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IMPLICATIONS OF POSITIVE CURVATURE TO SPECTRUM, ISOPERIMETRIC PROFILE AND TOPOLOGY OF A RIEMANNIAN MANIFOLD

Tesi di Laurea Magistrale in Geometria Riemanniana

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'Geometry, which is the only science that it
hath pleased God hitherto to bestow on mankind'
Thomas Hobbs

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Introduction

The first known isoperimetric problem is due to the legendary founder of Chartage, the Queen Dido. The legend says that Dido, forced to the exile by her brother, arrived to the coast of Tunisia asking the King of North Africa for asylum and a piece of land. The king answered he would have given her as much land as she could mark out with a bull's skin. With an amazing trick, Dido cut the bull's skin into thin strips and sewed them together to get a long string encircling the piece of land where she would have built Chartage.

This problem can be re-read in a rigorous mathematical context in the following way:

'What is the geometrical domain which maximizes the area among all the figures with the same perimeter?'

Intuitively, the answer is the circle domain, known since the Greek and rigorously proved only in the 19th century. By the time, this isoperimetric problem has been extended to higher dimensions and several fields.

In particular, one can consider intrinsically curved spaces, entering the realm of Riemannian Geometry. In this case, the hypothesis on the curvature play a fundamental role leading to different conclusions, or inequalities, in the case it is positive or negative.

In this dissertation we restrict to the positive case looking at the consequences of a strictly positive lower bound of the curvature.

This hypothesis has implications to the first eigenvalue of the Laplacian and to the isoperimetric profile of a Riemannian manifold of arbitrary dimension n. Not surprisingly, both of them are bounded from below by the round sphere of dimension n and, when equality holds there exists an isometry between the sphere and the manifold into consideration.

Since we have a bound from below, we can equivalently say that the objects of our survey are minimized on the $n-$dimensional sphere. In this sense, these results are considered as isoperimetric problems: both the first eigenvalue of the Laplacian and the isoperimetric profile contain information about the geometry of the manifold and we investigate how they can be bounded, finding out that they are minimized by the sphere.

Once this is established, we look at the cases of equality. As already said, it implies an isometry between the sphere and the manifold, therefore we ask how much the condition on the curvature can be weakened to get at least a homeomorphism between them. The answer is that if the sectional curvature K satisfies the condition $\frac{1}{4} < K \leq 1$, and M is simply connected and compact, then it is homeomorphic to a sphere, and since there is a link between the Ricci curvature and the sectional curvature, we get a restriction on our initial hypothesis.

All the material is organized in the following way.

The first four chapters contain all the necessary preliminaries. The reason why this introductory part is so long is that we want to make the reader familiar with the topic and we want to be sure that everything we are going to mention or talk about is already well defined and clear.

In Chapter 1 we describe how we move from a differential context to a Riemannian one giving the basic definitions on Riemannian manifolds such as affine and Levi-Civita connection, the metric function on a Riemannian manifold and the notion of geodesics along with the exponential map, the cut locus and the Theorem of Hopf-Rinow about the completeness. Chapter 2 is about curvature: after the definitions of sectional and Ricci curvature we turn to the variational formulas for length and energy, we state Bonnet-Myers Theorem, which can be seen as a first basic result under our hypothesis on the curvature, and finally, we discuss Jacobi fields. Moreover, as a concrete example, we calculate the sectional curvature of the complex projective space, that will be recalled in the last chapter. In Chapter 3 we show how the volume of a Riemannian manifold can be calculated presenting the complete calculation for the $n-$dimensional round sphere and some comparison theorems. Finally, in Chapter 4, we introduce the Laplace operator along with the eigenvalue problems and useful properties of the eigenvalues of the Laplacian.

After that, we have the main part of the dissertation.

In Chapter 5 we discuss some results about the first eigenvalue of the Laplacian. We first present the calculation of the spectrum of the sphere of radius r of dimension n and Bochner's Formula. Both of them are useful tools for the proof of Lichnerowicz' Theorem which gives a comparison with the sphere. Then we state and prove Cheeger's Inequality, giving a general bound for the first eigenvalue in the case of a compact Riemannian manifold.

In Chapter 6 we look at the isoperimetric profile. After the first definitions, we prove that the sphere minimizes the isoperimetric profile. The proof of this result needs some preliminaries: a particular way of calculating the volume of a manifold using the normal exponential map, Heintze-Karcher Inequality which gives a bound for the square root of the differential of the normal exponential map and the variational formulas for area and volume, which we will discuss in full detail.

Finally, in Chapter 7 we present the Sphere Theorem, which answers the question about the possibility of weakening the hypothesis on the curvature to conclude a homeomorphism with the sphere. We have chosen to present the proof only in the even dimensional case, since the odd dimensional one needs Morse Theory. The proof is based on some fundamental properties of the cut locus and on an important lower estimate of the injectivity radius which we will prove in full detail. To conclude this chapter, we show that the hypothesis of the theorem cannot be weakened by discussing the case of the complex projective space.

Chapter 1

From differentiable manifolds to Riemannian manifolds

The aim of this chapter is to introduce the space we are going to work with. A Riemannian manifold is built from a differentiable manifold defining an appropriate family of inner products at each point of the manifold. Since we start with a differentiable manifold, the Riemannian one inherits all its structures and objects.

In Section 1.1 we give a quick reminding of what an abstract manifold is and of its main structures along with their fundamental properties. Once it is done, in Section 1.2 we give a formal definition of Riemannian manifold and we introduce typical concepts of Riemannian manifolds such as the covariant derivative, the Levi-Civita connection, geodesics, exponential map and Jacobi fields.

1.1 Definitions and preliminaries about differentiable manifolds

For a complete introduction to differentiable manifolds and for all the missing materials we refer to [DoC] and [Lee] for a first approach and to [War] for a more abstract discussion.

Let M be a topological space. An atlas on M is a family $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in\mathcal{A}}$ such that:

i) $\bigcup_{\alpha\in\mathcal{A}} U_\alpha = M$ with $U_\alpha \subset M$;

ii) $\phi_\alpha : U_\alpha \longrightarrow V_\alpha \subset \mathbb{R}^n$ is a bijective map for every $\alpha \in \mathcal{A}$ with U_α and V_α open. ϕ_α is called coordinate chart;

iii) the composite $\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \longrightarrow \phi_\alpha(U_\alpha \cap U_\beta)$, called coordinate change, is a differentiable map between open sets of $\mathbb{R}^n$ for any $\alpha, \beta \in \mathcal{A}$ and n is the dimension of M.

A differentiable manifold M, i.e. a C^∞ manifold M, of dimension n is a topological space set which comes with a countable atlas $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in\mathcal{A}}$ and has the Hausdorff property, i.e. for every $x, y \in M$ with $x \neq y$ there are open neighbourhoods U_x and U_y such that $U_x \cap U_y = \emptyset$.

From now on M will be our differentiable manifold of dimension n with atlas $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in\mathcal{A}}$.

Sometimes we will replace 'differentiable' with 'smooth'.

A function $f : U \longrightarrow \mathbb{R}$, with U open set of M, is a C^∞ function if for every chart ϕ on U the composite $f \circ \phi^{-1}$ is a C^∞ function.

If $\tilde{M}$ is a differentiable manifold of dimension m, a map $g : M \longrightarrow \tilde{M}$ is differentiable at the point $p \in M$ if for any two charts (U, ϕ) on M and $(\tilde{U}, \psi)$ on $\tilde{M}$, the composite $\psi \circ g \circ \phi^{-1}$ is differentiable at $\phi(p)$.

It is a differentiable map if it is differentiable at each point $p \in M$.

Let $D(M, p) := \{f : M \longrightarrow \mathbb{R} \mid f \text{ is differentiable at } p \in M\}$.

A linear derivation $v : D(M, p) \longrightarrow \mathbb{R}$ is a tangent vector at $p \in M$.

The set of all tangent vectors at the point p is called tangent space and denoted by T_pM.

It can be shown, see [DoC, p. 7ff], that T_pM carries a structure of a real $n-$dimensional vector space.

A basis for T_pM is the set $\left\{\frac{\partial}{\partial x_1}\Big|_p, \ldots, \frac{\partial}{\partial x_n}\Big|_p\right\}$, choosing $\phi = (x_1, \ldots, x_n)$ as a chart on M.

The set of all tangent spaces of M at the point p, denoted by TM, is called tangent bundle.

Further, the tangent bundle TM is a manifold of dimension $2n$ and the foot-point projection $\pi : TM \longrightarrow M$ such that $\pi(v) = p$, if $v \in T_pM$, is a differentiable map.

The differential of a function $f : M \longrightarrow \tilde{M}$, with $\tilde{M}$ smooth manifold of dimension m, at the point $p \in M$ is the map

$$Df(p) : T_pM \longrightarrow T_{f(p)}\tilde{M}$$

such that for each curve $c : [a, b] \longrightarrow M$ with $c(0) = p$ and $c'(0) \in T_pM$ we have

$$Df(p)(c'(0)) = (f \circ c)'(0)$$

A vector field on M is a differentiable map $X : M \longrightarrow TM$ such that $p \longmapsto X(p) \in T_pM$.

We denote the vector space of all vector fields on M by $\mathcal{X}(M)$.

Taking a point $p \in U \subset M$ and a chart $\phi = (x_1, \ldots, x_n)$ on U, a vector field X on M is written as

$$X(p) = \sum_{i=1}^{n} \alpha_i(p) \frac{\partial}{\partial x_i}\Big|_p$$

where α_i are differentiable functions on U for all i.

Sometimes it is also convenient to consider the function $Xf : M \longrightarrow \mathbb{R}$ where $f \in C^\infty(M)$ which is defined as

$$(Xf)(p) = \sum_{i=1}^{n} \alpha_i(p) \frac{\partial f}{\partial x_i}\Big|_p$$

Finally, let X, Y be two vector fields on M. We define the Lie brackets of X and Y as the unique vector field $Z = [X, Y]$ such that for every $f \in C^\infty(M)$ we have

$$[X, Y](f) = X(Yf) - Y(Xf)$$

The Lie brackets of X and Y have the following properties:

i) $[X, Y] = -[Y, X]$

ii) $[aX + bY, Z] = a[X, Z] + b[Y, Z]$

iii) $[fX, gY] = fg[X, Y] + fX(g)Y - gY(f)X$

iv) $[[X, Y], Z] + [[Z, X], Y] + [[Y, Z], X] = 0$

where $Z \in \mathcal{X}(M)$, $f, g \in C^\infty(M)$ and $a, b \in \mathbb{R}$.

1.2 Riemannian manifolds

Definition 1.2.1. Let M be a differential manifold. A Riemannian metric $g = \{g_p\}_{p \in M}$ is a family of symmetric inner products

$$g_p : T_pM \times T_pM \to \mathbb{R}$$

such that for every two vector fields X, Y on M the map $p \mapsto g_p(X(p), Y(p))$ is differentiable.

The couple (M, g) is called Riemannian manifold.

Just a word of notation: we will always replace the scripture $g_p(X(p), Y(p))$ with $< X(p), Y(p) >_p$ (sometimes we will also drop the subscript 'p'), unless it is not clear what is the Riemannian metric used. In that case we will use the round brackets instead of the angle one.

For simplicity we will write M instead of (M, g) to indicate a Riemannian manifold, unless we need to specify the metric g.

Let (M, g) be a Riemannian manifold and let $\varphi = (x_1, \ldots, x_n)$ be a coordinate chart on $U \subset M$. This chart defines a family of differentiable functions

$$g_{i,j} : U \longrightarrow \mathbb{R}, \qquad g_{i,j}(p) :=< \frac{\partial}{\partial x_i}\Big|_p, \frac{\partial}{\partial x_j}\Big|_p >_p$$

1.2.1 Affine connection and Levi-Civita connection

We now turn to the notion of differentiation on a manifold. When we deal with smooth functions we already know how to work it out, but when it comes to vector fields we need to introduce the so called affine connection or covariant derivative and, more specifically, the Levi-Civita connection.

Definition 1.2.2. An affine connection or covariant derivative ∇ on a smooth manifold M is a map

$$\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \longrightarrow \mathcal{X}(M), \qquad (X, Y) \longmapsto \nabla_X Y$$

with the following properties:

i) $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$

ii) $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$

iii) $\nabla_X(fY) = f\nabla_X Y + (Xf)Y$

where $X, Y, Z \in \mathcal{X}(M)$ and f, g differentiable functions on M.

We remark that this definition is given for any smooth manifold.

Turning to the Riemannian contest we have the following result.

Theorem 1.2.3 ([DoC], Theorem 3.6, p. 55)**.** *Given a Riemannian manifold M, there exists a unique affine connection ∇, called the Levi-Civita connection, satisfying the following conditions for every $X, Y, Z \in \mathcal{X}(M)$:*

i) (Riemannian property) $X(< Y, Z >) =< \nabla_X Y, Z > + < Y, \nabla_X Z >$

ii) (Torsion freeness) $[X, Y] = \nabla_X Y - \nabla_Y X$

Since the theorem proves the uniqueness of the Levi-Civita connection we get a formula which defines it uniquely

$$< \nabla_X Y, Z >= \frac{1}{2}\Big(X < Y, Z > + Y < X, Z > - Z < X, Y >$$
$$- < X, [Y, Z] > - < Y, [X, Z] > - < Z, [Y, X] > \Big) \qquad (1.1)$$

One can also ask if it is possible to differentiate a vector field along a particular direction, i.e. along a given curve $c : [a, b] \longrightarrow M$. The answer is positive and given by the following definition.

Definition 1.2.4. Let M be a Riemannian manifold and $c : [a, b] \longrightarrow M$ be a curve on M. A differentiable map $X : [a, b] \longrightarrow TM$ such that $X(t) \in T_{c(t)}M$ is called a vector field along the curve c.

The set of all vector fields along a curve c form a vector space denoted by $\mathcal{X}_c(M)$.

Proposition 1.2.5 ([DoC],Proposition 2.2, p. 50)**.** *Let M be a differentiable Riemannian manifold, ∇ be the Levi-Civita connection on M, X be a vector field along the differentiable curve $c : [a, b] \to M$. There exists a unique vector field, denoted by $\frac{D}{dt}X$, along c such that for every $X, Y \in \mathcal{X}_c(M)$ and $f : [a, b] \longrightarrow \mathbb{R}$ differentiable map we have the following:*

i) $\frac{D}{dt}(X + Y) = \frac{D}{dt}X + \frac{D}{dt}Y$

ii) $\frac{D}{dt}(fX) = f'(t)X + f\frac{D}{dt}X$

iii) If X is induced by a vector field $Y \in \mathcal{X}(M)$, i.e. $X(t) = Y(c(t))$, then $\frac{D}{dt}X = \nabla_{c'(t)}Y$

This unique vector field is called the covariant derivative of X along the curve c.

By the covariant derivative we can define parallel vector fields along a given curve.

Definition 1.2.6. Let M be a Riemannian manifold, $c : [a, b] \longrightarrow M$ be a differentiable curve on M and $\frac{D}{dt}$ be the covariant derivative along c. A vector field X along c is called parallel if $\frac{D}{dt}X \equiv 0$.

The parallel vector field along a given curve c with a given initial value $X(t_0) = v$ is unique (see [Gal-Hul-Laf, Proposition 2.72, p. 75]).

Definition 1.2.7. Let M be a Riemannian manifold and let $c : [a, b] \longrightarrow M$ be a curve along M. The parallel transport P_c is a linear map

$$P_c : T_{c(a)}M \longrightarrow T_{c(b)}M, \quad \text{such that} \quad P_c(v) = X(b)$$

where $v \in T_{c(a)}M$ and $X \in \mathcal{X}_c(M)$ is the unique parallel along c such that $X(a) = v$.

Proposition 1.2.8 ([Gal-Hul-Laf], Proposition 2.74, p. 75)**.** *The parallel transport $P_c : T_{c(a)}M \longrightarrow T_{c(b)}M$ is a linear isometry.*

1.2.2 Riemannian manifolds as metric spaces

Our next goal is to define a distance function on a Riemannian manifold which allows us to talk about distance between points.

Definition 1.2.9. Let M be a Riemannian manifold and let $c : [a, b] \longrightarrow M$ be a curve on M. We define the length of the curve c as

$$l(c) = \int_a^b ||c'(t)||_{c(t)}\, dt$$

where $||v|| = \sqrt{< v, v >}$.
If c is a piecewise differentiable curve, i.e. c is continuous and there exists a partition $a = t_0 < t_1 < \ldots < t_n = b$ such that $c|_{[t_i,t_{i+1}]}$ is differentiable, then

$$l(c) = \sum_{i=0}^{n} l(c|_{[t_i,t_{i+1}]}) = \sum_{i=0}^{n} \int_{t_i}^{t_{i+1}} ||c'(t)||_{c(t)} dt$$

It can be shown that the length is invariant under reparametrization.

Definition 1.2.10. A differentiable curve $c : [a,b] \longrightarrow M$ is arc-length parametrised if $||c'(t)||_{c(t)} = 1$ for all $t \in [a,b]$

Every differentiable curve $c : [a,b] \longrightarrow M$ such that $c'(t) \neq 0$ for all $t \in [a,b]$ has an arc-length reparametrization, that is, it can be turned into a curve $\gamma : [0, l(c)] \longrightarrow M$ by a strictly monotone differentiable function $\psi : [0, l(c)] \longrightarrow [a,b]$ with $\psi(0) = a$ and $\psi(l(c)) = b$.

We can now define the distance function.

Definition 1.2.11. Let M be a connected Riemannian manifold. We define the distance function as the map

$$d : M \times M \longrightarrow \mathbb{R} \quad \text{such that} \quad d(p,q) = \inf\, l(c)$$

where the infimum runs all over the curves $c : [a,b] \longrightarrow M$, piecewise differentiable, such that $c(a) = p$ and $c(b) = q$.

Now it must be shown that this function is indeed a distance function, i.e it is non-negative, symmetric and the triangle inequality holds. For this we refer to [Gal-Hul-Laf, Definition-Proposition 2.91, p. 84].

Furthermore, T_pM inherits this metric function, therefore we can define the unit tangent space $S_pM = \{v \in T_pM \mid ||v|| = 1\}$, the unit metric ball into T_pM, and the unit tangent bundle $SM = \cup_{p\in M} S_pM$.

1.2.3 Geodesics

Definition and characterizations of geodesics

Definition 1.2.12. Let M be a Riemannian manifold and let $\frac{D}{dt}$ be the covariant derivative along the curve $\gamma : [a,b] \longrightarrow M$. γ is said to be a geodesic if for all $t \in [a,b]$ we have $\frac{D}{dt}\gamma'(t) = 0$.

Here we have stated the formal definition of geodesics, but they can also be characterized as the solution of a second order differential equation.

Further, they can be considered as critical points of the length function. We want to investigate the relation between this two points of view.

Theorem 1.2.13 ([Gal-Hul-Laf], Theorem 2.79, p.77)**.** *Let M be a Riemannian manifold, $p \in M$ and $U \subset M$ be an open neighbourhood of p. Choose a vector $v \in T_pM$, then there exists a unique geodesic $c_v : [-\epsilon, \epsilon] \longrightarrow M$ such that $c_v(0) = 0$ and $c_v'(0) = v$.*

From this theorem we derive the local existence and the uniqueness of a geodesic under certain conditions.

Now we turn to the other characterization of geodesics, but before we give the definition of length of a variation.

Definition 1.2.14. Let M be a Riemannian manifold and $c : [a, b] \longrightarrow M$ be a differentiable curve. A map $F : (-\epsilon, \epsilon) \times [a, b] \longrightarrow M$ is a variation of c if $F(0, t) = c(t)$ for every $t \in [a, b]$.

The variation is said to be proper if $F(s, a) = c(a)$ and $F(s, b) = c(b)$ for all $s \in (-\epsilon, \epsilon)$.

The variation is geodesic if all the curves $c(t) = F(0, t)$ are geodesics.

Definition 1.2.15. Let M be a Riemannian manifold, X be a vector field on M along the curve $c : [a, b] \longrightarrow M$ and $F : (-\epsilon, \epsilon) \times [a, b] \longrightarrow M$ be a variation of c. The vector field X is the variational vector field of F if $X(t) := \frac{\partial F}{\partial s}(0, t)$.

We note that if the variation F is proper then we have $X(a) = X(b) = 0$.

Definition 1.2.16. We define the length $l : (-\epsilon, \epsilon) \longrightarrow [0, \infty)$ of a given variation F along a curve $c : [a, b] \longrightarrow M$ with M Riemannian manifold as follow:

$$l(s) = \int_a^b \left\| \frac{\partial F}{\partial t}(s, t) \right\| dt$$

Now we can regard geodesics as critical points of the length function.

Theorem 1.2.17 ([Sak], Proposition 2.6, p. 38). *Let M be a Riemannian manifold. A curve $c : [-\epsilon, \epsilon] \longrightarrow M$ is a geodesic if and only if it is parametrized by arc-length and $l'(0) = 0$ for every proper variation of c.*

Actually for the proof of this theorem we need to use the first variation formula of length, but since it is just a reminding chapter and we are not interested in the proofs of our statement, we postpone this formula to the next chapter.

We have the following corollary.

Corollary 1.2.18. *Let M be a Riemannian manifold and $p, q \in M$. Let $c : [a, b] \longrightarrow M$ be the shortest curve with $c(a) = p$, $c(b) = q$ parametrized proportionally to arc-length. Then c is a geodesic.*

Exponential map

The notion of geodesic is useful to introduce the exponential map, a map from a subset of the tangent space of M at the point p into the manifold itself. Moreover, under certain condition it is a diffeomorphism which means that there exists its inverse function. So we can come and go from the tangent space into the manifold working with a space homeomorphic to an open set of $\mathbb{R}^n$ or with an open neighbourhood of p in the initial manifold according to the situation.

Definition 1.2.19. Let M be a Riemannian manifold and p be a point in M. Then let $v \in T_pM$ and c_v be the geodesic on M such that $c_v(0) = p$ and $c_v'(0) = v$. If $c_v(1)$ exists, we define the map

$$\exp_p(v) := c_v(1)$$

from a subset of T_pM into M. This map is called the exponential map of M at the point p.

Proposition 1.2.20 ([DoC],Proposition 2.9, p. 65). *Let M be a Riemannian manifold and $p \in M$. There exists an $\epsilon > 0$ such that $\exp_p : B_\epsilon(0_p) \longrightarrow M$,* *where $B_\epsilon(0_p) \subset T_pM$ denotes the ball of radius ϵ centred at the origin 0_p of T_pM, is a diffeomorphism into its image.*

To conclude this part we state the so called Gauss Lemma.

Theorem 1.2.21 (Gauss Lemma, [Sak], proposition 2.3, p. 36)**.** *Let M be a Riemannian manifold and $p \in M$. For $u \in T_pM$ suppose that a geodesic $\gamma(t) = \gamma_u(t)$ is defined for $0 \leq t \leq b$. Then $\exp_p$ is defined on an open neighborhood of $\{tu \mid t \in [0, b]\}$ in T_pM, and we get the following:*

i) $D\exp_p(tu)$ *maps* u *to* $\gamma'(t)$.

ii) if we regard $\xi \in T_pM$ also as a vector in $T_{tu}TM$ via the canonical identification, then the equality

$$< D\exp_p(tu)\xi, \gamma'(t) >=< u, \xi >$$

holds. In particular, $\|D\exp_p(tu)u\| = \|u\|$ and $D\exp_p(tu)\xi \perp \gamma'(t)$ if $\xi \perp u$.

Completeness

From analysis and topology we know what 'being complete' means for a space. Using geodesics we have another notion of completeness: 'being geodesically complete'. This is related to usual notion of completeness via Hopf-Rinow Thereom.

Definition 1.2.22. Let M be a Riemannian manifold and $\gamma : [a, b] \longrightarrow M$ be a geodesic with $\gamma(a) = p$ and $\gamma(b) = q$. γ is said to be minimal between p and q if $d(p, q) = l(\gamma)$.

Definition 1.2.23. Let M be a Riemannian manifold. M is said to be geodesically complete if any geodesic $\gamma : [a, b] \longrightarrow M$ can be extended to a geodesic defined on all $\mathbb{R}$.

Theorem 1.2.24 (Hopf-Rinow Theorem, [DoC], Theorem 2.8, p.146)**.** *Let M be a Riemannian manifold and let p be a point in M. Then the following are equivalent:*

i) $\exp_p$ *is defined on all* T_pM*;*

ii) every closed and bounded set of M *is compact;*

iii) M *is complete as a metric space;*

iv) M *is geodesically complete.*

In addition, if any of these statements holds then for each point q *in* M *there exists a minimal geodesic* γ *joining* p *and* q*.*

Cut locus

The last objects we recall about geodesics are the cut locus and the injectivity radius, which will be discussed in detail in Chapter 7.

Definition 1.2.25. Let M be a Rienmannian manifold and $\gamma : [0, T] \to M$ be a geodesic on M with $\gamma(0) = p \in M$ and $\gamma'(0) = v \in T_pM$.

Set $t_v = \sup\{t > 0 \mid \gamma \text{ is minimal between } p \text{ and } \gamma(t)\}$.

If $t_v < \infty$ we say that $\gamma(t_v)$ is the cut point of p along γ and we refer to t_v as its cut value along γ.

The set $C_m(p) = \{\gamma(t_v) \mid v \in T_pM \text{ and } t_v < \infty\}$ is the cut locus of p.

In other words, the cut point of a point p along a specific geodesic is the last point for which the geodesic is minimal.

Remark 1.2.26. If M is compact, then $diam(M) = \sup_{p,q\in M} d(p, q)$ is finite and every two points can be joined by a minimal geodesic according to Hopf-Rinow Theorem and so we get the existence of the cut point for every geodesic starting from p.

Finally we have this last definition.

Definition 1.2.27. The injectivity radius of M is defined as

$$i(M) = \inf_{p\in M} d(p, C_m(p))$$

Chapter 2

Curvature

The curvature is a key point in Riemannian Geometry. Speaking in simple words, we can say that Riemannian Geometry studies curved spaces and somehow the curvature measures how much the curved space is far away from a flat one. Therefore we can consider flat spaces, such as $\mathbb{R}$, as particular cases with curvature identically zero. Here we give a quick reminder of different kind of curvatures on a Riemannian manifold and of all the objects related to them.

In Section 2.1 we give the definitions of the sectional and Ricci curvature followed by the calculation of the curvature of the complex projective space. In Sections 2.2 and 2.3 we recall the first and second variational formula for length and energy and we state the Bonnet-Myers Theorem which relates a particular bound of the Ricci curvature with the diameter of a manifold. Finally, in Section 2.4 we introduce and discuss Jacobi fields.

2.1 Sectional and Ricci curvatures

We start from the definition of Riemannian curvature tensor and of sectional curvature.

Definition 2.1.1. Let M be a Riemannian manifold and ∇ be the Levi-Civita connection on M. The Riemannian curvature tensor is the map

$$R : \mathcal{X}(M) \times \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$$

such that

$$R(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z - \nabla_{[Y,X]} Z$$

For all the properties of the Riemannian curvature tensor we refer to [DoC, Section 2, Chapter 4].

Definition 2.1.2. Let M be a Riemannian manifold, p be a point in M and Σ be a $2-$dimensional subspace of T_pM. Then let $v_1, v_2 \in T_pM$ such that $\Sigma = span\{v_1, v_2\}$. Then the sectional curvature of Σ is defined as

$$K(\Sigma) = K(v_1, v_2) := \frac{< R(v_1, v_2)v_1, v_2 >}{||v_1||^2||v_2||^2 - < v_1, v_2 >^2}$$

Remark 2.1.3. It is easy to verify that this definition is independent on the choice of the vectors v_1 and v_2.

The second curvature we have is the Ricci curvature which is still defined with a tensor.

Definition 2.1.4. Let M be a Riemannian manifold of dimension n, $p \in M$ and v, w be two vectors in T_pM.

The function $R(v, \cdot)w : T_pM \longrightarrow T_pM$ such that $u \longmapsto R(v, u)w$ is linear.

The Ricci curvature tensor is defined as

$$\mathrm{Ric}(v, w) = \mathrm{tr}(R(v, \cdot)w) = \sum_{i=1}^{n} < R(v, v_i)w, v_i >$$

where $v_1, \ldots, v_n$ is an arbitrary orthonormal base of T_pM.

The Ricci curvature is defined as

$$\mathrm{Ric}(v) = \mathrm{Ric}(v, v) = \sum_{i=1}^{n} < R(v, v_i)v, v_i >$$

If v is a unit tangent vector and $\{v_1 = v, v_2, \ldots, v_n\}$ form an arbitrary orthonormal basis of T_pM, then

$$\mathrm{Ric}(v) = \sum_{i=2}^{n} < R(v_1, v_i)v_1, v_i >$$

We remark that this definition is independent on the choice of the basis of T_pM because the trace itself is independent on the basis chosen.

2.1.1 Example: Sectional curvature of $\mathbb{P}^n(\mathbb{C})$, the complex projective space

As a concrete example we calculate the sectional curvature of the complex projective space. For the calculation we use some results about Riemannian submersions for which we refer reader to [Sak, Section 6, Chapter 2, p. 74ff] for a quick introduction and to [Gal-Hul-Laf, pp. 63-65 and Section III.D] for a detailed discussion.

We consider $\mathbb{P}^n(\mathbb{C})$ as $\mathbb{C}^{n+1} - \{0\}/\sim$ where $\sim$ is the relation given by $z \sim w \Longrightarrow z = \lambda w$ with $\lambda \neq 0$.

Considering the unit sphere S^{2n+1} in $\mathbb{C}^{n+1}$, there exists a submersion $f : S^{2n+1} \longrightarrow \mathbb{P}^n(\mathbb{C})$.

Equipping S^{2n+1} with the metric g induced by $\mathbb{C}^{n+1} = \mathbb{R}^{2n+2}$ and considering the group of isometries $e^{i\theta} : S^{2n+1} \longrightarrow S^{2n+1}$, $0 \leq \theta \leq 2\pi$, we can make f a Riemannian submersion, i.e. a submersion whose differential is an isometry on the horizontal space into the tangent space of the complex projective plane in each point, defining uniquely a Riemannian metric h on $\mathbb{P}^n(\mathbb{C})$ by

$$h_q(u, v) = g_p(\overline{u}, \overline{v})$$

where $q = f(p)$ and $\overline{u}, \overline{v}$ are the horizontal lifts of u, v respectively, namely, $Df(p)(\overline{u}) = u$ and $Df(p)(\overline{v}) = v$.

Here the vertical space is the space of all tangent vectors in the direction of some fiber $f^{-1}(p)$ while the horizontal space contains the tangent vectors orthogonal to the fiber. Moreover the vertical directions form a one dimensional space at every point.

