac{\beta}{4\alpha} + \frac{5}{4}, \frac{3}{2}, \frac{1}{2}\alpha u^2\right)}{M\left(\frac{\beta}{4\alpha} + \frac{1}{4}, \frac{1}{2}, \frac{1}{2}\alpha u^2\right)} - 1 \right] \tag{3.1.12}$$

It is more instructive to investigate $C(u)$ in the phase plane.

$$D' = D^2 - v^2 - \gamma \quad \text{with} \quad D(0) = 0.$$

There are five cases to consider. In each case either $D = v$ or $D = -v$ is used as a reference line.

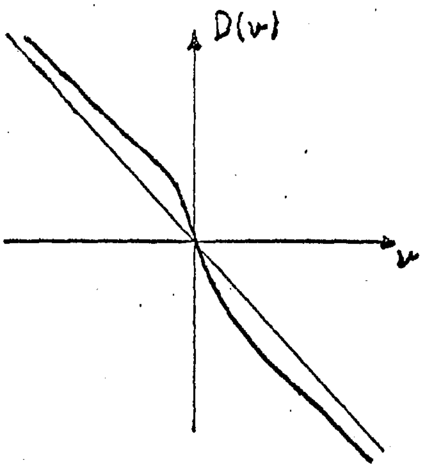
(i) $1 < \gamma$ (repulsive force)

(ii) $0 < \gamma < 1$ (repulsive force)

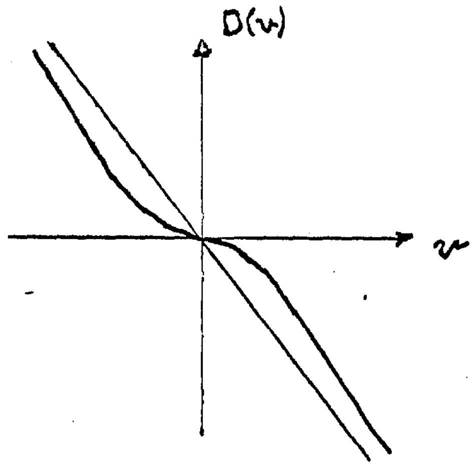
(iii) $-1 < \gamma < 0$ (attractive force for finite region only)

(iv) $\gamma = -1$ (attractive force - harmonically bound particle)

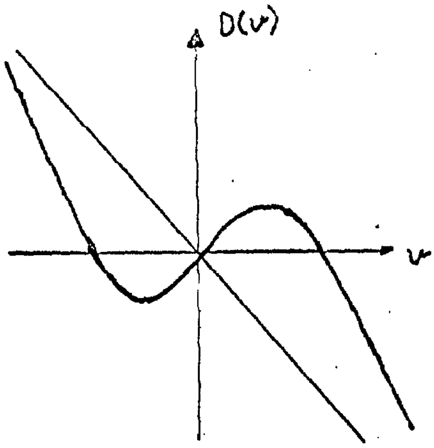
(v) $\gamma < -1$ (attractive force too large, leading to impossible situation)



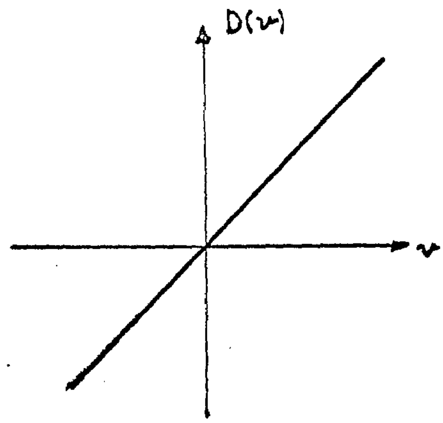
(i) $1 < \gamma$



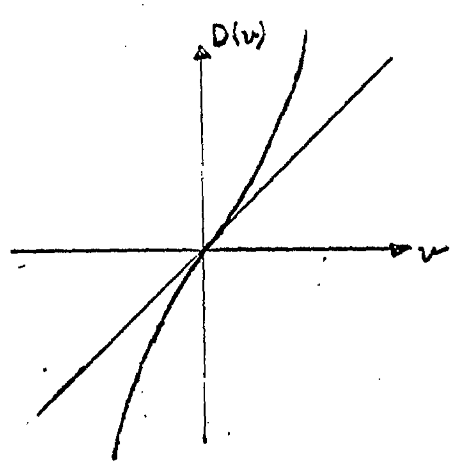
(ii) $0 < \gamma < 1$



(iii) $-1 < \gamma < 0$



(iv) $\gamma = -1$



(v) $\gamma < -1$

As $v \to 0$, in all cases $D(v) = -\gamma v + O(v^3)$.

As $v \to \infty$, in cases (i) to (iv) $D(v) \sim -v + \frac{1-\gamma}{2v}$.

For case (iii), as $\gamma \to 0$, the extrema of $D(v)$ occur at $v = \pm\sqrt{\gamma}\,[\,1 + O(\gamma^2)\,]$. Case (iv) corresponds to the situation where the extrema of case (iii) $\to \pm\infty$ as $\gamma \to -1$.

In case (v) if $\gamma << -1$, then $D(v) \to \infty$ when $v \to \frac{\pi}{2\sqrt{|\gamma|}}$ since $D(v) \sim \sqrt{|\gamma|-v^2}\ \tan(v\sqrt{|\gamma|-v^2})$ as $v \to \frac{\pi}{2\sqrt{|\gamma|}}$ so that the range R is $\sim \pi/\sqrt{|\gamma|}$. However it can be shown that here $\int_{-R/2}^{R/2} p(u,t)\,du = \mathrm{fn}(t)$ so that this case does not lead to a probability distribution.

We are now restricted to those cases where $\gamma = \frac{\beta}{\alpha} \geq -1$ and examine the condition $p(u, t_0) = \delta(u - u_0)$.

3.2 The solution of (3.1.1, a, b, c) and the computation of $\overline{u^2}$.

From (3.1.5'), (3.1.6'), (3.1.8), and (3.1.9) we obtain

$$\left\{\begin{aligned} T &= \lambda \\ U &= \kappa e^{\alpha t} + \nu e^{-\alpha t} \\ f &= \mu - \frac{C(u)}{2}\left[\kappa e^{\alpha t} + \nu e^{-\alpha t}\right] \\ &\quad - \frac{\alpha u}{2}\left[\kappa e^{\alpha t} - \nu e^{-\alpha t}\right] \end{aligned}\right\} \qquad (3.2.1)$$

where κ, λ, μ, ν, are arbitrary constants.

Since we must leave $t = t_0$ invariant, $\lambda = 0$

$$p'(u, t_0) = \delta(u'-u_0)$$

$$\Longrightarrow \quad \left\{ \begin{array}{l} U(u_0, t_0) = 0 \\ f(u_0, t_0) = -U_u(u_0, t_0) \end{array} \right\}$$

Thus

$$\mu - \frac{C(u_0)}{2}\left[\kappa e^{\alpha t_0} + \nu e^{-\alpha t_0}\right] - \frac{\alpha u_0}{2}\left[\kappa e^{\alpha t_0} - \nu e^{-\alpha t_0}\right] = 0$$

and

$$\kappa e^{\alpha t_0} + \nu e^{-\alpha t_0} = 0$$

Then

$$\frac{dp}{fp} = \frac{du}{U} = \frac{dt}{T}$$

$$\Longrightarrow \quad \frac{dp}{p\,[\alpha\{u_0 - u\cosh\alpha(t-t_0)\} - C(u)\sinh\alpha(t-t_0)]}$$

$$= \frac{du}{2\sinh\alpha(t-t_0)} = \frac{dt}{0}$$

Thus our similarity variable $\eta = t$.

Since $\dfrac{-C(u)\,du}{2} = \dfrac{dV(v)}{V(v)}$, we get

$$p(u,t) = g(t)\, M\!\left(\frac{\beta}{4\alpha} + \tfrac{1}{4}, \tfrac{1}{2}, \tfrac{1}{2}\alpha u^2\right) \cdot e^{-\frac{\alpha}{4}\left\{u^2[1+\coth\alpha(t-t_0)] - \frac{2u u_0}{\sinh\alpha(t-t_0)}\right\}} \tag{3.2.2}$$

where $g(t)$ is an arbitrary function of t.

Substitution of (3.2.2) into (3.1.1) leads to a first order ordinary differential equation for $g(t)$ whose solution is

$$g(t) = \frac{Ke^{-\left[\frac{\beta}{2}t + \frac{\alpha u_0^2}{4}\coth\alpha(t-t_0)\right]}}{\left[\sinh\alpha(t-t_0)\right]^{1/2}} \tag{3.2.3}$$

where K is some arbitrary constant to be fixed by the normalization condition (3.1.1c).

Let $t-t_0 = \tau$. Then

$$p(u,t) = \frac{Ke^{-\frac{\beta}{2}\tau}}{\left[\sinh\alpha\tau\right]^{1/2}} M\left(\frac{\beta}{4\alpha}+\tfrac{1}{4}, \tfrac{1}{2}, \tfrac{1}{2}\alpha u^2\right)\cdot \exp\left[\frac{-\alpha(1+\coth\alpha\tau)}{4}\{u-u_0e^{-\alpha\tau}\}^2\right] \tag{3.2.4}$$

We now use two relations

$$[9] \qquad M\left(-\frac{\nu}{2}, \tfrac{1}{2}, \tfrac{1}{2}v^2\right) = \frac{e^{\frac{v^2}{4}}\Gamma\left(\frac{1-\nu}{2}\right)}{\sqrt{\pi}} 2^{-\frac{\nu}{2}}\left\{D_\nu(v) + D_\nu(-v)\right\} \tag{3.2.5}$$

and

$$[10] \quad \int_{-\infty}^{\infty} e^{-\frac{(U-v)^2}{2\mu}} e^{\frac{v^2}{4}} D_\nu(v)dv = \sqrt{2\pi\mu}\,(1-\mu)^{\frac{1}{2}\nu}\, e^{\frac{U^2}{4(1-\mu)}} D_\nu\left[U(1-\mu)^{-1/2}\right] \tag{3.2.6}$$

where $D_\nu(v)$ is a parabolic cylinder function.

