ON THE EXISTENCE AND UNIQUENESS OF THE SOLUTION TO THE SMALL-SCALE NONLINEAR ANTI-PLANE SHEAR CRACK PROBLEM IN FINITE ELASTOSTATICS

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ABSTRACT

This thesis addresses the issue of existence and uniqueness of the solution to the small-scale nonlinear anti-plane shear crack problem in finite elastostatics. The hodograph transformation, commonly used in the theory of compressible fluid flows, plays an essential role. Existence is established by exhibiting an exact closed form solution, constructed via the hodograph transformation. Uniqueness is established by first proving the uniqueness of the solution to a related boundary-value problem, which is linear by virtue of the hodograph transformation, and then examining the implications of this result on the original problem. The possibility of making some of the conditions imposed on the solution to the small-scale nonlinear crack problem less restrictive is then investigated. This leads to several further results, including estimates of the nonvanishing shear stress component of the stress tensor along the crack faces.

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Introduction.

Boundary-value problems in the finite deformation theory of elasticity may or may not possess unique solutions. In fact, in certain cases one neither expects nor desires unqualified uniqueness. Take, for example, a slender elastic rod subject to compressive stresses at its ends. One would expect the theory to predict in addition to a uniformly compressed deformed configuration a number of "buckled" deformed configurations (see [1] for further examples and a discussion of nonuniqueness). When the uniqueness of the solution to a boundary-value problem is not assured, one faces the difficulty of determining the complete set of solutions and whether one solution may be preferred over some or all others on the grounds of a supplementary criterion, for example stability. A complete study of a boundary-value problem with multiple solutions presents a formidable task. Recently, Abeyaratne [2] has completed a study of the finite twisting of an incompressible elastic tube. Searching in a class of functions allowing the possibility of discontinuous deformation gradients, he found that there were infinitely many solutions to his boundary-value problem. He was, however, able to single out one solution using a stability criterion.

On the other hand, when the existence and uniqueness of the solution to a boundary-value problem are assured, the only difficulty that remains is finding that solution. Realizing the importance of existence and uniqueness theorems, several mathematicians (see [3] and [4], for references) have constructed existence and uniqueness proofs for a variety of boundary-value problems in finite elasticity. Among the more recent existence and uniqueness proofs, one may cite those of Ball [5], Antman [6,7], Spector [8], Gurtin and Spector [9], and John [10].

This paper focuses its attention on the existence and uniqueness issue for a crack problem in finite elastostatic anti-plane shear for incompressible materials. Consider an isotropic, homogeneous, incompressible elastic body in an unstressed state occupying a cylindrical region whose cross section is the whole plane with the exception of a line segment of finite length (the crack). Under appropriate loads all material points displace perpendicular to this plane. Such a deformation is referred to as one of anti-plane shear (or of mode III type in the terminology of fracture mechanics). Let the crack be traction-free while the displacement at infinity is one of out-of-plane simple shear. As shown by Knowles [11], the corresponding boundary-value problem for a class of incompressible elastic materials is analagous to the steady, irrotational, circulation-free flow of an inviscid, compressible fluid past a flat plate at a 90° angle of attack. In this fluid-mechanical analogy, the displacement function corresponds to the velocity potential while the stress-deformation constitutive law for the elastic material at hand plays the role of the density-speed relation.

Suppose for the moment that the crack is replaced by a hole; thus the cross section of the body is taken to lie outside a simple closed curve Γ. The resulting finite anti-plane shear boundary-value problem then has the same fluid mechanical analogy described previously, except that the flat plate is replaced by a profile Γ. *If the curve Γ is sufficiently smooth and if the governing* quasi-linear partial differential equation is uniformly elliptic, as it is for a certain class of elastic materials, then the solution to the problem exists and is unique (see [12], [13], [14], [15], and [16]). This suggests that the finite anti-plane shear crack problem has a unique solution.

The crack problem is one of considerable complexity not only because the governing partial differential equation is quasi-linear, but also because of the nature of the boundary, a line segment. If the prescribed shear at infinity is small, then it is reasonable to expect the displacement gradient to remain small throughout the body, except possibly near the crack tips. When the partial differential equation is linearized under the assumption of small displacement gradient, it becomes Laplace's equation for the out-of-plane displacement. This linearized crack problem corresponds to steady, irrotational, circulation-free flow of an incompressible, inviscid fluid past a flat plate at a 90° angle of attack. The exact solution to this problem can be found in many books (see, for example, [17] and [18]). It is easily verified that the displacement gradient calculated from the solution to the linearized problem is small away from the crack tips but becomes unbounded as the crack tips are approached, contrary to the underlying assumption that this gradient is small. Thus, while nonlinear effects are negligible away from the crack tips, they are dominant near the crack tips.

The above discussion suggests a way to determine an approximation to the solution of the crack problem for the case in which the prescribed shear at infinity is small. The solution to the linearized problem is expected to furnish a good approximation except near the crack tips, where nonlinear effects are important. The state of affairs near a crack tip, say the right one, is analyzed using the nonlinear theory by replacing the crack of finite length by a semi-infinite one and employing a suitable matching process to connect the nonlinear and linear asymptotic descriptions. To implement the matching, one assumes that there is an inner region near each crack tip where nonlinear effects dominate, an intermediate region in which the field is described by the near-tip approximation supplied by linear theory, and an outer region described by the full solution to the linearized problem. The mathematical problem generated by this matching process is called the small-scale nonlinear crack problem. In the setting of plasticity, (see [19] and [20], for example), it is often called the small scale yielding problem.

In this paper, the existence and uniqueness of the solution of the small-scale nonlinear crack problem is discussed. Existence will be established by explicitly exhibiting a solution. This solution was originally found by Knowles in [11] using the hodograph transformation, a special coordinate transformation that transforms the governing quasi-linear partial differential equation into a linear one. Although the boundary conditions for the small-scale problem stated subsequently in the present paper are much more restrictive than those imposed by Knowles, his solution is nevertheless shown to comply with these stronger conditions. The hodograph method is commonly used in compressible flow theory to construct solutions (see, for example, [12], [21], [22]). It had previously been used in crack problems by Hult and McClintock [23], Rice [19], Amazigo [24,25], and Freund and Douglas [26]. Uniqueness is proved here with the aid of the hodograph transformation. The fluid mechanical analogy suggests that some of the techniques used in proving uniqueness for compressible flows past two dimensional obstacles in the original plane of coordinates, some of which can be found in [12], [13], [14], [15], and [16] may be of use in the small-scale problem considered here. While this may be the case the use of the hodograph transformation allows the analysis to be carried out using concepts from calculus alone without having to resort to such theories as functional analysis, pseudo-analytic functions, and quasi-conformal mappings.

Section 1 begins with a brief discussion of the nonlinear equilibrium theory of homogeneous, isotropic, incompressible elastic solids. The special case of anti-plane shear is then introduced, followed by a discussion of the classes of elastic solids relevant to this study. The crack problem from which the small-scale nonlinear crack problem is subsequently extracted is formulated in Section 2. Two forms of the small-scale nonlinear crack problem are presented, the first being the one posed and solved exactly by Knowles in [11] and the second being the more restrictive version of the first alluded to earlier. Section 3 discusses the hodograph transformation. The exact solution to the first form of the small-scale nonlinear crack problem is presented and is confirmed to be a solution to the second form of the problem. It is also shown that any solution to the second form of the solution is related to a solution of a linear boundary-value problem in the hodograph plane. In Section 4, the solution to this linear problem is proved to be unique up to an additive constant. This in turn implies that the solution to the second problem is unique up to an additive constant. Finally, Section 5 is devoted to a study of the possibility of deriving the property that the modulus of the gradient of the solution becomes uniformly unbounded as the origin is approached, a property of the solution that was previously assumed in order to prove uniqueness. Although a lot can be proved about the behavior of $|\nabla u|$ near the origin, the analysis falls just short of its goal. The analysis is based on a comparison theorem for the second order elliptic operators. An interesting byproduct is a stress estimate along the faces of the crack.

1. Finite Elastostatics and Anti-Plane Shear for Incompressible Elastic Solids.

This section presents a brief discussion of finite elastostatics and anti-plane shear.[1] Let R be an open region in $\mathbb{R}^3$ occupied by the interior of a body in an undeformed configuration. Consider a deformation $\hat{\underset{\sim}{y}}: R \to \mathbb{R}^3$ represented by

$$\underset{\sim}{y} = \hat{\underset{\sim}{y}}(\underset{\sim}{x}) = \underset{\sim}{x} + \underset{\sim}{u}(\underset{\sim}{x}), \text{ for all } \underset{\sim}{x} \in R, \tag{1.1}$$

which maps R onto its deformation image $R^* = \hat{\underset{\sim}{y}}(R)$. Here $\underset{\sim}{x}$ and $\underset{\sim}{y}$ denote the position vectors relative to a common origin of a material point in the undeformed and deformed configurations, respectively. $\underset{\sim}{u}(\underset{\sim}{x})$ is the displacement of $\underset{\sim}{x}$. The mapping (1.1) is assumed to be twice continuously differentiable and invertible on R. The deformation gradient tensor field $\underset{\sim}{F}$ on R is given by the gradient of $\hat{\underset{\sim}{y}}$, i.e.

$$\underset{\sim}{F}(\underset{\sim}{x}) = \nabla\hat{\underset{\sim}{y}}(\underset{\sim}{x}) = \underset{\sim}{1} + \nabla\underset{\sim}{u}(\underset{\sim}{x}), \text{ for all } \underset{\sim}{x} \in R. \tag{1.2}$$

Deformations of an incompressible body are subject to the constraint

$$\det \underset{\sim}{F}(\underset{\sim}{x}) = 1 \text{ on } R, \tag{1.3}$$

which is necessary and sufficient for the deformation to be locally volume preserving. The left deformation tensor $\underset{\sim}{G}$ is given by

$$\underset{\sim}{G} = \underset{\sim}{F}\underset{\sim}{F}^T. \qquad (1.4)^2$$

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[1] For a more detailed discussion of finite elastostatics, see [3] and [4]; for finite anti-plane shear, see [11] and [27].

[2] The superscript T stands for transpose.

and its three fundamental scalar invarients I_1, I_2 , and I_3 are given by

$$I_1 = tr\underset{\sim}{G},\ I_2 = \frac{1}{2}[(tr\underset{\sim}{G})^2 - tr(\underset{\sim}{G}^2)],\ I_3 = \det\underset{\sim}{G} \equiv 1. \tag{1.5}$$ [3]

It is easily shown that $I_1 = I_2 = 3$ in the undeformed state, and for all deformations the inequalities $I_1 \geq 3$ and $I_2 \geq 3$ hold.

Let $\underset{\sim}{\tau}$ denote the actual (Cauchy) stress tensor field on R^*. Balance of angular momentum requires $\underset{\sim}{\tau}$ to be symmetric, i.e.

$$\underset{\sim}{\tau} = \underset{\sim}{\tau}^T. \tag{1.6}$$

Let $\underset{\sim}{\sigma}$ denote the nominal (Piola) stress field on R defined by

$$\underset{\sim}{\sigma} = \underset{\sim}{\tau}(\underset{\sim}{F}^T)^{-1}. \tag{1.7}$$

It is important to note that although $\underset{\sim}{\tau}$ is symmetric, $\underset{\sim}{\sigma}$ in general is not. Balance of linear momentum in the absence of body forces requires

$$div\ \underset{\sim}{\tau} = \underset{\sim}{0} \text{ on } R^*, \tag{1.8}$$

or, equivalently,

$$div\ \underset{\sim}{\sigma} = \underset{\sim}{0} \text{ on } R. \tag{1.9}$$

If S is a portion of the boundary of the region R with normal $\underset{\sim}{n}$, and if S^* is its deformation image with normal $\underset{\sim}{n}^*$, then it can be shown using (1.7) that

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[3] I_3, the third invariant of $\underset{\sim}{G}$, is identically unity due to the incompressibility constraint (1.3).

$$\underset{\sim}{\tau}\,\underset{\sim}{n}^{*} = \underset{\sim}{0} \text{ on } S^{*} \text{ if and only if } \underset{\sim}{\sigma}\,\underset{\sim}{n} = \underset{\sim}{0} \text{ on } S. \tag{1.10}$$

Thus, a portion of the boundary of R^{*} is free of actual tractions provided the nominal traction vanishes on its preimage in the boundary of R.

The mechanical response of the homogeneous, isotropic, incompressible elastic body is characterized by a strain energy density W per unit undeformed volume. More precisely, W is a scalar-valued function of the first two invariants of $\underset{\sim}{G}$, i.e. $W: [3,\infty) \times [3,\infty) \to [0,\infty)$, is three times continuously differentiable on its domain of definition, and satisfies $W(3,3) = 0$. The corresponding constitutive law is

$$\underset{\sim}{\tau} = 2\left[\frac{\partial W}{\partial I_1}\underset{\sim}{G} + \frac{\partial W}{\partial I_2}(I_1\underset{\sim}{1} - \underset{\sim}{G})\underset{\sim}{G}\right] - p\underset{\sim}{1}, \tag{1.11}$$ [4]

which, in view of (1.7), assumes the following equivalent form,

$$\underset{\sim}{\sigma} = 2\left[\frac{\partial W}{\partial I_1}\underset{\sim}{F} + \frac{\partial W}{\partial I_2}(I_1\underset{\sim}{1} - \underset{\sim}{G})\underset{\sim}{F}\right] - p(\underset{\sim}{F}^{T})^{-1}, \tag{1.12}$$

where $\underset{\sim}{1}$ is the unit tensor and p is an arbitrary scalar field needed to accomodate the incompressibility constraint. If (1.11) is substituted into (1.8), or (1.12) into (1.9), the result is a system of three coupled nonlinear partial differential equations for the three components of $\hat{\underset{\sim}{y}}(\underset{\sim}{x})$.

There are, however, special deformations which lead to simpler boundary-value problems. One of these is anti-plane shear deformations of cylindrical regions. Take R to be a cylindrical region and choose rectangular cartesian coordinates (x_1, x_2, x_3) with the x_3 -axis parallel to the generators of R. Let D

[4] See [3], Section 86, Eqn. (86.18).

denote the cross-section of the cylinder in the $x_3 = 0$ plane. A deformation (1.1) of the form

$$y_1 = x_1,\ y_2 = x_2,\ y_3 = x_3 + u(x_1,x_2) \tag{1.13}$$

is called an anti-plane shear. An example of an anti-plane shear is a simple shear, i.e.

$$u(x_1,x_2) = kx_2,\ k = \text{constant}. \tag{1.14}$$

The matrix of components $[F_{ij}]$ of the deformation gradient tensor $\underset{\sim}{F}$ and $[G_{ij}]$ of the left deformation tensor $\underset{\sim}{G}$ in the underlying cartesian frame in the special case of anti-plane shear are

$$[F_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ u_{,1} & u_{,2} & 1 \end{bmatrix},\ [G_{ij}] = \begin{bmatrix} 1 & 0 & u_{,1} \\ 0 & 1 & u_{,2} \\ u_{,1} & u_{,2} & 1+u_{,1}{}^2+u_{,2}{}^2 \end{bmatrix}. \qquad (1.15)^5$$

It is readily observed that the incompressibility constraint (1.3) is satisfied and the first invarient I_1 of $\underset{\sim}{G}$ is given by

$$I_1 = 3 + |\nabla u|^2, \tag{1.16}$$

where

$$|\nabla u|^2 = u_{,\alpha}u_{,\alpha}. \qquad (1.17)^6$$

[5] A subscript preceded by a comma indicates partial differentiation with respect to the corresponding $\mathbf{x}$-coordinate.

[6] Greek subscripts have the range 1,2 while Latin subscripts (to appear shortly) take the value 1,2,3. Repeated subscripts are summed over the appropriate range.

When the Cauchy or Piola stresses associated with the deformation (1.13) are determined from the constitutive law (1.11) or (1.12) and then substituted into the appropriate form of the equilibrium equations (1.8) or (1.9), a set of three differential equations for the two unknown scalar fields p and u results. These equations are not in general consistent unless the strain energy density function W is suitably restricted. Such restrictions are derived in [28].

A subclass of functions W that admits nontrivial states of anti-plane shear consists of those functions W that depend only on the first invarient I_1. The analysis presented in this paper will be restricted to those incompressible elastic materials whose strain energy functions W are of this type. When this is the case, the constitutive equations (1.11) and (1.12) reduce with the aid of (1.8) and (1.9) to

$$\left.\begin{aligned} p &= 2W'(I_1) + d_0(x_3 + u) + d_1, \\ \tau_{3\alpha} &= \tau_{\alpha 3} = \sigma_{3\alpha} = 2W'(I_1)u_{,\alpha}, \\ \sigma_{\alpha 3} &= [2W'(I_1) + d_0(x_3 + u) + d_1]u_{,\alpha}, \\ \tau_{\alpha\beta} &= \sigma_{\alpha\beta} = -[d_0(x_3 + u) + d_1]\delta_{\alpha\beta}, \\ \tau_{33} &= 2W'(I_1)|\nabla u|^2 - d_0(x_3 + u) - d_1, \\ \sigma_{33} &= -[d_0(x_3 + u) + d_1], \end{aligned}\right\} \qquad (1.18)^7$$

where d_1 and d_0 are constants and $\delta_{\alpha\beta}$ is the Kronecker delta. Moreover, the set of three differential equations for p and u can be consistently reduced to a single one for u on the two-dimensional domain D,

$$[2W'(3 + |\nabla u|^2)u_{,\alpha}]_{,\alpha} = d_0 \text{ on } D. \qquad (1.19)$$

[7]Henceforth, primes on W will indicate differentiations with respect to the argument of W. This notation will also be used for all functions of a single argument.

Thus far, the only requirement imposed on W is that it be three times continuously differentiable on the interval $[3, \infty)$ and the $W(3) = 0$. Further restrictions must be placed on W to ensure physically realistic stress fields. These restrictions are most readily interpreted if, for the moment, the region R is taken to be the unit cube lying in the first quadrant of a rectangular cartesian coordinate frame with one vertex at the origin as shown in Figure 1. Let the cube be subject to the simple shear (1.14). The shear stress τ_{32}, in view of (1.18), is given by

$$\tau_{32} = \hat{\tau}(k) = M(k)k, \; -\infty<k<\infty, \tag{1.20}$$

where

$$M(k) = 2W'(3 + k^2), \; M(0) = \mu, \tag{1.21}$$

is the (secant) modulus of shear at an amount of shear k and μ is the infinitesmal shear modulus (upon linearization for small k, (1.20) becomes $\tau = \mu k$). Intuitively, only a positive shearing stress τ_{32} can produce a positive amount of shear k in a solid. Thus, it is reasonable to require $M(k) > 0$ for all k, or, equivalently,

$$W'(I_1) > 0, \text{ for } I_1 \geq 3. \qquad (1.22)^8$$

This restriction on W will be in force throughout the analysis presented here. A material will be said to be softening in shear if $M'(k) < 0$ for all $k > 0$ and hardening in shear if $M'(k) > 0$ for all $k > 0$. A material will be called elliptic if

$$\hat{\tau}'(k) = 2W'(3 + k^2) + 4k^2 W''(3 + k^2) > 0, \; -\infty<k<\infty, \tag{1.23}$$

so that the function $\hat{\tau}(k)$ is monotone strictly increasing on its domain of definition. Condition (1.22) and (1.23) assure that the second order quasi-linear partial differential equation (1.19) is elliptic[9] at every solution u and at every point (x_1, x_2). A material will be called uniformly elliptic if its shear stress response function $\hat{\tau}(k)$ satisfies a slightly stronger condition than (1.23) above, namely,

$$k\hat{\tau}'(k) \geq c\hat{\tau}(k), \text{ for } all\ k \geq 0, \tag{1.24}$$

for some positive constant c. This condition implies that $\hat{\tau}'(k) > 0$ for all k and that $\hat{\tau}(\infty) = \infty$. When (1.22) and (1.24) hold $\tau = \hat{\tau}(k)$ can be inverted to give k as an odd, monotone, strictly increasing function of τ:$k = \hat{k}(\tau)$ with $k(\infty) = \infty$. In addition, when (1.22) and (1.24) hold, (1.19) is uniformly elliptic.[10]

In certain circumstances, W will be required to satisfy the following growth condition:

$$\lim_{I_1 \to \infty} W(I_1) = \infty. \tag{1.25}$$

This condition, roughly speaking, says that as the amount of deformation as measured by $|\nabla u|$ gets large, so does the strain energy density.