Therefore the map f is indeed

$$f : (S^{2n+1}, g) \longrightarrow (\mathbb{P}^n(\mathbb{C}), h)$$

Let $X, Y \in \mathcal{X}(\mathbb{P}^n(\mathbb{C}))$ be a pair of orthonormal vector fields. We observe that by the definition of h, the horizontal lifts $\overline{X}, \overline{Y}$ of X and Y, are a pair of orthonormal vector fields on S^{2n+1} as well.

Let $Z \in TS^{2n+1}$ such that $Z(p) = \frac{d}{d\theta} e^{i\theta} p = ip$. It turns out that the multiplication by i is an isometry and Z is tangent to the fiber $f^{-1}(p)$, i.e. a vertical vector field with norm 1.

Further, let $c : (-\epsilon, \epsilon) \longrightarrow S^{2n+1}$ be a curve such that $c(0) = p$ and $c'(0) = \overline{X}(p)$ and let $\tilde{\nabla}$ be the Levi-Civita connection on S^{2n+1}, we have

$$(\tilde{\nabla}_{\overline{X}} Z)(p) = \frac{d}{dt}\Big|_{t=0} (Z \circ c)(t) = ic'(0) = i\overline{X}(p)$$

We also note that $\tilde{\nabla}_{\overline{X}} Z = \tilde{\nabla}_Z \overline{X}$. In fact, $[\overline{X}, Z] = 0$ because $\overline{X}$ is an horizontal vector field and Z a vertical one and it is possible to find a local coordinate system such that $\overline{X}, Z$ are coordinate directions.

Moreover

$$g([\overline{X}, \overline{Y}], Z) = g(\tilde{\nabla}_{\overline{X}} \overline{Y}, Z) - g(\tilde{\nabla}_{\overline{Y}} \overline{X}, Z) =$$

$$= \overline{X}\Big(g(\overline{Y}, Z)\Big) - g(\overline{Y}, \tilde{\nabla}_{\overline{X}} Z) - \overline{Y}\Big(g(\overline{X}, Z)\Big) + g(\overline{X}, \tilde{\nabla}_{\overline{Y}} Z) =$$

$$= -g(\overline{Y}, \tilde{\nabla}_{\overline{X}} Z) + g(\overline{X}, \tilde{\nabla}_{\overline{Y}} Z) =$$

$$= -Z\Big(g(\overline{X}, \overline{Y})\Big) + g(\tilde{\nabla}_Z \overline{Y}, \overline{X}) + g(\overline{X}, \tilde{\nabla}_Z \overline{Y}) =$$

$$= 2g(\overline{X}, \tilde{\nabla}_Z \overline{Y}) = 2g(i\overline{Y}, \overline{X})$$

The vertical component of $[\overline{X}, \overline{Y}]$, denoted by $[\overline{X}, \overline{Y}]^v$, is given by:

$$[\overline{X}, \overline{Y}]^v = g([\overline{X}, \overline{Y}], Z)Z = 2g(i\overline{Y}, \overline{X})Z$$

Therefore

$$\|[\overline{X}, \overline{Y}]^v\|^2 = 4g(\overline{X}, i\overline{Y})^2 \|Z\|^2 = g(\overline{X}, i\overline{Y}) = 4\cos^2\varphi$$

where the last equality is due to $g(\overline{X}, i\overline{Y}) = \|\overline{X}\|^2 \cdot \|i\overline{Y}\|^2 \cos\varphi = \cos\varphi$ where φ is the angle between $\overline{X}$ and $\overline{Y}$.

At this point we can use O'Neill Formula (see [Gal-Hul-Laf, Theorem 3.61, p. 127]) which states that if $f : (\tilde{M}, \tilde{g}) \longrightarrow (M, g)$ is a Riemannian submersion, then for each pair of orthonormal vector fields $X, Y \in \mathcal{X}(M)$ and for their horizontal lifts $\overline{X}, \overline{Y} \in \mathcal{X}(\tilde{M})$, we have the following relation:

$$K(span\{X,Y\}) = K(span\{\overline{X},\overline{Y}\}) + \frac{3}{4}\|[\overline{X},\overline{Y}]^v\|^2$$

Hence, for the sectional curvature of $\mathbb{P}^n(\mathbb{C})$ we have:

$$1 \leq K(span\{X,Y\}) = 1 + 3\cos^2\varphi \leq 4$$

In particular, we can rescale the metric on $\mathbb{P}^n(\mathbb{C})$ in such a way that

$$\frac{1}{4} \leq K(span\{X,Y\}) \leq 1$$

2.2 Variational formulas of length and energy

For variational formulas we mean formulas about derivatives of a function defined on a variation. In our case we are going to recall them for the first and second derivative of length, introduced in Definition 1.2.23, and of energy that we define below.

Definition 2.2.1. Let $F : (-\epsilon, \epsilon) \times [a, b] \longrightarrow M$ be a variation on a Riemannian manifold M along the curve $c : [a, b] \longrightarrow M$. We define the energy as the map

$$E : (-\epsilon, \epsilon) \longrightarrow [0, \infty) \qquad \text{such that} \qquad E(s) = \frac{1}{2}\int_a^b \left\|\frac{\partial F}{\partial s}(s,t)\right\|^2 dt$$

Before starting with variational formulas, we want to show the relation between energy and length. In fact, taking a curve $c : [a, b] \longrightarrow M$ on a Riemannian manifold M and considering the associated length and energy, by Cauchy-Schwartz Inequality we get:

$$l(c)^2 \leq 2(b-a)E(c) \tag{2.1}$$

with equality if and only if the curve is parametrized proportional to arc-length.

This relation and the fact that geodesics minimize the length (see Corollary 1.2.18) allow the following lemma.

Lemma 2.2.2 ([DoC], Lemma 2.3, p. 194). *Let M be a Riemannian manifold and let $p, q \in M$. We consider the minimal geodesic $\gamma : [a, b] \longrightarrow M$ joining p and q. Then, for all curves $c : [a, b] \longrightarrow M$ joining p and q we have:*

$$E(\gamma) \leq E(c)$$

Equality holds if and only if c is a minimal geodesic.

As a consequence, all curves which minimize energy are parametrized proportionally by arc-length. In particular, minimizing curves for energy are minimal geodesics.

We can now introduce the variational formulas.

First and second variational formula for length

Theorem 2.2.3 (First variational formula of length). *Let M be a Riemannian manifold and $F : (-\epsilon, \epsilon) \times [a, b] \longrightarrow M$ be a variation of a curve $c : [a, b] \longrightarrow M$ with $c'(t) \neq 0$ for all $t \in [a, b]$. Let X be its variational vector field and $l : (-\epsilon, \epsilon) \longrightarrow [0, \infty)$ be the associated length. Then we have:*

$$l'(0) = \int_a^b \frac{1}{||c'(t)||} \frac{d}{dt} < X(t), c'(t) > dt - \int_a^b \frac{1}{||c'(t)||} < X(t), \frac{D}{dt} c'(t) > dt \tag{2.2}$$

Moreover if c is arc-length parametrized, then:

$$l'(0) = < X(b), c'(b) > - < X(a), c'(a) > - \int_a^b < X(t), \frac{D}{dt} c'(t) > dt \tag{2.3}$$

For its proof we refer to [Sak, Proposition 2.5, p. 38].

Remark 2.2.4. If the variation F is proper and the curve c is a geodesic then $l'(0) = 0$. Therefore geodesics are critical points for length.

Theorem 2.2.5 (Second variational formula for length)**.** *Let M be a Riemannian manifold and let $F : (-\epsilon, \epsilon) \times [a, b] \longrightarrow M$ be a proper variation of a geodesic $c : [a, b] \longrightarrow M$ with $||c'(t)|| = 1$ for all $t \in [a, b]$. Let X be its variational vector field and $X^{\perp} = X(t) - < X(t), c'(t) > c'(t)$ be the component of X orthogonal to c'. If $l : (-\epsilon, \epsilon) \longrightarrow M$ is the associated length, then $l'(0) = 0$ and*

$$l''(0) = \int_a^b \left\| \frac{D}{dt} X^{\perp} \right\|^2 - K(span\{c', X^{\perp}\}) \|X^{\perp}\|^2 \, dt \tag{2.4}$$

If $X^{\perp} = 0$, we set $K(span\{c', X^{\perp}\}) = 0$.

For its proof we refer to [Sak, p. 91]

First and second variational formula for energy

Theorem 2.2.6 (First variational formula for energy)**.** *Let M be a Riemannian manifold and $F : (-\epsilon, \epsilon) \times [a, b] \longrightarrow M$ be a variation of a differentiable curve $c : [a, b] \longrightarrow M$ with $c'(t) \neq 0$ for all $t \in [a, b]$. Let X be its variational vector field and consider $E : (-\epsilon, \epsilon) \longrightarrow M$ the associated energy. We have the following:*

$$E'(0) =< X(b), c'(b) > \; - \; < X(a), c'(a) > - \int_a^b < X(t), \frac{D}{dt} c'(t) \, dt \tag{2.5}$$

For its proof we refer to [DoC, Proposition 2.4, p. 195].

As a consequence we have the following.

Proposition 2.2.7 ([DoC], Proposition 2.5, p. 196)**.** *A differentiable curve $c : [a, b] \longrightarrow M$ on a Riemannian manifold M is a geodesic if and only if, for every proper variation F of c we have $E'(0) = 0$.*

That is, geodesics are critical points of the energy function for every proper variation.

Theorem 2.2.8 (Second variational formula for energy)**.** *Let M be a Riemannian manifold and let $F : (-\epsilon, \epsilon) \times [a, b] \longrightarrow M$ be a variation of a* *geodesic* $c : [a, b] \longrightarrow M$*. Let* X *be its variational vector field. Then we have:*

$$E''(0) = \int_a^b \left\| \frac{D}{dt} X \right\|^2 - < R(X, c')c', X > \, dt$$

$$+ < \frac{D}{ds}\Big|_{s=0} \frac{\partial F}{\partial s}(s, b), c'(b) > - < \frac{D}{ds}\Big|_{s=0} \frac{\partial F}{\partial s}(s, a), c'(a) > \tag{2.6}$$

In particular, if the variation is proper the last two terms vanish.

The reference for the proof is [DoC, Proposition 2.8 and following Remarks, p. 197ff].

2.3 Bonnet-Myers Theorem

Bonnet-Myers Theorem shows that the curvature, in this case the Ricci curvature, has implication on the diameter of a manifold. It can be proved by a variational argument using the second variational formula for length. For a complete proof see [DoC, Theorem 3.1, p. 200], .

Theorem 2.3.1 (Bonnet-Myers Theorem)**.** *Let* M *be a connected, complete,* $n-$*dimensional Riemannian manifold such that*

$$Ric(v) \geq \frac{n-1}{r^2}$$

for all $v \in SM = \{w \in TM \mid ||w|| = 1\}$*. Then we have*

$$diam(M) \leq \pi r$$

In particular, M *is closed and bounded and therefore it is compact.*

It can be shown that this estimate is optimal since equality holds for the round $n-$sphere of radius r, S^n_r. In fact, its sectional curvature is constant and equal to $\frac{1}{r^2}$ so the Ricci curvature is exactly equal to $\frac{n-1}{r^2}$, which means that the hypothesis of the theorem are satisfied. Moreover its diameter is πr and the theorem is satisfied with equality.

2.4 Jacobi fields

Definition 2.4.1. Let M be a Riemannian manifold, let $R(\cdot,\cdot)$ denote the Riemannian tensor and let $\gamma : [a,b] \longrightarrow M$ be a geodesic. A Jacobi field is a vector field J along γ satisfying the following equation for every $t \in [a,b]$:

$$\frac{D^2}{dt^2}J(t) + R(\gamma'(t), J(t))\gamma'(t) = 0 \tag{2.7}$$

Equation (2.7) is called Jacobi equation.

For simplicity we will write $J'' + R(\gamma', J)\gamma' = 0$ denoting with the double dash the second covariant derivative of the vector field J along the geodesic γ.

As for geodesics, Jacobi fields can be obtained solving a second order differential equation.

Theorem 2.4.2 ([Gal-Hul-Laf], Theorem 3.43, part (i), p. 115)**.** *Let M be a Riemannian manifold of dimension n and let $\gamma : [0,T] \longrightarrow M$ be a geodesic on M. Then for any $u, v \in T_{\gamma(0)}M$ there exists a unique Jacobi field along γ such that $J(0) = u$ and $J'(t) := \frac{D}{dt}J(t) = v$. If $J(0)$ and $J'(0)$ are orthogonal to $\gamma'(0)$ then $J(t)$ is orthogonal to $\gamma'(t)$ for every t.*

The vector space of all Jacobi fields has dimension $2n$ and the subspace of all the Jacobi fields which are normal to γ has dimension $2(n-1)$.

This is not the only way to get a Jacobi field: they also arise as variational vector fields of a geodesic variation.

Proposition 2.4.3 ([Gal-Hul-Laf], Proposition 3.45, p. 116)**.** *Let M be a Riemannian manifold, $\gamma : [a,b] \longrightarrow M$ be a geodesic on M and F be a geodesic variation of γ. Then $J(t) = \frac{\partial}{\partial s}F(0,t)$ is a Jacobi field along γ. Conversely every Jacobi field can be obtained in that way.*

2.4.1 Jacobi fields and the exponential map

There is a strict link between Jacobi fields, geodesics and curvature and as geodesics are linked to the exponential map, so are Jacobi fields.

Corollary 2.4.4 ([DoC], Corollary 2.5, p. 114). *Let $\gamma : [0,T] \longrightarrow M$ be a geodesic on Riemannian manifold M, then a Jacobi field J along γ with $J(0) = 0$ is given by $J(t) = D\exp_p(t\gamma'(0))(tJ'(0))$ for all $t \in [0,T]$.*

2.4.2 Conjugate points

Definition 2.4.5. Let M be a Riemannian manifold, $\gamma : [a,b] \longrightarrow M$ be a geodesic on M and $t_0, t_1 \in [a,b]$ with $t_0 < t_1$. The point $\gamma(t_1)$ is conjugate to $\gamma(t_0)$ if there exists a non vanishing Jacobi field J along γ such that $J(t_0) = J(t_1) = 0$. The maximum number of these linearly independent vector fields is called the multiplicity of the conjugate point $\gamma(t_1)$.

We observe that if $\gamma(t_1)$ is conjugate to $\gamma(t_0)$, then also $\gamma(t_0)$ is conjugate to $\gamma(t_1)$.

Proposition 2.4.6 ([DoC], Proposition 3.5, p. 117). *Let M be a Riemannian manifold and let $\gamma : [0,T] \longrightarrow M$ be a geodesic with $\gamma(0) = p$. The point $q = \gamma(t_0)$, $t_0 \in (0,T]$ is said to be conjugate to p if and only if $v_0 = t_0\gamma'(0) \in T_pM$ is a critical point of $\exp_p$. Moreover, the multiplicity of q as conjugate point is equal to the dimension of the kernel of the map $(D\exp_p)_{v_0}$.*

Therefore we have a characterization of conjugate points as critical point of the exponential map.

Moreover the conjugate points are related to minimal geodesics.

Proposition 2.4.7 ([DoC], Corollary 2.9, p. 248). *Let $\gamma : [0,a] \longrightarrow M$ be a geodesic segment on a Riemannian manifold M such that $\gamma(a)$ is not a conjugate point of $\gamma(0)$. Then γ has no conjugate points on $(0,a)$ if and only if for all proper variations of γ there exists $\delta > 0$ such that $E(s) < E(\delta)$ for $0 < |s| < \delta$. In particular, if γ is minimal, γ has no conjugate points on $(0,a)$.*

2.4.3 Rauch Comparison Theorem

Rauch Comparison Theorem gives a way to compare the length of Jacobi fields on two Riemannian manifolds M and $\tilde{M}$ when their sectional curvatures are such that $K_{\tilde{M}}(\tilde{\Sigma}) \leq K_M(\Sigma)$.

For the proof we refer to [DoC, Theorem 2.3, p. 215].

Theorem 2.4.8 (Rauch Comparison Theorem)**.** *Let M^n and $\tilde{M}^{n+k}$ be two Riemannian manifolds. Then let $c : [0,T] \longrightarrow M$, $\tilde{c} : [0,T] \longrightarrow \tilde{M}$ be two arc-length parametrized geodesics on M and $\tilde{M}$ and let $J : [0,T] \longrightarrow TM$, $\tilde{J} : [0,T] \longrightarrow T\tilde{M}$ be two orthogonal Jacobi fields along c and $\tilde{c}$ respectively, with $J(0) = \tilde{J}(0) = 0$ and $||\frac{D}{dt}J(0)|| = ||\frac{D}{dt}\tilde{J}(0)||$. Assume that $\tilde{c}$ does not have conjugate points on $(0,T]$ and that $K_{\tilde{M}}(\tilde{\Sigma}) \leq K_M(\Sigma)$ for all $t \in [0,T]$ and for any $2-$planes $\Sigma \subset T_{c(t)}M, \tilde{\Sigma} \subset T_{\tilde{c}(t)}\tilde{M}$. Then*

$$||J(t) \geq ||\tilde{J}(t)|| \qquad \forall\, t \in [0,T]$$

Moreover, if for some $t_0 \in (0,T]$ we have that $||\tilde{J}(t_0)|| = ||J(t_0)||$, then $K_{\tilde{M}}(\tilde{\Sigma}) = K_M(\Sigma)$ for all $t \in [0,t_0]$.

An application of this theorem gives information about the distance of a point p from its first conjugate point.

Proposition 2.4.9 ([DoC], Proposition 2.4, p. 218)**.** *Let M be a Riemannian manifold and let γ be a geodesic on M. If M has sectional curvature K such that $0 < L \leq K \leq H$, L, H constants, then the distance d along γ between two conjugate points is such that*

$$\frac{\pi}{\sqrt{H}} \leq d \leq \frac{\pi}{\sqrt{L}}$$

Anyway this is not the only consequence of the Rauch Comparison Thereom, for example it can also be used to compare length of curves.

The following proposition is a special version of [DoC, Proposition 2.5, p. 218].

Proposition 2.4.10. *Let M^n be an $n-$dimensional Riemannian manifold and let S^n be the $n-$dimensional round sphere of sectional curvature δ. Assume that the sectional curvature of M^n are all bigger or equal to δ, i.e. $K_{M^n} \geq \delta$. Let $p \in M$ and $\rho > 0$ such that $\exp_p : \overline{B_\rho(0_p)} \longrightarrow M$ is a diffeomorphism. For any triangle $\triangle p, q_1, q_2$ on M with $d(p, q_1), d(p, q_2) \leq \rho$ and $\sphericalangle q_1 p q_2 = \alpha < \pi$ and comparison triangle $\triangle \overline{p}\overline{q}_1\overline{q}_2$ on S^n with $d(\overline{p}, \overline{q}_1) = d(p, q_1)$ and $d(\overline{p}, \overline{q}_2) = d(p, q_2)$ and $\sphericalangle \overline{q}_1\overline{p}\overline{q}_2 = \alpha$, we have*

$$d_M(q_1, q_1) \leq d_{S^n}(\overline{q}_1, \overline{q}_2) \leq diam(S^n) = \frac{\pi}{2\sqrt{\delta}}$$

Chapter 3

Riemannian volume

One of the fundamental concepts in Riemannian Geometry is how to compute integrals on Riemannian manifolds. In this chapter we want to show what integrating a function on a manifold means and how we can calculate it. Once it is established, we can talk about volume.

In Section 3.1 we give the definition of the integral of a function defined on a manifold and of the volume of a manifold showing how it is calculated. Then, in Section 3.2 we focus on the volume of the $n-$sphere and finally, in Section 3.3 we give some comparison theorems about volume of metric balls of manifolds with certain conditions on the curvature.

3.1 Riemannian measure

Definition 3.1.1. Let (M, g) be a Riemannian manifold of dimension n with $\varphi = (x_1, \ldots, x_n) : U \longrightarrow V$ a local coordinate chart. Let $f : M \longrightarrow \mathbb{R}$ be a map such that $supp\, f \subset U$. The integral of f over M is defined as

$$\int_M f\, dvol_n = \int_U f\, dvol_n := \int_V f \circ \varphi^{-1}(x) \sqrt{\det g_{ij}} \circ \varphi^{-1}(x)\, dx$$

This means that $\sqrt{\det g_{ij}}\, dx$ defines a local density or a local measure on the manifold.

At a first glance, this definition seems to be dependent on the choice of the coordinate chart, but this is actually not true. In fact, let $\phi : U \longrightarrow V'$ another coordinate chart on U with $\phi = (y_1, \dots, y_n)$.

Then $F = \phi \circ \varphi^{-1} : V \longrightarrow V'$ is a diffeomorphism. Since V and V' are subset of $\mathbb{R}^n$, we have the following transformation rule:

$$\int_{V'} h(y)\, dy = \int_V (h \circ F)(x) |\det DF(x)|\, dx \tag{3.1}$$

Let $\tilde{g}_{ij}(p) =< \frac{\partial}{\partial y_i}\Big|_p, \frac{\partial}{\partial y_j}\Big|_p >$ be the Riemannian metric associated to the chart ϕ. There is the following relation:

$$g_{ij}(p) =< \frac{\partial}{\partial x_i}\Big|_p, \frac{\partial}{\partial x_j}\Big|_p >= \sum_{k,l} \frac{\partial y_k}{\partial x_i}(p) \frac{\partial y_l}{\partial x_j}(p) < \frac{\partial}{\partial y_k}\Big|_p, \frac{\partial}{\partial y_l}\Big|_p >$$

Setting $G = (g_{ij})$, $\tilde{G} = (\tilde{g}_{ij})$ and $\partial Y = \left(\left(\frac{\partial y_i}{\partial x_j}\right)_{ij} \right)$ we obtain the following matrix relation

$$G = (\partial Y)^T \tilde{G} (\partial Y)$$

Since the differential of F is the matrix $DF(x) = \left(\left(\frac{\partial (y_i \circ \phi^{-1})}{\partial x_j}(x) \right)_{ij} \right)$ and observing that $p = \phi^{-1}(x)$, we have

$$\det G(p) = (\det DF(\phi(x)))^2 (\det \tilde{G}(p))$$

Hence

$$\int_V f \circ \varphi^{-1}(x) \sqrt{\det g_{ij}} \circ \varphi^{-1}(x)\, dx = \int_V f \circ \varphi^{-1}(x) \sqrt{\det \tilde{g_{ij}}} \circ \varphi^{-1}(x) |\det D\, F(x)|\, dx =$$

$$= \int_V f \circ \phi^{-1} \circ F(x) \sqrt{\det \tilde{g_{ij}}} \circ \phi^{-1} \circ F(x) |\det D\, F(x)|\, dx =$$

$$= \int_V f \circ \phi^{-1}(y) \sqrt{\det \tilde{g_{ij}}} \circ \phi^{-1}(y)\, dy$$

However we need to remark that the Riemannian measure $dvol_n$ just introduced is only a local measure. To get a global Riemannian measure we need to define the integral of f over M using the partition of unity.

Definition 3.1.2. Let (M, g) be a Riemannian manifold of dimension n, $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\mathcal{A}}$ be a countable atlas and $\{\psi_\alpha\}_{\alpha\in\mathcal{A}}$ be a smooth subordinate partition of unity, i.e. $\psi_\alpha : V_\alpha \subset U_\alpha \longrightarrow [0, 1]$ and $\sum_{\alpha\in\mathcal{A}} \phi_\alpha = 1$. For a function $f : M \longrightarrow \mathbb{R}$ we define

$$\int_M f\, dvol_n = \sum_{\alpha\in\mathcal{A}} \int_{V_\alpha} f \cdot \psi_\alpha\, dvol_n$$

It can be checked that this definition is independent on the choice of the atlas and on the subordinated partition of unity.

Definition 3.1.3. Let (M, g) be a Riemannian manifold of dimension n and $\varphi = (x_1, \dots, x_n) : U \subset M \longrightarrow V \subset \mathbb{R}^n$ be a coordinate chart. The volume of a set $A \subset U$ is defined as

$$\mathrm{Vol}(A) = \int_M \chi_A\, dvol_n = \int_A dvol_n = \int_{\varphi(A)} \sqrt{\det g_{ij}} \circ \varphi^{-1}(x)\, dx$$

where χ_A is the characteristic function of A.
We say that $A \subset M$ is a set of measure zero if for every coordinate chart $\varphi : U \longrightarrow V$ we have $\mathrm{Vol}(U \cap A) = 0$.

Definition 3.1.4. Let (M, g) be a Riemannian manifold of dimension n. The volume of M is defined as

$$\mathrm{Vol}(M) = \int_M dvol_n \tag{3.2}$$

One can refer to the volume of any $n-$dimensional manifold M as the $n-$dimensional volume writing $\mathrm{Vol}_n(M)$. In fact, for $k > n$, $\mathrm{Vol}_k(M) = 0$ and for the $(n-1)-$dimensional volume of M we mean the volume of its boundary, which has dimension $(n-1)$. If M does not have boundary we set $\mathrm{Vol}_{n-1}(M) = \infty$.

At this point we can obtain an expression for the volume of a Riemannian manifold M of dimension n.

We will not follow the approach suggested by Definition 3.1.2, instead we choose an almost global chart, $\varphi : M\backslash Z \longrightarrow V \subset \mathbb{R}^n$ with Z a set of measure zero, to obtain

$$\int_M f\, dvol_n = \int_{M\backslash Z} f\, dvol_n$$

and then we write the last integral as a multiple integral in the Euclidean space.

With this guideline (3.2) is replaced by

$$\mathrm{Vol}(M) = \int_{M\backslash Z} dvol_n$$

The most suitable choice for the set Z is $C_m(p)$, the cut locus of a point $p \in M$. Then, the exponential map $\exp_p$ is a diffeomorphism into $M\backslash C_m(p)$ and, since T_pM is an $\mathbb{R}$-vector space with the same dimension of M, $\exp_p^{-1}$ can be viewed as a chart on $M\backslash C_m(p)$. Moreover, the Riemannian measure and the Euclidean measure agree on T_pM if we identify T_pM with the Euclidean $\mathbb{R}^n$ via an isometry.

Therefore we can write

$$\mathrm{Vol}(M) = \int_{U_p} \sqrt{\det(g_{ij} \circ \exp_p)}(x)\, dx \tag{3.3}$$

where $U_p = \exp_p^{-1}(M\backslash C_m(p)) \in T_pM$ and dx is the Riemannian measure on T_pM restricted to U_p.

To split (3.3) in a multiple integral in the Euclidean space we work as follow.

Set $V = \{(r, v) \in (0, \infty) \times S_pM \mid v \in S_pM \text{ and } 0 < r < t_v\}$ and we define the map

$$\theta : V \longrightarrow M\backslash C(p), \qquad \text{such that} \qquad (t, v) \longmapsto \exp_p tv$$

We observe that for v fixed, $\theta(t, v)$ is the geodesic starting at p with initial tangent vector v.

Then, we take $\{e_1 = v, e_2, \ldots, e_n\}$ an orthonormal basis of T_pM and Jacobi fields $J_i(t)$ for $i = 2, \ldots, n$ along the geodesic $c_v(t) = \exp_p tv$ with $J_i(0) = 0$ and $J_i'(0) = e_i$.

It holds, see Corollary 2.4.4, that $J_i(t) = t(D\exp_p)(tv)(e_i)$ for $i = 2, \ldots, n$. Further we have that $c_v'(t) = (D\exp_p)(tv)(e_1)$.

Henceforth, the entries of the matrix $(g_{ij} \circ \exp_p)$ at tv are given by

$$g_{ij}(\exp_p tv) = \begin{cases} < c_v'(t), c_v'(t) > & \text{for } i = j = 1 \\ t^{-1} < c_v'(t), J_j(t) > & \text{for } i = 1 \text{ and } j = 2, \ldots, n \\ t^{-2} < J_i(t), J_j(t) > & \text{for } i = 2, \ldots, n \text{ and } j = 2, \ldots, n \end{cases}$$

Therefore, taking the square root of its determinant, we have

$$\sqrt{\det(g_{ij} \circ \exp_p)}(tv) = t^{-(n-1)} \sqrt{\det(< J_i(t), J_j(t) >)_{ij}}$$

It remains to write the measure on U_p. Let $dvol_{n-1}$ denote the Riemannian measure on S_pM induced by the Euclidean measure on T_pM. We remark that we cannot talk about an induced Euclidean measure on S_pM since it is not diffeomorphic to a subset of $\mathbb{R}^n$.

We have the following relation:

$$dx = t^{n-1} dt\, dvol_{n-1}$$

Hence

$$\begin{aligned} \sqrt{\det(g_{ij} \circ \exp_p)}(tv)\, dx &= t^{n-1} \sqrt{\det(g_{ij} \circ \exp_p)}(tv)\, dt\, dvol_{n-1} \\ &= \sqrt{\det(< J_i(t), J_j(t) >)_{ij}}\, dt\, dvol_{n-1} \end{aligned}$$

Therefore (3.3) becomes:

$$\mathrm{Vol}(M) = \int_{S_pM} dvol_{n-1} \int_0^r \sqrt{\det(< J_i(t), J_j(t) >)_{ij}}\, dt \tag{3.4}$$

Of course, this procedure is applied to the calculation of the volume of any subset N of M.

In particular, we can write explicitly the volume of a metric ball $B_\rho(p)$ of radius ρ centred at p into M. If ρ is small enough, the exponential map is a diffeomorphism, see Proposition 1.2.20, and we don't have to cut out any set of measure zero.

This time the map θ is defined on $(0,\rho) \times S_pM$.

Hence, going through the same steps as before, we end up with

$$\mathrm{Vol}(B_\rho(q)) = \int_{S_pM} dvol_{n-1} \int_0^\rho t^{n-1} \sqrt{\det(< J_i(t), J_j(t) >)_{ij}}\, dt \tag{3.5}$$

3.2 Volume of the unit $n-$sphere

Let S^n be the unit $n-$dimensional sphere and let $p \in S^n$. Then let $\{E_1(t),\dots,E_n(t)\}$ parallel vector fields along the geodesic c_v with $v \in S_pM$ such that $E_i(0) = e_i$ with $\{e_1,\dots,e_n\}$ a basis for T_pS^n. Since geodesics on S^n are great circles, Jacobi fields on S^n are given by

$$J_i(t) = \sin t \cdot E_i(t)$$

Therefore, by (3.4) we have

$$\mathrm{Vol}(S^n) = \int_{S^{n-1}} dvol_{n-1} \int_0^\pi (\sin t)^{n-1} = \mathrm{Vol}(S^{n-1}) \int_0^\pi (\sin t)^{n-1} \tag{3.6}$$

where $dvol_{n-1}$ is the Riemannian measure on S^{n-1}.