Thus

$$\int_{-\infty}^{\infty} e^{-\frac{(U-v)^2}{2\mu}} M\left(-\frac{\nu}{2}, \tfrac{1}{2}, \tfrac{1}{2}v^2\right)dv = \sqrt{2\pi\mu}(1-\mu)^{\frac{\nu}{2}} M\left(-\frac{\nu}{2}, \tfrac{1}{2}, \tfrac{1}{2}\frac{U^2}{1-\mu}\right) \tag{3.2.7}$$

We now substitute into (3.2.7)

$$v = \sqrt{\alpha}\,u$$

$$U = \sqrt{\alpha}\,u_0 e^{-\alpha\tau}.$$

$$\nu = -\tfrac{1}{2}\left(\frac{\beta}{\alpha} + 1\right)$$

$$\mu = \frac{2}{1 + \coth\alpha\tau} = 2(\sinh\alpha\tau)e^{-\alpha\tau} = 1 - e^{-2\alpha\tau}$$

$$\Longrightarrow \quad 1 - \mu = e^{-2\alpha\tau}$$

$$\frac{U^2}{1-\mu} = \alpha u_0^2$$

Therefore, since we demand that

$$\int_{-\infty}^{\infty} p(u,t)du = 1, \qquad K = \sqrt{\frac{\alpha}{\pi}} \cdot \frac{1}{2M\left(\frac{\beta}{4\alpha} + \frac{1}{4}, \frac{1}{2}, \frac{1}{2}\alpha u_0^2\right)}$$

and

$p(u,t)$

$$= \tfrac{1}{2}\sqrt{\frac{\alpha}{\pi}}\, \frac{M\left(\frac{\beta}{4\alpha} + \frac{1}{4}, \frac{1}{2}, \frac{1}{2}\alpha u^2\right)}{M\left(\frac{\beta}{4\alpha} + \frac{1}{4}, \frac{1}{2}, \frac{1}{2}\alpha u_0^2\right)}\, \frac{e^{-\frac{\beta}{2}\tau}}{[\sinh\alpha\tau]^{1/2}} \cdot \exp\left[\frac{-\alpha(1+\coth\alpha\tau)}{4}\left\{u - u_0 e^{-\alpha\tau}\right\}^2\right] \tag{3.2.8}$$

Thus $p(u,t)$ is not a Gaussian distribution if $C(u)$ is nonlinear.

Next we compute the mean square displacement, $\overline{u^2}$. Let

$$M = M\left(-\frac{\nu}{2}, \frac{1}{2}, \frac{1}{2}\frac{U^2}{1-\mu}\right) \quad \text{and} \quad M' = \frac{d}{dz}\left[M\left(-\frac{\nu}{2}, \frac{1}{2}, z\right)\right]$$

Then after computing $\frac{\partial}{\partial U}$ of (3.2.7) and performing some simple manipulations, we get:

$$\int_{-\infty}^{\infty} v\, e^{-(U-v)^2/2\mu}\, M dv = U\sqrt{2\pi\mu}\,(1-\mu)^{\nu/2}\left[M + \frac{\mu}{1-\mu} M'\right] \tag{3.2.9}$$

Computing $\frac{\partial}{\partial U}$ of (3.2.9) and again manipulating, we obtain

$$I = \int_{-\infty}^{\infty} v^2 e^{-\frac{(U-v)^2}{2\mu}} M\,dv = \sqrt{2\pi\mu}\,(1-\mu)^{\frac{\nu}{2}} \Big[2(1-\mu)\,zM + \mu M + \frac{\mu^2}{1-\mu} M'$$

$$+ 2\mu z M' + \frac{2z\mu^2}{1-\mu} M''\Big]$$

Using the relation [8] $zM'' = -\frac{\nu}{2} M + (z - \frac{1}{2})M'$, we find that

$$I = \sqrt{2\pi\mu}\,(1-\mu)^{-2+\frac{\nu}{2}} \left[U^2(1-\mu)^2 M + (1-\mu)(\mu-\mu^2-\nu\mu^2)M + \mu U^2 M'\right]$$

Then after making the necessary substitutions and combining terms, we find that

$$\overline{u^2} = u_0^2\, e^{-2\alpha\tau} + \frac{2\beta}{\alpha^2}(\sinh\alpha\tau)^2 + \frac{\sinh 2\alpha\tau}{\alpha} + u_0^2\, \frac{M\left(\frac{5}{4}+\frac{\beta}{4\alpha}, \frac{3}{2}, \frac{1}{2}\alpha u_0^2\right)}{M\left(\frac{1}{4}+\frac{\beta}{4\alpha}, \frac{1}{2}, \frac{1}{2}\alpha u_0^2\right)} \cdot$$

$$\left[\frac{\beta}{\alpha} + 1\right]\left[\frac{e^{2\alpha\tau}-1}{2}\right] \tag{3.2.10}$$

In deriving (3.2.10) the relation $\frac{d}{dz} M(a,b,z) = \frac{a}{b} M(a+1,b+1,z)$ has been used. Other important relations for the Kummer functions are

$$(\gamma = -1) \qquad M(0, b, z) = 1$$

$$(\gamma = 1) \qquad M(b, b, z) = e^z$$

The case $\gamma = -1$ corresponds to a harmonically bounded particle and here we get the well known result for $C(u) = \alpha u$:

$$p(u,t) = \frac{1}{2}\sqrt{\frac{\alpha}{\pi}}\, \frac{e^{\frac{\alpha}{2}\tau}}{[\sinh\alpha\tau]^{1/2}} \cdot \exp\left[\frac{-\alpha(1+\coth\alpha\tau)}{4}\left\{u - u_0 e^{-\alpha\tau}\right\}^2\right]$$

with

$$\overline{u^2} = u_0^2\, e^{-2\alpha\tau} - \frac{2}{\alpha}(\sinh\alpha\tau)^2 + \frac{\sinh 2\alpha\tau}{\alpha} = \frac{1}{\alpha} + (u_0^2 - \frac{1}{\alpha})\, e^{-2\alpha\tau}$$

If $\gamma > -1$, as can be seen from the phase plane diagram for $C(u)$, or directly from (3.2.10), $\overline{u^2} \to \infty$ as $t \to \infty$ since $C(u)$ is repulsive.

Chapter IV

Application of Group Theory to a Particular Hyperbolic Equation

4.1 Derivation of the Group

In this chapter we shall find the Riemann function of $u_{xx} - u_{tt} + \frac{\lambda}{x} u_x = 0$, i.e., the solution of

$$u_{xx} - u_{tt} + \frac{\lambda}{x} u_x - \delta(x-\xi)\,\delta(t-\tau) = 0 \tag{4.1.1}$$

by group theoretical methods. This particular problem is discussed in [11]. First of all we find the classical group of (4.1.1).

Let

$$\left\{\begin{aligned} u' &= u + \epsilon f(x,t)\, u + O(\epsilon^2) \\ x' &= x + \epsilon X(x,t) + O(\epsilon^2) \\ t' &= t + \epsilon T(x,t) + O(\epsilon^2) \end{aligned}\right\} \tag{4.1.2}$$

leave (4.1.1) invariant. Then

$$\begin{aligned} &u'_{x'x'} - u'_{t't'} + \frac{\lambda}{x'} u'_{x'} - \delta(x'-\xi)\,\delta(t'-\tau) \\ &= u_{xx} - u_{tt} + \frac{\lambda}{x} u_x - \delta(x-\xi)\,\delta(t-\tau) \\ &+ \epsilon \Big\{ 2u_{tx}\left(X_t - T_x\right) + u_t\left(T_{tt} - 2f_t - T_{xx} - \frac{\lambda}{x} T_x\right) \\ &+ u\left(f_{xx} - f_{tt} + \frac{\lambda}{x} f_x\right) + u_{xx}\left(f - 2X_x\right) \\ &+ u_{tt}\left(2T_t - f\right) + u_x\Big(2f_x - X_{xx} + X_{tt} - \frac{\lambda}{x} X_x \\ &- \frac{\lambda}{x^2} X + \frac{\lambda}{x} f\Big) + \left(X_x + T_t\right)\delta(x-\xi)\,\delta(t-\tau) \Big\} \end{aligned} \tag{4.1.3}$$

with

$$X(\xi,\tau) = T(\xi,\tau) = 0 \tag{4.1.3a}$$

Substituting $u_{xx} = u_{tt} - \frac{\lambda}{x} u_x + \delta(x-\xi)\,\delta(t-\tau)$ into (4.1.3) and equating to zero the resulting coefficients of the different derivatives of u, we obtain the following determining equations for the classical group of (4.1.1):

$$T_x - X_t = 0 \tag{4.1.4}$$

$$T_{tt} - T_{xx} - 2f_t - \frac{\lambda}{x} T_x = 0 \tag{4.1.5}$$

$$f_{xx} - f_{tt} + \frac{\lambda}{x} f_x = 0 \tag{4.1.6}$$

$$T_t - X_x = 0 \tag{4.1.7}$$

$$2f_x - X_{xx} + X_{tt} + \frac{\lambda}{x} X_x - \frac{\lambda X}{x^2} = 0 \tag{4.1.8}$$

$$f(\xi,\tau) + T_t(\xi,\tau) - X_x(\xi,\tau) = 0 \tag{4.1.9}$$

(4.1.7) and (4.1.9) $\Longrightarrow$ $f(\xi,\tau) = 0$ (4.1.9')

Substituting (4.1.4) into (4.1.5) and using (4.1.7) we get:

$$2f + \frac{\lambda}{x} X = A(x) \tag{4.1.5'}$$

with arbitrary $A(x)$.