An example of a strain energy density function capable of conforming to the previous definitions and restrictions is the power law function first introduced in [11] and given by

[8]This assures that the Baker-Ericksen inequality holds for the materials considered. The Baker-Ericksen inequality holds if and only if the greatest principal stress occurs in the direction of the greatest principal stretch (see [3]). It reduces to $W'(I_1) > 0$ for $I_1 > 3$ in the present circumstances.
[9]This is in accordance with standard definition. See p. 203 of [29].
[10]See p. 203 of [29].

$$W(I_1) = \frac{\mu}{2b}\{[1 + \frac{b}{n}(I_1 - 3)]^n - 1\}, \tag{1.26}$$

where μ,b,n are positive constants. The functions $\hat{\tau}(k)$ and M(k) are given by

$$\hat{\tau}(k) = \mu(1 + \frac{b}{n}k^2)^{n-1}k, \; M(k) = \mu(1 + \frac{b}{n}k^2)^{n-1}, \tag{1.27}$$

and $\hat{\tau}'(k)$ and $M'(k)$ are given by

$$\left.\begin{aligned} \hat{\tau}'(k) &= \mu(1 + \frac{b}{n}k^2)^{n-2}[1 + (2n-1)\frac{b}{n}k^2], \\ M'(k) &= \mu(n-1)(1 + \frac{b}{n}k^2)^{n-2}\, 2k\frac{b}{n} \end{aligned}\right\} \tag{1.28}$$

Clearly (1.22) and (1.25) are satisfied. Also, the material hardens in shear if n > 1 and softens in shear if n < 1, the case n = 1 corresponding to a neo-Hookean material. Finally, the material is elliptic when $n \geq \frac{1}{2}$ but is uniformly elliptic only when $n > \frac{1}{2}$. Figure 2 shows a graph of $\hat{\tau}$(k) for various values of the hardening parameter n.

Finally, there is associated with equation (1.19) a path independent integral similar in structure to those used by Rice [30], Hutchinson [20] and [31], and Knowles and Sternberg [32] and [33]. If C is a simple closed curve which, together with its interior, lies entirely in the domain D in which u is a twice continuously differentiable solution of (1.19), it can easily be shown using the divergence theorem that

$$\int_C [\frac{1}{2}W(3 + |\nabla u|^2)n_\alpha - W'(3 + |\nabla u|^2)u_{,\alpha}\, u_{,\beta}\, n_\beta + d_0 u n_\alpha]ds = 0, \tag{1.29}$$

where s is the arc length and $\underset{\sim}{n}$ is the unit normal vector on C. This integral was applied by Knowles to the crack problem analyzed in [11] to determine an amplitude factor left undetermined from an asymptotic analysis of the displacement field near the tip of a crack, provided the remote loading is "small". It will be used in what follows to obtain information about $|\nabla u|$ in the vicinity of the crack tip.

2. The Crack Problem and the Associated Small-Scale Nonlinear Crack Problem

The two-dimensional region D is now taken to be the exterior of a line-segment of length 2c centered on the x_1- axis as shown in Figure 3. The crack problem consists of finding an anti-plane shear on R which leaves the plane surface of the crack traction free. In view of (1.10) and (1.18), this free-surface condition holds if and only if

$$d_0 = d_1 = 0, \tag{2.1}$$

and

$$u_{,2} = 0 \text{ at } x_2 = 0^{\pm},\ -c<x_1<c. \tag{2.2}$$

Equations (1.18) for the true and nominal stress reduce under (2.1) to

$$\left.\begin{array}{l} \tau_{3\alpha} = \tau_{\alpha 3} = \sigma_{3\alpha} = \sigma_{\alpha 3} = 2W'(3 + |\nabla u|^2)u_{,\alpha}, \\ \tau_{\alpha\beta} = \sigma_{\alpha\beta} = 0, \\ \tau_{33} = 2W'(3 + |\nabla u|^2)|\nabla u|^2,\ \sigma_{33} = 0 \end{array}\right\} \tag{2.3}$$

In addition to the free-surface condition the displacement field at infinity should approach that of a simple shear parallel to the crack surface and perpendicular to the cross-section D:

$$u = kx_2 + o(1),\ as\ x_\alpha x_\alpha \to \infty, \tag{2.4}$$

where k is a constant specifying the amount of shear at infinity. The governing partial differential equation (1.19) in accordance with (2.1) becomes

$$[2W'(3 + |\nabla u|^2)u_{,\alpha}]_{,\alpha} = 0 \text{ on } D. \tag{2.5}$$

The solution u to the boundary-value problem characterized by (2.2), (2.4), and (2.5) is required to be twice continuously differentiable on D and bounded within any circle of finite radius centered at either crack tip.

If one wishes to determine the effect of a traction-free hole instead of a crack on a field of finite simple shear, one retains (2.4) and (2.5) and replaces the boundary condition (2.2) of the crack problem by

$$\frac{\partial u}{\partial n} = 0 \text{ on } \Gamma, \qquad (2.6)$$

where Γ is the simple closed curve forming the boundary of the hole and $\frac{\partial u}{\partial n}$ is the derivative of u normal to Γ.

As explained by Knowles [11], the crack problem posed above has a fluid mechanical interpretation. Let q_0 be an arbitrary constant whose physical dimensions are those of velocity, and let Φ be the velocity potential defined by

$$\Phi = q_0 u \qquad (2.7)$$ [11]

It then follows from (2.2), (2.4), and (2.5) that Φ is a solution to the following boundary-value problem:

$$\left.\begin{aligned} &[W'(3 + q^2/q_0^2)\, \Phi_{,\alpha}]_{,\alpha} = 0 \text{ on } D, \\ &\Phi_{,2} = 0 \text{ at } x_2 = 0^{\pm}, -c < x_1 < c, \text{ and} \\ &\Phi = kq_0 u + o(1) \text{ as } x_\alpha x_\alpha \to \infty \end{aligned}\right\} \qquad (2.8)$$

where $q = |\nabla\Phi|$ is the speed of a fluid particle. The above boundary-value

[11]The fact that u, and hence Φ, are single-valued corresponds to circulation-free flow.

problem for Φ can also be derived in the context of compressible flow theory. Consider the steady, irrotational flow of an inviscid compressible fluid past a flat plate at a 90° angle of attack with a free stream speed of kq_0. If the density-speed relation that characterizes the fluid is given by

$$\rho = \frac{2}{q_0^2} W'(3 + q^2/q_0^2), \tag{2.9}$$

then the resulting boundary-value problem is precisely (2.8). In addition, the anti-plane shear problem in which the stress concentrator is a hole rather than a crack, so that (2.6) replaces (2.2), corresponds to flow past an obstacle of profile Γ.

When the stress concentrator is a sufficiently smooth hole and the material is uniformly elliptic, one can apply the results of Shiffman [16], Bers [13], and Finn and Gilbarg [15] to infer the existence and uniqueness of the solution to the problem. However, because their results apply to flows over profiles Γ that are, among other things, simple closed Jordan arcs, they cannot be applied to the crack problem. Moreover, the methods used by them, which include pseudo-analytic functions, quasi-conformal mappings, maximum and comparison principles, and variational principles, are not directly applicable to the crack problem due to the (anticipated) singularity of ∇u at the crack tips. A discussion and extensive list of references concerning boundary-value problems in unbounded domains may be found in Section 49 of [34].

The boundary-value problem (2.2), (2.4), and (2.5) is one of considerable complexity and has thus far eluded exact closed form solution for a general elliptic material. For a power law material defined by (1.26) the behavior of the solution near a crack tip has been analyzed by Knowles in [11]. At present, only the problem for a neo-Hookean material has an exact closed form global solution. Some understanding of the problem may be gained by formally linearizing the differential equation (2.5) under the assumption that $|\nabla u| << 1$. When this is done the resulting linear boundary-value problem is an elementary one for Laplace's equation[12] and is mathematically identical to the problem of steady, irrotational flow of an inviscid, incompressible fluid past a flat plate of width 2c at an angle of attack of 90°. In the flow, problem u is identified with the velocity potential and k with the free stream speed. The exact solution $\overset{\circ}{u}$ to the linearized problem may be found in standard textbooks such as [17] and [18]. If r and ϑ are polar coordinates at the right crack tip, the linearized solution exhibits the following asymptotic behavior:

$$\overset{\circ}{u} = k(2cr)^{1/2} \sin \frac{\vartheta}{2} + o(r^{1/2}),$$

$$\overset{\circ}{u}_{,1} = -kc(2cr)^{-1/2}\sin\frac{\vartheta}{2} + o(r^{-1/2}), \overset{\circ}{u}_{,2} = kc\ (2cr)^{-1/2}\cos\frac{\vartheta}{2} + o(r^{-1/2}), \quad (2.10)$$

$$\text{as } r \to 0, \text{ uniformly } in\ \vartheta \in [-\pi, \pi].$$

It is important to notice that $|\nabla \overset{\circ}{u}|$ becomes unbounded as r tends to zero, contradicting the underlying assumption that $|\nabla \overset{\circ}{u}| << 1$.

If the prescribed shear at infinity k appearing in (2.4) is small in comparison with unity, the linearized solution $\overset{\circ}{u}$ can be expected to furnish a good approximation except near the crack tips, where nonlinear effects must dominate. In order to investigate these nonlinear effects near a crack tip, say the

[12]When $W(I_1)$ is given by (1.26) with $n = 1$, (2.5) reduces exactly rather than approximately to Laplace's equation.

right one, the neighborhood of that crack tip is "magnified" by replacing the original domain D by a new domain $\tilde{D}$ consisting of the points exterior to the half-axis $x_2 = 0$, $x_1 \leq 0$. This half-axis, representing a semi-infinite crack, is required to be traction free so that

$$u_{,2} = 0 \text{ at } x_2 = 0^{\pm},\ x_1 < 0. \qquad (2.11)$$

The full nonlinear equation (2.5) is retained, i.e.

$$[2W'(3 + |\nabla u|^2)u_{,\alpha}]_{,\alpha} = 0 \text{ on } \tilde{D}. \qquad (2.12)$$

To connect the approximation to the original crack problem furnished by $\overset{\circ}{u}$ away from the crack tips with the approximation near the right tip furnished by a solution u to (2.11) and (2.12), the following matching process is used. When the amount of shear at infinity k is small, u and its first partial derivatives must satisfy in addition to (2.11) and (2.12) the condition that far away from the crack tip they must tend to the asymptotic forms (2.10) near the crack tip of $\overset{\circ}{u}$ and its first partial derivatives. Thus, if r,ϑ are polar coordinates at the origin of $\tilde{D}$, so that

$$x_1 = r \cos \vartheta,\ x_2 = r \sin \vartheta,\ r>0,\ -\pi \leq \vartheta \leq \pi, \qquad (2.13)^{13}$$

u is required to satisfy the matching conditions

$$u = k\,(2cr)^{1/2} \sin \frac{\vartheta}{2} + o(r^{1/2}),$$

$$u_{,1} = -kc(2cr)^{-1/2}\sin\frac{\vartheta}{2} + o(r^{-1/2}),\ u_{,2} = kc(2cr)^{-1/2}\cos\frac{\vartheta}{2} + o(r^{-1/2}). \qquad (2.14)$$

as $r \to \infty$, uniformly *in* $\vartheta \in [-\pi, \pi]$.

The problem of finding a solution u to (2.11), (2.12), and (2.14) which is twice continuously differentiable on $\tilde{D}$, bounded near the origin, and which has the property that u and $u_{,\alpha}$ tend to limiting values on the crack faces $x_2 = 0^{\pm}$, $x_1 < 0$ will be referred to as problem $P_1(u)$. This is the small-scale nonlinear crack problem posed by Knowles in [11] and solved exactly by him for elliptic materials, i.e. materials satisfying (1.23).

The uniqueness issue for problem $P_1(u)$ is still unresolved. It is, however, possible to prove uniqueness if u is required to satisfy conditions that are more restrictive than those of problem $P_1(u)$. Let $\tilde{D}^+ = \{ (x_1, x_2) \mid x_2 \geq 0,\ x_1^2 + x_2^2 \neq 0\}$ so that $\tilde{D}^+$ consists of all points lying on or above the x_1-axis except for the origin. Similarly, let $\tilde{D}^-$ be the set of points lying on or below the x_1-axis except for the origin. The more restrictive version of problem $P_1(u)$, to be referred to as problem $P_2(u)$ (see figure 4), consists of finding a scalar field u such that

$$\left.\begin{array}{l} u \in C^1(\tilde{D}^+) \cap C^1(\tilde{D}^-) \cap C^2(\tilde{D}), \\ \left[W'(3 + |\nabla u|^2)\, u_{,\alpha} \right]_{,\alpha} = 0 \text{ on } \tilde{D}, \text{ and} \\ u_{,2} = 0 \text{ at } x_2 = 0^{\pm},\ x_1 < 0, \end{array}\right\} \qquad (2.15)$$

satisfies the matching condition

$$\left.\begin{array}{c} u = k(2cr)^{1/2} \sin \dfrac{\vartheta}{2} + 0(1)\,, \\ u_{,1} = -kc(2cr)^{-1/2} \sin \dfrac{\vartheta}{2} + 0(r^{-1}), u_{,2} = kc(2cr)^{-1/2} \cos \dfrac{\vartheta}{2} + 0(r^{-1}), \\ \text{as } r \to \infty, \text{ uniformly in } \vartheta \in [-\pi, \pi], \end{array}\right\} \qquad (2.16)$$

and the crack tip conditions

[13] $\vartheta = \pi$ when $x_1 < 0$ and $x_2 = 0^+$, $\vartheta = -\pi$ when $x_1 < 0$ and $x_2 = 0^-$.

$$\text{for every } M>0, \text{ there exists a } \delta>0, \tag{2.17}$$

$$\text{such that } |\nabla u|>M \text{ for all } 0<r<\delta, \text{ and for all } \vartheta\in[-\pi,\pi],$$

where r, ϑ are the polar coordinates (2.13), and

$$2W'(3+|\nabla u|^2)|\nabla u|r^2\frac{\partial}{\partial r}\left(\frac{u}{r}\right)=0(1), \text{ as } r\to 0, \text{ uniformly in } \vartheta\in[-\pi,\pi]. \tag{2.18}$$

There is one final condition that the first derivatives of u must satisfy, which may be stated as follows. Given u satisfying the first of (2.15), compute its first partial derivatives $u_{,1}(x_1,x_2)$ and $u_{,2}(x_1,x_2)$, and using the notation $D^* = \{(r,\vartheta)|r>0,-\pi<\vartheta<\pi\}$ and $D^*_o = \{(r,\vartheta)|r>0,-\pi\le\vartheta\le\pi\}$ define the transformation $\underset{\sim}{T}:D^*_o \to \mathbb{R}^2$ by $\xi_1 = T_1(r,\vartheta) = u_{,1}(r\cos\vartheta, r\sin\vartheta)$, $\xi_2 = T_2(r,\vartheta) = u_{,2}(r\cos\vartheta, r\sin\vartheta)$. The transformation $\underset{\sim}{T}$, which is $C(D^*_o)\cap C^1(D^*)$ by the first of (2.15), is required to be one-to-one on D^*_o and have nonvanishing Jacobian at each point of D^*. This final condition on the first derivatives of u will be referred to as the hodograph condition. It follows from standard theorems that $\underset{\sim}{T}$ is an open mapping and the $\underset{\sim}{T}$ has an inverse $\underset{\sim}{T}^{-1}\in C(\underset{\sim}{T}(D^*_o))\cap C^1(\underset{\sim}{T}(D^*))$.

Conditions (2.15) are identical to those appearing in problem $P_1(u)$. The matching condition (2.16) is more restrictive than its counterpart (2.14) in problem $P_1(u)$ since the order of the error between u and its leading asymptotic behavior (and similarly for its first derivates) as $r\to\infty$ is more restrictive. The crack tip condition (2.17) says that $|\nabla u|$ is unbounded (uniformly in ϑ)in the neighborhood of the crack tip. The possibility of dropping this condition is investigated in Section 5. It will be shown for an elliptic material that (2.17) and (2.18) imply u is bounded near the crack tip. Thus, any solution to problem $P_2(u)$ will be a solution to problem $P_1(u)$. The hodograph condition makes it possible to use the hodograph transformation as a tool for proving uniqueness.

Several properties of any solution to problem $P_2(u)$ can be deduced directly from (2.15) through (2.18) and the hodograph condition.

Claim 1. Let u satisfy (2.17) and (2.18) and let the material be elliptic. Then u is bounded near the crack tip.

Proof. (2.17) implies that there exist positive constants a and $\bar{r}_1$ such that

$$|\nabla u(r,\vartheta)|>a, \text{ for } 0<r<\bar{r}_1, \ -\pi\leq\vartheta\leq\pi. \tag{2.19}$$

(2.18) can be stated as follows: there exist positive constants $\bar{r}_2$ and K such that

$$|2W'(3+|\nabla u|^2)|\nabla u|r^2\frac{\partial}{\partial r}(\frac{u}{r})|\leq K, \text{ for } 0<r<\bar{r}_2, \ -\pi\leq\vartheta\leq\pi. \tag{2.20}$$

(2.19) and (2.20) along with (1.20), (1.21), and (1.22) imply

$$|\frac{\partial}{\partial r}(\frac{u}{r})|\leq\frac{K}{r^2\hat{\tau}(|\nabla u|)}, \ 0<r<\bar{r}, \ -\pi\leq\vartheta\leq\pi, \tag{2.21}$$

where $\bar{r}=min(\bar{r}_1,\bar{r}_2)$. Since $\hat{\tau}$ (k) is a monotone increasing function of k, (2.19) and (2.21) imply

$$-\frac{K}{r^2\hat{\tau}(a)}\leq\frac{\partial}{\partial r}(\frac{u}{r})\leq\frac{K}{r^2\hat{\tau}(a)}, \ 0<r<\bar{r}, \ -\pi\leq\vartheta\leq\pi. \tag{2.22}$$

Let r_1 and r_2 be such that $0<r_1<r_2<\bar{r}$. Integrating (2.22) from r_1 to r_2 yields

$$-\frac{K}{\hat{\tau}(a)}(\frac{1}{r_1}-\frac{1}{r_2})\leq\frac{u(r_2,\vartheta)}{r_2}-\frac{u(r_1,\vartheta)}{r_1}\leq\frac{K}{\hat{\tau}(a)}(\frac{1}{r_1}-\frac{1}{r_2}), \tag{2.23}$$

$$0<r_1<r_2<\bar{r},\ -\pi\le\vartheta\le\pi,$$

which upon rearrangement yields

$$\left.\begin{aligned}&-K/\hat{\tau}(a)+\frac{r_1}{r_2}[u(r_2,\vartheta)+K/\hat{\tau}(a)]\le u(r_1,\vartheta)\\&\le K/\hat{\tau}(a)+\frac{r_1}{r_2}[u(r_2,\vartheta)-K/\hat{\tau}(a)],\\&0<r_1<r_2<\bar{r},\ -\pi\le\vartheta\le\pi\,.\end{aligned}\right\}\tag{2.24}$$

Fix r_2. Then (2.24) implies that $u(r_1,\vartheta)$ is bounded for $0<r_1<r_2$, $-\pi\le\vartheta\le\pi$.