Although this formula will be useful later, it is not the right approach to carry on the calculation.

A nicer way to calculate the volume of S^n is to observe that the Riemannian measure on S^n is the measure induced by the Euclidean one on $\mathbb{R}^{n+1}$, $dvol_{n+1} = t^n dt\, dvol_n$, where $dvol_n$ denotes the Riemannian measure on S^n, and to use the Gamma function

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt$$

We have that

$$\Big(\int_{\mathbb{R}} e^{-t^2}\, dt\Big)^{n+1} = \int_{\mathbb{R}^n} e^{-|x|^2}\, dvol_{n+1} = \int_{S^n} dvol_n \int_0^\infty e^{-t^2} t^n\, dt =$$

$$= \mathrm{Vol}(S^n)\int_0^\infty e^{-t^2}t^n\,dt = \mathrm{Vol}(S^n)\frac{\Gamma(\frac{n+1}{2})}{2}$$

From analysis we know that

$$\int_{\mathbb{R}} e^{-t^2} = \sqrt{\pi} \qquad \Longrightarrow \qquad \Big(\int_{\mathbb{R}} e^{-t^2}\,dt\Big)^{n+1} = \pi^{\frac{n+1}{2}}$$

Therefore we get

$$\mathrm{Vol}(S^n) = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})} \tag{3.7}$$

Moreover using (3.5), the volume of a metric ball $B_\rho(p) \subset S^n$ of radius ρ centred at p is given by:

$$\mathrm{Vol}(B_\rho(p)) = \int_{S^{n-1}} dvol_{n-1}\int_0^\rho (\sin t)^{n-1}\,dt = \mathrm{Vol}(S^{n-1})\int_0^\rho (\sin t)^{n-1}\,dt \tag{3.8}$$

3.3 Volume comparison theorems

In this section we introduce some comparison theorems about volumes.

The key point is the hypothesis on the curvature, sectional curvature or Ricci curvature according to the case, which has to be bounded from below for the Ricci curvature or above for the sectional curvature. In fact, these two different bounds lead to different, even opposite, conclusions.

Let (M, g) and $(\tilde{M}, \tilde{g})$ be two $n-$dimensional Riemannian manifolds and let $p \in M$ and $\tilde{p} \in \tilde{M}$. Then let c_v be the geodesic such that $c_v(0) = p$ and $c_v'(0) = v \in S_pM$. Let $c_v(T)$ be the first conjugate point of p along c_v.

We keep the notation of Section 3.1 and we set

$$j(t, v) = \sqrt{\det(g_{ij}(\exp_p tv))} \qquad \text{and} \qquad \Theta(t, v) = t^{n-1}j(t, v)$$

We observe that $\Theta(t, v) = \sqrt{\det(< J_i(t), J_j(t) >)_{ij}}$.

Now let $I : T_pM \longrightarrow T_{\tilde{p}}\tilde{M}$ be a linear isometry and let $\tilde{c}_{\tilde{v}}$ be the normal geodesic such that $\tilde{c}_{\tilde{v}}(0) = \tilde{p}$ and $\tilde{c}'_{\tilde{v}}(0) = \tilde{v} = I(v)$. We denote by $\tilde{c}_{\tilde{v}}(\tilde{T})$ the first cut point of $\tilde{p}$ along $\tilde{c}_{\tilde{v}}$. Moreover, for $i = 1, \ldots, n-1$ let $\tilde{J}_i(t)$ be Jacobi fields on $\tilde{M}$ along $\tilde{v}$ given by $\tilde{J}_i(0) = 0$ and $\tilde{J}'(0) = I(e_i)$.

Again, we set

$$\tilde{j}(t, \tilde{v}) = \sqrt{\det(\tilde{g}_{ij}(\exp_{\tilde{p}} t\tilde{v}))} \qquad \text{and} \qquad \tilde{\Theta}(t, \tilde{v}) = t^{n-1}\tilde{j}(t, \tilde{v})$$

Theorem 3.3.1 (Bishop's Comparison Theorem I)**.** *Let M and $\tilde{M}$ be two $n-$dimensional Riemannian manifolds with sectional curvature $K_M(\Sigma)$ and $K_{\tilde{M}}(\tilde{\Sigma})$ respectively. Suppose that for any $2-$planes $\tilde{\Sigma} \subset T_{\tilde{c}_{\tilde{v}}(t)}\tilde{M}$, $\Sigma \subset T_{c_v(t)}M$ and for $t \in [0, \tilde{T}]$ we have $K_{\tilde{M}}(\tilde{\Sigma}) \geq K_M(\Sigma)$. Then*

$$\frac{\partial}{\partial t}\Big(\frac{\Theta(t, v)}{\tilde{\Theta}(t, \tilde{v})}\Big) \geq 0 \qquad\qquad \frac{\partial}{\partial t}\Big(\frac{j(t, v)}{\tilde{j}(t, \tilde{v})}\Big) \geq 0 \tag{3.9}$$

for all $t \in [0, \tilde{T})$.

From here it follows that

$$\Theta(t, v) \geq \tilde{\Theta}(t, \tilde{v}) \qquad\qquad j(t, v) \geq \tilde{j}(t, \tilde{v}) \tag{3.10}$$

for all $t \in [0, \tilde{T}]$.

Further if equality holds for $\tau < \tilde{T}$, then equality holds for all $t \in [0, \tau]$ and it follows that $K(t) = \tilde{K}(t)$ and $\|J_i(t)\| = \|\tilde{J}_i(t)\|$.

Proof. Let

$A(t) = (< J_i(t), J_j(t) >)_{i,j=1,\ldots,n}$ and $\tilde{A}(t) = (< \tilde{J}_i(t), \tilde{J}_j(t) >)_{i,j=1,\ldots,n}$.

We observe that

$$\frac{\partial}{\partial t}\Big(\frac{\Theta(t, v)}{\tilde{\Theta}(t, \tilde{v})}\Big) = \Big(\sqrt{\frac{\det A(t)}{\det \tilde{A}(t)}}\Big)'$$

Developing the derivative, we note that to prove (3.9) it is enough to show

$$\Big(\frac{\det A(t)}{\det \tilde{A}(t)}\Big)' = \frac{\det \tilde{A}(\det A)' - (\det A)(\det \tilde{A})'}{(\det \tilde{A})^2} \geq 0 \tag{3.11}$$

Let $B(t) = A^*(t)A(t)$, where $A^*(t)$ is the adjoint of $A(t)$.

From now on we drop the argument of the matrices A and B to simplify the notation.

Then, $\det B = (\det A)^2$.

We have:

$$(\log \det A)' = \frac{1}{2}(\log \det B)' =$$
$$= \frac{1}{2}\operatorname{tr}(B^{-1}B') = \frac{1}{2}\sum_j \frac{< J_i(t), J_i'(t) >}{\|J_i(t)\|} \geq$$
$$\geq \frac{1}{2}\sum_j \frac{< \tilde{J}_i(t), \tilde{J}_i'(t) >}{\|\tilde{J}_i(t)\|} = (\log \det \tilde{A})'$$

where the inequality follows from the Rauch Comparison Theorem.

Therefore

$$\frac{(\det A)'}{\det A} = (\log \det A)' \geq (\log \det \tilde{A})' = \frac{\det \tilde{A}}{\det \tilde{A}}$$

Hence:

$$\det \tilde{A}(\det A)' - (\det A)\det \tilde{A} \geq 0$$

which implies (3.11).

Now (3.10) follows noticing that since the two derivatives are non-negative, then the functions are incresing on $[0, \tilde{T}]$.

We drop the case of equality because it can be easily handled. □

Remark 3.3.2. If $\tilde{M}$ has constant sectional k, then

$$\tilde{\Theta}(t, \tilde{v}) = t^{n-1}s_k^{n-1}(t), \qquad s_k(t) \begin{cases} \sin(\sqrt{k}t) & k > 0 \\ t & k = 0 \\ \sinh(\sqrt{-k}t) & k < 0 \end{cases}$$

Hence, for a manifold M with sectional curvature $K_M(\Sigma) \leq k$ for all $t \in [0, \tilde{T}]$, we have

$$\frac{\partial}{\partial t}\Big(\frac{\Theta(t, v)}{t^{n-1}s_k^{n-1}(t)}\Big) \geq 0 \qquad \frac{\partial}{\partial t}\Big(\frac{j(t, v)}{s_k^{n-1}(t)}\Big) \geq 0$$

for all $t \in [0, \tilde{T})$ and

$$\Theta(t, v) \geq t^{n-1}s_k^{n-1}(t) \qquad j(t, v) \geq s_k^{n-1}(t)$$

for all $t \in [0, \tilde{T}]$.

As a consequence we have an inequality on the volume of the metric balls.

Proposition 3.3.3 ([Sak], Corollary 3.2, part (1), p. 155). *Let M and $\tilde{M}$ two Riemannian manifold with sectional curvature as in Theorem 3.3.1 and let $B_r(p)$ and $B_r(\tilde{p})$ be metric balls in M and $\tilde{M}$ respectively. Then*

$$\mathrm{Vol}(B_r(\tilde{p})) \leq \mathrm{Vol}(B_r(p))$$

Moreover equality holds if and only if $B_r(p)$ is isometric to $B_r(\tilde{p})$.

We state the analogous comparison theorem considering the Ricci curvature.

Theorem 3.3.4 (Bishop's Comparison Theorem II,[Sak], Theorem 3.1, part (2), p. 154). *Let M_k be an Riemannian manifold of dimension n with constant sectional curvature k. Suppose that M is a Riemannian manifold with Ricci curvature such that $Ric(v) \geq (n-1)k$ for all $v \in SM$. Then*

$$\frac{\partial}{\partial t}\Big(\frac{\Theta(t,v)}{t^{n-1}s_k^{n-1}(t)}\Big) \leq 0 \qquad \frac{\partial}{\partial t}\Big(\frac{j(t,v)}{s_k^{n-1}(t)}\Big) \leq 0 \tag{3.12}$$

for all $t \in [0,T)$.

Therefore we have

$$\Theta(t,v) \leq t^{n-1}s_k^{n-1}(t) \qquad j(t,v) \leq s_k^{n-1}(t) \tag{3.13}$$

for all $t \in [0,T]$.

Moreover if equality holds for some $\tau \in [0,T)$ then equality holds for all $t \in [0,\tau]$ and we have $J_i(t) = s_k(t)E_i(t)$ for $i = 1,\dots,n$ where $E_i(t)$ are parallel vector fields along γ.

We do not give the proof of this theorem because it does not involve any other geometric method but it is based on rewriting (3.12) to get

$$\frac{\partial_t\Theta(t,v)}{\Theta(t,v)} \leq \frac{(ts_k(t))'}{ts_k(t)}$$

and defining two functions satisfied by the LHS and the RHS to be compared to get the claim.

Also in this case we have a consequence on the volume of the metric balls.

Proposition 3.3.5 ([Sak], Corollary 3.2, part (2), p. 155). *Let M be an $n-$dimensional Riemannian manifold and M_k be the $n-$dimensional Riemannian manifold with constant sectional curvature k. Suppose that the Ricci curvature of M is such that $Ric(v) \geq (n-1)k$ for all $v \in SM$. Let $B_r(p)$ be the metric ball in M and B_r^k be the metric ball in M_k with radius r and independent of the centre. Then*

$$\mathrm{Vol}(B_r(p)) \leq \mathrm{Vol}(B_r^k)$$

Chapter 4

The Laplacian

In the first three chapters we recalled and introduced basic concepts to work with a Riemannian manifold. Now we turn to the study of an operator, the Laplacian, for C^∞ functions on a Riemannian manifold.

We already know that the Laplacian is defined as $\Delta f = -\sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$ for functions $f \in C^\infty(\mathbb{R}^n)$. The minus sign is to make it a positive operator in spectral geometry.

Our aim is to transfer this operator on a Riemannian contest, giving a general understanding of how it can be calculated and in what kind of problems it is involved.

We begin with the main definitions in Section 4.1 to carry on with the well-known Divergence Theorem and Green's Formula in Section 4.2. We conclude with eigenvalue problems and main facts about eigenvalues of the Laplacian in Section 4.3.

For all the proofs we refer to [Cha1], unless differently specified.

4.1 First definitions

Definition 4.1.1. Let M be a Riemannian manifold and f be a smooth function on M, i.e. $f \in C^\infty(M)$. We define the gradient of f, ∇f, as the vector field such that for any $X \in \mathcal{X}(M)$

$$< \nabla f(p), X(p) >= X(f)(p)$$

To avoid confusion with the notation used for the gradient and the one used for the Levi-Civita connection, from now on we shall denote the Levi-Civita connection with D.

Definition 4.1.2. Let M be an $n-$dimensional Riemannian manifold and let X be a vector field on M. We define the divergence of X as follows:

$$\operatorname{div} X = \operatorname{tr} DX = \sum_{i=1}^{n} < D_{E_i} X, E_i >$$

where $E_1, \dots, E_n$ is an orthonormal frame.

The divergence of a vector field has the following properties. For every function f and vector fields X, Y on M:

i) $\operatorname{div}(X + Y) = \operatorname{div} X + \operatorname{div} Y$

ii) $\operatorname{div}(fX) = f(\operatorname{div} X) + < \nabla f, X >$

Definition 4.1.3. Let M be a Riemannian manifold and let $f \in C^{\infty}(M)$. We define the Laplacian of f as the function

$$\Delta f = -\operatorname{div}(\nabla f) = -\operatorname{tr}(D\nabla f) = -\sum_{i=1}^{n} < D_{E_i} \nabla f, E_i >$$

where $E_1, \dots, E_n$ is an orthonormal frame.

For f, h maps on M the following properties hold:

i) $\Delta\,(f + h) = \Delta\, f + \Delta\, h$

ii) $\operatorname{div}(h(\nabla f)) = -h(\Delta\, f) + < \nabla f, \nabla h >$

iii) $\Delta\,(fh) = h(\Delta\, f) - 2 < \nabla f, \nabla h > + f(\Delta\, h)$

The divergence and (henceforth) the Laplacian can be expressed in local coordinate.

Let U be an open set in M, $\psi : U \longrightarrow \mathbb{R}^n$ be a chart on M and $\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\}$ be a an orthonormal basis of T_pM at the point $p \in U$. The Riemannian metric g on M is given by the matrix:

$$G = (g_{jk}), \qquad g_{jk} =< \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k} >$$

and its inverse is

$$G^{-1} = (g^{jk}), \qquad g^{jk} = g_{jk}^{-1}$$

Consequently the vector fields ∇f and X are locally given by:

$$\nabla f = \sum_{j,k=1}^{n} \Big(g^{jk} \frac{\partial f}{\partial x_j} \Big) \frac{\partial}{\partial x_k} \tag{4.1}$$

$$X = \sum_{i=1}^{n} \alpha_i \frac{\partial}{\partial x_j} \tag{4.2}$$

Hence, we have (see [Cha1, p. 5ff] for the complete calculation):

$$\operatorname{div} X = \frac{1}{\sqrt{\det G}} \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \big(\alpha_i \sqrt{\det G} \big) \tag{4.3}$$

and

$$\Delta f = -\frac{1}{\sqrt{\det G}} \sum_{j,k=1}^{n} \frac{\partial}{\partial x_j} \Big(g^{jk} \sqrt{\det G} \frac{\partial f}{\partial x_k} \Big) \tag{4.4}$$

4.2 Divergence Theorem and Green's Formula

Here we state two important integral theorems, the Divergence Theorem and the Green Formula.

For their proofs we refer to [Sak, Theorem 5.11, p. 71] and [Sak, Corollary 5.13, p. 73], respectively.

Theorem 4.2.1 (Divergence Theorem). *(i) Let X be a C^1 vector field on a Riemannian manifold M with compact support on M, then*

$$\int_M (\operatorname{div} X)\, dvol_n = 0 \tag{4.5}$$

(ii) Let X be a C^1 vector field on $\overline{M}$ with compact support on $\overline{M}$. Then

$$\int_M (\operatorname{div} X)\, dvol_n = \int_{\partial M} < X, \nu > \, dvol_{n-1} \tag{4.6}$$

where $dvol_{n-1}$ denotes the measure on ∂M induced by the Riemannian measure on M and ν denotes the outward unit normal.

Theorem 4.2.2 (Green's Formula). *(i) Let $h \in C^1, f \in C^2$ be two maps on M such that $h(\nabla f)$ has compact support on M. Then*

$$\int_M \{-h\Delta\, f + < \nabla h, \nabla f >\}\, dvol_n = 0 \tag{4.7}$$

Moreover, if we suppose $h \in C^2$ and both f, h with compact support, then

$$\int_M \{h\Delta\, f - f\Delta\, h\}\, dvol_n = 0 \tag{4.8}$$

(ii) Let $h \in C^1(\overline{M}), f \in C^2(\overline{M})$ such that $h(\nabla f)$ has compact support on $\overline{M}$. Then

$$\int_M \{-h\Delta\, f + < \nabla h, \nabla f >\}\, dvol_n = \int_{\partial M} h(\nu f)\, dvol_{n-1} \tag{4.9}$$

Moreover if we assume $h \in C^2(\overline{M})$ and both f, h with compact support on $\overline{M}$, then

$$\int_M \{-h\Delta\, f + f\Delta\, h\}\, dvol_n = \int_{\partial M} \{h(\nu f) - f(\nu h)\}\, dvol_{n-1} \tag{4.10}$$

where $dvol_{n-1}$ denotes the measure on ∂M induced by the Riemannian measure on M and ν denotes the outward unit normal.

4.3 Eigenvalue problems

Considering the Laplacian of a function f, we are interesting in its eigenvales. This problem is known as the eigenvalue problem.

According to the conditions on the boundary of the manifold we talk about Dirichlet or Neumann eigenvalue problem.

Dirichlet Eigenvalue Problem: let $\overline{M}$ be a compact, connected, $n-$dimensional Riemannian manifold with $\partial M \neq \emptyset$ and let $f \in C^2(M) \cap C^0(\overline{M})$. We want to find all the real number λ such that

$$\begin{cases} \Delta f = \lambda f & \text{on } M \\ f = 0 & \text{on } \partial M \end{cases} \tag{4.11}$$

Neumann Eigenvalue Problem: let $\overline{M}$ be a compact, connected, $n-$dimensional Riemannian manifold with $\partial M \neq \emptyset$ and let $f \in C^2(M) \cap C^1(\overline{M})$. We want to find all the real number λ such that

$$\begin{cases} \Delta f = \lambda f & \text{on } M \\ \nu f = 0 & \text{on } \partial M \end{cases} \tag{4.12}$$

where ν is the outward normal vector field on ∂M.

The numbers λ are called eigenvalues of the Laplacian, the vector spaces of the solutions to (4.11) and (4.12) are called eigenspaces and the functions f are called eigenfunctions. Moreover the set of all eigenvalues on a manifold M is called the spectrum of M.

4.3.1 Basic facts about the eigenvalues of the Laplacian

We now give some basics results about the eigenvalues of a manifold which will be useful in the next chapters.

First of all we introduce the space $L^2(M)$ for functions on a Riemannian manifold M which we equip with an inner product.

Definition 4.3.1. Let M be an Riemannian manifold of dimension n. The space $L^2(M)$ is the Hilbert space given by all the function f on M such that

$$\int_M |f|^2 \, dvol_n < +\infty$$

with the inner product

$$(f, g) = \int_M f g \, dvol_n$$

We remark that the norm of a function $f \in L^2(M)$ is

$$\|f\|^2 = (f, f)$$

Definition 4.3.2. A function f is admissible for the Dirichlet eigenvalue problem if it belongs to the completion of C^∞ function compactly supported on M. It is admissible for the Neumann eigenvalue problem if it belongs to the completion of the space $\{f \in C^\infty(M) \mid \int_M |f|^2 + |\nabla f|^2 \, dvol_n < +\infty\}$.

Theorem 4.3.3 ([Cha1] Theorem 1, p. 8)**.** *For both the Dirichlet and Neumann eigenvalue problems, the set of all the eigenvalues can be arranged in the following sequence:*

$$0 \leq \lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots$$

converging to $+\infty$ and each associated eigenspace is finite dimensional.
Moreover, eigenspaces belonging to distinct eigenvalues are orthogonal in $L^2(M)$ and $L^2(M)$ is the direct sum of all the eigenspaces.
Furthermore, each eigenfunction is C^∞ on $\overline{M}$.

The reason of this theorem is all in the application of Green's Formula. In fact, taking ϕ an eigenfunction relative to the eigenvalue λ and using Green's Formula (4.7) or (4.9) according to the case, with $f = h = \phi$,we get

$$\lambda = \|\phi\|^{-1} \int_M |\nabla \phi|^2 dvol_n \geq 0 \tag{4.13}$$

Therefore, if λ is zero the eigenfunction has to be a constant function. Considering the first eigenvalue for the Durichlet eigenvalue problem, λ_1, from (4.5) we have $\lambda_1 > 0$. While for the Neumann eigenvalue problem we get $\lambda_1 = 0$.

The orthogonality is a consequence of Green's Formula as well.

We consider ϕ and ψ two eigenfunctions related to the eigenvalues λ_k and λ_j respectively and we apply Green's Formula (4.8) or (4.10) according to the case, with $\phi = f$ and $\psi = h$, we have

$$0 = \int_M \{\phi\Delta\,\psi - \psi\Delta\,\phi\}\, dvol_n = (\lambda_j - \lambda_k)\int_M \phi\psi\, dvol_n$$

from which orthogonality follows.

4.3.2 Rayleigh quotient

We will see in the next chapters that the first non trivial eigenvalue of the Laplacian can be bounded according to the hypothesis on the curvature

Here we just provide an upper bound using the Rayleigh quotient.

Definition 4.3.4. Let M be a Riemannian manifold and let $f \in C^\infty(M)$. We define the Rayleigh quotient as follow:

$$R(f) = \frac{\int_M \|\nabla f\|^2\, dvol_n}{\int_M f^2\, dvol_n}$$

The following theorem holds.

Theorem 4.3.5. *Let M be a connected Riemannian manifold with compact closure and nonempty piecewise C^∞ boundary and consider the eigenvalues $\lambda_1 \le \lambda_2 \le \dots,$ of the Dirichlet eigenvalue problem. Then, for any non-vanishing admissible function f for this problem we have:*

$$\lambda_1 \le R(f)$$

with equality if and only if f is an eigenfunction of λ_1.

For the proof and further investigations we refer to [Cha1, Section 5, Chapter 1]

Chapter 5

The first eigenvalue of the Laplacian

In this chapter we look at how we can bound the first eigenvalue of the Laplacian.

The first result we discuss is Lichnerowicz' Theorem. This theorem is similar to Bonnet-Myers Theorem stated in Section 2.3: the hypothesis on the curvature is the same but the diameter is replaced by the first eigenvalue of the Laplacian λ_1, and we obtain a lower bound instead of an upper one. Therefore, λ_1 contains information about the geometry of the manifold itself. This is also confirmed by the fact that when the inequality becomes sharp, the manifold is homeomorphic to the $n-$dimensional sphere and, vice versa, if the manifold is homeomorphic to the sphere, the equality is an inequality indeed.

In a second step we drop the condition on the curvature and we show that for a compact Riemannian manifold λ_1 can be bounded from below by Cheeger's constant.

We start with the complete calculation of the spectrum of the round sphere of radius r in Section 5.1: it will be useful to understand the equality case in Lichnerowicz' Theorem. In Section 5.2 we state and prove Bochner's Formula, the main ingredient of the proof of the Lichnerowicz' Theorem which is stated and proved in Section 5.3. Finally, in Section 5.4 we give the definition of Cheeger's constant and we state Cheeger's Inequality.

5.1 The spectrum of S^n and S^n_r

Let $S^n = \{x \in \mathbb{R}^{n+1} \mid |x|^2 = 1\}$ be the $n-$dimensional unit sphere.

The calculation of its spectrum is made up of two parts.

First of all we express the Laplacian on $\mathbb{R}^{n+1}$ in spherical coordinates and we apply this expression to an homogeneous harmonic polynomial obtaining a formula for the eigenvalues of S^n.

Secondly, we prove that the eigenvalues obtained are all the eigenvalues of S^n showing that the homogeneous harmonic polynomials restricted to S^n are dense in $L^2(S^n)$.

Lemma 5.1.1. *The Laplacian of a C^∞ function F on $\mathbb{R}^{n+1}$ in spherical coordinates is given by*

$$\Delta_{\mathbb{R}^{n+1}} F = \frac{1}{r^2}\Delta_{S^n}(F|_{S^n}) - \frac{\partial^2 F}{\partial r^2} - \frac{n}{r}\frac{\partial F}{\partial r} \tag{5.1}$$

Proof. We consider spherical coordinates in $\mathbb{R}^{n+1}$, i.e $\forall\, x \in \mathbb{R}^{n+1}$, $x = r\xi$ with $r \in (0, +\infty)$ and $\xi \in S^n$.

Let $\varphi = (y_1, \ldots, y_n) : U \longrightarrow \mathbb{R}^n$ be a chart on S^n whose associated Riemannian metric is $H = (h_{ij})$.

A chart on an open cone C in $\mathbb{R}^{n+1}$ is given by

$$\psi = (r, z_1, \ldots, z_n) : C \longrightarrow \mathbb{R}^{n+1}, \qquad \psi(r\xi) = r\varphi(\xi)$$

We construct the Riemannian metric $G = (g_{ij})$ associated to ψ.

We have that

$$\frac{\partial}{\partial r} = \xi \qquad\qquad \frac{\partial}{\partial z_i} = r\frac{\partial}{\partial y_i}$$

Therefore:

$$< \frac{\partial}{\partial r}, \frac{\partial}{\partial r} >= 1$$

$$< \frac{\partial}{\partial z_i}, \frac{\partial}{\partial r} >= r < \frac{\partial}{\partial y_i}, \xi >= 0$$

$$< \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j} >= r^2 < \frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j} >$$

and the elements of G are

$$g_{rr} = 1 \qquad g_{rj} = 0 \qquad g_{ij} = r^2 h_{ij} \qquad \forall i, j = 1, \ldots, n$$

and

$$\sqrt{\det G} = r^n \sqrt{\det H}$$

Now, let $F \in C^\infty(\mathbb{R}^{n+1})$, we have

$$\Delta_{\mathbb{R}^{n+1}} F = -\frac{1}{\sqrt{\det G}} \Big[\sum_{i,j} \Big(\frac{\partial}{\partial y_j} (g^{ij} \sqrt{\det G} \frac{\partial F}{\partial y_i}) \Big) + \frac{\partial}{\partial r} \Big(g^{rr} \sqrt{\det G} \frac{\partial F}{\partial r} \Big) \Big] =$$

$$= -\frac{1}{r^n \sqrt{\det H}} \Big[\sum_{i,j} \frac{\partial}{\partial z_j} \Big(r^{n-2} h^{ij} \sqrt{\det H} \frac{\partial F}{\partial z_i} \Big) + \frac{\partial}{\partial r} \Big(r^n \sqrt{\det H} \frac{\partial F}{\partial r} \Big) \Big] =$$

$$= -\frac{1}{r^2 \sqrt{\det H}} \sum_{i,j} \frac{\partial}{\partial z_j} \Big(h^{ij} \sqrt{\det H} \frac{\partial F}{\partial z_i}\Big|_{S^n} \Big) - \frac{1}{r^n} \frac{\partial}{\partial r} \Big(r^n \frac{\partial F}{\partial r} \Big) =$$

$$= \frac{1}{r^2} \Delta_{S^n}(F|_{S^n}) - \frac{\partial^2 F}{\partial r^2} - \frac{n}{r} \frac{\partial F}{\partial r}$$

□

We choose a function $F(x) = r^k \hat{F}(\xi)$. For this function we have

$$\Delta_{\mathbb{R}^{n+1}} F = r^{-2} \Delta_{S^n}(r^k \hat{F}(\xi)) - k(k-1) r^{k-2} \hat{F}(\xi) - nk r^{k-2} \hat{F}(\xi)$$

that is

$$\Delta_{\mathbb{R}^{n+1}} F = r^{k-2} \Delta_{S^n} \hat{F} - [k(k-1+n) r^{k-2}] \hat{F} \tag{5.2}$$

If F is an harmonic polynomial of degree k, i.e its Laplacian vanishes, then $\hat{F}$ is an eigenfunction of Δ_{S^n} of the eigenvalue $k(k-1+n)$.

Now let

$$\mathcal{P}_k := \{ \sum_{|I|=k} a_I x^I \mid a_I \text{ constants and } x \in \mathbb{R}^{n+1} \}$$

$$\mathcal{H}_k := \{\sum_{|I|=k} a_I x^I \in \mathcal{P}_k \mid \Delta_{\mathbb{R}^{n+1}}\Big(\sum_{|I|=k} a_I x^I\Big) = 0\}$$

Namely, $\mathcal{P}_k$ is the vector space of the homogeneous polynomial in $\mathbb{R}^{n+1}$ and $\mathcal{H}_k \subset \mathcal{P}_k$ is the subspace of the harmonic homogeneous polynomial in $\mathbb{R}^{n+1}$.

Further, we denote with $\tilde{\mathcal{P}}_k$ and $\tilde{\mathcal{H}}_k$ the restriction to S^n of $\mathcal{P}_k$ and $\mathcal{H}_k$ respectively.

We show that $\bigoplus_{k=1}^{\infty} \tilde{\mathcal{H}}_k$ is dense in $L^2(S^n)$. In order to do so, we introduce an inner product on $\bigoplus_{k=1}^{\infty} \tilde{\mathcal{P}}_k$ by

$$(P, Q) := \int_{S^n} \tilde{P}\tilde{Q}\, dvol_n \tag{5.3}$$

Lemma 5.1.2. *With respect to the inner product (5.3) we have the following orthogonal decompositions:*

$$\mathcal{P}_k = \mathcal{H}_{2k} \oplus r^2\mathcal{H}_{2k-2} \oplus \ldots \oplus r^{2k}\mathcal{H}_0$$

$$\mathcal{P}_{2k+1} = \mathcal{H}_{2k+1} \oplus r^2\mathcal{H}_{2k-1} \oplus \ldots \oplus r^{2k}\mathcal{H}_1$$

Proof. We note first that $\mathcal{H}_0 = \mathcal{P}_0$ contains all the constant functions and $\mathcal{H}_1 = \mathcal{P}_1$ contains all the homogeneous linear polynomial of degree 1.

Secondly, we point out that it is enough to prove

$$\mathcal{P}_{k+2} = \mathcal{H}_{k+2} \oplus r^2\mathcal{P}_k \quad \text{for } k = 0, 1, 2, \ldots \tag{5.4}$$

We proceed by induction.