(4.1.4) and (4.1.7) $\Longrightarrow$ $X_{xx} - X_{tt} = 0$. Thus, solving for f in (4.1.8) we find that $A(x) = \alpha$ = arbitrary constant. However, because of (4.1.3a), (4.1.9'), $\alpha = 0$.

$$\therefore \quad X = \frac{-2x}{\lambda} f \tag{4.1.10}$$

Next we compute

$$0 = X_{xx} - X_{tt} = \frac{2f_x}{\lambda} + \frac{xf_{xx}}{\lambda} - \frac{xf_{tt}}{\lambda}$$

Comparing this equation with (4.1.6) we see that

$$f_x = 0 \Longrightarrow f_{tt} = 0 .$$

Thus

$$f = \beta(t-\tau) \quad \text{using (4.1.9')}$$

$$X = -\frac{2x\beta}{\lambda}(t-\tau) \quad \text{from (4.1.10)}$$

(4.1.4) and (4.1.7) $\Longrightarrow$ $T = \frac{\beta}{\lambda}\left[2t\,\tau - t^2 - x^2\right] + \gamma.$

Using (4.1.3a), we finally get (setting $\beta = 1$):

$$\left\{\begin{array}{l} f = (t-\tau) \\[2ex] X = -\dfrac{2x}{\lambda}(t-\tau) \\[2ex] T = \dfrac{(\xi^2 - x^2) - (t-\tau)^2}{\lambda} \end{array}\right\} \qquad (4.1.11)$$

$$\therefore \quad \frac{du}{(t-\tau)u} = -\frac{\lambda dx}{2x(t-\tau)} = \frac{\lambda dt}{(\xi^2 - x^2) - (t-\tau)^2}$$

4.2 The Solution of (4.1.1)

First of all we find the similarity variable, defined by the integral of

$$\frac{dx}{dt} = \frac{2x(t-\tau)}{(t-\tau)^2 + (x^2 - \xi^2)}$$

Let $y = t-\tau$. Then $\frac{y^2 dx}{x^2} - \frac{2y dy}{x} + dx - \frac{\xi^2}{x^2}dx = 0$

$$\Longrightarrow \qquad x - \frac{y^2}{x} + \frac{\xi^2}{x} = \text{const} = z$$

$$\frac{du}{u} = -\frac{\lambda dx}{2x}$$

$$\Longrightarrow \qquad u = x^{-\frac{1}{2}\lambda}\, g(z) \qquad (4.2.1)$$

with $u(\xi, \tau) = 1$ and $g(z)$ an arbitrary function of z.

Substituting (4.2.1) into (4.1.1) we find that

$$g''\left[z^2 - 4\xi^2\right] + 2zg' + \frac{\lambda}{2}\left(1 - \frac{\lambda}{2}\right) g = 0 \qquad (4.2.2)$$

Letting $\varphi = \frac{z + 2\xi}{4\xi}$, (4.2.2) becomes the hypergeometric equation

$$\varphi(1-\varphi)\frac{d^2 g}{d\varphi^2} + (1-2\varphi)\frac{dg}{d\varphi} - \frac{\lambda}{2}\left(1 - \frac{\lambda}{2}\right) g = 0 \qquad (4.2.3)$$

The solution of (4.2.3) having the desired properties is $g(\varphi) = A\, F\left(\frac{\lambda}{2},\ 1 - \frac{\lambda}{2}\ ;\ 1\ ;\ \varphi\right)$ where A is some arbitrary constant to be determined by normalization.

We now use two relations

$$F(a, b\,; c; z) = (1-z)^{-a}\, F\left(a, c-b; c; \frac{z}{z-1}\right)$$

and

$$F(a, b; c; z) = \frac{\Gamma(c)\,\Gamma(b-a)}{\Gamma(b)\,\Gamma(c-a)}(-z)^{-a}\, F(a, 1-c + a; 1-b + a; z^{-1})$$

$$+ (-z)^{-b}\frac{\Gamma(c)\,\Gamma(a-b)}{\Gamma(a)\,\Gamma(c-b)}\, F(b, 1-c + b; 1-a + b; z^{-1})$$

Thus

$$u = B\,(x\varphi)^{-\lambda/2}\; F\left(\frac{\lambda}{2}, \frac{\lambda}{2}; 1; \frac{\varphi-1}{\varphi}\right) \qquad \text{with} \quad B\xi^{-\lambda/2} = 1$$

$$\Longrightarrow \qquad u(x,t) = \frac{(2\xi)^{\lambda}}{\left[(x+\xi)^2 - (t-\tau)^2\right]^{\frac{1}{2}\lambda}} \; F\left(\frac{\lambda}{2}, \frac{\lambda}{2}; 1; \frac{(x-\xi)^2 - (t-\tau)^2}{(x+\xi)^2 - (t-\tau)^2}\right)$$

The result is the same as that found in [11] by different methods.

Chapter V

Nonlinear Heat Conduction

5.1 Derivation of the Classical Group for the Equation of Nonlinear Heat Conduction

In this chapter we shall study the equation of nonlinear heat conduction

$$\frac{\partial}{\partial x}\left(C(u)\frac{\partial u}{\partial x}\right) = \frac{\partial u}{\partial t} \qquad (5.1.1)$$

and find the classical set of functions $C(u)$ for which (5.1.1) can be reduced to an ordinary differential equation. Of course we assume $C(u) \neq$ constant. Ovsjannikov [4], [13], was the first to work on the group theoretical aspects of (5.1.1). He derived the classical group of (5.1.1) but did not proceed to find the resulting ordinary differential equations.

Let

$$\left\{\begin{array}{l} u' = u + \epsilon U(x,t,u) + O(\epsilon^2) \\ t' = t + \epsilon T(x,t,u) + O(\epsilon^2) \\ x' = x + \epsilon X(x,t,u) + O(\epsilon^2) \end{array}\right\} \qquad (5.1.2)$$

represent, as previously, a Lie group of transformations leaving (5.1.1) invariant. Then

$$\frac{\partial}{\partial x'}\left(C(u')\frac{\partial u'}{\partial x'}\right) - \frac{\partial u'}{\partial t'} = \frac{\partial}{\partial x}\left(C(u)\frac{\partial u}{\partial x}\right) - \frac{\partial u}{\partial t}$$

$$+ \epsilon \left\{ u_x\left(X_t + 2C'(u)U_x - C(u)X_{xx} + 2C(u)U_{xu}\right)\right.$$

$$+ \left(C(u)U_{xx} - U_t\right) - 2u_{tx}\, C(u)\, T_x$$

$$- 2u_x\, u_{tx}\, C(u)\, T_u + u_x\, u_t\left(X_u - 2C'(u)\, T_x - 2C(u)\, T_{xu}\right)$$

$$- u_x u_x u_t\left(2C'(u)\, T_u + C(u)\, T_{uu}\right)$$

$$- u_x u_x u_x\left(2C'(u)\, X_u + C(u)\, X_{uu}\right)$$

$$- u_{xx} u_t C(u) T_u - 3u_x u_{xx} C(u) X_u + u_t\left(T_t - U_u - C(u) T_{xx}\right) + u_t u_t T_u$$

$$+ u_x u_x\left(UC''(u) - 2C'(u)X_x + 2C'(u)U_u - 2C(u)X_{xu} + C(u)U_{uu}\right)$$

$$+ u_{xx}\left(C(u)\, U_u + UC'(u) - 2C(u)\, X_x\right)\Big\} + O(\epsilon^2) \qquad (5.1.3)$$

Next we substitute $u_t = C(u)\, u_{xx} + C'(u)\, u_x u_x$ into (5.1.3) and solve for the classical group by the method outlined in 2.2.

Setting the coefficients of u_{tx} and $u_x u_{tx}$ equal to zero, we obtain

$$T = T(t)$$

Equating to zero the coefficient of $u_x u_{xx}$, we have

$$X_u = 0 \qquad .$$

Now equating to zero successively the coefficients of u_x, u_{xx}, $u_x u_x$, and the remaining terms, we are led to the following relations:

$$X_t + 2C'(u)\, U_x - C(u)\, X_{xx} + 2C(u)\, U_{xu} = 0 \qquad (5.1.4)$$

$$C(u)\, T'(t) + UC'(u) - 2C(u)\, X_x = 0 \qquad (5.1.5)$$

$$C'(u)T'(t) + C'(u)U_u + UC''(u) - 2C'(u)X_x + C(u)U_{uu} = 0 \qquad (5.1.6)$$

$$C(u)\,U_{xx} - U_t = 0 \qquad (5.1.7)$$

$$(5.1.5) \implies \qquad U = \frac{C(u)}{C'(u)}\left[2X_x - T'(t)\right]$$

Substituting for U in (5.1.7), we obtain

$$\left\{\begin{aligned} X &= \frac{x}{2}T'(t) + \alpha x^2 + \beta x + \gamma \\ T &= T(t) \\ U &= \frac{C(u)}{C'(u)}\left[4\alpha x + 2\beta\right] \end{aligned}\right\} \qquad (5.1.8)$$

where α, β, γ are arbitrary constants.

Substituting (5.1.5) into (5.1.6) and using (5.1.8) we find that

$$\left(\frac{C}{C'}\right)'' = 0$$

Thus, if one of $\alpha, \beta \neq 0$, classically,

$$C(u) = \lambda(u + \kappa)^{\nu} \qquad (5.1.9)$$

where λ, κ, ν are arbitrary constants.