Claim 2. Let u satisfy (2.17) and (2.18) and let the material be uniformly elliptic. Then $u = o(1)$ as $r\to 0$, uniformly in $\vartheta\in[-\pi,\pi]$.

Proof. (2.17), (2.18), and $\hat{\tau}(\infty)=\infty$ imply that

$$r^2\,\frac{\partial}{\partial r}\left(\frac{u}{r}\right)=o(1)\ \text{as}\ r\to 0,\ \text{uniformly}\ in\ \vartheta\in[-\pi,\pi].\tag{2.25}$$

Let $\varepsilon>0$ be arbitrary. By (2.25) and (2.17) there exist $\bar{r}>0$ and $a>0$ such that

$$\left.\begin{aligned}&-\varepsilon/2\le r^2\,\frac{\partial}{\partial r}\left(\frac{u}{r}\right)\le\varepsilon/2,\ \text{and}\\&|\nabla u(r,\vartheta)|>a,\ \text{for}\ 0<r<\bar{r},\ -\pi\le\vartheta\le\pi.\end{aligned}\right\}\tag{2.26}$$

Let r_1 and r_2 be such that $0<r_1<r_2<\bar{r}$. Dividing (2.26) through by r^2 and integrating from r_1 to r_2 yields,

$$-\frac{\varepsilon}{2}\left(-\frac{1}{r_2}+\frac{1}{r_1}\right)\le\frac{u(r_2,\vartheta)}{r_2}-\frac{u(r_1,\vartheta)}{r_1}\le\frac{\varepsilon}{2}\left(-\frac{1}{r_2}+\frac{1}{r_1}\right),\tag{2.27}$$

$$0<r_1<r_2<\bar{r},\ -\pi\le\vartheta\le\pi.$$

Rearrangement of (2.27) yields

$$\frac{r_1}{r_2}u(r_2,\vartheta)-\frac{\varepsilon}{2}(1-\frac{r_1}{r_2})\le u(r_1,\vartheta)\le\frac{r_1}{r_2}u(r_2,\vartheta)+\frac{\varepsilon}{2}(1-\frac{r_1}{r_2}), \qquad (2.28)$$

$$0<r_1<r_2<\bar{r},\ -\pi\le\vartheta\le\pi.$$

Using the inequalities

$$\left.\begin{aligned}-\frac{r_1}{r_2}|u(r_2,\vartheta)|-\frac{\varepsilon}{2}(1-\frac{r_1}{r_2})&\le\frac{r_1}{r_2}u(r_2,\vartheta)-\frac{\varepsilon}{2}(1-\frac{r_1}{r_2}),\\ \frac{r_1}{r_2}u(r_2,\vartheta)+\frac{\varepsilon}{2}(1-\frac{r_1}{r_2})&\le\frac{r_1}{r_2}\,|u(r_2,\vartheta)|+\frac{\varepsilon}{2}(1-\frac{r_1}{r_2}),\end{aligned}\right\} \qquad (2.29)$$

with (2.28) yields

$$|u(r_1,\vartheta)|\le\frac{r_1}{r_2}|u(r_2,\vartheta)|+\frac{\varepsilon}{2}(1-\frac{r_1}{r_2})\le r_1\frac{|u(r_2,\vartheta)|}{r_2}+\frac{\varepsilon}{2}, \qquad (2.30)$$

$$0<r_1<r_2<\bar{r},\ -\pi\le\vartheta\le\pi.$$

Fix r_2. Let $K=\max\limits_{-\pi\le\vartheta\le\pi}|\nabla u(r_2,\vartheta)|>0$ by the second of (2.26). Then, (2.30) implies

$$|u(r_1,\vartheta)|\le r_1K+\varepsilon/2,\ 0<r_1<r_2,\ -\pi\le\vartheta\le\pi. \qquad (2.31)$$

This in turn implies

$$|u(r_1,\vartheta)|\le\varepsilon \text{ for } 0<r_1<\min(\frac{\varepsilon}{2K},r_2),\ -\pi\le\vartheta\le\pi, \qquad (2.32)$$

which completes the proof.

Claim 3. Let u satisfy (2.17) and (2.18) and let the material be uniformly elliptic. Moreover, let the shear modulus M(k) defined by (1.21) be bounded for all k. Then $\sigma_{rz} = o(r^{-1})$ as $r \to 0$, uniformly in $\vartheta \in [-\pi, \pi]$, where $\sigma_{rz} = 2W'(3+|\nabla u|^2)\frac{\partial u}{\partial r} = \underset{\sim}{e}_r \bullet \underset{\sim\sim}{\sigma} \underset{\sim}{e}_z$ is a component of the nominal stress tensor in the orthonormal basis $(\underset{\sim}{e}_r, \underset{\sim}{e}_\vartheta, \underset{\sim}{e}_z)$ associated with a cylindrical coordinate system.[14]

Before proceeding with a proof of this claim, we make two remarks. First, M(k) is bounded for all softening materials since (1.21), (1.22), and $M'(k)<0$ for all k>0 imply that $0<M(k)\leq \mu$ for all k. Not all hardening materials, however, have a bounded M(k). The power law material given by (1.26), for example, hardens for n>1, but M(k) is not bounded for all k. Second, if the forces on a wedge in the domain $\tilde{D}$ with vertex at the crack tip are considered, the result of claim 3 implies that there is no concentrated shear force at the tip of the wedge. More precisely, it the wedge is described by $\{(r,\vartheta)\,|\,0<r<\bar{r},\ \vartheta_1<\vartheta<\vartheta_2\}$ where r, ϑ are plane polar coordinates and $\bar{r} > 0$ and $-\pi\leq\vartheta_1<\vartheta_2\leq\pi$ are constants, then claim 3 implies that the shear force at the tip of the wedge $F = \lim_{r\to 0}\int_{\vartheta_1}^{\vartheta_2} \sigma_{rz}\, r d\vartheta = 0$.

Proof. (2.17) and (2.18) imply there exist constants K>0, a>0, and $\bar{r}$>0 such that

$$\left.\begin{aligned} &-K \leq 2W'(3+|\nabla u|^2)\,|\nabla u|\,(r\frac{\partial u}{\partial r} - u) \leq K, \\ &|\nabla u(r,\vartheta)| > a, \text{ for } 0<r<\bar{r}, -\pi\leq\vartheta\leq\pi. \end{aligned}\right\} \qquad (2.33)$$

[14]A discussion of orthogonal curvilinear coordinates and transformation formulas to them from cartesian coordinates can be found in Appendix II of [35].

Rearrangement of the above and (1.21) imply

$$-K/|\nabla u| + M(|\nabla u|)u \le \sigma_{rz}r \le K/|\nabla u| + M(|\nabla u|)u, \tag{2.34}$$

$$0<r<\bar{r},\ -\pi\le\vartheta\le\pi.$$

This in turn implies

$$|\sigma_{rz}r| \le K/|\nabla u| + M(|\nabla u|)|u|. \tag{2.35}$$

The right hand side of the above inequality is o(1) as $r\to 0$, uniformly in $\vartheta\in[-\pi,\pi]$, since $M(|\nabla u|)$ is bounded and both $1/|\nabla u|$ and $|u|$, by applying claim 2, are o(1) as $r\to 0$, uniformly in $\vartheta\in[-\pi,\pi]$. Hence, $\sigma_{rz}=o(\frac{1}{r})$ as $r\to 0$, uniformly in $\vartheta\in[-\pi,\pi]$.

<u>Claim 4.</u> Let (2.15) and (2.16) hold. Then

$$\frac{1}{4}\mu ck^2 \le \int_{-\pi}^{\pi}[\frac{1}{2}W(3+|\nabla u(r,\vartheta)|^2) + W'(3+|\nabla u(r,\vartheta)|^2)|\nabla u(r,\vartheta)|^2]r\,d\vartheta, \tag{2.36}$$

$$\text{for all } r>0.$$

Moreover, if W is an elliptic power law material so that (1.26) holds with $n\ge\frac{1}{2}$, then the assumption that $|\nabla u| = o(r^{-1/2n})$ contradicts (2.36), and hence this behavior of $|\nabla u|$ near the crack tip is excluded.

Inequality (2.36) clearly implies that $|\nabla u|$ cannot be bounded near the origin.[15] Moreover, it restricts the rate of growth of $|\nabla u|$ as $r\to 0$, and this

[15]This does not exclude the possibility that $|\nabla u(r,\vartheta)|$ remains bounded as the origin is approached along certain rays ϑ = constant and hence does not imply (2.17).

restriction does depend on the function W. When W is given by (1.26), the restriction can be explicitly determined.

Proof. Since (2.15) holds, (1.29) holds with $d_0 = 0$. Choose the simple closed curve C to consist of circles $C_{\bar{r}}$ and $C_{\underline{r}}$ of radii $\bar{r}>\underline{r}>0$, respectively, centered at the crack tip, "slit" by the crack, and connected by linear portions Γ_+ running along the upper crack face and Γ_- running along the lower crack face, as shown in Figure 5. Make the following definition

$$J(r) = \int_{-\pi}^{\pi}[\frac{1}{2}W(3+|\nabla u(r,\vartheta)|^2)\cos\vartheta \tag{2.37}$$

$$-W'(3+|\nabla u(r,\vartheta)|^2)u_{,1}(r,\vartheta)\frac{\partial u}{\partial r}(r,\vartheta)]r\,d\vartheta.$$

Then, (2.37), the third of (2.15), and (1.29) with $\alpha=1$ and C the path just described imply

$$J(\bar{r}) = J(\underline{r}),\ \text{for any}\ \bar{r}>\underline{r}>0, \tag{2.38}$$

so that $J(r)$ is independent of r. $J(r)$ can be calculated by using (2.16) in (2.37) and taking the limit as r tends to infinity. Doing this yields

$$J(r) = \frac{1}{4}\mu c k^2 (>0). \tag{2.39}$$

The use of (1.17), (1.22), (2.38), (2.39), and standard inequalities yield the following chain of inequalities

$$0<\frac{1}{4}\mu c k^2 = J(r)$$

$$=\left|\int_{-\pi}^{\pi}\left[\frac{1}{2}W(3+|\nabla u(r,\vartheta)|^2)\cos\vartheta - W'(3+|\nabla u(r_1\vartheta)|^2)u_{,1}(r,\vartheta)\frac{\partial u}{\partial r}(r,\vartheta)\right]r\,d\vartheta\right|$$

$$\leq\int_{-\pi}^{\pi}\left|\frac{1}{2}W(3+|\nabla u(r,\vartheta)|^2)\cos\vartheta - W'(3+|\nabla u(r,\vartheta)|^2)u_{,1}(r,\vartheta)\frac{\partial u}{\partial r}(r,\vartheta)\right|r\,d\vartheta$$

$$\leq\int_{-\pi}^{\pi}\Big[\left|\frac{1}{2}W(3+|\nabla u(r,\vartheta)|^2)\cos\vartheta\right|$$

$$+\left|-W'(3+|\nabla u(r,\vartheta)|^2)u_{,1}(r,\vartheta)\frac{\partial u}{\partial r}(r,\vartheta)\right|\Big]r\,d\vartheta$$

$$=\int_{-\pi}^{\pi}\left[\frac{1}{2}W(3+|\nabla u(r,\vartheta)|^2)|\cos\vartheta| + W'(3+|\nabla u(r,\vartheta)|^2)\left|-u_{,1}(r,\vartheta)\frac{\partial u}{\partial r}(r,\vartheta)\right|\right]r\,d\vartheta$$

$$\leq\int_{-\pi}^{\pi}\left[\frac{1}{2}W(3+|\nabla u(r,\vartheta)|^2) + W'(3+|\nabla u(r,\vartheta)|^2)|\nabla u(r,\vartheta)|^2\right]r\,d\vartheta, \qquad (2.40)$$

which implies (2.36)

Suppose now that W is given by (1.26) with $n\geq\frac{1}{2}$ and $|\nabla u(r,\vartheta)|=o(r^{-1/2n})$ as $r\to0$, uniformly in $\vartheta\in[-\pi,\pi]$. Let $\varepsilon>0$ be arbitrary. Then there exists $\bar{r}>0$ such that

$$|\nabla u(r,\vartheta)|\leq(\varepsilon/r)^{1/2n},\ \text{for } 0<r<\bar{r},\ -\pi\leq\vartheta\leq\pi. \qquad (2.41)$$

Also, make the following definition

$$G(k)=\frac{1}{2}W(3+k^2)+\frac{1}{2}\hat{\tau}(k)k,\ k\geq0, \qquad (2.42)$$

where $\hat{\tau}(k)$ is given by (1.20). By direct calculation

$$G'(k)=W'(3+k^2)k+\frac{1}{2}\hat{\tau}'(k)k+\frac{1}{2}\hat{\tau}(k)\geq0,\ \text{for } k\geq0, \qquad (2.43)$$

where equality holds only if k=0. Thus, G(k) is a nondecreasing function for $k \geq 0$.

This along with (2.41) implies

$$\left.\begin{aligned} &G(|\nabla u(r,\vartheta)|) \leq G((\varepsilon/r)^{1/2n}) \\ &= \frac{\mu}{4b}\{[1+\frac{b}{n}(\frac{\varepsilon}{r})^{1/n}]^n-1\} + \frac{\mu}{2}[1+\frac{b}{n}(\frac{\varepsilon}{r})^{1/n}]^{n-1}(\frac{\varepsilon}{r})^{1/n}, \\ &= \frac{\mu}{4b}(\frac{b}{n})^n\frac{\varepsilon}{r} + \frac{\mu}{2}(\frac{b}{n})^{n-1}\frac{\varepsilon}{r} + g(r) \end{aligned}\right\} \qquad (2.44)$$

where g(r) has the property that $\lim_{r \to 0} rg(r) = 0$. Thus

$$\int_{-\pi}^{\pi} G(|\nabla u|)r d\vartheta \leq 2\pi[\frac{\mu}{4b}(\frac{b}{n})^n + \frac{\mu}{2}(\frac{b}{n})^{n-1}]\,\varepsilon \qquad (2.45)$$

$$+\ 2\pi r g(r),\ 0<r<\bar{r},\ -\pi \leq \vartheta \leq \pi.$$

The right hand side of (2.45) can be made arbitrarily small by first choosing ε and then r sufficiently small. This contradicts (2.36), completing the proof.

Claim 5. Let u be a solution to problem $P_2(u)$ for a uniformly elliptic material. Also, assume that u is antisymmetric with respect to the x_1-axis, i.e. $u(x_1,-x_2) = -u(x_1,x_2)$ on $\tilde{D}$ and $u(x_1,0^-) = -u(x_1,0^+)$ for $x_1<0$. Define u^- on $H^- = \{(x_1,x_2) \mid -\infty<x_1<\infty,\ x_2 \leq 0\}$ as follows:

$$u^- = \left\{\begin{array}{l} u,\ \text{for } (x_1,x_2) \in H^- \cap \tilde{D}, \\ u(x_1,0^-),\ \text{for } x_1<0,\ x_2=0, \\ 0,\ \text{for } x_1=x_2=0. \end{array}\right\} \qquad (2.46)$$

Then $u^{-}\in C(H^{-})$ and $u^{-}\leq 0$ on H^{-}.

The proof of this claim makes use of the following theorem of Herzog [36][14] :

Theorem. Let H be an unbounded domain contained in a half space of $\mathbb{R}^2$. Let $a_{\alpha\beta}(p_1,p_2)\in C^1(\mathbb{R}^2)$ be four functions satisfying

$$a_{\alpha\beta}(p_1,p_2)\mu_\alpha\mu_\beta>0 \text{ for all real } \mu_1,\mu_2 \text{ (not all zero)} \quad (2.47)$$

$$\text{and all } (p_1,p_2)\in\mathbb{R}^2,$$

and

$$a_{\alpha\beta} = a_{\beta\alpha} \text{ and the determinant of the } [a_{\alpha\beta}] \quad (2.48)$$

$$\text{is identically one on } \mathbb{R}^2.$$

Assume $u\in C^2$ (H) satisfies the inequality

$$a_{\alpha\beta}(u_{,1}(x_1,x_2),u_{,2}(x_1,x_2))u_{,\alpha\beta}(x_1,x_2)\geq 0 \text{ in } H, \quad (2.49)$$

and the upper limit of $u(x_1,x_2)$ is nonpositive as (x_1,x_2) approaches any point on the boundary of H. Then, if

$$\limsup_{r\to\infty} \frac{M(r)}{r} \leq 0, \quad (2.50)$$

where $M(r)=\inf u(x_1,x_2)$ for $(x_1{}^2+x_2^2)^{1/2} = r$, $(x_1,x_2)\in H$, it follows that $u(x_1,x_2)\leq 0$ throughout H. The proof of this theorem and extensions to unbounded domains

[14]For a list of references on theorems of Phragmen-Lindelöf type, see [37].

H of higher dimensions can be found in [36]. The proof relies on a general comparison theorem for second order quasi-linear elliptic operators (a discussion of comparison principles may be found in [29] and [38]).

Proof of Claim 5. The first of (2.15) and claim 2 imply $u^- \in C(H^-)$. The condition of antisymmetry implies that $u(x_1,0) = 0$ for $x_1>0$. It will be shown in the next section that $u_{,1}(x_1,0^-)>0$ for $x_1<0$. Taking this for granted implies that $u_{,1}^-(x_1,0)>0$ for $x_1<0$ so that u^- is strictly increasing on $x_1<0$. This in turn implies $u^-(x_1,0)<0$ on $x_1<0$ (for if $u^-(\overset{\circ}{x}_1,0)\geq 0$ for some $\overset{\circ}{x}_1<0$ then by the mean value theorem $u_{,1}^-(\bar{x}_1,0) = \dfrac{u^-(\overset{\circ}{x}_1,0)-u^-(0,0)}{\overset{\circ}{x}_1-0} \leq 0$ for some $\bar{x}_1 \in (\overset{\circ}{x}_1,0)$, a contradiction). Thus, $u^- \leq 0$ on the boundary of H^-. Moreover, in view of the first of (2.16), (2.50) is trivially satisfied.

It remains to show that (2.47), (2.48), and (2.49) are satisfied. The second of (2.15) can be written in the form

$$\bar{a}_{\alpha\beta}(u_{,1},u_{,2})u_{,\alpha\beta} = \text{ on } \tilde{D}, \tag{2.51}$$

where

$$[\bar{a}_{\alpha\beta}] = \begin{bmatrix} 2W'(I_1)+4W''(I_1)u_{,1}^2 & 4W''(I_1)u_{,1}u_{,2} \\ 4W''(I_1)u_{,1}u_{,2} & 2W'(I_1)+4W''(I_1)u_{,2}^2 \end{bmatrix}, \tag{2.52}$$

and I_1 is given by (1.16). The two eigenvalues λ_1 λ_2 of the matrix (2.52) are

$$\left.\begin{aligned} \lambda_1 &= 2W'(3+|\nabla u|^2)>0, \\ \lambda_2 &= 2W'(3+|\nabla u|^2)+4W''(3+|\nabla u|^2)|\nabla u|^2>0, \end{aligned}\right\} \tag{2.53}$$

which are positive in view of (1.22) and (1.23). The determinant of the matrix is

$$\det[\bar{a}_{\alpha\beta}] = 4W'(3+|\nabla u|^2)[W'(3+|\nabla u|^2) + 2W''(3+|\nabla u|^2)|\nabla u|^2], \qquad (2.54)$$

which is also positive from (1.22) and (1.23). Let

$$[a_{\gamma\delta}] = (\det[\bar{a}_{\alpha\beta}])^{-1/2}[\bar{a}_{\gamma\delta}]. \qquad (2.55)$$

Then, u satisfies

$$a_{\gamma\delta}(u_{,1},u_{,2})u_{,\gamma\delta} = 0 \text{ on } \tilde{D}\,, \qquad (2.56)$$

and the $a_{\gamma\delta}$ satisfy conditions (2.47) and (2.48). The theorem of Herzog can now be applied to u^- with $H = \{(x_1,x_2)|\ -\infty<x_1<\infty, x_2<0\}$ and the proof is complete.