Let $k = 0$, picking up a $P \in \mathcal{H}_2$ we have:

$$(P, r^2) = \int_{S^n} \tilde{\mathcal{P}}\, dvol_n = \frac{1}{2(n-1)} \int_{S^n} \Delta\tilde{P}\, dvol_n = 0$$

Since P is arbitrary, it means that $\mathcal{H}_2 \perp r^2\mathcal{P}_0$.

We have $\mathcal{H}_2 \oplus r^2\mathcal{P}_0 \subset \mathcal{P}_2$.

On the other hand, if $P \in \mathcal{P}_2$ and $P \perp r^2\mathcal{P}_0$, then $\int_{S^n} \tilde{P}\, dvol_n = (P, r^2) = 0$. Moreover, by (5.2) we get

$$\widetilde{\Delta_{\mathbb{R}^{n+1}} P} = \Delta_{S^n}\tilde{P} - 2(n+1)\tilde{P}$$

Integration over S^n gives

$$\int_{S^n} \widetilde{\Delta_{\mathbb{R}^{n+1}} P}\, dvol_n = 0$$

Since $\widetilde{\Delta_{\mathbb{R}^{n+1}} P} \in \mathcal{P}_0$, i.e. it is a constant, and its integral is zero, it must be zero itself. Therefore P is homogeneous and $P \in \mathcal{H}_2$.

Hence, $\mathcal{P}_2 \subset \mathcal{H}_2 \oplus r^2\mathcal{P}_0$.

Therefore we have proved $\mathcal{P}_2 = \mathcal{H}_2 \oplus r^2\mathcal{P}_0$.

Now we suppose (5.4) holds up to $k-1$, we prove it for k. The argument follows the same steps as for $k = 0$.

We have:

$$\mathcal{P}_k = \mathcal{H}_k \oplus r^2\mathcal{P}_{k-2} = \mathcal{H} \oplus r^2\mathcal{H}_{k-2} \oplus r^4\mathcal{P}_{k-4} = \ldots = \mathcal{H}_k \oplus r^2\mathcal{H}_{k-2} \oplus \ldots \oplus r^k\mathcal{H}_0$$

If $P \in \mathcal{H}_{k+2}$ and $Q \in \mathcal{H}_{k-2l}$ where $l = 0, \ldots, \frac{k}{2}$, we have the following:

$$(P, r^{2l+2}Q) = \int_{S^n} \tilde{P}\tilde{Q}\, dvol_n$$

$$\Delta_{S^n}\tilde{P} = (k+2)(n+k+1)\tilde{P}$$

$$\Delta_{S^n}\tilde{Q} = (k-2l)(n+k-2l-1)\tilde{Q}$$

We get:

$$(\Delta_{\mathbb{R}^{n+1}}P, Q) = \int_{S^n} \widetilde{\Delta_{\mathbb{R}^{n+1}} P}\tilde{Q}\, dvol_n =$$

$$= (k+2)(n+k+1)\int_{S^n} \tilde{P}\tilde{Q}\, dvol_n = (k+2)(n+k+1)(P, r^{2l+2}Q)$$

and

$$(P, \Delta_{\mathbb{R}^{n+1}}Q) = \int_{S^n} \tilde{P}\widetilde{\Delta_{\mathbb{R}^{n+1}} Q}\, dvol_n =$$

$$= (k-2l)(n+k-2l-1)\int_{S^n} \tilde{P}\tilde{Q}\, dvol_n = (k-2l)(n+k-2l-1)(P, r^{2l+2}Q)$$

Hence

$$(P, r^{2l+2}Q) = \frac{(k-2l)(n+k-2l-1)}{(k+2)(n+k+1)}(P, r^{2l+2}Q) \quad \Longrightarrow \quad (P, r^{2l+2}Q) = 0$$

This means that $\mathcal{H}_{k+2} \perp r^2\mathcal{P}_k$, since P and Q are arbitrary.

Then, $\mathcal{P}_{k+2} \supset \mathcal{H}_{k+2} \oplus r^2\mathcal{P}_k$.

On the other hand, if $P \in \mathcal{P}_{k+2}$ and $P \perp r^2\mathcal{P}_k$, for any $Q \in \mathcal{H}_{k-2l}$, $l = 0, \dots, \frac{k}{2}$, we have

$$(\Delta_{\mathbb{R}^{n+1}} P, r^{2l+2}Q) = \int_{S^n} \Delta_{S^n}\tilde{P}\tilde{Q}\, dvol_n - (k+2)(n+k+1)\int_{S^n} \tilde{P}\tilde{Q}\, dvol_n =$$

$$= \int_{S^n} \tilde{P}\Delta_{S^n}\tilde{Q}\, dvol_n = (k-2l)(n+k-2l-1)(P, r^{2l+2}Q) = 0$$

where the first equality comes from (5.2).

This means that $\Delta_{\mathbb{R}^n}\tilde{P} \perp r^2\mathcal{P}_k$ and, since it is an element of $\mathcal{P}_k$, it has to be zero. Therefore $P \in \mathcal{H}_{k+2}$.

Hence, we have proved that $\mathcal{P}_{k+2} \subset \mathcal{H}_{k+2} \oplus r^2\mathcal{P}_k$.

Finally we have $\mathcal{P}_{k+2} = \mathcal{H}_{k+2} \oplus r^2\mathcal{P}_k$, which concludes the proof. $\square$

With the above decompositions we have the following equality

$$\bigoplus_{k=1}^{\infty} \tilde{\mathcal{H}}_k = \bigoplus_{k=1}^{\infty} \tilde{\mathcal{P}}_k$$

and the RHS has the following properties:

i) it contains constant functions, in fact they are in $\tilde{\mathcal{H}}_0$;

ii) there exists a $P \in \bigoplus_{k=1}^{\infty} \tilde{\mathcal{H}}_k$ such that $P(p) \neq P(q)$ for $p \neq q$, in fact we may choose it as a polynomial of degree one.

By Stone-Weierstrass Theorem (see [Hew-Str, Proposition 7.30, p. 95]) we have that $\bigoplus_{k=1}^{\infty} \tilde{\mathcal{H}}_k$ is dense in $C^\infty(S^n)$ and so in $L^2(S^n)$. Then we can state the following.

Proposition 5.1.3. *The eigenvalues of the eigenvalue Dirichlet problem on the $n-$dimensional sphere of radius one are given by*

$$\lambda_k = k(n+k-1) \quad \textit{for} \quad k = 0, 1, 2, \dots$$

Moreover the eigenspace of λ_k is $\tilde{\mathcal{H}}_k$ and its multiplicity is

$$\binom{n+k}{k} - \binom{n+k-2}{k-2}$$

Proof. The formula for the eigenvalues of S^n and the fact that their eigenspaces are the $\tilde{\mathcal{H}}_k$s come from the discussion above. We just need to check their multiplicity which is equal to the dimension of the eigenspace.

We have:

$$\dim \tilde{\mathcal{H}}_k = \dim \mathcal{H}_k = \dim \mathcal{P}_k - \dim \mathcal{P}_{k-2} = \binom{n+k}{k} - \binom{n+k-2}{k-2}$$

□

Now we look at the spectrum of $S^n_r = \{x \in \mathbb{R}^{n+1} \mid |x|^2 = r\}$ the $n-$dimensional sphere of radius r.

Proposition 5.1.4. *The eigenvalues of the eigenvalue Dirichlet problem on the $n-$dimensional sphere of radius r are given by*

$$\lambda_k = \frac{1}{r^2} k(n+k-1) \quad \textit{for} \quad k = 0, 1, 2, \ldots$$

The proof comes applying the following scaling argument to the previous discussion.

Lemma 5.1.5. *Let S^n and S^n_r be the $n-$dimensional sphere of radius 1 and r respectively, and let g be the Riemannian metric on S^n and $\tilde{g}$ be the Riemannian metric on S^n_r respectively. Then the two metrics are related in the following way:*

$$\tilde{g}_{ij} = r^2 g_{ij} \qquad \sqrt{\det \tilde{g}} = r^n \sqrt{\det g}$$

Proof. Let U and V be two open set in S^n and S^n_r respectively. Then let ϕ be a chart on S^n and φ be a chart on S^n_r, i.e.

$$\phi = (x_1, \ldots, x_n) : U \longrightarrow \mathbb{R}^n, \qquad \xi \longmapsto \phi(\xi)$$

$$\varphi = (y_1, \ldots, y_n) : V \longrightarrow \mathbb{R}^n, \qquad \eta := r\xi \longmapsto \varphi(\eta) := r\phi(\xi)$$

We have:

$$\frac{\partial}{\partial y_i} = r \frac{\partial}{\partial x_i}$$

This implies the following relations:

$$< \frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j} >= r^2 < \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} > \qquad \Longrightarrow \qquad \tilde{g}_{ij} = r^2 g_{ij}$$

$$\tilde{g} = r^2 g \qquad \Longrightarrow \qquad \sqrt{\det \tilde{g}} = r^n \sqrt{\det g}$$

as we claimed. □

Therefore we have:

$$\Delta_{(S_r^n, \tilde{g})} = -\frac{1}{r^n \sqrt{\det g}} \sum_{ij} \frac{\partial}{\partial x_i} \Big(r^{-2} g^{ij} r^n \sqrt{\det g} \frac{\partial}{\partial x_j} \Big) = \frac{1}{r^2} \Delta_{(S^n, g)}$$

and equations (5.1) and (5.2) become:

$$\Delta_{\mathbb{R}^{n+1}} F = \Delta_{S_r^n} (F|_{S_r^n}) - \frac{\partial^2 F}{\partial r^2} - \frac{n}{r} \frac{\partial F}{\partial r}$$

$$\Delta_{\mathbb{R}^{n+1}} F = r^k \Delta_{S_r^n} \hat{F} - [k(k-1+n) r^{k-2}] \hat{F}$$

and going again through the proof of Lemma 5.1.2 considering S_r^n instead of S^n we get the result.

5.2 Bochner's Formula

We state Bochner's Formula in the following way:

Theorem 5.2.1 (Bochner's Formula)**.** *Let (M, g) be a complete Riemannian manifold. Then for any $f \in C^\infty(M)$ we have*

$$-\frac{1}{2} \Delta(\|\nabla f\|^2) = \| \operatorname{Hess} f \|^2 - < \nabla f, \nabla(\Delta f) > + \operatorname{Ric}(\nabla f) \tag{5.5}$$

The proof is postponed since first we need to understand the meaning of $\| \operatorname{Hess} f \|^2$.

We start with the following definition.

Definition 5.2.2. Let M be a Riemannian manifold and $f \in C^\infty(M)$. We define the Hessian of f as

$$\operatorname{Hess} f(X,Y) =< D_X \nabla f, Y >= \tag{5.6}$$

$$= X(Yf) - < \nabla f, D_X Y > \tag{5.7}$$

Remark 5.2.3. The Hessian of a function f can also be defined in a tensorial manner as the second covariant derivative in X and Y of the function f, i.e. $\operatorname{Hess} f(X,Y) = D^2_{X,Y} f$.

Lemma 5.2.4. *Let M be a Riemannian manifold and let $f \in C^\infty(M)$. The Hessian of f,* $\operatorname{Hess} f$, *is symmetric.*

Proof.

$$\begin{aligned}
\operatorname{Hess} f(X,Y) &= X(Y(f)) - < \nabla f, D_X Y >= \\
&= X(Y(f)) - < \nabla f, D_Y X > - < \nabla f, [X,Y] >= \\
&= X(Y(f)) - < \nabla f, D_Y X > - [X,Y](f) = \\
&= Y(X(f)) - < \nabla f, D_Y X >= \\
&= \operatorname{Hess} f(Y,X)
\end{aligned}$$

□

Definition 5.2.5. Let M be a $n-$dimensional Riemannian manifold and $f \in C^\infty(M)$. The norm of $\operatorname{Hess} f$ is defined as follows:

$$\| \operatorname{Hess} f \| = \sqrt{\sum_{i=1}^{n} < D_{E_i} \nabla f, D_{E_i} \nabla f >}$$

where $E_1, \ldots, E_n$ is an orthonormal frame.

Remark 5.2.6. This definition is independent of the choice of the orthonormal frame.

Remark 5.2.7. We observe that

$$\Delta f = -\operatorname{tr}(\operatorname{Hess} f) = -\sum_{i=1}^{n} \operatorname{Hess} f(E_i, E_i)$$

where $E_1, \ldots, E_n$ is an orthonormal frame.

Now we are ready to prove (5.5).

Proof of Theorem 5.2.1. Fix a point $p \in M$ and consider a local orthonormal frame $E_1, \ldots, E_n$ such that $< E_i, E_j >= \delta_{ij}$ and $D_{E_i} E_j(p) = 0$ for all i and j. The following calculation is carried out at the point p.

$-\frac{1}{2}\Delta(\|\nabla f\|^2) = \frac{1}{2}\operatorname{tr}(\operatorname{Hess} f\|\nabla f\|^2)$

$= \frac{1}{2}\sum_{i=1}^{n} \operatorname{Hess} \|\nabla f\|^2(E_i, E_i) =$

$= \frac{1}{2}\sum_{i=1}^{n} E_i(E_i(< \nabla f, \nabla f >)) - < \nabla(\|\nabla f\|^2), D_{E_i} E_i >=$ (by (5.7))

$= \frac{1}{2}\sum_{i=1}^{n} E_i(E_i(< \nabla f, \nabla f >)) =$ (because $D_{E_i} E_i = 0$)

$= \sum_{i=1}^{n} E_i(< D_{E_i}\nabla f, \nabla f >) =$ (by the Riemannian property)

$= \sum_{i=1}^{n} E_i(\operatorname{Hess} f(E_i, \nabla f)) =$ (by (5.6))

$= \sum_{i=1}^{n} E_i(\operatorname{Hess} f(\nabla f, E_i)) =$ (by the simmetry of the Hessian)

$= \sum_{i=0}^{n} E_i(< D_{\nabla f}\nabla f, E_i >) =$ (by (5.6))

$= \sum_{i=1}^{n} < D_{E_i} D_{\nabla f}\nabla f, E_i > - < D_{\nabla f}\nabla f, D_{E_i} E_i >=$ (by the Riemannian property)

$= \sum_{i=1}^{n} < D_{E_i} D_{\nabla f}\nabla f, E_i >=$ (by $D_{E_i} E_i = 0$)

$$= \sum_{i=1}^{n} < R(\nabla f, E_i)\nabla f, E_i > + \sum_{i=1}^{n} < D_{\nabla f} D_{E_i}\nabla f, E_i > + \sum_{i=1}^{n} < D_{[E_i,\nabla f]}\nabla f, E_i >$$

We analyse each term on its own.

The first sum is

$$\sum_{i=1}^{n} < R(\nabla f, E_i)\nabla f, E_i >= \mathrm{Ric}(\nabla f)$$

For the second sum, by the Riemannian property, we have

$$\sum_{i=1}^{n} < D_{\nabla f} D_{E_i} \nabla f, E_i >= \sum_{i=1}^{n} \nabla f(< D_{E_i}\nabla f, E_i >) - < D_{E_i}\nabla f, D_{\nabla f} E_i >$$

Since $\nabla f = \sum_{i=1}^{n} E_i(f)E_i$, we have at p

$$D_{\nabla f} E_i = \sum_{j=1}^{n} E_j(f) D_{E_j} E_i = 0 \tag{5.8}$$

Therefore $< D_{E_i}\nabla f, D_{\nabla f} E_i >= 0$.

On the other hand:

$\sum_{i=1}^{n} \nabla f(< D_{E_i}\nabla f, E_i >) = \nabla f\Big(\sum_{i=1}^{n} < D_{E_i}\nabla f, E_i > \Big) =$

$\nabla f\Big(\sum_{i=1}^{n} \mathrm{Hess}\, f(E_i, E_i)\Big) =$ (by (5.6))

$= \nabla f(\mathrm{tr}\,\mathrm{Hess}\, f) = -(\nabla f)(\Delta f) =$

$= - < \nabla(\Delta f), \nabla f >= - < \nabla f, \nabla(\Delta f) >$

Therefore

$$\sum_{i=1}^{n} < D_{\nabla f} D_{E_i} \nabla f, E_i >= - < \nabla f, \nabla(\Delta f) >$$

For the third sum we have

$\sum_{i=1}^{n} < D_{[E_i,\nabla f]}\nabla f, E_i >= \sum_{i=1}^{n} \mathrm{Hess}\, f([E_i, \nabla f], E_i) =$

$= \sum_{i=1}^{n} < D_{D_{E_i}\nabla f}\nabla f, E_i > - < D_{D_{\nabla f}E_i}\nabla f, E_i >=$

$= \sum_{i=1}^{n} < D_{D_{E_i}\nabla f}\nabla f, E_i >=$ (by (5.8))

$= \sum_{i=1}^{n} \operatorname{Hess} f(D_{E_i}\nabla f, E_i) =$

$\sum_{i=1}^{n} \operatorname{Hess} f(E_i, D_{E_i}\nabla f) =$ (by the simmetry of the Hessian)

$= \sum_{i=1}^{n} < D_{E_i}\nabla f, D_{E_i}\nabla f >= \| \operatorname{Hess} f\|^2$

Therefore we get

$$-\frac{1}{2}\Delta(\|\nabla f\|^2) = \| \operatorname{Hess} f\|^2 - < \nabla f, \nabla(\Delta f) > + \operatorname{Ric}(\nabla f)$$

□

5.3 Lichnerowicz' Theorem

We can now state and prove Lichnerowicz' Theorem

Theorem 5.3.1 (Lichnerowicz' Theorem)**.** *Let M be a complete Riemannian manifold of dimension n such that $\operatorname{Ric}(u) \geq (n-1)\delta$ for all $u \in SM$ and $\delta > 0$ constant. Then, M is compact and for the first eigenvalue of the Laplacian we have*

$$\lambda_1(M) \geq n\delta$$

Proof. We first note that the compactness of M comes from Bonnet-Myers Theorem, so we just need to care about the estimate of $\lambda_1(M)$.

Let f be an eigenfunction of Δ relative to the eigenvalue λ_1, that is, $\Delta f = \lambda_1 f$.

Substituting this expression of the Laplacian in Bochner's Formula (5.5), we have

$$-\frac{1}{2}\Delta(\|\nabla f\|^2) = \| \operatorname{Hess} f\|^2 - \lambda_1\|\nabla f\|^2 + \operatorname{Ric}(\nabla f) \tag{5.9}$$

For $\mathrm{Ric}(\nabla f)$ we have the following inequality

$$\mathrm{Ric}(\nabla f) \geq \|\nabla f\|^2(n-1)\delta \tag{5.10}$$

In fact, setting $v = \frac{\nabla f}{\|\nabla f\|}$, we have $\|v\|^2 = 1$ and

$$\mathrm{Ric}(\nabla f) = \mathrm{Ric}(\|\nabla f\| v) = \sum_{i=1}^{n} < R(E_i, \|\nabla f\| v)\|\nabla f\| v, E_i >=$$

$$= \sum_{i=1}^{n} \|\nabla f\|^2 < R(E_i, v)v, E_i >= \|\nabla f\|^2 \mathrm{Ric}(v) \geq \|\nabla f\|^2(n-1)\delta$$

Moreover, for $\|\operatorname{Hess} f\|^2$ it holds that

$$\|\operatorname{Hess} f\|^2 \geq \frac{(\operatorname{tr}\operatorname{Hess} f)^2}{n} \tag{5.11}$$

In fact:

$$(\operatorname{tr}\operatorname{Hess} f)^2 = \Big(\sum_{i=1}^{n} < D_{E_i}\nabla f, E_i > \Big)^2 \leq \Big(\sum_{i=1}^{n} \|D_{E_i}\nabla f\| \cdot \|E_i\|\Big)^2 \leq$$

$$\leq n \sum_{i=1}^{n} \|D_{E_i}\nabla f\|^2 \leq n \sum_{i=1}^{n} < D_{E_i}\nabla f, D_{E_i}\nabla f >= n\|\operatorname{Hess} f\|^2$$

where the second inequality is due to $\Big(\sum_{i=1}^{n} a_i\Big)^2 \leq n\sum_{i=1}^{n} a_i^2$.
We also note that, since $\Delta f = \lambda_1 f$, we get

$$\|\operatorname{Hess} f\|^2 \geq \frac{(\operatorname{tr}\operatorname{Hess} f)^2}{n} = \frac{(\Delta f)^2}{n} = \frac{\lambda_1^2 f^2}{n} \tag{5.12}$$

Applying (5.10) and (5.12) to (5.9) we have the following inequality

$$-\frac{1}{2}\Delta(\|\nabla f\|^2) \geq \frac{\lambda_1^2 f^2}{n} - \lambda_1\|\nabla f\|^2 = (n-1)\|\nabla f\|^2$$

Integration over M gives

$$\int_M -\frac{1}{2}\Delta(\|\nabla f\|^2)\, dvol_n \geq \int_M \frac{\lambda_1^2 f^2}{n} + [-\lambda_1 + (n-1)\delta]\|\nabla f\|^2\, dvol_n \tag{5.13}$$

By the Divergence Theorem (see Theorem 4.2.1) we have

$$\int_M -\frac{1}{2}\Delta(\|\nabla f\|^2)\, dvol_n = \frac{1}{2}\int_M \operatorname{div}\big(\nabla(\|\nabla f\|^2)\big)\, dvol_n = 0$$

Therefore, from (5.13) we obtain

$$[\lambda_1 - (n-1)\delta] \int_M \|\nabla f\|^2 \, dvol_n \geq \int_M \frac{\lambda_1^2 f^2}{n} \, dvol_n$$

and it follows that

$$[\lambda_1 - (n-1)\delta] R(f) \geq \frac{\lambda_1^2}{n}$$

where $R(f)$ is the Rayleigh quotient of the eigenfunction f which is equal to the first eigenvalue (see Theorem 4.3.5).

Then

$$[\lambda_1 - (n-1)\delta] \geq \frac{\lambda_1}{n} \quad \Longrightarrow \quad \lambda_1(1 - \frac{1}{n}) \geq (n-1)\delta \quad \Longrightarrow \quad \lambda_1 \geq n\delta$$

□

Remark 5.3.2. The statement of this theorem is optimal since equality is obtained in the case of the $n-$dimensional round sphere (see Section 5.1).

This remark can be restate in the following proposition.

Proposition 5.3.3. *Let M be a complete $n-$dimensional Riemannian manifold such that $Ric(u) \geq (n-1)\delta$ for all $u \in SM$ and $\delta > 0$ constant. If M is isometric to S^n_r then $\lambda_1(M) = \frac{n}{r^2}$.*

Indeed, we also have the following rigidity result.

Proposition 5.3.4. *Let M be a complete $n-$dimensional Riemannian manifold such that $Ric(u) \geq (n-1)\delta$ for all $u \in SM$ and $\delta > 0$ constant. If $\lambda_1(M) = \frac{n}{r^2}$ then M is isometric to S^n_r.*

The idea of the proof is to show that with these hypothesis the manifold has maximal diameter, i.e $diam(M) = \frac{\pi}{\sqrt{\delta}}$, and then to use Obata-Toponogov Theorem which states that if M is a manifold as in Proposition 5.3.4 and its diameter is exactly $\frac{\pi}{\sqrt{\delta}}$ then it is isometric to S^n_r (for the proof of Obata-Toponogov Theorem we refer to [Cha1, p. 83] and for a complete proof of Proposition 5.3.4 we refer to [Sak, p. 275ff]).

5.4 Cheeger's Inequality

In this section we look at another possible bound for λ_1 when the Riemannian manifold is complete and we do not have assumptions on the curvature.

Definition 5.4.1. Let M be a complete Riemannian manifold.

If M is compact, we define Cheeger's constant as

$$h_c(M) = \inf_S \left\{ \frac{\mathrm{Vol}_{n-1}(S)}{\min\{\mathrm{Vol}_n(M_1), \mathrm{Vol}_n(M_2)\}} \right\}$$

where S runs all over the $(n-1)-$dimensional compact submanifolds that divides M into two disjoint submanifolds M_1, M_2 whose boundary is S.

If M is non compact, we define

$$h_c(M) = \inf_S \left\{ \frac{\mathrm{Vol}_{n-1}(S)}{\mathrm{Vol}_n(M_1)} \right\}$$

where S runs through all smooth boundary of all open submanifold M_1 with compact closure.

Cheeger's constant is used to bound from below the first eigenvalue of the Laplacian for a compact Riemannian manifold and the proof of this fact is based on the Co-Area Formula.

Theorem 5.4.2 (Co-Area Formula, [Cha1], Theorem 1, p. 86)**.** *Let M be a compact Riemannian manifold with boundary and let $f \in L_1^1(M)$. Then for any non-negative measurable function g on M we have*

$$\int_M g\, dvol_n = \int_{-\infty}^{+\infty} \left(\int_{\{f=t\}} \frac{g}{\|\nabla f\|}\, dvol_{n-1} \right) dt$$

We can now prove Cheeger's Inequality.

Theorem 5.4.3 (Cheeger's Inequality)**.** *Let M be a compact Riemannian manifold. We have*

$$\lambda_1(M) \geq \frac{1}{4} h_c(M)^2 \tag{5.14}$$

Proof. Let f be an eigenfunction of $\lambda_1(M)$.

We have that $\nabla(f^2) = 2f\nabla f$ and, by Cauchy-Schwartz Inequality, we obtain

$$\Big(\frac{\int_M \|\nabla(f^2)\|\,dvol_n}{\int_M f^2\,dvol_n}\Big)^2 = \frac{\big(\int_M \|2f\nabla f\|\,dvol_n\big)^2}{\big(\int_M f^2\,dvol_n\big)^2} \leq$$
$$\leq 4\,\frac{\int_M f^2\,dvol_n \cdot \int_M \|\nabla f\|^2\,dvol_n}{\big(\int_M f^2\,dvol_n\big)^2} = 4\,\frac{\int_M \|\nabla f\|^2\,dvol_n}{\int_M f^2\,dvol_n} \tag{5.15}$$

Moreover, it holds that $\Delta f = \lambda_1 f$ and so $f\Delta f = \lambda_1 f^2$. Integration by parts gives

$$\int_M \|\nabla f\|^2\,dvol_n = \int_M f\Delta f\,dvol_n = \lambda_1 \int_M f^2\,dvol_n \tag{5.16}$$

where the first equality is Green's Formula (see Theorem 4.2.2)

Combining equations (5.15) and (5.16) we conclude

$$\lambda_1 \geq \frac{1}{4}\Big(\frac{\int_M \|\nabla(f^2)\|\,dvol_n}{\int_M f^2\,dvol_n}\Big) \tag{5.17}$$

By the Co-Area Formula we have:

$$\int_M \|\nabla(f^2)\|\,dvol_n = \int_0^{+\infty}\Big(\int_{\{f^2=t\}} 1\,dvol_{n-1}\Big)\,dt =$$
$$= \int_0^{+\infty} \mathrm{Vol}_{n-1}(\{f^2=t\})\,dt = \int_0^{+\infty} \frac{\mathrm{Vol}_{n-1}(\{f^2=t\})}{\mathrm{Vol}_n(\{f^2\geq t\})}\,\mathrm{Vol}_n(\{f^2\geq t\})\,dt \geq$$
$$\geq \inf_t \frac{\mathrm{Vol}_{n-1}(\{f^2=t\})}{\mathrm{Vol}_n(\{f^2\geq t\})}\int_0^{+\infty}\mathrm{Vol}_n(\{f^2\geq t\})\,dt \geq h_c(M)\int_0^{+\infty}\mathrm{Vol}_n(\{f^2\geq t\})\,dt$$

Now, we set $V(t) = \mathrm{Vol}_n(\{f^2 \geq t\})$. Integrating by parts and again using the Co-Area Formula we have

$$\int_0^{+\infty} V(t)\,dt = tV(t)|_0^{+\infty} - \int_0^{+\infty} tV'(t)\,dt = \int_0^{+\infty} t\Big(-\int_{\{f^2\geq t\}} 1\,dvol_n\Big)\,dt =$$
$$= \int_0^{+\infty} t\Big(-\int_{\{f=t\}} \frac{1}{\|\nabla f\|}dvol_n\Big)\,dt = \int_0^{+\infty}\int_{\{f^2=t\}} \frac{f^2}{\|\nabla f\|}\,dvol_n\,dt = \int_M f^2\,dvol_n$$

Therefore we obtain

$$\int_M \|\nabla(f^2)\| \, dvol_n \geq h_c(M) \int_M f^2 \, dvol_n \tag{5.18}$$

By equations (5.17) and (5.18) we get

$$\lambda_1(M) \geq \frac{1}{4} h_c^2(M)$$

Now, since f as an eigenfunction of λ_1, the submanifold $S = \{x | f(x) = 0\}$ divides M into two $n-$dimensional submanifolds $M_1 = \{x | f(x) \geq 0\}$ and $M_2 = \{x | f(x) \leq 0\}$ whose boundary is S, since f is non constant because λ_1 is non-trivial.

Let $h_c(M_1)$, $h_c(M_2)$, $h_c(M)$ denote Cheeger's constants of the manifolds M_1, M_2, M respectively. For M_1 and M_2 equation (5.14) holds.

We suppose that $\mathrm{Vol}_n(M_1) \leq \mathrm{Vol}_n(M_2)$, then $h_c(M_1) \geq h_c(M)$.

Since $\lambda_1(M) \geq \frac{1}{4} h_c^2(M_1)$, it holds again that $\lambda_1(M) \geq \frac{1}{4} h_c^2(M)$. $\square$

A consequence of Cheeger's Inequality is Mc Kean's Inequality.

Theorem 5.4.4 (Mc Kean's Inequality)**.** *Let (M, g) be a complete, simply connected $n-$dimensional Riemannian manifold, all of whose sectional curvatures are less than or equal to $k < 0$, then we have*

$$\lambda_1(M) \geq -(n-1)^2 \frac{k}{4}$$

Proof. We consider $p \in M$ and let $t(q) = d(p, q)$ where q is another point of M.

By equation (4.4), the Laplacian of the function t is given by:

$$\Delta t = -\frac{1}{\sqrt{\det G}} \Big[\sum_{\alpha,\beta} \frac{\partial}{\partial \alpha} \big(g^{\alpha,\beta} \sqrt{\det G} \partial_\beta t \big) + \frac{\partial}{\partial t} \big(g^{tt} \sqrt{\det G} \partial_t t \big) \Big]$$

where α and β are radial directions and there are no mixed term because of Gauss Lemma (see p. 16).