Now we investigate (5.1.4). Substituting (5.1.8) into this equation, we obtain

$$\frac{x}{2}T''(t) + 2\alpha\, C(u)\left[7 - \frac{4C(u)\,C''(u)}{(C'(u))^2}\right] = 0$$

Therefore for arbitrary $C(u)$, $T''(t) = 0$, $\alpha = 0$

$\implies$ $T(t) = 2A + 2Bt$, where A, B are arbitrary constants.

If $\alpha \neq 0$, we have an additional group which corresponds to functions $C(u)$ satisfying

$$7\left(C'(u)\right)^2 - 4C(u)\,C''(u) = 0$$

$\Longrightarrow$ $C(u) = \lambda(u+\kappa)^{-4/3}$. Thus for $\alpha \neq 0$, $\nu = -\frac{4}{3}$ in (5.1.9).

In summary we have the following three cases:

<u>Case I</u> <u>$C(u)$ arbitrary</u>

$$\left\{\begin{aligned} X &= Bx + \gamma \\ T &= 2A + 2Bt \\ U &= 0 \end{aligned}\right\} \qquad (5.1.10)$$

<u>Case II</u> <u>$C(u) = \lambda(u+\kappa)^{\nu}$</u>

$$\left\{\begin{aligned} X &= (\beta + B)\,x + \gamma \\ T &= 2A + 2Bt \\ U &= \frac{2\beta}{\nu}(u+\kappa) \end{aligned}\right\} \qquad (5.1.11)$$

A limiting case here is $C(u) = e^{u}$

<u>Case III</u> <u>$C(u) = \lambda(u+\kappa)^{-4/3}$</u>

$$\left\{\begin{aligned} X &= (\beta + B)x + \alpha x^2 + \gamma \\ T &= 2A + 2Bt \\ U &= -\tfrac{3}{2}(u+\kappa)\left[2\alpha x + \beta\right] \end{aligned}\right\} \qquad (5.1.12)$$

For each case we consider the resulting ordinary differential equation.

5.2 C(u) arbitrary

If we assume that $B \neq 0$, then

$$\frac{dx}{x + \gamma} = \frac{dt}{2(A + t)} = \frac{du}{0}$$

$\therefore$ the similarity variable is $\eta = \dfrac{x + \gamma}{(A + t)^{1/2}}$

$$u = F(\eta) \quad \Longrightarrow \quad C(u) = D(F)$$

The resulting or dinary differential equation is

$$\frac{\eta}{2}\frac{dF}{d\eta} + \frac{d}{d\eta}\left(D(F)\ \frac{dF}{d\eta}\right) = 0$$

5.3 $C(u) = \lambda(u + \kappa)^{\nu}$

Again, we assume $B \neq 0$. Thus

$$\frac{dx}{(1 + \beta)x + \gamma} = \frac{dt}{2(A + t)} = \frac{du}{2\frac{\beta}{\nu}(u + \kappa)}$$

Here the similarity variable is

$$\eta = \frac{\left(x + \frac{\gamma}{1 + \beta}\right)}{\left(A + t\right)^{\frac{\beta + 1}{2}}}$$

with $u + \kappa = (A + t)^{\beta/\nu}\ F(\eta)$.

The resulting ordinary differential equation is

$$\frac{\beta + 1}{2}\ \eta\ \frac{dF}{d\eta} - \frac{\beta}{\nu} F(\eta) + \lambda \frac{d}{d\eta}\left[\left[F(\eta)\right]^{\nu} \frac{dF}{d\eta}\right] = 0 \qquad (5.3.1)$$

(5.3.1) is invariant under the following one parameter group of transformations

$$\left\{\begin{aligned} \eta' &= \mu\eta \\ F' &= \mu^{2/\nu} F \end{aligned}\right\} \qquad (5.3.2)$$

Therefore as new variables we choose the invariants

$$\left\{\begin{aligned} g &= \frac{F^\nu}{\eta^2} \quad \text{and} \\ h &= h(g) = \frac{F^{\nu-1}}{\eta} \frac{dF}{d\eta} \end{aligned}\right\} \qquad (5.3.3)$$

Then $$\frac{dg}{d\eta} = \frac{1}{\eta}\left[\nu h - 2g\right]$$

$$\frac{dh}{d\eta} = \frac{1}{\eta}\left[\frac{\beta}{\lambda\nu} - \left(\frac{h^2}{g} + \frac{\beta+1}{2\lambda}\,\frac{h}{g} + h\right)\right]$$

$\therefore$ (5.3.1) is reduced to the following first order ordinary differential equation:

$$\frac{dh}{dg} = \frac{2\beta g - \nu h\left[2\lambda(h+g) + (\beta+1)\right]}{2\lambda\nu g\left[\nu h - 2g\right]} \qquad (5.3.4)$$

which may be solved by looking at the phase plane and picking the path appropriate to given data.

After solving (5.3.4), to pass from the (h,g)-plane to the (U,η)-plane, we solve

$$\frac{d\eta}{\eta} = \frac{dg}{\nu h - 2g}$$

5.4 $\underline{C(u) = \lambda(u+\kappa)^{-4/3}}$

We assume that $B \neq 0$

Here $$\frac{dx}{(\beta+1)x + \alpha x^2 + \gamma} = \frac{dt}{2(A+t)} = \frac{du}{-\frac{3}{2}(u+\kappa)(2\alpha x+\beta)}$$

For illustrative purposes, we consider the special case $(\beta + 1)^{\nu} = 4\alpha\gamma$. Then the similarity variable

$$\eta = \frac{e^{-\frac{2}{2\alpha x + \beta + 1}}}{(A + t)^{1/2}}$$

and

$$u + \kappa = \frac{F(\eta)}{\left(x + \frac{\beta + 1}{2\alpha}\right)^3} \; e^{-\frac{3}{2\alpha x + \beta + 1}}$$

The resulting ordinary differential equation is:

$$F'' - \frac{4}{3}\,\frac{F'F'}{F} - \frac{3}{4}\,\frac{F}{\eta^2} + \frac{\alpha^2}{2\lambda}\,\eta\, F^{4/3}\, F' = 0 \qquad (5.4.1)$$

Let

$$\left\{\begin{aligned} g &= F\eta^{3/2} \\ h &= F'\eta^{5/2} \end{aligned}\right\} \qquad (5.4.2)$$

Then

$$\frac{dg}{d\eta} = \frac{1}{\eta}\left[\frac{3}{2}g + h\right]$$

$$\frac{dh}{d\eta} = \frac{1}{\eta}\left[\frac{4}{3}\,\frac{h^2}{g} + \frac{3}{4}g + \frac{5}{2}h - \frac{\alpha^2}{2\lambda}\, g^{4/3}\, h\right].$$

Thus (5.4.1) is reduced to

$$\frac{dh}{dg} = \frac{16h^2 + 9g^2 + 30gh - \frac{6\alpha^2}{\lambda}\, g^{5/3}\, h}{6g(3g + 2h)} \qquad \text{with} \qquad \frac{d\eta}{\eta} = \frac{2dg}{3g + 2h}$$

CHAPTER VI. THE CONFORMAL GROUP

In this chapter we discuss the conformal group, $C(n,m)$, which is a subgroup of the classical group of transformations leaving invariant the equation

$$\left[\sum_{i=1}^{n} - \sum_{i=n+1}^{n+m}\right] \frac{\partial^2 u}{\partial x_i^2} = 0$$

We show that $C(n,m)$ is locally isomorphic to the group $SO(n+1,m+1)$ which leaves invariant the quadratic form

$$\left[\sum_{i=1}^{n+1} - \sum_{i=n+2}^{n+m+2}\right] x_i^2 .$$

A consequence of this result for Laplace's equation in n dimensions is the existence of closed bounded geometries other than concentric spherical boundaries for which the number of variables may be reduced by one since compact transformation other than rotations leave the equation invariant.

The following notation will be used in this chapter:

(1) $A(i) = \begin{cases} \{1, 2, \ldots, n\} & \text{if } i = 1, 2, \ldots, n \\ \{n+1, n+2, \ldots, n+m\} & \text{if } i = n+1, n+2, \ldots, n+m \end{cases}$

(2) $\partial_i = \dfrac{\partial}{\partial x_i}$

(3) Repeated indices $\Longrightarrow$ summation over the index.

(4) $\sum = \left(\sum_{i=1}^{n} - \sum_{i=n+1}^{n+m}\right)$

(5) We shall be using the space $R^{n,m}$ with metric $\vec{r}\cdot\vec{r} = \sum x_i^2$.

(6) $\nabla^2_{n,m} = \sum \partial_i^{\,2}$

6.1 Matrix Groups

Up to now (cf. Chapter I) we have only considered group transformations on functions. In this section we give a brief introduction to matrix groups. A matrix group is simply a set of matrices which satisfy the group axioms with matrix multiplication as the law of combination.

Let α be a linear transformation which leaves invariant the real non-singular bilinear form $G_{\ell m}x_\ell x_m$. This is equivalent to finding a matrix α such that $(\alpha\vec{e}_\ell, \alpha\vec{e}_m) = (\vec{e}_\ell, \vec{e}_m) = G_{\ell m}$ where the $\vec{e}_\ell$ are basis vectors and G is the metric matrix,

$$\text{i.e.,}\quad \alpha^T G \alpha = G \tag{6.1.1}$$

where $(\alpha^T)_{ij} = \alpha_{ji}$. The set of all α satisfying (6.1.1) form a matrix group.