If W is given by (1.26) with n = 1 corresponding to a neo-Hookean material so that the second of (2.15) reduces to Laplace's equation for the out-of-plane displacement u, the uniqueness of the solution to problem $P_2(u)$ can be established without the use of the hodograph transformation. In fact, in this case, uniqueness can be established under much less restrictive conditions as follows.

Claim 6. Let the material be neo-Hookean, so that the governing partial differential equation for u is Laplace's equation. In addition to (2.15), require u to be bounded near the crack tip and to satisfy only the first of (2.16). Then the solution u to this boundary-value problem is unique up to an additive constant.

The proof of this theorem uses techniques similar to those used by Knowles and Pucik [39] in proving the uniqueness of the solution to a plane strain problem in linear elasticity.

Proof. Let u_1 and u_2 be two solutions to the boundary-value problem. Then the difference of these two solutions $u = u_1 - u_2$ satisfies

$$\left.\begin{array}{l} u \in C^1(\tilde{D}^+) \cap C^1(\tilde{D}^-) \cap C^2(D), \\ u_{,11} + u_{,22} = 0 \text{ on } \tilde{D}, \\ u_{,2} = 0 \text{ at } x_2 = 0^\pm,\ x_1 < 0, \\ |u| < M \text{ on } \tilde{D}, \end{array}\right\} \tag{2.57}$$

for some M > 0. Let r, ϑ be polar coordinates in the plane given by (2.13) and for any $0 < \underline{r} < \bar{r} < \infty$ let $A_{\underline{r}}^{\bar{r}} = \{(r,\vartheta) \mid \underline{r} < r < \bar{r},\ -\pi < \vartheta < \pi\}$. Let the simple closed curve C forming the boundary of $A_{\underline{r}}^{\bar{r}}$ be oriented as shown in Figure 5. Then, using the second and third of (2.57) and the divergence theorem,

$$E(\bar{r}, \underline{r}) \equiv \int_{\underline{r}}^{\bar{r}} \int_{-\pi}^{\pi} \{[\frac{\partial u}{\partial r}(r,\vartheta)]^2 + [\frac{1}{r}\frac{\partial u}{\partial \vartheta}(r,\vartheta)]^2\} r\, d\vartheta\, dr$$

$$= \int_{A_{\underline{r}}^{\bar{r}}} \{[\frac{\partial u}{\partial x_1}(x_1,x_2)]^2 + [\frac{\partial u}{\partial x_2}(x_1,x_2)]^2 + u(x_1,x_2)[\frac{\partial^2 u}{\partial x_1^2}(x_1,x_2)$$

$$+ \frac{\partial^2 u}{\partial x_2^2}(x_1,x_2)]\} dA$$

$$= \int_{A_{\underline{r}}^{\bar{r}}} \{\frac{\partial}{\partial x_1}[u(x_1,x_2)\frac{\partial u}{\partial x_1}(x_1,x_2)] + \frac{\partial}{\partial x_2}[u(x_1,x_2)\frac{\partial u}{\partial x_2}(x_1,x_2)]\} dA$$

$$= \int_C \{n_1(x_1,x_2)[u(x_1,x_2)\frac{\partial u}{\partial x_1}(x_1,x_2)] + n_2(x_1,x_2)[u(x_1,x_2)\frac{\partial u}{\partial x_2}(x_1,x_2)]\} dA$$

$$= f(\bar{r}) - f(\underline{r}) \geq 0, \tag{2.58}$$

where n_1 and n_2 are the components of the unit outward normal to $A_{\underline{r}}^{\bar{r}}$ and

$$f(r) = \int_{-\pi}^{\pi} u(r,\vartheta)\frac{\partial u}{\partial r}(r,\vartheta) r\, d\vartheta. \tag{2.59}$$

Note that (2.58) implies that f(r) is nondecreasing as r increases. By direct calculation

$$\frac{\partial E}{\partial \bar{r}}(\bar{r},\underline{r}) = \int_{-\pi}^{\pi}\{[\frac{\partial u}{\partial r}(\bar{r},\vartheta)]^2 + [\frac{1}{\bar{r}}\frac{\partial u}{\partial \vartheta}(\bar{r},\vartheta)]^2\}\bar{r}d\vartheta = f'(\bar{r}) \tag{2.60}$$

$$\geq \int_{\pi}^{\pi}[\frac{\partial u}{\partial r}(\bar{r},\vartheta)]^2\bar{r}d\vartheta,$$

and

$$-\frac{\partial E}{\partial \underline{r}}(\bar{r},\underline{r}) = \int_{-\pi}^{\pi}\{[\frac{\partial u}{\partial r}(\underline{r},\vartheta)]^2 + [\frac{1}{\underline{r}}\frac{\partial u}{\partial \vartheta}(\underline{r},\vartheta)]^2\}\underline{r}d\vartheta = f'(\underline{r}) \tag{2.61}$$

$$\geq \int_{-\pi}^{\pi}[\frac{\partial u}{\partial r}(\underline{r},\vartheta)]^2\underline{r}d\vartheta.$$

Also, by the Schwartz inequality,

$$|f(r)|\leq|\int_{-\pi}^{\pi}r[u(r,\vartheta)]^2d\vartheta|^{1/2}|\int_{-\pi}^{\pi}r[\frac{\partial u}{\partial r}(r,\vartheta)]^2d\vartheta|^{1/2}. \tag{2.62}$$

Now, f(r) = 0 for all r is shown in two steps.

Step 1. $f(r) \leq 0$ for all $r > 0$.

Assume there exists $\mathring{r} > 0$ such that $f(\mathring{r}) > 0$. Then, since f(r) is nondecreasing, $f(r)\geq f(\mathring{r})>0$ for all $r\geq\mathring{r}$. For $\bar{r}>\underline{r}\geq\mathring{r}$, the fourth of (2.57), (2.58), (2.60), and (2.62) imply

$$E^2(\bar{r},\underline{r})\leq f^2(\bar{r})\leq|\int_{-\pi}^{\pi}\bar{r}[u(\bar{r},\vartheta)]^2d\vartheta||\int_{-\pi}^{\pi}\bar{r}[\frac{\partial u}{\partial r}(\bar{r},\vartheta)]^2d\vartheta| \tag{2.63}$$

$$\leq 2\pi M^2\bar{r}f'(\bar{r}).$$

This implies

$$\frac{1}{2\pi M^2\bar{r}}\leq\frac{f'(\bar{r})}{f^2(\bar{r})}. \tag{2.64}$$

Integration of the above differential inequality from r_1 to r_2 where $r_2>r_1>\underline{r}\geq\mathring{r}$ yields

$$\frac{1}{2\pi M^2}\log\frac{r_2}{r_1}\leq\left[\frac{-1}{f(r_2)}+\frac{1}{f(r_1)}\right]\leq\frac{1}{f(r_1)}, \tag{2.65}$$

which can be violated by keeping r_1 fixed and taking r_2 sufficiently large.

Step 2. $f(r)\geq 0$ for all $r>0$.

Assume there exists $\mathring{r}$ such that $f(\mathring{r})<0$. Then, since f(r) is nondecreasing, $0>f(\mathring{r})\geq f(r)$ for all $\mathring{r}\geq r$. For $\mathring{r}\geq\bar{r}>\underline{r}>0$ the fourth of (2.57), (2.58), (2.61), and (2.62) imply

$$E^2(\bar{r},\underline{r})\leq f^2(\underline{r})\leq\left|\int_{-\pi}^{\pi}\underline{r}[u(\underline{r},\vartheta)]^2d\vartheta\right|\left|\int_{-\pi}^{\pi}\underline{r}\left[\frac{\partial u}{\partial r}(\underline{r},\vartheta)\right]^2d\vartheta\right| \tag{2.66}$$

$$\leq\underline{r}\,2\pi M^2 f'(\underline{r}).$$

This implies

$$\frac{1}{2\pi M^2\underline{r}}\leq\frac{f'(\underline{r})}{f^2(\underline{r})} \tag{2.67}$$

Integration from r_1 to r_2 where $\mathring{r}\geq\bar{r}>r_2>r_1$ yields

$$\frac{1}{2\pi M^2}\log\frac{r_2}{r_1}\leq\left[-\frac{1}{f(r_2)}+\frac{1}{f(r_1)}\right]\leq-\frac{1}{f(r_2)}. \tag{2.68}$$

which can be violated by keeping r_2 fixed and taking r_1 sufficiently small. The results of steps 1 and 2 imply f(r) = 0 for all r > 0. Thus, $E(\bar{r},\underline{r}) = 0$ for all $\bar{r}>\underline{r}>0$. This implies $\frac{\partial u}{\partial r}(r,\vartheta) = \frac{\partial u}{\partial \vartheta}(r,\vartheta) = 0$ for all r > 0, $-\pi \le \vartheta \le \pi$, which in turn implies u = constant, completing the proof of the claim.

The techniques used in the preceding proof cannot be applied directly to the general case of problem $P_2(u)$, since there the governing partial differential equation is nonlinear. However, with the aid of the hodograph transformation, which is the topic of the next section, these techniques can be used to prove the uniqueness of the solution to a related linear boundary-value problem which in turn will imply the uniqueness up to an additive constant of the solution to problem $P_2(u)$.

3. The Hodograph Transformation

The hodograph transformation is commonly used in the study of second order quasi-linear partial differential equations. In particular, it has been applied extensively in the theory of compressible flow (see [12], [22] and the references cited there). It has been employed in the analysis of crack problems in [11] and [40]-[42].

Let u be any solution of problem $P_2(u)$ on the domain $\tilde{D}$ in the (x_1,x_2) -plane, and let

$$\xi_\alpha = \hat{\xi}_\alpha(x_1,x_2) = u_{,\alpha}(x_1,x_2),\ (x_1,x_2)\in\tilde{D}, \tag{3.1}$$

represent a transformation from $\tilde{D}$ into the "hodograph plane" in which ξ_1,ξ_2 are cartesian coordinates. By the hodograph condition of Section 2, this transformation is one-to-one; let Δ be the image of $\tilde{D}$ in the hodograph plane. It will be shown below that Δ is the upper half-plane $\xi_2>0$, and that the positive (negative) ξ_1-axis corresponds to the lower (upper) crack face, and that the origin and the point at infinity in the hodograph plane are the respective images of the point at infinity and the origin in the physical plane.

Let the Legendre transform U of u be defined by Δ by

$$\left.\begin{aligned} &U(\xi_1,\xi_2) = x_\beta U_{,\beta}(x_1,x_2) - u(x_1,x_1), \\ &x_\beta = \hat{x}_\beta(\xi_1,\xi_2),\ (\xi_1,\xi_2)\in\Delta, \end{aligned}\right\} \tag{3.2}$$

where $\hat{x}_\beta$ represent the inverse functions associated with (3.1). From (3.1), (3.2) it in fact follows that

$$x_\alpha = \hat{x}_\alpha(\xi_1,\xi_2) = \frac{\partial U}{\partial \xi_\alpha}(\xi_1,\xi_2),\ (\xi_1,\xi_2)\in\Delta, \tag{3.3}$$

and hence from (3.1) - (3.3) that

$$u(x_1,x_2) = \xi_\beta \frac{\partial U}{\partial \xi_\beta}(\xi_1,\xi_2) - U(\xi_1,\xi_2),\ (\xi_1,\xi_2)\in\Delta. \tag{3.4}$$

Moreover,

$$\left.\begin{aligned} &\frac{\partial \xi_\alpha}{\partial x_\beta}(x_1,x_2) = u_{,\alpha\beta}(x_1,x_2) \\ &= H(x_1,x_2)\varepsilon_{\alpha\lambda}\varepsilon_{\beta\mu}\frac{\partial^2 U}{\partial \xi_\lambda \partial \xi_\mu}(\hat{\xi}_1(x_1,x_2),\ \hat{\xi}_2(x_1,x_2)), \\ &(x_1,x_2)\in\tilde{D}, \end{aligned}\right\} \tag{3.5}$$

where $\varepsilon_{\alpha\lambda}$ are the components of the two-dimensional alternator ($\varepsilon_{11} = \varepsilon_{22} = 0$, $\varepsilon_{12} = -\varepsilon_{21} = 1$), and

$$H = u_{,11}u_{,22} - u_{,12}{}^2 \text{ on } \tilde{D} \tag{3.6}$$

is the Jacobian of the transformation (3.1). The hodograph condition requires that $H \neq 0$ on D.

The foregoing relations between x_α, u, and ξ_α, U may be used to transform problem $P_2(u)$ for u into a problem $P(U)$ for U in the hodograph plane. Let (R,φ) be polar coordinates in the latter plane:

$$\xi_1 = R\cos\varphi,\ \xi_2 = R\sin\varphi,\ R>0,\ 0\leq\varphi\leq\pi, \tag{3.7}$$

and let the relations between (r,ϑ) and (R,φ) induced by (3.1) be denoted by

$$R = \hat{R}(r,\vartheta),\ \varphi = \hat{\varphi}(r,\vartheta),\ r = \hat{r}(R,\varphi),\ \vartheta = \hat{\vartheta}(R,\varphi). \tag{3.8}$$

From (3.7), (3.1) and (2.16) it follows that

$$R = \hat{R}(r,\vartheta) = |\nabla u(r,\vartheta)| = (2r/ck^2)^{-1/2} + 0(r^{-1}) \text{ as } r\to\infty, \tag{3.9}$$

$$\cos\hat{\varphi}(r,\vartheta) = -\sin\frac{\vartheta}{2} + 0(r^{-1/2}), \text{ as } r\to\infty, \tag{3.10}$$

$$\sin\hat{\varphi}(r,\vartheta) = \cos\frac{\vartheta}{2} + 0(r^{-1/2}), \text{ as } r\to\infty, \tag{3.11}$$

uniformly in $\vartheta\in[-\pi,\pi]$.

In view of (3.1) and the free-surface conditions of (2.15), we conclude that $\xi_2=0$ on the hodograph image of each crack face. Conditions (3.9)-(3.11) then show that in fact the upper crack face maps onto the negative ξ_1-axis $R>0,\varphi=\pi$, while the lower crack face is carried onto the positive ξ_1-axis. Further, (3.2)-(3.11) show that a neighborhood of infinity in the (x_1,x_2)-plane is carried onto the intersection of a neighborhood of $\xi_1 = \xi_2 = 0$ with the upper half of the hodograph plane. Thus, the image Δ of the cut physical plane $\tilde{D}$ must be a subset of the upper half of the hodograph plane. We defer until the end of this section a proof of the fact that Δ coincides with $\xi_2>0$, $-\infty<\xi_1<\infty$.

Momentarily taking this result for granted, one may now pose the problem which must be satisfied by U on Δ. From (3.9)-(3.11), (3.2) and the matching condition (2.16), one infers that

$$\left.\begin{aligned} &U(\hat{R}(r,\vartheta)\cos\hat{\varphi}(r,\vartheta),\hat{R}(r,\vartheta)\sin\hat{\varphi}(r,\vartheta)) \\ &\quad = \frac{1}{2}ck^2\frac{\cos\hat{\varphi}(r,\vartheta)}{\hat{R}(r,\vartheta)} + 0(1), \\ &\text{as } r\to\infty, \text{ uniformly in } \vartheta\in[-\pi,\pi]. \end{aligned}\right\} \tag{3.12}$$

It follows immediately that

$$\left.\begin{aligned} &\overline{U}(R,\varphi) \equiv U(R\cos\varphi,R\sin\varphi) = \frac{1}{2}ck^2\frac{\cos\varphi}{R} + 0(1), \\ &\text{as } R\to 0, \text{ uniformly in } \varphi\in[0,\pi]. \end{aligned}\right\} \quad (3.13)^{16}$$

The free surface condition of (2.15) are readily shown from (3.3) to lead to

$$\frac{\partial \overline{U}}{\partial \varphi}(R,\varphi) = 0,\ \varphi = 0, R>0 \text{ and } \varphi=\pi, R>0, \tag{3.14}$$

while the crack tip condition (2.18) becomes

$$\hat{\tau}(R)\,\overline{U}(R,\varphi) = 0(1) \text{ as } R\to\infty, \text{ uniformly in } \varphi\in[0,\pi]. \tag{3.15}$$

Finally, the differential equation of (2.15) for u may be transformed with the help of (3.1), (3.5) and the hodograph condition to a differential equation for $\overline{U}$. One finds this equation to be

$$\left.\begin{array}{c}\hat{\tau}(R)\dfrac{\partial^2 \overline{U}}{\partial R^2}(R,\varphi) + \hat{\tau}'(R)\dfrac{\partial \overline{U}}{\partial R}(R,\varphi) + \\ \dfrac{\hat{\tau}'(R)}{R}\ \dfrac{\partial^2 \overline{U}}{\partial \varphi^2}(R,\varphi) = 0, \\ (R,\varphi)\in\{(R,\varphi)\,|\,R>0,\ 0<\varphi<\pi\}.\end{array}\right\} \tag{3.16}$$

Let $\Delta_o = \{(\xi_1,\xi_2)\,|\,-\infty<\xi_1<\infty,\ \xi_2\geq 0,\ \xi_1^2+\xi_2^2\neq 0\}$. One shows readily that U inherits from u the following smoothness: $U\in C^1(\Delta_o)\cap C^2(\Delta)$. The problem for $\overline{U}$ represented by (3.12)-(3.16) will be referred to as problem $P(U)$.

A solution of problem $P(U)$ was constructed in [11] for a special choice of W and hence for a special $\hat{\tau}(R)$. We generalize this solution here to the case of any W for which the associated $\hat{\tau}(R)$ satisfies the ellipticity condition (1.23). Let $\overline{U}(R,\varphi)$ be defined by

[16] In order to avoid unduly cumbersome notation, throughout most of this paper the same functional symbol is employed for functions on $\tilde{D}$ and their corresponding altered forms when expressed in terms of polar coordinates. Here the distinction is critical and so different functional symbols are used.

$$\overline{U}(R,\varphi) = RI(R)\cos\varphi \text{ on } \Delta_0, \tag{3.17}$$

where

$$I(R) = \mu ck^2 \int_R^\infty \frac{dt}{t^2\hat{\tau}(t)},\ R>0. \qquad (3.18)^{17}$$

Since (1.23), (3.18) imply

$$I(R) \le \frac{\mu ck^2}{R\hat{\tau}(R)}, \tag{3.19}$$

one has

$$|\overline{U}(R,\varphi)| \le \frac{\mu ck^2}{\hat{\tau}(R)}, \tag{3.20}$$

so that (3.15) is satisfied. From (3.18) and the nature of $\hat{\tau}(R)$ for small R, one can show that

$$I(R) = ck^2[1/(2R^2) + (\gamma/\mu)\log R + 0(1)],\text{as } R\to 0, \tag{3.21}$$

where

$$\gamma = 2W''(3). \tag{3.22}$$

It follows from (3.21), (3.17) that (3.13) holds. It is easy to verify directly that the boundary conditions (3.14) and the differential equation (3.16) are satisfied when $\overline{U}$ is given by (3.17), (3.18). Thus, $\overline{U}$ is a solution of problem $P(U)$.