Since $g^{tt} = 1$, $\partial_\beta t = 0$ and $\partial_t t = 1$, the above equation becomes

$$\Delta t = -\frac{1}{\sqrt{\det G}} \partial_t (\sqrt{\det G}) \tag{5.19}$$

We want to get a bound for Δt using Theorem 3.3.1, hence

$$0 \leq \partial_t\Big(\frac{\sqrt{\det G}}{s_k^{n-1}}\Big) = \frac{\partial_t(\sqrt{\det G})s_k^{n-1} - (n-1)s_k^{n-2}\partial_t(s_k)\sqrt{\det G}}{s_k^{2(n-1)}} =$$

$$= \frac{\partial_t(\sqrt{\det G})}{s_k^{n-1}} - (n-1)\frac{\partial_t(s_k)\sqrt{\det G}}{s_k^{n}}$$

where s_k is defined in Remark 3.3.2.

That is,

$$\frac{\partial_t(\sqrt{\det G})}{\sqrt{\det G}} \geq (n-1)\frac{\partial_t(s_k)}{s_k} \tag{5.20}$$

Since k is negative, $s_k = \frac{1}{\sqrt{-k}}\sinh(t\sqrt{-k})$, therefore

$$(n-1)\frac{\partial_t(s_k)}{s_k} = (n-1)\sqrt{-k}\coth(t\sqrt{-k}) \tag{5.21}$$

From (5.19), (5.20) and (5.21) we obtain

$$-\Delta t \geq (n-1)\sqrt{-k}\coth(t\sqrt{-k}) \geq (n-1)\sqrt{-k}$$

where the last inequality is because of the behaviour of coth.

Hence, for all submanifolds Ω in M with smooth boundary we have

$$(n-1)\sqrt{-k}\,\mathrm{Vol}_n(\Omega) \leq \int_\Omega -\Delta t\, dvol_n = \int_{\partial\Omega} < \nabla t, \nu > \, dvol_{n-1} \leq \mathrm{Vol}_{n-1}(\partial\Omega)$$

where the equality is obtained by the Theorem 4.2.1 (Divergence Theorem).

Therefore,

$$\frac{\mathrm{Vol}_{n-1}(\partial\Omega)}{\mathrm{Vol}_n(\Omega)} \geq (n-1)\sqrt{-k}$$

which implies

$$h_c(M) \geq (n-1)\sqrt{-k} \tag{5.22}$$

Now, by Cheeger's Inequality and (5.22)

$$\lambda_1(M) \geq -\frac{1}{4}(n-1)^2 k$$

□

This inequality can be used to prove that Cheeger's Inequality is sharp when we deal with manifolds with negative constant sectional curvature. However, if one works with positive sectional curvature, Cheeger's Inequality is not sharp.

In fact, let M be a manifold with constant sectional curvature -1 with geodesic disk of radius δ denoted by $D_{-1}(\delta)$ having lowest Dirichlet eigenvalue $\lambda_{-1}(\delta)$, then:

$$\frac{(n-1)^2}{4} \leq \frac{1}{4} h_c^2(D_{-1}(\delta)) \leq \lambda_{-1}(\delta) \leq \frac{(n-1)^2}{4} + o(\delta)$$

for $\delta \to \infty$. Here the first inequality follows from taking the square of (5.22), the second inequality is Mc Kean's Inequality and the last inequality follows from the bottom of the spectrum of the hyperbolic plane for which we refer to [Cha1, p.46].

Chapter 6

The isoperimetric profile

6.1 Definitions and examples

We begin giving the definition of the isoperimetric profile and presenting the isoperimetric profile of the unit sphere of dimension n.

Definition 6.1.1. Let M be a compact, $n-$dimensional Riemannian manifold and let $\beta \in (0,1)$. We define the isoperimetric profile of M as the function $\beta \longmapsto h_M(\beta)$ such that

$$h_M(\beta) = \inf_{D \subset M} \left\{ \frac{\operatorname{Vol}_{n-1}(\partial D)}{\operatorname{Vol}_n(D)} \quad | \quad \operatorname{Vol}_n(D) = \beta \operatorname{Vol}_n(M) \right\}$$

Remark 6.1.2. Let $\beta' = 1 - \beta$. We note that $\operatorname{Vol}_n(D) = \beta \operatorname{Vol}_n(M)$ implies that $\operatorname{Vol}_n(D^c) = \beta' \operatorname{Vol}_n(M)$ and $\partial D = \partial D^c$. Therefore we conclude:

$$h_M(\beta) = h_M(1 - \beta)$$

Remark 6.1.3. If M is a closed Riemannian manifold, the isoperimetric profile and Cheeger's constant are related by the following

$$h_c(M) = \inf_{\beta \in (0, \frac{1}{2}]} h(\beta)$$

The isoperimetric profile can not be easy to calculate if we haven't clues about what kind of domains we have to take into account to get the infimum, but with well-known manifolds can become easy. As an example we present the isoperimetric profile of the unit sphere S^n.

Isoperimetric profile of S^n

By the Spherical Isoperimetric Inequality (see [Sak, Subsection 1.1, Chapter 6]) we know that the infimum of the isoperimetric profile is realized by a metric ball of radius r inside the sphere.

We start analysing two low dimensional cases, $n = 2, 3$, and then we turn to the general case.

Let B be a ball of radius r inside the unit $n-$sphere.

- Dimension $n = 2$

  By equations (3.7) and (3.8) we have:

  $\mathrm{Vol}_2(S^2) = 4\pi$

  $\mathrm{Vol}_2(B) = 2\pi(1 - \cos r)$

  $\mathrm{Vol}_1(\partial B) = \mathrm{Vol}_1(S^1)\sin r = 2\pi \sin r.$

  We require that B has given volume $\mathrm{Vol}_2(B) = \beta 4\pi$. Hence:

$$\frac{\mathrm{Vol}_1(\partial B)}{\mathrm{Vol}_2(B)} = \frac{\sin r}{2\beta}$$

  The radius r has to be calculated according to the volume required for B solving

$$4\beta\pi = 2\pi(1 - \cos r) \qquad \Longrightarrow \qquad r = \arccos(1 - 2\beta)$$

  Therefore:

$$h_{S^2}(\beta) = \frac{\sin(\arccos(1-2\beta)}{2\beta} = \frac{\sqrt{1 - \cos^2(\arccos(1-2\beta))}}{2\beta} =$$

$$= \frac{\sqrt{(1-2\beta)^2}}{2\beta} = \sqrt{\frac{\beta - 1}{\beta}}$$

- Dimension $n = 3$

  As before:

  $\mathrm{Vol}_3(S^3) = \pi^2$

$\mathrm{Vol}_3(B) = \mathrm{Vol}_2(S^2)\int_0^r (\sin t)^2 dt = 4\pi(\frac{r}{2} - \frac{\sin 2r}{4})$

$\mathrm{Vol}_2(\partial B) = (\sin r)^2 4\pi$

Requiring that B has given volume $\mathrm{Vol}_3(B) = \beta\pi^2$, the isoperimetric profile is

$$h_{S^3}(\beta) = \frac{4(\sin r)^2}{\beta\pi}$$

In this case the radius r is given solving $\beta\pi^2 = 4\pi(\frac{r}{2} - \frac{\sin 2r}{4})$

- General case

  Referring to the formulas we recalled at the beginning, we have

$$h_{S^n}(\beta) = \frac{(\sin r)^{n-1}\Gamma(\frac{n+1}{2})}{2\beta\sqrt{\pi}\,\Gamma(\frac{n}{2})}$$

  where r is given solving $\beta\sqrt{\pi}\,\Gamma(\frac{n+1}{2}) = \Gamma(\frac{n}{2})\int_0^r (\sin t)^{n-1} dt$ and $\Gamma(\cdot)$ denotes the Gamma function.

6.2 A comparison theorem for the isoperimetric profile

The aim of this section is to show that the isoperimetric profile of the unit sphere of dimension n is a lower bound for the isoperimetric profile of any other manifold with the same dimension and Ricci curvature bounded from below by a positive constant.

The proof of this claim involves different results and it is based on a calculation of volume which uses hypersurfaces immersed in the manifold and on a comparison theorem similar to the Bishop's Comparison Theorems.

We start by giving all the necessary notions that we will use to prove all the results.

6.2.1 Basic definitions for submanifolds

Isometric immersions

Let M be Riemannian manifold of dimension n and let $N \subset M$ be a submanifold of dimension $k < n$ isometrically immersed in M, i.e. there exists the inclusion map $f : N \longrightarrow M$ such that given the metric g on M, metric $\tilde{g}$ on N is defined via $\tilde{g}(v_1, v_2) = g(Df(p)(v_1), Df(p)(v_2))$.

For each $p \in N$ we have $T_pM = T_pN \oplus (T_pN)^\perp$ where $(T_pN)^\perp$ is the orthogonal complement of T_pN in T_pM. Therefore, any vector $v \in T_pM$ is written as $v = v^T + v^N$ where $v^T \in T_pN$ is called tangent component and $v^N \in (T_pN)^\perp$ is called normal component.

Let ∇, $\widetilde{\nabla}$ be the Levi-Civita connections on M and N respectively and let X, Y be vector fields on N with extensions $\overline{X}, \overline{Y}$ on M. $\tilde{\nabla}$ and ∇ satisfy $\widetilde{\nabla}_X Y = (\nabla_{\overline{X}}\overline{Y})^T$.

Definition 6.2.1. Let X, Y be vector fields on N and let $\overline{X}, \overline{Y}$ be their extensions on M. We define the map

$$B : \mathcal{X}(N) \times \mathcal{X}(N) \longrightarrow \mathcal{X}(N)^\perp, \qquad B(X,Y) = \nabla_{\overline{X}}\overline{Y} - \widetilde{\nabla}_X Y = (\nabla_{\overline{X}}\overline{Y})^N$$

where $\mathcal{X}(N)^\perp$ is the set of all vector fields normal to N.

Remark 6.2.2. The map B is bilinear and symmetric.

Definition 6.2.3. Let $p \in N$ and let $v \in (T_pN)^\perp$. We define the map

$$H_v : T_pN \times T_pN \longrightarrow \mathbb{R} \quad \text{such that} \quad H_v(w_1, w_2) =< B(w_1, w_2), v >$$

The second fundamental form of N is given by

$$II_v(w) = H_v(w, w)$$

Definition 6.2.4. Let $p \in N$ and let $v \in (T_pN)^\perp$. We define the shape operator of N as the map

$$S_v : T_pN \longrightarrow T_pN \quad \text{such that} \quad < S_v(w), u >=< B(w, u), v >$$

We point out that S_v is a symmetric linear operator and it can also be described by

$$S_v(w) = -(\nabla_{\overline{W}} V)^T \tag{6.1}$$

where $\overline{W}$ is an extension of w on M and V is an extension of v normal to N.

Definition 6.2.5. We define the mean curvature vector of N as

$$H = \frac{1}{k}\sum_{i=1}^{n-k}(\operatorname{tr} S_{E_i})E_i$$

where $\{E_1, \ldots, E_{n-k}\}$ is a local orthonormal frame of $(TN)^\perp$, that is $H(p) \in (T_pN)^\perp$ for all $p \in N$.

If N is an hypersurface of M, i.e $\dim N = n-1$, then the mean curvature is given by

$$\eta = \frac{1}{n-1}(\operatorname{tr} S_\nu) = \frac{1}{n-1}\sum_{i=1}^{n-1} < S_\nu(e_i), e_i >$$

with ν the outward unit normal and $\{e_1, \ldots, e_{n-1}\}$ an orthonormal basis for T_pN. In this case $H = \eta\nu$.

Remark 6.2.6. Since S_ν is a symmetric operator, it can be diagonalized, that is $S_\nu = diag(\lambda_1, \ldots, \lambda_{n-1})$ where λ_i are the eigenvalues. Therefore the mean curvature can also be written as

$$\eta = \frac{1}{n-1}\sum_{i=1}^{n-1}\lambda_i \tag{6.2}$$

Normal exponential map, $N-$Jacobi fields and focal points

Keeping the same notation of the previous subsection, we now look at Jacobi fields associated to each submanifold N.

Let $F : (-\epsilon, \epsilon) \times [0, T] \longrightarrow M$ be a geodesic variation of the geodesic $c_v : [0, T] \longrightarrow M$ with $c_v(0) = p \in N$ and $c_v'(0) = v \in (T_pN)^\perp$ such that for all $s \in (-\epsilon, \epsilon)$, the curve $\alpha(s) = F(s, 0)$ is into N and $A(s) = \frac{\partial F}{\partial t}(s, 0) \in (T_{\alpha(s)}N)^\perp$ (see Fig. 6.1).

The variational vector field $J(t) = \frac{\partial F}{\partial s}(t,0)$ is a Jacobi field along c_v. Moreover $J(t)$ satisfies the initial conditions

$$J(0) \in T_pN \qquad J'(0) + S_v(J(0)) \in (T_pN)^\perp \tag{6.3}$$

In fact, the variation can be expressed as $F(s,t) = \exp_{\alpha(s)} tA(s)$ and, by the Symmetry Lemma ([DoC, Lemma 3.4, p. 68]), easy consequence of the torsion freeness of ∇, we have

$$J'(t) = \frac{D}{dt}\Big|_t \frac{\partial F}{\partial s}\Big|_{s=0}(s,t) = \frac{D}{ds}\Big|_{s=0} \frac{\partial F}{\partial t}(s,t)$$

At $t = 0$ we get

$$J'(0) = \frac{D}{ds}\Big|_{s=0} \frac{\partial F}{\partial t}\Big|_{t=0}(s,t) = \nabla_{\alpha'(0)}A(s) = (\nabla_{\alpha'(0)}A(s))^T + (\nabla_{\alpha'(0)}A(s))^N$$

Since $-(\nabla_{\alpha'(0)}A(s))^T = S_v(\alpha'(0)) = S_v(J(0))$, we get

$$J'(0) + S_v(J(0)) = J'(0) - (\nabla_{\alpha'(0)}A(s))^T = (\nabla_{\alpha'(0)}A(s))^N \in (T_pN)^\perp$$

On the other hand, any Jacobi fields satisfying (6.3) arises from such a variation (see [DoC, Lemma 4.1, p. 227]).

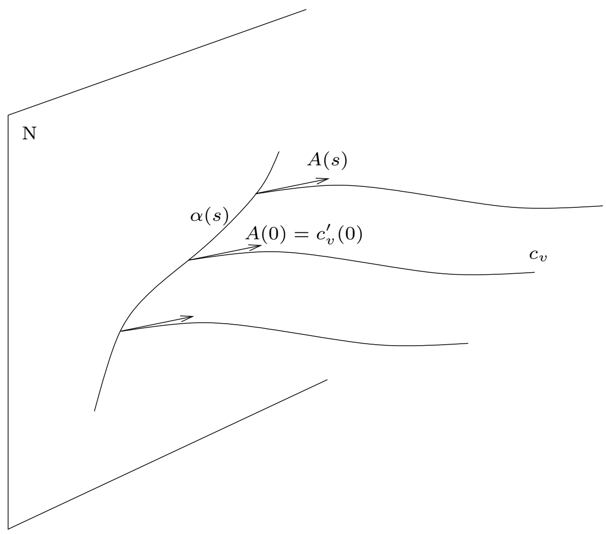

Figure 6.1: The variation F

These Jacobi fields are also called $N-$Jacobi fields.

There is an analogous version of conjugate points for $N-$Jacobi fields: focal points.

Definition 6.2.7. Let N be a submanifold of a Riemannian manifold M. A point $q \in M$ is a focal point of N if there exists a geodesic $c_v : [0, T] \longrightarrow M$ such that $c_v(0) = p \in N$, $c_v(T) = q$ and $c_v'(0) \in (T_pN)^\perp$ and a non zero $N-$Jacobi field satisfying $J(T) = 0$.

Moreover, as the Jacobi fields are linked to the exponential map, the $N-$Jacobi fields are linked to the normal exponential map.

Let $(TN)^\perp = \bigcup_{p\in N}(T_pN)^\perp$ be the normal tangent bundle.

We define $(TM)_N = \bigcup_{p\in N} T_pM$.

We have the decomposition $(TM)_N = TN \oplus (TN)^\perp$. The exponential map can be seen as a map $\exp : (TM)_N = TN \oplus (TN)^\perp \longrightarrow M$ with $\exp(p, v) = \exp_p v$.

We define the normal exponential map as the exponential map restricted to the normal tangent bundle, that is

$$\exp^\perp : (TN)^\perp \longrightarrow M$$

We have the following results identifying the tangent space $T_w(TN)^\perp$, $w \in (T_qN)^\perp$, with vectors in $T_qN \oplus (T_qN)^\perp$ as described in [Sak, p. 58]

Lemma 6.2.8. *Let $N \subset M$ be a submanifold of a Riemannian manifold M. If $J(t)$ is an $N-$Jacobi field such that $J(0) = v \in T_pN$ and $J'(0) + S_w(v) = u \in (T_pN)^\perp$, then $D\exp^\perp(tw)(v, tu)$ where (v, tu) is an element in $T_{tw}(TN)^\perp$.*

Proposition 6.2.9 ([DoC], Proposition 4.4, p. 231)**.** *The point $q \in M$ is a focal point of N if and only if it is a critical value of* $\exp^\perp$.

To conclude this subsection we introduce the index form for an $N-$Jacobi field $J(t)$.

Definition 6.2.10. The index form of an $N-$Jacobi field $J(t)$ along the geodesic $c_v : [0,T] \longrightarrow M$, with $c_v(0) = p \in N$ and $c_v'(0) = v \in (T_pN)^{\perp}$, at the point $t \in [0,T]$ is given by

$$I_t(J,J) =< J(0), -S_v(J(0)) > + \int_0^t \|J'(s)\|^2 - < R(J', c_v')c_v', J' > (s)\, ds$$

Remark 6.2.11. Since $J(t)$ is defined by the initial conditions $J(0) \in T_pN$ and $J'(0) + S_v(J(0)) = w \in (T_pN)^{\perp}$, we have:

$$< J(0), -S_v(J(0)) >=< J(0), J'(0) - w >=< J(0), J'(0) >$$

Therefore

$$I_t(J,J) =< J(0), J'(0) > + \int_0^t \|J'(s)\|^2 - < R(J', c_v')c_v', J' > (s)\, ds \qquad (6.4)$$

Remark 6.2.12. The index form can also be expressed as

$$I_t(J,J) =< J(t), J'(t) > \qquad (6.5)$$

In fact:

$$I_t(J,J) =< J'(0), J(0) > + \int_0^t \|J'(s)\|^2 - < R(J, c_v')c_v', J > (s)ds =$$

$$=< J'(0), J(0) > + \int_0^t (< J', J >)'(s) - < J'', J > (s) + < J'', J > (s)ds =$$

$$=< J'(0), J(0) > - < J'(0), J(0) > + < J'(t), J(t) >=< J'(t), J(t) >$$

We can compare the index form of a piecewise C^{∞} vector field X along a geodesic with no focal points of N and such that $X(0) \in N$ with the index form of an $N-$Jacobi field Y along the same geodesic.

Lemma 6.2.13 ([Sak], Lemma 2.10, p. 95)**.** *Let $c_v : [0,T] \longrightarrow M$ be a geodesic such that $c_v(0) = p \in N$ and $c_v'(0) = v \in (T_pN)^{\perp}$ with no focal points of N on $c_v([0,T])$. For any piecewise C^{∞} vector field X along a geodesic c_v with $X(0) \in T_pN$ there exists a unique $N-$Jacobi field Y along c_v satisfying conditions (6.3) with $Y(T) = X(T)$. Moreover $I_t(Y,Y) \leq I_t(X,X)$. Equality holds if and only if $Y = X$.*

6.2.2 Heintze-Karcher Inequality

Here we show how we can use the fact that $N \subset M$ is a submanifold of M to calculate the volume of M.

The idea is to use the inverse of the normal exponential map as a coordinate chart on M and to split the Riemannian measure on M into the Riemannian measure on N and $(T_pN)^\perp$ for all $p \in N$.

Since from now on we deal with hypersurfaces of a Riemannian manifold M, we restrict to the case $N \subset M$ submanifold of dimension $n-1$. For the general case we refer to [Cha2, Section 6, Chapter 3].

The following statements can be proved with the same arguments used for the exponential map (see [Cha2, Section 6, Chapter 3] or [Sak, p. 59ff]).

i) There exists an open neighbourhood U of $(TN)^\perp$ on which the normal exponential map is a diffeomorphism onto an open set of M.

ii) (Gauss Lemma) Let $c_w : [0,T] \longrightarrow M$ be a geodesic normal to N such that $c_w(0) = p$ and $c'_w(0) = w \in (T_pN)^\perp$ and let $(w,u) \in T_w(TN)^\perp$. We have that $D\exp^\perp(tw)(0,tw) = tc'_w(t)$ and in particular $\|D\exp^\perp(w)(0,w)\| = \|w\|$.

   Moreover $< D\exp^\perp(tw)(v,tu), c'_w(t) >=< u,w > t$.

iii) Let $w \in (T_pN)^\perp$ and suppose the normal exponential map is defined on an open neighbourhood U of $\{tw \mid 0 \leq t \leq T\} \in (TN)^\perp$, the same introduced in i). For any curve φ in U starting from N and ending at Tw, we set $\gamma(t) = \exp^\perp \varphi(t)$ and we have $l(c_w) \leq l(\gamma)$ where $c_w(t) : [0,T] \longrightarrow M$ is the geodesic normal to N such that $c_w(0) = p$ and $c'_w(0) = w \in (T_pN)^\perp$.

iv) Suppose that N is a closed submanifold. Then for any $q \in \exp^\perp(U)$ there exists a unique minimal geodesic c parametrized by arc-length from a point of N to q that realizes the distance $d(q,N)$.

Now, since the normal exponential map is a diffeomorphism, its inverse can be used as a coordinate chart on M.

The Riemannian measure on M can be written in a suitable neighbourhood of N, as

$$dvol_n = \sqrt{\det g_{ij} \circ \exp^{\perp}(t\nu)} dt dvol_{n-1}$$

where ν is a smooth normal frame along N and t is the coordinate that represents the signed distance from N.

Furthermore, the square root of this determinant can be calculated in terms of $N-$Jacobi fields.

In fact, let $\{e_1, \ldots, e_{n-1}\}$ be an orthonormal basis for T_pN and let $J_i(t)$ for $i = 1, \ldots n-1$ be $N-$Jacobi fields along the geodesic $c_\nu(t)$ with $c_\nu(0) = p$ and $c'_\nu(0) = \nu$ perpendicular to N such that

$$J_i(0) = e_i \qquad J'_i(0) = -S_\nu(e_i) \tag{6.6}$$

for all $i = 1, \ldots, n-1$.

Note that $\{e_1, \ldots, e_{n-1}\}$ can be chosen as the eigenvectors of S_ν, in which case $J'_i(0) = -\lambda_i e_i$ where λ_i are the eigenvalues of S_ν.

Then

$$\sqrt{\det g_{ij} \circ \exp^{\perp}(t\nu)} = \sqrt{\det(< J_i(t), J_j(t) >)_{i,j=1,\ldots,n-1}}$$

Therefore, in case N is connected we have (see [Cha2, p. 144]):

$$\mathrm{Vol}_n(M) = \int_M dvol_n = \int_N dvol_{n-1}(p) \int_{-r(-\nu_p)}^{r(\nu_p)} \sqrt{\det(< J_i(t), J_j(t) >)_{i,j=1,\ldots,n-1}} dt \tag{6.7}$$

where $r(\nu_p)$ is the first focal distance in direction $\nu_p = \nu(p)$ for all $p \in N$.

If N is the connected boundary of M and $-\nu$ is the inward unit vector field, then (6.7) can be replaced by

$$\mathrm{Vol}_n(M) = \int_M dvol_n = \int_N dvol_{n-1}(p) \int_0^{r(-\nu_p)} \sqrt{\det(< J_i(t), J_j(t) >)_{i,j=1,\ldots,n-1}} dt \tag{6.8}$$

Remark 6.2.14. If M has constant sectional curvature $\delta > 0$, then there is an explicit formula for the volume of M.

Let N be an hypersurface of M, the $N-$Jacobi fields satisfying the initial conditions $J_i(0) = e_i$ and $J_i'(0) = -\lambda_i e_i$, with $\{e_1, \dots, e_n\}$ orthonormal basis for T_pN, are given by:

$$J_i(t) = \Big(\cos(\sqrt{\delta}t) - \frac{\lambda_i}{\sqrt{\delta}}\sin(\sqrt{\delta}t)\Big)E_i(t)$$

with $\{E_1(t), \dots, E_{n-1}(t)\}$ an orthonormal frame such that $E_i(0) = e_i$ for all i.

Hence:

$$\sqrt{\det(< J_i(t), J_j(t) >)_{i,j=1,\dots,n-1}} = \prod_{i=1}^{n-1}\Big(\cos(\sqrt{\delta}t) - \frac{\lambda_i}{\sqrt{\delta}}\sin(\sqrt{\delta}t)\Big)$$

and equation (6.7) becomes

$$\mathrm{Vol}_n(M) = \int_N dvol_{n-1}\int_{-r(-\nu_p)}^{r(\nu_p)}\prod_{i=1}^{n-1}\Big(\cos(\sqrt{\delta}t) - \frac{\lambda_i}{\sqrt{\delta}}\sin(\sqrt{\delta}t)\Big)dt \tag{6.9}$$

Now we can state Heintze and Karcher inequality which gives us a bound for the integrand function in the formula for the volume of M.

Proposition 6.2.15. *Let M be a Riemannian manifold of dimension n with connected boundary N and with $\mathrm{Ric}(v) \geq (n-1)$ for all $v \in SM$. Then, let $\eta \in \mathbb{R}$ be a constant such that the mean curvature of N satisfies the trace condition $\operatorname{tr} S_{-\nu} \geq -\eta(n-1)$, where $-\nu$ is the inward unit normal. Then for $t \leq t_0$, with t_0 the first focal distance of N in direction $-\nu$, we have*

$$\sqrt{\det D\exp^{\perp}(t(-\nu))} \leq (\cos t + \eta\sin t)^{n-1} \tag{6.10}$$

Moreover, the first zero of the LHS function (first focal distance of N in direction $-\nu$) does not occur before the first zero of the RHS function.

Proof. We compare M with a Riemannian manifold $\overline{M}$ of dimension n with constant sectional curvature 1. In $\overline{M}$ we pick up a submanifold $\overline{N}$ of dimension $(n-1)$ and totally umbilic with $\operatorname{tr}\overline{S}_{-\overline{\nu}} = -\eta(n-1)$, where $-\overline{\nu}$ is the inward unit normal of $\overline{N}$.

We observe that the manifold $\overline{M}$ is the unit $n-$dimensional sphere and the submanifold $\overline{N}$ is the boundary of a metric ball inside the sphere.

Let $\{e_1,\dots,e_{n-1}\}$ be an orthonormal basis for T_pN and let $\{J_1(t),\dots,J_{n-1}(t)\}$ be $N-$Jacobi fields along $c(t)=\exp^{\perp}(t(-\nu))$, with $c(0)=p$ and $c'(0)=-\nu$, perpendicular to N and satisfying $J_i(0)=e_i$ and $J_i'(0)=-S_{-\nu}(e_i)$ for all $i=1,\dots,n-1$.

These $N-$Jacobi fields describe the normal exponential map on M, so

$$\sqrt{\det D\exp^{\perp}(t(-\nu))}=\sqrt{\det(< J_i(t),J_j(t) >)_{i,j=1,\dots,n-1}} \tag{6.11}$$

In the same way let $\{\overline{e}_1,\dots,\overline{e}_{n-1}\}$ be an orthonormal basis for $T_{\overline{p}}\overline{N}$ and let $\{\overline{J}_1(t),\dots,\overline{J}_{n-1}(t)\}$ be $\overline{N}-$Jacobi fields along $\overline{c}(t)=\exp^{\perp}(t(-\overline{\nu}))$, with $\overline{c}(0)=\overline{p}$ and $\overline{c}'(0)=\overline{\nu}$, perpendicular to $\overline{N}$ and satisfying $\overline{J}_i(0)=\overline{e}_i$ and $\overline{J}_i'(0)=-\overline{S}_{-\overline{\nu}}(\overline{e}_i)$ for all $i=1,\dots,n-1$.

In particular, setting $\overline{\lambda}_i$ for $i=1,\dots,n-1$ the eigenvalues of $\overline{S}_{\overline{\nu}}$, we have that $-\overline{\lambda}_i$ for $i=1,\dots,n-1$ are the eigenvalues of $\overline{S}_{-\overline{\nu}}$.

Since $\overline{N}$ is totally umbilic, all $\overline{\lambda}_i$ agree and are equal to η.

Choosing $\overline{e}_i$ as the eigenvectors of $\overline{S}_{\overline{\nu}}$ we have $\overline{J}_i'(0)=\overline{\lambda}_i\overline{e}_i=\eta\overline{e}_i$ for all $i=1,\dots,n-1$.

Moreover, by Remark 6.2.14 we have

$$\overline{J}_i(t)=(\cos t+\overline{\lambda}_i\sin t)E_i(t)=(\cos t+\eta\sin t)E_i(t)$$

providing $\{E_1,\dots,E_{n-1}\}$ parallel vector fields along $\overline{c}$ with $E_i(0)=\overline{e}_i$ for all $i=1,\dots,n-1$.

Again, these orthogonal Jacobi fields describe the normal exponential map. Therefore

$$\sqrt{\det D\exp^{\perp}(t(-\overline{\nu}))}=\sqrt{\det(< \overline{J}_i(t),\overline{J}_j(t) >)_{i,j=1,\dots,n-1}} \tag{6.12}$$

In particular,

$$\sqrt{\det(< \overline{J}_i(t),\overline{J}_j(t) >)_{i=1,\dots,n-1}}=\prod_{k=1}^{n-1}(\cos t+\overline{\lambda}_k\sin t)=(\cos t+\eta\sin t)^{n-1} \tag{6.13}$$

Therefore, to prove the theorem we need to show that

$$\sqrt{\det(< J_i(t), J_j(t) >)_{i,j=1,\dots,n-1}} \leq \sqrt{\det(< \overline{J}_i(t), \overline{J}_j(t) >)_{i,j=1,\dots,n-1}} \quad (6.14)$$

Indeed, setting

$$f(t) = \sqrt{\det A(t)} \qquad \text{with} \qquad A(t) = (< J_i(t), J_j(t) >)_{i,j=1,\dots,n-1}$$

$$\overline{f}(t) = \sqrt{\det B(t)} \qquad \text{with} \qquad B(t) = (< \overline{J}_i(t), \overline{J}_j(t) >)_{i,j=1,\dots,n-1}$$

it is enough to show

$$(\log f)'(t) \leq (\log \overline{f})'(t) \qquad \text{for } t \leq t_0 \tag{6.15}$$

where t_0 is the first focal distance of N in direction $-\nu$.

In fact, if it holds we have

$$\log f(\tilde{t}) - \log f(0) = \int_0^{\tilde{t}} (\log f)'(t)dt \leq \int_0^{\tilde{t}} (\log \overline{f})'(t)dt = \log \overline{f}(\tilde{t}) - \log \overline{f}(0)$$

with $\tilde{t} \leq t_0$.