For Euclidean space, R^n, one may choose G = Identity matrix = I and $\{\alpha\}$ = {orthogonal matrices}.

For our purposes we are only interested in connected, continuous matrix groups, whose study may be reduced to considering one-parameter sub-groups which are canonical in their respective parameters, i.e., we have a set of matrices $\gamma(t)$ such that

$$\left.\begin{array}{ll} (1) & \gamma(0) = I \\ (2) & \gamma(t_1)\,\gamma(t_2) = \gamma(t_1+t_2) \\ \Longrightarrow & \gamma^{-1}(t) = \gamma(-t) \end{array}\right\} \tag{6.1.2}$$

$$(6.1.2) \implies \gamma(t) = e^{t\gamma'(0)}$$

$B = \gamma'(0)$ is called the infinitesimal generator of the one-parameter matrix group $\{\gamma(t)\}$.

In order to give some interpretation to B, as an example we consider the rotation matrices

$$\gamma(t) = \begin{pmatrix} \cos\omega t & -\sin\omega t & 0 \\ \sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which represent rotation about the z-axis with angular velocity ω.

Let

$$\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$$

and say $\vec{r}(t) = \gamma(t)\,\vec{r}(0)$.

Then $\vec{r}\,'(0) = \gamma'(0)\,\vec{r}(0)$

$$= \omega \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \vec{r}(0)$$

$$= \vec{\omega} \times \vec{r}(0) \quad \text{where} \quad \vec{\omega} = \omega\vec{k}\,.$$

Thus in this example $\gamma'(0) = \vec{\omega}\times$ and corresponds to angular velocity.

6.2 Lie Algebras

A linear space $\mathcal{L}$ is called a Lie algebra with $\circ$ as the law of combination if for any $A, B, C \in \mathcal{L}$, α, β scalars:

(1) $A \circ B \in \mathcal{L}$

(2) $(\alpha A + \beta B)\circ C = \alpha(A\circ C) + \beta(B\circ C)$

(3) $A\circ B = -B\circ A$

(4) $A\circ(B\circ C) + B\circ(C\circ A) + C\circ(A\circ B) = 0$

A familiar example of a Lie algebra is the set of all vectors in R^3 with $\times$ (vector product symbol) as the law of combination.

Both the infinitesimal operators corresponding to group transformations on functions and the infinitesimal generators of a matrix group form Lie algebras with

$$A\circ B = [A,B] = AB - BA$$

$[A,B]$ is known as the commutator of the elements A and B.

6.3 The Lie Algebra of the Group SO(n,m)

For small t, $\gamma(t) = I + tB + O(t^2)$ $\therefore$ from (6.1.1) we obtain

$$B^T G + GB = 0 \qquad (6.3.1)$$

Now we specialize G to the space $R^{n,m}$ (n "space-like" and m "time-like" dimensions).

Here $G_{\alpha\beta} = \epsilon_\alpha \delta_{\alpha\beta}$ where

$$\epsilon_\alpha = \begin{cases} 1 & \text{if } \alpha \in A(1) \\ -1 & \text{if } \alpha \in A\, n{+}1) \end{cases}$$

$$\Longrightarrow \epsilon_k B_{ki} + \epsilon_i B_{ik} = 0 \qquad (6.3.2)$$

$\therefore$ if $A(k) = A(i)$, i.e., if k and i are relatively space-like, then $B_{ki} = -B_{ik}$ (elliptic rotation); if $A(k) \neq A(i)$, then $B_{ki} = B_{ik}$ (hyperbolic rotations). . The B_{ik} satisfying (6.3.2) constitute a basis for the Lie algebra of $SO(n,m)$.

6.4 Derivation of the Conformal Group

In this section for the space $R^{n,m}$ we derive the conformal group, which is a subgroup of the Lie group of transformations leaving invariant

$$\nabla^2_{n,m} u = 0 \tag{6.4.1}$$

Let

$$\left\{\begin{array}{l} x'_i = x_i + \epsilon X_i + O(\epsilon^2), \ i = 1, 2, \ldots, n+m \\ \\ u' = u + \epsilon f u + O(\epsilon^2) \end{array}\right\} \tag{6.4.2}$$

where

$$f = f(x_1, x_2 \ldots, x_{n+m})$$

$$X_i = X_i(x_1, x_2, \ldots, x_{n+m})$$

represent the classical group of transformations leaving (6.4.1) invariant. Then

$$\partial'^2_i u' = \partial^2_i u + \epsilon\left[- 2X_{j,i}\partial_i\partial_j - X_{j,ii}\partial_j + 2f_{,i}\partial_i + f_{,ii} + f\partial^2_i\right]u + O(\epsilon^2) \tag{6.4.3}$$

In (6.4.3) there is no summation over i.

From (6.4.3) we obtain:

$$\nabla'^2_{n,m} u' = \sum \partial'^2_i u' = \sum \left(\partial_i^2 + \epsilon\left[f\partial_i^2 - 2X_{j,i}\,\partial_i\partial_j - X_{j,ii}\partial_j + 2f,_i\partial_i + f,_{ii}\right]\right)u + O(\epsilon^2) \qquad (6.4.4)$$

Then the classical group results from solving the following determining equations:

$$\left\{\begin{array}{ll} X_{i,j} = X_{j,i} & \text{if} \quad A(i) \neq A(j) \\ X_{i,j} = \delta_{ij} f - X_{j,i} & \text{if} \quad A(i) = A(j) \end{array}\right\} \qquad (6.4.5a)$$

$$\left\{\begin{array}{ll} 2f,_j = \sum X_{j,ii} & \text{if} \quad j \in A(1) \\ 2f,_j = -\sum X_{j,ii} & \text{if} \quad j \in A(n+1) \end{array}\right\} \qquad (6.4.5b)$$

$$\sum f,_{ii} = 0 \qquad (6.4.5c)$$

We assume that $\ell = n+m \geqslant 3$.

In order to systematically analyze the determining equations it is convenient to first consider $X_{i,jk}$ where $i \neq j \neq k \neq i$

(a) $i, j, k \in A(i)$

Then $X_{i,jk} = -X_{j,ik} = X_{k,ji} = -X_{i,jk} = 0$

(b) $i, j \in A(i), \quad A(k) \neq A(i)$

Then $X_{i,jk} = -X_{j,ik} = -X_{k,ij} = -X_{i,jk} = 0$

Next we consider $X_{i,ijj} = \frac{1}{2} f,_{jj}$ where there is no summation over i and j.

If $A(j) = A(i)$, then $X_{i,ijj} = -X_{j,jii} = -\frac{1}{2} f,_{ii}$.

If $k \neq i,j$ but $A(k) = A(i)$, then $f,_{kk} = -f,_{ii}$ and $f,_{kk} = -f,_{jj} \Longrightarrow f,_{ii} = 0$.

If $k \neq i,j$ and $A(k) \neq A(i)$, then $X_{i,ikk} = X_{k,ikk}$ $\Longrightarrow f,_{kk} = f,_{ii} = f,_{jj} = 0$ since $f,_{ii} = -f,_{jj}$ $\therefore$ if $\ell \geq 3$

$$\left\{\begin{array}{l} X_{i,ijj} = 0 \\ \\ X_{i,jk} = 0 \quad \text{if} \quad i \neq j \neq k \neq i \end{array}\right\} \tag{6.4.6}$$

with no summation over i and j.

We note that 3 indices were needed to derive (6.4.6).

$$(6.4.6) \Longrightarrow \quad X_k = \tilde{a}_k + \tilde{b}_{kj} x_j + \tilde{C}_{kj} x_k x_j + \tilde{d}_{kj} x_j x_j \tag{6.4.7}$$

Applying (6.4.5a) to (6.4.7), we find that

(1) $\tilde{b}_{kk} = g$ for $k = 1, 2, \ldots, \ell$.

(2) $\tilde{C}_{kj} = 2C_j$ for each $k = 1, 2, \ldots, \ell$.

(3) $\tilde{d}_{kj} = -C_k$ if $A(k) = A(j)$

(4) $\tilde{d}_{kj} = C_k$ if $A(k) \neq A(j)$

(5) $\tilde{b}_{kj} = b_{kj} = -b_{jk}$ if $A(k) = A(j)$, $k \neq j$

(6) $\tilde{b}_{kj} = b_{kj} = b_{jk}$ if $A(k) \neq A(j)$

where g, C_j, b_{jk} $(j < k)$ are arbitrary constants. Then

$$X_k = \left\{\begin{array}{ll} a_k + b_{kj} x_j - C_k \sum x_i^2 + 2C_j x_k x_j + g x_k & \text{if} \quad k \in A(1) \\ \\ a_k + b_{kj} x_j + C_k \sum x_i^2 + 2C_j x_k x_j + g x_k & \text{if} \quad k \in A(n+1) \end{array}\right\} \tag{6.4.8}$$

Then (6.4.5b) $\Longrightarrow$

$$f = (2-\ell)C_k x_k + h \tag{6.4.9}$$

where h is an arbitrary constant.

The conformal group, $C(n,m)$ is defined to be the subgroup for which $h = \frac{(2-\ell)g}{2}$. We must select this value for h in order that $C(n,m)$ be isomorphic to the usual conformal group which leaves invariant $\sum dx_i^2 = 0$.

Next we consider the infinitesimal transformations individually and form their corresponding infinitesimal operators. This is accomplished by setting one of our constants equal to one, and the rest equal to zero.