To obtain the solution u of problem $P_2(u)$ from U, one first uses (3.3) and the relation between cartesian and polar coordinates in each plane to find that

[17] The convergence of the improper integral is assured by the ellipticity condition (1.22).

$$
\left.\begin{aligned}
r\cos\vartheta &= I(R) + RI'(R)\cos^2\varphi, \\
r\sin\vartheta &= RI'(R)\sin\varphi\cos\varphi,
\end{aligned}\right\} \tag{3.23}
$$

It is possible to prove that (3.23), (3.18) represent a one-to-one map between Δ_0 and the (x_1,x_2)-plane cut along the nonpositive x_1-axis. The polar-coordinate version of (3.4) is

$$
\left.\begin{aligned}
u(r\cos\vartheta, r\sin\vartheta) &= \hat{R}(r,\vartheta)\frac{\partial\overline{U}}{\partial R}(\hat{R}(r,\vartheta),\ \hat{\varphi}(r,\vartheta)) \\
&-\overline{U}(\hat{R}(r,\vartheta),\ \hat{\varphi}(r,\vartheta)).
\end{aligned}\right\} \tag{3.24}
$$

Equations (3.23), (3.24) determine u implicitly. One can then verify directly that u is a solution of problem $P_2(u)$. Details are similar to those sketched in [11]. In the following section, we investigate the uniqueness of this solution.

We turn now to the proof that the hodograph image of the physical domain $\tilde{D}$ is the upper half of the (ξ_1,ξ_2)-plane, regardless of which solution of problem $P_2(u)$ is considered, should there be more than one. To this end, we first construct a sequence $\{C_n\}_{n=1}^{\infty}$ of circles centered at the crack tip $x_1=x_2=0$ whose radii r_n decrease monotonically to zero as $n\to\infty$, while the minimum value of $|\nabla u|$ on C_n tends to infinity as $n\to\infty$. This is possible by (2.17). It follows that the image curves $\{C'_n\}_{n=1}^{\infty}$ in the hodograph plane of the circles $\{C_n\}_{n=1}^{\infty}$ have the following properties: (i) C'_n is a smooth curve with one end point on the negative ξ_2-axis, the other on the positive ξ_1-axis; (ii) C'_n and C'_m do not intersect if $n\neq m$; (iii) the minimum distance from points on C'_n to the origin $\xi_1=\xi_2=0$ is a monotone increasing function of n which tends to infinity as $n\to\infty$.

To construct $\{C_n\}_{n=1}^{\infty}$, let C_1 be a circle of radius $r_1=1$ centered at the crack tip, i.e. $C_1=\{(r,\vartheta)\,|\,r=r_1,\ -\pi\le\vartheta\le\pi\}$. Let $m_1=\min_{C_1}|\nabla u|$, $M_1=\max_{C_1}|\nabla u|$, and (r_1,ϑ_1) be a point on C_1 at which m_1 is attained. By (2.17) there exists a positive constant δ_1 such that $|\nabla u|>M_1$ for all $0<r<\delta_1$ and for all $-\pi\le\vartheta\le\pi$. Note that necessarily $\delta_1\le r_1$. Now, choose a number r_2 such that $0<r_2<\min(\delta_1,\frac{1}{2})\le\min(r_1,\frac{1}{2})$. Let C_2 be a circle of radius r_2 centered at the crack tip. Let $m_2=\min_{C_2}|\nabla u|$, $M_2=\max_{C_2}|\nabla u|$, and (r_2,ϑ_2) be a point on c_2 at which m_2 is attained. Note that $m_2>m_1$. Again by (2.17) there exists a positive constant δ_2 such that $|\nabla u|>M_2$ for all $0<r<\delta_2$ and for all $-\pi\le\vartheta\le\pi$. Let C_3 be a circle of radius $0<r_3<\min(\delta_2,\frac{1}{3})\le\min(r_2,\frac{1}{3})$. Define m_3, M_3, and (r_3,ϑ_3) analogous to m_2, M_2, and (r_2,ϑ_2). Continue this procedure, constructing circles C_n with radii r_n that decrease monotonically as n increases.

As a result of the above procedure, one obtains a sequence of points $\{(r_n,\vartheta_n)\}_{n=1}^{\infty}$, the polar coordinates of a sequence of points in the physical plane, each lying on C_n. Since $0<r_n<\min(r_{n-1},\frac{1}{n})$ for each n, the sequence of numbers $\{r_n\}_{n=1}^{\infty}$ decreases strictly to zero. In view of (2.17), this in turn implies that the strictly increasing sequence $\{m_n\}_{n=1}^{\infty}$ approaches infinity. The final result of the above procedure is the construction of a sequence of circles with properties (i)-(iii) above.

A similar argument using the large r conditions (3.9)-(3.11) can be used to construct a sequence of circles Γ_n centered at the crack tip, whose radii strictly <u>increase</u> from one (thus Γ_1 coincides with C_1) to infinity and on which the maximum value of the modulus of the gradient attained tends monotonically to zero as $n\to\infty$. The image of each circle Γ_n is a simple smooth curve Γ'_n in the hodograph plane with one end on the positive ξ_1-axis, the other in $\xi_1<0$, $\xi_2=0$. Furthermore, if $B>M_1$, then the minimum distance between the semicircle $C'_B=\{(\xi_1,\xi_2)|(\xi_1^2+\xi_2^2)^{1/2}=B,\ \xi_2\ge0\}$ and Γ'_n strictly increases with n from $(B-M_1)$ to B.

Figure 6 shows the circles C_n and Γ_n, while Figure 7 shows their images C'_n and Γ'_n in the hodograph plane. It then follows from the assumed properties of the hodograph transformation that Δ must be the upper half of the hodograph plane.

4. Uniqueness

In the previous section, it was shown that any solution to problem $P_2(u)$ posed in the physical plane has a Legendre transform U satisfying problem $P(U)$ in the upper half of the hodograph plane. In this section the uniqueness of the solution up to an additive constant to problems $P(U)$ and $P_2(u)$ both for an elliptic material satisfying the growth condition (1.25) will be established.

Let the material be elliptic, so that (1.23) holds, and satisfy the growth condition (1.25). Suppose u_1 and u_2 are both solutions to problem $P_2(u)$. It then follows that their respective Legendre transforms U_1 and U_2 are solutions to problem $P(U)$ and their difference $U = U_1 - U_2$ is a solution to problem $P(U)$ with boundary condition (3.12) being replaced by $U = O(1)$ as $R \to 0$, uniformly in $\varphi \in [0,\pi]$. Hence, to prove that the solution to problem $P(U)$ is unique up to an additive constant, it suffices to prove the following:

Theorem.

Let

$$\Delta = \{(\xi_1,\xi_2) \mid -\infty<\xi_1<\infty,\ \xi_2>0\}$$

and

$$\Delta_o = \{(\xi_1,\xi_2) \mid -\infty<\xi_1<\infty,\ \xi_2\geq 0,\ \xi_1{}^2+\xi_2{}^2\neq 0\}.$$

Let

$$U\in C^1(\Delta_0)\cap C^2(\Delta)$$

be a scalar valued function satisfying the following four conditions,

$$(\alpha)\quad 2W'(3+R^2)R\frac{\partial^2 U}{\partial R^2}(R,\varphi) + [2W'(3+R^2) + 4R^2W''(3+R^2)]\frac{\partial U}{\partial R}(R,\varphi) \tag{4.1}$$

$$+\frac{1}{R}[2W'(3+R^2) + 4R^2W''(3+R^2)]\frac{\partial^2 U}{\partial\varphi^2}(R,\varphi) = 0\ ,$$

$$(b)\quad \frac{\partial U}{\partial\varphi}(R,\varphi) = 0 \text{ for } \varphi = 0,\pi\ ,\ R>0\ , \tag{4.2}$$

$$(c)\quad U(R,\varphi) = 0(1),\ \text{as } R\to 0,\ \text{uniformly in } \varphi\in[0,\pi]\ , \tag{4.3}$$

$$(d)\quad 2W'(3+R^2)RU(R,\varphi) = 0(1),\ \text{as } R\to\infty,\ \text{uniformly in } \varphi\in[0,\pi]\ , \tag{4.4}$$

where (R,φ) are the polar coordinates of (ξ_1,ξ_2) given by (3.7) and $W\in C^2([3,\infty))$ satisfies (1.21), (1.23), and (1.25). Then U is constant on Δ. The proof uses the "energy integral" approach and shares some features with the proof presented in [39] for plane crack problems in linear elastostatics. It is, however, essentially different from that in [39] and contains several new twists.

Proof. The partial differential equation (4.1) can be rewritten in the form

$$L[U] \equiv \frac{\partial}{\partial R}[2W'(3+R^2)R\frac{\partial U}{\partial R}(R,\varphi)] + \frac{\partial}{\partial\varphi}\{[2W'(3+R^2) \tag{4.5}$$

$$+ 4R^2W''(3+R^2)]\frac{1}{R}\frac{\partial U}{\partial\varphi}(R,\varphi)\} = 0\ .$$

Let $\bar{R}$ and $\underline{R}$ satisfy $0<\underline{R}<\bar{R}<\infty$ and let $A_{\underline{R}}^{\bar{R}} = \{(R,\varphi)\,|\,\underline{R}<R<\bar{R},\ 0<\varphi<\pi\}$. Using (4.5), (4.2), and the divergence theorem gives

$$\int_{A_{\underline{R}}^{\bar{R}}}\{2W'(3+R^2)R[\frac{\partial U}{\partial R}(R,\varphi)]^2 + \frac{2}{R}[W'(3+R^2)+2R^2W''(3+R^2)][\frac{\partial U}{\partial\varphi}(R,\varphi)]^2\}dA$$

$$= \int_{A_{\underline{R}}^{\bar{R}}}\{2W'(3+R^2)R[\frac{\partial U}{\partial R}(R,\varphi)]^2 + \frac{2}{R}[W'(3+R^2)+2R^2W''(3+R^2)][\frac{\partial U}{\partial\varphi}(R,\varphi)]^2$$

$$+ UL[U]\}dA$$

$$= \int_{A_{\underline{R}}^{\bar{R}}}\Big\langle\frac{\partial}{\partial R}[2W'(3+R^2)RU(R,\varphi)\frac{\partial U}{\partial R}(R,\varphi)]$$

$$+\ \frac{\partial}{\partial\varphi}\{[2W'(3+R^2)+4R^2W''(3+R^2)]\frac{1}{R}U(R,\varphi)\frac{\partial U}{\partial\varphi}(R,\varphi)\}\rangle dA$$

$$=\int_0^\pi 2W'(3+\bar{R}^2)\bar{R}\frac{\partial U}{\partial R}(\bar{R},\varphi)\,U(\bar{R},\varphi)d\varphi-\int_0^\pi 2W'(3+\underline{R}^2)\underline{R}\frac{\partial U}{\partial R}(\underline{R},\varphi)\,U(\underline{R},\varphi)d\varphi. \tag{4.6}$$

Make the following two definitions:

$$e(R,\varphi)=2W'(3+R^2)R[\frac{\partial U}{\partial R}(R,\varphi)]^2+\frac{2}{R}[W'(3+R^2) \tag{4.7}$$

$$+\ 2R^2W''(3+R^2)][\frac{\partial U}{\partial\varphi}(R,\varphi)]^2\,,$$

$$F(R)=\int_0^\pi 2W'(3+R^2)R\frac{\partial U}{\partial R}(R,\varphi)U(R,\varphi)d\varphi\,. \tag{4.8}$$

These definitions used in (4.6) give

$$E(\bar{R},\underline{R})\equiv\int_{\underline{R}}^{\bar{R}}[\int_0^\pi e(\rho,\varphi)d\varphi]d\rho=F(\bar{R})-F(\underline{R})\geq 0\ ,\ 0<\underline{R}<\bar{R}\,, \tag{4.9}$$

since the restrictions (1.22) and (1.23) on W imply that $e(R,\varphi)\geq 0$ on Δ_0 and $e(R,\varphi)=0$ if and only if $\frac{\partial U}{\partial R}(R,\varphi)=\frac{\partial U}{\partial\varphi}(R,\varphi)=0$. (4.9) in turn implies

$$F(\bar{R})\geq F(\underline{R})\ ,\ 0<\underline{R}<\bar{R}\ ;\ \textit{i.e.}\ \text{F is nondecreasing on }(0,\infty)\,, \tag{4.10}$$

$$\frac{\partial E}{\partial\bar{R}}(\bar{R},\underline{R})=\int_0^\pi e(\bar{R},\varphi)d\varphi=F'(\bar{R})$$

$$\geq\int_0^\pi[\frac{\partial U}{\partial R}(\bar{R},\varphi)]^2 2W'(3+\bar{R}^2)\bar{R}d\varphi\geq 0\ ,\ \bar{R}>0\,, \tag{4.11}$$

$$-\frac{\partial E}{\partial\underline{R}}(\bar{R},\underline{R})=\int_0^\pi e(\underline{R},\varphi)d\varphi=F'(\underline{R})$$

$$\geq \int_0^{\pi} \left[\frac{\partial U}{\partial R}(\underline{R},\varphi) \right]^2 2W'(3+\underline{R}^2)\underline{R}\,d\varphi \geq 0 \;,\; \underline{R}>0 \;. \tag{4.12}$$

Also, it follows from the Schwarz inequality that

$$|F(R)| \leq \left[\int_0^{\pi} R\left(\frac{\partial U}{\partial R}(R,\varphi) \right)^2 2W'(3+R^2)\,d\varphi \right]^{1/2} \tag{4.13}$$

$$\times \left[\int_0^{\pi} R\left(U(R,\varphi) \right)^2 2W'(3+R^2)\,d\varphi \right]^{1/2} \;,\; R>0 \;.$$

The next step is to show that $F(R) \geq 0$ for all $R>0$. This is done by contradiction. Thus, assume there exist $\bar{R}>0$ such that $F(\bar{R})<0$. Then, since (4.10) implies $F(\underline{R}) \leq F(\bar{R})<0$ for all $\underline{R} \in (0,\bar{R})$ it follows that

$$0 \leq E(\bar{R},\underline{R}) = F(\bar{R}) - F(\underline{R}) < -F(\underline{R}) \text{ for all } \underline{R} \in (0,\bar{R}) \;. \tag{4.14}$$

Squaring yields,

$$0 \leq E^2(\bar{R},\underline{R}) < F^2(\underline{R})$$

$$\leq \left[\int_0^{\pi} \underline{R}\left(\frac{\partial U}{\partial R}(\underline{R},\varphi) \right)^2 2W'(3+\underline{R}^2)\,d\varphi \right] \left[\int_0^{\pi} \underline{R}\left(U(\underline{R},\varphi) \right)^2 2W'(3+\underline{R}^2)\,d\varphi \right] ,$$

using (4.13);

$$\leq \left[-\frac{\partial E}{\partial \underline{R}}(\bar{R},\underline{R}) \right] \left[\int_0^{\pi} \underline{R}\left(U(\underline{R},\varphi) \right)^2 2W'(3+\underline{R}^2)\,d\varphi \right] ,$$

using (4.12);

$$= \left[F'(\underline{R}) \right] \left[\int_0^{\pi} \underline{R}\left(U(\underline{R},\varphi) \right)^2 2W'(3+\underline{R}^2)\,d\varphi \right] ,$$

using (4.12) again;

$$\leq [F'(\underline{R})][\underline{R}\pi M^2 b_{\bar{R}}], \tag{4.15}$$

where

$$b_{\bar{R}} = \sup_{\underline{R}\in(0,\bar{R})} 2W'(3+\underline{R}^2) > 0 , \tag{4.16}$$

$$M = \sup_{\substack{\underline{R}\in(0,\bar{R}) \\ \varphi\in[0,\pi]}} |U(\underline{R},\varphi)| \geq 0 . \tag{4.17}$$

Note that the existence of M (as a finite real number) is assured by (4.3). Rearrangement of terms in (4.15) gives

$$\frac{1}{\underline{R}\pi b_{\bar{R}}} \leq \frac{F'(\underline{R})}{F^2(\underline{R})} M^2 , \; \underline{R}\in(0,\bar{R}) . \tag{4.18}$$

Integrating (4.18) from R_1 to R_2 where $0<R_1<R_2<\bar{R}$ yields

$$\frac{1}{\pi b_{\bar{R}}} \log\frac{R_2}{R_1} \leq -\frac{M^2}{F(R_2)} + \frac{M^2}{F(R_1)} \leq -\frac{M^2}{F(R_2)} , \tag{4.19}$$

or,

$$0 < -F(R_2) \leq \frac{\pi M^2 b_{\bar{R}}}{\log\dfrac{R_2}{R_1}} . \tag{4.20}$$

Inequality (4.20) is violated if $M = 0$ or can violated if $M > 0$ by taking R_1 sufficiently small. Thus,

$$F(R) \geq 0 , \text{ for all } R > 0 . \tag{4.21}$$

The final step involves showing that $E(\bar{R},\underline{R}) = 0$ for all $0 < \underline{R} < \bar{R}$, from which it immediately follows from the definition of $E(\bar{R},\underline{R})$ given by (4.9) that $e(R,\varphi) = 0$ on Δ_0, which in turn implies U is constant on Δ_0. To this end (4.9) and (4.21) imply

$$0 \leq E(\bar{R},\underline{R}) = F(\bar{R}) - F(\underline{R}) \leq F(\bar{R}) \ , \ \bar{R}\in(\underline{R},\infty) \ . \tag{4.22}$$

Squaring (4.22) and using (4.11) and (4.13) yields the chain of inequalities

$$\begin{aligned}
&0 \leq E^2(\bar{R},\underline{R}) \leq F^2(\bar{R}) \\
&\leq [\int_0^\pi \bar{R}((\frac{\partial U}{\partial R}(\bar{R},\varphi))^2 2W'(3+\bar{R}^2)d\varphi][\int_0^\pi \bar{R}(U(\bar{R},\varphi))^2 2W'(3+\bar{R}^2)d\varphi] \\
&\leq [\frac{\partial E}{\partial \bar{R}}(\bar{R},\underline{R})][\int_0^\pi \bar{R}(U(\bar{R},\varphi))^2 2W'(3+\bar{R}^2)d\varphi] \\
&= [\frac{\partial E}{\partial \bar{R}}(\bar{R},\underline{R})] \ \frac{1}{2W'(3+\bar{R}^2)\bar{R}} \int_0^\pi [2W'(3+\bar{R}^2)\bar{R}U(\bar{R},\varphi)]^2 d\varphi \ , \\
&\leq \frac{\partial E}{\partial \bar{R}}(\bar{R},\underline{R}) \ \frac{m^2\pi}{2W'(3+\bar{R}^2)\bar{R}},
\end{aligned} \tag{4.23}$$

where

$$m = \sup_{\substack{\bar{R}\in(\underline{R},\infty) \\ \varphi\in[0,\pi]}} 2W'(3+\bar{R}^2)\bar{R}|U(\bar{R},\varphi)| \geq 0. \tag{4.24}$$

Again, note that the existence of m (as a finite real number) is assured by (4.4). The differential inequality (4.23) implies $E(\bar{R},\underline{R})=0$ for all $0<\underline{R}<\bar{R}$. For if there exists $R_1>\underline{R}$ such that $E(R_1,\underline{R})>0$, then (4.11) implies $E(\bar{R},\underline{R})\geq E(R_1,\underline{R})$ for all $\bar{R}\geq R_1$ so that for any $R_2>R_1$, integrating the following rearrangement of the differential inequality (4.23)

$$2W'(3+\bar{R}^2)\bar{R} \le m^2\pi \; \frac{\frac{\partial E}{\partial \bar{r}}(\bar{R},\underline{R})}{E^2(\bar{R},\underline{R})}, \tag{4.25}$$

gives

$$W(3+\bar{R}^2)\Big|_{\bar{R}=R_1}^{\bar{R}=R_2} \le m^2\pi\Big[-\frac{1}{E(\bar{R},\underline{R})}\Big]\Big|_{\bar{R}=R_1}^{\bar{R}=R_2}$$

$$= m^2\pi\Big[\frac{1}{E(R_1,\underline{R})} - \frac{1}{E(R_2,\underline{R})}\Big] \le \frac{m^2\pi}{E(R_1,\underline{R})} \tag{4.26}$$

or equivalently,

$$0 < E(R_1,\underline{R}) \le \frac{m^2\pi}{W(3+R_2{}^2)-W(3+R_1{}^2)}. \tag{4.27}$$

Inequality (4.27) is violated if $m = 0$ or can be violated if $m > 0$ by taking R_2 sufficiently large. Thus, $E(\bar{R},\underline{R})=0$ for all $0<\underline{R}<\bar{R}$, and the theorem is proved.