By the initial condition $\log f(0) = \log \overline{f}(0) = 1$, we have

$$\log f(\tilde{t}) \leq \log \overline{f}(\tilde{t}) \qquad \Longrightarrow \qquad f(\tilde{t}) \leq \overline{f}(\tilde{t})$$

and using equations (6.11), (6.12) and (6.13) we finish the proof.

To prove equation (6.15) we proceed as follow.

$$(\log f)'(t) = \left(\log \sqrt{\det A(t)}\right)' = \frac{(\det A(t))'}{2 \det A(t)} =$$

$$= \frac{\det A(t) \cdot \operatorname{tr}\left(A^{-1}(t)A'(t)\right)}{2 \det A(t)} = \frac{1}{2} \operatorname{tr}\left(A^{-1}(t)A'(t)\right) \tag{6.16}$$

With the same computation

$$(\log \overline{f})'(t) = \frac{1}{2} \operatorname{tr}\left(B^{-1}(t)B'(t)\right) \tag{6.17}$$

We note that $(\log f)'$ and $(\log \overline{f})'$ does not change if we choose instead of J_i (respectively $\overline{J}_i$) a set of $(n-1)$ linearly independent $N-$Jacobi fields (respectively $\overline{N}-$Jacobi fields).

In fact, if $Y_1(t), \ldots, Y_{n-1}(t)$ are such that $Y_i(t) = \sum_{l=1}^{n-1} a_{il} J_l(t)$ with a_{il} elements of a $(n-1) \times (n-1)$ matrix with constant coefficients, then

$$\det(< Y_i(t), Y_j(t) >)_{i,j=1,\ldots,n-1} = (\det(a_{ij})_{i,j=1,\ldots,n-1})^2 \det(< J_i(t), J_j(t) >)_{i,j=1,\ldots,n-1} =$$

$$= (\det(a_{ij})_{i,j=1,\ldots,n-1})^2 \det A(t)$$

Let $f_Y(t) = \sqrt{\det(< J_i(t), J_j(t) >)_{i,j=1,\ldots,n-1}}$.
We have $f_Y(t) = |\det(a_{ij})_{i,j=1,\ldots,n-1}| f(t)$.
Therefore $(\log f_Y)'(t) = \big(\log(|\det(a_{ij})_{i,j=1,\ldots,n-1}|) + \log f\big)'(t) = (\log f)'(t)$.

We fix $\hat{t} \in [0, T]$ and we choose the $\overline{J}_i$s to be orthonormal at $\hat{t}$, then $B(\hat{t}) = I$ and $B'(\hat{t}) = (< \overline{J}_i'(\hat{t}), \overline{J}_j(\hat{t}) > + < \overline{J}_i(\hat{t}), \overline{J}_j'(\hat{t}) >)_{i,j=1,\ldots,n-1}$.
Equation (6.17) becomes

$$(\log \overline{f})'(\hat{t}) = \sum_{i=1}^{n-1} < \overline{J}_i'(\hat{t}), \overline{J}_i(\hat{t}) >= \sum_{i=1}^{n-1} I_{\hat{t}}(\overline{J}_i, \overline{J}_i)$$

where the last equality comes from (6.5).
In the same manner, i.e. choosing the J_is to be orthonormal at $\hat{t}$, we get

$$(\log f)'(\hat{t}) = \sum_{i=1}^{n-1} < J_i'(\hat{t}), J_i(\hat{t}) >= \sum_{i=1}^{n-1} I_{\hat{t}}(J_i, J_i)$$

It remains to show that

$$\sum_{i=1}^{n-1} I_{\hat{t}}(J_i, J_i) \leq \sum_{i=1}^{n-1} I_{\hat{t}}(\overline{J}_i, \overline{J}_i) \tag{6.18}$$

We define a linear isometry

$$i_p^{\hat{t}} : T_{\overline{p}}\overline{M} \longrightarrow T_p M$$

such that

$$i_p^{\hat{t}}(\overline{c}'(0)) = c'(0) \qquad i_p^{\hat{t}}(T_{\overline{p}}\overline{N}) = T_p N \qquad (P_{\hat{t}}^c \circ i_p^{\hat{t}} \circ P_{-\hat{t}}^{\overline{c}})(\overline{J}_i(\hat{t})) = J_i(\hat{t})$$

Here $P_{\hat{t}}^c$ and $P_{-\hat{t}}^{\overline{c}}$ denotes the parallel transport along the curve c from 0 to $\hat{t}$ and along the curve $\overline{c}$ from $\hat{t}$ to 0 respectively.

For $i = 1, \ldots, n-1$ and for $0 \le s \le \hat{t}$, we define vector fields $W_i(s)$ along c as

$$W_i(s) = (P^c_s \circ i^{\hat{t}}_p \circ P^{\overline{c}}_{-s})(\overline{J}_i(s))$$

Each $W_i(s)$ satisfies

$$W_i(\hat{t}) = J_i(\hat{t}) \qquad \text{and} \qquad \|W_i(s)\| = \|\overline{J}_i(s)\|, \;\; \|W_i'(s)\| = \|\overline{J}_i'(s)\| \quad \forall\, s \in [0, \hat{t}]$$

By Lemma 6.2.13 we have

$$I_{\hat{t}}(J_i, J_i) \le I_{\hat{t}}(W_i, W_i) \qquad \forall\, i = 1, \ldots, n-1$$

Therefore

$$\sum_{i=1}^{n-1} I_{\hat{t}}(J_i, J_i) \le \sum_{i=1}^{n-1} I_{\hat{t}}(W_i, W_i)$$

To conclude the proof of (6.18) it remains to prove that

$$\sum_{i=1}^{n-1} I_{\hat{t}}(W_i, W_i) \le \sum_{i=1}^{n-1} I_{\hat{t}}(\overline{J}_i, \overline{J}_i) \tag{6.19}$$

Since $W_i(s)$ are orthonormal up to a common factor and using the trace assumption, we conclude

$$\sum_{i=1}^{n-1} < W_i(0), S_\nu(W_i(0)) > \le \sum_{i=1}^{n-1} < \overline{J}_i(0), \overline{S}_{\overline{\nu}}(\overline{J}_i(0)) >$$

Further

$$\sum_{i=1}^{n-1} \|W_i'(s)\|^2 = \sum_{i=1}^{n-1} \|\overline{J}_i'(s)\|^2$$

and

$$\sum_{i=1}^{n-1} \int_0^{\hat{t}} < R(W_i, c')c', W_i > (s)\, ds = \int_0^{\hat{t}} \sum_{i=1}^{n-1} < R(W_i, c')c', W_i > (s)\, ds =$$

$$= \int_0^{\hat{t}} \mathrm{Ric}_M(c') \|W_i(s)\|^2\, ds \ge \int_0^{\hat{t}} \mathrm{Ric}_{\overline{M}}(\overline{c}') \|\overline{J}_i(s)\|^2\, ds = \sum_{i=1}^{n-1} \int_0^{\hat{t}} < R(\overline{J}_i, \overline{c}')\overline{c}', \overline{J}_i > (s)\, ds$$

We obtain

$$\sum_{i=1}^{n-1} < W_i(0), S_\nu(W_i(0)) > + \int_0^{\hat{t}} \|W_i'(s)\| - < R(W_i, c')c', W_i > (s)\, ds \le$$

$$\leq \sum_{i=1}^{n-1} < \overline{J}_i(0), S_\nu(\overline{J}_i(0)) > + \int_0^{\hat{t}} \|\overline{J}_i'(s)\| - < R(\overline{J}_i, \overline{c}')\overline{c}', \overline{J}_i > (s)\, ds$$

which means

$$\sum_{i=1}^{n-1} I_{\hat{t}}(W_i, W_i) = \sum_{i=1}^{n-1} I_{\hat{t}}(\overline{J}_i, \overline{J}_i)$$

Since (6.19) is proved, then the proof of the initial inequality is finished.

Concerning the first zero, we observe that the two functions in (6.10) start from the same point and the first derivative of the LHS function is less than the one of the RHS function. Therefore the first zero of the LHS cannt occur before the first zero of the RHS. □

Remark 6.2.16. If we had worked with the outward unit normal ν, we would have ended up with the following inequality

$$\sqrt{\det D \exp^{\perp}(t\nu)} \leq (\cos t - \eta \sin t)^{n-1} \tag{6.20}$$

6.2.3 Variational formula for area and volume

Here we work out another important step for our final goal. We present and prove the variational formulas for area and volume stating a condition for a domain inside a manifold to have constant mean curvature.

Let D be a domain inside a Riemannian manifold M of dimension n with smooth boundary ∂D and let ν be the outward unit normal vector field to ∂D. We set $\nu(p) = \nu_p$. For a C^∞ function u on ∂D with compact support, we consider $\alpha_t(p) = \exp^{\perp}(tu(p)\nu_p)$ and $\partial D_t = \{\alpha_t(p) \mid p \in \partial D\}$.

∂D_t is a variation of ∂D and we consider D_t, the corresponding domain with boundary ∂D_t.

It is clear that $\partial D_0 = \partial D$ and $D_0 = D$.

Proposition 6.2.17. *In the above situation we have*

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{Vol}_n(D_t) = \int_{\partial D} u\, dvol_{n-1} \tag{6.21}$$

$$\frac{d}{dt}\Big|_{t=0} \mathrm{Vol}_{n-1}(\partial D_t) = -(n-1)\int_{\partial D} \eta(p)u(p)dvol_{n-1} \qquad (6.22)$$

where $\eta(p)$ is the mean curvature vector of ∂D at $p \in \partial D$.

Proof. Let $\{e_1, \ldots, e_{n-1}\}$ be an orthonormal basis of $T_p\partial D$ and let $\{J_1(t), \ldots, J_{n-1}(t)\}$ be $\partial D-$Jacobi fields along the geodesic $c_{\nu_p}(tu(p))$ perpendicular to ∂D and such that $J_i(0) = e_i$ and $J_i'(0) = -S_{\nu_p}(e_i)$.

Denoting by $dvol_{n-1,t}$ the Riemannian measure of ∂D_t with respect to the induced metric at $c_{\nu_p}(tu(p))$, we have

$$dvol_{n-1,t} = \sqrt{\det(< J_i(tu(p)), J_j(tu(p)) >)_{ij=1,\ldots,n-1}}dvol_{n-1}$$

Hence

$$\mathrm{Vol}_n(D_t) = \int_{D_t} dvol_{n,t} = \mathrm{Vol}_n(D) + \int_{\partial D}\int_0^{tu(p)} \sqrt{\det(< J_i(s), J_j(s) >)_{ij}}ds\, dvol_{n-1}$$

and

$$\mathrm{Vol}_{n-1}(\partial D_t) = \int_{\partial D_t} dvol_{n-1,t} = \int_{\partial D} \sqrt{\det(< J_i(tu(p)), J_j(tu(p)) >)_{ij}}dvol_{n-1}$$

Therefore

$$\frac{d}{dt}\Big|_{t=0} \mathrm{Vol}_n(D_t) = \int_{\partial D} u\sqrt{\det(< J_i(0), J_j(0)) >)_{ij}}dvol_{n-1} = \int_{\partial D} u\, dvol_{n-1}$$

which proves (6.21).

Taking the derivative of the $(n-1)$ volume of ∂D_t we have

$$\frac{d}{dt}\Big|_{t=0} \mathrm{Vol}_{n-1}(\partial D_t) = \int_{\partial D} \frac{d}{dt}\Big|_{t=0} \sqrt{\det(< J_i(tu(p)), J_j(tu(p)) >)_{ij}}dvol_{n-1} \qquad (6.23)$$

and we need to calculate the derivative inside the integral.

We set $A(t) = (< J_i(t), J_j(t) >)_{i,j=1,\ldots,n-1}$, then

$$\frac{d}{dt}\Big|_{t=0} \sqrt{\det A(t)} = \frac{1}{2}\,\mathrm{tr}(A^{-1}(0)A'(0)) \qquad (6.24)$$

Now,

$A(0) = I$ so $A^{-1}(0) = I$ and

$A'(0) = (< J_i'(0), J_j(0) > + < J_i(0), J_j'(0) >)_{i,j=1,\ldots,n-1}$

For the initial conditions of these $\partial D-$Jacobi fields we have

$$A'(0) = (-2 < S_{\nu_p}(e_i), e_j >)_{i,j=1,\ldots,n-1}$$

Hence, equation (6.24) becomes

$$\frac{d}{dt}\Big|_{t=0} \sqrt{\det A(t)} = -\sum_{i=1}^{n-1} < S_{\nu_p}(e_i), e_i >= -\eta(p)(n-1)$$

and substituting this expression into (6.23) we conclude

$$\frac{d}{dt}\Big|_{t=0} \mathrm{Vol}_{n-1}(\partial D_t) = -(n-1)\int_{\partial D} \eta(p)u(p)dvol_{n-1}$$

which gives us (6.22). □

As a consequence we have the following corollary.

Corollary 6.2.18. *Assume the same situation of the previous proposition. If ∂D minimizes the $(n-1)-$dimensional volume among all the $(n-1)-$dimensional submanifolds of M that are smooth boundaries of domains with the same volume as that of D, then ∂D has constant mean curvature.*

Proof. The minimizing assumption implies

$$0 = \frac{d}{dt}\Big|_{t=0} \mathrm{Vol}_{n-1}(\partial D_t) = \int_{\partial D} \eta(p)u(p)dvol_{n-1}$$

for all $u \in C^\infty(\partial D)$ satisfying

$$\int_{\partial D} u(p)dvol_{n-1} = 0 \tag{6.25}$$

We choose

$$u = \eta - \frac{1}{\mathrm{Vol}_{n-1}(\partial D)}\int_{\partial D} \eta \; dvol_{n-1}$$

We observe that this choice of u satisfies (6.25), in fact

$$\int_{\partial D} udvol_{n-1} = \int_{\partial D} \eta dvol_{n-1} - \frac{1}{\mathrm{Vol}_{n-1}(\partial D)}\mathrm{Vol}_{n-1}(\partial D)\int_{\partial D} \eta(p)dvol_{n-1} = 0$$

Now we consider

$$u^2 = \eta^2 - \frac{2}{\mathrm{Vol}_{n-1}(\partial D)}\eta \int_{\partial D} \eta dvol_{n-1} + \Big(\frac{1}{\mathrm{Vol}_n \, \partial D} \int_{\partial D} \eta dvol_{n-1}\Big)^2$$

and

$$\int_{\partial D} u^2 dvol_{n-1} = \int_{\partial D} \eta^2 dvol_{n-1} - \frac{1}{\mathrm{Vol}_{n-1}(\partial D)}\Big(\eta dvol_{n-1}\Big)^2$$

By the Cauchy-Schwartz Inequality:

$$0 = \Big(\int_{\partial D} u \cdot 1 dvol_{n-1}\Big)^2 \leq \mathrm{Vol}_{n-1}(\partial D) \int_{\partial D} u^2 dvol_{n-1} =$$

$$= \mathrm{Vol}_{n-1}(\partial D)\Big[\int_{\partial D} \eta^2 dvol_{n-1} - \frac{1}{\mathrm{Vol}_{n-1}(\partial D)}\Big(\int_{\partial D} \eta dvol_{n-1}\Big)^2\Big] =$$

$$= \mathrm{Vol}_{n-1}(\partial D)\Big[\int_{\partial D} \eta^2 dvol_{n-1} - \frac{1}{\mathrm{Vol}_{n-1}(\partial D)} \int_{\partial D} \eta \Big(\int_{\partial D} \eta dvol_{n-1}\Big) dvol_{n-1}\Big] =$$

$$= \mathrm{Vol}_{n-1}(\partial D) \int_{\partial D} u(p)\eta(p) dvol_{n-1} = 0$$

Therefore the Cauchy-Schwartz Inequality is an equality indeed. This means that the function u is a multiple of 1, i.e. it is constant. Since its integral over ∂D is zero, $u = 0$.

Hence:

$$0 = \eta - \frac{1}{\mathrm{Vol}_{n-1}(\partial D)} \int_{\partial D} \eta dvol_{n-1} \quad \Longrightarrow \quad \eta = \frac{1}{\mathrm{Vol}_{n-1}(\partial D)} \int_{\partial D} \eta dvol_{n-1}$$

This means that η is constant. □

6.2.4 Estimating the isoperimetric profile

At this point we have all the ingredients to prove the main result. As a warm up, we deal with the volume of a metric ball of radius r, B_r, into the unit $n-$sphere and discuss this situation in detail.

First of all, we calculate the mean curvature η and the shape operator of ∂B_r.

By Corollary 6.2.18 we know that η is constant.

We take the variation $\alpha_t(p) = \exp^{\perp}(t\nu_p)$, for all $p \in \partial B_r$, which takes ∂B_r to ∂B_{r+t}. Then by equation (6.22) of Proposition 6.2.17 we have

$$\frac{d}{dt}\Big|_{t=0} \mathrm{Vol}_{n-1}(\partial B_{r+t}) = -(n-1)\eta \, \mathrm{Vol}_{n-1}(\partial B_r)$$

Moreover

$$\frac{d}{dt}\Big|_{t=0} \mathrm{Vol}_{n-1}(\partial B_{r+t}) = \frac{d}{dt}\Big|_{t=0} (\sin r)^{n-1} \, \mathrm{Vol}_n(S^n) = (n-1)(\sin r)^{n-2} \cos r \, \mathrm{Vol}_n(S^n)$$

and

$$-(n-1)\eta \, \mathrm{Vol}_{n-1}(\partial B_r) = -(n-1)\eta(\sin r)^{n-1} \, \mathrm{Vol}_n(S^n)$$

using formula (3.8).

Combining the last two formulas we obtain $\eta = -\cot r$.

We note that, using the inward unit normal instead, we would end up with $\eta = \cot r$.

Now we turn to the shape operator, finding an explicit expression for $S_\nu(v)$ with $v \in T_p \partial B_r$.

We first consider the case $n = 2$ (see Fig. 6.2), since it can easily be extended to arbitrary dimensions.

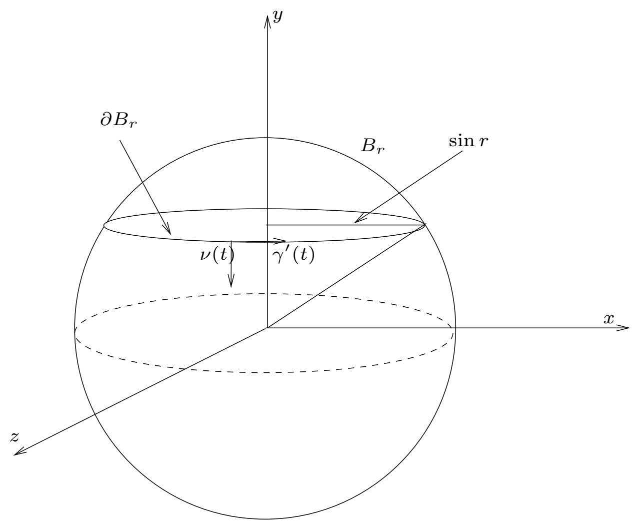

Figure 6.2: S^2

We have that $S_\nu(v) = -(\nabla_v N)^\perp$ by equation (6.1), with N a local extension of ν in B_r which is also a local extension in S^2.

We parametrize S^2 by $\varphi^{-1}(x_1, x_2) = (\sin x_1 \sin x_2, \sin x_1 \cos x_2, \cos x_1)$ with $x_1 \in [0, \pi]$ and $x_2 \in [-\pi, \pi]$.

A curve in ∂B_r is given by $\gamma(t) = (\sin r \sin t, \sin r \cos t, \cos r) = \varphi^{-1}(r, t)$ and $\gamma'(t) = (\sin r \cos t, -\sin r \sin t, 0) = \frac{\partial}{\partial x_2}\big|_{\gamma(t)}$ and $\|\gamma'(t)\| = \sin r$.

The unit outward normal vector of B_r in S^2 is given by
$\nu(t) = (\cos r \sin t, \cos r \cos t, -\sin r) = \frac{\partial}{\partial x_1}\big|_{\gamma(t)}$ and the extension N takes the same form.

Therefore, setting $v = \gamma'(t)$, we have

$$\nabla_v N = \nabla_{\gamma'(t)} N = \nabla_{\frac{\partial}{\partial x_2}\big|_{\gamma(t)}} \frac{\partial}{\partial x_1} = \Gamma^2_{21} \frac{\partial}{\partial x_2}|_{\gamma(t)} = \cot r \cdot \gamma'(t) = \cot r \cdot v$$

and so $S_\nu(v) = -\cot r \cdot v$. Therefore $S_{-\nu}(v) = \cot r \cdot v$.

If $n > 2$, a similar calculation yields $S_\nu(v) = -\cot r \cdot v$ again.

Now we can consider the volume of B_r. The metric ball B_r inherits the metric on S^n, so it has constant sectional curvature $\delta = 1$.

We take $p \in \partial B_r$ and we consider the unit inward normal $-\nu$. The ∂B_r-Jacobi fields $\{J_i(t), \dots, J_{n-1}(t)\}$ along the geodesic $c_{-\nu}(t) = \exp^\perp(t(-\nu))$ with $c_{-\nu}(0) = p$ and $c'_{-\nu}(0) = -\nu$ and perpendicular to ∂B_r are defined by the initial conditions $J_i(0) = e_i$ and $J'_i(0) = -S_{-\nu}(e_i) = S_\nu(e_i)$ for all i with $\{e_1, \dots, e_{n-1}\}$ an orthonormal basis of $T_p\partial B_r$.

By Remark 6.2.14, these Jacobi fields are

$$J_i(t) = (\cos t - \cot r \cdot \sin t)E_i(t)$$

for all i with $\{E_i(t), \dots, E_{n-1}(t)\}$ an orthonormal frame of $\{e_1, \dots, e_{n-1}\}$.

Moreover

$$\sqrt{\det g_{ij} \circ \exp^\perp(t\nu)} = \sqrt{\det(< J_i(t), J_j(t) >_{ij})} = (\cos t - \cot r \cdot \sin t)^{n-1}$$

so the volume of the ball is given by

$$\mathrm{Vol}_n(B_r) = \int_{\partial B_r} \int_0^r (\cos t - \cot r \cdot \sin t)^{n-1} dt = \mathrm{Vol}_{n-1}(\partial B_r) \int_0^r (\cos t - \cot r \cdot \sin t)^{n-1} dt \tag{6.26}$$

where r is the first focal distance in direction $-\nu$ which is equal to the radius of B_r. Moreover, it is also the first zero of the integrand function.

After this explicit calculation for the metric ball in S^2, we now turn to the estimate of the isoperimetric profile.

Theorem 6.2.19. *Let M be an connected complete Riemannian manifold of dimension n with $\mathrm{Ric}(v) \geq (n-1)$ for all $v \in SM$ and let S^n be the unit $n-$dimensional sphere. Then*

$$h_M(\beta) \geq h_{S^n}(\beta)$$

Proof. Using deep results of Geometric Measure Theory, there exists a domain $D \subset M$ such that

i) $\mathrm{Vol}_n(D) = \beta \, \mathrm{Vol}_n(M)$ with $\beta \in (0, 1]$;

ii) its boundary, ∂D, minimizes the volume among all the domains with same volume of D.

This means that the isoperimetric profile of M is given by

$$h_M(\beta) = \frac{\mathrm{Vol}_{n-1}(\partial D)}{\mathrm{Vol}_n(M)} \tag{6.27}$$

Unfortunately, 'these result of Geometric Measure Theory are very hard to locate and to find in an handy form'[1]. The reader can refer to [Mor] for a firts introduction to the subject.

By its properties, ∂D has constant mean curvature (see Corollary 6.2.18) $-\eta$ since we choose to work with the inward unit normal.

We now calculate the volume of D and regard ∂D as an hypersurface isometrically immersed in D. Choosing an orthonormal basis $\{e_1, \ldots, e_{n-1}\}$ of $T_p\partial D$ and $\{J_1(t), \ldots, J_{n-1}(t)\}$ $\partial D-$Jacobi fields along $c(t) = \exp^{\perp}(t(-\nu))$

[1] M. Berger, *A panoramic view of Riemannian Geometry*, Springer, 2003, p. 320

such that $J_i(0) = e_i$ and $J_i'(0) = -S_{-\nu}(e_i) = \lambda_i e_i$ for all $i = 1, \dots, n-1$ where λ_i are the eigenvalues of the shape operator S_ν, we have:

$$\mathrm{Vol}_n(D) = \int_{\partial D} \int_0^{t_1} \sqrt{\det(< J_i(t), J_j(t) >)_{i,j=1,\dots,n-1}} dt \, dvol_{n-1} \tag{6.28}$$

where $dvol_{n-1}$ is the Riemannian mesure on ∂D induced by D and t_1 is the first focal distance of ∂D in direction $-\nu$

Using Proposition 6.2.15, we have that

$$\sqrt{\det(< J_i(t), J_j(t) >)_{i,j=1,\dots,n-1}} \leq (\cos t - (-\eta) \sin t)^{n-1} = (\cos t + \eta \sin t)^{n-1}$$

and $t_1 \leq t_0$ with t_0 the first zero of the RHS function.

Therefore (6.28) becomes:

$$\mathrm{Vol}_n(D) \leq \int_{\partial D} \int_0^{t_0} (\cos t + \eta \sin t)^{n-1} dt \, dvol_{n-1} = \mathrm{Vol}_{n-1}(\partial D) \int_0^{t_0} (\cos t + \eta \sin t)^{n-1} \tag{6.29}$$

Now we turn to the case of the sphere S^n. As we did for M when we chose D, we are able to find a ball $B_r \subset S^n$ of radius r with given volume $\mathrm{Vol}_n(B_r) = \beta \, \mathrm{Vol}_n(S^n)$ and whose boundary minimizes the volume among all the domain with the same volume of B_r, that is, the isoperimetric profile of S^n is realized by

$$h_{S^n}(\beta) = \frac{\mathrm{Vol}_{n-1}(\partial B_r)}{\mathrm{Vol}_n(S^n)} \tag{6.30}$$

Further, ∂B_r has constant mean curvature that we denote by $-\eta^*$, again choosing the inward unit normal.

Looking at the volume of B_r we have that:

$$\mathrm{Vol}_n(B_r) = \mathrm{Vol}_{n-1}(\partial B_r) \int_0^r (\cos t + \eta^* \sin t)^{n-1} dt \tag{6.31}$$

We first assume that $\eta < \eta^*$.

Clearly, $(\cos t + \eta \sin t)^{n-1} < (\cos t + \eta^* \sin t)^{n-1}$, for all $t \leq t_0$.

Moreover, t_0 does not occur before the first zero of the RHS, which is r, the radius of the metric ball.

Hence, equation (6.29) becomes:

$$\mathrm{Vol}_n(D) \leq \mathrm{Vol}_{n-1}(\partial D) \int_0^r (\cos t + \eta^* \sin t)^{n-1} dt \tag{6.32}$$

Therefore:

$$\frac{\mathrm{Vol}_n(D)}{\mathrm{Vol}_{n-1}(\partial D)} \leq \int_0^r (\cos t + \eta^* \sin t)^{n-1} dt = \frac{\mathrm{Vol}_n(B_r)}{\mathrm{Vol}_{n-1}(\partial B_r)} \tag{6.33}$$

Which implies

$$\frac{\mathrm{Vol}_{n-1}(\partial D)}{\mathrm{Vol}_n(D)} \geq \frac{\mathrm{Vol}_{n-1}(\partial B_r)}{\mathrm{Vol}_n(B_r)} \tag{6.34}$$

Recalling that $\mathrm{Vol}_n(D) = \beta \,\mathrm{Vol}_n(M)$ and $\mathrm{Vol}_n(B_r) = \beta \,\mathrm{Vol}_n(S^n)$, we obtain:

$$\frac{\mathrm{Vol}_n(\partial D)}{\beta \,\mathrm{Vol}_n(M)} \geq \frac{\mathrm{Vol}_{n-1}(\partial B_r)}{\beta \,\mathrm{Vol}_n(S^n)} \tag{6.35}$$

which implies

$$h_M(\beta) \geq h_{S^n}(\beta)$$

On the other hand, if $\eta \geq \eta^*$, we look at D^c and B_r^c, the complements of D and B_r in M and S^n respectively.

We note that $\mathrm{Vol}_n(D^c) = (1-\beta)\,\mathrm{Vol}_n(M)$ and $\mathrm{Vol}_n(B_r^c) = (1-\beta)\,\mathrm{Vol}_n(S^n)$.

Furthermore, the unit normals point in the opposite direction, so now the mean curvatures are η and η^* respectively.

Looking at the volume and imitating equations (6.29) and (6.31) we get

$$\mathrm{Vol}_n(D^c) = \mathrm{Vol}_{n-1}(\partial D^c) \int_0^{t_0'} (\cos t - \eta \sin t)^{n-1} dt\, dvol_{n-1} =$$

$$= \mathrm{Vol}_{n-1}(\partial D) \int_0^{t_0'} (\cos t - \eta \sin t)^{n-1} dt \tag{6.36}$$

where t_0' is the first zero of the integrand function, and

$$\mathrm{Vol}_n(B_r) = \mathrm{Vol}_{n-1}(\partial B_r^c) \int_0^{\pi-r} (\cos t - \eta^* \sin t)^{n-1} dt. \tag{6.37}$$

Again we have $(\cos t - \eta \sin t)^{n-1} \leq (\cos t - \eta^* \sin t)^{n-1}$ and $t_0' \leq \pi - r$.

Hence

$$\mathrm{Vol}_n(D^c) \leq \mathrm{Vol}_{n-1}(\partial D) \int_0^{\pi-r} (\cos t - \eta^* \sin t)^{n-1} dt, \tag{6.38}$$

and, therefore

$$\frac{\mathrm{Vol}_{n-1}(\partial D)}{(1-\beta)\,\mathrm{Vol}_n(M)} = \frac{\mathrm{Vol}_{n-1}(\partial D)}{\mathrm{Vol}_n(D^c)} \geq \frac{\mathrm{Vol}_{n-1}(\partial B_r)}{\mathrm{Vol}_n(B_r^c)} = \frac{\mathrm{Vol}_{n-1}(\partial B_r)}{(1-\beta)\,\mathrm{Vol}_n(S^n)} \tag{6.39}$$

which implies

$$h_M(1-\beta) \geq h_{S^n}(1-\beta)$$

and, by Remark 6.1.2, we have again

$$h_M(\beta) \geq h_{S^n}(\beta)$$

□

Remark 6.2.20. If M is a closed Riemannian manifold, this result implies the same inequality involving Cheeger's constants of M and S^n via Remark 6.1.3.