(1) $\ell = n+m$ translations

(a) n "space" translations for which $X_j = \delta_{ji}$, $j = 1, 2, \ldots, n$, infinitesimal operators $T_i = \frac{\partial}{\partial x_i}$ (6.4.10)

(b) m "time" translations for which $X_j = \delta_{ji}$, $j = n+1, n+2, \ldots, n+m$, with infinitesimal operators $t_i = \frac{\partial}{\partial x_i}$ (6.4.11)

(2) 1 stretching corresponding to

$$X_i = x_i\,, \quad \text{for each } i$$

$$f = \left(\frac{2-\ell}{2}\right)$$

The infinitesimal operator is

$$S = x_k \frac{\partial}{\partial x_k} + \left(\frac{2-\ell}{2}\right) u \frac{\partial}{\partial u} \tag{6.4.12}$$

(3) $\frac{\ell(\ell-1)}{2}$ "rotations"

(a) $\frac{n(n-1)}{2}$ elliptic rotations for $i, j \in A(1)$

$$\left.\begin{aligned} X_i &= x_j \\ X_j &= -x_i \end{aligned}\right\} \quad i, j = 1, 2, \ldots, n, \quad i \neq j$$

The infinitesimal operators are

$$U_{ij} = \left[x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} \right] \tag{6.4.13}$$

(b) $\frac{m(m-1)}{2}$ elliptic rotations for $i, j \in A(n+1)$

Here the infinitesimal operators are

$$u_{ij} = \left[x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} \right] \tag{6.4.14}$$

for $i, j = n+1, n+2, \ldots, n+m$.

(c) $n\,m$ hyperbolic rotations corresponding to $A(i) \neq A(j)$

$$\left.\begin{aligned} X_i &= x_j \\ X_j &= x_j \end{aligned}\right\} \quad \begin{aligned} &i = 1, 2, \ldots, n \\ &j = n+1, n+2, \ldots, n+m \end{aligned}$$

The infinitesimal operators are

$$V_{ij} = \left[x_i \frac{\partial}{\partial x_j} + x_j \frac{\partial}{\partial x_i} \right] \tag{6.4.15}$$

(4) ℓ distortions

(a) n "space" distortions

$$X_k = x_k x_k - \frac{\vec{r} \cdot \vec{r}}{2}$$

$$X_j = x_k x_j \quad \text{if} \quad j \neq k$$

$$f = \left(\frac{2-\ell}{2}\right) x_k, \quad k = 1, 2, \ldots, n$$

Of course we are not summing over k. The infinitesimal operators are

$$D_k = -\tfrac{1}{2}(\vec{r}\cdot\vec{r})\frac{\partial}{\partial x_k} + x_k x_j \frac{\partial}{\partial x_j} + \frac{2-\ell}{2} x_k u \frac{\partial}{\partial u} \tag{6.4.16}$$

(b) m "time" distortions

$$X_k = x_k x_k + \frac{(\vec{r}\cdot\vec{r})}{2}$$

$$X_j = x_k x_j, \quad \text{if} \quad j \neq k$$

$$f = \left(\frac{2-\ell}{2}\right) x_k, \quad k = n+1,\ n+2,\ \dots,\ n+m$$

with infinitesimal operators

$$d_k = \tfrac{1}{2}(\vec{r}\cdot\vec{r})\frac{\partial}{\partial x_k} + x_k x_j \frac{\partial}{\partial x_j} + \left(\frac{2-\ell}{2}\right) x_k u \frac{\partial}{\partial u} \tag{6.4.17}$$

We note that, in all, we have $\frac{(\ell+1)(\ell+2)}{2}$ independent infinitesimals. The basis of the Lie algebra of $SO(n+1, m+1)$ has the same number of elements.

6.5 The Distortion Transformations

Let $\vec{r} = (x_1, x_2, \dots, x_\ell)$ represent a vector in $R^{n,m}$ and let $\vec{I}_i$ represent a unit vector for the i^{th} component so that

$$\vec{I}_i\cdot\vec{r} = x_i \quad \text{if} \quad i \in A(1),$$

$$\vec{I}_i\cdot\vec{r} = -x_i \quad \text{if} \quad i \in A(n+1).$$

In our manifold $R^{n,m}$, we define a "hypersphere" of radius α about the origin $\underbrace{(0,0,\dots,0)}_{\ell}$ as $\{(x_1, x_2, \dots, x_\ell)\}$ such that

$$\sum x_i^2 = \left[\sum_{i=1}^{n} - \sum_{i=n+1}^{n+m}\right] x_i^2 = \vec{r}\cdot\vec{r} = \alpha^2 = \text{constant} \geqslant 0$$

Let $\mathcal{I}$ denote inversion about the unit hypersphere. Then $\mathcal{I}(\vec{r}) = \frac{\vec{r}}{\vec{r}\cdot\vec{r}}$. In addition a transformation in the large will be denoted by a tilde. Thus, $e^{\epsilon D_i} = \tilde{D}_i(\epsilon)$, $e^{\epsilon T_i} = \tilde{T}_i(\epsilon)$, $e^{\epsilon S} = \tilde{S}(\epsilon)$, etc. It can be easily shown that

$$\tilde{T}_i(\epsilon)\,(x_1, x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_\ell)$$

$$= (x_1, x_2, \ldots, x_{i-1}, x_i + \epsilon, x_{i+1}, \ldots, x_\ell)$$

and $$\tilde{S}(\epsilon)\,(\vec{r}) = e^{\epsilon}\,\vec{r}$$

We now prove that in the large a space distortion $\tilde{D}_i(\epsilon)$ corresponds to an inversion, followed by a space translation of x_i by $-\frac{1}{2}\epsilon$, followed by another inversion, i.e.,

(a) $$\tilde{D}_i(\epsilon) = \mathcal{I}\,\tilde{T}_i(-\tfrac{1}{2}\epsilon)\,\mathcal{I}$$

Similarly it can be proved that

(b) $$\tilde{d}_i(\epsilon) = \mathcal{I}\,\tilde{t}_i(-\tfrac{1}{2}\epsilon)\,\mathcal{I}$$

<u>Proof</u>

(a) $$\mathcal{I}\,\tilde{T}_i(-\tfrac{1}{2}\epsilon)\,\mathcal{I}(\vec{r})$$

$$= \mathcal{I}\,\tilde{T}_i(-\tfrac{1}{2}\epsilon)\left(\frac{\vec{r}}{\vec{r}\cdot\vec{r}}\right)$$

$$= \mathcal{I}\left(\frac{\vec{r} - \frac{1}{2}\epsilon\,\vec{1_i}}{(\vec{r}\cdot\vec{r}) - \epsilon x_i + \frac{\epsilon^2}{4}}\right)$$

$$= \frac{\left(\frac{\vec{r}}{\vec{r}\cdot\vec{r}} - \frac{1}{2}\epsilon\,\vec{1}_i\right)}{\left(\frac{1}{\vec{r}\cdot\vec{r}} - \frac{\epsilon x_i}{\vec{r}\cdot\vec{r}} + \frac{\epsilon^2}{4}\right)}$$

$$= \frac{\vec{r} - \frac{1}{2}\epsilon\,(\vec{r}\cdot\vec{r})\,\vec{1}_i}{1 - \epsilon x_i + \frac{1}{4}\epsilon^2(\vec{r}\cdot\vec{r})}$$

$$= \vec{r} + \tfrac{1}{2}\epsilon\left[-(\vec{r}\cdot\vec{r})\,\vec{1}_i + 2x_i\,\vec{r}\,\right] + O(\epsilon^2)$$

which agrees with (6.4.16).

6.6 The Conformal Group for R^2

It is easily shown that for R^2, in terms of complex variables, the local conformal group corresponds to all analytic functions. However we now could consider those conformal transformations which we have derived for $R^{n,m}(n+m \geqslant 3)$ in R^2. We prove that in R^2, these correspond to the 6 parameter Möbius (bilinear) group which is represented by the complex valued functions $\ell(z) = \frac{az+b}{cz+d}$, $ad-bc \neq 0$.

It is well known that the Möbius group can be built up from the following fundamental transformations:

(a) $\quad \ell_1(z) = z + \alpha \quad$ (α complex) (translation)

(b) $\quad \ell_2(z) = \alpha z \quad$ (α complex) (rotation and stretching)

(c) $\quad \ell_3(z) = \frac{1}{z} = \mathcal{J}(z) \quad$ (inversion about the unit circle)

Using the results of 6.5, in order to prove the equivalence of $C(2,0)$ and the Möbius group, we only need to prove that $\mathcal{J}(z)$ can be obtained from stretchings, translations and distortions.

Proof

One can easily show that

$$\tilde{D}_1(2\epsilon)(z) = \frac{z}{1-\epsilon z}$$

$$\tilde{D}_2(2\epsilon)(z) = \frac{z}{1+i\epsilon z}$$

Consider

$$\tilde{S}(\log\frac{1}{\epsilon^2})\,\tilde{T}_1(\epsilon)\,\tilde{D}_1(\frac{2}{\epsilon})\,\tilde{T}_1(\epsilon)(z)$$

$$= \tilde{S}(\log\frac{1}{\epsilon^2})\,\tilde{T}_1(\epsilon)\,\tilde{D}_1(\frac{2}{\epsilon})\,(z+\epsilon)$$

$$= \tilde{S}(\log\frac{1}{\epsilon^2})\,\tilde{T}_1(\epsilon)\,(\frac{\epsilon^2}{\epsilon - z})$$

$$= \tilde{S}(\log\frac{1}{\epsilon^2})\,(\frac{-\epsilon^2}{z})$$

$= -\frac{1}{z}$. Then after rotating $z = r\,e^{i\theta}$ by angle π, we finally get $\frac{1}{z} = \mathcal{J}(z)$

6.7 $C(n,m) \cong SO(n+1), m+1)$

In this section we prove that $C(n,m)$ is locally isomorphic to $SO(n+1, m+1)$ by showing that their respective Lie algebras satisfy the same commutation relations.