Now, if two solutions U_1 and U_2 to problem $P(U)$ differ by a constant and upon inversion yield two solutions u_1 and u_2 to problem $P_2(u)$, then u_1 and u_2 must differ by that same constant. A glance at (3.3) shows that the functions relating points in the physical plane to points in the hodograph plane are invariant and respect to the addition of a constant to a solution U. In addition (3.4) shows that adding a constant to a solution U results in adding that same constant to the function u. Thus, two solutions to problem $P_2(u)$ can differ by at most a constant, completing the proof of uniqueness for the solution of problem $P_2(u)$.

5. Estimates for the Modulus of the Displacement Gradient

The uniqueness theorem for problem $P_2(u)$ would be considerably more valuable if condition (2.17) could be derived instead of assumed. Recall that condition (2.17) made it possible to determine the region Δ into which the region $\tilde{D}$ is transformed by (3.1). The present section investigates the possibility of dropping condition (2.17) in problem $P_2(u)$. The principal tool used in this investigation is a comparison principle for second order quasi-linear elliptic operator (a discussion of comparison principles may be found in [29] and [38]). Comparison principles have been used extensively in the theory of subsonic flow and in some cases have been used to establish the uniqueness of boundary-value problems directly, i.e. without the use of the hodograph transformation, (see, for example, [43] and [44] and the references cited therein). They have recently been used in finite anti-plane shear problems to derive boundary stress estimates. In [45], for example, an estimate for the nonvanishing component of the stress tensor τ_{31} at the traction free long sides of a semi-infinite strip loaded at the short side was derived. In [46] an estimate of $\tau = (\tau_{31}{}^2 + \tau_{32}{}^2)^{1/2}$ at the free surface of a circular hole in an infinite body was derived. In what follows, a stress estimate like the preceding examples is derived along the crack faces, and this in turn yields information concerning the behavior of $|\nabla u|$ along the crack faces. Certain facts about the behavior of $|\nabla u|$ as the crack tip is approached are also derived. The investigation, however, falls just short of its ultimate goal of deriving (2.17).

Denote by problem $P_3(u)$ problem $P_2(u)$ with condition (2.17) deleted and (2.16) replaced by

$$u = k(2cr)^{1/2} \sin\frac{\vartheta}{2} + o(1)\ ,$$

$$u_{,1} = -kc(2cr)^{-1/2}\sin\frac{\vartheta}{2} + o(\frac{1}{r}),\ u_{,2} = kc(2cr)^{-1/2}\cos\frac{\vartheta}{2} + o(\frac{1}{r}), \quad (5.1)$$

all as $r \to \infty$, uniformly in ϑ.

These slightly stronger conditions are needed in order to apply the comparison principle mentioned previously. One can verify that the solution to $P_2(u)$ given by (3.23) and (3.24) satisfies (5.1).

Application of the comparison principle requires Dirichlet boundary conditions, and therefore it cannot be applied directly to problem $P_3(u)$. However, as done in [45] and [46] problem $P_3(u)$ for a uniformly elliptic material with a bounded shear modulus M(k) can be converted to an equivalent problem for a related function v which is of the Dirichlet type. For a uniformly elliptic material with unbounded M(k) the conversion cannot be achieved without a further restriction on u near the crack tip. It follows from the second of (2.15) that there is a function $v \in C^1(\tilde{D}^+) \cap C^1(\tilde{D}^-) \cap C^2(\tilde{D})$ which satisfies

$$\tau_{3\alpha} = 2W'(3 + |\nabla u|^2)u_{,\alpha} = \varepsilon_{\alpha\beta}v_{,\beta} \text{ on } \tilde{D}. \quad (5.2)$$

The function v admits an explicit representation in terms of the path independent integral[18]

$$v(x_1,x_2) = \int_{(x_1,x_2)}^{(\overset{o}{x}_1,\overset{o}{x}_2)} [2W'(3+|\nabla u(\bar{x}_1,\bar{x}_2)|^2)u_{,2}(\bar{x}_1,\bar{x}_2)d\bar{x}_1 \quad (5.3)$$

$$- 2W'(3+|\nabla u(\bar{x}_1,\bar{x}_2)|^2)u_{,1}(\bar{x}_1,\bar{x}_2)d\bar{x}_2],$$

[18] The path independence of the integral (5.3) follows from the second of (2.15).

where $(\overset{o}{x}_1, \overset{o}{x}_2)$ is an arbitrary fixed reference point at which $v(\overset{o}{x}_1, \overset{o}{x}_2)$ is assigned the value zero. From (5.2) and (1.20), one infers

$$\hat{\tau}(|\nabla u|) = |\nabla v|, \tag{5.4}$$

which upon inversion yields

$$|\nabla u| = \hat{k}(|\nabla v|). \tag{5.5}$$

Equations (5.2) can now be rewritten as

$$u_{,\alpha} = \frac{1}{2W'(3+\hat{k}^2(|\nabla v|))}\varepsilon_{\alpha\beta}v_{,\beta}. \tag{5.6}$$

It then follows that v satisfies the differential equation

$$Qv \equiv [\frac{1}{2W'(3+\hat{k}^2(|\nabla v|))}v_{,\alpha}]_{,\alpha} = 0 \tag{5.7}$$

which is elliptic on $\tilde{D}$ since (1.22) and (1.24) hold. For future purposes it is convenient to express (5.1), (5.2), and (5.7) in polar coordinates (2.13). These expressions are, respectively,

$$\left.\begin{array}{l} u = Ar^{1/2}\sin\frac{\vartheta}{2} + o(1), \\ \frac{\partial u}{\partial r} = \frac{A}{2}r^{-1/2}\sin\frac{\vartheta}{2} + o(\frac{1}{r}),\ \frac{\partial u}{\partial \vartheta} = \frac{A}{2}r^{1/2}\cos\frac{\vartheta}{2} + o(1), \\ \text{all as } r\to\infty, \text{ uniformly in } \vartheta\in[-\pi,\pi], \text{ where } A = k(2c)^{1/2}, \end{array}\right\} \tag{5.8}$$

$$\left.\begin{array}{l} 2W'(3+|\nabla u|^2)\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \vartheta}, \\ 2W'(3+|\nabla u|^2)\frac{1}{r}\frac{\partial u}{\partial \vartheta} = -\frac{\partial v}{\partial r}, \end{array}\right\} \tag{5.9}$$

$$Qv \equiv \frac{\partial}{\partial r}\left[\frac{1}{2W'(3+\hat{k}^2(|\nabla v|))}\frac{\partial v}{\partial r}\right] + \frac{1}{r}\frac{1}{2W'(3+\hat{k}^2(|\nabla v|))}\frac{\partial v}{\partial r} \tag{5.10}$$

$$+\frac{1}{r^2}\frac{\partial}{\partial \vartheta}\left[\frac{1}{2W'(3+\hat{k}^2(|\nabla v|))}\frac{\partial v}{\partial \vartheta}\right] = 0,$$

The free surface condition $u_{,2}(x_1,0^{\pm})$, $x_1<0$, along with (5.2) implies

$$v_{,1}(x_1,0^{\pm}) = 0,\ x_1<0, \tag{5.11}$$

so that

$$v(x_1,0^+) = c^+,\ v(x_1,0^-) = c^-,\ x_1<0, \tag{5.12}$$

where c^+ and c^- are constants. That $c^+ = c^-$ can be shown as follows. Let C be the closed contour consisting of two circles $C_{\bar{r}}$ and $C_{\underline{r}}$ of radii $\bar{r}$ and $\underline{r}$, respectively, with $\bar{r} > \underline{r} > 0$ centered at the crack tip and regarded as slit by the negative x_1-axis, joined by a line segment Γ^+ running along the upper crack face and a line segment Γ_- running along the lower crack face. Let C be oriented as shown in Figure 5 and let $A_{\underline{r}}^{\bar{r}}$ be the area enclosed by C. Integrating the second of (2.15) over $A_{\underline{r}}^{\bar{r}}$ and using the divergence theorem with the third of (2.15) yields

$$0 = \int_{A_{\underline{r}}^{\bar{r}}}[2W'(3+|\nabla u|^2)u_{,\alpha}]_{,\alpha}dA = \int_{C}[2W'(3+|\nabla u|^2)u_{,\alpha}]n_\alpha ds \tag{5.13}$$

$$= \int_{C_{\bar{r}}} 2W'(3+|\nabla u|^2)u_{,\alpha}n_\alpha\, ds + \int_{C_{\underline{r}}} 2W'(3+|\nabla u|^2)u_{,\alpha}n_\alpha ds,$$

where n_α are the components of the outward normal to $A_{\underline{r}}^{\bar{r}}$. The conditions (5.8) imply

$$|\nabla u(r,\vartheta)| = \frac{A^2}{4r} + o(r^{-3/2}),\ \text{as } r\to\infty,\ \text{uniformly in } \vartheta\in[-\pi,\pi], \tag{5.14}$$

so that

$$2W'(3+|\nabla u(r,\vartheta)|^2) = \mu + 0(\frac{1}{r}),\text{ as } r\to\infty,\text{uniformly in } \vartheta\in[-\pi,\pi], \qquad (5.15)$$

and thus

$$\int_{c_{\bar r}} 2W'(3+|\nabla u|^2)u_{,a}n_a ds = \int_{\pi}^{-\pi} 2W'(3+|\nabla u(\bar r,\vartheta)|^2)\ \frac{\partial u}{\partial r}(\bar r,\vartheta)\bar r d\vartheta$$

$$= \mu \int_{\pi}^{-\pi}[1+O(\frac{1}{\bar r})][\frac{A}{2}(\bar r)^{-1/2}\sin\frac{\vartheta}{2}+o(\frac{1}{\bar r})]\bar r d\vartheta \qquad (5.16)$$

$$= \mu\int_{\pi}^{-\pi}[\frac{A}{2}(\bar r)^{1/2}\sin\frac{\vartheta}{2} + o(1)]d\vartheta = o(1)$$

all as $\bar r\to\infty$, uniformly in $\vartheta\in[-\pi,\pi]$.

(5.16) and (5.13) then imply

$$\int_{-\pi}^{\pi} 2W'(3+|\nabla u(r,\vartheta)|^2)r\ \frac{\partial u}{\partial r}(r,\vartheta)d\vartheta = 0,\text{ for all } r>0. \qquad (5.17)$$

Now, (5.12) is equivalent to the polar coordinate form $v(r,\pi) = c^+$, $v(r,-\pi) = c^-$, $r>0$. This implies

$$\left.\begin{aligned} \int_{-\pi}^{\vartheta} \frac{\partial v}{\partial \vartheta}(r,\bar\vartheta)d\bar\vartheta &= v(r,\vartheta)-c^-,\\ \int_{\vartheta}^{\pi} \frac{\partial v}{\partial \vartheta}(r,\bar\vartheta)d\bar\vartheta &= -v(r,\vartheta)+c^+, \end{aligned}\right\}\text{for all } r>0. \qquad (5.18)$$

Adding the above two equations, making use of the first of (5.9), and then (5.17) gives

$$c^+ - c^- = \int_{-\pi}^{\pi} \frac{\partial v}{\partial \vartheta}(r,\bar{\vartheta})d\bar{\vartheta} = \int_{-\pi}^{\pi} 2W'(3+|\nabla u(r,\bar{\vartheta})|^2)r \, \frac{\partial u}{\partial r}(r,\bar{\vartheta})d\bar{\vartheta} = 0. \qquad (5.19)$$

Thus, $c^+ = c^-$ as claimed. For convenience, the reference point $(\overset{o}{x}_1,\overset{o}{x}_2)$ will be taken to be a point on the upper crack face. It then follows that

$$v(x_1,0^+) = v(x_1,0^-) = 0 \ , \ x_1<0. \qquad (5.20)$$

It would be desirable to show for all uniformly elliptic materials that

$$\lim_{\substack{(x_1,x_2)\to(0,0) \\ (x_1,x_2)\in\bar{D}}} v(x_1,x_2) = 0, \qquad (5.21)$$

so that v would then satisfy

$$v(x_1,0) = 0, \ x_1 \le 0. \qquad (5.22)$$

While (5.21) can be derived for a uniformly elliptic material with bounded shear modulus M(k), it cannot be derived for a uniformly elliptic material with unbounded M(k). For this case (5.21) must be assumed, and this in turn puts an additional restriction on the behavior of ∇u near the origin. Fortunately, the solution to problem $P_3(u)$ given by (3.23) and (3.24) does yield a function v which satisfies (5.21).

To determine $v(x_1,x_2)$ choose a path between (x_1,x_2) and $(\overset{o}{x}_1,\overset{o}{x}_2)$ like this: go from (x_1,x_2) to a point on the upper crack face along a circular arc Γ_r centered at the origin of radius r and then proceed to $(\overset{o}{x}_1,\overset{o}{x}_2)$ along the upper crack face. This linear portion of the path contributes nothing to the integral (5.3) defining v due to the free surface condition. Thus, in terms of polar coordinates, v is given by

$$v(r,\vartheta) = -\int_{\vartheta}^{\pi} 2W'(3+|\nabla u(r,\bar{\vartheta})|^2)r \frac{\partial u}{\partial r}(r,\bar{\vartheta})d\bar{\vartheta}. \qquad (5.23)$$

If M(k) is bounded, Claim 3, Section 2, and (5.23) above imply (5.21). For any uniformly elliptic material the solution given by (3.23) and (3.24) can be shown to satisfy (5.21) as follows:

$$2W'(3+|\nabla u(r,\vartheta)|^2)r\left|\frac{\partial u}{\partial r}(r,\vartheta)\right| \le 2W'(3+|\nabla u(r,\vartheta)|^2)\ |\nabla u(r,\vartheta)|r \qquad (5.24)$$

$$= \hat{\tau}(|\nabla u(r,\vartheta)|)r = \hat{\tau}(R)\hat{r}(R,\varphi),$$

where (R,ϑ) are polar coordinates in the hodograph plane determined by (3.23) and $\hat{\tau}$ is given by (1.20). Using (3.23) gives

$$\hat{r}^2(R,\varphi) = [I(R)+RI'(R)\cos^2\varphi]^2 + [RI'(R)\sin\varphi\cos\varphi]^2 \qquad (5.25)$$

$$\le I^2(R) + 2R^2[I'(R)]^2,$$

since I(R) given by (3.18) has the properties I(R) > 0 and I'(R) < 0 for all $R > 0$. Simple computation of $I'(R)$ and (3.19) yield

$$I^2(R)+2R^2[I'(R)]^2 \le [\frac{\mu ck^2}{R\hat{\tau}(R)}]^2 + 2R^2[-\frac{\mu ck^2}{R^2}\ \frac{1}{\hat{\tau}(R)}]^2 = 3[\frac{\mu ck^2}{R\hat{\tau}(R)}]^2. \quad (5.26)$$

(5.25) and (5.26) together yield

$$\hat{r}(R,\varphi)\hat{\tau}(R) \le \frac{\mu ck^2\sqrt{3}}{R} = o(1),\ \text{as}\ R\to\infty,\ \text{uniformly in}\ \varphi\in[0,\pi], \qquad (5.27)$$

which upon inversion to the physical plane implies

$$\hat{\tau}(|\nabla u(r,\vartheta)|)r = o(1)\ \text{as}\ r\to 0,\ \text{uniformly in}\ \vartheta\in[-\pi,\pi]. \qquad (5.28)$$

Finally, (5.28) along with (5.23) imply (5.21), the desired result.

The conversion of the condition (5.8) as r $\to\infty$ to a condition on v is straight forward. Integration of the first of (5.9) from $-\pi$ to ϑ yields

$$v(r,\vartheta) - v(r,-\pi) = \int_{-\pi}^{\vartheta} \frac{A\mu}{2} r^{1/2} \sin\frac{\bar{\vartheta}}{2} d\bar{\vartheta} \tag{5.29}$$

$$+ \int_{-\pi}^{\vartheta} [2W'(3+|\nabla u(r,\bar{\vartheta})|^2) r \frac{\partial u}{\partial r}(r,\bar{\vartheta}) - \frac{A\mu}{2} r^{1/2}\sin\frac{\bar{\vartheta}}{2}] d\bar{\vartheta}.$$

Now, (5.14), (5.15), and (5.8) imply

$$2W'(3+|\nabla u(r,\vartheta)|^2) r \frac{\partial u}{\partial r}(r,\vartheta) - \frac{A\mu}{2} r^{1/2}\sin\frac{\vartheta}{2} = o(1), \tag{5.30}$$

as $r\to\infty$, uniformly in $\vartheta\in[-\pi,\pi]$,

so that the second integral appearing in (5.29) is o(1) as $r\to\infty$, uniformly in $\vartheta\in[-\pi,\pi]$. This combined with the fact that $v(r,-\pi) = 0$ gives the following far field condition for v:

$$v(r,\vartheta) = -A\mu r^{1/2}\cos\frac{\vartheta}{2} + o(1), \text{ as } r\to\infty, \text{ uniformly in } \vartheta\in[-\pi,\pi]. \tag{5.31}$$

Thus, the differential equation (5.10) together with the boundary conditions (5.22) and (5.31) compose a boundary-value problem for v on $\tilde{D}$, henceforth referred to as problem $P(v)$.

Having arrived at a Dirichlet problem for v, the stage is now set for the application of a comparison principle for second order quasi-linear elliptic operators, namely Theorem 9.2, p. 207, of [29]. A version of that theorem sufficient for the present purpose is as follows:

<u>Comparison</u> <u>Principle.</u> Let Ω be a bounded domain in $\mathbb{R}^2$. Let $Qw \equiv a_{\alpha\beta}(w_{,1},w_{,2})w_{,\alpha\beta}$ be elliptic at every point in Ω and for every function $w \in C^2(\Omega)$ where $a_{\alpha\beta} \in C^1(\mathbb{R}^2)$. If $u,v \in C(\bar{\Omega}) \cap C^2(\Omega)$ satisfy $Qu \geq Qv$ in Ω and $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω.[19]

Define the comparison function w^- by

$$\left.\begin{aligned} w^-(x_1,x_2) &= -A\mu\tilde{r}^{1/2}(x_1,x_2)\cos\frac{\tilde{\vartheta}(x_1,x_2)}{2} \text{ on } \tilde{D}, \\ w^-(x_1,x_2) &= 0 \text{ on } \partial\tilde{D}, \end{aligned}\right\} \tag{5.32}$$

where $r=\tilde{r}(x_1,x_2)$ and $\vartheta=\tilde{\vartheta}(x_1,x_2)$ are the polar coordinates (2.13) of x_1 and x_2 and $\partial\tilde{D}=\{(x_1,x_2)|\ x_1 \leq 0,\ x_2=0\}$. Note $w^- \in C(\overline{\tilde{D}}) \cap C^2(\tilde{D})$ and $w^-=v$ on $\partial\tilde{D}$. Using the definition of the quasi-linear elliptic operator Q given by (5.10), it is easily shown that

$$Qw^- = \psi\left(\frac{A\mu}{2r^{1/2}}\right) \frac{1}{2W'\left(3+\hat{k}^2\left(\frac{A\mu}{2r^{1/2}}\right)\right)} \frac{A\mu}{4r^{3/2}} \cos\frac{\vartheta}{2} \text{ on } \tilde{D}, \tag{5.33}$$

where

$$\psi(\tau) = -\frac{4W''(3+\hat{k}^2(\tau))}{2W'(3+\hat{k}^2(\tau))}\tau\hat{k}(\tau)\hat{k}'(\tau). \tag{5.34}$$

The properties of the function $\hat{k}(\tau)$, (1.22), and the definitions of a hardening and softening material imply that $\psi(\tau)$ has the following two properties:

$$\left.\begin{aligned} &(i) \text{ for a hardening material, } \psi(\tau)<0 \text{ for all } \tau>0, \\ &(ii) \text{ for a softening material, } \psi(\tau)>0 \text{ for all } \tau>0. \end{aligned}\right\} \tag{5.35}$$

[19] Ω and $\partial\Omega$ denote the closure and boundary, respectively, of the point set Ω. It can be shown that $\Omega = \Omega \cap \partial\Omega$.