Chapter 7

The Sphere Theorem

In all the previous chapters, the guideline was analysing the consequences of a lower bound of the Ricci curvature on the geometry of a manifold. In particular, we found some conditions for which we have isometry between the $n-$sphere and a Riemannian manifold of dimension n.

Now, we want to weaken the condition on the curvature to know when the manifold is at least homoeomorphic with the $n-$sphere.

The answer is in the Sphere Theorem stating that when the sectional curvature K of a complete, simply connected $n-$dimensional manifold is such that $\frac{1}{4} < K \leq 1$, then it is homeomorphic to the $n-$sphere.

Here we prove this theorem and we discuss the case of $\mathbb{P}^n(\mathbb{C})$, the complex projective plane, showing that the lower bound of the sectional curvature cannot be weakened.

Since the proof involves a deeper knowledge about the cut locus and the injectivity radius, Section 7.1 and 7.2 are dedicated to some important properties of the cut locus and the estimate of the injectivity radius in even dimension respectively, while in Section 7.3 we give the proof of the Sphere Theorem along with the discussion of $\mathbb{P}^n(\mathbb{C})$.

All the manifolds we consider in this chapter are complete and the geodesics are arc-length parametrized, unless stated otherwise.

7.1 Some properties of the cut locus

We start giving a characterization of cut points.

Proposition 7.1.1. *Let M be a Riemannian manifold and $\gamma : [0,T] \longrightarrow M$ be a geodesic on M. Suppose that $\gamma(t_0)$, $t_0 \in [0,T]$, is the cut point of $p = \gamma(0)$ along γ. Then one of the following holds true:*

i) $\gamma(t_0)$ is the first conjugate point of $\gamma(0)$ along γ;

ii) there exists a geodesic $\sigma \neq \gamma$ from p to $\gamma(t_0)$ such that $l(\gamma) = l(\sigma)$.

Conversely if (i) or (ii) holds, then there exists $\tau \in [0, t_0]$ such that $\gamma(\tau)$ is the cut point of p along γ.

Proof. Let $t_0 \in [0,T]$ as in the hypothesis and let $\{t_0 + \epsilon_i\}_{i\in I}$ be a sequence such that $\epsilon_i > 0$ for all i and $\epsilon_i \to 0$ as $i \to +\infty$. Then let $\{\sigma_i\}_{i\in I}$ be a sequence of minimal geodesics joining p to $\gamma(t_0+\epsilon_i)$ and let $\{\sigma_i'(0)\}_{i\in I} \in T_pM$ be the sequence of the corresponding tangent vectors.

Each $\sigma_i'(0)$ is contained in the unit ball S_pM. Since S_pM is compact and taking a subsequence if necessary, we can suppose that $\{\sigma_i'(0)\}_{i\in I}$ converges to v, where v is the tangent vector at the point p of the geodesic σ_v. By continuity σ_v is a minimal geodesic joining p and $\gamma(t_0)$.

In fact, γ is minimal up to $\gamma(t_0)$ while each σ_i is minimal up to $\gamma(t_0+\epsilon_i)$, so $\sigma_i(t_0+\delta_i) = \gamma(t_0+\epsilon_i)$ for $0 \le \delta_i < \epsilon_i$. By continuity, for $i \to \infty$ we have $\sigma_v(t_0) = \lim_{i\to\infty} \sigma_i(t_0+\delta_i) = \gamma(t_0)$ and, hence, $l(\sigma_v) = l(\gamma) = t_0$.

If $\sigma_v \neq \gamma$ then (ii) is satisfied.

Otherwise, if $\sigma_v = \gamma$ then (i) holds.

To prove this last claim we show that $t_0\gamma'(0)$ is a singular point of $D\exp_p$. It will follow that $\gamma(t_0)$ is the first conjugate point of p (see Proposition 2.4.6).

Working by contradiction we suppose that $t_0\gamma'(0)$ is not a singular point of $D\exp_p$. Hence, choosing an open neighbourhood U of $t_0\gamma'(0)$, $D\exp_p$ is a local isomorphism and, therefore, a local diffeomorphism on U (Fig. 7.1).

Since $\gamma(t_0+\epsilon_i)=\sigma_i(t_0+\delta_i)$ for $\delta_i<\epsilon_i$ and taking ϵ_i small enough such that $(t_0+\delta_i)\sigma_i'(0)$ belongs to U, we have

$$\exp_p(t_0+\epsilon_i)\gamma'(0)=\gamma(t_0+\epsilon_i)=\sigma_i(t_0+\delta_i)=\exp_p(t_0+\delta_i)\sigma_i'(0)$$

Hence, for i large enough

$$(t_0+\epsilon_i)\gamma'(0)=(t_0+\delta_i)\sigma_i'(0)$$

which means

$$\gamma'(0)=\sigma_i'(0)\qquad\text{and}\qquad\epsilon_i=\delta_i$$

Therefore the cut point of p along γ is $t_0+\epsilon_i$ and no longer t_0, which is a contradiction.

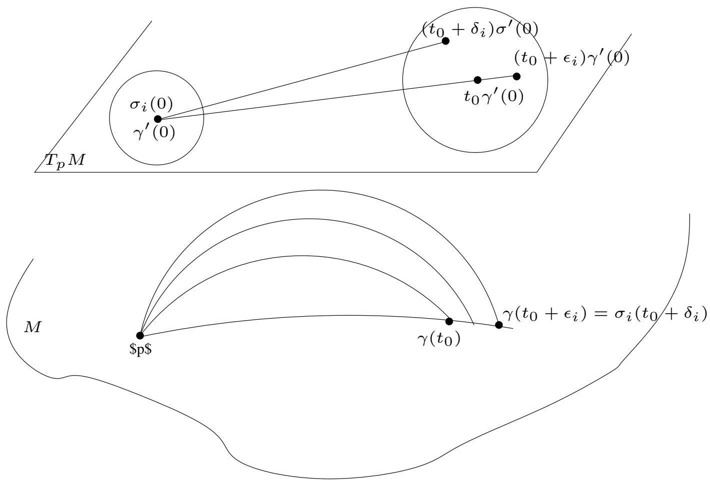

Figure 7.1: case $\sigma_v=\gamma$

Conversely, we first suppose that condition (i) holds. Since a geodesic does not minimize the distance after the first conjugate point (see Proposition 2.4.7), the cut point of p along γ occurs at latest at $\gamma(t_0)$. Hence there exists a point $\tau\in[0,t_0]$ such that $\gamma(\tau)$ is the cut point of p along γ.

Now we suppose that condition (i) does not hold but condition (ii) does. Let $\epsilon>0$ such that $\exp_{\gamma(t_0)}$ is a diffeomorphism in a neighbourhood U' of $t_0\gamma(0)\in T_{\gamma(t_0)}M$ and let $\sigma(t_0-\epsilon)$ and $\gamma(t_0+\epsilon)$ belong to $\exp_p(U')=V$.

Further, let τ be the unique minimal geodesic joining $\sigma(t_0 - \epsilon)$ and $\gamma(t_0 + \epsilon)$ (Fig. 7.2). The curve given by the union of σ from p to $\sigma(t_0 + \epsilon)$ and τ has arc length strictly less then $t_0 - \epsilon$. Therefore the cut point of p along γ occurs at $\gamma(\tau)$ with $\tau \leq t_0$.

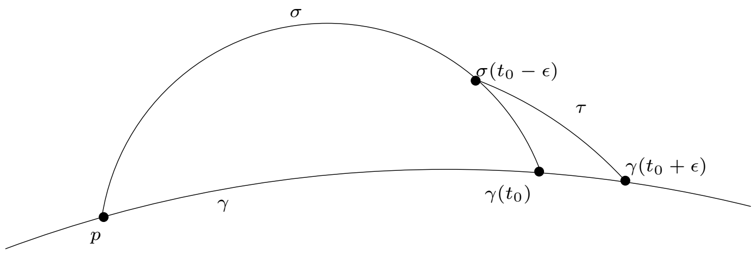

Figure 7.2: case when condition (ii) holds

□

For a better understanding we present two examples: the $n-$sphere and the flat torus.

Example 7.1.2. Let S^n be the n-dimensional sphere and let $p \in S^n$. The cut locus of p is $-p$, its antipodal point. In this case both (i) and (ii) occur.

In fact $-p$ is the first conjugate point of p and two different minimal geodesics between p and $-p$ can be found rolling down on opposite side.

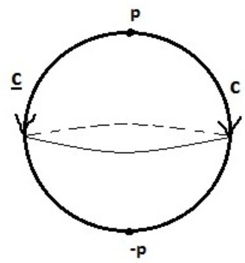

Figure 7.3: cut locus of the sphere

Example 7.1.3. Let T^n be the flat torus, i.e. $T^n = \mathbb{R}^n/\mathbb{Z}^n$. In this case the torus has the same Riemannian metric of $\mathbb{R}^n$ and the geodesics are straight lines. For example the cut locus of the point A is made up of the edges of the square. This time only situation (ii) can occur. In fact, identifying $\mathbb{R}^n$ with T_pT^n, the Riemannian covering $f : \mathbb{R}^n \longrightarrow T^n$, $f(t) = p + tv$ coincides with the exponential map. Moreover, taking the geodesic $c_v : [0, 1] \longrightarrow T^n$ with $c_v(0) = p$ and $c'_v(0) = v$, we know that $D\exp_p(v)(w) = J(1)$ with $J : [0, 1] \to T^n$ the unique Jacobi field along c_v with $J(0) = 0$ and $J'(0) = w$. In this case we have a geodesic variation $F(s,t) = \exp_p(t(v+sw)) = t(v+sw)$ because of the above identification and $J(t) = \frac{\partial F}{\partial s}(0,t) = tw$. But we never get a $t_0 \neq 0$ for which $J(t_0) = 0$.

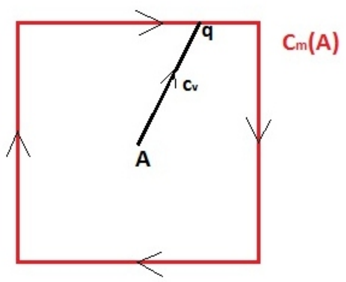

Figure 7.4: cut locus of the flat torus at the point A

As consequences of Proposition 7.1.1 we have the following corollaries.

Corollary 7.1.4. *Let M be a Riemannian manifold and let $\gamma : [0, T] \longrightarrow M$ be a geodesic on M. If q is the cut point of p along γ, then p is the cut point of q along $-\gamma$.*

In particular $q \in C_m(p)$ if and only if $p \in C_m(q)$.

Proof. If q is the cut point of p along γ then either q is the conjugate point of p along γ or there exists a geodesic σ different from γ with $l(\sigma) = l(\gamma) = d(p, q)$.

Considering the geodesic $-\gamma$, we get that either p is conjugate to q or there exists a geodesic $\tilde{\sigma}$ different form $-\gamma$ and with the same length. In particular $\tilde{\sigma} = -\sigma$. In both cases the cut point of q along $-\gamma$ occurs at latest at p.

Hence p is the cut point of q along $-\gamma$ and $p \in C_m(p)$.

Exchanging the roles of p and q, we have that if $p \in C_m(p)$ then $q \in C_m(p)$ and the proof is complete. □

Corollary 7.1.5. *Let M be a Riemannian manifold and let p and q points in M. If $q \in M \backslash C_m(p)$ then there exists a unique minimal geodesic joining p and q.*

Proof. We can always find a minimal geodesic γ joining p and q, moreover by Proposition 7.1.1, neither q is the first conjugate point of p along γ, nor there exists a geodesic σ different from γ which is distance realizing as well.

Therefore the minimal geodesic between p and q is unique. □

It follows that the exponential map is injective on an open ball centred at p with radius $r = d(p, C_m(p))$.

Moreover, $M \backslash C_m(p)$ is homeomorphic to an euclidean ball through the exponential map.

With this characterization of cut points in mind, we turn to the analysis of the function f which maps a point $p \in M$ to its cut value along a chosen geodesic. In particular, we show it is continuous and, as a consequence, we get the compactness of the cut locus.

Proposition 7.1.6. *Let M be a Riemannian manifold and $\gamma : [0, T] \longrightarrow M$ be a geodesic on M with $\gamma(0) = p \in M$ and $\gamma'(0) = v \in S_pM$. The function f, defined as*

$$f : SM \to \mathbb{R} \cup \infty$$

$$f(p, v) = \begin{cases} t_0, & \textit{if } \gamma(t_0) \textit{ is the cut point of } p \textit{ along } \gamma; \\ \infty, & \textit{if the cut point along } \gamma \textit{ does not exist.} \end{cases} \tag{7.1}$$

is continuous.

Proof. Let $\{\gamma_i'(0)\}_{i \in I}$ be a sequence in SM. Since it is compact, $\gamma_i'(0) \to \gamma'(0)$ as $i \to +\infty$. Therefore the sequence $\{\gamma_i(0)\}_{i \in I}$ in M converges to $\gamma(0)$.

Then let $\gamma_i(t_0^i)$ and $\gamma(t_0)$ be the cut points of $\gamma_i(0)$ and $\gamma(0)$ along γ_i and γ respectively with $t_0^i, t_0 \in \mathbb{R} \cup \infty$.

To prove the continuity of the function f we show that $\lim_{i\to\infty} t_0^i = t_0$, that is

$$\limsup_{i\to\infty} t_0^i = \liminf_{i\to\infty} t_0^i = t_0$$

First we prove that $\limsup_{i\to\infty} t_0^i \le t_0$.

If $t_0 = \infty$ there is nothing to prove. Hence suppose $t_0 < \infty$ and let $\epsilon > 0$ small enough.

There not exist infinitely many j such that $t_0 + \epsilon < t_0^j$ otherwise we would have $d(\gamma_j(0), \gamma_j(t_0 + \epsilon)) = t_0 + \epsilon$ and so $d(\gamma(0), \gamma(t_0 + \epsilon)) = t_0 + \epsilon$, by the continuity of the distance function, which contradicts the definition of $\gamma(t_0)$.

Therefore

$$\limsup_{i\to\infty} t_0^i \le t_0 + \epsilon$$

and for $\epsilon \to 0$,

$$\limsup_{i\to\infty} t_0^i \le t_0$$

Now set $\bar{t} = \liminf_{i\to\infty} t_0^i$. We have

$$\bar{t} = \liminf_{i\to\infty} t_0^i \le \limsup_{i\to\infty} t_0^i \le t_0$$

We show that $\bar{t} \ge t_0$ to complete the proof.

If $\bar{t} = \infty$ there is nothing to prove, so we take $\bar{t} < \infty$ and we consider the subsequence $\{t_0^j\}_{j\in J}$ of $\{t_0^i\}_{i\in I}$ which converges to $\bar{t}$. We can either suppose that for all t_0^j, $\gamma_j(t_0^j)$ is conjugate to $\gamma_j(0)$ along γ_j or not all of them are conjugate to $\gamma_j(0)$ along γ_j.

In the first case, by continuity, $\gamma(\bar{t})$ is conjugate to $\gamma(0)$ because the limiting point of conjugate points is a conjugate point as well. Hence $\bar{t} \ge t_0$.

In the second case, by Proposition 7.1.1 there exists a geodesic $\sigma_j \ne \gamma_j$ such that $\sigma_j(0) = \gamma_j(0)$, $\sigma_j(t_0^j) = \gamma_j(t_0^j)$ and $l(\sigma_j) = l(\gamma_j)$. We suppose that the sequence of $\sigma_j s$ converges to σ, where σ is a geodesic joining $\gamma(0)$ and $\gamma(\bar{t})$.

If $\sigma \ne \gamma$, then $t_0 \le \bar{t}$ by Proposition 7.1.1.

If $\sigma = \gamma$ we show that $\gamma(\bar{t})$ is conjugate to $\gamma(0)$ arguing as in Proposition 7.1.1. Hence $\bar{t} \ge t_0$.

Now the claim follows. $\square$

Corollary 7.1.7. *Let M be a Riemannian manifold. For all $p \in M$ the cut locus at p, $C_m(p)$, is closed. In particular if M is compact, then $C_m(p)$ is compact.*

Proof. It is clear that

$$C_m(p) = \{\gamma(t) \mid t = f(p, \gamma'(0)) < \infty\}$$

Now, if q is an accumulation point of $C_m(p)$, there exists a sequence $\{\gamma_j(t_j)\}_{j\in J}$ with $t_j = f(p, \gamma_j'(0))$ such that $\gamma_j(t_j) \to q$ as $j \to \infty$.

Since SM is compact, we have $\gamma_j'(0) \to v \in SM$ as $j \to \infty$ (choosing a subsequence if needed).

Let γ be the geodesic with $\gamma(0) = p$ ad $\gamma'(0) = v$.

We have:

$$q = \lim_{j\to\infty} \gamma_j(t_j) = \lim_{j\to\infty} \gamma_j(f(p, \gamma'(0))) = \lim_{j\to\infty} \exp_p(f(p, \gamma_j'(0))\gamma_j'(0)) =$$

$$= \exp_p(f(p, v)v) = \gamma(f(p, v)) \in C_m(p).$$

Therefore $C_m(p)$ is closed.

Since M is compact, it is closed and bounded and since $C_m(p) \subset M$, then $C_m(p)$ is bounded too. Therefore it is compact. $\square$

We end this section with a proposition similar in spirit to Proposition 7.1.1.

Proposition 7.1.8. *Let M be a Riemannian manifold and $p \in M$. Suppose there exists a point $q \in C_m(p)$ which realizes the distance from p to $C_m(p)$. Then:*

i) either there exists a minimal geodesic γ from p to q along which q is conjugate to p;

ii) or there exists exactly two minimal geodesic γ and σ from p to q with $\gamma'(l) = -\sigma'(l)$ and $l = d(p, q)$.

Proof. Let $\gamma : [0, l] \to M$ be a minimal geodesic joining p and q.

By Proposition 7.1.1 either q is conjugate to p along γ or there exists a geodesic $\sigma \neq \gamma$ joining p and q such that $l(\sigma) = l(\gamma)$.

In the first case condition (i) holds.

If q is not conjugate to p we show that condition (ii) holds (see Fig. 7.5).

Working by contradiction we suppose $\gamma'(l) \neq -\sigma'(l)$.

We claim it is always possible to find a $w \in T_qM$ such that

$$< w, \gamma'(l) > < 0 \qquad \text{and} \qquad < w, \sigma'(0) > < 0$$

In fact, the angle α between $\gamma'(l)$ and $\sigma'(l)$ is smaller than π, so the angle $\beta = 2\pi - \alpha$ is bigger than π. Then it is always possible to find a w such that the angles between w and $\gamma'(l)$ and between w and $\sigma'(l)$ are equal to $\frac{\beta}{2}$.

Therefore, $\cos\frac{\beta}{2} < 0$ and since $< w, \gamma'(l) >= \|w\| \cdot \|\gamma'(l)\| \cos\frac{\beta}{2}$, and the same holds for $\sigma'(l)$, we have the claim.

Now let $c : (-\epsilon, \epsilon) \longrightarrow M$ be a curve such that $c(0) = q$ and $c'(0) = w$.

Since q is not conjugate to p there exists an open neighbourhood $U \subset T_pM$ of $l\gamma'(0)$ on which the exponential map $\exp_p$ is a diffeomorphism.

Let $\tilde{c} : (-\epsilon, \epsilon) \longrightarrow U$ be a curve such that $\exp_p \tilde{c}(s) = c(s)$, $s \in (-\epsilon, \epsilon)$.

The function $\gamma_s(t) = \exp_p\left(\frac{t}{l}\tilde{c}(s)\right)$ is a variation of γ.

In fact,

$$\gamma_0(t) = \exp_p\left(\frac{t}{l}\tilde{c}(0)\right) = \exp_p\left(\frac{t}{l}l\gamma'(0)\right) = \exp_p(t\gamma'(0)) = \gamma(t)$$

Let $X(t) = \frac{\partial}{\partial s}(\gamma_s(t))\big|_{s=0} = \frac{\partial}{\partial s}\Big(\exp_p\left(\frac{t}{l}\tilde{c}(s)\right)\Big)\big|_{s=0}$ be the variational vector field of $\gamma_s(t)$.

We have the following:

$\int_0^l < X(t), \frac{D}{dt}\gamma'(t) > dt = 0$ since γ is a geodesic;

$< X(0), \gamma'(0) >= 0$ because $\gamma_s(0) = \exp_p 0_p = p$ for all s, that is, the variation leaves p invariant.

$X(l) = \frac{\partial}{\partial s}\big|_{s=0}(\exp_p \tilde{c}(s)) = c'(0) = w$

Applying the formula of first variation of length (2.3)we have

$$\frac{d}{ds}\Big|_{s=0} l(\gamma_s(t)) =< w, \gamma'(0) > < 0$$

Going through the same argument using σ we get the same, i.e.

$$\frac{d}{ds}\bigg|_{s=0} l(\sigma_s(t)) =< w, \sigma'(0) >< 0$$

Hence for $s > 0$ small enough we have $l(\gamma_s) < l(\gamma)$ and $l(\sigma_s) < l(\sigma)$.

Now we suppose $l(\gamma_s) = l(\sigma_s)$.

We note that $\sigma_s(l) = c(s) = \gamma_s(l)$ and, by Proposition 7.1.1, $\gamma_s(l) = c(s)$ is the cut point of p.

Since $d(p, \gamma_s(l)) = l(\gamma_s) < d(p, C_m(p))$ for all s, then $q = c(0) = \gamma_0(l)$ does not realize the distance between p and $C_m(p)$, which is a contradiction.

Instead, if $l(\gamma_s) < l(\sigma_s)$, then σ_s is not minimal so there exists a cut point $\sigma_s(\bar{t})$ with $\bar{t} < l$ of p along σ_s. But this is again a contradiction to the same fact.

The case $l(\gamma_s) > l(\sigma_s)$ is similar.

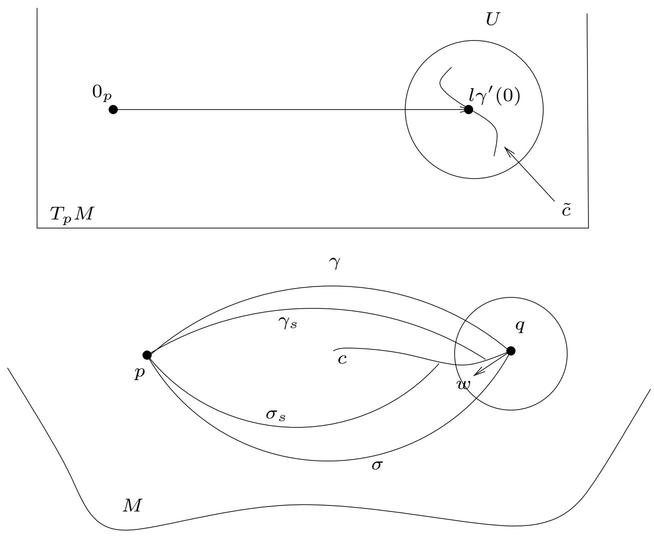

Figure 7.5: case when condition (ii) holds

$\square$

7.2 The estimate of the injectivity radius

Here we prove the estimate of the injectivity radius which is bounded from below by π. The proof is given in the even dimensional case.

First of all we need some preliminary results.

Lemma 7.2.1. *Let M be a compact Riemannian manifold. Then there exist two points $p, q \in M$ with $q \in C_m(p)$ such that $d(p,q) = i(M)$.*

Proof. Since SM is compact and the function f in (7.1) is continuous, then $f(SM) \in (\mathbb{R} \cup \infty)$ is compact. Hence f admits a minimum.

Let $\gamma : [0,l] \longrightarrow M$ be a geodesic joining $\gamma(0) = x \in M$ with its first cut point along γ and such that $\gamma'(0) = v \in S_xM$. We have:

$$i(M) = \inf_{x \in M} d(x, C_m(x)) = \inf_{x \in M} \inf_{\gamma} l(\gamma) = \inf_{x \in M, v \in S_xM} f(x,v) = \min_{x \in M, v \in S_xM} f(x,v) =$$

$$= f(p,w) = d(p, C_m(p)) = \inf_{y \in C_m(p)} d(p,y) = d(p,q)$$

where the last inequality is because $C_m(p)$ is compact (see Corollary 7.1.7) and the distance function is continuous. □

Proposition 7.2.2. *If the sectional curvature of a complete Riemannian manifold M is such that $0 < K_{min} \leq K \leq K_{max}$ with K_{min}, K_{max} constants, then one of the following holds:*

i)

$$i(M) \geq \frac{\pi}{\sqrt{K_{max}}}$$

ii) there exists a closed geodesic γ in M, whose length is less than or equal to the length of any other closed geodesic on M, which is such that

$$i(M) = \frac{1}{2} l(\gamma)$$

Proof. Since M is complete, then M is compact by Hopf-Rinow Theorem and by Lemma 7.2.1, there exists two point $p \in M$ and $q \in C_m(p)$ such that $d(p,q) = i(M)$.

First we suppose that q is conjugate to p, then by Proposition 2.4.9,

$$i(M) = d(p,q) \geq \frac{\pi}{\sqrt{K_{max}}}$$

So condition (i) holds.

Now we suppose that p and q are not conjugate. In this case there exist two minimal geodesics μ and σ joining p and q such that $\mu'(l) = -\sigma'(l)$ where $l = d(p,q)$.

On the other hand, since $q \in C_m(p)$, then $p \in C_m(q)$. Hence p realizes the distance from q to $C_m(q)$ and so $\mu'(0) = -\sigma'(0)$.

Therefore, μ and σ form a closed geodesic γ and

$$i(M) = \frac{1}{2} l(\gamma)$$

that is, condition (ii) holds. □

Now, we are in position to prove the estimate of the injectivity radius.

Proposition 7.2.3. *Let M be a compact, orientable, even dimensional Riemannian manifold. If its sectional curvature K satisfies $0 < K \leq 1$, then*

$$i(M) \geq \pi$$

Proof. By Lemma 7.2.1 there exist two points $p \in M$ and $q \in C_m(p)$ in M such that $d(p,q) = i(M)$.

Working by contradiction we suppose that $d(p,q) < \pi$.

If p and q are conjugate points then by Proposition 2.4.9, $d(p,q) \geq \pi$ and we have a contradiction.

Therefore, we suppose that p and q are not conjugate. There exist two minimal geodesics γ_1 and γ_2 from p to q such that $\gamma_1'(l) = -\gamma_2'(l)$ with $l = d(p,q)$.

Their union gives a closed geodesic γ with $l(\gamma) = 2i(M) < 2\pi$.

We consider the parallel transport $P_\gamma : T_pM \to T_pM$ along γ. P_γ preserves the orientation and the orthogonal complement of $\gamma'(0) = \gamma'(l)$ in T_pM is invariant under the parallel transport. In fact, P_γ can be consider as an element in $SO(n-1)$ with $n-1$ odd and since each matrix in $SO(k)$, k odd, has an invariant eigenvector, P_γ leaves invariant a vector w orthogonal to γ'.

Let F be a variation of γ with variational vector field $X(t) = \frac{\partial F}{\partial s}$ such that $X(0) = X(l) = w$ and $X(t)$ parallel along γ.

Now:

$\|\frac{D}{dt}X\|^2 = 0$ since X is parallel along γ,

$< \frac{D}{ds}\big|_{s=0}\frac{\partial F}{\partial s}(s,l), \gamma'(l) > - < \frac{D}{ds}\big|_{s=0}\frac{\partial F}{\partial s}(s,0), \gamma'(0) >= 0.$

Therefore, by the formula of second variation of energy (2.6) we have

$$E''_w(0) = -\int_0^l < R(X,\gamma')\gamma', X > dt$$

It follows that

$$E''_w(0) < 0$$

In fact

$$0 < K(span\{X(t),\gamma'(t)\}) = \frac{< R(X(t),\gamma'(t))\gamma'(t), X(t) >}{\|X(t)\|^2} \leq 1$$

That is, $< R(X,\gamma')\gamma', X >$ positive, so its integral with the negative sign is negative. Moreover, $E'_w(0) = 0$ (see (2.5)) because γ is a geodesic and $< X(0), \gamma'(0) >=< X(l), \gamma'(l) >$.

Therefore, the closed geodesic γ is a maximum for energy and for length because of (2.1).

Hence it is possible to find a variation through regular closed curves $\gamma_s(t)$ of γ with $s \in [0,\epsilon]$ such that $l(\gamma_s) < l(\gamma) \quad \forall s \neq 0$.

Let q_s be a point at maximum distance from $\gamma_s(0)$.

Since $d(\gamma_s(0), q_s) < d(p,q)$, there exists a unique minimal geodesic σ_s joining $q_s = \sigma_s(0)$ and $\gamma_s(0)$ and $\lim_{s\to 0}\sigma_s(0) = q$ because q is the unique point of γ at maximum distance from p (Fig. 7.6). Moreover, since TM is locally compact, there exists a vector $v \in T_qM$ accumulation point of $\sigma'_s(0)$ for all s.

By continuity, the curve $\sigma_s(t) = \exp_q tv$ joins q and p and it is minimal.

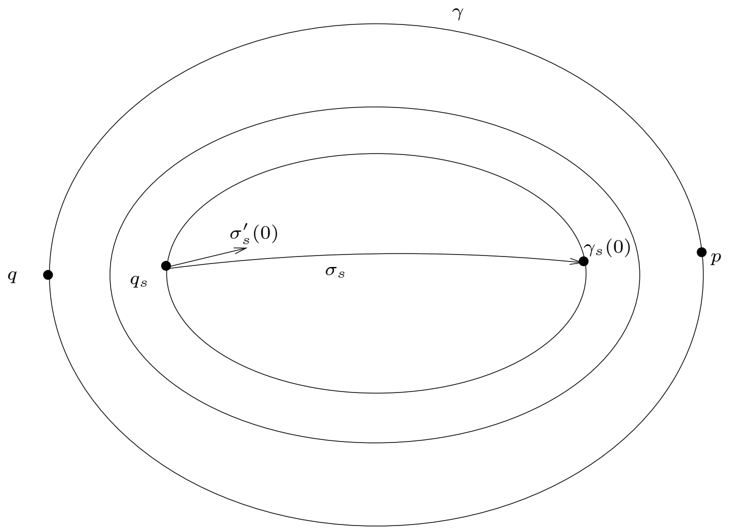

Figure 7.6: the variation σ_s

Now let $\sigma_{s,t}$ be the minimal geodesic joining $\gamma_s(0)$ and $\gamma_s(t)$ close to q_s (Fig. 7.7). It turns that $\sigma_{s,t}(\tau)$ with $t \in (\bar{t}-\epsilon, \bar{t}+\epsilon)$, $\tau \in [0,l]$ where $\bar{t} = \frac{1}{2}l(\gamma_s)$ and $\gamma_s(\bar{t}) = q_s$ is a variation of σ_s.