The commutation relations between the infinitesimal operators of $C(n,m)$ are:

$$[T_i, T_j] = [T_i, t_j] = [t_i, t_j] = [D_i, D_j] = [D_i, d_j] = [d_i, d_j] =$$

$$= [T_i, u_{jk}] = [D_i, u_{jk}] = [t_i, U_{jk}] = [d_i, U_{jk}] = 0\,;$$

$$[T_i, U_{jk}] = \delta_{ij}T_k - \delta_{ik}T_j\,;$$

$$[D_i, U_{jk}] = \delta_{ij}D_k - \delta_{ik}D_j\,;$$

$[t_i, u_{jk}] = \delta_{ij}t_k - \delta_{ik}t_j$;

$[d_i, u_{jk}] = \delta_{ij}d_k - \delta_{ik}d_j$;

$[T_i, V_{jk}] = \delta_{ij}t_k + \delta_{ik}t_j$;

$[D_i, V_{jk}] = -[\delta_{ij}d_k + \delta_{ik}d_j]$;

$[t_i, V_{jk}] = \delta_{ij}T_k + \delta_{ik}T_j$;

$[d_i, V_{jk}] = -[\delta_{ij}D_k + \delta_{ik}D_j]$;

$[T_i, S] = T_i$; $[t_i, S] = t_i$;

$[D_i, S] = -D_i$; $[d_i, S] = -d_i$;

$[T_j, d_i] = [t_i, D_j] = V_{ij}$;

$[t_i, d_j] = \delta_{ij}\, S - u_{ij}$;

$[T_i, D_j] = \delta_{ij}\, S - U_{ij}$;

$[A_{ij}, A_{k\ell}] = \delta_{jk}A_{i\ell} - \delta_{j\ell}A_{ik} + \delta_{i\ell}A_{jk} - \delta_{ik}A_{j\ell}$;

$[V_{ij}, V_{k\ell}] = \delta_{jk}A_{i\ell} + \delta_{j\ell}A_{ik} + \delta_{i\ell}A_{jk} + \delta_{ik}A_{j\ell}$;

$[A_{ij}, V_{k\ell}] = \delta_{jk}V_{i\ell} + \delta_{j\ell}V_{ik} - \delta_{i\ell}V_{jk} - \delta_{ik}V_{j\ell}$;

$[A_{ij}, S] = [V_{ij}, S] = 0$; where $A = u$ or U.

Let E_{ij} represent an elementary $(\ell+2)\times(\ell+2)$ matrix, i.e., the only nonzero element in the matrix E_{ij} is a 1 in the i^{th} row and j^{th} column. Then a basis for the Lie algebra of $SO(n+1, m+1)$ is:

$$E_{ij} - E_{ji}, \quad i,j = 1, 2, \ldots, n$$

$$E_{ij} + E_{ji}, \text{ for each } i,j = 1, 2, \ldots, \ell \text{ with } A(i) \neq A(j)$$

$$E_{ij} - E_{ji}, \quad i,j = n+1, n+2, \ldots, \ell$$

$$E_{\ell+1, \ell+2} + E_{\ell+2, \ell+1}$$

$$E_{i,\ell+2} - E_{\ell+2,i}, \quad i = n+1, n+2, \ldots, \ell$$

$$E_{i,\ell+1} - E_{\ell+1,i}, \quad i = 1, 2, \ldots, n$$

$$E_{i,\ell+2} + E_{\ell+2,i}, \quad i = 1, 2, \ldots, n$$

$$E_{i,\ell+1} + E_{\ell+1,i}, \quad i = n+1, n+2, \ldots, n+m$$

We see that $(\ell+1)$ acts like a "space" index and $(\ell+2)$ like a "time" index.

We show that $C(n,m)$ is locally isomorphic (the global structure of the group will not be discussed here) to $SO(n+1, m+1)$ by letting:

$$U_{ij} = [E_{ij} - E_{ji}], \quad i,j = 1, 2, \ldots, n$$

$$V_{ij} = [E_{ij} + E_{ji}], \quad \text{for each } i,j = 1, 2, \ldots, \ell \text{ such that. } A(i) \neq A(j)$$

$$u_{ij} = [E_{ij} - E_{ji}], \quad i,j = n+1, n+2, \ldots, \ell$$

$$S = E_{\ell+1, \ell+2} + E_{\ell+2, \ell+1}$$

$$T_i = \frac{1}{\sqrt{2}} \left\{ [E_{\ell+1,i} - E_{i,\ell+1}] - [E_{\ell+2,i} + E_{i,\ell+2}] \right\}, \quad i+1, 2, \ldots, n$$

$$t_i = \frac{1}{\sqrt{2}} \left\{ [E_{i,\ell+2} - E_{\ell+2,i}] + [E_{\ell+1,i} + E_{i,\ell+1}] \right\}, \quad i = n+1, n+2, \ldots, \ell$$

$$D_i = \frac{1}{\sqrt{2}} \left\{ [E_{\ell+1,i} - E_{i,\ell+1}] + [E_{\ell+2,i} + E_{i,\ell+2}] \right\} \quad , \; i = 1, 2, \ldots, n \,.$$

$$d_i = \frac{1}{\sqrt{2}} \left\{ [E_{i,\ell+2} - E_{\ell+2,i}] - [E_{\ell+1,i} + E_{i,\ell+1}] \right\} \quad , \; i = n+1, n+2, \ldots, \ell.$$

6.8 The Compact Operators $T_i + D_i$, $t_i + d_i$

From the proof of relations 6.7 we note that ℓ more locally compact operators $T_i + D_i$, $t_i + d_i$ exist in addition to the elliptic rotation operators U_{ij}, u_{ij} since $E_{i,\ell+2} - E_{\ell+2,i}$ and $E_{i,\ell+2} - E_{\ell+1,i}$ represent elliptic rotations. We now find the surface which $T_i + D_i$ leaves invariant for the case $n = \ell$, i.e., for Laplace's equation in n dimensions we try to find additional closed boundaries (besides concentric spheres) for which the number of variables may be reduced by one. As an example we take $T_1 + D_1$. To find the invariant surface, we solve

$$\frac{dx_1}{1 + \frac{1}{2}[x_1{}^2 - x_2{}^2 - x_3{}^2 - \ldots - x_n{}^2]} = \frac{dx_2}{x_1 x_2} = \frac{dx_3}{x_1 x_3} = \ldots = \frac{dx_n}{x_1 x_n}$$

$$\therefore \quad x_n = \alpha_n x_2 \quad \text{for} \quad n \geqslant 3.$$

Let $\beta = -\frac{1}{2}[1 + \alpha_3{}^2 + \alpha_4{}^2 + \ldots + \alpha_n{}^2]$

Then $$\frac{dx_1}{dx_2} = \frac{1}{x_1 x_2} + \frac{1}{2}\frac{x_1}{x_2} + \beta \frac{x_2}{x_1}$$

$$\Longrightarrow \qquad x_1{}^2 = -2 + 2\beta x_2{}^2 + x_2 \text{ fn} \left(\frac{x_3}{x_2}, \frac{x_4}{x_2}, \ldots, \frac{x_n}{x_2} \right)$$

Taking the arbitrary fn to be $2\gamma \frac{x_k}{x^2}$ and using the formulas for β and the α's one finds that the following set of surfaces is left invariant $(k \geqslant 2)$:

$$x_1{}^2 + x_2{}^2 + \ldots + x_{k-1}^2 + (x_k - \gamma)^2 + x_{k+1}^2 + \ldots + x_n^2 = \gamma^2 - 2$$

which corresponds to a family of non-concentric spheres. This implies that the number of variables in Laplace's equation may be reduced by one for special boundary conditions imposed on <u>any</u> two non-intersecting spheres (since after rotations and stretchings they may be made members of the above family):

Consider
$$\frac{\partial^2 u}{\partial x_i^2} = 0 \tag{6.8.1}$$

We now derive the reduced form of (6.8.1) due to invariance under $T_1 + D_1$.

Our similarity variables will be

$$\xi_i = \frac{x_i}{x_2}, \quad i \geqslant 3$$

$$\eta = \frac{\left(\sum_{i=1}^{n} x_i^2\right) + 2}{x_2}$$

$$u = x_2^{\frac{2-n}{2}} \quad U(\xi_3, \xi_4, \ldots, \xi_n, \eta)$$

with U satisfying the following partial differential equation:

$$(\eta^2 - \gamma)\frac{\partial^2 U}{\partial \eta^2} + \frac{\partial^2 U}{\partial \xi_k^2} + 2\eta\, \xi_k \frac{\partial^2 U}{\partial \xi_k \partial \eta} + \xi_k \xi_j \frac{\partial^2 U}{\partial \xi_j \partial \xi_k} + n\eta \frac{\partial U}{\partial \eta} + n\xi_k \frac{\partial U}{\partial \xi_k}$$
$$+ \frac{n(n-2)}{4} U = 0 \qquad \text{where } \eta \geqslant 2\sqrt{2}$$

We consider the special case when $U = U(\eta)$. Then

$$(\eta^2 - 8)\frac{d^2 U}{d\eta^2} + n\eta \frac{dU}{d\eta} + \frac{n(n-2)}{4} U = 0$$

If $\quad n = 2, \quad$ then $\quad U = \lambda \log\left(\frac{\eta - 2\sqrt{2}}{\eta + 2\sqrt{2}}\right) + \mu$

If $\quad n \geqslant 3, \quad$ then $\quad U = \dfrac{\lambda}{(\eta - 2\sqrt{2})^{\frac{n-2}{2}}} + \dfrac{\mu}{(\eta + 2\sqrt{2})^{\frac{n+2}{2}}}$