For a power law material whose strain energy function W is given by (1.26),

$$\Psi(k) = \psi(\hat{\tau}(k)) = 2(1-n)\frac{bk^3}{n+(2n-1)bk^2} , \tag{5.36}$$

which for a hardening material (n > 1) is negative for all k and for a softening material ($\frac{1}{2} \le n < 1$) is positive for all k. Henceforth restricting the analysis involving w^- to hardening materials, (5.33) - (5.35) imply

$$Qw^- < 0 \text{ on } \tilde{D}. \tag{5.37}$$

Since $v(r,\vartheta) - w^-(r,\vartheta) = o(1)$ as $r \to \infty$, uniformly in $\vartheta \in [-\pi,\pi]$, there exists for any given $\varepsilon > 0$, a number $r_\varepsilon > 0$ such that

$$w^-(r,\vartheta) + \varepsilon \ge v(r,\vartheta), \text{ for } r \ge r_\varepsilon,\ -\pi \le \vartheta \le \pi. \tag{5.38}$$

Let $\tilde{D}_\varepsilon = \tilde{D} \cap \{(x_1,x_2) | 0 < (x_1^2+x_2^2)^{1/2} < r_\varepsilon\}$. Then, it easily follows that

$$w^-(r,\vartheta) + \varepsilon \ge v(r,\vartheta) \text{ on } \partial\tilde{D}_\varepsilon, \tag{5.39}$$

where $\partial\tilde{D}_\varepsilon = \{(x_1,x_2) |\ (x_1^2+x_2^2)^{1/2} = r_\varepsilon \text{ or } -r_\varepsilon \le x_1 \le 0,\ x_2 = 0\}$. Also, (5.10) and (5.37) imply

$$Q(w^- + \varepsilon) = Qw^- < 0 \text{ on } \tilde{D}. \tag{5.40}$$

(5.39) and (5.40), the assumption that v satisfies problem $P(v)$, and the comparison principle imply

$$w^-(r,\vartheta) + \varepsilon \ge v(r,\vartheta) \text{ on } \overline{\tilde{D}}_\varepsilon, \tag{5.41}$$

which along with (5.38) implies

$$w^-(r,\vartheta) + \varepsilon \ge v(r,\vartheta) \text{ on } \overline{\tilde{D}}. \tag{5.42}$$

Since $\varepsilon > 0$ is arbitrary, it follows from (5.42) that

$$w^{-}(r,\vartheta) \geq v(r,\vartheta) \text{ on } \overline{\tilde{D}}. \tag{5.43}$$

This inequality will be analyzed later concerning its implications for the behavior of $|\nabla u|$ as the origin is approached from within $\tilde{D}$. To derive boundary gradient and stress estimates from (5.43) let $\Delta>0$ be given and formulate the difference quotient

$$\frac{v(r,\pi) - v(r,\pi-\Delta)}{\Delta} = -\frac{v(r,\pi-\Delta)}{\Delta} \geq A\mu r^{1/2}\frac{\cos\frac{\pi-\Delta}{2}}{\Delta}, \; r>0. \tag{5.44}$$

Taking the limit as Δ goes to zero through positive values yields

$$\frac{1}{r}\frac{\partial v}{\partial \vartheta}(r,\pi) \geq \frac{A\mu}{2} r^{-1/2}, \; r>0. \tag{5.45}$$

In view of (5.5), this implies $\lim\limits_{r\to 0} |\nabla u(r,\pi)| = \infty$. Repeating the above procedure at $\vartheta=-\pi$ yields $\lim\limits_{r\to 0} |\nabla u(r,-\pi)| = \infty$. an estimate for the stress τ_{31} on the upper crack face is obtained by combining (5.45), (2.3), and the third of (2.15) with the first of (5.9).

$$-\tau_{31}(r,\pi) = 2W'(3+(\frac{\partial u}{\partial r}(r,\pi))^2)\frac{\partial u}{\partial r}(r,\pi) \geq \frac{A\mu}{2r^{1/2}} \tag{5.46}$$

$$= -\tau_{31}^{neo-Hookean}(r,\pi), \; r>0,$$

where $\tau_{31}^{neo-Hookean}$ is the τ_{31} component of the stress tensor associated with the solution of problem $P_3(u)$ given by (3.23) and (3.24) for a neo-Hookean material. Thus, (5.46) is a statement of the physically plausible result that $|\tau_{31}|$ along the upper crack face is larger for a hardening material than for a neo-Hookean material.

To derive estimates similar to (5.45) and (5.46) for a softening material, one must consider the comparison function $w^{+}\in C(\overline{\tilde{D}})\cap C^{2}(\tilde{D})$ defined by

$$w^{+}(x_1,x_2) = A\mu \tilde{r}^{1/2}(x_1,x_2)\cos\frac{\tilde{\vartheta}(x_1,x_2)}{2} \text{ on } \tilde{D}, \tag{5.47}$$

$$w^{+}(x_1,x_2) = 0 \text{ on } \partial\tilde{D},$$

where, as before, $r = \tilde{r}(x_1,x_2)$ and $\vartheta = \tilde{\vartheta}(x_1,x_2)$ are the polar coordinates (2.13) of x_1 and x_2. It can be easily shown that

$$Qw^{+} = -\psi\left(\frac{A\mu}{2r^{1/2}}\right)\frac{1}{2W'(3+\hat{k}^{2}(\frac{A\mu}{2r^{1/2}}))}\frac{A\mu}{4r^{3/2}}\cos\frac{\vartheta}{2} \text{ on } \tilde{D}, \tag{5.48}$$

which along with the second of (5.35) implies

$$Qw^{+}< 0 \text{ on } \tilde{D}, \tag{5.49}$$

for a softening material. In view of (5.31) and (5.47), there exists for any given $\varepsilon>0$, a number $r_{\varepsilon}> 0$ such that

$$w^{+}(r,\vartheta)+\varepsilon \geq -v(r,\vartheta), \text{ for } r\geq r_{\varepsilon},\ -\pi\leq\vartheta\leq\pi. \tag{5.50}$$

Defining $\tilde{D}_{\varepsilon}$ and $\partial\tilde{D}_{\varepsilon}$ as before (see the discussion adjacent to (5.39)) one has

$$w^{+}(r,\vartheta)+\varepsilon \geq -v(r,\vartheta) \text{ on } \partial\tilde{D}_{\varepsilon}. \tag{5.51}$$

Also, (5.10) and (5.49) imply

$$Q(w^{+} + \varepsilon) = Qw^{+}< 0 \text{ on } \tilde{D}, \tag{5.52}$$

while the assumption that v satisfies problem $P(v)$ implies

$$Q(-v) = Qv = 0 \text{ on } \tilde{D}. \tag{5.53}$$

(5.51) and (5.52), the assumption that v satisfies problem $P(v)$, (5.53), and the comparison principle imply

$$w^+(r,\vartheta) + \varepsilon \geq -v(r,\vartheta) \text{ on } \overline{\tilde{D}}_\varepsilon, \tag{5.54}$$

which along with (5.50) and the fact that $\varepsilon>0$ is arbitrary yields

$$v(r,\vartheta) \geq -w^+(r,\vartheta) \text{ on } \overline{\tilde{D}}. \tag{5.55}$$

To obtain an upper bound for v, one observes from (5.47) and (5.31) that there exists a number $r_\varepsilon>0$ such that

$$w^+(r,\vartheta) \geq v(r,\vartheta),\ r\geq r_\varepsilon, -\pi\leq\vartheta\leq\pi, \tag{5.56}$$

so that

$$w^+(r,\vartheta) \geq v(r,\vartheta) \text{ on } \partial\tilde{D}_\varepsilon. \tag{5.57}$$

(5.52), (5.53), (5.56), (5.57), the assumption that v satisfies problem $P(v)$, and the comparison principle imply

$$w^+(r,\vartheta) \geq v(r,\vartheta) \text{ on } \overline{\tilde{D}}. \tag{5.58}$$

Following the same procedure leading to (5.45), (5.55) and (5.58) gives

$$\frac{A\mu}{2r^{1/2}} \geq \frac{1}{r}\frac{\partial v}{\partial\vartheta}(r,\pi) \geq -\frac{A\mu}{2r^{1/2}},\ r>0. \tag{5.59}$$

It must be noted that (5.59) implies that $|\frac{1}{r}\frac{\partial v}{\partial\vartheta}(r,\pi)|$ can "blow up" at a rate no greater than $r^{-1/2}$ as $r \to 0$; $\left|\frac{1}{r}\frac{\partial v}{\partial \vartheta}(r,\pi)\right|$ can remain bounded as $r \to 0$. Thus, whereas for a hardening material it can be concluded that $\left|\frac{1}{r}\frac{\partial v}{\partial \vartheta}(r,\pi)\right|, |\nabla v(r,\pi)|$, and $|\nabla u(r,\pi)|$ all become unbounded as $r \to 0$, no such conclusion can be reached for a softening material. (5.59) and the first of (5.9) yield the estimate for τ_{31} on the upper crack face

$$|\tau_{31}(r,\pi)| \leq \frac{A\mu}{2r^{1/2}} = |\tau_{31}^{neo-Hookean}|. \tag{5.60}$$

The above estimate, which holds for a softening material, is again consistent with physical intuition.

At this point it would be desirable to show that $|\nabla u|$ tends to infinity as the origin is approached from within $\tilde{D}$ using standard techniques for obtaining interior gradient estimates, some of which are described in [29]. Attempts at doing this have so far proved unsuccessful. What can be shown is that along each ray eminating from the origin and extending into $\tilde{D}$ (i.e. for each $\vartheta \in (-\pi,\pi)$) there exists a sequence of points whose polar coordinates are given by $\{(r_n,\vartheta)\}_{n=1}^{\infty}$ tending to the origin with the property that $|\nabla u(r_n,\vartheta)|$ becomes unbounded as n tends to infinity. The proof is based on (5.43) and proceeds as follows. Since ϑ is fixed, $v(r,\vartheta)$ and $\frac{\partial v}{\partial r}(r,\vartheta)$ can be considered as functions of the single variable r. Then, $v \in C([0,\infty)) \cap C^1((0,\infty))$ with $v(0) = 0$. Let $\{(\bar{r}_n,\vartheta)\}_{n=1}^{\infty}$ be the polar coordinates of a sequence of points tending to the origin along a particular ray. Thus, $\{\bar{r}_n\}_{n=1}^{\infty}$ is a sequence of real numbers tending to zero as n tends to infinity. By the mean value theorem

$$v(\bar{r}_n) - v(0) = v(\bar{r}_n) = \frac{\partial v}{\partial r}(r_n)\bar{r}_n, \text{ for some } r_n \in (0,\bar{r}_n). \tag{5.61}$$

It is crucial to remark that r_n in general will depend both on ϑ and $\bar{r}_n$. (5.61) along with (5.43) yields

$$\frac{\partial v}{\partial r}(r_n) = \frac{v(\bar{r}_n)}{\bar{r}_n} \leq -A\mu(\bar{r}_n)^{-1/2}\cos\frac{\vartheta}{2}. \tag{5.62}$$

Since $\lim_{n\to\infty} \bar{r}_n = 0$ and $0<r_n<\bar{r}_n$ hold, $\lim_{n\to\infty} r_n = 0$, implying with (5.62) that $\lim_{n\to\infty} \frac{\partial V}{\partial r}(r_n,\vartheta) = -\infty$. The relation between $|\nabla u|$ and $|\nabla v|$ given by (5.5) and the formula $|\nabla v| = [(\frac{\partial v}{\partial r})^2 + (\frac{1}{r}\frac{\partial V}{\partial \vartheta})^2]^{1/2}$ then imply that $\lim_{n\to\infty}|\nabla u(r_n,\vartheta)| = \infty$.

The preceding result only makes use of the fact that the function v satisfies (5.43). The fact that v is a solution to problem $P(v)$ was not used directly. It may be possible to use (5.43) in conjunction with other properties of solutions to problem $P(v)$ to obtain the desired result. This possibility has not been thoroughly investigated.

Conclusion

The existence and uniqueness of the solution to a more restrictive version of the small-scale nonlinear crack problem first introduced by Knowles [11] has been established. These restrictions allow the uniqueness part of the proof to be carried out using simple techniques. The exact solution to the original version of the small-scale nonlinear crack problem was shown to be an exact solution to the more restrictive version.

The ultimate goal would be to establish the existence and uniqueness of the solution to the original version of the small-scale nonlinear crack problem with the fewest additional restrictions. Dropping the requirement that $|\nabla \mathbf{u}|$ becomes unbounded at the crack tip seems most promising. It would also be desirable to impose a crack tip condition that has a direct physical interpretation such as the requirement of bounded displacement. Perhaps the uniqueness of the solution to the original small-scale nonlinear crack problem can be established directly, without the use of the hodograph transformation, by first applying the theory of quasi-conformal mappings and pseudo-analytic functions to a sufficiently nice portion of the cracked body to avoid difficulties presented by potential singularities and then letting the area of that part increase, eventually covering the entire body. It may also be possible to extend the results obtained in the context of compressible flow over a sufficiently smooth bounded body to the present case. Still other approaches may be possible.

The small-scale nonlinear crack problem is a very difficult one to analyze because of the geometry of the body and the singularities at the crack tip. Although the results of this investigation rely on a large number of hypotheses, some of which ideally should be derived rather than assumed, they do provide a first step toward resolving a very important issue.

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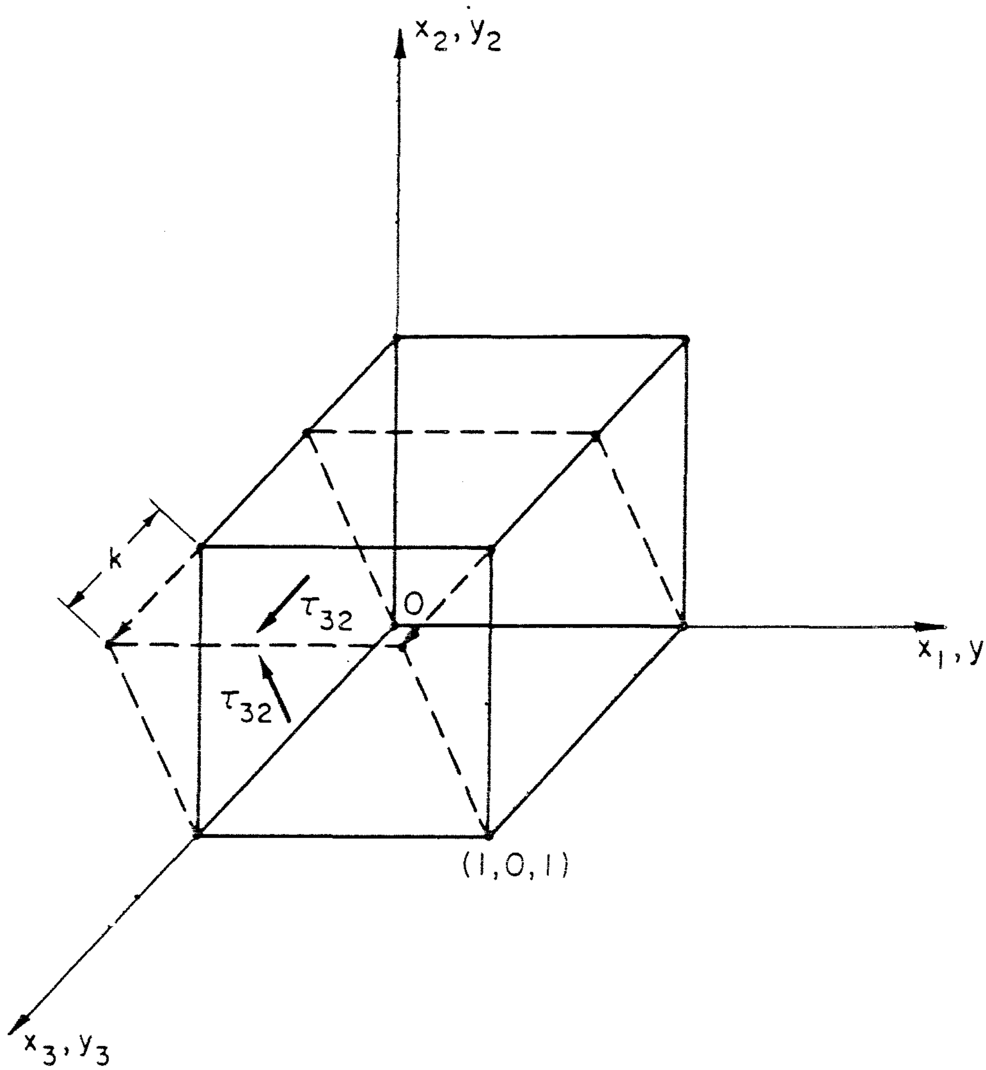

Figure 1. Simple shear of the unit cube.

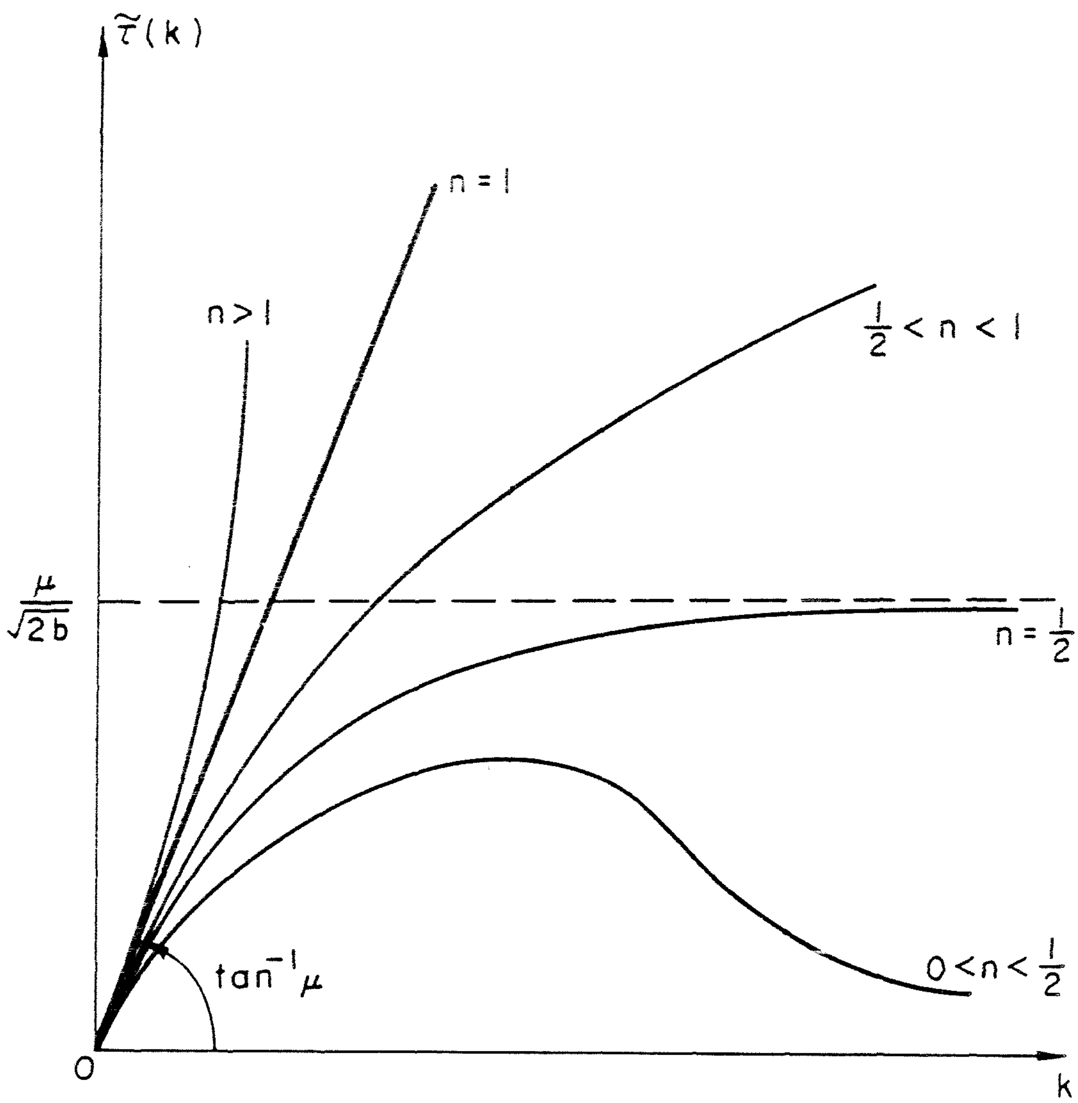

Figure 2. Shear stress $\hat{\tau}$ vs. amount of shear k in simple shear for power law materials.