In fact, fixing $s \in [0,\epsilon]$, $\sigma_{s,\bar{t}}$ joins $\gamma_s(0)$ and $\gamma_s(\bar{t}) = q_s$ (Fig. 7.7). Since both σ_s and $\sigma_{s,\bar{t}}$ are the unique minimal geodesics between $\sigma_s(0)$ and q_s, they must coincide.

We remark that this variation leaves the end point $\gamma_s(0)$ invariant, also its variational vector field is $X(\tau) = \frac{\partial}{\partial t}\big|_{t=\bar{t}}\sigma_{s,t}$. Moreover:

$$X(0) = \frac{\partial}{\partial t}\Big|_{t=\bar{t}}\sigma_{s,t}(0) = \frac{\partial}{\partial t}\Big|_{t=\bar{t}}\gamma_s(t) = \gamma_s'(\bar{t}), \qquad X(l) = \frac{\partial}{\partial t}\Big|_{t=\bar{t}}\sigma_{s,t}(l) = 0.$$

Applying the first formula of variation of energy(2.5) we get

$$\frac{d}{dt}\Big|_{t=\bar{t}}E(\sigma_{s,t}) = -<\gamma_s'(\bar{t}), \sigma_s'(0)>$$

Since $\sigma_{s,t}$ is a variation of geodesics, it minimizes the energy (see Lemma 2.2.2), i.e $\frac{d}{dt}\big|_{t=\bar{t}}E(\sigma_{s,t}) = 0$ and we end up with

$$<\gamma_s'(\bar{t}), \sigma_s'(0)>= 0$$

Hence, for all $s \in [0,\epsilon]$, $\sigma_s'(0)$ is orthogonal to γ_s' in q_s. It follows that $\sigma_w'(0)$ is orthogonal to γ' in q.

Therefore we have three minimal geodesics between p and q and this is a contradiction according to Proposition 7.1.8.

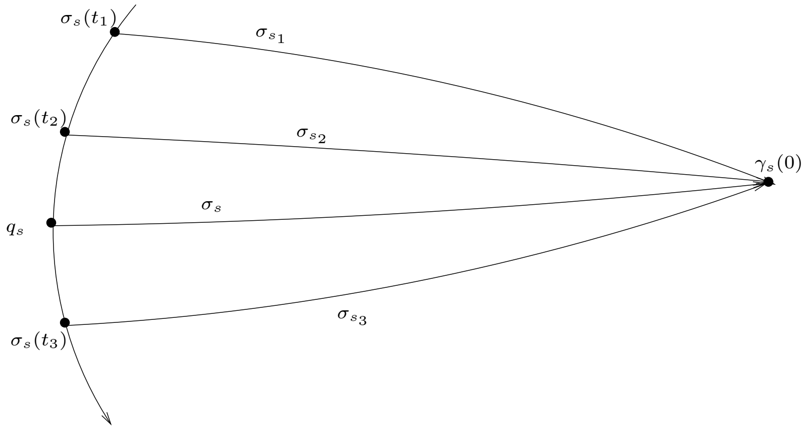

Figure 7.7: the variation $\sigma_{s,t}$

□

7.3 The Sphere Theorem

Keeping in mind the lower bound for the injectivity radius, we can start the proof of the Sphere Theorem. It follows from a geometrical argument based on the fact that the manifold M can be covered by two open balls homeomorphic to the two hemispheres of the sphere.

We begin with some lemmas to show how we can cover a manifold with two metric balls.

Lemma 7.3.1. *Let M be a compact Riemannian manifold and let p and q be points of M such that $d(p,q) = diam(M)$. Then for all $w \in T_pM$ there exists a minimal geodesic γ from $p = \gamma(0)$ to q with $< \gamma'(0), w > \geq 0$.*

Proof. Let $\lambda(t) = \exp_p tw$, i.e. $\lambda'(0) = w$, and let $\gamma_t : [0,l] \to M$ be a minimal geodesic between $\gamma_t(0) = \lambda(t)$ and $\gamma_t(l) = q$.

First we suppose that for all $n \in \mathbb{N}$ there exists a t_n such that

$$< \gamma'_{t_n}(0), \lambda'(t_n) > \geq 0 \qquad \text{for all } 0 \leq t_n \leq \frac{1}{n}$$

Taking a subsequence if necessary, we get that γ_{t_n} converges to a minimal geodesic γ such that

$$0 \leq < \gamma'(0), \lambda'(0) > = < \gamma'(0), w >$$

and the proof would be concluded in this case.

As a second case we suppose that there exists an $n \in \mathbb{N}$ such that

$$< \gamma_t'(0), \lambda'(0) > < 0 \qquad \text{for all } 0 \leq t \leq \frac{1}{n}$$

We work by contradiction using the inequality just stated.

We consider $V \subset T_{\lambda(t)}M$ on which $\exp_{\lambda(t)}$ is a diffeomorphism. We set $U = \exp_{\lambda(t)} V \subset M$ an open neighbourhood of $\lambda(t)$ and we take $q_0 \in U$ a point of γ_t. Then let σ_s be a minimal geodesic from q_0 to $\lambda(s)$, $s \in (t-\epsilon, t+\epsilon)$ and $\epsilon > 0$ small enough (Fig. 7.8).

For $s = t$, $\sigma_t = \gamma_t$. So $\sigma_t' = -\gamma_t'$ at $\lambda(t)$.

Moreover it turns that $\sigma_s : (t-\epsilon, t+\epsilon) \times [0, \bar{t}] \to M$, where $\bar{t}$ is such that $\gamma_t(\bar{t}) = q_0$, is a variation of σ_t which leaves q_0 invariant. Hence, the variational vector field $X(\tau) = \frac{\partial}{\partial s}\big|_{s=t}\sigma_s(\tau)$ is zero for $\tau = 0$.

We have that

$X(\bar{t}) = \frac{\partial}{\partial s}\big|_{s=t}\sigma_s(\bar{t}) = \frac{\partial}{\partial s}\big|_{s=t}\gamma_s(0) = \frac{\partial}{\partial s}\lambda(s) = \lambda'(t),$

$< X(0), \sigma_t'(0) > = 0,$

$\int_0^{\bar{t}} < X(t), \frac{D}{ds}\sigma_s(t) > dt = 0.$

Applying the first formula of variation of length (2.3) we get

$$\frac{d}{ds}\Big|_{s=t} l(\sigma_s) = < X(\bar{t}), \sigma_t'(\bar{t}) > = - < \lambda'(t), \gamma_t(0) > > 0$$

Hence, for all $s < t$ and close to t, we have that $l(\sigma_s) < l(\sigma_t)$. Therefore

$$d(q_0, \lambda(s)) < d(q_0, \lambda(t))$$

and

$$d(q, \lambda(s)) < d(q, q_0) + d(q_0, \lambda(s)) < d(q, q_0) + d(q_0, \lambda(t)) = d(q, \lambda(t))$$

Moreover, since p is at maximum distance form q, we have

$$d(q, \lambda(s)) < d(q, \lambda(t)) \leq d(q, p) = d(q, \lambda(0))$$

Now, t can move towards zero and it can be as close to zero as we want. In this case, it can happen that s takes value zero and we would end up with

$$d(q, \lambda(0)) < d(q, \lambda(0))$$

which is a contradiction.

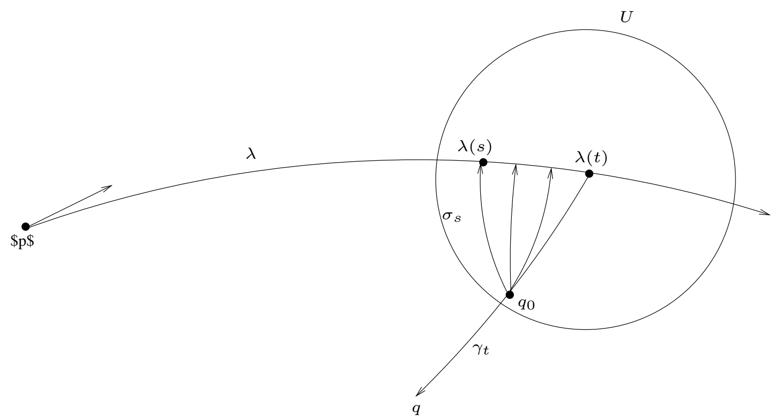

Figure 7.8: situation of Lemma 7.3.1

□

From now on, the dimension n of the manifold is always even.

Lemma 7.3.2. *Let M be a $n-$dimensional, compact, simply connected Riemannian manifold whose sectional curvature K satisfies $\frac{1}{4} < \delta \leq K \leq 1$ and let $p, q \in M$ such that diam$(M) = d(p, q)$. Then*

$$M = B_\rho(p) \bigcup B_\rho(q)$$

where $B_\rho(p) \subset M$ is such that $B_\rho(p) = \exp_p B_\rho(0_p)$ and $\frac{\pi}{2\sqrt{\delta}} < \rho < \pi$

Proof. From Proposition 7.2.3, $i(M) \geq \pi$.

Therefore, for $\frac{\pi}{2\sqrt{\delta}} < \rho < \pi$, $B_\rho(p)$ is diffeomorphic to an Euclidean ball through the exponential map because it does not contain points of the cut locus of p.

We set $B_\rho(p) = \exp_p B_\rho(0_p)$ and the same holds for $B_\rho(q)$.

We work by contradiction and we suppose that there exists a point $x \in M$ which is contained neither in $B_\rho(p)$ nor in $B_\rho(q)$, namely, $d(p,x) \geq \rho$ and $d(q,x) \geq \rho$. In particular we can assume

$$d(p,x) \geq d(q,x) \geq \rho$$

The minimal geodesic between q and x intersects $\partial B_\rho(q)$, the boundary of $B_\rho(q)$, in q' and $q' \notin B_\rho(p)$ because otherwise we would have

$$d(x,q') > d(x, B_\rho(p)) \geq d(x, B_\rho(q)) = d(x,q')$$

which is a contradiction.

Moreover, the minimal geodesic from q to p intersects $\partial B_\rho(q)$ in a point q'' which lies inside $B_\rho(p)$.

In fact, $d(p,q) = diam(M) \leq \frac{\pi}{\sqrt{\delta}} < 2\rho$ by Bonnet-Myers Theorem and we have

$$d(p,q'') = d(p,q) - d(q,q'') < 2\rho - \rho = \rho$$

Now, since $B_\rho(q)$ is homeomorphic to an Euclidean ball, $\partial B_\rho(q)$ is path connected and we can move from $q'' \in B_\rho(p)$ to $q' \notin B_\rho(p)$ along a curve on $\partial B_\rho(q)$. At some point this curve has to hit $\partial B_\rho(p)$, therefore there exists $r_0 \in \partial B_\rho(p) \cap \partial B_\rho(q)$ such that

$$d(r_0,p) = d(r_0,q) = \rho$$

Let λ be a minimal geodesic joining p to r_0.

By Lemma 7.3.1 there exists a minimal geodesic γ from p to q such that $< \gamma'(0), \lambda'(0) > \geq 0$.

Let s be a point of γ such that $d(p,s) = \rho$ (see Fig 7.3).

We compare M with the $n-$sphere S^n of constant curvature δ.

We observe that the angle $\sphericalangle r_0 p s$ is less than or equal to $\frac{\pi}{2}$.

Since $d(p,s) = d(p,r_0) = \rho$, by Proposition 2.4.10 we can conclude

$$d(r_0,s) \leq \frac{\pi}{2\sqrt{\delta}} < \rho$$

Furthermore, there exists a point s_0 in the interior of $B_\rho(p)$ such that $d(r_0,\gamma) = d(r_0,s_0)$ because $d(r_0,p) = d(r_0,q) = \rho$ and $d(r_0,s) < \rho$ with $s \in \gamma$ (see Fig. 7.3).

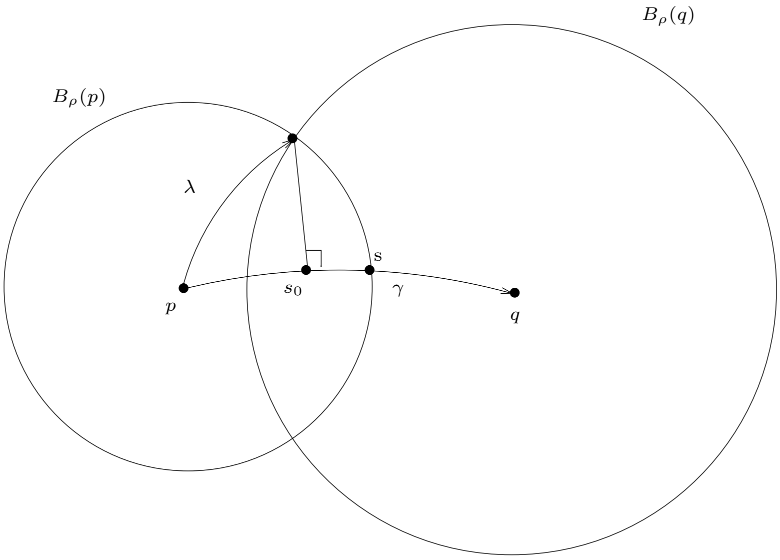

Figure 7.9: spherical triangle between r_0, p and s

The minimal geodesic from r_0 to s_0 is orthogonal to γ, hence

$$d(r_0,s_0) \leq \frac{\pi}{2\sqrt{\delta}}$$

Since $d(p,q) \leq \frac{\pi}{\sqrt{\delta}}$, then

$$\text{either} \qquad d(p,s_0) \leq \frac{\pi}{2\sqrt{\delta}} \qquad \text{or} \qquad d(q,s_0) \leq \frac{\pi}{2\sqrt{\delta}}$$

In the first case we have

$$d(r_0,s_0) \leq \frac{\pi}{2\sqrt{\delta}} \qquad \text{and} \qquad \sphericalangle p s_0 r_0 = \frac{\pi}{2}$$

Then, again by Proposition 2.4.10, we conclude that

$$d(p,r_0) \leq \frac{\pi}{2\sqrt{\delta}} < \rho$$

which is a contradiction since $d(p, r_0) = \rho$.

The second case leads to contradiction using the same argument. □

Lemma 7.3.3. *Let M be a compact, simply connected Riemannian manifold of dimension n with sectional curvature K such that $\frac{1}{4} < \delta \leq K \leq 1$. Let $p, q \in M$ be two points such that $diam(M) = d(p, q)$ and let $M = B_\rho(p) \bigcup B_\rho(q)$ where $B_\rho(p) \subset M$ with $\frac{\pi}{2\sqrt{\delta}} < \rho < \pi$. Then, on each geodesic of length ρ starting from p there exists a unique point m such that*

$$d(p, m) = d(q, m) < \rho$$

Similarly, on each geodesic starting from q there exists a unique point n equidistant from p and q.

Proof. Let $\gamma(s)$ be a geodesic such that $\gamma(0) = p$ and we consider the function

$$f(s) = d(q, \gamma(s)) - d(p, \gamma(s))$$

Since d is continuous, f is continuous and we observe that $f(0) = d(p, q) > 0$.

Let $s_0 \in M$ be a point such that $\gamma(s_0)$ is the cut point of p along γ. By Proposition 7.2.3 we have $i(M) \geq \pi$, hence

$$d(p, \gamma(s_0)) \geq \pi > \rho$$

Consequently, since M is covered by two balls of radius ρ we have

$$d(q, \gamma(s_0)) < \rho$$

Then:

$$f(s_0) = d(q, \gamma(s_0)) - d(p, \gamma(s_0)) < \rho - \rho = 0$$

Therefore there exists a point $s_1 \in (0, s_0)$ such that $f(s_1) = 0$, that is, $d(q, \gamma(s_1)) = d(p, \gamma(s_1))$.

From here it follows the existence of a point m setting $m = \gamma(s_1)$.

To prove the uniqueness we work by contradiction and we take two points $m_1 \neq m_2$ on γ both equidistant from p and q. We can assume that m_1 is between p and m_2.

We have:

$$d(q, m_2) = d(p, m_2) = d(p, m_1) + d(m_1, m_2) = d(q, m_1) + d(m_1, m_2)$$

Let σ_1 be the unique (we are inside the injectivity radius) minimal geodesic between q and m_1.

By the equality above σ_1 coincides with γ, hence q belongs to γ.

Since $d(p, m_1) = d(q, m_1)$, $d(p, m_2) = d(q, m_2)$ and $m_1 \neq m_2$ we conclude that $p = q$, which is a contradiction.

The case for the point q follows in the same way. □

Remark 7.3.4. Since the point m is unique on each geodesic γ with $\gamma(0) = p$, it depends continuously on the initial direction $\gamma'(0)$.

These lemmas are the final ingredients to present the proof of the Sphere Theorem which follows easily by a direct construction of the homeomorphism.

Theorem 7.3.5 (Sphere Theorem)**.** *Let M be a compact, simply connected Riemannian manifold of even dimension such that its sectional curvature K satisfies $0 < \frac{1}{4} < \delta \leq K \leq 1$, then M is homeomorphic to a sphere.*

Proof. Let $p, q \in M$ be two points such that $diam(M) = d(p, q)$.

Let D_1 and D_2 be two subsets of M formed by all geodesic segments $\overline{pm}$ and $\overline{qn}$ with m and n points as in Lemma 7.3.3.

By Remark 7.3.4 and the compactness of M, we conclude that D_1 and D_2 are closed subsets.

We claim that

$$M = D_1 \cup D_2 \tag{7.2}$$

and

$$\partial D_1 = \partial D_2 = D_1 \cap D_2 \tag{7.3}$$

We start showing (7.2).

Obviously $D_1 \cup D_2 \subset M$, then we just need to show the other inclusion.

Let $r \in M$, as shown in the proof of Lemma 7.3.2, either $d(p, r) < \rho$ or $d(q, r) < \rho$.

We consider the first case, the second being analogous.

Since $d(p, C_m(p)) \geq i(M) \geq \pi > \rho$ there exists a unique minimal geodesic γ between p and r and, by Lemma 7.3.3, there exists a unique point m along γ such that $d(p, m) = d(q, m) < \rho$.

If $d(p, r) < d(q, r)$, then $r \in \overline{pm}$, hence $r \in D_1$.

If $d(p, r) = d(q, r)$, by the uniqueness of m, $r = m$ and so $r \in \partial D_1$.

If $d(p, r) > d(q, r)$ then $r \in \overline{qn}$ where n is the unique point along γ minimal geodesic between q and r equidistant from p and q, hence $r \in D_2$. Therefore if $r \in M$ then $r \in D_1 \cup D_2$, i.e $M \subset D_1 \cup D_2$ and so (7.2) holds.

Now we prove (7.3).

If $r \in \partial D_1 \subset D_1$ then $d(p, r) = d(q, r)$ and so, $r = m = n$ using the previous argument. Therefore $r \in \partial D_2 \subset D_2$ and $\partial D_1 \subset \partial D_2$. In particular $r \in D_1 \cap D_2$.

Swapping D_1 and D_2 we work in the same way and we have that $\partial D_2 \subset \partial D_1$. Hence $r \in \partial D_1 = \partial D_2 \subset D_1 \cap D_2$.

Now if $r \in D_1 \cap D_2$, with the same argument, $r = m = n$ and $r \in \partial D_1 = \partial D_2$, therefore (7.3) holds.

We are now ready to define the homeomorphism between S^n and M.

Let $\varphi : S^n \to M$.

Let $N, S \in S^n$ be the north pole and the south pole of the sphere, respectively.

We set $\varphi(N) = p$ and $\varphi(S) = q$.

Then we choose a linear isometry $I : T_N S^n \to T_p M^n$.

For each point e in the equator E relative to N we consider the geodesic $\gamma(s)$, $0 \leq s \leq \pi$, given by $\gamma(0) = N$ and $\gamma(\frac{\pi}{2}) = e$.

Let $m \in M$ be a point as in Lemma 7.3.3 on the geodesic starting from p with initial tangent vector $I(\gamma'(0)) = w$.

We define:

$$\begin{cases} \varphi(\gamma(s)) = \exp_p\left(s\frac{2}{\pi}d(p, m)(w)\right) & 0 < s \leq \frac{\pi}{2} \\ \varphi(\gamma(s)) = \exp_q\left((2 - \frac{2s}{\pi})d(q, m)v\right) & \frac{\pi}{2} \leq s < \pi \end{cases}$$

where v is the unit tangent vector at q of the unique minimal geodesic from q to m (see Fig. 7.10).

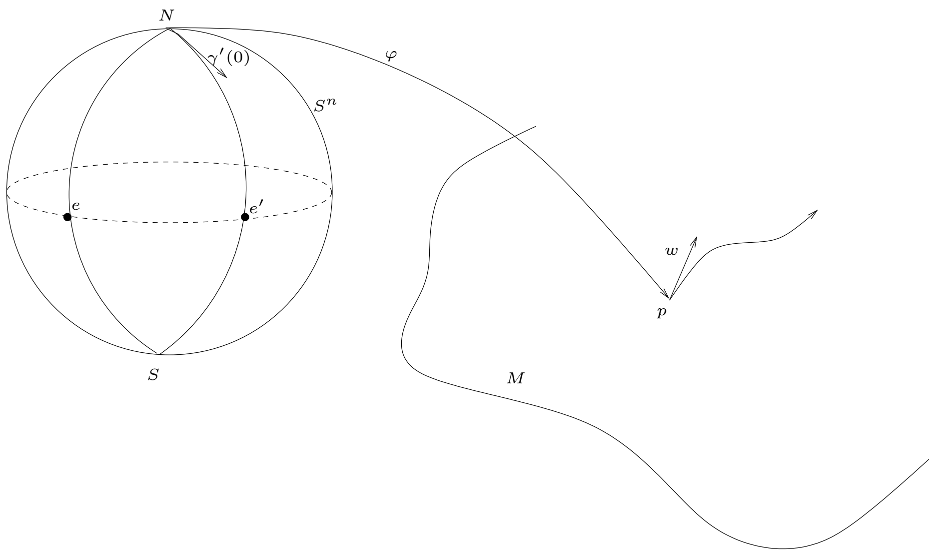

Figure 7.10: homeomorphism between M and S^n

With this definition it is easy to see that the closed northern hemisphere, the closed southern hemisphere and the equator E are mapped bijectively onto D_1, D_2 and $\partial D_1 = \partial D_2 = D_1 \cap D_2$ respectively.

Now by uniqueness of the points m and n as in Lemma 7.3.3, φ is continuous.

Further, φ is surjective because $M = D_1 \cup D_2$ and it is injective because $D_1 \cap D_2 = \partial D_1 = \partial D_2 = \varphi(E)$. Therefore φ is a bijection.

Finally, since φ is a continuous bijection between a compact and an Hausdorff space, it is a homeomorphism. $\square$

We want to remark that the hypothesis on the curvature is fundamental and it can not be weakened.

In fact, as soon as $\delta = \frac{1}{4}$ the theorem is no longer valid.

This is because the key point of the argument is Lemma 7.3.2 where we suppose $\frac{\pi}{2\sqrt{\delta}} < \rho < \pi$. Clearly, if $\delta = \frac{1}{4}$ we have a contradiction since we would end up with $\pi < \rho < \pi$.

Therefore, the real question is why we need such a bound for ρ.

The upper bound is due to the estimate of the injectivity radius $i(M) \geq \pi$. In fact, to make all the proofs work, we need to be strictly inside the injectivity radius to consider the exponential map as a diffeomorphism and have the existence and uniqueness of minimal geodesics between two points, which is the basic assumption of every proof.

The lower bound is due to the Bonnet-Myers Theorem. In fact, since M is compact, $\operatorname{diam} M \leq \frac{\pi}{\sqrt{\delta}}$. Moreover we must have $\operatorname{diam} M < \operatorname{diam} B_\rho(p) + \operatorname{diam} B_\rho(q) = 2\rho$, where the inequality is strict because metric balls are open in M and to cover it completely they have to overlap a little bit. Therefore it is required $\rho > \frac{\pi}{2\sqrt{\delta}}$.

As a concrete example we discuss the case of $\mathbb{P}^n(\mathbb{C})$.

It is compact because it is the quotient space of S^n which is compact.

It is also simply connected, in fact it is path-connected (it comes from being the quotient space of S^n) and its fundamental group is zero. The calculation of the fundamental group is done using the long exact sequence generated by the Hopf fibration $f : S^n \longrightarrow \mathbb{P}^n(\mathbb{C})$ according to [Hat, Theorem 4.41, p. 376] and taking into account the homotopy groups of the sphere. It seems that $\mathbb{P}^n(\mathbb{C})$ is a suitable space to apply the Sphere Theorem but its sectional curvature is $\frac{1}{4} \leq K < 1$ (see Section 2.1.1) and the theorem is no longer valid.

This statement can be confirmed looking at their homotopy groups (see [Hat, Corollary 2.14, p. 114] and [Hat, p.140])

$$H_k(S^n) = \begin{cases} \mathbb{Z} & k = n \\ 0 & \text{otherwise} \end{cases} \qquad\qquad H_k(\mathbb{P}^n(\mathbb{C})) = \begin{cases} \mathbb{Z} & k \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

If there were a homeomorphism between the two spaces it would induce an isomorphism between the homology groups in each dimension which is not possible in this case.

Conclusion and further developments

In this work we analysed how the assumption of positive Ricci curvature of a Riemannian manifold (M, g) leads to a comparison of geometric/spectral invariants with the round sphere in the same dimension.

The dissertation is mostly self-contained. We like to mention that some of the results discussed are restrictions of more general arguments. For example the comparison theorem in Section 6.2.2 can be extended to arbitrary dimension and the hypothesis on the curvature can be modified to obtain related results. For this general results we address the reader to the original articles quoted in the bibliography. We also like to mention that the estimate of the isoperimetric profile has implications on the first eigenvalue of the Laplacian since Lichnerowicz' Theorem is obtained as a corollary and improved in the case M not isometric to the sphere (see [Ber-Bes-Gal]). This fact supports the statement that the first eigenvalue of the Laplacian contains information about the geometry of the manifold.

The Sphere Theorem and its original argument open up a variety of research fields.

First of all, we can consider the case when the sectional curvature K is such that $\frac{1}{4} \leq K \leq 1$ instead of a strict lower bound. In this case Berger proved that the manifold is either homeomorphic to S^n or isometric to a symmetric space ([Br, Thereom 1.1, p. 6]). One can also look at manifolds with sectional curvature $K \geq 1$, in which case Grove and Shiohama proved that, under the additional hypothesis of dimension bigger than or equal to 4 and diameter strictly bigger than $\frac{\pi}{2}$, the manifold is homeomorphich to S^n ([Br, Theorem 1.15, p. 8]). Other authors consider assumptions on the Riemannian curvature tensor rather than on the curvature itself.

However, the most interesting question is when one can substitute the property 'homeomorphic' with 'diffeomorphic'. The natural question is: 'Are there differentiable manifolds that are homeomorphic but not diffeomorphic?'. The answer is 'yes, the exotic sphere'.

This question is known as the 'Differentiable Sphere Theorem' and the most striking result is by S. Brendle and R. Shoen in 2007 when they proved the following.

Theorem 1 [Br, Theorem 1.23, p. 14]. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 4$. Suppose that $R(\xi, \eta, \tilde{\xi}, \tilde{\eta}) > 0$ for all $p \in M$ and for all linearly independent vectors in $T_p^{\mathbb{C}}M$ such that $g(\xi, \xi)g(\eta, \eta) - g(\xi, \eta)^2 = 0$. Then M is diffeomorphic to a spherical space form.*

A space form is a complete Riemannian manifold with constant sectional curvature. If it is spherical, the sectional curvature is 1.

The consequence is:

Corollary 1 [Br, Corollary 1.24, p. 14]. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 4$ and strictly $\frac{1}{4}-$pinched in the pointwise sense. Then M is diffeomorphich to spherical space form.*

Here, a strictly $\alpha-$pinched manifold in the pointwise sense is a manifold such that for all $p \in M$ the sectional curvatures of any $2-$planes $\Sigma_1, \Sigma_2 \in T_pM$ satisfies $0 < \alpha K(\Sigma_1) < K(\Sigma_2)$.

Weakening the condition on the curvature assumption they obtained a rigidity result.

Theorem 2 [Br, Theorem 1.25, p. 14]. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 4$ and weakly $\frac{1}{4}-$pinched in the pointwise sense. Then M is either diffeomorphich to spherical space form or isometric* *to a locally symmetric space.*

Again, a weakly $\alpha-$pinched manifold in the pointwise sense is a manifold such that for all $p \in M$ the sectional curvatures of any $2-$planes $\Sigma_1, \Sigma_2 \in T_pM$ satisfies $0 \leq \alpha K(\Sigma_1) <\leq (\Sigma_2)$.

The main technique in the proofs of these theorems is the Ricci flow. This flow was first introduced by Hamilton in the 80s with the aim of finding canonical metrics. It connects PDEs with dynamical systems and it describes how a one parameter family of Riemannian metrics $g(t)$, with $t \in [0, T)$ on M, evolves with the help of the Ricci tensor. Namely, $g(t)$ is a solution to the Ricci flow if it satisfies $\frac{\partial}{\partial t} g(t) = -2 \operatorname{Ric}_{g(t)}$.

If M is an Eistein manifold (i.e. it is has constant Ricci tensor), then the geometry of M does not change except for rescaling.

Since its introduction, it has been used with great success, for example in Perelman's solution of the Poincarè conjecture, and various convergence theorems have been established.

These results can also be used in the proof of the Differentiable Sphere Theorem.

The first achievement was made by Hamilton when he proved that a compact three-manifold with positive Ricci curvature is diffeomorphic to a spherical space form (see [Br, Theorem 1.22, p. 13]) and a more general result in dimension bigger than or equal to 3 with some extra conditions. (see [Br, Theorem 5.23, p. 65]).

The proof of Theorem 1 consists of two parts: first we need to show the existence of a suitable set of algebraic tensors, then we use the results of Hamilton. Corollary 1 is obtained showing that the hypothesis of Theorem 1 are satisfied when the manifold is strictly $\frac{1}{4}-$pinched ([Br, Proposition 8.13 and Corollary 8.14, p. 116ff]).

Instead, the rigidity result follows from the classification of weakly $\frac{1}{4}-$pinched manifolds, derived from different convergence results (see [Br, Section 9.8, p. 149ff]).

As one can see, the Ricci flow technique is a very powerful tool to approach deep open questions in Riemannian Geometry. Of course, what we explain it is just a hint of how they work and can be used and we did not mention many important facts like the existence and uniqueness of the solution or the evolution of the connections and tensors under the Ricci flow. Also, we did not mention what is behind the construction of the set of algebraic tensors. Nevertheless, we hope the reader may be inspired by these hints and we recommend the surveys [Br] and [Br-Sh] for further investigations in this direction.

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