By performing a simple translation in η these solutions contain, as special cases, the Green's functions for Laplace's equation in a half-space or in an n-dimensional sphere which are usually found by the method of images and application of the Kelvin transformation. [11]

Having found the invariant surfaces for $T + D$, we now show what $T + D$ represents in the large. Since from the commutation relations T, D, S span a Lie algebra (which in fact is nothing but $C(1)$), we expect to be able to express $e^{\epsilon(T+D)}$ as a product of $\tilde{T}, \tilde{S}, \tilde{D}$. Thus we shall assume that we can find functions $\theta = \theta(\epsilon)$, $y = y(\epsilon)$, such that

$$e^{\epsilon(T+D)} = \tilde{T}(\theta)\,\tilde{S}(y)\,\tilde{D}(\theta)$$

Differentiating both sides of this equation with respect to ϵ we find that

$$(T+D)\,e^{\epsilon(T+D)} = \theta' T\,\tilde{T}(\theta)\tilde{S}(y)\tilde{D}(\theta) + y'\,\tilde{T}(\theta)S\tilde{S}(y)\tilde{D}(\theta) + \theta'\tilde{T}(\theta)\tilde{S}(y)D\tilde{D}(\theta)$$

From the commutation relations,

$$TS = ST + T$$

$$\Longrightarrow \qquad T^n S = S\,T^n + nT^n$$

Thus
$$\tilde{T}(\theta)S = [S + \theta T]\,\tilde{T}(\theta)$$

Next, we consider

$$SD = DS + D = D\,[S + I]$$

$$\Longrightarrow \qquad S^n D = D\,[S + I]^n$$

Thus $\qquad \tilde{S}(y)\,D \;=\; e^{y}\tilde{S}(y)$

Finally, we need to consider

$$TD = DT + S$$

$$\Longrightarrow \qquad T^{n}D = DT^{n} + nST^{n-1} + \frac{n(n-1)}{2}\,T^{n-1}$$

$$\therefore \qquad \tilde{T}(\theta)\,D = \left[D + \theta S + \frac{\theta^2}{2}\,T\right]\tilde{T}(\theta)$$

$$\therefore \qquad T + D = \theta' T + y'\left[S + \theta T\right] + \theta' e^{y}\left[D + \theta S + \frac{\theta^2}{2}T\right]$$

$$\Longrightarrow \qquad y' + \theta'\theta\, e^{y} \;=\; 0 \qquad (6.8.2)$$

$$\theta' e^{y} = 1 \qquad (6.8.3)$$

$$\theta' + y'\theta + \frac{\theta^2}{2}\,\theta'\, e^{y} = 1 \qquad (6.8.4)$$

Substituting (6.8.3) into (6.8.2) we find that $y' + \theta = 0$. Differentiating this expression and again substituting in (6.8.3) we obtain

$$y'' + e^{-y} = 0 \quad \text{with} \quad y(0) = y'(0) = 0$$

$$\Longrightarrow \; y \;=\; \log\cos^2 \frac{\epsilon}{\sqrt{2}}$$

$$\Longrightarrow \; \theta \;=\; \sqrt{2}\tan\frac{\epsilon}{\sqrt{2}}$$

(It can be shown that (6.8.2) and (6.8.3) $\Longrightarrow$ (6.8.4)). Thus

$$e^{2\epsilon(T_i + D_i)} = \tilde{T}_i(\sqrt{2}\tan\sqrt{2}\,\epsilon)\,\tilde{S}(\log\cos^2\sqrt{2}\,\epsilon)\,\tilde{D}_i(\sqrt{2}\tan\sqrt{2}\,\epsilon) \qquad (6.8.5)$$

applied to a point $\vec{x}$ involves an inversion about the unit hypersphere, then a translation of x_i by $-\frac{1}{\sqrt{2}}\tan\sqrt{2}\,\epsilon$, followed by another inversion about the unit hypersphere, then a stretching of coordinates by $\cos^2\sqrt{2}\epsilon$, and finally a translation of x_i by $\sqrt{2}\tan\sqrt{2}\epsilon$.

An alternative approach for checking (6.8.5) (we take the two-dimensional case) involves finding the group in the large by solving for the integral curves of the group. Thus we solve $(i = 1)$

$$\frac{dx'}{2 + (x'^2 - y'^2)} = \frac{dy'}{2x'y'} = d\epsilon$$

Letting $z' = x' + iy'$, we have

$$\frac{dz'}{2 + z'^2} = d\epsilon \quad \text{where} \quad z'(\epsilon = 0) = z$$

Thus

$$z' = \sqrt{2}\tan(\sqrt{2}\,\epsilon + \alpha).$$

Using the initial condition, we find that

$$z = \sqrt{2}\tan\alpha$$

and finally

$$z' = \sqrt{2}\left(\frac{\sqrt{2}\tan\sqrt{2}\,\epsilon + z}{\sqrt{2} - z\tan\sqrt{2}\epsilon}\right)$$

Now applying the right side of (6.8.5) to z, we have

$$z' = \tilde{T}_1(\sqrt{2}\tan\sqrt{2}\,\epsilon)\,\tilde{S}(\log\cos^2\sqrt{2}\epsilon)\,\tilde{D}_1(\sqrt{2}\tan\sqrt{2}\,\epsilon)(z)$$

$$= \tilde{T}_1(\sqrt{2}\tan\sqrt{2}\,\epsilon)\,\tilde{S}(\log\cos^2\sqrt{2}\,\epsilon)\left(\frac{z}{1 - \frac{z}{\sqrt{2}}\tan\sqrt{2}\,\epsilon}\right)$$

$$= \tilde{T}_1 \left(\sqrt{2}\tan\sqrt{2}\epsilon\right) \left(\frac{z \cos^2\sqrt{2}\,\epsilon}{1 - \frac{z}{\sqrt{2}} \sin\sqrt{2}\,\epsilon \cos\sqrt{2}\,\epsilon} \right)$$

$$= \frac{z \cos^2\sqrt{2}\,\epsilon + \sqrt{2}\sin\sqrt{2}\,\epsilon \cos\sqrt{2}\,\epsilon}{1 - \frac{z}{\sqrt{2}} \sin\sqrt{2}\epsilon \cos\sqrt{2}\epsilon - \sin^2\sqrt{2}\,\epsilon}$$

$$= \sqrt{2} \left(\frac{z + \sqrt{2}\tan\sqrt{2}\,\epsilon}{\sqrt{2} - z\tan\sqrt{2}\epsilon} \right)$$

This method may be extended to higher dimensions.

It is also rather interesting to check (6.8.5) algebraically by using the well-known Pauli spin matrices:

$$\sigma_1 = \sigma_x = \tfrac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_2 = \sigma_y = \tfrac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_3 = \sigma_z = \tfrac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_+ = \sigma_x + i\sigma_y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\sigma_- = \sigma_x - i\sigma_y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Then
$$\left[\frac{i\sigma_-}{\sqrt{2}}, \frac{i\sigma_+}{\sqrt{2}}\right] = \sigma_z$$

$$\left[\frac{i\sigma_-}{\sqrt{2}}, \sigma_z\right] = \frac{i\sigma_-}{\sqrt{2}}$$

$$\left[\frac{i\sigma_+}{\sqrt{2}}, \sigma_z\right] = \frac{-i\sigma_+}{\sqrt{2}}$$

Thus we have an isomorphic representation of the Lie algebra of $\{T, D, S\}$ by associating:

$$D \longleftrightarrow \frac{i\sigma_+}{\sqrt{2}}$$

$$T \longleftrightarrow \frac{i\sigma_-}{\sqrt{2}}$$

$$S \longleftrightarrow \sigma_z$$

$$\Longrightarrow \quad 2(T+D) \longleftrightarrow i\sqrt{2}\,[\sigma_+ + \sigma_-] = i\,2\sqrt{2}\,\sigma_x$$

In fact the Lie algebra of $\{T, D, S\}$ is isomorphic to the Lie algebra of the Lorentz group $SO(2,1)$. This incidentally demonstrates the theorem of 6.7 for the case $n = 1$, $m = 0$.

We make use of the following easily proved relations:

$$e^{i2a\sigma_k} = (\cos a)I + 2i(\sin a)\sigma_k \qquad \text{for} \quad k = 1, 2, 3 .$$

$$e^{ib\sigma_{\pm}} = 1 + ib\sigma_{\pm}$$

Setting $\quad e^{i\epsilon 2\sqrt{2}\,\sigma_x} = e^{\frac{ib}{\sqrt{2}}\sigma_-} \; e^{2\alpha\sigma_z} \; e^{\frac{ib}{\sqrt{2}}\sigma_+}$,

we find that

$$\begin{pmatrix} \cos\sqrt{2}\,\epsilon & i\sin\sqrt{2}\,\epsilon \\ i\sin\sqrt{2}\,\epsilon & \cos\sqrt{2}\,\epsilon \end{pmatrix} = \begin{pmatrix} e^{\alpha} & \frac{ib}{\sqrt{2}}\,e^{\alpha} \\ \frac{ib}{\sqrt{2}}\,e^{\alpha} & \frac{-b^2}{2}\,e^{\alpha} + e^{-\alpha} \end{pmatrix}$$

$$\therefore \quad \cos\sqrt{2}\,\epsilon = e^{\alpha} = -\frac{b^2}{2}\,e^{\alpha} + e^{-\alpha}$$

$$\sin\sqrt{2}\,\epsilon = \frac{be^{\alpha}}{\sqrt{2}}$$

$$\Longrightarrow \quad \left\{ \begin{array}{l} a = \log\cos\sqrt{2}\,\epsilon \\ b = \sqrt{2}\tan\sqrt{2}\,\epsilon \end{array} \right\}$$

which checks (6.8.5)

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