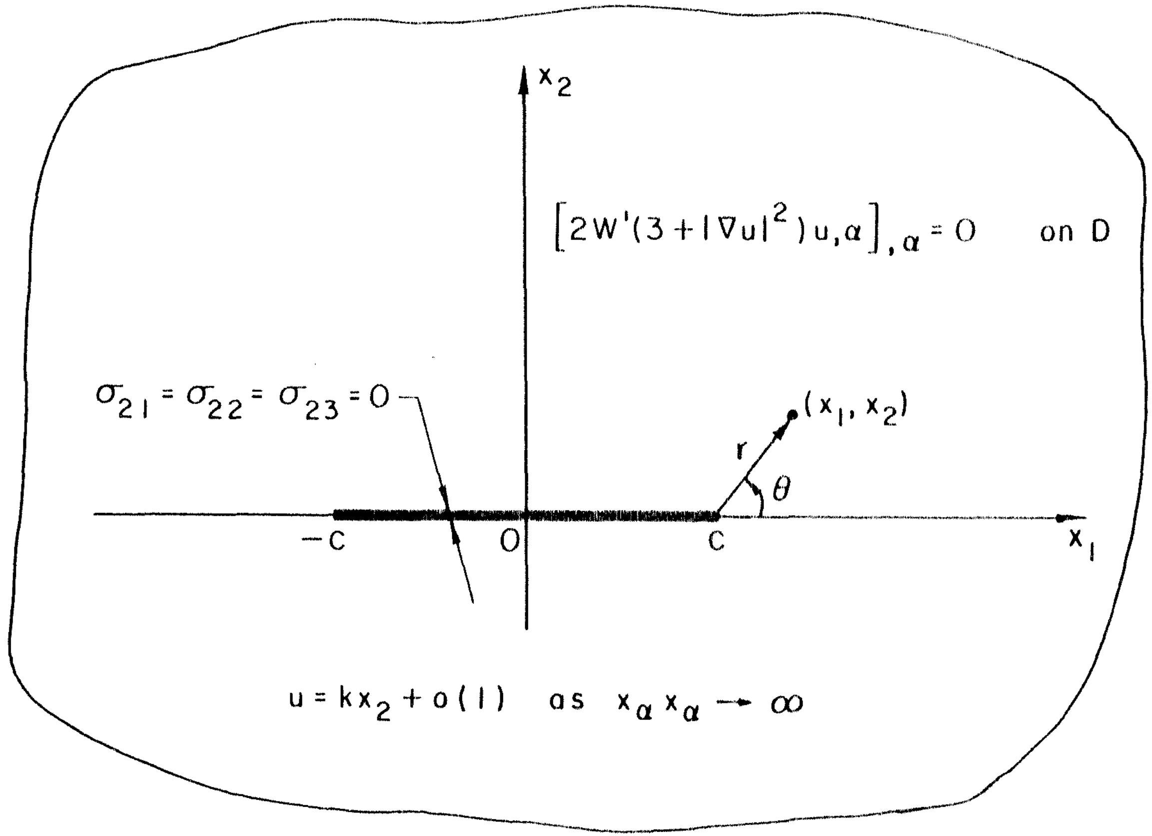

Figure 3. The crack problem.

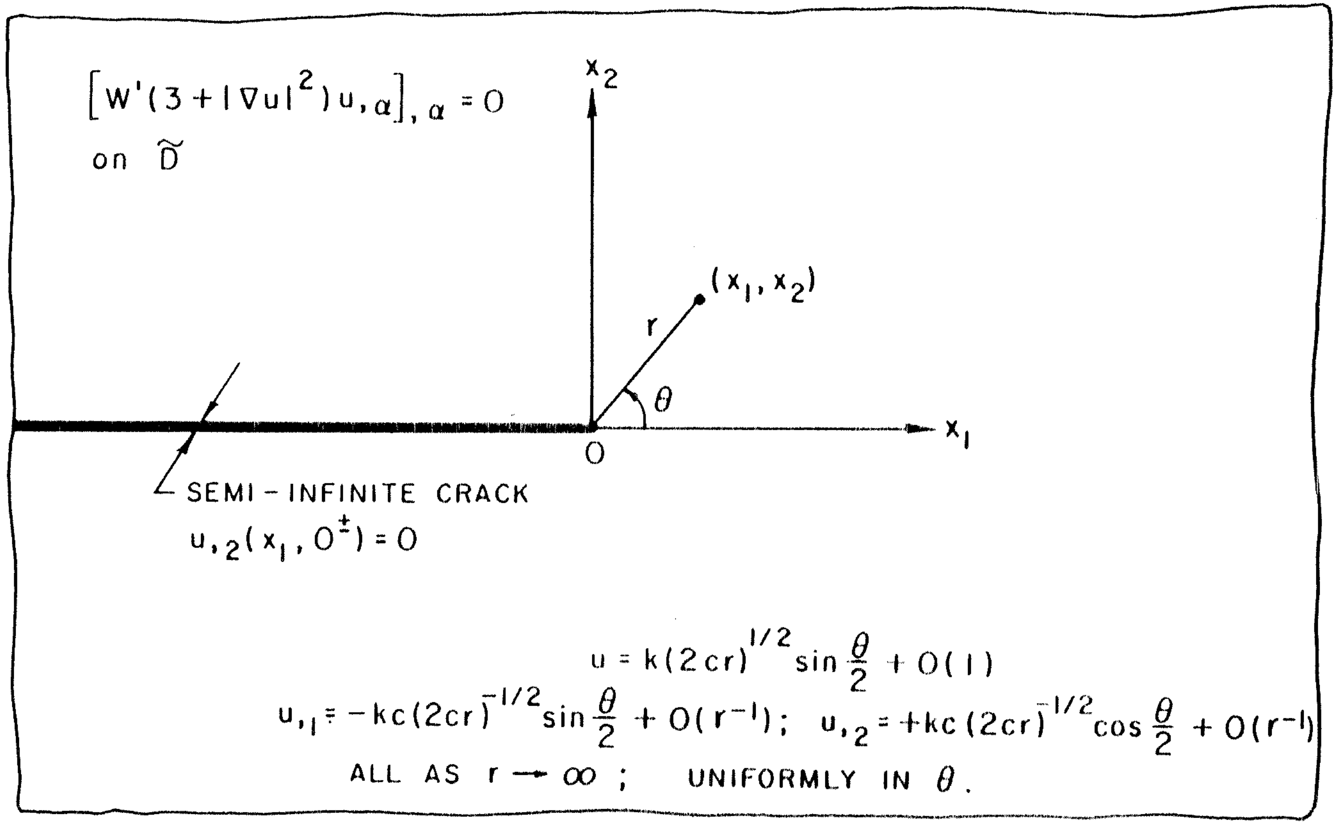

Figure 4. Cross-section of body and coordinates for the small-scale nonlinear crack problem $P_2(u)$.

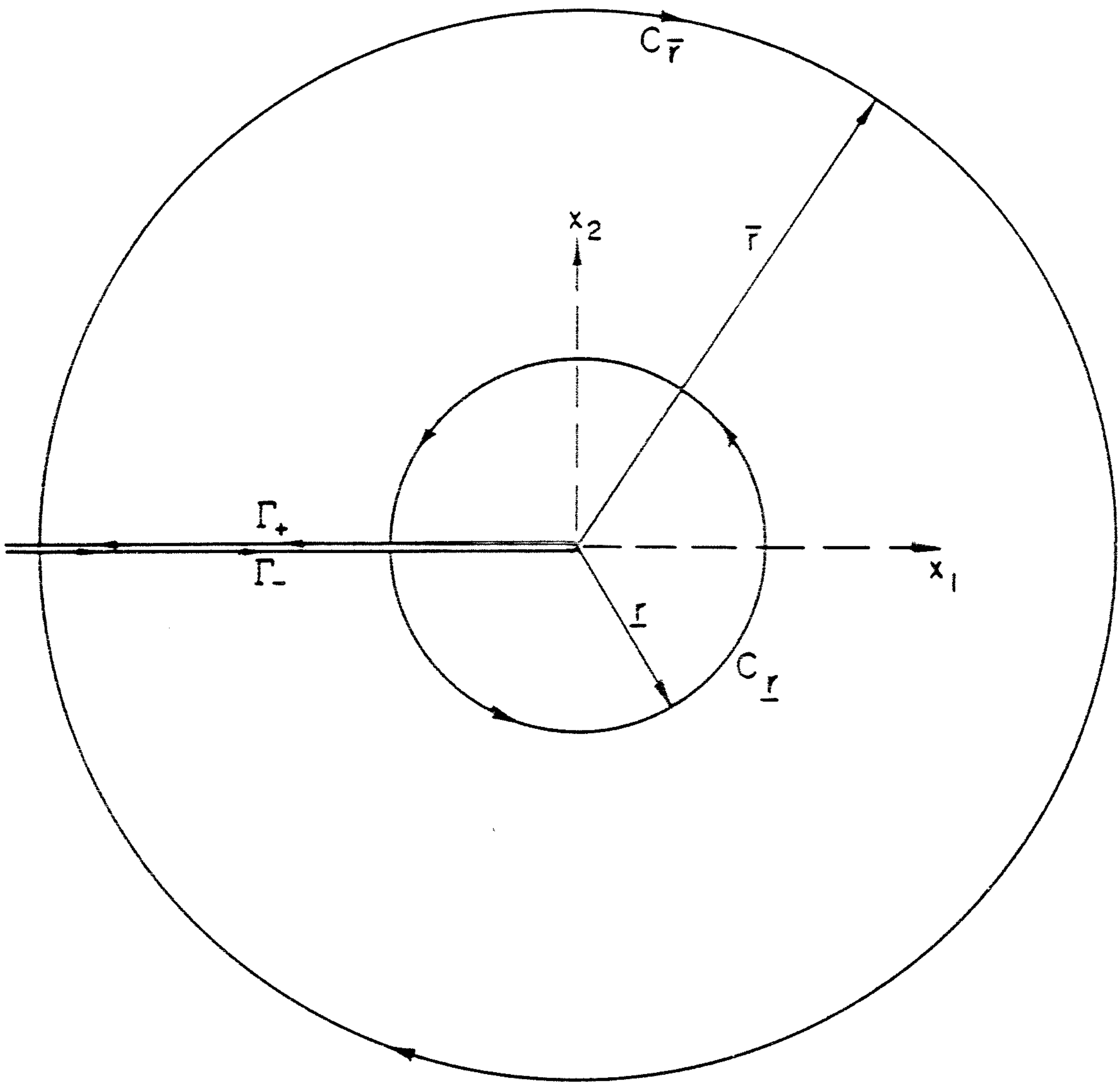

Figure 5. Paths $C_{\underline{r}}$, Γ_-, $C_{\overline{r}}$, and Γ_+ which compose the simple closed curve C.

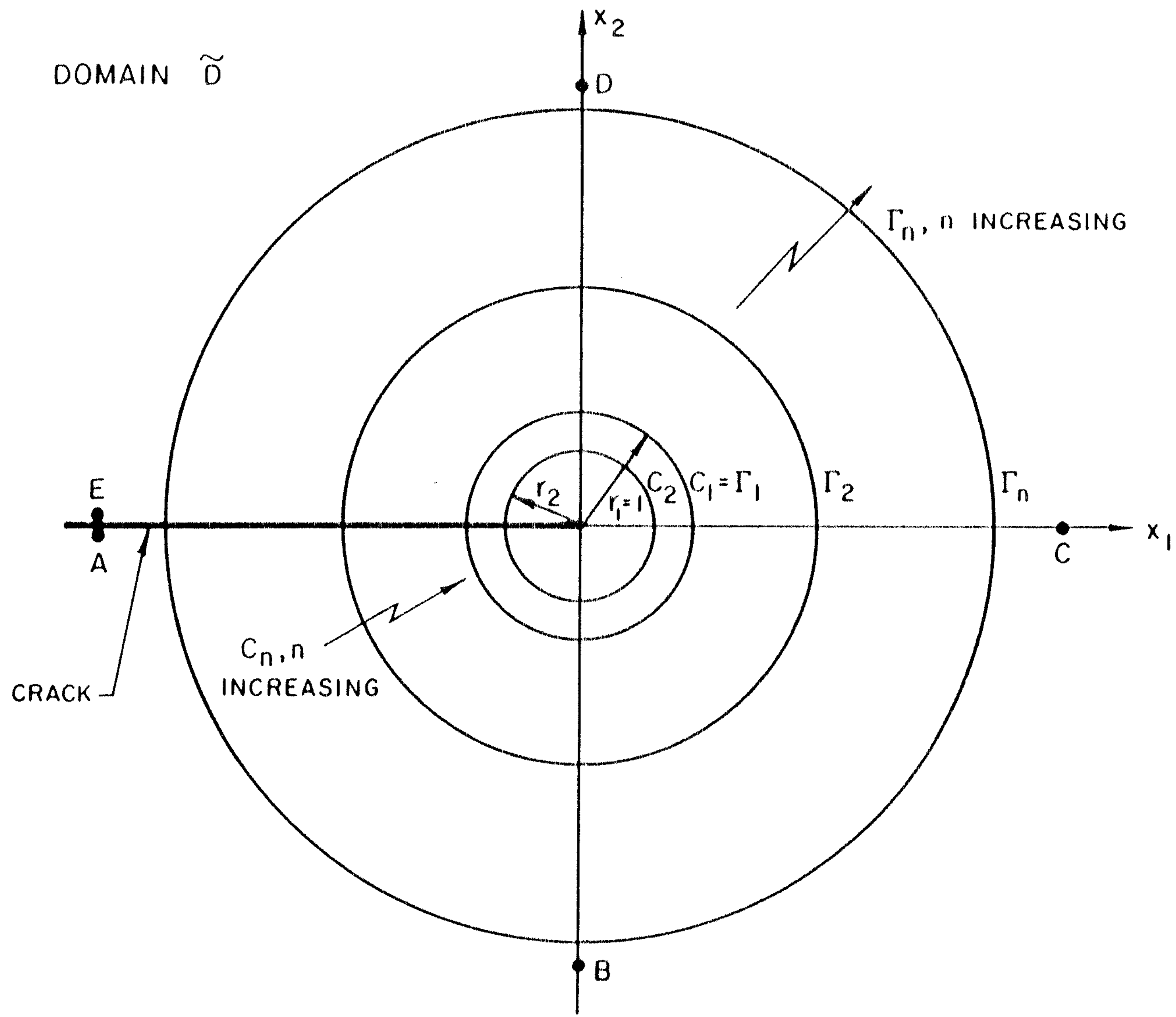

Figure 6. Circles $\{C_n\}$ and $\{\Gamma_n\}$ in the physical plane.

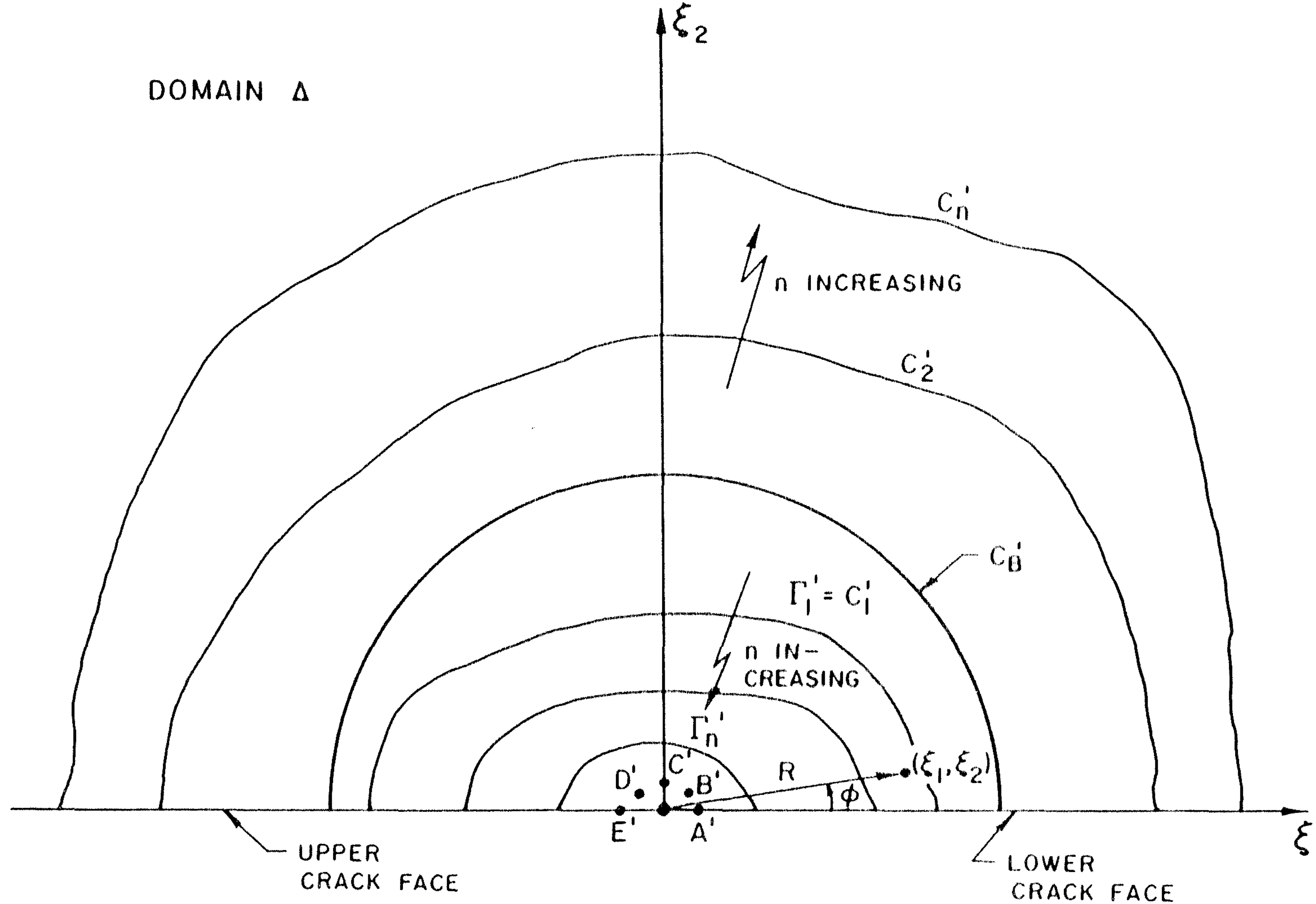

Figure 7. Curves $\{C_n'\}$ and $\{\Gamma_n'\}$ in the hodograph plane.